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HOLOMORPHIC EXTENSION OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We prove sharp analytic estimates and holomorphic extensions in sectors of $\mathbb{C}^d$ for the solutions of a wide class of semilinear elliptic differential and pseudodifferential equations in $\mathbb{R}^d$, improving earlier results where the extension was shown for a strip. Moreover, we derive exponential decay estimates for such extended solutions. The results presented apply in particular to solitary wave solutions of many classical nonlinear evolution equations as Kdv-type, long-wave type and Schrödinger equations.

1. Introduction

The main concern in this paper is the study of holomorphic extensions of the solutions of semilinear elliptic equations in $\mathbb{R}^d$. Broadly speaking, we deal with equations of the form

$$Pu = F[u], \tag{1.1}$$

where P is a linear elliptic differential, or even pseudodifferential, operator in $\mathbb{R}^d$ and $F[u]$ is a nonlinearity, possibly involving the derivatives of u. For a wide class of equations of this type it is known that every solution u sufficiently regular and with a certain decay at infinity, actually is analytic on $\mathbb{R}^d$ and it extends to a holomorphic function in a strip of $\mathbb{C}^d$ of the form

$$\{z = x + iy \in \mathbb{C}^d : \ |y| < \varepsilon\}, \tag{1.2}$$

for some $\varepsilon > 0$, satisfying there the estimates

$$|u(x + iy)| \leq Ce^{-c|x|} \tag{1.3}$$

for some $C > 0$, $c > 0$. A pioneering work on this subject was the paper by Kato and Masuda [24]. Later the problem of the holomorphic extension in a strip has been intensively studied in connection with the applications to solitary wave equations. In particular it was noticed in dimension 1 that several model equations like the Korteweg-de Vries equations and its generalizations, Schrödinger-type equations,

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long wave-type equations admit solitary wave solutions which extend to meromorphic functions with poles out of a strip of the form (1.2) and having a decay of type (1.3). Among the main contributions in this sense we recall the papers by Bona and Li [8], [9], Grujic' and Kalisch [21], Bona, Grujic' and Kalisch [6], Bona and Weissler [10]. We also recall the paper [22] by Hayashi on well-posedness of the generalized KdV equation in Bergman-Szegö spaces of holomorphic functions in a sector. Apart from its interest "per se" in the analysis of regularity properties of the solutions of differential equations, the study of *complex* singularities of solitary waves could give information also on the onset of real blowup; we refer to Bona and Weissler [10] for a fascinating discussion in this connection.

Recently, the properties described above have been proved for some general classes of semilinear elliptic equations in any dimension, even with variable coefficients; see for example Biagioni and Gramchev [5], Gramchev [20] and Gramchev, Cappiello and Rodino [11, 12, 13, 14, 15]. The results in these papers have been stated and proved in terms of estimates in the Gelfand-Shilov spaces of type S, cf. [19]. We refer to Nicola and Rodino [27, Chapter 6] for a self-contained account of these results.

Nevertheless in the above mentioned papers some relevant issues remained unexplored. The first one is the identification of a *maximal* holomorphic extension. In other words, the problem is to understand what is the biggest complex domain on which a holomorphic extension is possible. The second one is related to a dual aspect and concerns the identification of the maximal domain on which the decay properties on $\mathbb{R}^d$ of solutions remain valid.

To be precise, let us introduce a class of operators which the above results and those in the present paper apply to. First, one can consider differential operators with polynomial coefficients

$$P = \sum_{|\alpha|\leq m,\, |\beta|\leq n} c_{\alpha\beta} x^\beta D^\alpha, \tag{1.4}$$

$m \geq 1$, $n \geq 0$, $c_{\alpha\beta} \in \mathbb{C}$, with symbol

$$p(x,\xi) = \sum_{|\alpha|\leq m,\, |\beta|\leq n} c_{\alpha\beta} x^\beta \xi^\alpha \tag{1.5}$$

of *G-elliptic* type, namely satisfying the following global version of ellipticity:

$$\sum_{|\alpha|=m,\, |\beta|\leq n} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0, \quad x \in \mathbb{R}^d,\ \xi \neq 0, \tag{1.6}$$

$$\sum_{|\alpha|\leq m,\, |\beta|=n} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0, \quad x \neq 0,\ \xi \in \mathbb{R}^d, \tag{1.7}$$

$$\sum_{|\alpha|=m,\,|\beta|=n} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0, \quad x \neq 0,\ \xi \neq 0. \tag{1.8}$$

If P has constant coefficients (that is $n = 0$) then (1.6), (1.7), (1.8) are satisfied if and only if P is elliptic and its symbol $p(\xi)$ satisfies $p(\xi) \neq 0$ in $\mathbb{R}^d$. As a relevant model one can consider the operator $P = -\Delta + \lambda$, $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{R}_- \cup \{0\}$.

More generally we deal with pseudodifferential operators. Namely, given real numbers $m > 0$, $n \geq 0$ we consider symbols $p(x,\xi)$ satisfying the following estimates

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C^{|\alpha|+|\beta|+1} \alpha!\beta! \langle \xi \rangle^{m-|\alpha|} \langle x \rangle^{n-|\beta|} \tag{1.9}$$

for every $(x,\xi) \in \mathbb{R}^{2d}, \alpha, \beta \in \mathbb{N}^d$ and for some positive constant C independent of α, β (we denote as usual $\langle x \rangle = (1 + |x|^2)^{1/2}$). Notice that every polynomial $p(x,\xi)$ as in (1.5) satisfies such estimates. For this general class the condition of G-ellipticity can be stated by requiring that

$$|p(x,\xi)| \geq c \langle \xi \rangle^m \langle x \rangle^n, \qquad |x| + |\xi| \geq C, \tag{1.10}$$

for some positive constants c, C. It is known that a symbol of the form (1.5) satisfies (1.10) if and only if it satisfies the three conditions (1.6), (1.7), (1.8) simultaneously (see [32, Proposition 1.4.37]) or [27, Theorem 3.2.9]).

Finally, concerning the nonlinear term $F[u]$ in (1.1), we assume here that it is of the form

$$F[u] = \sum_{h,l} \sum_{\rho_1,\dots,\rho_l} F_{h,l,\rho_1,\dots,\rho_l} x^h \prod_{k=1}^{l} \partial^{\rho_k} u, \tag{1.11}$$

where $h \in \mathbb{N}^d$, with $0 \leq |h| \leq \max\{n-1, 0\}$, $\rho_k \in \mathbb{N}^d$ with $0 \leq |\rho_k| \leq \max\{m-1, 0\}$, $l \in \mathbb{N}$, $l \geq 2$ and $F_{h,l,\rho_1,\dots,\rho_l} \in \mathbb{C}$ (the above sum being finite). Moreover, we allow some of the factors in (1.11) to be replaced by their complex conjugates.

As a simple example, consider the equation

$$-\Delta u + u = |u|^{l-1} u, \tag{1.12}$$

with $l \in \mathbb{N}$, $l > 2$ odd, which arises e.g. when looking for standing wave solutions for the Klein-Gordon or Schrödinger equation, as well as travelling wave solutions for the Klein-Gordon equation (cf. Berestycki and Lions [4]). The existence of solutions in $H^1(\mathbb{R}^d)$ and their exponential decay was studied in [4], whereas the possibility of extending them holomorphically on a strip has been recently shown in [13] (incidentally, the exponential decay generally drops for elliptic equations which are not globally elliptic, such as $-\Delta u = |u|^{l-1} u$).

The above more general class of operators includes, in dimension $d = 1$, the solitary wave counterpart of several evolution equations of Korteweg-de Vries type, as well as of long-wave type; see [9] and below. This also motivated the statements of the results for pseudodifferential operators, or at least for Fourier multipliers (e.g. $p(\xi) = \xi \operatorname{Coth} \xi + \lambda$, $\lambda > -1$, for the intermediate-long-wave equation; see [9]).

Now, it is known from [9] and [13, Theorem 7.3] that (for classes of nonlinearities that intersect the above one), if P is G-elliptic, then all the solutions u of (1.1) with $u \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + m - 1$ for $n > 0$ (respectively $\langle x \rangle^{\varepsilon_o} u \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + m - 1$ for $n = 0$) actually decay at infinity like $e^{-c|x|}, c > 0$, and extend holomorphically on a strip of the form (1.2).

Here we shall improve this result by showing that the holomorphic extension and the exponential decay actually hold in a *sector* of the complex domain. Namely, we have the following result, which seems new even for the simplest equation (1.12).

Theorem 1.1. *Let P be a pseudodifferential operator with symbol p satisfying* (1.9)*, $m > 0$, $n \geq 0$, and assume that p is G-elliptic, that is* (1.10) *is satisfied. Let $F[u]$ be of the form* (1.11) *(possibly with some factors in the product replaced by their conjugates). Assume moreover that $u \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + \max\{|\rho_k|\}$, is a solution of* (1.1)*. In the case $n = 0$ assume further $\langle x \rangle^{\varepsilon_0} u \in L^2(\mathbb{R}^d)$, for some $\varepsilon_0 > 0$. Then u extends to a holomorphic function in the sector of $\mathbb{C}^d$*

$$\mathcal{C}_\varepsilon = \{z = x + iy \in \mathbb{C}^d : \ |y| < \varepsilon(1 + |x|)\}, \tag{1.13}$$

for some $\varepsilon > 0$, satisfying there the estimates (1.3) *for some constants $C > 0, c > 0$.*

It will follow from the proof that the width of the sector depends in general on the solution u considered, in fact on upper bounds for the norm $\|u\|_{H^s}$ (and $\|\langle x \rangle^{\varepsilon_0} u\|_{L^2}$ if $n = 0$), although for nonlinearities of special type one could even replace these norms by others, corresponding to a lower regularity (see Section 5.1 below). In any case, for a given nonlinear equation the width of the sector cannot be expected to be the same for all the solutions; this is best seen for autonomous equations, where the width in fact *must* depend on u, because one can exploit the invariance with respect to translations to move the complex singularities parallel to $\mathbb{R}^d$.

The shape of the domain of holomorphic extension as a sector is sharp, in the sense that, even in dimension $d = 1$, for any angle $\theta \neq 0, \pi$ we can find G-elliptic equations with constant coefficients admitting Schwartz solutions whose meromorphic extensions have a sequence of poles along the ray $\arg z = \theta$. We refer to Section 5 below for details on this point and also for remarks on the a priori regularity assumptions in Theorem 1.1.

The linear case ($F[u] = 0$) deserves a separate discussion. Indeed, the analysis of the linear equation $Pu = 0$ is important for the holomorphic extension and the decay of eigenfunctions of G-elliptic operators and their powers; see Maniccia and Panarese [25] and Schrohe [31]. Moreover in the linear case it is possible to relax the assumptions on the regularity and decay of u, admitting solutions with a priori algebraic growth and in that case the width of the sector is independent on the particular solution considered; see Theorem 5.1 below.

Finally we present an application of the above result to solitary waves. Following [9], we consider the following class of Korteweg-de Vries-type equation

$$v_t + v_x + F[v]_x - (Mv)_x = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.14}$$

and long-wave-type equations

$$v_t + v_x + F[v]_x + (Mv)_t = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.15}$$

where $M = p(D)$ is a Fourier multiplier, $F[v]$ is a polynomial with real coefficients, $F(0) = F'(0) = 0$, and subscripts denote derivatives. We look for solutions $v(t,x)$ of solitary wave type, i.e. $v(t,x) = u(x - Vt)$, for some function u of one variable and some constant velocity V. We have the following result.

Theorem 1.2. *Let $p(\xi)$ satisfy the analytic symbol estimates of order $m \in \mathbb{R}$, namely*

$$|\partial^\alpha p(\xi)| \le A^{\alpha+1}\alpha!\langle\xi\rangle^{m-\alpha}, \quad \xi \in \mathbb{R},\ \alpha \in \mathbb{N} \tag{1.16}$$

for some constant $A > 0$, as well as the lower bounds

$$p(\xi) \ge 0,\ \xi \in \mathbb{R}, \qquad p(\xi) \ge C^{-1}|\xi|^m,\ |\xi| \ge C, \tag{1.17}$$

for some constant $C > 0$. Suppose moreover $m \ge 1$.

