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Sub-exponential decay and uniform holomorphic extensions for semilinear pseudodifferential equations

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Abstract

The goal of the present paper is to derive a simultaneous description of the decay and the regularity properties for elliptic equations in $\mathbb{R}^n$ with coefficients admitting irregular decay at infinity of the type $O(|x|^{-\sigma}), \sigma > 0$, filling the gap between the case of Cordes globally elliptic operators and the case of regular/Fuchs behaviour at infinity. Representative examples in $\mathbb{R}^n$ are the equations

$$-\Delta u + \frac{\omega(x)}{\langle x\rangle^\sigma} u = f + F[u], \qquad x \in \mathbb{R}^n,$$

where $0 < \sigma < 2, \langle x\rangle = (1+|x|^2)^{1/2}, \omega(x)$ a bounded smooth function, f given and $F[u]$ a polynomial in u, and similar Schrödinger equations at the endpoint of the spectrum. Other relevant examples are given by linear and nonlinear ordinary differential equations with irregular type of singularity for $x \to \infty$, admitting solutions $y(x)$ with holomorphic extension in a strip and sub-exponential decay of type $|y(x)| \leq Ce^{-\varepsilon|x|^r}, 0 < r < 1$. Sobolev estimates for the linear case are proved in the frame of a suitable pseudodifferential calculus; decay and uniform holomorphic extensions are then obtained in terms of Gelfand-Shilov spaces by an inductive technique. The same technique allows to extend the results to the semilinear case.

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1 Introduction

The main goal of the present paper is to study global regularity and decay at infinity for linear (pseudo)-differential equations in $\mathbb{R}^n$

$$P(x,D)u = f(x), \qquad x \in \mathbb{R}^n \tag{1.1}$$

and for semilinear perturbations

$$P(x,D)u = f(x) + F[u], \qquad x \in \mathbb{R}^n, \tag{1.2}$$

where the nonlinear term $F[u]$ is typically a polynomial of u and the source term f belongs to some functional space of smooth or analytic-Gevrey functions having sub-exponential decay at infinity. The main novelty (and difficulty) is related to the fact that we consider globally in $\mathbb{R}^n$ operators which are locally elliptic, but with coefficients admitting "irregular" type of singularity for $|x| \to \infty$.
As a motivating model operator we exhibit

$$P \quad = \quad A_m(D) + \frac{\omega(x)}{\langle x \rangle^\sigma} \tag{1.3}$$

where $A_m(D)$ is an elliptic homogeneous linear partial differential operator with constant coefficients and real valued symbol $A_m(\xi)$ of order $m \in \mathbb{N}$ and $\omega \in C^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{x \in \mathbb{R}^n} \left(\langle x \rangle^{|\alpha|} |\partial_x^\alpha \omega(x)| \right) =: A_\alpha < +\infty, \qquad \alpha \in \mathbb{Z}_+^n. \tag{1.4}$$

In (1.3), (1.4) we denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. In particular, if we take $m = 2$ and $A_2(D) = -\Delta$, we have

$$P \quad = \quad -\Delta + \frac{\omega(x)}{\langle x \rangle^\sigma}. \tag{1.5}$$

Our attention in this paper will be fixed on the case of the irregular type singularity $0 < \sigma < m$ in (1.3), that is $0 < \sigma < 2$ in (1.5). In fact, thinking of the one dimensional case, the assumption $\sigma > m$ implies regularity at infinity for the ordinary differential operator P, whereas $\sigma = m$ corresponds to the classical Fuchs condition at infinity. As a counterpart in $\mathbb{R}^n, n > 1$, we address to Lockhart and McOwen [22], [23], [24] and references there for Fuchs-type operators. In particular, in [22], [23], [24] the authors carried out a comprehensive analysis of elliptic operators in $\mathbb{R}^n$ under the two assumptions (for (1.3)) $\sigma = m$ and $\lim_{|x| \to \infty} \omega(x) = 0$. The case $\sigma \leq 0$, also corresponding to irregular-type singularity at infinity in the language of the ordinary differential operators, has been extensively studied in literature. Namely, if $\sigma < 0$ we read in (1.5) a potential with algebraic growth at infinity and P is then included in the theory of Shubin [33] and related generalizations, see [3]. For $\sigma = 0$, Cordes [13] (see also Parenti [26] and Schrohe [31]) developed a complete theory on the so called md-elliptic (or SG-elliptic) pseudodifferential operators in $\mathbb{R}^n$, in the framework of the L^2-based weighted Sobolev spaces $H_{s_1,s_2}(\mathbb{R}^n)$ with norm $\|\langle x \rangle^{s_2} \langle D \rangle^{s_1} u\|_{L^2}$. Note that SG-ellipticity in (1.3) reads as

$$\sigma = 0 \quad \text{ and } \quad |A_m(\xi) + \omega(x)| \geq C \langle \xi \rangle^m \tag{1.6}$$

for $C > 0$ and large $|x| + |\xi|$, satisfied by (1.3) if $\sigma = 0$ and

$$A_m(\xi) > 0 \quad \textit{for} \quad \xi \neq 0, \qquad \Re\omega(x) \geq C' > 0 \qquad \textit{for} \quad |x| \geq R' > 0 \tag{1.7}$$

or else

$$A_m(\xi) \in \mathbb{R} \quad \textit{for} \quad \xi \in \mathbb{R}^n, \qquad |\Im\omega(x)| \geq C' > 0 \qquad \textit{for} \quad |x| \geq R' > 0 \tag{1.8}$$

for some positive C', R'.
Somewhat surprisingly such a natural issue - the complementary case

$$\sigma \in]0, m[\tag{1.9}$$

seems to be (as far as we know) not investigated in detail.

Our program is, in short, the following. First, we want to embed (1.3) under assumptions (1.4), (1.7), (1.8) into a pseudodifferential calculus, and to derive Sobolev estimates for the corresponding operators. This turns out to be a generalization of Cordes [13] and, on the other hand, a particularization of the Weyl-Hörmander calculus, cf. [20]. Then, we look for holomorphic extensions and decay properties of solutions of the hypoelliptic equations. In fact, taking (1.3) as model in the one-dimensional case, i.e. $A_m(D) = D^m$, cf. (1.23) below, we expect a decay as $e^{-\varepsilon|x|^{1-\sigma/m}}$, $\varepsilon > 0$, that is sub-exponential decay, in the sense that $0 < 1 - \sigma/m < 1$ by (1.9). This analysis will be the core of our paper; it will be performed in the language of the Gelfand-Shilov spaces, see below. Finally we shall extend the previous results to the semilinear case (1.2).

Let us state our main results. First we recall the basic notions about the functional frame. The Gelfand–Shilov spaces $S^\mu_\nu(\mathbb{R}^n)$, $\mu > 0$, $\nu > 0$, $\mu + \nu \geq 1$, are defined as the set of all $f \in C^\infty(\mathbb{R}^n)$ satisfying the following estimates: there exist positive constants C, ε such that

$$|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^\mu e^{-\varepsilon|x|^{1/\nu}}, \qquad x \in \mathbb{R}^n, \tag{1.10}$$

cf. the book of Gelfand and Shilov [18] (see also Mityagin [25], Pilipovic [27]). We notice that for $\mu = 1$, functions from $S^\mu_\nu(\mathbb{R}^n)$ are real analytic and admit a holomorphic extension in a strip of the form $\{z \in \mathbb{C} : |\Im z| < T\}, T > 0$. We also remind that the Fourier transformation $\mathcal{F}$ acts as an isomorphism

$$\mathcal{F} : S^\mu_\nu(\mathbb{R}^n) \longrightarrow S^\nu_\mu(\mathbb{R}^n). \tag{1.11}$$

Gelfand-Shilov spaces were already used by the authors in [8], [10] for semilinear Shubin equations, i.e. $\sigma < 0$ in (1.3), (1.5), giving for the solutions estimates of the form (1.10) with $\mu \geq 1/2, \nu \geq 1/2$, and in [9] for semilinear SG-elliptic equations, i.e. $\sigma = 0$ in (1.3),(1.5); in this case exponential decay of the type $e^{-\varepsilon|x|}, \varepsilon > 0$, was proved.

To state our results in full generality, let us refer to the following class of pseudodifferential operators.
Given $\overline{m} = (m_1, m_2) \in \mathbb{R}^2, \delta \in [0, 1[$, we denote by $\Gamma^{\overline{m},\delta} = \Gamma^{\overline{m},\delta}(\mathbb{R}^n)$ the space of all functions $p(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m_1 - |\alpha|} \langle x\rangle^{m_2 - |\beta| + \delta|\alpha|} \tag{1.12}$$

for all $(x, \xi) \in \mathbb{R}^n, \alpha, \beta \in \mathbb{Z}^n_+$ and for some positive constant $C_{\alpha\beta}$. We shall also denote by $OP\Gamma^{\overline{m},\delta}$ the class of pseudodifferential operator $P = p(x, D)$ defined by a symbol $p \in \Gamma^{\overline{m},\delta}$.

We introduce fundamental hypotheses which turn out to be crucial for the global hypoellipticity in the weighted Sobolev spaces $H_{s_1,s_2}(\mathbb{R}^n)$: there exist $\overline{m}' = (m_1', m_2')$ with $m_1' \leq m_1, m_2' \leq m_2$ and $R > 0$ such that

$$\inf_{|x|+|\xi|\geq R}(\langle\xi\rangle^{-m_1'}\langle x\rangle^{-m_2'}|p(x,\xi)|) =: C_1 > 0 \tag{1.13}$$

and for every $\alpha, \beta \in \mathbb{Z}_+^n$ one cand find $C'_{\alpha\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C'_{\alpha\beta}|p(x,\xi)|\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|+\delta|\alpha|} \tag{1.14}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$ and for all $(x,\xi) \in \mathbb{R}^{2n}$ with $|x| + |\xi| \geq R$.

Notice that if $\delta = 0$, then $\Gamma^{\overline{m},0}$ coincides with the class of SG pseudodifferential operators studied in [13], [26], [31], [32], and if we assume further $m_1' = m_1, m_2' = m_2$ in (1.19), the symbol p is SG-elliptic (or md-elliptic).
The metric $\langle x\rangle^{-2}|dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2\langle x\rangle^{2\delta}, 0 \leq \delta < 1$, is an admissible metric for the Weyl-Hörmander calculus in [20] and we may regard the preceding pseudodifferential operators in this frame. For globally hypoelliptic operators we have then easily the following result, see also [6] for details.

Theorem 1.1. *Let $P = p(x,D)$ with $p \in \Gamma^{\overline{m},\delta}$ satisfying* (1.13), (1.14)*. Then the operator P admits a parametrix $E \in OP\Gamma^{-\overline{m}',\delta}$ satisfying*

$$E \circ P = I + R_1, \qquad P \circ E = I + R_2, \tag{1.15}$$

with R_j, $j = 1,2$, being $\mathcal{S}$-regularizing, i.e.,

$$R_j : \mathcal{S}'(\mathbb{R}^n) \mapsto \mathcal{S}(\mathbb{R}^n), \qquad j = 1,2, \tag{1.16}$$

and

$$E : H_{s_1,s_2}(\mathbb{R}^n) \mapsto H_{s_1+m_1',s_2+m_2'}(\mathbb{R}^n), \tag{1.17}$$

for all $s_1, s_2 \in \mathbb{R}$. Hence, $Pu = f \in \mathcal{S}(\mathbb{R}^n), u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{S}(\mathbb{R}^n)$. The operator P is Fredholm in $\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)$, cf. [33], Definition 2.54. In particular, the solutions $u \in \mathcal{S}'(\mathbb{R}^n)$ of $Pu = 0$ are a finite dimensional subspace of $\mathcal{S}(\mathbb{R}^n)$.

The information about decay and regularity given by the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in Theorem 1.1 is not sharp for the equation $Pu = 0$. Namely our purpose is to identify sub-exponential decay and analytic regularity of the solutions in the framework of Gelfand-Shilov spaces under suitable additional assumptions on the regularity of the symbol of P. Let us then introduce a Gevrey-analytic variant of the class $\Gamma^{\overline{m},\delta}$ defined above. We shall limit to consider the case $\overline{m} = (m,0)$ for a given $m \geq 1$.
Let then $m \geq 1, \delta \in [0,1[, \mu \geq 1$. We denote by $\Gamma_\mu^{m,\delta} = \Gamma_\mu^{m,\delta}(\mathbb{R}^n)$ the class of all symbols $p \in C^\infty(\mathbb{R}^{2n})$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C^{|\alpha|+|\beta|+1}\alpha!(\beta!)^\mu\langle\xi\rangle^{m-|\alpha|}\langle x\rangle^{-|\beta|+\delta|\alpha|} \tag{1.18}$$

for some constant $C > 0$ independent of $\alpha, \beta \in \mathbb{Z}_+^n$ and by $OP\Gamma_\mu^{m,\delta}$ the class of pseudodifferential operators with symbol in $\Gamma_\mu^{m,\delta}$. Simplifying further and approaching the notation in the model (1.3), we assume (1.13) is satisfied with $\overline{m}' = (m, -\sigma)$ for some $\sigma \geq 0$, namely

$$\inf_{|x|+|\xi|\geq R} (\langle \xi \rangle^{-m} \langle x \rangle^{\sigma} |p(x,\xi)|) =: C_1 > 0. \tag{1.19}$$

Moreover we shall assume the following variant of the condition (1.14): there exist $C_2, R > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_2^{|\alpha|+|\beta|+1} \alpha! (\beta!)^\mu |p(x,\xi)| \langle \xi \rangle^{-|\alpha|} \langle x \rangle^{-|\beta|+\delta|\alpha|} \tag{1.20}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$ and for $|x| + |\xi| \geq R$.
We have the following result.

