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BRAIDED BIALGEBRAS OF HECKE-TYPE

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Abstract. The paper is devoted to prove a version of Milnor-Moore Theorem for connected braided bialgebras that are infinitesimally cocommutative. Namely in characteristic different from 2, we prove that, for a given connected braided bialgebra $(A, \mathfrak{c}_A)$ which is infinitesimally λ-cocommutative for some element $\lambda \neq 0$ that is not a root of one in the base field, then the infinitesimal braiding of A is of Hecke-type of mark λ and A is isomorphic as a braided bialgebra to the symmetric algebra of the braided subspace of its primitive elements.

Introduction

The structure of cocommutative connected bialgebras is well-understood in characteristic zero. By Milnor-Moore Theorem [MM] such a bialgebra is the enveloping algebra of its primitive part, regarded as a Lie algebra in a canonical way. This result is one of the most important ingredients in the proof of Cartier-Gabriel-Kostant Theorem [Di, Sw], that characterizes cocommutative pointed bialgebras in characteristic zero as "products" between enveloping algebras and group algebras.

Versions of Milnor-Moore Theorem and Cartier-Gabriel-Kostant Theorem for graded bialgebras (over $\mathbb{Z}$ and $\mathbb{Z}_2$) can be also found in the work of Kostant, Leray and Milnor-Moore. For other more recent results, analogous to Milnor-Moore Theorem, see [Go, Kh, LR, Ma2, R1, R2, St].

The $\mathbb{Z}_2$-graded bialgebras, nowadays called superbialgebras, appeared independently in the work of Milnor-Moore and MacLane. From a modern point of view, superalgebras can be seen as bialgebras in a braided monoidal category. These structures play an increasing role not only in algebra (classification of finite dimensional Hopf algebras or theory of quantum groups), but also in other fields of mathematics (algebraic topology, algebraic groups, Lie algebras, etc.) and physics. Braided monoidal categories were formally defined by Joyal and Street in the seminal paper [JS], while (bi)algebras in a braided category were introduced in [Ma1]. By definition, $(A, \nabla, u, \Delta, \varepsilon)$ is a bialgebra in a braided category $\mathfrak{M}$, if ∇ is an associative multiplication on A with unit u and Δ is a coassociative comultiplication on A with counit ε such that ε and Δ are morphism of algebras where the multiplication on $A \otimes A$ is defined via the braiding. In other words, the last property of Δ rereads as follows

$$\Delta\nabla = (\nabla \otimes \nabla)(A \otimes \mathfrak{c}_{A,A} \otimes A)(\Delta \otimes \Delta), \tag{1}$$

where $\mathfrak{c}_{X,Y} : X \otimes Y \to Y \otimes X$ denotes the braiding in $\mathfrak{M}$. The relation above had already appeared in a natural way in [Ra], where Hopf algebras with a projection are characterized. More precisely, let A be a Hopf algebra and let $p : A \to A$ be a morphism of Hopf algebras such that $p^2 = p$. To these data, Radford associates an ordinary Hopf algebra $B := \mathrm{Im}(p)$ and a Hopf algebra $R = \{a \in A \mid (A \otimes p)\Delta(a) = a \otimes 1\}$ in the braided category ${}^B_B\mathcal{YD}$ of Yetter-Drinfeld modules. Then he shows that $A \simeq R \otimes B$, where on $R \otimes B$ one puts the tensor product algebra and tensor product coalgebra of R and B in the category ${}^B_B\mathcal{YD}$ (the braiding of ${}^B_B\mathcal{YD}$ is used to twist the elements of R and B). It is worth to notice that the above Hopf algebra structure on $R \otimes B$ can be constructed for an arbitrary Hopf algebra R in ${}^B_B\mathcal{YD}$. It is called the *bosonization* of R and it is denoted by $R\#B$. Notably, the "product" that appears in Cartier-Gabriel-Kostant Theorem is precisely the bosonization of an enveloping Lie algebra, regarded as a bialgebra in the category of

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Yetter-Drinfeld modules over its coradical. The result of Radford was extended for more general classes of bialgebras with a projection in [AMS1, AMS2].

Bosonization also plays a very important role in the "lifting" method for the classification of finite dimensional pointed Hopf algebras, introduced by N. Andruskiewitsch and H.J. Schneider. Roughly speaking, the lifting method requires two steps. If A is a pointed Hopf algebra, then $\mathrm{gr}A$, the graded associated of A with respect to the coradical filtration, is a Hopf algebra with projection onto the homogeneous component of degree 0. Hence, by Radford's result, $\mathrm{gr}A$ is the bosonization of a graded *connected* Hopf algebra R in ${}^B_B\mathcal{YD}$, where B is the coradical of A. Accordingly to the lifting method, first one has to classify all connected and graded Hopf algebras R in ${}^B_B\mathcal{YD}$ such that $\dim R = \dim A/\dim B$. Then one has to find all Hopf algebras A such that $A \simeq R\#B$, with R as in the first step.