Let $v(t,x) = u(x - Vt)$ be a weak solution of (1.14) *or* (1.15)*, with $V > 1$, $u \in L^\infty(\mathbb{R})$, $\lim_{x\to\infty} u(x) = 0$. Then u extends to a holomorphic function $u(x+iy)$ in the sector*

$$\{z = x + iy \in \mathbb{C} :\ |y| < \varepsilon(1 + |x|)\} \tag{1.18}$$

for some $\varepsilon > 0$, satisfying there the estimates (1.3) *for some constants $C > 0, c > 0$.*

Notice that the estimates (1.16) are satisfied by any polynomial $p(\xi)$ of degree m. More generally, the condition (1.16) is equivalent to saying that $p(\xi)$ extends to a holomorphic function $p(\xi + i\eta)$ in a sector of the type (1.18), and satisfies there the bounds $|p(\xi + i\eta)| \le C'\langle\xi\rangle^m$ (see Proposition 5.2 below). This remark makes (1.16) very easy to check in concrete situations, where typically $p(\xi)$ is expressed in terms of elementary functions.

We also observe that, under the hypotheses of Theorem 1.2, we already know from [9] that u extends to a holomorphic function on a strip and displays there an exponential decay of type (1.3), hence Theorem 1.2 can be regarded as an improvement of this result. For the *existence* of solitary waves for these equations we refer to the detailed analysis in Albert, Bona and Saut [1], Amick and Toland [2], Benjamin, Bona and Bose [3], Weinstein [34].

Finally we mention that similar extensions results should hopefully be valid for other classes of non-linear elliptic equations, e.g. with linear part $P = -\Delta +$

$|x|^2$ (cf. [11]). Similarly, even non-elliptic hypoelliptic operators and dispersion managed solitons could share similar properties, possibly with the sector replaced by a smaller domain (but larger than a strip). We plan to investigate these issues in a subsequent paper.

The paper is organized as follows. Section 2 collects notation and some preliminary results about G-pseudodifferential operators (composition, boundedness on Sobolev spaces, parametrices, etc.). In Section 3 we introduce a suitable space of analytic functions on $\mathbb{R}^d$, which admit a holomorphic extension to sectors in $\mathbb{C}^d$, and we prove some relevant properties used in the sequel. In Section 4 we prove Theorem 1.1. The proof is based on the application of an iterative Picard scheme in the space of analytic functions defined in Section 3. In Section 5 we prove Theorem 1.2 and treat in detail the linear case $F[u] = 0$, see Theorem 5.1. We moreover show some other examples and counterexamples and test on them the sharpness of our results.

2. Notation and preliminary results

2.1. **Factorial and binomial coefficients.** We use the usual multi-index notation for factorial and binomial coefficients. Hence, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ we set $\alpha! = \alpha_1! \ldots \alpha_d!$ and for $\beta, \alpha \in \mathbb{N}^d$, $\beta \leq \alpha$, we set $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

The following inequality is standard and used often in the sequel:

$$\binom{\alpha}{\beta} \leq 2^{|\alpha|}. \tag{2.1}$$

Also, we recall the identity

$$\sum_{\substack{|\alpha'|=j \\ \alpha' \leq \alpha}} \binom{\alpha}{\alpha'} = \binom{|\alpha|}{j}, \qquad j = 0, 1, \ldots, |\alpha|,$$

which follows from $\prod_{i=1}^d (1+t)^{\alpha_i} = (1+t)^{|\alpha|}$, and gives in particular

$$\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}, \quad \alpha, \beta \in \mathbb{N}^d,\ \beta \leq \alpha. \tag{2.2}$$

The last estimate implies in turn, by induction,

$$\frac{\alpha!}{\delta_1! \ldots \delta_j!} \leq \frac{|\alpha|!}{|\delta_1|! \ldots |\delta_j|!}, \quad \alpha = \delta_1 + \ldots + \delta_j, \tag{2.3}$$

as well as

$$\frac{\alpha!}{(\alpha-\beta)!} \leq \frac{|\alpha|!}{|\alpha-\beta|!}, \quad \beta \leq \alpha. \tag{2.4}$$

Finally we recall the so-called inverse Leibniz' formula:

$$x^\beta \partial^\alpha u(x) = \sum_{\gamma\le\beta,\,\gamma\le\alpha} \frac{(-1)^{|\gamma|}\beta!}{(\beta-\gamma)!}\binom{\alpha}{\gamma}\partial^{\alpha-\gamma}(x^{\beta-\gamma}u(x)). \tag{2.5}$$

2.2. **G-Pseudo-differential operators.** Pseudo-differential operators are formally represented as integral operators of the type

$$p(x,D)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\xi} p(x,\xi)\widehat{u}(\xi)\,d\xi, \tag{2.6}$$

where

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi}u(x)\,dx$$

denotes the Fourier transform of u and $p(x,\xi)$ is the so-called *symbol* of $p(x,D)$. According to the symbol spaces which p belongs to, one can consider u in several classes of functions or distributions and symbolic calculi and boundedness results on Sobolev spaces are available. We briefly recall this for the class of the so-called G-pseudodifferential operators (also named SG or *scattering* pseudodifferential operators in the literature). They are defined by the formula (2.6), where $p(x,\xi)$ satisfies, for some $m,n\in\mathbb{R}$, the following estimates: for every $\alpha,\beta\in\mathbb{N}^d$ there exists a constant $C_{\alpha,\beta}>0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{n-|\beta|}\langle\xi\rangle^{m-|\alpha|} \tag{2.7}$$

for every $x,\xi\in\mathbb{R}^d$. The space of functions satisfying these estimates is denoted by $G^{m,n}(\mathbb{R}^d)$, whereas we set $\mathrm{OP}G^{m,n}(\mathbb{R}^d)$ for the corresponding operators. We endow $G^{m,n}(\mathbb{R}^d)$ with the topology defined by the seminorms

$$\|p\|_N^{(G)} = \sup_{|\alpha|+|\beta|\le N}\ \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left\{|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\langle x\rangle^{-n+|\beta|}\langle\xi\rangle^{-m+|\alpha|}\right\},\quad N\in\mathbb{N}.$$

As a prototype one can take $P=-\Delta+\lambda$, $\lambda\in\mathbb{C}$, of order $m=2$, $n=0$. More generally, the case of Fourier multipliers, where the symbol $p(\xi)$ depends only on ξ (hence $n=0$) is of great interest, mostly for applications to solitary waves and ground state equations, see [13]. As another example, we have $x^\beta\partial^\alpha\in\mathrm{OP}G^{|\alpha|,|\beta|}(\mathbb{R}^d)$.

The classes $\mathrm{OP}G^{m,n}(\mathbb{R}^d)$ were introduced in [28] and studied in detail in [16], [17], [26], [30], [32]. They are in fact a particular case of the general Hörmander's classes, see [23, Chapter XVIII], and turn out to be very convenient for a series of problems involving global aspects of partial differential equations in $\mathbb{R}^d$.

We now summarize some properties which will be useful for us later on; beside the above mentioned papers, we refer to [27, Chapter 3] for a detailed and self-contained presentation.

First, if $p \in G^{m,n}(\mathbb{R}^d)$ then $p(x,D)$ defines a continuous map $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ which extends to a continuous map $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. The composition of two such operators is therefore well defined in $\mathcal{S}(\mathbb{R}^d)$ and in $\mathcal{S}'(\mathbb{R}^d)$; more precisely, if $p_1 \in G^{m_1,n_1}(\mathbb{R}^d)$ and $p_2 \in G^{m_2,n_2}(\mathbb{R}^d)$, then $p_1(x,D)p_2(x,D) = p_3(x,D)$ with $p_3 \in G^{m_1+m_2,n_1+n_2}(\mathbb{R}^d)$ and the map $(p_1,p_2) \mapsto p_3$ is continuous $G^{m_1,n_1}(\mathbb{R}^d) \times G^{m_2,n_2}(\mathbb{R}^d) \to G^{m_1+m_2,n_1+n_2}(\mathbb{R}^d)$.

It is useful to consider the action of such operators on the standard Sobolev spaces

$$H^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) :\ \|u\|_s := \left(\int |\widehat{u}(\xi)|^2(1+|\xi|^2)^s\, d\xi\right)^{1/2} < \infty\},$$

and on the weighted versions

$$H^{s_1,s_2}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) :\ \|u\|_{s_1,s_2} := \|\langle x\rangle^{s_2} u\|_{s_1} < \infty\}.$$

Notice that

$$\bigcup_{s_1,s_2\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d) \qquad \textit{and} \qquad \bigcap_{s_1,s_2\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d).$$

Indeed, if $p \in G^{m,n}(\mathbb{R}^d)$ then

$$p(x,D) : H^{s_1,s_2}(\mathbb{R}^d) \to H^{s_1-m,s_2-n}(\mathbb{R}^d) \tag{2.8}$$

continuously, and

$$\|p(x,D)\|_{\mathcal{B}(H^{s_1,s_2}(\mathbb{R}^d),H^{s_1-m_1,s_2-m_2}(\mathbb{R}^d))} \le C\|p\|_N^{(G)}$$

for suitable $C > 0$, $N \in \mathbb{N}$ depending only on s_1, s_2, m_1, m_2 and on the dimension d (see [27, Theorem 3.1.5]). In particular, for $s_2 = 0$ we see that, if $n \le 0$ then $p(x,D) : H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d)$ continuously for every $s \in \mathbb{R}$. We also recall that $\bigcap_{m,n\in\mathbb{R}} G^{m,n}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$. In particular, operators with Schwartz symbols are (globally) regularizing, i.e. they map continuously $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$.

Remark 2.1. *The complex interpolation for the spaces $H^{s_1,s_2}(\mathbb{R}^d)$ works as one expects, i.e. for $s_1, s_2, t_1, t_2 \in \mathbb{R}$, $0 < \theta < 1$,*

$$[H^{s_1,t_1}(\mathbb{R}^d), H^{s_2,t_2}(\mathbb{R}^d)]_\theta = H^{s,t}(\mathbb{R}^d), \quad s = (1-\theta)s_1+\theta s_2,\ t = (1-\theta)t_1+\theta t_2, \tag{2.9}$$

see for example [18, 33]. *The property* (2.9) *will be useful in the sequel in view of the following consequence: suppose that $u \in H^s(\mathbb{R}^d)$ and $\langle x\rangle^{\varepsilon_0} u \in L^2(\mathbb{R}^d)$ for some $\varepsilon_0 > 0$. Then for every $s' < s$ there exists $\varepsilon > 0$ such that $\langle x\rangle^{\varepsilon} u \in H^{s'}(\mathbb{R}^d)$.*

A symbol $p \in G^{m,n}(\mathbb{R}^d)$ (and the corresponding operator) is called *G-elliptic* if it satisfies (1.10) for some constants $C, c > 0$. For example, $P = -\Delta + \lambda$ is G-elliptic if and only if $\lambda \notin \mathbb{R}_- \cup \{0\}$. More generally, as we mentioned before, for an operator with polynomial coefficients as in (1.4), G-ellipticity is equivalent to (1.6), (1.7), (1.8) to hold simultaneously (see [32, Proposition 1.4.37] or [27, Theorem 3.2.9]).

The importance of G-ellipticity in the subsequent arguments relies in the fact that this condition guaranties the existence of a parametrix $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$ of $P = p(x, D)$. Namely we have the following result, see [16, 28] for the proof.

Proposition 2.2. *Let $p \in G^{m,n}(\mathbb{R}^d)$ be G-elliptic. Then there exists an operator $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$ such that $EP = I + R$ and $PE = I + R'$, where R, R' are (globally) regularizing pseudodifferential operators, i.e. with Schwartz symbols. Hence R and R' are continuous maps $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. The operator E is said to be a parametrix for P.*

Remark 2.3. *We emphasize the fact that in the following (see Proposition 4.3 below) actually we shall not use the fact that the remainder R in Proposition 2.2 has arbitrarily small orders. To prove our results it is sufficient to consider it as a symbol in $G^{-1,-1}(\mathbb{R}^d)$.*

Finally we point out for further reference the following formulas, which can be verified by a direct computation: for $\alpha, \beta \in \mathbb{N}^d$,

$$x^\beta P u = \sum_{\gamma \le \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} (D_\xi^\gamma p)(x, D)(x^{\beta-\gamma} u), \tag{2.10}$$

$$\partial^\alpha P u = \sum_{\delta \le \alpha} \binom{\alpha}{\delta} (\partial_x^\delta p)(x, D) \partial^{\alpha-\delta} u. \tag{2.11}$$

3. A space of analytic functions

We introduce here a space of analytic functions in $\mathbb{R}^d$, already considered in [12] (and denoted there by $S_1^{1\star}(\mathbb{R}^d)$), which is tailored to the problem of the holomorphic extension to subsets of $\mathbb{C}^d$ of the form (1.13), as shown by the subsequent Theorem 3.2.