Theorem 1.2. *Let $\mu \geq 1, \nu \geq 1$ and let $f \in S_\nu^\mu(\mathbb{R}^n)$. Let P be a pseudodifferential operator with symbol $p \in \Gamma_\mu^{m,\delta}$ satisfying* (1.19), (1.20). *Then, if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of the linear equation* (1.1), *then $u \in S_{\nu'}^\mu(\mathbb{R}^n)$, where $\nu' = \max\{\nu, \frac{1}{1-\delta}\}$. In particular, every solution $u \in \mathcal{S}'(\mathbb{R}^n)$ of the equation $Pu = 0$ satisfies the following estimate*

$$|\partial_x^\alpha u(x)| \leq C^{|\alpha|+1} (\alpha!)^\mu e^{-\varepsilon |x|^{1-\delta}}$$

for all $x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n$ and for some positive constants C, ε independent of α.

Example 1. Note that under the assumptions (1.4), (1.7), (1.9), the symbol $p(x,\xi)$ of the operator P in (1.3) satisfies the conditions (1.14), (1.19). In fact we have the following estimates

$$|p(x,\xi)| = \left| A_m(\xi) + \omega(x)\langle x \rangle^{-\sigma} \right| \geq C \langle \xi \rangle^m \langle x \rangle^{-\sigma} \quad \textit{for} \quad |x| + |\xi| \quad \textit{large.}$$

Moreover, it is easy to see that the derivatives of p with respect to x satisfy (1.14) for $\delta = 0$. Nevertheless, ξ-derivatives require $\delta > 0$. Limit for simplicity attention to the expected estimate

$$\left| \partial_{\xi_j}^m p(x,\xi) \right| = \textit{const} \leq C \left| A_m(\xi) + \omega(x)\langle x \rangle^{-\sigma} \right| \langle \xi \rangle^{-m} \langle x \rangle^{m\delta} \quad \textit{for} \quad |x| + |\xi| \quad \textit{large}$$

which is satisfied if and only if $\delta \geq \sigma/m \in]0,1[$. Hence Theorem 1.1 applies to P in (1.3), (1.5). Similarly, if ω satisfies (1.4) for $A_\alpha = C^{|\alpha|+1}(\alpha!)^\mu$, then $p \in \Gamma_\mu^{m,\sigma/m}$ and the condition (1.20) is fulfilled. Then Theorem 1.2 gives for the solutions $u(x) \in \mathcal{S}'(\mathbb{R}^n)$ of

$$Pu = A_m(D)u + \frac{\omega(x)}{\langle x \rangle^\sigma} u = 0$$

the regularity $u \in S_{m/(m-\sigma)}^\mu(\mathbb{R}^n)$, that is a sub-exponential decay $u(x) \sim e^{-\varepsilon|x|^{1-\sigma/m}}$ and uniform Gevrey regularity of order μ. The pointwise decay rate is sharp (see below). We note that if $\delta = \sigma = 0$ the theorem above reduces to the known statements for SG elliptic operators, cf. [9], Thm. 7.13. Finally, if $\mu = 1$, then u admits a holomorphic extension in a strip of the form $\{z \in \mathbb{C} : |\Im z| < T\}$ for some $T > 0$. Consider in particular the equation

$$-\Delta u + \frac{\omega(x)}{\langle x \rangle^\sigma} u = 0 \tag{1.21}$$

with $0 < \sigma < 2$, $\omega(x)$ satisfying (1.4) of the form $\omega(x) = 1 + \omega_o(x)$, with $\lim_{|x|\to\infty} \omega_o(x) = 0$. For $\omega_o(x) \equiv 0$ solutions do not exist because of the positivity of the operator. Taking for instance $\omega_o(x) = (1 - n + \sigma/2)\langle x\rangle^{\frac{\sigma}{2}-1} - (\sigma/2+1)\langle x\rangle^{-3+\frac{\sigma}{2}} - \langle x\rangle^{-2}$, we may easily verify that $u(x) = e^{-\frac{\langle x\rangle^{1-\sigma/2}}{1-\sigma/2}} \in S^1_{2/(2-\sigma)}(\mathbb{R}^n)$ is a solution of (1.21).

Next, we deal with semilinear perturbations. We suppose that the nonlinear term is of the form

$$F[u] = \sum_{j=\ell}^{d} F_j u^j, \qquad F_j \in \mathbb{C}, \tag{1.22}$$

for some integers $d \geq \ell \geq 2$.

We have the following result.

Theorem 1.3. *Let $\mu \geq 1, \nu \geq 1$ and let $f \in S^\mu_\nu(\mathbb{R}^n)$. Let P be a pseudodifferential operator with symbol $p \in \Gamma^{m,\delta}_\mu$ satisfying (1.19), (1.20) and F be of the form (1.22). If u is a solution of (1.2) such that $\langle x\rangle^{\varepsilon_o} u \in H^s(\mathbb{R}^n)$ for some $s > n/2, \varepsilon_o > \sigma/(\ell-1)$, then $u \in S^\mu_{\nu'}(\mathbb{R}^n)$, with $\nu' = \max\{\nu, \frac{1}{1-\delta}\}$.*

Concerning ordinary differential equations, i.e. $n = 1$ in Theorems 1.2 and 1.3, our results in their general form can be seen in the spirit of the classical analysis on regularity and/or asymptotic behaviour at infinity (e.g. see Wasow [34]) and also intersect recent results on Gevrey regularity for nonlinear equations proved by Djakov and Mityagin [15], [16]. They apply to a large class of equations described in detail in Section 5. The simplest model in this frame is given by the operator

$$L = \frac{d}{dx} + x(1+x^2)^{-\gamma} \qquad x \in \mathbb{R}, \tag{1.23}$$

with $\gamma > 0$. If $\gamma \geq 1$, the equation is Fuchsian or regular type at infinity, so let us further assume $\gamma < 1$. After multiplication by $-i$, we recognize in (1.23) an operator of the form (1.3) with $m = n = 1, A_1(D) = D, \omega(x) = -ix/\langle x\rangle, \sigma = 2\gamma - 1$. The solutions of $Ly = 0$ are given by

$$y(x) = const \cdot \exp\left[\frac{1}{2(\gamma-1)}(1+x^2)^{1-\gamma}\right]. \tag{1.24}$$

The conditions (1.4), (1.8) are verified, so L is SG-elliptic for $\gamma = 1/2$. The results in the present paper refer to the case $1/2 < \gamma < 1$; in particular Theorem 1.2 applies. We are then exactly in the frame of Example 1, where now $\mu = 1, \delta = \sigma = 2\gamma - 1$, so that we expect $y \in S^1_{1/(1-\delta)}(\mathbb{R})$ that is the regularity we may test in (1.24). We may now give the nonlinear version of Example 1, taking for simplicity L in (1.23) as linear part.

Example 2. Consider the ordinary differential equation

$$Ly = y' + x(1+x^2)^{-\gamma} y = y^\ell, \qquad x \in \mathbb{R}, \ell \geq 2, \tag{1.25}$$

with $1/2 < \gamma < 1$. Theorem 1.3 applies and we have that all the solutions of (1.25) such that $\langle x\rangle^{\varepsilon_o} y(x) \in H^s(\mathbb{R})$, for some $s > 1/2$ and $\varepsilon_o > (2\gamma - 1)/(\ell - 1)$, are analytic and decay at infinity like $\exp(-|x|^{2(1-\gamma)})$. This will be tested on the explicit expression of the solutions given by (5.16) in Section 5. Notice that with respect to the linear case (1.1), we ask an a priori decay on the solution. Such assumption is necessary to obtain sub-exponential decay. In fact in Section 5 we shall check that the equation (1.25) admits two types of homoclinics: one with only algebraic decay $y(x) \sim x^{(1-2\gamma)/(\ell-1)}$ for $x \to +\infty$, which does not satisfy the required a priori bound; other homoclinics, with $\langle x\rangle^{\varepsilon_o} y(x) \in H^s(\mathbb{R}), s > 1/2, \varepsilon_o > (2\gamma - 1)/(\ell - 1)$, which have the expected sub-exponential decay. Moreover we may check that $\frac{2\gamma-1}{\ell-1}$ is indeed a sharp lower bound for ε_o.

In conclusion, we would like to observe that the problems of the asymptotic decay and the holomorphic extensions of solutions, apart from the interest "per se" in the general theory of differential equations (both ordinary and partial), arise in different contexts in Mathematical Physics, e.g. for analytic regularity and exponential decay of travelling wave type solutions, cf. the fundamental work by Bona and Li [4] (see also [2]), for the exponential decay of eigenfunctions of Schrödinger operators appearing in Quantum Mechanics, starting from the celebrated work of Agmon [1] (see also [5], [14], [19], [29]) and more generally, for solutions of second order elliptic equations, cf. [28] and the references therein.
The paper is organized as follows. In Section 2, we introduce some scales of Sobolev norms providing suitable characterizations of the space $S^\mu_\nu(\mathbb{R}^n)$, which will be instrumental in the proofs of our statements. In Sections 3 and 4, we prove sub-exponential decay estimates and uniform regularity respectively, for the solutions of the equations (1.1), (1.2). As a consequence we obtain Theorems 1.2 and 1.3. In Section 5, we fix the attention on a class of ordinary differential operators including (1.23) and also check the sharpness of our results on the solutions of (1.25). In the proofs of Sections 3 and 4, we shall use the classical theorems of pseudodifferential calculus for the class $\Gamma^{m,\delta}_\mu$ (composition, adjoints, construction of parametrices). Unlike the case of $\Gamma^{\overline{m},\delta}$, we are not aware of an existing specific calculus for $\Gamma^{m,\delta}_\mu$ in Gelfand-Shilov classes, hence we proved these statements in the present paper and for a more general class including $\Gamma^{m,\delta}_\mu$. Nevertheless, in order to introduce immediately the reader to the proofs of the main results of the paper we postponed the pseudodifferential calculus in an Appendix at the end of the paper.

2 Preliminaries

For any $s \in \mathbb{R}$, we shall denote by $H^s(\mathbb{R}^n)$ the Sobolev space

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\},$$

endowed with the standard norm $\|\langle\cdot\rangle^s \hat{u}(\cdot)\|_{L^2}$, where $\hat{u}$ denotes the Fourier transform of u. Let us now introduce some scales of Sobolev norms defining the Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^n)$ in (1.10). First of all we recall a result obtained in [12] which provides a useful characterization of $S^\mu_\nu(\mathbb{R}^n)$.

Proposition 2.1. *Let $\mu > 0, \nu > 0$ with $\mu + \nu \geq 1$ and let $f \in C^\infty(\mathbb{R}^n)$. Then the following conditions are equivalent:*
i) $f \in S^\mu_\nu(\mathbb{R}^n)$;
ii) There exist positive constants A, B such that

$$\sup_{x\in\mathbb{R}^n} |x^k f(x)| \leq A^{|k|+1}(k!)^\nu \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^n} |\partial_x^j f(x)| \leq B^{|j|+1}(j!)^\mu \tag{2.1}$$

for all $j, k \in \mathbb{Z}^n_+$;
iii) There exist positive constants a, T such that

$$\sup_{x\in\mathbb{R}^n}\Big(\exp(a|x|^{1/\nu})|f(x)|\Big) < +\infty \quad \text{and} \quad \sup_{j\in\mathbb{Z}^n_+} T^{-|j|} j!^{-\mu} \sup_{x\in\mathbb{R}^n} |\partial_x^j f(x)| < +\infty.$$

Proposition 2.1 states that to prove that the solution of (1.2) belongs to $S^\mu_\nu(\mathbb{R}^n)$ we can prove decay and regularity estimates separately. This will be the approach we shall follow in the next Sections 3 and 4.

Taking into account Proposition 2.1, we introduce norms which describe only the decay or the regularity properties. Precisely, let us set

$$|||u|||_{s,\nu;\varepsilon} = \sum_{k\in\mathbb{Z}^n_+} \frac{\varepsilon^{|k|}}{|k|!^\nu}\|x^k u\|_s$$

and denote

$$H^{s,\nu;\varepsilon}_N[u] = \sum_{\substack{k\in\mathbb{Z}^n_+\\|k|\leq N}} \frac{\varepsilon^{|k|}}{|k|!^\nu}\|x^k u\|_s.$$

By Sobolev embedding estimates, it is obvious that if $|||u|||_{s,\nu;\varepsilon} < +\infty$ for some $\nu > 0, s > n/2, \varepsilon > 0$, then u satisfies the first inequality in (2.1). Similarly, we can define

$$|||u|||_{\{s,\mu;T\}} = \sum_{j\in\mathbb{Z}^n_+} \frac{T^{|j|}}{j!^\mu}\|\partial_x^j u\|_s.$$

It is easy to verify that if $|||u|||_{\{s,\mu;T\}} < +\infty$ for some $T > 0, s \geq 0$, then u satisfies the second inequality in (2.1). In fact, for technical reasons that will be clear in the next sections, we shall use a slightly different scale of norms to prove regularity estimates for nonlinear equations. Precisely, fixed $\varepsilon_o \geq 0$ we shall consider the norm

$$|||u|||_{\{s,\mu;T,\varepsilon_o\}} = \sum_{j\in\mathbb{Z}^n_+} \frac{T^{|j|}}{j!^\mu}\|\langle x\rangle^{\varepsilon_o}\partial_x^j u\|_s \tag{2.2}$$

and denote the corresponding partial sum as follows

$$\mathcal{E}^{s,\mu;T,\varepsilon_o}_N[u] = \sum_{\substack{j\in\mathbb{Z}^n_+\\|j|\leq N}} \frac{T^{|j|}}{j!^\mu}\|\langle x\rangle^{\varepsilon_o}\partial_x^j u\|_s. \tag{2.3}$$

We shall write $\mathcal{E}^{s,\mu;T}_N[u]$ for $\mathcal{E}^{s,\mu;T,0}_N[u]$.