Therefore, in many cases, for proving a certain property of Hopf algebras, it suffices to do it in the connected case. The price that one has to pay is that we have to work with Hopf algebras in a braided category (usually ${}^B_B\mathcal{YD}$), and not with ordinary Hopf algebras. Motivated by this observation, in this paper we will investigate connected and cocommutative bialgebras in a braided category. Actually nowadays people recognize that it is more appropriate to work with braided bialgebras, that were introduced in [Ta] (see e.g. [Gu3] and [Kh]).

To define a braided bialgebra we first need a braided vector space, that is a pair $(A, \mathfrak{c})$, where A is a vector space and $\mathfrak{c}$ is a solution of the braid equation (2). Then we need an algebra (A, ∇, u) and a coalgebra (A, Δ, ϵ) which are compatible with the braiding (see Definitions 1.2 and 1.4). Now, for defining braided bialgebras, one can proceed as in the classical case; see Definition 1.6.

The prototype braided bialgebra is $T := T(V, \mathfrak{c})$, the tensor algebra of a braided K-vector space $(V, \mathfrak{c})$. The braiding $\mathfrak{c}$ lifts uniquely to an operator $\mathfrak{c}_T : T \otimes T \to T \otimes T$. The usual algebra structure on $T(V)$ is compatible with $\mathfrak{c}_T$, so $T \otimes T$ is an algebra (for the multiplication, of course, we use $\mathfrak{c}_T$ and not the usual flip map). Therefore, there is a unique coalgebra structure on T, so that V is included in the space of primitive elements of T and the comultiplication is an algebra map.

For constructing other examples of braided bialgebras, we focus on the case when $(V, \mathfrak{c})$ is a braided vector space such that $\mathfrak{c}$ is a braiding of Hecke-type of mark $\lambda \in K^*$, that is $\mathfrak{c}$ is a root of $(X + 1)(X - \lambda)$. Then, to every K-linear map $\mathfrak{b} : V \otimes V \to V$ which is compatible with $\mathfrak{c}$ (i.e. a so called braided bracket, see Definition 23) we associate a new braided bialgebra $U(V, \mathfrak{c}, \mathfrak{b})$, called enveloping algebra, as follows. The set

$$X_{\mathfrak{c},\mathfrak{b}} = \{\mathfrak{c}(z) - \lambda z - \mathfrak{b}(z) \mid z \in V \otimes V = T^2(V)\}$$

contains only primitive elements in $T(V, \mathfrak{c})$, so the ideal $I_{\mathfrak{c},\mathfrak{b}}$ generated by $X_{\mathfrak{c},\mathfrak{b}}$ in $T(V, \mathfrak{c})$ is a coideal too. Hence the quotient $U(V, \mathfrak{c}, \mathfrak{b})$ of $T(V, \mathfrak{c})$ through $I_{\mathfrak{c},\mathfrak{b}}$ is a braided bialgebra. As a particular case we obtain the braided symmetric algebra $S(V, \mathfrak{c}) := U(V, \mathfrak{c}, 0)$.

It is worthwhile noticing that the braided subspace $(P, \mathfrak{c}_P)$ of primitive elements of a connected braided bialgebra $(A, \mathfrak{c}_A)$ can always be endowed with a braided bracket $\mathfrak{b}_P : P \otimes P \to P$ whenever $\mathfrak{c}_P$ is a braiding of Hecke-type. In this case one can consider the corresponding enveloping algebra (see Theorem (2.4)).

The main result of the paper is Theorem 5.5. In fact, we prove that, for a given field K with $\mathrm{char}\, K \neq 2$, if $(A, \mathfrak{c}_A)$ is a connected braided bialgebra which is infinitesimally λ-cocommutative for some regular element $\lambda \neq 0$ in K, then

- the infinitesimal braiding $\mathfrak{c}_P$ of A is of Hecke-type of mark λ and
- A is isomorphic as a braided bialgebra to the symmetric algebra $S(P, \mathfrak{c}_P)$ of $(P, \mathfrak{c}_P)$ whenever $\lambda \neq 1$.

We would like to stress that, since the symmetric algebra is indeed a universal enveloping algebra with trivial bracket, the foregoing result could be thought as a strong version of Milnor-Moore Theorem for connected braided bialgebras that are infinitesimally cocommutative.

We also point out that, when $\lambda = 1$, A is isomorphic to the symmetric algebra $S(P, \mathfrak{c}_P)$ just as coalgebra by means of Kharchenko's results (see Remark 5.7 and [Kh, Theorem 7.2]).