Definition 3.1. *We denote by $\mathcal{A}_{sect}(\mathbb{R}^d)$ the space of all functions $f \in C^\infty(\mathbb{R}^d)$ satisfying the following condition: there exists a constant $C > 0$ such that*

$$|x^\beta \partial^\alpha f(x)| \le C^{|\alpha|+|\beta|+1} \max\{|\alpha|, |\beta|\}!, \quad \text{for all } \alpha,\ \beta \in \mathbb{N}^d. \tag{3.1}$$

Theorem 3.2. *Let $f \in \mathcal{A}_{sect}(\mathbb{R}^d)$. Then f extends to a holomorphic function $f(x + iy)$ in the sector of $\mathbb{C}^d$*

$$\mathcal{C}_\varepsilon = \{z = x + iy \in \mathbb{C}^d : \ |y| < \varepsilon(1 + |x|)\} \tag{3.2}$$

for some $\varepsilon > 0$, satisfying there the estimates

$$|f(x + iy)| \le C e^{-c|x|}, \tag{3.3}$$

for some constants $C > 0$, $c > 0$.

Proof. First we show the estimates

$$|x^\beta\partial^\alpha f(x)|\leq C^{|\alpha|+1}|\alpha|!e^{-c|x|},\quad \text{for } |\beta|\leq|\alpha|. \tag{3.4}$$

Indeed, since $|x|^n\leq k^n\sum_{|\gamma|=n}|x^\gamma|$ for a constant $k>0$ depending only on the dimension d, by (3.1) we have (assuming $C\geq 1$ in (3.1))

$$\begin{aligned} e^{c|x|}|x^\beta\partial^\alpha f(x)| &= \sum_{n=0}^{\infty}\frac{(c|x|)^n}{n!}|x^\beta\partial^\alpha f(x)| \\ &\leq \sum_{n=0}^{\infty}(ck)^n\sum_{|\gamma|=n}\frac{1}{|\gamma|!}|x^{\beta+\gamma}\partial^\alpha f(x)| \\ &\leq \sum_{n=0}^{\infty}(ck)^n\sum_{|\gamma|=n}C^{2|\alpha|+|\gamma|+1}\frac{(|\alpha|+|\gamma|)!}{|\gamma|!}. \end{aligned}$$

Since the number of multi-indices γ satisfying $|\gamma|=n$ does not exceed 2^{d+n-1}, an application of (2.1) gives (3.4) for a new constant C, if c is small enough.

Now, (3.4) and the estimate $|\alpha|!\leq d^{|\alpha|}\alpha!$ give

$$|\partial^\alpha f(x)|\leq C^{|\alpha|+1}\alpha!\langle x\rangle^{-|\alpha|}e^{-c|x|}, \tag{3.5}$$

for a new constant $C>0$. This shows that the power series

$$\sum_\alpha \frac{\partial^\alpha f(x)}{\alpha!}(z-x)^\alpha, \tag{3.6}$$

for any fixed $x\in\mathbb{R}^d$ converges in a polydisc in $\mathbb{C}^d$ defined by $|z_k-x_k|<\frac{\langle x\rangle}{2C}$, $1\leq k\leq d$. The union of such polydiscs, when x varies in $\mathbb{R}^d$, cover a subset $\mathcal{C}_\varepsilon\subset\mathbb{C}^d$ of the type (3.2), for some $\varepsilon>0$. Since on the intersection (when not empty) of two such polydiscs these extensions agree ($\mathbb{R}^d\subset\mathbb{C}^d$ is totally real), $f(x)$ extends to a holomorphic function on $\mathcal{C}_\varepsilon$. For $z\in\mathcal{C}_\varepsilon$, using the representation (3.6) as a power series with $x=\operatorname{Re} z$ and (3.5), we also get the desired estimate (3.3) for a new constant $C>0$. ∎

In the sequel we will use the following characterization of the space $\mathcal{A}_{sect}(\mathbb{R}^d)$ in terms of H^s-based norms.

Set, for $f\in\mathcal{S}'(\mathbb{R}^d)$,

$$S_\infty^{s,\varepsilon}[f]=\sum_{\alpha,\beta}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|x^\beta\partial^\alpha f\|_s. \tag{3.7}$$

Proposition 3.3. *Let $f\in\mathcal{A}_{sect}(\mathbb{R}^d)$. Then for every $s\geq 0$ there exists $\varepsilon>0$ such that $S_\infty^{s,\varepsilon}[f]<\infty$.*

In the opposite direction, if for some $s \geq 0$ there exists $\varepsilon > 0$ such that $S^{s,\varepsilon}_\infty[f] < \infty$ then $f \in \mathcal{A}_{sect}(\mathbb{R}^d)$.

Proof. Assume $f \in \mathcal{A}_{sect}(\mathbb{R}^d)$. It suffices to argue when $s = k$ is integer. Then

$$\|x^\beta \partial^\alpha f\|_k \leq C' \sum_{|\gamma|\leq k} \|\partial^\gamma\big(x^\beta\partial^\alpha f\big)\|_{L^2}.$$

Now, if $M \in \mathbb{N}$ satisfies $M > d/4$ we have

$$\|\partial^\gamma\big(x^\beta\partial^\alpha f\big)\|_{L^2} \leq C''\|(1+|x|^2)^M\partial^\gamma\big(x^\beta\partial^\alpha f\big)\|_{L^\infty}. \tag{3.8}$$

On the other hand we have, by Leibniz' formula and (3.1),

$$\begin{aligned}\|(1+|x|^2)^M\partial^\gamma\big(x^\beta\partial^\alpha f\big)\|_{L^\infty} &\leq \sum_{\sigma\leq\gamma,\,\sigma\leq\beta}\binom{\gamma}{\sigma}\frac{\beta!}{(\beta-\sigma)!}\|(1+|x|^2)^M x^{\beta-\sigma}\partial^{\alpha+\gamma-\sigma}f\|_{L^\infty}\\ &\leq C_\gamma|\beta|^{|\gamma|}C^{|\alpha|+|\beta|}\max\{2M+|\beta|,|\alpha|+|\gamma|\}!.\end{aligned}$$

Since $\max\{2M+|\beta|,|\alpha|+|\gamma|\} \leq \max\{|\beta|,|\alpha|\}+2M+|\gamma|$ and $|\beta|^{|\gamma|} \leq \tilde{C}_\gamma^{|\beta|}$, by (2.1) we get

$$\|x^\beta\partial^\alpha f\|_k \leq C_k^{|\alpha|+|\beta|+1}\max\{|\alpha|,|\beta|\}!$$

for some constant $C_k > 0$. Hence $S^{s,\varepsilon}_\infty[f] < \infty$ if $\varepsilon < C_k^{-1}$.

In the opposite direction, we may take $s = 0$; hence assume $S^{0,\varepsilon}_\infty[f] < \infty$ for some $\varepsilon > 0$. Then (3.1) holds with the L^∞ norm replaced by the L^2 norm. If M is an integer, $M > d/2$, we have

$$\|x^\beta\partial^\alpha f\|_{L^\infty} \leq C\sum_{|\gamma|\leq M}\|\partial^\gamma\big(x^\beta\partial^\alpha f\big)\|_{L^2}.$$

Hence an application of Leibniz' formula and the same arguments as above show that $f \in \mathcal{A}_{sect}(\mathbb{R}^d)$. □

4. Proof of the main result (Theorem 1.1)

In this section we prove Theorem 1.1. In fact we shall state and prove this result for the more general non-homogeneous equation

$$Pu = f + F[u], \tag{4.1}$$

where P and $F[u]$ satisfy the assumptions of Theorem 1.1 and f is a function in the space $\mathcal{A}_{sect}(\mathbb{R}^d)$ defined in Section 3. Moreover we shall restate our result in terms of estimates in $\mathcal{A}_{sect}(\mathbb{R}^d)$. Namely, in view of Theorem 3.2, it will be sufficient to prove the following theorem.

Theorem 4.1. *Let P be a pseudodifferential operator with symbol p satisfying* (1.9)*, $m>0, n\geq 0$ and assume that p is G-elliptic, that is* (1.10) *is satisfied. Let $F[u]$ be of the form* (1.11) *(possibly with some factors in the product replaced by their conjugates) and $f\in\mathcal{A}_{sect}(\mathbb{R}^d)$. Assume moreover that $u\in H^s(\mathbb{R}^d)$, $s>\frac{d}{2}+\max\{|\rho_k|\}$, is a solution of* (4.1)*. In the case $n=0$ assume further $\langle x\rangle^{\varepsilon_0}u\in L^2(\mathbb{R}^d)$ for some $\varepsilon_0>0$. Then $u\in\mathcal{A}_{sect}(\mathbb{R}^d)$.*

In fact we always assume that $F[u]$ has the form in (1.11), and we leave to the reader the easy changes when some factors of the product in (1.11) are replaced by their conjugates.

We start by showing that, under the assumptions of Theorem 4.1, u is in fact a Schwartz function.

Lemma 4.2. *Let P, m, n, $F[u]$, f be as in Theorem 4.1. Let u be a solution of* (4.1) *satisfying $\langle x\rangle^{\varepsilon_0}u\in H^s(\mathbb{R}^d)$ for some $s>\frac{d}{2}+\max\{|\rho_k|\}$, $\varepsilon_0\geq 0$. Then, when $n>0$, we have $\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}u\in H^s(\mathbb{R}^d)$ for every $\sigma_1\leq\min\{m,1\}$, and $\sigma_2\leq\min\{n,1\}$. If $n=0$ and we assume in addition $\varepsilon_0>0$, then we have $\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}u\in H^s(\mathbb{R}^d)$ for every $\sigma_1\leq\min\{m,1\}$ and $\sigma_2\leq\min\{\varepsilon_0,1\}$.*

Proof. We first consider the case $n>0$. Let $E\in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$ be a parametrix for P; hence $R:=EP-I\in\mathrm{OP}G^{-1,-1}(\mathbb{R}^d)$. We have from (4.1)

$$\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}u=\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}Ef-\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}Ru+\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}EF[u]. \tag{4.2}$$

Since $\sigma_1\leq 1$ and $\sigma_2\leq 1$, the operator $\langle x\rangle^{\sigma_2}\langle D\rangle^{\sigma_1}R\in\mathrm{OP}G^{0,0}(\mathbb{R}^d)$ is bounded on $H^{s,\varepsilon_0}(\mathbb{R}^d)$; cf. (2.8). Taking also into account the assumptions on f it follows therefore that the H^s-norm of the first two terms in the right-hand side of (4.2) is finite. Concerning the last term, observe that, by the assumptions on σ_1,σ_2 and h, the operator $\langle x\rangle^{\sigma_2}\langle D\rangle^{\sigma_1}\circ E\circ x^h\in\mathrm{OP}G^{-m+\sigma_1,-n+\sigma_2+|h|}(\mathbb{R}^d)$ belongs in fact to $\mathrm{OP}G^{-\max\{m-1,0\},0}(\mathbb{R}^d)$. As a consequence, since $M:=\max\{|\rho_k|\}\leq\max\{m-1,0\}$, it is bounded $H^{s-M,\varepsilon_0}(\mathbb{R}^d)\to H^{s,\varepsilon_0}(\mathbb{R}^d)$. Hence by Schauder's estimates (recall that $s>d/2+M$) we have

$$\begin{aligned}\|\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}EF[u]\|_s&\leq C_s\sum_l\sum_{\rho_1,\dots,\rho_l}\|\langle x\rangle^{\varepsilon_0}\prod_{k=1}^l\partial^{\rho_k}u\|_{s-M}\\&\leq C'_s\sum_l\sum_{\rho_1,\dots,\rho_l}\|\langle x\rangle^{l\varepsilon_0}\prod_{k=1}^l\partial^{\rho_k}u\|_{s-M}\\&\leq C''_s\sum_l\sum_{\rho_1,\dots,\rho_l}\prod_{k=1}^l\|\langle x\rangle^{\varepsilon_0}\partial^{\rho_k}u\|_{s-M}\leq C'''_s\sum_l\|\langle x\rangle^{\varepsilon_0}u\|_s^l<\infty.\end{aligned}$$