3 Decay estimates

The main goal of the present section is to derive sharp decay estimates for the solutions of the equations (1.1), (1.2), where F is of the form (1.22) and P is a pseudodifferential operator with symbol $p \in \Gamma_{\mu}^{m,\delta}$ satisfying the conditions (1.19), (1.20). The approach will be the same for the linear and the semilinear case, but the latter case requires some a priori restrictions on the behavior at infinity of the solution. Let us then start from the linear case $F[u] = 0$.

If $u \in \mathcal{S}(\mathbb{R}^n)$ is a solution of $Pu = f$, then for every $k \in \mathbb{Z}_+^n, \varepsilon > 0, \nu \geq 1$, we can write

$$\frac{\varepsilon^{|k|}}{|k|!^{\nu}} x^k Pu(x) = \frac{\varepsilon^{|k|}}{|k|!^{\nu}} x^k f(x).$$

from which we get

$$\frac{\varepsilon^{|k|}}{|k|!^{\nu}} P(x^k u) = \frac{\varepsilon^{|k|}}{|k|!^{\nu}} x^k f(x) + \frac{\varepsilon^{|k|}}{|k|!^{\nu}} [P, x^k] u.$$

Now, since P satisfies (1.19) and (1.20), by Proposition A.13 there exists a left parametrix E for P. Then we have

$$\frac{\varepsilon^{|k|}}{|k|!^{\nu}} x^k u = \frac{\varepsilon^{|k|}}{|k|!^{\nu}} E(x^k f) + \frac{\varepsilon^{|k|}}{|k|!^{\nu}} R(x^k u) + \frac{\varepsilon^{|k|}}{|k|!^{\nu}} E([P, x^k] u).$$

where R is a regularizing operator mapping $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, cf. Remark A.8. Taking Sobolev norms and summing up for $|k| \leq N, N \in \mathbb{Z}_+$ we obtain

$$\begin{aligned} H_N^{s,\nu;\varepsilon}[u] \quad \leq \quad & \sum_{|k| \leq N} \frac{\varepsilon^{|k|}}{|k|!^{\nu}} \|E(x^k f)\|_s \\ & + \sum_{|k| \leq N} \frac{\varepsilon^{|k|}}{|k|!^{\nu}} \|R(x^k u)\|_s \\ & + \sum_{0 < |k| \leq N} \frac{\varepsilon^{|k|}}{|k|!^{\nu}} \|E([P, x^k] u)\|_s. \end{aligned} \tag{3.1}$$

We have the following result.

Theorem 3.1. *Let $P = p(x, D) \in OP\Gamma_{\mu}^{m,\delta}$ satisfy the assumptions of Theorem 1.2. Assume moreover that $f \in \mathcal{S}(\mathbb{R}^n)$ is such that $|||\langle x \rangle^{\sigma} f|||_{s,\nu;\varepsilon'} < \infty$ for some $\nu \geq 1, \varepsilon' > 0, s > n/2$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of $Pu = f$, then there exists $\varepsilon > 0$ such that $|||u|||_{s,\nu';\varepsilon} < +\infty$, where $\nu' = \max\{\nu, \frac{1}{1-\delta}\}$. In particular, there exist positive constants C, c such that*

$$|u(x)| \leq C e^{-c|x|^{1/\nu'}} \tag{3.2}$$

for every $x \in \mathbb{R}^n$.

In order to prove Theorem 3.1 we want to show that for some $\varepsilon > 0$ the left-hand side of (3.1) converges for $N \to +\infty$. To do this we need to estimate properly the three terms in the right-hand side. The most delicate term is the one containing commutators for which some preliminary steps are necessary.

Lemma 3.2. *Let $\delta \in]0,1[$ and $r > 0$. Then*

$$t^{\beta\delta} \le rt^{\beta} + (1-\delta)\left(\frac{\delta}{r}\right)^{\delta/(1-\delta)}, \qquad t \ge 0. \tag{3.3}$$

for all $\beta \in \mathbb{N}$.

Proof. Clearly we can assume $\beta = 1$, setting $t^{\beta} = z$. Set $g(z) = z^{\delta} - rz$, $z \ge 0$. Since $g'(z) = \delta z^{\delta-1} - r = 0$ iff $z = z_{\delta,r} = (\delta/r)^{1/(1-\delta)}$ we readily obtain that

$$\sup_{z\ge 0} g(z) = g(z_{\delta,r}) = \left(\frac{\delta}{r}\right)^{\delta/(1-\delta)} - r\left(\frac{\delta}{r}\right)^{1/(1-\delta)} = (1-\delta)\left(\frac{\delta}{r}\right)^{\delta/(1-\delta)}.$$

The proof is complete. □

Lemma 3.3. *Let $\delta \in]0,1[$, $\nu \ge 1$, $\gamma, \eta > 0$. Then*

$$\frac{|x|^{|\beta|\delta}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}} \le \frac{\eta}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu}}\gamma^{(1-\delta)|\beta|}|x|^{|\beta|} + \frac{(1-\delta)\gamma^{-\delta|\beta|}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1/(1-\delta)}}\left(\frac{\delta}{\eta}\right)^{\delta/(1-\delta)} \tag{3.4}$$

for all $x \in \mathbb{R}^n$, $k, \beta \in \mathbb{Z}_+^n$, $|\beta| \le |k|$.

Proof. We set $r = \eta/(|k|(|k|-1)\dots(|k-\beta|+1))$ and $t = \gamma|x|$. Then (3.4) follows by (3.3) and straightforward calculation. □

Lemma 3.4. *Let $\delta \in]0,1[$, $\nu \ge 1$. Then there exists $C_0 > 0$ such that for every $\gamma \in]0,1[, \eta > 0$ the following estimate holds:*

$$\frac{|\langle x\rangle^{|\beta|\delta} x^{k-\beta}|}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}(|k-\beta|)!^{\nu}} \le C_0^{|\beta|}\frac{\eta}{|k|!^{\nu}}\sum_{q=1}^{n}\gamma^{(1-\delta)|\beta|}\,|x^{k-\beta+|\beta|e_q}| + C_0^{|\beta|}\gamma^{-\delta|\beta|}\mathcal{D}_{\nu,\delta,\eta}^{|k-\beta|}\frac{|x^{k-\beta}|}{|k-\beta|!^{\nu}}, \tag{3.5}$$

where

$$\mathcal{D}_{\nu,\delta,\eta}^{|k-\beta|} := \frac{1}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}} + n\frac{1-\delta}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1/(1-\delta)}}\left(\frac{\delta}{\eta}\right)^{\delta/(1-\delta)}, \tag{3.6}$$

for all $x \in \mathbb{R}^n$, $k, \beta \in \mathbb{Z}_+^n$, $\beta \le k$.

Proof. Since $\delta \in]0,1[$ we have

$$\langle x\rangle^{|\beta|\delta} \le (1+|x|^{\delta})^{|\beta|} \le (n+2)^{|\beta|}(1+\sum_{q=1}^{n}|x_q|^{|\beta|\delta}). \tag{3.7}$$

Next, we estimate by (3.4) and derive

$$\frac{|x_q^{|\beta|\delta} x^{k-\beta}|}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}} \leq \frac{\eta\gamma^{(1-\delta)|\beta|}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu}}|x^{k-\beta+|\beta|e_q}|$$

$$+\frac{(1-\delta)\gamma^{-\delta|\beta|}|x^{k-\beta}|}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1/(1-\delta)}}\left(\frac{\delta}{\eta}\right)^{\delta/(1-\delta)} \tag{3.8}$$

for $q=1,\dots,n$, $x\in\mathbb{R}^n$. Combining (3.7) and (3.8) and summing over q we get (3.5) and (3.6). □

The next lemma states some crucial estimates for the operator P in (1.1), (1.2). Since the proof is based on some results contained in the Appendix, we give here only the statement and refer the reader to the Appendix for the proof.

Lemma 3.5. *Let $P = p(x,D)$ with $p \in \Gamma_\mu^{m,\delta}$ satisfying* (1.19), (1.20) *and let E be a left parametrix for P as in Proposition A.13. Then for every $s\in\mathbb{R}$ there exist positive constants A_s, C_s such that for every $u\in\mathcal{S}(\mathbb{R}^n)$ we have*

$$\|Eu\|_s \leq C_s\|\langle x\rangle^\sigma u\|_{s-m} \tag{3.9}$$

and

$$\frac{1}{|k|!^\nu}\|E[P,x^k]u\|_s \leq \sum_{\beta\leq k,\beta\neq 0}\frac{A_s^{|\beta|}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}}\cdot\frac{\|\langle x\rangle^{\delta|\beta|}x^{k-\beta}u\|_s}{|k-\beta|!^\nu} \tag{3.10}$$

for all $k\in\mathbb{Z}_+^n$, $k\neq 0, \nu\geq 1$.

Taking into account Lemma 3.4 and Lemma 3.5 we can now estimate the commutator in the right-hand side of (3.1).

Proposition 3.6. *Let $\nu \geq \frac{1}{1-\delta}, s\in\mathbb{Z}_+$. Then, there exist positive constants ε, C_s such that for every $\eta>0$ the following estimate holds*

$$\sum_{\substack{k\in\mathbb{Z}_+^n\\ s\leq|k|\leq N}}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E[P,x^k]u\|_s \leq C_s(\eta H_N^{s,\nu;\varepsilon}[u]+\varepsilon H_{N-1}^{s,\nu;\varepsilon}[u]). \tag{3.11}$$

for every $N\in\mathbb{Z}_+$ with $N\geq s$.

Proof. In view of Lemma 3.5 we have

$$\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E[P,x^k]u\|_s \leq \varepsilon^{|k|}\sum_{\beta\leq k,\beta\neq 0}\frac{A_s^{|\beta|}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}}\frac{\|\langle x\rangle^{\delta|\beta|}x^{k-\beta}u\|_s}{|k-\beta|!^\nu}$$

$$=\varepsilon^{|k|}\sum_{\beta\leq k,\beta\neq 0}\frac{A_s^{|\beta|}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}}\times$$

$$\times \sum_{|\alpha|\le s} \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha \\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\cdot \frac{(k-\beta)!}{(k-\beta-\alpha_2)!} \|(\partial_x^{\alpha_1}u)\cdot x^{k-\beta-\alpha_2}\partial_x^{\alpha_3}\langle x\rangle^{|\beta|\delta}\|_{L^2}$$

$$\le \varepsilon^{|k|} \sum_{\beta\le k,\beta\neq 0} \frac{(A_sC)^{|\beta|}}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}}\times$$

$$\times \sum_{|\alpha|\le s} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\cdot \frac{(k-\beta)!}{(k-\beta-\alpha_2)!} \|x^{k-\beta-\alpha_2}\langle x\rangle^{|\beta|\delta}\partial_x^{\alpha_1}u\|_{L^2}$$

using the fact that $|\partial_x^{\alpha_3}\langle x\rangle^{|\beta|\delta}| \le C^{|\alpha_3|+|\beta|+1}\alpha_3!\langle x\rangle^{|\beta|\delta}$. Then, by Lemma 3.4, we get for any $\eta > 0, \gamma \in]0,1[$:

$$\begin{aligned}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E[P,x^k]u\|_s \quad \le \quad & \eta \sum_{\beta\le k,\beta\neq 0} (M\gamma^{(1-\delta)})^{|\beta|} \sum_{|\alpha|\le s} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\times \\ & \times \varepsilon^{|\alpha_2|}\frac{(k-\beta)!}{(k-\beta-\alpha_2)!}\sum_{q=1}^{n} \frac{\varepsilon^{|k-\alpha_2|}\|x^{k-\beta-\alpha_2+|\beta|e_q}\partial_x^{\alpha_1}u\|_{L^2}}{|k|!^\nu} \\ & +\mathcal{D}_0 \sum_{\beta\le k,\beta\neq 0} (M\gamma^{-\delta})^{|\beta|} \sum_{|\alpha|\le s} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\varepsilon^{|\beta+\alpha_2|}\times \\ & \times \frac{(k-\beta)!}{(k-\beta-\alpha_2)!}\cdot \frac{\varepsilon^{|k-\beta-\alpha_2|}\|x^{k-\beta-\alpha_2}\partial_x^{\alpha_1}u\|_{L^2}}{|k-\beta|!^\nu} \end{aligned} \tag{3.12}$$

where M is a positive constant independent of $\varepsilon, k, \beta, \gamma$ and

$$D_0 \quad := \quad \sup_{k,\beta\in\mathbb{Z}_+^n\setminus 0,\beta\le k} \mathcal{D}_{\nu,\delta,\eta}^{|k-\beta|} < +\infty \tag{3.13}$$

in view of the condition $\nu \ge \frac{1}{1-\delta}$. Now, observing that

$$\frac{(k-\beta)!}{(k-\beta-\alpha_2)!}\cdot\frac{1}{|k|!^\nu} \le \frac{1}{|k-\alpha_2|!^\nu}$$

and

$$\frac{(k-\beta)!}{(k-\beta-\alpha_2)!}\cdot\frac{1}{|k-\beta|!^\nu} \le \frac{1}{|k-\beta-\alpha_2|!^\nu}$$

and choosing $\gamma < M^{-1/(1-\delta)}, \varepsilon < 1$ we obtain

$$\begin{aligned}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E[P,x^k]u\|_s \quad \le \quad & \eta \sum_{\beta\le k,\beta\neq 0} \sum_{|\alpha|\le s} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\varepsilon^{|\alpha_2|}\times \\ & \times \sum_{q=1}^{n} \frac{\varepsilon^{k-|\alpha_2|}\|x^{k-\beta-\alpha_2+|\beta|e_q}\partial_x^{\alpha_1}u\|_{L^2}}{|k-\alpha_2|!^\nu} \\ & +\mathcal{D}_0\varepsilon \sum_{\beta\le k,\beta\neq 0} (M\gamma^{-\delta})^{|\beta|}\times \\ & \times \sum_{|\alpha|\le s} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\varepsilon^{|k-\beta-\alpha_2|}\|x^{k-\beta-\alpha_2}\partial_x^{\alpha_1}u\|_{L^2}}{|k-\beta-\alpha_2|!^\nu}. \end{aligned} \tag{3.14}$$