To achieve our result we characterize bialgebras of type one with infinitesimal braiding of Hecke type (see Theorem 2.15). Moreover, for a given braided vector space $(V, \mathfrak{c})$ of Hecke-type of mark

$\lambda \neq 0, 1$, we show that $\mathfrak{b} = 0$ is the unique $\mathfrak{c}$-bracket on $(V, \mathfrak{c})$ for which the K-linear canonical map $\iota_{\mathfrak{c},\mathfrak{b}} : V \to U(V, \mathfrak{c}, \mathfrak{b})$ is injective whenever $(3)!_\lambda \neq 0$ (see Theorem 4.3).

It is worth noticing that, as a consequence of Theorem 2.15, we can prove (see Remark 5.6) that for a connected braided Hopf algebra (H, c_H) and $\lambda \in K^*$, the following assertions are equivalent:

- H is cosymmetric in the sense of [Kh, Definition 3.1] and its infinitesimal braiding is of Hecke-type of mark λ,
- H is infinitesimally λ-cocommutative.

On the other hand, as a consequence of Theorem 4.3, we get (see Remark 4.6) that a braided Lie algebra introduced by Gurevich in [Gu2] has trivial bracket in the Hecke case.

1. Braided bialgebras

Throughout this paper K will denote a field. All vector spaces will be defined over K and the tensor product of two vector spaces will be denoted by $\otimes$.

In this section we define the main notion that we will deal with, namely braided bialgebras. We also introduce one of the basic examples, namely the tensor algebra of a braided vector space.

Definition 1.1. A pair $(V, \mathfrak{c})$ is called *braided vector space* if $\mathfrak{c} : V \otimes V \to V \otimes V$ is a solution of the braid equation

$$\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_1 = \mathfrak{c}_2\mathfrak{c}_1\mathfrak{c}_2 \tag{2}$$

where $\mathfrak{c}_1 = \mathfrak{c} \otimes V$ and $\mathfrak{c}_2 = V \otimes \mathfrak{c}$. A morphism of braided vector spaces $(V, \mathfrak{c}_V)$ and $(W, \mathfrak{c}_W)$ is a K-linear map $f : V \to W$ such that $\mathfrak{c}_W(f \otimes f) = (f \otimes f)\mathfrak{c}_V$.

Note that, for every braided vector space $(V, \mathfrak{c})$ and every $\lambda \in K$, the pair $(V, \lambda\mathfrak{c})$ is a braided vector space too. A general method for producing braided vector spaces is to take an arbitrary braided category $(\mathcal{M}, \otimes, K, a, l, r, c)$, which is a monoidal subcategory of the category of K-vector spaces. Hence any object $V \in \mathcal{M}$ can be regarded as a braided vector space with respect to $\mathfrak{c} := \mathfrak{c}_{V,V}$. Here, $\mathfrak{c}_{X,Y} : X \otimes Y \to Y \otimes X$ denotes the braiding in $\mathcal{M}$. The category of comodules over a coquasitriangular Hopf algebra and the category of Yetter-Drinfeld modules are examples of such categories. More particularly, every bicharacter of a group G induces a braiding on the category of G -graded vector spaces.

Definition 1.2 (Baez, [Ba]). A *braided algebra*, or $\mathfrak{c}$-*algebra* is a quadruple $(A, \nabla, 1, \mathfrak{c})$ where $(A, \mathfrak{c})$ is a braided vector space and $(A, \nabla, 1)$ is an associative unital algebra such that ∇ and u commute with $\mathfrak{c}$, that is the following conditions hold:

$$\mathfrak{c}(\nabla \otimes A) = (A \otimes \nabla)(\mathfrak{c} \otimes A)(A \otimes \mathfrak{c}), \tag{3}$$

$$\mathfrak{c}(A \otimes \nabla) = (\nabla \otimes A)(A \otimes \mathfrak{c})(\mathfrak{c} \otimes A), \tag{4}$$

$$\mathfrak{c}(1 \otimes a) = a \otimes 1, \qquad \mathfrak{c}(a \otimes 1) = 1 \otimes a \qquad \text{for all } a \in A. \tag{5}$$

A morphism of braided algebras is, by definition, a morphism of ordinary algebras which, in addition, is a morphism of braided vector spaces.

Remark 1.3. 1) Let $(A, \nabla, u, \mathfrak{c})$ is a braided algebra. Then $A \otimes A$ is an associative algebra with multiplication $\nabla_{A\otimes A} := (\nabla \otimes \nabla)(A \otimes \mathfrak{c} \otimes A)$ and unit $1 \otimes 1$. Moreover, $A \otimes A$ is a $\mathfrak{c}_{A\otimes A}$-algebra, where $\mathfrak{c}_{A\otimes A} = (A \otimes \mathfrak{c} \otimes A)(\mathfrak{c} \otimes \mathfrak{c})(A \otimes \mathfrak{c} \otimes A)$. This algebra structure on $A \otimes A$ will be denoted by $A \otimes_{\mathfrak{c}} A$.