We treat now the case $n = 0$, hence $h = 0$ in the nonlinearity (1.11). We consider again (4.2). For the terms $\langle x\rangle^{\varepsilon_o+\sigma_2}\langle D\rangle^{\sigma_1}Ef$ and $\langle x\rangle^{\varepsilon_o+\sigma_2}\langle D\rangle^{\sigma_1}Ru$ we argue as before. For the nonlinear term observe that, since $M := \max\{|\rho_k|\} \le \max\{m-1, 0\}$, we have that $\langle D\rangle^{\sigma_1}E \in \mathrm{OP}G^{-\max\{m-1,0\},0}(\mathbb{R}^d)$ is bounded $H^{s-M,l\varepsilon_0}(\mathbb{R}^d) \to H^{s,l\varepsilon_0}(\mathbb{R}^d)$, for every l; see (2.8). Hence for $\sigma_2 \le \varepsilon_0 \le \varepsilon_0(l-1)$ we get

$$\begin{aligned}
\|\langle x\rangle^{\varepsilon_0+\sigma_2}\langle D\rangle^{\sigma_1}EF[u]\|_s &\le C_s\sum_l\sum_{\rho_1,\dots,\rho_l}\|\langle x\rangle^{l\varepsilon_0}\langle D\rangle^{\sigma_1}E\prod_{k=1}^l\partial^{\rho_k}u\|_s \\
&\le C'_s\sum_l\sum_{\rho_1,\dots,\rho_l}\|\langle x\rangle^{l\varepsilon_0}\prod_{k=1}^l\partial^{\rho_k}u\|_{s-M} \\
&\le C''_s\sum_l\sum_{\rho_1,\dots,\rho_l}\prod_{k=1}^l\|\langle x\rangle^{\varepsilon_0}\partial^{\rho_k}u\|_{s-M} \\
&\le C'''_s\sum_l\|\langle x\rangle^{\varepsilon_0}u\|_s^l < \infty,
\end{aligned}$$

where we applied again Schauder's estimate and, in the last inequality, (2.8) to ∂^{ρ_k}. ∎

Let us observe that, when $n > 0$, an iterated application of Lemma 4.2 shows that, under the assumptions of Theorem 4.1, $\langle x\rangle^{\tau_2}\langle D\rangle^{\tau_1}u \in H^s(\mathbb{R}^d)$ for every $\tau_1 > 0, \tau_2 > 0$, that is $u \in \mathcal{S}(\mathbb{R}^d)$. The same is true when $n = 0$, because the assumptions $u \in H^s(\mathbb{R}^d)$, $s > d/2 + \max\{|\rho_k|\}$, and $\langle x\rangle^{\varepsilon_0}u \in L^2(\mathbb{R}^d)$, $\varepsilon_0 > 0$, imply that for new values of s and ε_0 as above we have $\langle x\rangle^{\varepsilon_0}u \in H^s(\mathbb{R}^d)$ (see Remark 2.1), and Lemma 4.2 still allows us to upgrade regularity and decay.

In particular, the sum $S_N^{s,\varepsilon}[u]$ is finite for every $N \in \mathbb{N}$.

In order to prove Theorem 4.1 it suffices to verify that $S_\infty^{s,\varepsilon}[u] < \infty$, in view of Proposition 3.3. This will be achieved by an iteration argument involving the partial sum of the series in (3.7), that is

$$S_N^{s,\varepsilon}[f] = \sum_{|\alpha|+|\beta|\le N}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|x^\beta\partial^\alpha f\|_s. \tag{4.3}$$

We shall treat separately the cases $m \ge 1$ and $0 < m < 1$, since the study of the nonlinearity requires different arguments.

4.1. **Proof of Theorem 4.1: the case** $m \ge 1$**.** We need several estimates to which we address now. It is understood that they hold for arbitrary $N \ge 1$, with constants independent on N.

Proposition 4.3. *Let $R \in \mathrm{OP}G^{-1,-1}(\mathbb{R}^d)$. Then for every $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that, for every $\varepsilon \leq 1, N \in \mathbb{N}$ and $u \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\sum_{0<|\alpha|+|\beta|\leq N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|R(x^\beta \partial^\alpha u)\|_s \leq C_s \varepsilon S^{s,\varepsilon}_{N-1}[u].$$

Proof. We first estimate the terms with $\alpha = 0$, hence $\beta \neq 0$. Let $j \in \{1,\dots,d\}$ such that $\beta_j \neq 0$. Since $R \circ x_j \in \mathrm{OP}G^{-1,0}(\mathbb{R}^d)$ is bounded on $H^s(\mathbb{R}^d)$ we have[1]

$$\frac{\varepsilon^{|\beta|}}{|\beta|!}\|R(x^\beta u)\|_s \leq C_s \varepsilon \frac{\varepsilon^{|\beta|-1}}{|\beta|!}\|x^{\beta-e_j}u\|_s.$$

Similarly one argues if $\beta = 0$, $\alpha \neq 0$. If finally $\alpha \neq 0$, $\beta \neq 0$, hence for some $j,k \in \{1,\dots,d\}$, we have $\alpha_j \neq 0$, $\beta_k \neq 0$, we write

$$x^\beta \partial^\alpha = \partial_j \circ x_k x^{\beta-e_k}\partial^{\alpha-e_j} - \beta_j x^{\beta-e_j}\partial^{\alpha-e_j}$$

and use the fact that $R\,\partial_j \circ x_k \in \mathrm{OP}G^{0,0}(\mathbb{R}^d)$ is bounded on $H^s(\mathbb{R}^d)$. We get

$$\begin{aligned}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|R(x^\beta\partial^\alpha u)\|_s \leq{}& C_s\varepsilon^2 \frac{\varepsilon^{|\alpha|+|\beta|-2}}{\max\{|\alpha|,|\beta|\}!}\|x^{\beta-e_k}\partial^{\alpha-e_j}u\|_s \\ &+ C_s\varepsilon^2\frac{\varepsilon^{|\alpha|+|\beta|-2}}{\max\{|\alpha|-1,|\beta|-1\}!}\|x^{\beta-e_j}\partial^{\alpha-e_j}u\|_s,\end{aligned}$$

(we understand that the second term in the right-hand side is omitted if $\beta_j = 0$). These estimates give at once the desired result if $\varepsilon \leq 1$. ∎

Proposition 4.4. *Let $P = p(x,D)$ be a pseudodifferential with symbol $p(x,\xi)$ satisfying the estimates* (1.9)*, with $m \geq 0$, $n \geq 0$. Let $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$. Then for every $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that, for every ε small enough, $N \in \mathbb{N}$ and $u \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\sum_{0<|\alpha|+|\beta|\leq N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|E[P,x^\beta\partial^\alpha]u\|_s \leq C_s\varepsilon S^{s,\varepsilon}_{N-1}[u]. \tag{4.4}$$

Proof. We have

$$[P,x^\beta\partial^\alpha] = [P,x^\beta]\partial^\alpha + x^\beta[P,\partial^\alpha].$$

[1]We denote by e_j the jth vector of the standard basis of $\mathbb{R}^d$.

Hence, using (2.10), (2.11), we get

$$[P, x^\beta \partial^\alpha]u = \sum_{0\neq\gamma_0\leq\beta} (-1)^{|\gamma_0|+1}\binom{\beta}{\gamma_0}(D_\xi^{\gamma_0}p)(x,D)(x^{\beta-\gamma_0}\partial^\alpha u) - \sum_{0\neq\delta\leq\alpha}\binom{\alpha}{\delta} x^\beta \partial_x^\delta p(x,D)\partial^{\alpha-\delta}u. \tag{4.5}$$

Given β, δ, let $\tilde{\delta}$ be a multi-index of maximal length among those satisfying $|\tilde{\delta}| \leq |\delta|$ and $\tilde{\delta} \leq \beta$ (hence $|\tilde{\delta}| = |\delta|$ unless $\beta - \tilde{\delta} = 0$). Writing $x^\beta = x^{\tilde{\delta}} x^{\beta-\tilde{\delta}}$ in the last term of (4.5) and using again (2.10) we get

(4.6)

$$[P, x^\beta \partial^\alpha]u = \sum_{\delta\leq\alpha} \sum_{\substack{\gamma_0\leq\beta-\tilde{\delta} \\ (\delta,\gamma_0)\neq(0,0)}} (-1)^{|\gamma_0|+1}\binom{\beta-\tilde{\delta}}{\gamma_0}\binom{\alpha}{\delta} x^{\tilde{\delta}}(D_\xi^{\gamma_0}\partial_x^\delta p)(x,D)(x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta}u).$$

We now look at the operator $x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta}$. Given γ_0, α, δ, let $\tilde{\gamma}_0$ be a multi-index of maximal length among those satisfying $|\tilde{\gamma}_0| \leq |\gamma_0|$ and $\tilde{\gamma}_0 \leq \alpha - \delta$. We write, by the inverse Leibniz formula (2.5),

$$\begin{aligned} x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta} &= x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\tilde{\gamma}_0}\partial^{\alpha-\delta-\tilde{\gamma}_0} = \partial^{\tilde{\gamma}_0} \circ x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta-\tilde{\gamma}_0} \\ &+ \sum_{\substack{0\neq\gamma_1\leq\beta-\tilde{\delta}-\gamma_0 \\ \gamma_1\leq\tilde{\gamma}_0}} \frac{(-1)^{|\gamma_1|}(\beta-\tilde{\delta}-\gamma_0)!}{(\beta-\tilde{\delta}-\gamma_0-\gamma_1)!}\binom{\tilde{\gamma}_0}{\gamma_1}\partial^{\tilde{\gamma}_0-\gamma_1} \circ x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0}. \end{aligned} \tag{4.7}$$

We now look at the operator $x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0}$. We denote by $\tilde{\gamma}_1$ a multi-index of maximal length among those satisfying $|\tilde{\gamma}_1| \leq |\gamma_1|$, $\tilde{\gamma}_1 \leq \alpha - \delta - \tilde{\gamma}_0$. Again by the inverse Leibniz formula we have

$$\begin{aligned} x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0} &= x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\tilde{\gamma}_1}\partial^{\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1} \\ &= \partial^{\tilde{\gamma}_1} \circ x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1} \\ &+ \sum_{\substack{0\neq\gamma_2\leq\beta-\tilde{\delta}-\gamma_0-\gamma_1 \\ \gamma_2\leq\tilde{\gamma}_1}} \frac{(-1)^{|\gamma_2|}(\beta-\tilde{\delta}-\gamma_0-\gamma_1)!}{(\beta-\tilde{\delta}-\gamma_0-\gamma_1-\gamma_2)!}\binom{\tilde{\gamma}_1}{\gamma_2}\partial^{\tilde{\gamma}_1-\gamma_2} \circ x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1-\gamma_2}\partial^{\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1}. \end{aligned} \tag{4.8}$$

Continuing in this way and substituting all in (4.6) we get

$$[P, x^\beta\partial^\alpha]u = \sum_{\delta\le\alpha}\sum_{j=0}^{h}\sum_{\substack{\gamma_0\le\beta-\tilde\delta\\(\delta,\gamma_0)\ne(0,0)}}\ \sum_{\substack{0\ne\gamma_1\le\beta-\tilde\delta-\gamma_0\\ \gamma_1\le\tilde\gamma_0}}\cdots\sum_{\substack{0\ne\gamma_j\le\beta-\tilde\delta-\gamma_0-\ldots-\gamma_{j-1}\\ \gamma_j\le\tilde\gamma_{j-1}}} C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}$$
$$\times\, p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,D)\big(x^{\beta-\tilde\delta-\gamma_0-\ldots-\gamma_j}\partial^{\alpha-\delta-\tilde\gamma_0-\ldots-\tilde\gamma_j}u\big),$$

where $\tilde\gamma_j$ is defined inductively as a multi-index of maximal length among those satisfying $|\tilde\gamma_j|\le|\gamma_j|$ and $\tilde\gamma_j\le\alpha-\delta-\tilde\gamma_0-\ldots-\tilde\gamma_{j-1}$,

$$|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}| = \frac{\alpha!(\beta-\tilde\delta)!}{(\alpha-\delta)!\delta!\gamma_0!(\beta-\tilde\delta-\gamma_0-\ldots-\gamma_j)!}\prod_{k=1}^{j}\binom{\tilde\gamma_{k-1}}{\gamma_k}$$
$$\le \frac{|\alpha|!|\beta-\tilde\delta|!}{|\alpha-\delta|!\delta!\gamma_0!|\beta-\tilde\delta-\gamma_0-\ldots-\gamma_j|!}2^{|\tilde\gamma_0+\ldots+\tilde\gamma_{j-1}|}, \tag{4.9}$$

cf. (2.4) and (2.1), and

$$p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,\xi) = x^{\tilde\delta}\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x,\xi)\xi^{\tilde\gamma_0-\gamma_1+\tilde\gamma_1-\ldots-\gamma_j+\tilde\gamma_j},\quad j\ge0, \tag{4.10}$$

(if $j=0$ in (4.9) we mean that there are not the binomial factors, nor the power of 2). Observe that, since we have $\gamma_j\ne0$ for every $j\ge1$, this procedure in fact stops after a finite number of steps.