We observe now that in the first term in the right-hand side of (3.14), if $s \le |k| \le N$, we have $0 \le |k-\beta-\alpha_2+|\beta|e_q| = |k-\alpha_2| \le N$. Then, rescaling indices in the sums we obtain that

$$\eta \sum_{s\le|k|\le N}\sum_{\beta\le k,\beta\neq 0}\sum_{|\alpha|\le s}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\le k-\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\varepsilon^{|\alpha_2|}\sum_{q=1}^{n}\frac{\varepsilon^{k-|\alpha_2|}\|x^{k-\beta-\alpha_2+|\beta|e_q}\partial_x^{\alpha_1}u\|_{L^2}}{|k-\alpha_2|!^{\nu}}$$

$$\le C_s\eta \sum_{|\alpha_1\le s}\sum_{0\le|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^{\nu}}\|x^k\partial_x^{\alpha_1}u\|_{L^2}.$$

Similarly, choosing $\varepsilon < (M^{-1}\gamma^{\delta})$ and taking into account the fact that in the second term in the right-hand side of (3.14) we have $0 \le |k-\beta-\alpha_2| \le N-1$ since $\beta \neq 0$, we obtain the following estimate:

$$\mathcal{D}_0\varepsilon \sum_{s\le|k|\le N}\sum_{\beta\le k,\beta\neq 0}(M\gamma^{-\delta})^{|\beta|}\sum_{|\alpha|\le s}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\le k-\beta}}\frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\varepsilon^{|k-\beta-\alpha_2|}\|x^{k-\beta-\alpha_2}\partial_x^{\alpha_1}u\|_{L^2}}{|k-\beta-\alpha_2|!^{\nu}}$$

$$\le C_s\mathcal{D}_0\varepsilon \sum_{|\alpha_1|\le s}\sum_{0\le|k|\le N-1}\frac{\varepsilon^{|k|}}{|k|!^{\nu}}\|x^k\partial_x^{\alpha_1}u\|_{L^2}.$$

From the last two estimates we easily obtain (3.11) observing that

$$x^k\partial_x^{\alpha_1}u = \sum_{\substack{j\le\alpha_1\\ j\le k}}\frac{k!}{(k-j)!}\binom{\alpha_1}{j}(-1)^{|j|}\partial_x^{\alpha_1-j}(x^{k-j}u),$$

cf. [8], Lemma 3.2. □

Proof of Theorem 3.1. We first observe that under the assumptions of Theorem 3.1, we already know that $u \in \mathcal{S}(\mathbb{R}^n)$, cf. Theorem 1.1. Now by (3.9) we have, for any $\varepsilon \in]0,\varepsilon']$:

$$\sum_{|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^{\nu'}}\|E(x^k f)\|_s \le C_s\sum_{|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^{\nu'}}\|x^k\langle x\rangle^{\sigma}f\|_s \le C_s|\!|\!|\langle x\rangle^{\sigma}f|\!|\!|_{s,\nu;\varepsilon'} < +\infty.$$

Moreover, since R is $\mathcal{S}$-regularizing, also $R\circ x_j$ is $\mathcal{S}$-regularizing for every $j = 1,\dots,n$. Fixed $k\neq 0$, there exists $j = j_k \in \{1,\dots,n\}$ such that $R\circ x^k = R\circ x_{j_k}\circ x^{k-e_{j_k}}$. Then

$$\sum_{|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^{\nu'}}\|R(x^k u)\|_s \le \|u\|_s + C_s\varepsilon\sum_{0<|k|\le N}\frac{\varepsilon^{|k|-1}}{|k|!^{\nu'}}\|x^{k-e_{j_k}}u\|_s.$$

We also observe that by (3.10), we have

$$\sum_{0<|k|\le s-1}\frac{\varepsilon^{|k|}}{|k|!^{\nu'}}\|E[P,x^k]u\|_s \le C_s\|\langle x\rangle^{s-2+\delta}u\|_s.$$

Finally, applying Proposition 3.6 we get for any $\eta > 0$ and for some $\varepsilon \in]0, \varepsilon']$:

$$H_N^{s,\nu';\varepsilon}[u] \le C_s(\|u\|_s + \|\langle x\rangle^{s-2+\delta}u\|_s + \eta H_N^{s,\nu';\varepsilon}[u] + \varepsilon H_{N-1}^{s,\nu';\varepsilon}[u] + \| \langle x\rangle^\sigma f \|_{s,\nu;\varepsilon'}).$$

Now, choosing η sufficiently small, we obtain

$$H_N^{s,\nu';\varepsilon}[u] \le C_s'(\|u\|_s + \|\langle x\rangle^{s-2+\delta}u\|_s + \varepsilon H_{N-1}^{s,\nu';\varepsilon}[u] + \| \langle x\rangle^\sigma f \|_{s,\nu;\varepsilon'}). \tag{3.15}$$

Then, possibly shrinking ε and iterating estimate (3.15), it follows that $H_N^{s,\nu';\varepsilon}[u]$ is bounded from above with respect to N. Then for $N \to +\infty$ we obtain that $\| u \|_{s,\nu';\varepsilon} < +\infty$.

To treat the nonlinear case, we shall suppose without loss of generality that $F[u] = u^\ell$ for some integer $\ell \ge 2$. With respect to the linear case, here we need to assume some a priori decay on u.

Theorem 3.7. *Let $P = p(x, D) \in OP\Gamma_\mu^{m,\delta}$ satisfy the assumptions of Theorem 1.3. Let u be a solution of (1.2), such that $\langle x\rangle^{\varepsilon_o}u \in H^s(\mathbb{R}^n), s \in \mathbb{Z}_+, s > n/2$ for some $\varepsilon_o > \sigma/(\ell-1)$. Assume moreover that $\| \langle x\rangle^\sigma f \|_{s,\nu;\varepsilon'} < \infty$ for some $\varepsilon' > 0, \nu \ge 1$. Then there exists $\varepsilon > 0$ such that $\| u \|_{s,\nu';\varepsilon} < +\infty$, where $\nu' = \max\{\nu, \frac{1}{1-\delta}\}$.*

Lemma 3.8. *Under the assumptions of Theorem 3.7 we have $\langle x\rangle^{\varepsilon_o+\rho}u \in H^s(\mathbb{R}^n)$ for every $\rho \le \min\{1-\delta, (\ell-1)\varepsilon_o - \sigma\}$.*

Proof. By (1.2) we have

$$\langle x\rangle^{\varepsilon_o+\rho}Pu = \langle x\rangle^{\varepsilon_o+\rho}f + \langle x\rangle^{\varepsilon_o+\rho}u^\ell$$

from which

$$\langle x\rangle^{\varepsilon_o+\rho}u = E(\langle x\rangle^{\varepsilon_o+\rho}f) + R(\langle x\rangle^{\varepsilon_o+\rho}u) + E[P, \langle x\rangle^{\varepsilon_o+\rho}]u + E(\langle x\rangle^{\varepsilon_o+\rho}u^\ell) \tag{3.16}$$

for some regularizing operator R mapping $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. Clearly, the assumption on f and (3.9) imply that the Sobolev norm of the first term in the right-hand side of (3.16) is finite. Furthermore, as a consequence of Theorem A.11 and Lemma A.16, the operator $E[P, \langle x\rangle^{\varepsilon_o+\rho}]\langle x\rangle^{-\varepsilon_o-\rho-\delta+1}$ maps $H^s(\mathbb{R}^n)$ into itself. Hence

$$\|E[P, \langle x\rangle^{\varepsilon_o+\rho}]u\|_s \le C_s\|\langle x\rangle^{\varepsilon_o+\rho+\delta-1}u\|_s < +\infty$$

since $\rho \le 1-\delta$. Finally, we have

$$\begin{aligned} \|E\langle x\rangle^{\varepsilon_o+\rho}u^\ell\|_s &\le C_s\|\langle x\rangle^{\varepsilon_o+\rho+\sigma}u^\ell\|_s \\ &= C_s\|\langle x\rangle^{\varepsilon_o}u \cdot (\langle x\rangle^{\frac{\sigma+\rho}{\ell-1}}u)^{\ell-1}\|_s \\ &\le C_s'\|\langle x\rangle^{\varepsilon_o}u\|_s \cdot \|\langle x\rangle^{\frac{\sigma+\rho}{\ell-1}}u\|_s^{\ell-1} < +\infty \end{aligned}$$

applying Schauder's lemma. The proof is complete. □

Iterating Lemma 3.8 we obtain that $\langle x\rangle^\tau u \in H^s(\mathbb{R}^n)$ for all $\tau > 0$.

Lemma 3.9. *Under the assumptions of Theorem 3.7, the following estimate holds:*

$$\sum_{\substack{k\in\mathbb{Z}_+^n\\ 0<|k|\le N}} \frac{\varepsilon^{|k|}}{|k|!^\nu}\|E(x^k u^\ell)\|_s \le C_s''\varepsilon\|\langle x\rangle^{\frac{\sigma+1}{\ell-1}}u\|_s^{\ell-1}\cdot H_{N-1}^{s,\nu;\varepsilon}[u] \tag{3.17}$$

for every $N\in\mathbb{Z}_+$.

Proof. By (3.9), applying Schauder's lemma, we obtain for every $k\in\mathbb{Z}_+^n, k\neq 0$:

$$\begin{aligned}
\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E(x^k u^\ell)\|_s &\le C_s\frac{\varepsilon^{|k|}}{|k|!^\nu}\|x^k\langle x\rangle^\sigma u^\ell\|_s \\
&\le C_s'\varepsilon\|\langle x\rangle^\sigma x_{j_k}u^{\ell-1}\|_s\cdot\frac{\varepsilon^{|k|-1}\|x^{k-e_{j_k}}u\|_s}{(|k|-1)!^\nu} \\
&\le C_s''\varepsilon\|\langle x\rangle^{\frac{\sigma+1}{\ell-1}}u\|_s^{\ell-1}\cdot\frac{\varepsilon^{|k|-1}\|x^{k-e_{j_k}}u\|_s}{(|k|-1)!^\nu}
\end{aligned}$$

from which we obtain (3.17). □

Proof of Theorem 3.7. Starting from the equation (1.2) and arguing as for (3.1) we obtain that the solution u satisfies

$$\begin{aligned}
H_N^{s,\nu;\varepsilon}[u] \le& \sum_{|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E(x^k f)\|_s + \sum_{|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|R(x^k u)\|_s \\
&+\sum_{0<|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E([P,x^k]u)\|_s + \sum_{|k|\le N}\frac{\varepsilon^{|k|}}{|k|!^\nu}\|E(x^k u^\ell)\|_s.
\end{aligned}\tag{3.18}$$

The first three terms can be estimated as in the linear case, the last one using Lemma 3.9. Then we conclude as in the proof of Theorem 3.1. □

Remark 3.10. *We remark that Lemma 3.8 can be applied also in the linear case for* $\rho\le 1-\delta$. *In this way we can obtain a variant of Theorem 3.1 in the case in which* $f\notin\mathcal{S}(\mathbb{R}^n)$ *but simply* $|||\langle x\rangle^\sigma f|||_{s,\nu,\varepsilon'}<+\infty$ *for some* $s>n/2,\varepsilon'>0$. *Namely, in this case, if* u *is a solution of* $Pu=f$ *and* $u\in H^s(\mathbb{R}^n)$, *then* $|||u|||_{s,\nu';\varepsilon}<+\infty$ *for some* $\varepsilon>0$, *with* ν' *as in Theorem 3.1.*

4 Regularity estimates

In this section, we derive regularity estimates for the solutions of (1.1), (1.2). As in the previous section we first consider the linear case $F[u]=0$. If $u\in\mathcal{S}(\mathbb{R}^n)$ is a solution of the equation $Pu=f$, then for every $j\in\mathbb{Z}_+^n, T>0,\mu\ge 1$ we have the identity

$$\frac{T^{|j|}}{j!^\mu}\partial_x^j Pu(x)=\frac{T^{|j|}}{j!^\mu}\partial_x^j f(x)$$

from which

$$\frac{T^{|j|}}{j!^\mu}\partial_x^j u=\frac{T^{|j|}}{j!^\mu}E\left(\partial_x^j f\right)+\frac{T^{|j|}}{j!^\mu}R\left(\partial_x^j u\right)+\frac{T^{|j|}}{j!^\mu}E([P,\partial_x^j]u), \tag{4.1}$$

where E is a left parametrix of P and R is a $\mathcal{S}$-regularizing operator.