2)If A is an object in a braided monoidal category $\mathcal{M}$ and $\mathfrak{c} := \mathfrak{c}_{A,A}$ then the above four compatibility relations hold automatically, as the braiding c is a natural morphism.

Definition 1.4. A *braided coalgebra* (or $\mathfrak{c}$-*coalgebra*) is a quadruple $(C, \Delta, \varepsilon, \mathfrak{c})$ where $(C, \mathfrak{c})$ is a braided vector space and (C, Δ, ε) is a coassociative counital coalgebra such that the comultiplication Δ and the counit ε commute with $\mathfrak{c}$, that is the following relations hold:

$$(C \otimes \Delta)\mathfrak{c} = (\mathfrak{c} \otimes C)(C \otimes \mathfrak{c})(\Delta \otimes C), \tag{6}$$

$$(\Delta \otimes C)\mathfrak{c} = (C \otimes \mathfrak{c})(\mathfrak{c} \otimes C)(C \otimes \Delta), \tag{7}$$

$$(\varepsilon \otimes C)\mathfrak{c}(c \otimes d) = \varepsilon(d)c = (C \otimes \varepsilon)\mathfrak{c}(d \otimes c) \qquad \text{for all } c, d \in C. \tag{8}$$

A morphism of braided coalgebras is, by definition, a morphism of ordinary coalgebras which, in addition, is a morphism of braided vector spaces.

1.5. Recall that a coalgebra C is called *connected* if the coradical C_0 of C is one dimensional. In this case there is a unique group-like element $g \in C$ such that $C_0 = Kg$. Sometimes, we will write (C, g), to emphasize the group-like element g. We also ask that $f(g_C) = g_D$, for any morphism $f : (C, g_C) \to (D, g_D)$ of connected coalgebras.

By definition, a $\mathfrak{c}$-coalgebra C is *connected* if $C_0 = Kg$ and, for any $x \in C$,

$$\mathfrak{c}(x \otimes g) = g \otimes x, \qquad \mathfrak{c}(g \otimes x) = x \otimes g. \tag{9}$$

Definition 1.6 (Takeuchi, [Ta]). A *braided bialgebra* is a sextuple $(A, \nabla, 1, \Delta, \varepsilon, \mathfrak{c})$ where

- $(A, \nabla, 1, \mathfrak{c})$ is a braided algebra,
- $(A, \Delta, \varepsilon, \mathfrak{c})$ is a braided coalgebra,
- Δ and ε are morphisms of algebras (on the vector space $A \otimes A$ we take the algebra structure $A \otimes_{\mathfrak{c}} A$).

Remark 1.7. Note that $\Delta : A \to A \otimes_{\mathfrak{c}} A$ is multiplicative if and only if

$$\Delta\nabla = (\nabla \otimes \nabla)(A \otimes \mathfrak{c} \otimes A)(\Delta \otimes \Delta). \tag{10}$$

1.8. We will need graded versions of braided algebras, coalgebras and bialgebras. By definition, a braided algebra $(A, \nabla, 1, \mathfrak{c})$ is graded if $A = \bigoplus_{n\in\mathbb{N}} A^n$ and $\nabla(A^n \otimes A^m) \subseteq A^{n+m}$. The braiding $\mathfrak{c}$ is assumed to satisfy $\mathfrak{c}(A^n \otimes A^m) \subseteq A^m \otimes A^n$. In this case, it is easy to see that $1 \in A^0$.

Therefore a graded braided algebra can be defined by means of maps $\nabla^{n,m} : A^n \otimes A^m \to A^{n+m}$ and $\mathfrak{c}^{n,m} : A^n \otimes A^m \to A^m \otimes A^n$, and an element $1 \in A^0$ such that:

$$\nabla^{n+m,p}(\nabla^{n,m} \otimes A^p) = \nabla^{n,m+p}(A^n \otimes \nabla^{m,p}), \qquad \text{for all } n, m, p \in \mathbb{N}, \tag{11}$$

$$\nabla^{0,n}(1 \otimes a) = a = \nabla^{n,0}(a \otimes 1) \qquad \text{for all } a \in A^n, n \in \mathbb{N}, \tag{12}$$

$$\mathfrak{c}^{n+m,p}(\nabla^{n,m} \otimes A^p) = (A^p \otimes \nabla^{n,m})(\mathfrak{c}^{n,p} \otimes A^m)(A^n \otimes \mathfrak{c}^{m,p}) \qquad \text{for all } n, m, p \in \mathbb{N}, \tag{13}$$

$$\mathfrak{c}^{n,m+p}(A^n \otimes \nabla^{m,p}) = (\nabla^{m,p} \otimes A^n)(A^m \otimes \mathfrak{c}^{n,p})(\mathfrak{c}^{n,m} \otimes A^p) \qquad \text{for all } n, m, p \in \mathbb{N}, \tag{14}$$

$$\mathfrak{c}^{0,n}(1 \otimes a) = a \otimes 1 \qquad \text{and} \qquad \mathfrak{c}^{n,0}(a \otimes 1) = 1 \otimes a \qquad \text{for all } a \in A^n, n \in \mathbb{N}. \tag{15}$$