Now we observe that, by (1.9), (2.1), and Leibniz' formula, for every $\theta,\sigma\in\mathbb{N}^d$ we have

$$|\partial_\xi^\theta\partial_x^\sigma p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,\xi)| \le C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!\langle x\rangle^{n-|\sigma|}\langle\xi\rangle^{m-|\theta|}, \tag{4.11}$$

for some constant C depending only on θ and σ. In fact $|\tilde\delta|\le|\delta|$, $|\tilde\gamma_0-\gamma_1+\tilde\gamma_1-\ldots-\gamma_j+\tilde\gamma_j|\le|\tilde\gamma_0|\le|\gamma_0|$, and the powers of $|\delta|$ and $|\gamma_0|$ which arise can be estimated by $C^{|\gamma_0|+|\delta|+1}$ for some $C>0$.

We now use these last bounds to estimate $E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,D)$. To this end, observe that this operator belongs to $\mathrm{OP}G^{0,0}(\mathbb{R}^d)$, and therefore its norm as a bounded operator on $H^s(\mathbb{R}^d)$ is estimated by a seminorm of its symbol in $G^{0,0}(\mathbb{R}^d)$, depending only on s and d. Such a seminorm is in turn estimated by the product of a seminorm of the symbol of E in $G^{-m,-n}(\mathbb{R}^d)$ and a seminorm of $p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}$ in $G^{m,n}(\mathbb{R}^d)$, again depending only on s,d. Hence we see from (4.11) that

$$\|E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,D)\|_{\mathcal{B}(H^s(\mathbb{R}^d))} \le C_s^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!. \tag{4.12}$$

Let now $|\beta|\ge|\alpha|$. Then $|\tilde\delta|=|\delta|$. Using moreover the estimate

$$\frac{|\alpha|!|\beta-\tilde\delta|!}{|\beta|!|\alpha-\delta|!} = \frac{|\alpha|!(|\beta|-|\delta|)!}{|\beta|!(|\alpha|-|\delta|)!}\le 1,$$

together with (4.9) and (4.12), we obtain

(4.13)
$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\beta|!}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\|E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)(x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j}u)\|_s$$
$$\le C_s(C_s\varepsilon)^{|\delta|+|\gamma_0+\dots+\gamma_j|+|\tilde{\gamma}_0+\dots+\tilde{\gamma}_{j-1}|}\frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\gamma_0+\dots+\gamma_j|-|\tilde{\gamma}_0+\dots+\tilde{\gamma}_{j-1}|}}{|\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|!}$$
$$\times\|x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j}u\|_s.$$

Similarly, if $|\alpha|\ge|\beta|$ we have $|\tilde{\gamma}_k|=|\gamma_k|$, $0\le k\le j$, and

$$\frac{|\beta-\tilde{\delta}|!|\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j|!}{|\alpha-\delta|!|\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|!}\le 1, \tag{4.14}$$

(recall that if $|\tilde{\delta}|<|\delta|$ then $\beta-\tilde{\delta}=\gamma_0=\dots=\gamma_j=\tilde{\gamma}_0=\dots=\tilde{\gamma}_j=0$).

By (4.9), (4.12) (4.14), we get in this case

(4.15)
$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\|E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)(x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j}u)\|_s$$
$$\le C_s(C_s\varepsilon)^{|\delta|+|\gamma_0+\dots+\gamma_j|+|\tilde{\gamma}_0+\dots+\tilde{\gamma}_{j-1}|}\frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\gamma_0+\dots+\gamma_j|-|\tilde{\gamma}_0+\dots+\tilde{\gamma}_{j-1}|}}{|\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j|!}$$
$$\times\|x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j}u\|_s.$$

Since, if $|\beta|\ge|\alpha|$, we have

$$\max\{|\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|,|\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j|\}=|\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|,$$

whereas if $|\alpha|\ge|\beta|$ it turns out

$$\max\{|\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|,|\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j|\}=|\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j|,$$

we deduce from (4.13) and (4.15) that, if ε is small enough,

$$\sum_{|\alpha|+|\beta|\le N}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|E[P,x^\beta\partial^\alpha]u\|_s$$
$$\le C_s\sum_{|\tilde{\alpha}|+|\tilde{\beta}|\le N-1}\frac{\varepsilon^{|\tilde{\alpha}|+|\tilde{\beta}|}}{\max\{|\tilde{\alpha}|,|\tilde{\beta}|\}!}\|x^{\tilde{\beta}}\partial^{\tilde{\alpha}}u\|_s\sum_{j=0}^{h}\sum_{\delta}\sum_{\substack{\gamma_1\neq0,\dots,\gamma_j\neq0\\ \gamma_0:\,(\delta,\gamma_0)\neq(0,0)}}(C_s\varepsilon)^{|\delta|+|\gamma_0+\gamma_1+\dots+\gamma_j|}$$
$$\le S^{s,\varepsilon}_{N-1}[u]\sum_{j=0}^{h}(C'_s\varepsilon)^{j+1}\le C''_s\varepsilon S^{s,\varepsilon}_{N-1}[u].$$

□

We now turn the attention to the nonlinear term.

Proposition 4.5. *Let* $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$*,* $m \geq 1$*,* $n \geq 0$*, and* $h \in \mathbb{N}^d$*,* $|h| \leq \max\{n-1, 0\}$*,* $\rho_k \in \mathbb{N}^d$*,* $|\rho_k| \leq m-1$*, for* $1 \leq k \leq l$*,* $l \geq 2$*.*

Then for every $s > d/2 + \max_k\{|\rho_k|\}$ *there exists a constant* $C_s > 0$ *such that, for every* ε *small enough,* $N \in \mathbb{N}$ *and* $u \in \mathcal{S}(\mathbb{R}^d)$*, the following estimates hold:*

$$\sum_{0<|\alpha|+|\beta|\leq N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!} \|E\big(x^\beta \partial^\alpha \big(x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big)\big)\|_s \leq C_s \varepsilon \big(\|u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u] + (S_{N-1}^{s,\varepsilon}[u])^l\big) \tag{4.16}$$

if $n \geq 1$ *and*

$$\sum_{0<|\alpha|+|\beta|\leq N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!} \|E\big(x^\beta \partial^\alpha \prod_{k=1}^{l} \partial^{\rho_k} u\big)\|_s \leq C_s \varepsilon \big(\|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u] + (S_{N-1}^{s,\varepsilon}[u])^l\big). \tag{4.17}$$

if $0 \leq n < 1$.

Proof. Let $n \geq 1$, (hence $|h| \leq n-1$). In the sum (4.16) we consider the terms with $\alpha = 0$. Namely, we prove that

$$\sum_{0\neq|\beta|\leq N} \frac{\varepsilon^{|\beta|}}{|\beta|!} \|E\big(x^\beta x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big)\|_s \leq C_s \varepsilon \|u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u]. \tag{4.18}$$

Given $\beta \neq 0$, let $j \in \{1, \ldots, d\}$ such that $\beta_j \neq 0$. Since $E \circ x_j x^h \in \mathrm{OP}G^{-m,-n+|h|+1}(\mathbb{R}^d)$ is bounded $H^{s-M}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$, with $M = \max_k\{|\rho_k|\}$ (because $|h| \leq n-1$, $|\rho_k| \leq m-1$), and applying Schauder's estimates (recall that $s - M > d/2$) we have

$$\frac{\varepsilon^{|\beta|}}{|\beta|!} \|E\big(x^\beta x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big)\|_s \leq C_s \varepsilon \frac{\varepsilon^{|\beta|-1}}{(|\beta|-1)!} \|x^{\beta-e_j} \partial^{\rho_1} u\|_{s-M} \|u\|_s^{l-1}.$$

Then (4.18) follows by writing

$$x^{\beta-e_j} \partial^{\rho_1} u = \sum_{\substack{\gamma \leq \beta - e_j \\ \gamma \leq \rho_1}} \frac{(-1)^{|\gamma|}(\beta-e_j)!}{(\beta-e_j-\gamma)!} \binom{\rho_1}{\gamma} \partial^{\rho_1-\gamma}(x^{\beta-e_j-\gamma} u). \tag{4.19}$$

in view of(2.5), and using

$$\frac{(\beta-e_j)!}{(|\beta|-1)!(\beta-e_j-\gamma)!} \leq \frac{1}{(|\beta|-1-|\gamma|)!}, \tag{4.20}$$

cf. (2.4).

For $0 \leq n < 1$, consider first the terms in (4.17) with $\alpha = 0$. We prove that

$$\sum_{0 \neq |\beta| \leq N} \frac{\varepsilon^{|\beta|}}{|\beta|!} \|E\big(x^\beta \prod_{k=1}^{l} \partial^{\rho_k} u\big)\|_s \leq C_s \varepsilon \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u].$$

To this end, given $\beta \neq 0$, let $j \in \{1, \ldots, d\}$ such that $\beta_j \neq 0$. Since E is bounded $H^{s-M}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$, $M = \max_k\{|\rho_k|\}$, by Schauder's estimates we have

$$\begin{aligned} \frac{\varepsilon^{|\beta|}}{|\beta|!} \|E\big(x^\beta \prod_{k=1}^{l} \partial^{\rho_k} u\big)\|_s &\leq C_s' \varepsilon \frac{\varepsilon^{|\beta|-1}}{(|\beta|-1)!} \|x^{\beta-e_j} \partial^{\rho_1} u\|_{s-M} \|x_j \prod_{k=2}^{l} \partial^{\rho_k} u\|_{s-M} \\ &\leq C_s'' \varepsilon \frac{\varepsilon^{|\beta|-1}}{(|\beta|-1)!} \|x^{\beta-e_j} \partial^{\rho_1} u\|_{s-M} \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1}, \end{aligned}$$

cf. the action on weighted Sobolev spaces described in Section 2. Then the claim follows by applying again (4.19) and (4.20).

We now treat the terms with $\alpha \neq 0$ in the sums (4.16) and (4.17). Namely, we prove that (both in the cases $0 \leq n < 1$ and $n \geq 1$)

$$\sum_{\substack{0<|\alpha|+|\beta|\leq N \\ \alpha\neq 0}} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|, |\beta|\}!} \|E\big(x^\beta \partial^\alpha \big(x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big)\big)\|_s \leq C_s \varepsilon (S_{N-1}^{s,\varepsilon}[u])^l. \tag{4.21}$$

Let $\alpha \neq 0$ and $j \in \{1, \ldots, d\}$ such that $\alpha_j \neq 0$. We can write

$$x^\beta \partial^\alpha \big(x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big) = Q_1^{\alpha,\beta}[u] + Q_2^{\alpha,\beta}[u],$$

with

$$\begin{aligned} Q_1^{\alpha,\beta}[u] &= \partial_{x_j} x^\beta \partial^{\alpha-e_j} \big(x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big), \\ Q_2^{\alpha,\beta}[u] &= -\beta_j x^{\beta-e_j} \partial^{\alpha-e_j} \big(x^h \prod_{k=1}^{l} \partial^{\rho_k} u\big). \end{aligned}$$

Now we estimate $\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!} \|EQ_1^{\alpha,\beta}[u]\|_s$. To this end observe that, by Leibniz' formula,

$$Q_1^{\alpha,\beta}[u] = \partial_{x_j} \sum_{\substack{\delta_0+\delta_1+\ldots+\delta_l=\alpha-e_j \\ \delta_0 \leq h}} \frac{(\alpha-e_j)!}{\delta_0!\delta_1!\ldots\delta_l!} \frac{h!}{(h-\delta_0)!} x^{h-\delta_0} x^\beta \prod_{k=1}^{l} \partial^{\delta_k+\rho_k} u.$$

Let now $\tilde{\delta}_0$ be a multi-index of maximal length among those satisfying $|\tilde{\delta}_0| \leq |\delta_0|$ and $\tilde{\delta}_0 \leq \beta$. Observe that $E\partial_{x_{j_\alpha}} \circ x^{h-\delta_0} x^{\tilde{\delta}_0} \in \mathrm{OP}G^{-m+1,-n+|h|}(\mathbb{R}^d)$ is bounded

$H^{s-M}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ with $M = \max_k\{|\rho_k|\}$ (because $|\rho_k| \leq m-1$ for $1 \leq k \leq l$ and $|h| \leq \max\{n-1, 0\} \leq n$). Hence

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|EQ_1^{\alpha,\beta}[u]\|_s \leq C_s \sum_{\substack{\delta_0+\delta_1+\ldots+\delta_l=\alpha-e_j\\ \delta_0\leq h}} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\frac{(\alpha-e_j)!}{\delta_0!\delta_1!\ldots\delta_l!}\frac{h!}{(h-\delta_0)!} \times \|x^{\beta-\tilde{\delta}_0}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u\|_{s-M}.$$

We can now write

$$x^{\beta-\tilde{\delta}_0}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u = \prod_{k=1}^{l}x^{\gamma_k}\partial^{\delta_k+\rho_k}u, \tag{4.22}$$

where $\gamma_1+\ldots+\gamma_l = \beta-\tilde{\delta}_0$ and, if $|\beta| \leq |\alpha|-1$, with $|\gamma_k| \leq |\delta_k|$ for $1 \leq k \leq l$ (which is possible because in that case $|\beta-\tilde{\delta}_0| \leq |\alpha-\delta_0|-1$; observe that if $|\tilde{\delta}_0| < |\delta_0|$ then $\beta-\tilde{\delta}_0 = 0$), whereas, if $|\beta| \geq |\alpha|$, with $|\gamma_k| \geq |\delta_k|$ for $1 \leq k \leq l$ (which is possible because in that case $|\tilde{\delta}_0| = |\delta_0|$ and $|\beta-\tilde{\delta}_0| \geq |\alpha-\delta_0| \geq |\alpha-\delta_0|-1$).