Lemma 4.1. *Let P satisfy the assumptions of Theorems 1.2 and 1.3. Then, for every $\varepsilon_o \geq 0$ there exists a constant $B > 0$ such that*

$$\|E[P, \langle x\rangle^{\varepsilon_o}\partial_x^j]u\|_s \leq Cj!^\mu \sum_{0\neq\gamma\leq j} \frac{B^{|\gamma|+1}}{(j-\gamma)!^\mu}\|\langle x\rangle^{\varepsilon_o}\partial_x^{j-\gamma}u\|_s. \tag{4.2}$$

Proof. See Appendix.

Theorem 4.2. *Let $P = p(x,D) \in OP\Gamma_\mu^{m,\delta}$ satisfy the assumptions of Theorem 1.2. Assume moreover that $f \in \mathcal{S}(\mathbb{R}^n)$ is such that $|||f|||_{\{0,\mu;T',\sigma\}} < +\infty$ for some $\mu \geq 1, T' > 0$ and let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a solution of the equation (1.1). Then there exists $T > 0$ such that $|||u|||_{\{0,\mu;T\}} < +\infty$.*

Proof. As in the proof of Theorem 3.1 we know that u is actually in $\mathcal{S}(\mathbb{R}^n)$. By (4.1) we can write

$$\begin{aligned}\sum_{|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|\partial_x^j u\|_{L^2} \quad \leq \quad & \sum_{|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|E(\partial_x^j f)\|_{L^2} \\ & + \sum_{|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|R(\partial_x^j u)\|_{L^2} \\ & + \sum_{0<|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|E([P,\partial_x^j]u)\|_{L^2}.\end{aligned}$$

By the assumption on f and by (3.9) we can estimate for any $T \in]0, T']$:

$$\sum_{|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|E(\partial_x^j f)\|_{L^2} \leq C\sum_{|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|\langle x\rangle^\sigma\partial_x^j f\|_{L^2} \leq C|||f|||_{\{0,\mu;T',\sigma\}} < +\infty. \tag{4.3}$$

Moreover

$$\sum_{|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|R(\partial_x^j u)\|_{L^2} \leq C(\|u\|_{L^2} + T\mathcal{E}_{N-1}^{0,\mu;T}[u]). \tag{4.4}$$

Finally, by Lemma 4.1 for $s = 0$ we obtain

$$\begin{aligned}\sum_{0<|j|\leq N}\frac{T^{|j|}}{j!^\mu}\|E[P,\partial_x^j]u\|_{L^2} \quad & \leq \quad B^2T\sum_{0<|j|\leq N}\sum_{0\neq\gamma\leq j}(BT)^{|\gamma|-1}\left\|T^{|j-\gamma|}\frac{\partial_x^{j-\gamma}u}{(j-\gamma)!^\mu}\right\|_{L^2} \\ & \leq \quad CT\mathcal{E}_{N-1}^{0,\mu;T}[u],\end{aligned} \tag{4.5}$$

choosing $T \leq \min\{B^{-1}, T'\}$. Then, by (4.3), (4.4), (4.5) we obtain that

$$\mathcal{E}_N^{0,\mu;T}[u] \leq C\left(\|u\|_{L^2} + T\mathcal{E}_{N-1}^{0,\mu;T}[u] + |||f|||_{\{0,\mu;T',\sigma\}}\right). \tag{4.6}$$

Hence, possibly shrinking T and iterating (4.6) we deduce that $|||u|||_{\{0,\mu;T\}} < +\infty$. □

As a direct consequence of Theorems 3.1 and 4.2 we can prove Theorem 1.2.

Proof of Theorem 1.2. If $f \in S_\nu^\mu(\mathbb{R}^n)$ then it satisfies the assumptions of both Theorems 3.1 and 4.2. By Theorem 3.1, it follows that

$$\sup_{k\in\mathbb{Z}_+^n} \varepsilon^{|k|}(k!)^{-\nu'} \sup_{x\in\mathbb{R}^n} |x^k u(x)| < +\infty$$

for some $\varepsilon > 0$, with $\nu' = \max\{\nu, \frac{1}{1-\delta}\}$. On the other hand, by Theorem 4.2, we deduce that

$$\sup_{j\in\mathbb{Z}_+^n} T^{|j|} j!^{-\mu} \sup_{x\in\mathbb{R}^n} |\partial_x^j u(x)| < +\infty.$$

Then, invoking Proposition 2.1, we obtain that $u \in S_{\nu'}^\mu(\mathbb{R}^n)$. □

As well as for decay estimates, in order to obtain regularity estimates for the nonlinear case, we have to assume some a priori decay on u and to use the sums (2.2) with $\varepsilon_o > 0, s > n/2$. We have the following result.

Theorem 4.3. *Let $P = p(x,D) \in OP\Gamma_\mu^{m,\delta}$ satisfy the assumptions of Theorem 1.3. Let u be a solution of (1.2) with $\langle x\rangle^{\varepsilon_o} u \in H^s(\mathbb{R}^n), s > n/2$ for some $\varepsilon_o > \sigma/(\ell-1)$ and assume moreover that $|||f|||_{\{s,\mu;T',\sigma+\varepsilon_o\}} < +\infty$ for some $\mu \geq 1, T' > 0$. Then, there exists $T > 0$ such that $|||u|||_{s,\mu;T,\varepsilon_o} < +\infty$.*

Lemma 4.4. *Under the assumptions of Theorem 4.3, the following estimate holds true:*

$$\sum_{|j|\leq N} \frac{T^{|j|}}{j!^\mu} \|E(\langle x\rangle^{\varepsilon_o}\partial_x^j u^\ell)\|_s \leq C_s \left(\|\langle x\rangle^{\frac{\varepsilon_o+\sigma}{\ell}} u\|_s^\ell + T(\mathcal{E}_{N-1}^{s,\mu;T,\varepsilon_o}[u])^\ell \right). \tag{4.7}$$

Proof. Let $j \in \mathbb{Z}_+^n, j \neq 0$. Then $j_q \neq 0$ for some $q \in \{1,...,n\}$. By (3.9) since $m \geq 1$, we have

$$\begin{aligned} \|E(\langle x\rangle^{\varepsilon_o}\partial_x^j u^\ell)\|_s &\leq C_s' \|\langle x\rangle^{\varepsilon_o+\sigma}\partial_x^j u^\ell\|_{s-m} \\ &\leq C_s' \left\| \partial_{x_q}\left(\langle x\rangle^{\varepsilon_o+\sigma}\partial_x^{j-e_q}u^\ell\right)\right\|_{s-m} \\ &\quad + C_s' \left\| [\langle x\rangle^{\varepsilon_o+\sigma}, \partial_{x_q}]\partial_x^{j-e_q}u^\ell \right\|_s \\ &\leq C_s'' \left\| \left(\langle x\rangle^{\varepsilon_o+\sigma}\partial_x^{j-e_q}u^\ell\right)\right\|_s \\ &\quad + C_s'' \left\| [\langle x\rangle^{\varepsilon_o+\sigma}, \partial_{x_q}]\partial_x^{j-e_q}u^\ell \right\|_s . \end{aligned} \tag{4.8}$$

Since we can estimate

$$\left\| [\langle x\rangle^{\varepsilon_o+\sigma}, \partial_{x_q}]\partial_x^{j-e_q}u^\ell \right\|_s \leq C_s''' \left\| \langle x\rangle^{\varepsilon_o+\sigma-1}\partial_x^{j-e_q}u^\ell \right\|_s,$$

we obtain

$$\|E(\langle x\rangle^{\varepsilon_o}\partial_x^j u^\ell)\|_s \leq C_s \|\langle x\rangle^{\varepsilon_o+\sigma}\partial_x^{j-e_q}u^\ell\|_s. \tag{4.9}$$

Now, applying Leibniz formula, we can write

$$\langle x\rangle^{\varepsilon_o+\sigma}\partial_x^{j-e_q}u^\ell = \sum_{j_1+\ldots+j_\ell=j-e_q}\frac{(j-e_q)!}{j_1!\ldots j_\ell!}\left(\langle x\rangle^{\varepsilon_o}\partial_x^{j_1}u\right)\prod_{k=2}^{\ell}\left(\langle x\rangle^{\frac{\sigma}{\ell-1}}\partial_x^{j_k}u\right). \quad (4.10)$$

Then, since $\mu\geq 1$ we obtain

$$\begin{aligned}\frac{T^{|j|}}{j!^\mu}\|\langle x\rangle^{\varepsilon_o+\sigma}\partial_x^{j-e_q}u^\ell\|_s &\leq \frac{C_sT}{j!^\mu}\sum_{j_1+\ldots+j_\ell=j-e_q}\frac{(j-e_q)!}{(j_1!\ldots j_\ell!)^{1-\mu}}\left(\frac{T^{|j_1|}}{j_1!^\mu}\left\|\langle x\rangle^{\varepsilon_o}\partial_x^{j_1}u\right\|\right)\times\\ &\quad\times\prod_{k=2}^{\ell}\left(\frac{T^{|j_k|}}{j_k!^\mu}\left\|\langle x\rangle^{\frac{\sigma}{\ell-1}}\partial_x^{j_k}u\right\|\right)\\ &\leq C_sT\sum_{j_1+\ldots+j_\ell=j-e_q}\left(\frac{T^{|j_1|}}{j_1!^\mu}\left\|\langle x\rangle^{\varepsilon_o}\partial_x^{j_1}u\right\|_s\right)\times\\ &\quad\times\prod_{k=2}^{\ell}\left(\frac{T^{|j_k|}}{j_k!^\mu}\left\|\langle x\rangle^{\frac{\sigma}{\ell-1}}\partial_x^{j_k}u\right\|_s\right)\end{aligned}$$

applying Schauder's lemma and using the condition $\varepsilon_o>\sigma/(\ell-1)$. Using the last estimate, summing up over j we obtain (4.7). □

Proof of Theorem 4.3. First observe that by an inductive argument similar to the one adopted in Lemma 3.8, we have that $\langle x\rangle^{\varepsilon_o}\partial_x^j u\in H^s(\mathbb{R}^n)$ for every $j\in\mathbb{Z}_+^n$. Then, arguing as in the proof of Theorem 4.2, we obtain that

$$\begin{aligned}\mathcal{E}_N^{s,\mu;T,\varepsilon_o}[u] &\leq C_s'(\|\langle x\rangle^{\varepsilon_o+\sigma}u\|_s+T\mathcal{E}_{N-1}^{s,\mu;T,\varepsilon_o}[u]+|\!|\!|f|\!|\!|_{\{s,\mu;T,\sigma+\varepsilon_o\}})\\ &\quad+\sum_{0\neq|j|\leq N}\frac{T^{|j|}}{j!^\mu}\left\|E(\langle x\rangle^{\varepsilon_o}\partial_x^j u^\ell)\right\|_s.\end{aligned}$$

Then, applying Lemma 4.4, we get for any $T\leq\min\{B^{-1},T'\}$:

$$\begin{aligned}\mathcal{E}_N^{s,\mu;T,\varepsilon_o}[u] &\leq C_s\Big(\|\langle x\rangle^{\varepsilon_o+\sigma}u\|_s+\|\langle x\rangle^{\frac{\varepsilon_o+\sigma}{\ell}}u\|_s^\ell+\\ &\quad+T\mathcal{E}_{N-1}^{s,\mu;T,\varepsilon_o}[u]+T(\mathcal{E}_{N-1}^{s,\mu,T,\varepsilon_o}[u])^\ell+|\!|\!|f|\!|\!|_{\{s,\mu;T,\varepsilon_o+\sigma\}}\Big)\end{aligned}$$

from which we obtain that $|\!|\!|u|\!|\!|_{\{s,\mu;T,\varepsilon_o\}}<+\infty$.

Similarly as for the linear case, Theorem 1.3 can be easily obtained combining Theorems 3.7 and 4.3. We leave the details to the reader.