The multiplication ∇ can be recovered from $(\nabla^{n,m})_{n,m\in\mathbb{N}}$ as the unique K-linear map such that $\nabla(x \otimes y) = \nabla^{p,q}(x \otimes y)$ for all $p, q \in \mathbb{N}, x \in A^p, y \in A^q$. Analogously, the braiding $\mathfrak{c}$ is uniquely defined by $\mathfrak{c}(x \otimes y) = \mathfrak{c}^{p,q}(x \otimes y)$ for all $p, q \in \mathbb{N}, x \in A^p, y \in A^q$. We will say that $\nabla^{n,m}$ and $\mathfrak{c}^{n,m}$ are the (n, m)-homogeneous components of ∇ and $\mathfrak{c}$, respectively.

Graded braided coalgebras can by described in a similar way. By definition a braided coalgebra $(C, \Delta, \varepsilon, \mathfrak{c})$ is graded if $C = \bigoplus_{n\in\mathbb{N}} C^n, \Delta(C^n) \subseteq \sum_{p+q=n} C^p \otimes C^q$, $\mathfrak{c}(C^n \otimes C^m) \subseteq C^m \otimes C^n$ and $\varepsilon|_{C_n} = 0$, for $n > 0$. If π^p denotes the projection onto C^p then the comultiplication Δ is uniquely defined by the maps $\Delta^{p,q} : C^{p+q} \to C^p \otimes C^q$, where $\Delta^{p,q} := (\pi^p \otimes \pi^q)\Delta|_{C^{p+q}}$. The counit is given by a map $\varepsilon^0 : C^0 \to K$, while the braiding $\mathfrak{c}$ is uniquely determined by a family $(\mathfrak{c}^{n,m})_{n,m\in\mathbb{N}}$, as for braided algebras. The families $(\Delta^{n,m})_{n,m\in\mathbb{N}}$, $(\mathfrak{c}^{n,m})_{n,m\in\mathbb{N}}$ and ε^0 has to satisfy the relations that are dual to (11) – (15), namely:

$$(\Delta^{n,m} \otimes C^p)\Delta^{n+m,p} = (C^n \otimes \Delta^{m,p})\Delta^{n,m+p} \qquad \text{for all } n, m, p \in \mathbb{N}, \tag{16}$$

$$(\varepsilon^0 \otimes C^n)\Delta^{0,n}(c) = c = (C^n \otimes \varepsilon^0)\Delta^{n,0}(c) \qquad \text{for all } c \in C^n, n \in \mathbb{N}, \tag{17}$$

$$(C^p \otimes \Delta^{n,m})\mathfrak{c}^{n+m,p} = (\mathfrak{c}^{n,p} \otimes C^m)(C^n \otimes \mathfrak{c}^{m,p})(\Delta^{n,m} \otimes C^p) \qquad \text{for all } n, m, p \in \mathbb{N}, \tag{18}$$

$$(\Delta^{m,p} \otimes C^n)\mathfrak{c}^{n,m+p} = (C^m \otimes \mathfrak{c}^{n,p})(\mathfrak{c}^{n,m} \otimes C^p)(C^n \otimes \Delta^{m,p}) \qquad \text{for all } n, m, p \in \mathbb{N}, \tag{19}$$

$$(\varepsilon^0 \otimes C)\mathfrak{c}(c \otimes d) = \varepsilon^0(d)c = (C \otimes \varepsilon^0)\mathfrak{c}(d \otimes c) \qquad \text{for all } c \in C^n, d \in C^0. \tag{20}$$

We will say that $\Delta^{n,m}$ is the (n, m)-homogeneous component of Δ.

A **graded braided bialgebra** is a braided bialgebra which is graded both as an algebra and as a coalgebra.

REMARK 1.9. Let $C = \bigoplus_{n\in\mathbb{N}} C^n$ be a graded braided coalgebra. By [Sw, Proposition 11.1.1], if $(C_n)_{n\in\mathbb{N}}$ is the coradical filtration, then $C_n \subseteq \bigoplus_{m\leq n} C^m$. Therefore, if C^0 is one dimensional then C is connected.

DEFINITION 1.10. A graded braided coalgebra will be called 0-*connected* if its homogeneous component of degree 0 is of dimension one.