Hence we get by Schauder's estimates

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|EQ_1^{\alpha,\beta}[u]\|_s \leq C_s\varepsilon \sum_{\substack{\delta_0+\delta_1+\ldots+\delta_l=\alpha-e_j\\ \delta_0\leq h}} \prod_{k=1}^{l}\frac{\varepsilon^{|\gamma_k|+|\delta_k|}}{\max\{|\gamma_k|,|\delta_k|\}!}\|x^{\gamma_k}\partial^{\delta_k+\rho_k}u\|_{s-M}, \tag{4.23}$$

where if $|\beta| \leq |\alpha|-1$ we used the inequality

$$\frac{1}{(|\alpha|-1)!}\frac{(\alpha-e_j)!}{\delta_0!\delta_1!\ldots\delta_l!} \leq \frac{1}{|\delta_0|!|\delta_1|!\ldots|\delta_l|!}, \tag{4.24}$$

which is (2.3), whereas if $|\alpha| \leq |\beta|$ we applied

$$\frac{1}{|\beta|!}\frac{(\alpha-e_j)!}{\delta_0!\ldots\delta_l!} \leq \frac{1}{|\tilde{\delta}_0|!|\gamma_1|!\ldots|\gamma_l|!}, \tag{4.25}$$

which also follows at once from (2.3).

Now, we write $x^{\gamma_k}\partial^{\delta_k+\rho_k}u = \partial^{\rho_k}\big(x^{\gamma_k}\partial^{\delta_k}u\big) + [x^{\gamma_k}\partial^{\delta_k}, \partial^{\rho_k}]u$ in the last term of (4.23), so that

$$\|x^{\gamma_k}\partial^{\delta_k+\rho_k}u\|_{s-M} \leq \|x^{\gamma_k}\partial^{\delta_k}u\|_s + \|\langle D\rangle^{-|\rho_k|}[x^{\gamma_k}\partial^{\delta_k}, \partial^{\rho_k}]u\|_s.$$

Using this last estimate we get

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|EQ_1^{\alpha,\beta}[u]\|_s$$

$$\leq C_s\varepsilon \sum_{\substack{\delta_0+\delta_1+\ldots+\delta_l=\alpha-e_j\\ \delta_0\leq h}} \prod_{k=1}^{l} \frac{\varepsilon^{|\gamma_k|+|\delta_k|}}{\max\{|\gamma_k|,|\delta_k|\}!}\Big\{\|x^{\gamma_k}\partial^{\delta_k}u\|_s + \sum_{|\gamma|\leq m-1}\|\langle D\rangle^{-|\gamma|}[x^{\gamma_k}\partial^{\delta_k},\partial^\gamma]u\|_s\Big\},$$

(recall that the γ_k's depend on $\beta,\delta_1,\ldots,\delta_l$). We now sum the above expression over $|\alpha|+|\beta|\leq N$, $\alpha\neq 0$. When α and β vary, every term in the above product also appears in the development of

$$\Big\{\sum_{|\tilde\alpha|+|\tilde\beta|\leq N-1}\frac{\varepsilon^{|\tilde\alpha|+|\tilde\beta|}}{\max\{|\tilde\alpha|,|\tilde\beta|\}!}\Big\{\|x^{\tilde\beta}\partial^{\tilde\alpha}u\|_s + \sum_{|\gamma|\leq m-1}\|\langle D\rangle^{-|\gamma|}[x^{\tilde\beta}\partial^{\tilde\alpha},\partial^\gamma]u\|_s\Big\}\Big\}^l,$$

and is repeated at most, say, L times, with L depending only on h and the dimension d. Hence we obtain

$$\sum_{\substack{|\alpha|+|\beta|\leq N\\ \alpha\neq 0}}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|EQ_1^{\alpha,\beta}[u]\|_s$$

$$\leq C_s''\varepsilon\Big\{\sum_{|\tilde\alpha|+|\tilde\beta|\leq N-1}\frac{\varepsilon^{|\tilde\alpha|+|\tilde\beta|}}{\max\{|\tilde\alpha|,|\tilde\beta|\}!}\Big\{\|x^{\tilde\beta}\partial^{\tilde\alpha}u\|_s + \sum_{|\gamma|\leq m-1}\|\langle D\rangle^{-|\gamma|}[x^{\tilde\beta}\partial^{\tilde\alpha},\partial^\gamma]u\|_s\Big\}\Big\}^l$$

$$\leq C_s''\varepsilon\big\{S_{N-1}^{s,\varepsilon}[u]+C_s'''\varepsilon S_{N-2}^{s,\varepsilon}[u]\big\}^l \leq C_s''''\varepsilon(S_{N-1}^{s,\varepsilon}[u])^l,$$

where we used Proposition 4.4 applied with ∂^γ and $\langle D\rangle^{-|\gamma|}$ in place of P and E respectively, and we understand $S_{-1}^{s,\varepsilon}[u]=0$.

We finally show that

$$\sum_{\substack{|\alpha|+|\beta|\leq N\\ \alpha\neq 0}}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|EQ_2^{\alpha,\beta}[u]\|_s \leq C_s\varepsilon(S_{N-1}^{s,\varepsilon}[u])^l. \tag{4.26}$$

Since the arguments are similar to the previous ones, we give only a sketch of the proof.

We can write

$$Q_2^{\alpha,\beta}[u]=\beta_j\sum_{\substack{\delta_0+\delta_1+\ldots+\delta_l=\alpha-e_j\\ \delta_0\leq h}}\frac{(\alpha-e_j)!}{\delta_0!\delta_1!\ldots\delta_l!}\frac{h!}{(h-\delta_0)!}x^{h-\delta_0}x^{\beta-e_j}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u.$$

If $\beta_j\neq 0$, we choose a multi-index $\tilde\delta_0$ of maximal length among those satisfying $|\tilde\delta_0|\leq|\delta_0|$ and $\tilde\delta_0\leq\beta-e_j$. Next, we use the fact that $E\circ x^{h-\delta_0}x^{\tilde\delta_0}\in$

$\mathrm{OP}G^{-m,-n+|h|}(\mathbb{R}^d)$ is bounded $H^{s-M}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ with $M = \max_k\{|\rho_k|\}$ and we apply the decomposition

$$x^{\beta-e_j-\tilde{\delta}_0}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u = \prod_{k=1}^{l}x^{\gamma_k}\partial^{\delta_k+\rho_k}u,$$

with $\gamma_1+\ldots+\gamma_l = \beta - e_j - \tilde{\delta}_0$, and $|\gamma_k| \le |\delta_k|$ for $1 \le k \le l$, if $|\beta| \le |\alpha|$, or $|\gamma_k| \ge |\delta_k|$ for $1 \le k \le l$, if $|\alpha| \le |\beta|$. Moreover, if $|\beta| \le |\alpha|$ one uses

$$\frac{\beta_j}{|\alpha|}\frac{1}{(|\alpha|-1)!}\frac{(\alpha-e_j)!}{\delta_0!\delta_1!\ldots\delta_l!} \le \frac{1}{|\delta_0|!|\delta_1|!\ldots|\delta_l|!},$$

in place of (4.24), whereas if $|\alpha| \ge |\beta|$ (hence $|\tilde{\delta}_0| = |\delta_0|$) one uses

$$\frac{\beta_j}{|\beta|}\frac{1}{(|\beta|-1)!}\frac{(\alpha-e_j)!}{\delta_0!\ldots\delta_l!} \le \frac{1}{|\tilde{\delta}_0|!|\gamma_1|!\ldots|\gamma_l|!},$$

in place of (4.25). Therefore we get the same formula (4.23) with $Q_2^{\alpha,\beta}$ in place of $Q_1^{\alpha,\beta}$. The proof then proceeds as that for $Q_1^{\alpha,\beta}$ without other modifications. □

We are now ready to conclude the proof of Theorem 4.1.

End of the proof of Theorem 4.1 (the case $m \ge 1$). It follows from (4.1) that, for $\alpha, \beta \in \mathbb{N}^d$, $\varepsilon > 0$,

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}x^\beta\partial^\alpha Pu = \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}x^\beta\partial^\alpha f + \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}x^\beta\partial^\alpha F[u],$$

so that

$$\begin{aligned}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}P(x^\beta\partial^\alpha u) = \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}[P,x^\beta\partial^\alpha]u + \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}x^\beta\partial^\alpha f \\ + \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}x^\beta\partial^\alpha F[u].\end{aligned}$$

We now apply to both sides the parametrix E of P. With $R = EP - I \in \mathrm{OP}G^{-1,-1}(\mathbb{R}^d)$ we get

$$\begin{aligned}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}x^\beta\partial^\alpha u = -\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}R(x^\beta\partial^\alpha u) + \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}E[P,x^\beta\partial^\alpha]u \\ + \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}E(x^\beta\partial^\alpha f) + \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}E(x^\beta\partial^\alpha F[u]).\end{aligned}$$

Taking the H^s norms and summing over $|\alpha|+|\beta|\le N$ give

$$
\begin{aligned}
S_N^{s,\varepsilon}[u] \;\le\; & \|u\|_s + \sum_{0<|\alpha|+|\beta|\le N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|R(x^\beta\partial^\alpha u)\|_s \\
& + \sum_{0<|\alpha|+|\beta|\le N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|E[P,x^\beta\partial^\alpha]u\|_s \\
& + \sum_{0<|\alpha|+|\beta|\le N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|E(x^\beta\partial^\alpha f)\|_s \\
& + \sum_{0<|\alpha|+|\beta|\le N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|E(x^\beta\partial^\alpha F[u])\|_s.
\end{aligned}
\tag{4.27}
$$

The second and the third term in the right-hand side of (4.27) can be estimated using Propositions 4.3 and 4.4 while the term containing f is obviously dominated by $S_\infty^{s,\varepsilon}[f]$. For the last term we can apply Proposition 4.5. Hence, for $n\ge 1$, we have that, for ε small enough,

$$
S_N^{s,\varepsilon}[u] \le \|u\|_s + C_s S_\infty^{s,\varepsilon}[f] + C_s\varepsilon\Big(S_{N-1}^{s,\varepsilon}[u] + \sum_l \big((S_{N-1}^{s,\varepsilon}[u])^l + \|u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u]\big)\Big),
$$

whereas if $0\le n<1$ we get

$$
\begin{aligned}
S_N^{s,\varepsilon}[u] \le \|u\|_s + C_s S_\infty^{s,\varepsilon}[f] + C_s\varepsilon\Big(S_{N-1}^{s,\varepsilon}[u] + \sum_l \big((S_{N-1}^{s,\varepsilon}[u])^l & \\
+ \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u]\big)\Big). &
\end{aligned}
$$

In both cases we obtain $S_\infty^{s,\varepsilon}[u] < \infty$ if ε is small enough, which implies $u \in \mathcal{A}_{sect}(\mathbb{R}^d)$ by Proposition 3.3 (or, more simply, by the standard Sobolev embeddings, since $s > d/2$).