5 The case of ordinary differential operators

In this section we apply the results obtained in the previous sections to a class of ordinary differential operators including (1.23) as example. Consider the operator

$$P=\frac{1}{\kappa^m(x)}\left[\left(\kappa(x)\frac{d}{dx}\right)^m+a_1(x)\left(\kappa(x)\frac{d}{dx}\right)^{m-1}+\ldots+a_m(x)\right]. \quad (5.1)$$

The hypotheses on the coefficients of P are the following: $\kappa(x)$ is even, $\kappa(x) > 0$ for all $x \in \mathbb{R}$, and there exist $C_o, \kappa_o > 0$ such that for $\mu \geq 1, 0 < \delta < 1$:

$$\left|D_x^j \kappa(x)\right| \leq C_o^{j+1}(j!)^\mu \langle x\rangle^{\delta-j}, \qquad x \in \mathbb{R}, j \in \mathbb{Z}_+, \tag{5.2}$$

$$\kappa(x) = \kappa_o |x|^\delta (1 + o(1)) \quad for \quad x \to \pm\infty. \tag{5.3}$$

Concerning $a_j(x), j = 1, ..., m$, we assume they satisfy estimates of type (5.2) with $\delta = 0$ and $a_j(x) = a_{j0}^\pm + o(1), a_{j0}^\pm \in \mathbb{C}$, for $x \to \pm\infty$. It is easy to prove that P can be re-written as

$$P = i^m (D_x^m + b_1(x) D_x^{m-1} + ... + b_m(x)), \tag{5.4}$$

where for $j = 1, ..., m$

$$\left|D_x^k b_j(x)\right| \leq C^{k+1}(k!)^\mu \langle x\rangle^{-j\delta-k}$$

and

$$b_j(x) = (-i)^j a_j(x) \kappa^{-j}(x) + O(\langle x\rangle^{-j\delta-1}),$$

so that

$$b_j(x) = b_{j0}^\pm |x|^{-j\delta}(1 + o(1)) \quad for \quad x \to \pm\infty, \tag{5.5}$$

where $b_{j0}^\pm = (-i)^j a_{j0}^\pm \kappa_o^{-j}$. At this moment, we consider the two algebraic equations

$$\mathcal{L}^\pm(\lambda) = \lambda^m + b_{10}^\pm \lambda^{m-1} + ... + b_{m0}^\pm = 0$$

and we assume

$$\Im\lambda \neq 0 \quad for\ every \quad \lambda \quad such\ that \quad \mathcal{L}^\pm(\lambda) = 0. \tag{5.6}$$

Proposition 5.1. *Under the previous assumptions, disregarding the factor* i^m *in* (5.4)*, we consider* P *in* (5.1) *as a pseudodifferential operator with symbol*

$$p(x, \xi) = \xi^m + b_1(x)\xi^{m-1} + ... + b_m(x). \tag{5.7}$$

Then, $p(x, \xi)$ *can be seen as symbol in* $\Gamma_\mu^{m,\delta}$ *globally hypoelliptic satisfying* (1.19), (1.20) *for* $\sigma = m\delta$.

Proof. First observe that

$$c\langle\xi\rangle^m \langle x\rangle^{-m\delta} \leq |p(x, \xi)| \leq C\langle\xi\rangle^m \quad for \quad |x| + |\xi| \geq R. \tag{5.8}$$

for some positive constants C, c, R. The second estimate is obvious. To prove the estimate in the left-hand side, observe that under our assumptions

$$|\mathcal{L}^\pm(\lambda)| \geq c(1 + |\lambda|^m), \tag{5.9}$$

hence

$$p_o^\pm(x, \xi) = \frac{1}{\langle x\rangle^{m\delta}} \mathcal{L}^\pm(\langle x\rangle^\delta \xi) \geq c \frac{1 + \langle x\rangle^{m\delta}|\xi|^m}{\langle x\rangle^{m\delta}}. \tag{5.10}$$

Argue first in the region $x > 0$. Write there

$$p(x, \xi) = p_o^+(x, \xi) + p(x, \xi) - p_o^+(x, \xi).$$

In view of (5.5), given $\varepsilon > 0$, for $x > R$ we can estimate:

$$\left|p(x,\xi) - p_o^+(x,\xi)\right| \leq \varepsilon \frac{1 + \langle x\rangle^{m\delta}|\xi|^m}{\langle x\rangle^{m\delta}}.$$

Applying (5.10) and taking ε sufficiently small, we get for a new constant $c > 0$

$$|p(x,\xi)| \geq c\frac{1 + \langle x\rangle^{m\delta}|\xi|^m}{\langle x\rangle^{m\delta}} \quad \textit{for} \quad x > r, \xi \in \mathbb{R}. \tag{5.11}$$

Arguing similarly for $x < 0$, we obtain the same estimate for $x < -R$. On the other hand, for $|x| \leq R$, the estimates (5.11) are trivial provided $|\xi|$ is large, so we have proved (5.11) for $|x| + |\xi| \geq R$. At this moment we observe that

$$\frac{1 + \langle x\rangle^{m\delta}|\xi|^m}{\langle x\rangle^{m\delta}} \geq \langle x\rangle^{-m\delta}\langle \xi\rangle^m$$

and we get the left-hand side of (5.8). So we have proved that p satisfies (1.19) with $\sigma = m\delta$. It remains to check the hypoellipticity condition (1.20). We first estimate

$$|\partial_\xi p(x,\xi)| \leq C(|\xi|^{m-1} + \langle x\rangle^{-\delta}|\xi|^{m-2} + ... + \langle x\rangle^{-(m-1)\delta}). \tag{5.12}$$

To proceed, it is convenient to use an equivalent version of (5.11) for $|x| + |\xi| \geq R$, namely

$$|p(x,\xi)| \geq c\sum_{j=0}^{m} H_j(x,\xi) \tag{5.13}$$

with $H_j(x,\xi) = \langle x\rangle^{-(m-j)\delta}|\xi|^j$, which follows easily from the previous arguments. Let us estimate the generic term in the right-hand side of (5.12). We have to prove that

$$\langle x\rangle^{-(m-j)\delta}|\xi|^{j-1} \leq C|p(x,\xi)|\langle\xi\rangle^{-1}\langle x\rangle^{\delta}, \quad j = 1, ..., m. \tag{5.14}$$

Arguing for small $|\xi|$, we observe that

$$\langle x\rangle^{-(m-j)\delta}|\xi|^{j-1} \leq CH_{j-1}(x,\xi)\langle\xi\rangle^{-1}\langle x\rangle^{\delta},$$

and in view of (5.13) we obtain (5.14). For large $|\xi|$ we use the inequality

$$\langle x\rangle^{-(m-j)\delta}|\xi|^{j-1} \leq CH_j(x,\xi)\langle\xi\rangle^{-1}\langle x\rangle^{\delta},$$

and again in view of (5.13), we deduce (5.14). We leave to the reader similar estimates of the other derivatives. □

We may then construct a parametrix for P in (5.1). Then for $n = 1$, Theorems 1.2, 1.3 apply to (5.1) under the assumptions (5.2), (5.3). To be definite, for the solutions $y(x), x \in \mathbb{R}$, of the semilinear homogeneous equation (i.e. $f = 0$) we obtain the estimates

$$|y^{(\alpha)}(x)| \leq C^{|\alpha|+1}(\alpha!)^\mu e^{-\varepsilon|x|^{1-\delta}}, \qquad x \in \mathbb{R}^n.$$

We notice that in the particular case in which the coefficients a_j in (5.1) are constant, the operator P, besides being globally hypoelliptic, admits even a left inverse P^{-1}.

We also notice that the example (1.23) in the Introduction is included in the class described in this section. The same conclusions then apply to (1.23) with $\delta = 2\gamma - 1$ as we observed in the Introduction.

To conclude the section, let us write down the solutions of (1.25), and check on them that the assumption on ε_o in Theorem 1.3 is sharp in this case. In fact, the ordinary differential equation

$$y' + x(1+x^2)^{-\gamma} y = y^\ell, \quad \ell \geq 2, \frac{1}{2} < \gamma < 1, \tag{5.15}$$

is a Bernoulli equation, which we can treat explicitly. Namely, let us write

$$\psi(x) = -\frac{\ell - 1}{2(1-\gamma)}(1+x^2)^{1-\gamma}$$

and

$$A = (\ell - 1) \int_0^{+\infty} e^{\psi(x)} dx.$$

Fixing for simplicity attention on the solution $y(x)$ for which $y(0) = y_o > 0$, we have

$$y(x) = \left(\frac{e^{\psi(x)}}{\lambda + (\ell - 1) \int_x^{+\infty} e^{\psi(t)} dt} \right)^{\frac{1}{\ell - 1}} \tag{5.16}$$

with $\lambda = y_o^{1-\ell} e^{\psi(0)} - A$. Here and in the following, roots are defined to be positive for positive numbers, with continuous extension in the complex domain, i.e. we take principal branches. To study the behaviour of the solutions, let us observe that

$$E(x) = (\ell - 1) \int_x^{+\infty} e^{\psi(t)} dt$$

is positive and decreasing on the real axis, with

$$\lim_{x \to -\infty} E(x) = 2A, \quad E(0) = A, \quad \lim_{x \to +\infty} E(x) = 0 \tag{5.17}$$

and asymptotic expansion for $x \to +\infty$

$$E(x) = e^{\psi(x)} (x^{2\gamma - 1} + o(1)). \tag{5.18}$$

We may easily prove (5.18) by applying the classical De l'Hôpital rule. Let us test Theorem 1.3 on (5.16). We distinguish three cases.

i) $\left(\frac{A}{e^{\psi(0)}}\right)^{1/(1-\ell)} < y_o < +\infty$, that is $-A < \lambda < 0$. Then the solution blows up at the point $x_o > 0$, uniquely defined by imposing $E(x_o) = -\lambda$, cf. (5.17).

ii) $y_o = \left(\frac{A}{e^{\psi(0)}}\right)^{1/(1-\ell)}$, that is $\lambda = 0$. Then the solution $y(x) = \left(\frac{e^{\psi(x)}}{E(x)}\right)^{\frac{1}{\ell-1}}$ is well defined analytic in $\mathbb{R}$. The decay at $-\infty$ is sub-exponential, whereas from (5.18) we get

$$y(x) \sim x^{\frac{1-2\gamma}{\ell - 1}} \quad for \quad x \to +\infty.$$

Note that $y(x)$ is then homoclinic, in the sense that $\lim_{x\to\pm\infty} y(x) = 0$, but Theorem 1.3 cannot be applied, since for $\varepsilon_o > \sigma/(\ell-1)$, with $\sigma = 2\gamma - 1$ in the present case, we have $\langle x\rangle^{\varepsilon_o} y(x) \notin L^\infty(\mathbb{R})$, hence $\langle x\rangle^{\varepsilon_o} y(x) \notin H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$ for $s > n/2$.

iii) $0 < y_o < \left(\frac{A}{e^{\psi(0)}}\right)^{1/(1-\ell)}$, that is $\lambda > 0$. In this case, since

$$0 < \lambda < \lambda + E(x) < \lambda + 2A$$

in view of (5.17), the solution $y(x)$ is well defined analytic in $\mathbb{R}$ and

$$0 < y(x) < \lambda^{1/(1-\ell)} e^{\psi(x)} \le c_1 e^{-c_2|x|^{2-2\gamma}}$$

for positive constants c_1, c_2. Similar sub-exponential bound is satisfied by $y'(x)$, hence $\langle x\rangle^{\varepsilon_o} y(x) \in H^1(\mathbb{R})$ for every $\varepsilon_o \in \mathbb{R}$. Therefore, Theorem 1.3 applies and gives the more precise information $y \in S^1_{\frac{1}{2-2\gamma}}(\mathbb{R})$.

A Appendix: Pseudodifferential operators on Gelfand-Shilov spaces

In the sequel, we will use the following notation:

$$e_1 = (1,0), e_2 = (0,1), e = (1,1).$$

Moreover, we will denote as standard $D_x^\alpha = (-i)^{|\alpha|}\partial_x^\alpha$ for all $\alpha \in \mathbb{Z}_+^n$.

Let $\overline{m} = (m_1, m_2) \in \mathbb{R}^2$ and let μ, ν be real numbers such that $\mu \ge 1, \nu \ge 1$. Let also $\varrho_1, \varrho_2, \delta_1, \delta_2$ be real numbers with $0 \le \delta_j < \varrho_j \le 1, j = 1, 2$ and denote $\bar\varrho = (\varrho_1, \varrho_2), \bar\delta = (\delta_1, \delta_2)$.

Definition A.1. *We shall denote by $\Gamma_{\nu,\mu}^{\overline{m},\bar\varrho,\bar\delta}$ the space of all functions $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition: there exists a positive constant C such that*

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \le C^{|\alpha|+|\beta|+1}(\alpha!)^\nu(\beta!)^\mu \langle\xi\rangle^{m_1-\varrho_1|\alpha|+\delta_1|\beta|}\langle x\rangle^{m_2-\varrho_2|\beta|+\delta_2|\alpha|} \tag{A.1}$$

for every $(x,\xi) \in \mathbb{R}^{2n}$ and $\alpha, \beta \in \mathbb{Z}_+^n$. We will denote by $OP\Gamma_{\nu,\mu}^{\overline{m},\bar\varrho,\bar\delta}$ the space of all operators (A.2) *with symbol in $\Gamma_{\nu,\mu}^{\overline{m},\bar\varrho,\bar\delta}$.*

We shall denote by $\Gamma_{\nu,\mu}^{0,\bar\varrho,\bar\delta}$ the class $\Gamma_{\nu,\mu}^{(0,0),\bar\varrho,\bar\delta}$.

Given $p \in \Gamma_{\nu,\mu}^{\overline{m},\bar\varrho,\bar\delta}$, we can consider the pseudodifferential operator defined as standard by

$$Pu(x) = p(x,D)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat u(\xi)d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{A.2}$$

where $\hat u$ denotes the Fourier transform of u.

Remark A.2. *In this Appendix we shall construct a calculus for the class* $OP\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *on the Gelfand-Shilov spaces* $S^{\mu}_{\nu}(\mathbb{R}^n)$*. First of all we notice that the class* $\Gamma^{m,\delta}_{\mu}$ *considered in the previous sections corresponds in the notation of this section to the class* $\Gamma^{me_1,e,\delta e_2}_{1,\mu}$*, so all the results presented here apply to* $\Gamma^{m,\delta}_{\mu}$*. We observe that most part of the results in the sequel can be proved following the same arguments used in other similar contexts, cf. [7], [9], [11]. For this reason some proofs will be just sketched or omitted for the sake of brevity.*

We start by giving a continuity theorem on Sobolev spaces for operators from $OP\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ which gives precise factorial estimates for the norm of the operators. This is an obvious consequence of the Weyl-Hörmander calculus, see [20].