LEMMA 1.11. *Let $(C, \mathfrak{c})$ be a connected braided coalgebra. Then $\mathfrak{c}$ induces a canonical braiding $\mathfrak{c}_{\mathrm{gr}\, C}$ on $\mathrm{gr}\, C$ such that $(\mathrm{gr}\, C, \mathfrak{c}_{\mathrm{gr}\, C})$ is a 0-connected graded braided coalgebra, where $\mathrm{gr}\, C$ is constructed with respect to the coradical filtration on C.*

Proof. Let $(C_n)_{n\in\mathbb{N}}$ be the coradical filtration. Since C is connected, we have $C_0 = Kg$, where g is the unique group-like element of C. We claim that $\mathfrak{c}(C_n \otimes C_m) \subseteq C_m \otimes C_n$. For $n = 0$ this relation holds true as, by definition, $\mathfrak{c}(x \otimes g) = g \otimes x$ for all $x \in C$. We choose a basis $\{y_i \mid i \in I\}$ on C_m and we assume that the above inclusion is true for n. Let $x \in C_{n+1}$ and $y \in C_m$. Since $C_{n+1} = {C_{n+1}}^+ \oplus Kg$, by [Sw, Proposition 10.0.2], $\Delta(x) = x \otimes g + g \otimes x + \sum_{k=1}^q x_k' \otimes x_k''$, where $x_k', x_k'' \in C_n$. Moreover,

$$\mathfrak{c}(x \otimes y) = \sum\nolimits_{i\in I} y_i \otimes x_i,$$

with $x_i \in C$, and the set $\{i \in I \mid x_i \neq 0\}$ finite. Since $(C \otimes \Delta)\mathfrak{c} = (\mathfrak{c} \otimes C)(C \otimes \mathfrak{c})(\Delta \otimes C)$ we get

$$\sum\nolimits_{i=1}^p y_i \otimes \Delta(x_i) = \sum\nolimits_{i=1}^p y_i \otimes x_i \otimes g + \sum\nolimits_{i=1}^p y_i \otimes g \otimes x_i + (\mathfrak{c} \otimes C)(C \otimes \mathfrak{c})(\sum\nolimits_{k=1}^q x_k' \otimes x_k'' \otimes y).$$

By induction hypothesis, $(\mathfrak{c} \otimes C)(C \otimes \mathfrak{c})(\sum_{k=1}^q x_k' \otimes x_k'' \otimes y) \in C_m \otimes C_n \otimes C_n$, so this element can be written as $\sum_{i\in I} y_i \otimes z_i$, with $z_i \in C_n \otimes C_n$ and the set $\{i \in I \mid z_i \neq 0\}$ is finite. Hence, for all $i \in I$, we have

$$\Delta(x_i) = x_i \otimes g + g \otimes x_i + z_i.$$

Thus $x_i \in C_{n+1}$, so $\mathfrak{c}(x \otimes y) \in C_m \otimes C_{n+1}$. Hence, by induction, $\mathfrak{c}(C_n \otimes C_m) \subseteq C_m \otimes C_n$, so

$$\mathfrak{c}(C_{n-1} \otimes C_m + C_n \otimes C_{m-1}) \subseteq C_m \otimes C_{n-1} + C_{m-1} \otimes C_n.$$

Therefore $\mathfrak{c}$ induces a unique K-linear map $\mathfrak{c}_{\mathrm{gr}\, C}^{n,m} : \mathrm{gr}^{\,n} C \otimes \mathrm{gr}^{\,m} C \to \mathrm{gr}^{\,m} C \otimes \mathrm{gr}^{\,n} C$. We define $\mathfrak{c}_{\mathrm{gr}\, C} := \bigoplus_{n,m} \mathfrak{c}_{\mathrm{gr}\, C}^{n,m}$. Now it is easy to see that $(\mathrm{gr}\, C, \mathfrak{c}_{\mathrm{gr}\, C})$ is a graded braided coalgebra. □

REMARK 1.12. If $(C, \mathfrak{c}, g)$ is a connected braided coalgebra then $\mathfrak{c}$ induces a canonical braiding $\mathfrak{c}_P$ on the space $P(C) = \{c \in C \mid \Delta(c) = c \otimes g + g \otimes c\}$, of primitives elements in C. Indeed, by [Sw, Proposition 10.0.1] we have $P(C) = (\mathrm{Ker}\, \varepsilon) \cap C_1$. Thus $\mathfrak{c}$ maps $P(C) \otimes P(C)$ to itself, see the proof of the preceding lemma.