□

4.2. **Proof of Theorem 4.1: the case** $0<m<1$. In this case the nonlinearity has the form

$$
F[u] = \sum_{h,l} F_{h,l} x^h u^l, \tag{4.28}
$$

where $l\in\mathbb{N}$, $l\ge 2$, $h\in\mathbb{N}^d$, with $|h|\le \max\{n-1,0\}$ and $F_{h,l}\in\mathbb{C}$, the above sum being finite.

We follow the same argument used for the case $m \geq 1$. In particular we can estimate the first four terms in the right-hand side of (4.27) as before since Propositions 4.3 and 4.4 hold in general for $m > 0$. Hence, to conclude, it is sufficient to prove an estimate for the nonlinear term. We have the following result.

Proposition 4.6. *Let P satisfy the assumptions of Theorem 4.1 for $0 < m < 1$ and let E be a parametrix of P. Let $l \in \mathbb{N}$, $l \geq 2$, $h \in \mathbb{N}^d$, $|h| \leq \max\{n-1, 0\}$. Then, there exists a constant $C'_s > 0$ and, for every $\tau > 0$, there exists $C_\tau > 0$ such that, for every ε small enough, $N \in \mathbb{N}$ and $u \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$\sum_{0<|\alpha+\beta|\leq N} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|, |\beta|\}!} \|E(x^\beta \partial^\alpha (x^h u^l))\|_s \leq \tau C'_s \|u\|_s^{l-1} S_N^{s,\varepsilon}[u]$$

$$+ C'_s(\varepsilon C_\tau + \tau + \varepsilon)(S_{N-1}^{s,\varepsilon}[u])^l + C'_s \varepsilon \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} S_{N-1}^{s,\varepsilon}[u]. \tag{4.29}$$

Proof. We first consider the terms in (4.29) with $\alpha = 0$. Since $E \circ x^h \in \mathrm{OP}G^{-m,0}(\mathbb{R}^d)$ is bounded on $H^s(\mathbb{R}^d)$, we have, by Schauder's estimates:

$$\|E(x^{h+\beta} u^l)\|_s \leq C'_s \|x^\beta u^l\|_s \leq C''_s \|x^{\beta - e_j} u\|_s \cdot \|x_j u^{l-1}\|_s$$

if, say, $\beta_j \neq 0$. Then we get

$$\sum_{0<|\beta|\leq N} \frac{\varepsilon^{|\beta|}}{|\beta|!} \|E(x^\beta u^l)\|_s \leq C'''_s \varepsilon \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} \cdot S_{N-1}^{s,\varepsilon}[u]. \tag{4.30}$$

Consider now the terms in (4.29) with $\alpha \neq 0$. We may write

$$\begin{aligned} x^\beta \partial^\alpha (x^h u^l) &= x^{h+\beta} \partial^\alpha (u^l) + \sum_{\substack{0 \neq \gamma \leq \alpha \\ \gamma \leq h}} \binom{h}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} x^{h+\beta-\gamma} \partial^{\alpha-\gamma}(u^l) \\ &= x^h \partial_j (x^\beta \partial^{\alpha - e_j}(u^l)) - \beta_j x^{h+\beta-e_j} \partial^{\alpha-e_j}(u^l) \\ &\quad + \sum_{\substack{0 \neq \gamma \leq \alpha \\ \gamma \leq h}} \binom{h}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} x^{h+\beta-\gamma} \partial^{\alpha-\gamma}(u^l). \end{aligned} \tag{4.31}$$

Since E is bounded $H^{s-m}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$, we then obtain

$$\begin{aligned} \|E(x^\beta \partial^\alpha (x^h u^l))\|_s &\leq C'_s \|\partial_j (x^\beta \partial^{\alpha-e_j}(u^l))\|_{s-m} + C'_s \beta_j \|x^{\beta-e_j} \partial^{\alpha-e_j}(u^l)\|_{s-m} \\ &\quad + C'_s \sum_{\substack{0 \neq \gamma \leq \alpha \\ \gamma \leq h}} \binom{h}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} \|x^{\beta-\gamma} \partial^{\alpha-\gamma}(u^l)\|_{s-m}. \end{aligned} \tag{4.32}$$

Let us estimate the first term in the right-hand side of (4.32). We observe that for every $\tau > 0$ there exists a constant $C_\tau > 0$ such that

$$\langle \xi \rangle^{-m} |\xi_j| \leq \tau |\xi_j| + C_\tau.$$

Hence

$$\begin{aligned}\|\partial_j(x^\beta\partial^{\alpha-e_j}(u^l))\|_{s-m} &= \|\langle D\rangle^{-m}\partial_j(x^\beta\partial^{\alpha-e_j}(u^l))\|_s\\ &\leq \tau\|\partial_j(x^\beta\partial^{\alpha-e_j}(u^l))\|_s + C_\tau\|x^\beta\partial^{\alpha-e_j}(u^l)\|_s\\ &\leq \tau\beta_j\|x^{\beta-e_j}\partial^{\alpha-e_j}(u^l)\|_s + \tau\|x^\beta\partial^\alpha(u^l)\|_s\\ &\quad + C_\tau\|x^\beta\partial^{\alpha-e_j}(u^l)\|_s.\end{aligned}$$

Now we replace

$$\partial^\alpha(u^l) = lu^{l-1}\partial^\alpha u + \sum_{\substack{\delta_1+\ldots+\delta_l=\alpha\\ \delta_k\neq\alpha\,\forall k}} \frac{\alpha!}{\delta_1!\ldots\delta_l!}\prod_{k=1}^{l}\partial^{\delta_k}u$$

in the last estimate and we come back to (4.32). We get, for a new constant $C'_s > 0$,

$$\begin{aligned}\|E(x^\beta\partial^\alpha(x^h u^l))\|_s \leq\ & C'_sC_\tau\|x^\beta\partial^{\alpha-e_j}(u^l)\|_s + C'_s(1+\tau)\beta_j\|x^{\beta-e_j}\partial^{\alpha-e_j}(u^l)\|_s\\ &+\tau C'_s\|u\|_s^{l-1}\|x^\beta\partial^\alpha u\|_s\\ &+\tau C'_s\sum_{\substack{\delta_1+\ldots+\delta_l=\alpha\\ \delta_k\neq\alpha\,\forall k}}\frac{\alpha!}{\delta_1!\ldots\delta_l!}\|x^\beta\prod_{k=1}^{l}\partial^{\delta_k}u\|_s\\ &+C'_s\sum_{\substack{0\neq\gamma\leq\alpha\\ \gamma\leq h}}\binom{h}{\gamma}\frac{\alpha!}{(\alpha-\gamma)!}\|x^{\beta-\gamma}\partial^{\alpha-\gamma}(u^l)\|_{s-m}.\end{aligned}\tag{4.33}$$

We have now to estimate the terms in the right-hand side of (4.33). Concerning the first one, applying Leibniz' formula we obtain

$$\|x^\beta\partial^{\alpha-e_j}(u^l)\|_s \leq \sum_{\delta_1+\ldots+\delta_l=\alpha-e_j}\frac{(\alpha-e_j)!}{\delta_1!\ldots\delta_l!}\|x^\beta\prod_{k=1}^{l}\partial^{\delta_k}u\|_s.$$

If $|\beta|\geq|\alpha|$, then we can argue as in the previous section and find $\gamma_1,\ldots,\gamma_l\in\mathbb{N}^d$ such that $\gamma_1+\ldots+\gamma_l=\beta$ and $|\gamma_k|\geq|\delta_k|$ for every $k=1,\ldots,l$. Moreover, we observe that the following estimate holds:

$$\frac{1}{|\beta|!}\cdot\frac{(\alpha-e_j)!}{\delta_1!\ldots\delta_l!}\leq\frac{1}{|\gamma_1|!\ldots|\gamma_l|!}.\tag{4.34}$$

Then

(4.35)

$$\begin{aligned}\sum_{\substack{|\alpha|+|\beta|\leq N\\ 0<|\alpha|\leq|\beta|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|x^\beta\partial^{\alpha-e_j}(u^l)\|_s \leq C''_s\varepsilon\sum_{\substack{|\alpha|+|\beta|\leq N\\ 0<|\alpha|\leq|\beta|}}\ \sum_{\delta_1+\ldots+\delta_l=\alpha-e_j}\ \prod_{k=1}^{l}\frac{\varepsilon^{|\gamma_k|+|\delta_k|}}{|\gamma_k|!}\\ \times\|x^{\gamma_k}\partial^{\delta_k}u\|_s \leq C'''_s\varepsilon(S^{s,\varepsilon}_{N-1}[u])^l.\end{aligned}$$

On the other hand, for $|\beta| \leq |\alpha| - 1$ we can choose multi-indices $\gamma_1, \dots, \gamma_l$ such that $\gamma_1 + \dots + \gamma_l = \beta$ and $|\gamma_k| \leq |\delta_k|$ for any $k = 1, \dots, l$ and observe that

$$\frac{1}{(|\alpha|-1)!}\frac{(\alpha-e_j)!}{\delta_1!\dots\delta_l!} \leq \frac{1}{|\delta_1|!\dots|\delta_l|!}. \tag{4.36}$$

Then

$$\sum_{\substack{|\alpha|+|\beta|\leq N\\ 0<|\beta|\leq|\alpha|-1}} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|x^\beta\partial^{\alpha-e_j}(u^l)\|_s \tag{4.37}$$
$$\leq C_s'' \sum_{\substack{|\alpha|+|\beta|\leq N\\ 0<|\beta|\leq|\alpha|-1}} \sum_{\delta_1+\dots+\delta_l=\alpha-e_j} \prod_{k=1}^{l} \frac{\varepsilon^{|\gamma_k|+|\delta_k|}}{|\delta_k|!}\|x^{\gamma_k}\partial^{\delta_k}u\|_s \leq C_s'''\varepsilon(S_{N-1}^{s,\varepsilon}[u])^l,$$

for new constants C_s'' and C_s'''.

For the second term in the right-hand side of (4.33) we can argue as before, with $\gamma_1 + \dots + \gamma_l = \beta - e_j$ and $|\gamma_k| \leq |\delta_k|$ for $k = 1, \dots, l$, if $|\beta| \leq |\alpha|$ or $|\gamma_k| \geq |\delta_k|$ for $k = 1, \dots, l$, if $|\beta| \geq |\alpha|$, using the estimates

$$\frac{\beta_j}{|\beta|}\frac{1}{(|\beta|-1)!}\frac{(\alpha-e_j)!}{\delta_1!\dots\delta_l!} \leq \frac{1}{|\gamma_1|!\dots|\gamma_l|!}$$

respectively,

$$\frac{\beta_j}{|\alpha|}\frac{1}{(|\alpha|-1)!}\frac{(\alpha-e_j)!}{\delta_1!\dots\delta_l!} \leq \frac{1}{|\delta_1|!\dots|\delta_l|!}$$

instead of (4.34), respectively (4.36). We obtain, for a new constant $C_s' > 0$,

$$\sum_{\substack{|\alpha|+|\beta|\leq N\\ \alpha\neq 0}} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!}\|\beta_j x^{\beta-e_j}\partial^{\alpha-e_j}(u^l)\|_s \leq C_s'\varepsilon(S_{N-1}^{s,\varepsilon}[u])^l. \tag{4.38}$$

Concerning the fourth term in (4.33), we can decompose similarly $\beta = \gamma_1 + \dots + \gamma_l$ and argue as before, taking into account that now $|\gamma_1+\dots+\gamma_l+\delta_1+\dots+\delta_l| = |\alpha+\beta|$, so that we do not longer gain ε as a factor in the estimate. Hence we get

$$\sum_{\substack{|\alpha+\beta|\leq N\\ \alpha\neq 0}} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!} \sum_{\substack{\delta_1+\dots+\delta_l=\alpha\\ \delta_k\neq\alpha\,\forall k}} \frac{\alpha!}{\delta_1!\dots\delta_l!}\|x^\beta\prod_{k=1}^{l}\partial^{\delta_k}u\|_s \leq C_s'(S_{N-1}^{s,\varepsilon}[u])^l \tag{4.39}$$

for a new constant C_s'. Finally, for the last term in (4.33), we first observe that $\max\{|\alpha-\gamma|, |\beta-\gamma|\} = \max\{|\alpha|,|\beta|\} - |\gamma|$. Then we can argue as before obtaining

the estimate
(4.40)

$$\sum_{\substack{|\alpha+\beta|\le N\\ \alpha\neq 0}} \frac{\varepsilon^{|\alpha|+|\beta|}}{\max\{|\alpha|,|\beta|\}!} \sum_{\substack{0\neq\gamma\le\alpha\\ \gamma\le h}} \binom{h}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} \|x^{\beta-\gamma}\partial^{\alpha-\gamma}(u^l)\|_{s-m} \le C_s\varepsilon(S^{s,\varepsilon}_{N-1}[u])^l.$$