Theorem A.3. *Given* $p \in \Gamma^{0,\bar{\varrho},\bar{\delta}}_{\nu,\mu}$*, the operator* $p(x,D)$ *defined by* (A.2) *is linear and continuous from* $H^s(\mathbb{R}^n)$ *to* $H^s(\mathbb{R}^n)$ *for every* $s \in \mathbb{R}$ *and*

$$\|p(x,D)\|_{\mathcal{L}(H^s,H^s)} \leq K \max_{|\alpha|+|\beta|\leq N} C^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu},$$

where C *is the constant appearing in* (A.1) *and the constants* K, N *depend only on* s *and on the dimension* n.

The next result states the action of the operators defined above on the Gelfand-Shilov spaces.

Theorem A.4. *Given* $p \in \Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$*, the operator* P *defined by* (A.2) *is linear and continuous from* $S^{\mu'}_{\nu'}(\mathbb{R}^n)$ *into itself for any* μ', ν' *with* $\mu' \geq \mu/(1-\delta_1), \nu' \geq \nu/(1-\delta_2)$*. Furthermore,* P *can be extended to a linear and continuous map from* $(S^{\mu'}_{\nu'}(\mathbb{R}^n))'$ *into itself.*

Proof. For any $\alpha, \beta \in \mathbb{Z}^n_+$ and for any positive integer N, we can write:

$$x^\alpha D^\beta_x Pu(x) = (2\pi)^{-n} x^\alpha \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \xi^{\beta_1} D^{\beta_2}_x p(x,\xi)\hat{u}(\xi)d\xi$$

$$= (2\pi)^{-n} x^\alpha \langle x\rangle^{-2N} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (1-\Delta_\xi)^N [\xi^{\beta_1} D^{\beta_2}_x p(x,\xi)\hat{u}(\xi)]d\xi.$$

Choosing $N = \left[\frac{|\alpha|+m_2}{2}\right] + 1$, by (1.11), (A.1) and by standard factorial inequalities, we obtain

$$\langle x\rangle^{|\alpha|-2N} \left|(1-\Delta_\xi)^N [\xi^{\beta_1} D^{\beta_2}_x p(x,\xi)\hat{u}(\xi)]\right|$$
$$\leq C^{|\alpha|+|\beta|+1}(\alpha!)^{\nu+\delta_2\nu'}(\beta_1!)^{\mu'}(\beta_2!)^{\mu+\delta_1\mu'} e^{-a\langle\xi\rangle^{1/\mu'}}$$

for some positive constants C, a. Then, by the conditions $\mu' \geq \mu/(1-\delta_1), \nu' \geq \nu/(1-\delta_2)$, it follows that P is continuous from $S^{\mu'}_{\nu'}(\mathbb{R}^n)$ into itself. By standard arguments we can extend P on the dual space $(S^{\mu'}_{\nu'})'(\mathbb{R}^n)$, cf. Theorem 2.2 in [11]. □

For $t \geq 0$, denote by Q_t the set

$$Q_t = \{(x,\xi) \in \mathbb{R}^{2n} : \langle \xi \rangle^{\varrho_1-\delta_1} < t \quad and \quad \langle x \rangle^{\varrho_2-\delta_2} < t\}$$

and

$$Q_t^e = \mathbb{R}^{2n} \setminus Q_t.$$

Definition A.5. *We denote by $FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ the space of all formal sums $\sum_{j\geq 0} p_j$ such that $p_j \in C^\infty(\mathbb{R}^{2n})$ for all $j \geq 0$ and there exist positive constants B, C such that $\forall j \geq 0$:*

$$\begin{aligned} \left|\partial_\xi^\alpha \partial_x^\beta p_j(x,\xi)\right| &\leq C^{|\alpha|+|\beta|+2j+1}(\alpha!)^\nu(\beta!)^\mu(j!)^{\mu+\nu-1} \\ &\times \langle \xi \rangle^{m_1-\varrho_1|\alpha|+\delta_1|\beta|-(\varrho_1-\delta_1)j}\langle x \rangle^{m_2-\varrho_2|\beta|+\delta_2|\alpha|-(\varrho_2-\delta_2)j} \end{aligned}$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$ and for all $(x,\xi) \in Q^e_{Bj^{\mu+\nu-1}}$.

Definition A.6. *We say that two sums $\sum_{j\geq 0} p_j, \sum_{j\geq 0} q_j \in FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ are equivalent if there exist positive constants B, C such that for every $N = 1, 2, ...$*

$$\begin{aligned} \left|\partial_\xi^\alpha \partial_x^\beta \sum_{j<N}(p_j - q_j)\right| &\leq C^{|\alpha|+|\beta|+2N+1}(\alpha!)^\nu(\beta!)^\mu(N!)^{\mu+\nu-1} \\ &\times \langle \xi \rangle^{m_1-\varrho_1|\alpha|+\delta_1|\beta|-(\varrho_1-\delta_1)N}\langle x \rangle^{m_2-\varrho_2|\beta|+\delta_2|\alpha|-(\varrho_2-\delta_2)N} \end{aligned}$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$ and for all $(x,\xi) \in Q^e_{BN^{\mu+\nu-1}}$. In this case, we write $\sum_{j\geq 0} p_j \sim \sum_{j\geq 0} q_j$.

Proposition A.7. *Let $p \in \Gamma^{0,\bar{\varrho},\bar{\delta}}_{\nu,\mu}$. If $p \sim 0$, then the operator P is $S^{\mu'}_{\nu'}$-regularizing, i.e. it extends to a continuous linear map from $(S^{\mu'}_{\nu'}(\mathbb{R}^n))'$ into $S^{\mu'}_{\nu'}(\mathbb{R}^n)$, for every μ', ν' such that $\min\{\mu',\nu'\} \geq \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$.*

Proof. In view of Proposition 2.11 in [7], it is sufficient to show that $p \in S^\theta_\theta(\mathbb{R}^{2n})$, with $\theta = \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$. Now, if $p \sim 0$, then there exist positive constants B, C such that

$$\begin{aligned} \left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| &\leq C^{|\alpha|+|\beta|+1}(\alpha!)^\nu(\beta!)^\mu\langle \xi \rangle^{-\varrho_1|\alpha|+\delta_1|\beta|}\langle x \rangle^{-\varrho_2|\beta|+\delta_2|\alpha|} \\ &\times \inf_{0\leq N\leq B(\max\{\langle \xi \rangle^{\varrho_1-\delta_1},\langle x \rangle^{\varrho_2-\delta_2}\})^{\frac{1}{\mu+\nu-1}}} \frac{C^{2N}(N!)^{\mu+\nu-1}}{(\langle \xi \rangle^{\varrho_1-\delta_1}\langle x \rangle^{\varrho_2-\delta_2})^N} \\ &\leq C^{|\alpha|+|\beta|+1}(\alpha!)^\nu(\beta!)^\mu\langle \xi \rangle^{-\varrho_1|\alpha|+\delta_1|\beta|}\langle x \rangle^{-\varrho_2|\beta|+\delta_2|\alpha|} \\ &\times \inf_{0\leq N\leq B(\max\{\langle \xi \rangle^{\varrho_1-\delta_1},\langle x \rangle^{\varrho_2-\delta_2}\})^{\frac{1}{\mu+\nu-1}}} \frac{C^{2N}(N!)^{\mu+\nu-1}}{(\max\{\langle \xi \rangle^{\varrho_1-\delta_1},\langle x \rangle^{\varrho_2-\delta_2}\})^N}. \end{aligned}$$

Furthermore, by standard arguments (see for example [30], Lemma 3.2.4), we have the following estimate:

$$(\alpha!)^\nu(\beta!)^\mu\langle \xi \rangle^{\delta_1|\beta|}\langle x \rangle^{\delta_2|\alpha|} \inf_{0\leq N\leq B(\max\{\langle \xi \rangle^{\varrho_1-\delta_1},\langle x \rangle^{\varrho_2-\delta_2}\})^{\frac{1}{\mu+\nu-1}}} \frac{C^{2N}(N!)^{\mu+\nu-1}}{(\max\{\langle \xi \rangle^{\varrho_1-\delta_1},\langle x \rangle^{\varrho_2-\delta_2}\})^N}$$

$$\leq C_2^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta \exp[-a((\max\{\langle\xi\rangle^{\varrho_1-\delta_1},\langle x\rangle^{\varrho_2-\delta_2}\})^{\frac{1}{\mu+\nu-1}}]$$
$$\leq C_2^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta \exp[-\frac{a}{2}(\langle x\rangle^{1/\theta}+\langle\xi\rangle^{1/\theta})] \tag{A.3}$$

for some constants $C_2, a > 0$. Then, $p \in S^\theta_\theta(\mathbb{R}^{2n})$. □

Remark A.8. *Notice that if R is S^μ_ν-regularizing, then in particular it is $\mathcal{S}$-regularizing, i.e. it maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

Every symbol $p \in \Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ can be identified with a sum $\sum_{j\geq 0} p_j \in FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$, by setting $p_0 = p$ and $p_j = 0 \quad \forall j \geq 1$. In order to construct a symbol in $\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ starting from a formal sum in $FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$, some restictions on μ,ν are necessary. In fact, the arguments in the following require the use of Gevrey cut-off functions of order μ and ν. This leads to assume the non-analyticity condition:

$$\mu > 1, \nu > 1. \tag{A.4}$$

Hence, the next results of this section hold for analytic symbols of $\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{1,1}$ only considering them as elements of $\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ for any choice of $\mu > 1, \nu > 1$.

With the same argument used in [7], Theorem 2.14, it is easy to prove the following result.

Proposition A.9. *Let $\sum_{j\geq 0} p_j \in FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$, with $\mu > 1, \nu > 1$. Then, for every fixed $R > 0$, we can find a sequence of nonnegative functions $\varphi_j \in C^\infty(\mathbb{R}^{2n})$ satisfying the following conditions:*

$$\varphi_0(x,\xi) = 1 \quad in \quad \mathbb{R}^{2n}, \tag{A.5}$$

$$\varphi_j(x,\xi) = 0 \quad in \quad Q_{2Rj^{\mu+\nu-1}} \quad and \quad \varphi_j(x,\xi) = 1 \quad in \quad Q^e_{3Rj^{\mu+\nu-1}}, \tag{A.6}$$

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} \left|\partial^\alpha_\xi \partial^\beta_x \varphi_j(x,\xi)\right| \leq C^{|\alpha|+|\beta|+1}(\alpha!)^\nu(\beta!)^\mu \left[Rj^{\mu+\nu-1}\right]^{-|\alpha|-|\beta|}, \quad j \geq 1 \tag{A.7}$$

for some positive constant C and such that the function

$$p(x,\xi) = \sum_{j\geq 0} \varphi_j(x,\xi) p_j(x,\xi) \tag{A.8}$$

is in $\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ and $p \sim \sum_{j\geq 0} p_j$ in $FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ for R sufficiently large.

Using the same arguments as in [11], we obtain the following results about the transpose of an operator from $OP\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ and the composition of two operators. We omit the proofs for the sake of brevity.

Proposition A.10. *Let $P = p(x,D) \in OP\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ and let ${}^tP$ be its transpose defined by*

$$\langle {}^tPu, v\rangle = \langle u, Pv\rangle, \quad u \in (S^{\mu'}_{\nu'}(\mathbb{R}^n))', v \in S^{\mu'}_{\nu'}(\mathbb{R}^n), \tag{A.9}$$

with $\mu' \geq \mu/(1-\delta_1), \nu' \geq \nu/(1-\delta_2)$ *as in Theorem A.4. Then,* ${}^tP = Q + R$*, where* $Q = q(x,D)$ *is in* $OP\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *with*

$$q(x,\xi) \sim \sum_{j\geq 0} \sum_{|\alpha|=j} (\alpha!)^{-1} \partial_\xi^\alpha D_x^\alpha p(x,-\xi)$$

in $FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *and* R *is a* $S^{\mu'}_{\nu'}$*-regularizing operator for any* μ', ν' *with* $\min\{\mu',\nu'\} \geq \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$.

Theorem A.11. *Let* $p \in \Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}, q \in \Gamma^{\overline{m}',\bar{\varrho},\bar{\delta}}_{\nu,\mu}$*. Then, there exists a symbol* $s \in \Gamma^{\overline{m}+\overline{m}',\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *such that* $p(x,D)q(x,D) = s(x,D) + R$ *for some* $S^{\mu'}_{\nu'}$*-regularizing operator* R*, with* μ', ν' *as in Proposition A.10. Moreover, if* $p \sim \sum_{j\geq 0} p_j$ *in* $FS^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *and* $q \sim \sum_{j\geq 0} q_j$ *in* $FS^{\overline{m}',\bar{\varrho},\bar{\delta}}_{\nu,\mu}$*, then*

$$s(x,\xi) \sim \sum_{j\geq 0} \sum_{h+k+|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha p_h(x,\xi) D_x^\alpha q_k(x,\xi) \quad in \quad FS^{\overline{m}+\overline{m}',\bar{\varrho},\bar{\delta}}_{\nu,\mu}. \tag{A.10}$$

Similarly, the commutator $[P,Q] = c(x,D) \in OP\Gamma^{\overline{m}+\overline{m}'-\bar{\varrho}+\bar{\delta},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *with*

$$c(x,\xi) \sim \sum_{\alpha\neq 0} \frac{1}{\alpha!} \left(\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) - \partial_\xi^\alpha q(x,\xi) D_x^\alpha p(x,\xi) \right)$$

in $FS^{\overline{m}+\overline{m}'-\bar{\varrho}+\bar{\delta},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$.