LEMMA 1.13. *Let $(A, \nabla, u, \mathfrak{c})$ be a $\mathfrak{c}$-algebra and let $\Delta : A \to A \otimes_{\mathfrak{c}} A$ be a morphism of algebras. We fix $x, a \in A$ such that $\Delta(a) = a \otimes 1 + 1 \otimes a$. Then*

$$\Delta(xa) = (\nabla \otimes A)(A \otimes \mathfrak{c})[\Delta(x) \otimes a] + (A \otimes \nabla)[\Delta(x) \otimes a]. \tag{21}$$

Proof. Let $\Delta(x) = \sum_{i=1}^p x_i' \otimes x_i''$. Then

$$\begin{aligned} \Delta(xa) &= \Delta(x) \cdot (a \otimes 1 + 1 \otimes a) = \sum\nolimits_{i=1}^p (x_i' \otimes x_i'') \cdot (a \otimes 1 + 1 \otimes a) \\ &= \sum\nolimits_{i=1}^p x_i' \mathfrak{c}(x_i'' \otimes a) + \sum\nolimits_{i=1}^p x_i' \otimes x_i'' a \\ &= (\nabla \otimes A)(A \otimes \mathfrak{c})[\Delta(x) \otimes a] + (A \otimes \nabla)[\Delta(x) \otimes a]. \end{aligned}$$

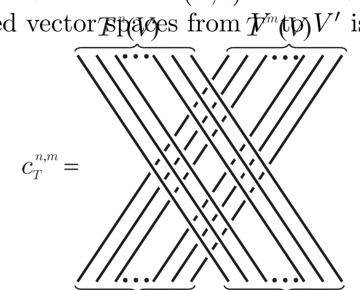
□

PROPOSITION 1.14. *To any braided vector space $(V, \mathfrak{c})$ we can associate a 0-connected graded braided bialgebra $(T = T(V, \mathfrak{c}), \nabla_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{c}_T)$ where*

- *$(T = T(V, \mathfrak{c}), \nabla_T, 1_T)$ is the tensor algebra $T(V)$ i.e. the free algebra generated by V.*
- *$\mathfrak{c}_T$ is constructed iteratively from $\mathfrak{c}$.*
- *$\Delta_T : T \to T \otimes_{\mathfrak{c}_T} T$ is the unique algebra homomorphism defined by setting $\Delta_T(v) = 1_T \otimes v + v \otimes 1_T$ for every $v \in V$.*

- *$\varepsilon_T : T \to K$ is the unique algebra homomorphism defined by setting $\varepsilon_T(v) = 0$ for every $v \in V$.*

REMARK 1.15. Note that Δ_T is the dual construction of the quantum shuffle product, introduced by Rosso in [Ro]. The K-linear map $\mathfrak{c}_T^{n,m}$ is given in the figure below:



where each crossing represents a copy of $\mathfrak{c}$.

REMARK 1.16. If $A = \bigoplus_{n\in\mathbb{N}} A^n$ is a 0-connected graded $\mathfrak{c}$-bialgebra then $\Delta^{n,0} = \Delta^{0,n} = \mathrm{Id}_{A^n}$. Indeed, the proof given in the case $A = T(V)$ works for an arbitrary connected graded $\mathfrak{c}$-bialgebra.

THEOREM 1.17. *Let $(V, \mathfrak{c})$ be a braided vector space. Then $i_V : V \to T(V)$ is a morphism of braided vector spaces. If $(A, \nabla_A, 1_A, \mathfrak{c}_A)$ is a braided algebra and $f : V \to A$ is a morphism of braided vector spaces then there is a unique morphism $\widetilde{f} : T(V, \mathfrak{c}) \to A$ of braided algebras that lifts f. If, in addition, A is a braided bialgebra and $f(V)$ is contained in $P(A)$, the set of primitive elements of A, then $\widetilde{f}$ is a morphism of braided bialgebras.*

1.18. Recall ([Ka, page 74]) that the X-binomial coefficients $\binom{n}{k}_X$ are defined as follows. We set $(0)_X = (0)!_X = 1$ and, for $n > 0$, define $(n)_X := 1+X+\cdots+X^{n-1}$ and $(n)!_X = (1)_X(2)_X \cdots (n)_X$. Then:

$$\binom{n}{k}_X = \frac{(n)!_X}{(k)!_X\,(n-k)!_X}. \tag{22}$$

It is well known that $\binom{n}{k}_X$ is a polynomial. Therefore we may specialize X at an arbitrary element $\lambda \in K$. In this way we get an element $\binom{n}{k}_\lambda \in K$. Note that, if λ is a root a unity, then $\binom{n}{k}_\lambda$ may be zero. If $\mathrm{char}(K) = 0$ and $\lambda = 1$, the formula shows that $\binom{n}{k}_1$ is the classical binomial coefficient.

DEFINITION 1.19. We say that a braided vector space $(V, \mathfrak{c})$ is of *Hecke-type (or that* $\mathfrak{c}$ *is of Hecke-type) of mark* $\mathfrak{m}(\mathfrak{c}) = \lambda$ if

$$(\mathfrak{c} + \mathrm{Id}_{V^{\otimes 2}})(\mathfrak{c} - \lambda\mathrm{Id}_{V^{\otimes 2}}) = 0.$$

REMARK 1.20. For every Hecke-type braiding $\mathfrak{c}$ of mark λ, the operator $\check{\mathfrak{c}} := -\lambda^{-1}\mathfrak{c}$ is also of Hecke-type. We have $\mathfrak{m}(\check{\mathfrak{c}}) = \lambda^{-1}$. Note that, for $\mathfrak{d} = \check{\mathfrak{c}}$, one has $\check{\mathfrak{d}} = \mathfrak{c}$.