The estimates (4.30), (4.35), (4.37), (4.38), (4.39), (4.40) applied in (4.33) yield (4.29). ∎

End of the proof of Theorem 4.1 (the case $0 < m < 1$*).* Using the same argument as in the case $m \ge 1$, by Propositions 3.3, 4.3, 4.4, 4.6 we obtain

$$S^{s,\varepsilon}_N[u] \le \|u\|_s + C'_s S^{s,\varepsilon}_\infty[f] + C'_s\varepsilon S^{s,\varepsilon}_{N-1}[u] + \sum_l \Big(\tau C'_s \|u\|_s^{l-1} S^{s,\varepsilon}_N[u] \\ + C'_s(\varepsilon C_\tau + \tau + \varepsilon)(S^{s,\varepsilon}_{N-1}[u])^l + C'_s\varepsilon \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} S^{s,\varepsilon}_{N-1}[u]\Big)$$

for every $N \ge 1$ and ε small enough. Now, choosing $\tau < (2\sum_l C'_s\|u\|_s^{l-1})^{-1}$ we obtain

$$S^{s,\varepsilon}_N[u] \le 2\|u\|_s + 2C'_s S^{s,\varepsilon}_\infty[f] + 2C'_s\varepsilon S^{s,\varepsilon}_{N-1}[u] + \\ + \sum_l \Big(2C'_s(\varepsilon C_\tau + \tau + \varepsilon)(S^{s,\varepsilon}_{N-1}[u])^l + 2C'_s\varepsilon \|\langle x\rangle^{\frac{1}{l-1}} u\|_s^{l-1} S^{s,\varepsilon}_{N-1}[u]\Big).$$

Then we can iterate the last estimate observing that, shrinking τ and then ε, the quantity $\varepsilon C_\tau + \tau + \varepsilon$ can be taken arbitrarily small. This gives $S^{s,\varepsilon}_\infty[u] < \infty$ and therefore $u \in \mathcal{A}_{sect}(\mathbb{R}^d)$.

5. Remarks and applications

5.1. **Lower a priori regularity.** For special nonlinearities, in Theorem 1.1 we can assume lower a priori regularity on the solution u. For example, if $F[u] = (\partial^\rho u)^l$, $|\rho| \le \min\{m-1, 0\}$, $l \in \mathbb{N}$, $l \ge 2$, or even $F[u] = |\partial^\rho u|^{l-1}\partial^\rho u$, $|\rho| \le \min\{m-1, 0\}$, $l \in \mathbb{N}$, $l > 2$ odd (as in (1.12)), then we can assume $u \in H^s(\mathbb{R}^d)$, with $s > \frac{d}{2} - \frac{m-|\rho|}{l-1}$, $s \ge |\rho|$. Indeed, such a solution is actually in $H^\infty(\mathbb{R}^d)$; see e.g. [5, Lemma 4.1, Remark 4.1] (where that threshold is also proved to be sharp). We also refer to [5] for other types of non-linearities.

5.2. **Eigenfunctions of G-elliptic operators.** In the linear case, the assumptions on the a priori regularity of u can be relaxed assuming $u \in \mathcal{S}'(\mathbb{R}^d)$. Then, we have the following result.

Theorem 5.1. *Let P be a G-elliptic pseudodifferential operator with a symbol $p(x,\xi)$ satisfying* (1.9)*. Then there exists $\varepsilon > 0$ such that every solution $u \in \mathcal{S}'(\mathbb{R}^d)$*

of the equation $Pu = 0$ extends to a holomorphic function in the sector of $\mathbb{C}^d$

$$\mathcal{C}_\varepsilon = \{z = x + iy \in \mathbb{C}^d : \ |y| \leq \varepsilon(1 + |x|)\},$$

satisfying there the estimates (1.3) *for some constants $C > 0, c > 0$.*

Proof. It follows from the existence of a parametrix (see Section 2) that any solution $u \in \mathcal{S}'(\mathbb{R}^d)$ of $Pu = 0$ is in fact a Schwartz function. Hence we can apply Theorem 1.1 directly without using Lemma 4.2. Moreover it follows from the classical Fredholm theory of *globally* regular operators (see e.g. [27, Theorem 3.1.6]) that the kernel of P is a finite dimensional subspace of $\mathcal{S}(\mathbb{R}^d)$, which implies that there exists a sector where all the solutions extend holomorphically. ∎

The main application of Theorem 5.1 concerns eigenfunctions of G-elliptic operators of orders $m > 0$, $n > 0$. Indeed, in that case, if P is G-elliptic also $P - \lambda$ is G-elliptic for every $\lambda \in \mathbb{C}$, and one can apply Theorem 5.1 to $P - \lambda$. As regards existence, we recall that if $P \in \mathrm{OP}G^{m,n}(\mathbb{R}^d)$, $m > 0$, $n > 0$, is formally self-adjoint (i.e. symmetric when regarded as an operator in $L^2(\mathbb{R}^d)$ with domain $\mathcal{S}(\mathbb{R}^d)$) then it has a sequence of real eigenvalues either diverging to $+\infty$ or $-\infty$, and $L^2(\mathbb{R}^d)$ has an orthonormal basis made of eigenfunctions of P (cf. e.g [25] or [27, Theorem 4.2.9]). As an example in dimension 1, one can consider the operator $Pu = -(1 + x^2)u + x^2u - 2xu'$, $x \in \mathbb{R}$.

5.3. **Solitary waves.** The present subsection is devoted to some applications to solitary waves, in particular to the proof of Theorem 1.2. First we report the following useful characterization of the condition (1.16).

Proposition 5.2. *The estimates* (1.16) *are equivalent to requiring that $p(\xi)$ extends to a holomorphic function $p(\xi + i\eta)$ in a sector of the type* (1.18)*, and satisfies there the bound $|p(\xi + i\eta)| \leq C'\langle\xi\rangle^m$.*

Proof. The sufficiency of (1.16) for the holomorphic extension with the desired bound follows exactly as in the last part of the proof of Theorem 3.2, where in (3.5) the exponential factor is now replaced by $\langle\xi\rangle^m$.

In the opposite direction, we obtain (1.16) from Cauchy's estimates applied to a disc in $\mathbb{C}$ with center at ξ and radius $\varepsilon'\langle\xi\rangle$ for some small $\varepsilon' > 0$ (independent of ξ). ∎

Proof of theorem 1.2. Consider first the equation (1.14). We observe that u satisfies the equation

$$Mu + (V - 1)u = F[u], \tag{5.1}$$

cf. the proof of [9, Theorem 3.2.1]. By (1.17) and the condition $V > 1$ the symbol of the linear part of the equation (5.1), that is $p(\xi) + V - 1$, is G-elliptic: for some constant $c > 0$

$$p(\xi) + V - 1 \geq c\langle\xi\rangle^m, \quad \xi \in \mathbb{R}. \tag{5.2}$$

Moreover by (1.16) it satisfies the analytic symbol estimates (1.9) (with $n = 0$). To conclude the proof it is sufficient to show that $u \in H^s(\mathbb{R}^d)$ and $\langle x\rangle^{\varepsilon_0} u \in L^2(\mathbb{R}^d)$ for some $\varepsilon_0 > 0$, $s > d/2$, and to apply Theorem 1.1. The fact that u enjoys the above properties will follow from [9, Theorem 3.1.2, Corollary 4.1.6] once we observe that the function

$$K(\xi) = \frac{1}{p(\xi) + V - 1}$$

satisfies $|K(\xi)| \leq C\langle\xi\rangle^{-m}$ for some $C > 0$ and belongs to $H^\infty(\mathbb{R}^d)$. This is clear, because (5.2) and (1.16) give

$$|\partial^\alpha K(\xi)| \leq C_\alpha \langle\xi\rangle^{-m-|\alpha|},$$

and $m \geq 1$.

The case of the equation (1.15) is completely similar: in place of (5.2) one just has $V\,Mu + V - 1 = F[u]$, and the above arguments apply to the function $K(\xi) = (Vp(\xi) + V - 1)^{-1}$.

The theorem is then proved. □

As an example where the solutions are known in closed form, consider the generalized Korteweg-de Vries equation

$$v_t + v_x + v^l v_x + v_{xxx} = 0, \tag{5.3}$$

where $l \geq 1$ is a positive integer. Here we have $p(\xi) = \xi^2$. The solitary wave solutions have the form $v(x,t) = u(x - Vt)$, where $V > 1$ and

$$u(x) = \sqrt[l]{\frac{(l+1)(l+2)(V-1)}{2}}\,\mathrm{Cosh}^{-2/l}\Big(\frac{\sqrt{V-1}}{2} lx\Big),$$

which singularities at the points $z = i\frac{(2k+1)\pi}{l\sqrt{V-1}}$, $k \in \mathbb{Z}$. Also, the exponential decay in sectors containing the real axis predicted by Theorem 1.2 is confirmed.

During the years 1990-2000, several papers were devoted to 5-th order and 7-th order generalization of KdV, see for example Porubov [29, Chapter 1]. The corresponding stationary equation is of the type

$$\sum_{j=0}^{m} a_j u^{(j)} + Q[u] = 0, \tag{5.4}$$

where Q is a polynomial, $Q[u] = \sum_{j=2}^{M} b_j u^j$ and $a_0 \neq 0$. Because of physical assumptions, the equation $\sum_{j=0}^{m} a_j \lambda^j = 0$ has no purely imaginary roots, and then all the solutions of the corresponding linear equation have exponential decay/growth. This condition can be read as G-ellipticity of the symbol of the linear part of the corresponding stationary equation: $\sum_{j=0}^{m} a_j (i\xi)^j \neq 0$ for $\xi \in \mathbb{R}$, in particular $\xi^2 + V - 1 \neq 0$ in the case of(5.3). Non-trivial solutions u of (5.4) with $u(x) \to 0$ as $x \to \pm\infty$ may exist or not, according to the coefficients a_j, b_j, and when they exist, in general they do not have an explicit analytic expression. Holomorphic extension and exponential decay on a sector are granted anyhow by Theorem 1.2.

5.4. **Standing wave solutions of the Schrödinger equation.** Consider the Schrödinger equation in $\mathbb{R}^d$,

$$i\partial_t v + \Delta v = \mu |v|^{l-1} v, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

with $l \in \mathbb{N}$, $l > 2$ odd, $\mu \in \mathbb{C}$, and look at standing wave solutions, i.e. $v(t, x) = e^{i\omega t} u(x)$, $\omega > 0$. The corresponding equation for u is

$$\Delta u - \omega u = \mu |u|^{l-1} u.$$

Since the operator $\Delta - \omega$ is G-elliptic (because $\omega > 0$), when solutions u exist, with $u \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2} - \frac{2}{l-1}$, $s \geq 0$, and $\langle x \rangle^{\varepsilon_0} u \in L^2(\mathbb{R}^d)$, for some $\varepsilon_0 > 0$, then Theorem 1.1 and the remark in Subsection 5.1 assure that u extends to a holomorphic function on a sector of the type (1.13) and displays there an exponential decay of type (1.3). This applies, in particular, to the bound states in $H^1(\mathbb{R}^d)$ exhibited in [4] when $l < \frac{d+2}{d-2}$, $d \geq 3$.

5.5. **Sharpness of the results.** Here we show the sharpness of Theorem 1.1 as far as the shape of the domain of holomorphic extension is concerned.

Consider in dimension $d = 1$ the equation

$$-u'' + e^{-2i\theta} u = \frac{e^{-2i\theta}}{2} u^2,$$

where $-\pi < \theta \leq \pi$, $|\theta| \neq \frac{\pi}{2}$. This equation is G-elliptic, since it is elliptic and $\xi^2 + e^{-2i\theta} \neq 0$ for every $\xi \in \mathbb{R}$. An explicit Schwartz solution is given by

$$u(x) = 3\mathrm{Cosh}^{-2}\Big(\frac{e^{-i\theta}}{2} x\Big).$$

The function u extends to a meromorphic function in the complex plane with poles at $z = e^{i(\theta + \pi/2)}(2k + 1)\pi$, $k \in \mathbb{Z}$. This shows that in Theorem 1.1 we cannot replace the sector (1.13), e.g., with a larger set of the type

$$\{z = x + iy \in \mathbb{C}^d : \ |y| \leq \varepsilon(1 + |x|)\psi(x)\},$$

for any continuous function $\psi(x) > 0$, with $\psi(x) \to +\infty$ as $|x| \to +\infty$.

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