We now formulate the global hypoellipticity conditions in their general form for the class $\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$.

Definition A.12. *A symbol* $p \in \Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *is said to be globally hypoelliptic if there exist* $B, C_1, C_2 > 0$ *and* $m' = (m_1', m_2') \in \mathbb{R}^2$ *such that*

$$\inf_{(x,\xi)\in Q_B^e} \langle\xi\rangle^{-m_1'} \langle x\rangle^{-m_2'} |p(x,\xi)| = C_1 > 0 \tag{A.11}$$

and

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_2^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu |p(x,\xi)| \langle\xi\rangle^{-\varrho_1|\alpha|+\delta_1|\beta|} \langle x\rangle^{-\varrho_2|\beta|+\delta_2|\alpha|} \tag{A.12}$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$ *and* $(x,\xi) \in Q_B^e$.

Proposition A.13. *Let* p *be a globally hypoelliptic symbol in* $\Gamma^{\overline{m},\bar{\varrho},\bar{\delta}}_{\nu,\mu}$*. Then, there exists a left parametrix for* P *i.e. an operator* E *with symbol in* $\Gamma^{-\overline{m}',\bar{\varrho},\bar{\delta}}_{\nu,\mu}$ *such that* $EP = I + R$*, where* I *is the identity operator and* R *is a* $S^{\mu'}_{\nu'}$*-regularizing operator for every* μ', ν' *such that* $\min\{\mu',\nu'\} \geq \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$.

Proof. As standard, we construct the symbol $e(x,\xi)$ of E starting from its asymptotic expansion and applying Proposition A.9. Define

$$e_0(x,\xi) = p(x,\xi)^{-1}(1-\omega(x,\xi)) \tag{A.13}$$

where ω is a Gevrey function of order $\sigma = \min\{\mu,\nu\}$ with compact support such that $\omega = 1$ in a neighborhood of Q_B. It is easy to prove by induction on $|\alpha+\beta|$ that

$$\left|\partial_\xi^\alpha \partial_x^\beta e_0(x,\xi)\right| \leq C^{|\alpha|+|\beta|}(\alpha!)^\nu(\beta!)^\mu \langle\xi\rangle^{-\varrho_1|\alpha|+\delta_1|\beta|}\langle x\rangle^{-\varrho_2|\beta|+\delta_2|\alpha|}|e_0(x,\xi)| \tag{A.14}$$

for every $(x,\xi) \in Q_B^e$ and $\alpha,\beta \in \mathbb{Z}_+^n$. For $j = 1,2,...$ we can define by induction

$$e_j(x,\xi) = -e_0(x,\xi) \sum_{0<|\alpha|\leq j} \partial_\xi^\alpha e_{j-|\alpha|}(x,\xi) D_x^\alpha p(x,\xi). \tag{A.15}$$

Using (A.14) and arguing by induction on j, we deduce that $\sum_{j\geq 0} e_j \in FS_{\nu,\mu}^{-\overline{m}',\bar{\varrho},\bar{\delta}}$. Then, by Proposition A.9 we can find a symbol $e \in \Gamma_{\nu,\mu}^{-\overline{m}',\bar{\varrho},\bar{\delta}}$ such that $e \sim \sum_{j\geq 0} e_j$. Moreover, in view of (A.10), the symbol of $EP-I$ is equivalent to 0. We conclude applying Proposition A.7. □

As an immediate consequence of Proposition A.13 we obtain the following result of hypoellipticity for linear equations in Gelfand-Shilov spaces.

Theorem A.14. *Let p be a globally hypoelliptic symbol in $\Gamma_{\nu,\mu}^{\overline{m},\bar{\varrho},\bar{\delta}}$ and let $f \in S_{\nu'}^{\mu'}(\mathbb{R}^n)$, for some μ',ν' with $\min\{\mu',\nu'\} \geq \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$. Then, if $u \in (S_{\nu'}^{\mu'}(\mathbb{R}^n))'$ is a solution of the equation*

$$Pu = f,$$

then $u \in S_{\nu'}^{\mu'}(\mathbb{R}^n)$.

Proof. By Proposition A.13 there exists an operator E with symbol in $\Gamma_{\nu,\mu}^{-\overline{m}',\bar{\varrho},\bar{\delta}}$ such that

$$u = Ru + Ef$$

for some $S_{\nu'}^{\mu'}$-regularizing operator R. Then, $u \in S_{\nu'}^{\mu'}(\mathbb{R}^n)$. □

Remark A.15. *Notice that due to the restrictions on μ,ν in the pseudodifferential calculus, cf. Proposition A.9, Theorem A.14 does not cover the case $\mu = 1$ and/or $\nu = 1$. Hence, if we compare Theorem 1.2 with Theorem A.14, we notice that the latter result, though it is valid for the more general class $\Gamma_{\nu,\mu}^{\overline{m},\bar{\varrho},\bar{\delta}}$, does not give sharp estimates. This justifies the alternative approach used in the previous Sections 3 and 4 also for linear equations.*

We conclude the Appendix giving the proofs of Lemmas 3.5 and 4.1. To do this we need a preliminary technical result which could be stated in general for a symbol $p \in \Gamma_{\nu,\mu}^{\overline{m},\bar{\varrho},\bar{\delta}}$. For simplicity we shall prove it only for the class $\Gamma_\mu^{m,\delta}$ treated in the previous sections, since this is enough for our purposes.

Lemma A.16. *Let* $P = p(x, D)$ *with* $p \in \Gamma_{\mu}^{m,\delta}$ *satisfying* (1.19), (1.20) *and let* E *be its left parametrix. For every* $\alpha, \beta \in \mathbb{Z}_+^n$, *denote by* $r_{\alpha\beta}$ *the symbol of the operator* $E(\partial_\xi^\alpha D_x^\beta p)(x, D)$. *Then, for every* $\gamma, \theta \in \mathbb{Z}_+^n$, *there exists a positive constant* $C = C(\gamma, \theta)$ *independent of* α, β *such that*

$$|\partial_\xi^\theta \partial_x^\gamma r_{\alpha\beta}(x, \xi)| \leq C^{|\alpha|+|\beta|} \alpha! (\beta!)^\mu \langle \xi \rangle^{-|\alpha|-|\theta|} \langle x \rangle^{-|\beta|+\delta|\alpha|-|\gamma|+\delta|\theta|} \tag{A.16}$$

for all $(x, \xi) \in \mathbb{R}^{2n}$.

Proof. By Theorem A.11 and Proposition A.13, we have $r_{\alpha\beta} \sim \sum_{j \geq 0} r_{\alpha\beta j}$ with

$$r_{\alpha\beta j}(x, \xi) = \sum_{h+|\eta|=j} \frac{1}{\eta!} \partial_\xi^\eta e_h(x, \xi) \partial_\xi^\alpha D_x^{\beta+\eta} p(x, \xi),$$

where the functions e_h are defined by (A.13), (A.15). Then, by (A.14), (A.15) and by Leibniz rule it is easy to verify that

$$\begin{aligned} |\partial_\xi^\theta \partial_x^\gamma r_{\alpha\beta j}(x, \xi)| \quad \leq \quad & C^{|\alpha|+|\beta|+|\theta|+|\gamma|+2j+1} \alpha! (\beta!)^\mu (\theta!)^{\nu'} (\gamma!)^{\mu'} (j!)^{\mu'+\nu'-1} \\ & \times \langle \xi \rangle^{-|\alpha|-|\theta|-j} \langle x \rangle^{-|\beta|+\delta|\alpha|-|\gamma|+\delta|\theta|-(1-\delta)j}. \end{aligned} \tag{A.17}$$

for some $\mu' > 1, \nu' > 1, \mu' \geq \mu$. Then, in particular we deduce that $\sum_{j \geq 0} r_{\alpha\beta j} \in FS_{\nu',\mu'}^{(-|\alpha|, -|\beta|+\delta|\alpha|), e, \delta e_2}$. Finally we apply Proposition A.9, taking in (A.8) cut-off functions $\varphi_j(x, \xi)$ independent of α, β and obtain (A.16). □

Proof of Lemma 3.5. Estimate (3.9) is obvious by the previous arguments. Concerning (3.10), we can write

$$\begin{aligned} x^k P u(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} x^k p(x, \xi) \hat{u}(\xi) d\xi \\ &= (2\pi)^{-n} (-1)^{|k|} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} D_\xi^k \left(p(x, \xi) \hat{u}(\xi) \right) d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \sum_{\beta \leq k} \binom{k}{\beta} (-D_\xi^\beta) p(x, \xi) (-D_\xi)^{k-\beta} \hat{u}(\xi) d\xi \\ &= \sum_{\beta \leq k} \binom{k}{\beta} \left((-D_\xi)^\beta p \right) (x, D) (x^{k-\beta} u). \end{aligned}$$

Hence

$$\frac{1}{|k|!^\nu} E[P, x^k] u = -\frac{1}{|k|!^\nu} \sum_{0 \neq \beta \leq k} \binom{k}{\beta} E \left((-D_\xi)^\beta p \right) (x, D) (x^{k-\beta} u). \tag{A.18}$$

Therefore

$$\frac{1}{|k|!^\nu} \|E[P, x^k] u\|_s \leq \sum_{0 \neq \beta \leq k} \frac{1}{|k|!^\nu} \binom{k}{\beta} \left\| E(\partial_\xi^\beta p)(x, D) \langle x \rangle^{-\delta|\beta|} (\langle x \rangle^{\delta|\beta|} x^{k-\beta} u) \right\|_s.$$

At this moment observe that

$$\|E(\partial_\xi^\beta p)(x,D)\langle x\rangle^{-\delta|\beta|}\|_{\mathcal{L}(H^s,H^s)} \le C^{|\beta|+1}\beta!. \tag{A.19}$$

In fact, from Lemma A.16 we know that $E(\partial_\xi^\beta p)(x,D) = r_\beta(x,D)$ with $r_\beta(x,\xi)$ satisfying

$$|\partial_\xi^\theta \partial_x^\gamma r_\beta(x,\xi)| \le C^{|\beta|+1}\beta!\langle\xi\rangle^{-|\beta|-|\theta|}\langle x\rangle^{-|\gamma|+\delta|\beta|+\delta|\theta|} \tag{A.20}$$

for every $\theta,\gamma \in \mathbb{Z}_+^n$ and for some constant $C = C(\theta,\gamma,s) > 0$. At this moment we consider the operator $r_\beta(x,D)\langle x\rangle^{-\delta|\beta|}$ and its transpose, cf. Proposition A.10, with symbol given by $s_\beta(x,\xi) = \langle x\rangle^{-\delta|\beta|}\tilde{r}_\beta(x,\xi)$, with $\tilde{r}_\beta(x,D) = {}^t r_\beta(x,D)$. It is easy to recognize that also $\tilde{r}_\beta$ satisfies (A.20) and hence

$$|\partial_\xi^\theta \partial_x^\gamma s_\beta(x,\xi)| \le C_{\theta\gamma}C^{|\beta|}\beta!\langle\xi\rangle^{-|\beta|-|\theta|}\langle x\rangle^{|\gamma|+\delta|\theta|}.$$

Then, by Theorem A.3

$$\|s_\beta(x,D)\|_{\mathcal{L}(H^s,H^s)} \le KC^{|\beta|}\beta!,$$

with $K = \max_{|\theta|+|\gamma|\le N} C_{\theta\gamma}$ and we deduce (A.19). Summing up, we obtain

$$\frac{1}{|k|!^\nu}\|E[P,x^k]u\|_s \le \sum_{0\neq\beta\le k}\frac{1}{|k|!^\nu}\binom{k}{\beta}A_s^{|\beta|+1}\beta!\left\|\langle x\rangle^{\delta|\beta|}x^{k-\beta}u\right\|_s$$

for some constant A_s depending only on s and on the dimension n. Then we conclude observing that

$$\frac{1}{|k|!^\nu}\binom{k}{\beta}\beta! \le \frac{1}{(|k|(|k|-1)\dots(|k-\beta|+1))^{\nu-1}|k-\beta|!^\nu}.$$

□

Proof of Lemma 4.1. Assume initially $\varepsilon_o = 0$. To treat $[P,\partial_x^j]$ we write

$$\partial_x^j P = \sum_{\gamma\le j}\binom{j}{\gamma}(\partial_x^\gamma p)(x,D)\partial_x^{j-\gamma}.$$

Hence

$$E[P,\partial_x^j]u = \sum_{0\neq\gamma\le j}\binom{j}{\gamma}E(\partial_x^\gamma p)(x,D)\partial_x^{j-\gamma}u.$$

Applying Theorem A.3 and Lemma A.16, we obtain

$$\left\|E[P,\partial_x^j]u\right\|_s \le \sum_{0\neq\gamma\le j}\binom{j}{\gamma}C^{|\gamma|+1}(\gamma!)^\mu\|\partial_x^{j-\gamma}u\|_s$$

from which we deduce (4.2) for $\varepsilon_o = 0$. The case $\varepsilon_o > 0$ can be derived similarly. We leave details to the reader. □

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