2. BRAIDED ENVELOPING ALGEBRAS

In this section we introduce the main example of $\mathfrak{c}$-bialgebras that we will deal with, namely the enveloping algebra of a $\mathfrak{c}$-Lie algebra.

Given a vector spaces V, W and a K-linear map $\alpha : V \otimes V \to W$, we will denote $\alpha \otimes V$ and $V \otimes \alpha$ by α_1 and α_2 respectively.

DEFINITION 2.1. Let $(V, \mathfrak{c})$ be a braided vector space. We say that a K-linear map $\mathfrak{b} : V \otimes V \to V$ is a $\mathfrak{c}$-*bracket*, or *braided bracket*, if the following compatibility conditions hold true:

$$\mathfrak{c}\mathfrak{b}_1 = \mathfrak{b}_2\mathfrak{c}_1\mathfrak{c}_2 \qquad \text{and} \qquad \mathfrak{c}\mathfrak{b}_2 = \mathfrak{b}_1\mathfrak{c}_2\mathfrak{c}_1 \tag{23}$$

Let $\mathfrak{b}$ be a $\mathfrak{c}$-bracket on $(V, \mathfrak{c})$ and let $\mathfrak{b}'$ be a $\mathfrak{c}'$-bracket on $(V', \mathfrak{c}')$. We will say that a morphism of braided vector spaces from V to V' is *a morphism of braided brackets* if $f\mathfrak{b} = \mathfrak{b}'(f \otimes f)$.

Let $\mathfrak{b}$ be a bracket on a braided vector space $(V, \mathfrak{c})$ of Hecke-type with mark λ. Let $I_{\mathfrak{c},\mathfrak{b}}$ is the two-sided ideal generated by the set

$$X_{\mathfrak{c},\mathfrak{b}} = \{\mathfrak{c}(z) - \lambda z - \mathfrak{b}(z) \mid z \in V \otimes V = T^2(V)\}.$$

The *enveloping algebra of* $(V, \mathfrak{c}, \mathfrak{b})$ is by definition the algebra

$$U = U(V, \mathfrak{c}, \mathfrak{b}) := \frac{T(V, \mathfrak{c})}{I_{\mathfrak{c},\mathfrak{b}}}.$$

We will denote by $\pi_{\mathfrak{c},\mathfrak{b}} : T(V, \mathfrak{c}) \to U(V, \mathfrak{c}, \mathfrak{b})$ the canonical projection.

When $\mathfrak{b} = 0$, the enveloping algebra of $(V, \mathfrak{c}, 0)$ is called the $\mathfrak{c}$-*symmetric algebra*, or *braided symmetric algebra*, if there is no danger of confusion. It will be denoted by $S(V, \mathfrak{c})$.

Proposition 2.2. *Let $\mathfrak{b}$ be a bracket on a braided vector space $(V, \mathfrak{c})$ of Hecke-type with mark λ. Then $I_{\mathfrak{c},\mathfrak{b}}$ is a coideal in $T(V, \mathfrak{c})$. Moreover, on the quotient algebra (and coalgebra) there is a natural braiding $\mathfrak{c}_U$ such that $(U(V, \mathfrak{c}, \mathfrak{b}), \mathfrak{c}_U)$ is a braided bialgebra.*

Proof. We denote $T(V, \mathfrak{c})$ and $T^n(V, \mathfrak{c})$ by T and T^n, respectively. We first prove that $\mathfrak{c}_T$ maps $I_{\mathfrak{c},\mathfrak{b}} \otimes T$ into $T \otimes I_{\mathfrak{c},\mathfrak{b}}$ and $T \otimes I_{\mathfrak{c},\mathfrak{b}}$ into $I_{\mathfrak{c},\mathfrak{b}} \otimes T$. Let $x \in T^n$, $y \in T^m$, $t \in T^p$ and $z \in T^2$. Since $\mathfrak{c}$ verifies the braid equation and (23) we get:

$$\begin{aligned} \mathfrak{c}^{n+m+2,p}(T^n \otimes \mathfrak{c} \otimes T^m \otimes T^p) &= (T^p \otimes T^n \otimes \mathfrak{c} \otimes T^m)\mathfrak{c}^{n+m+2,p}, \\ \mathfrak{c}^{n+m+1,p}(T^n \otimes \mathfrak{b} \otimes T^m \otimes T^p) &= (T^p \otimes T^n \otimes \mathfrak{b} \otimes T^m)\mathfrak{c}^{n+m+2,p}. \end{aligned}$$

Then

$$\mathfrak{c}_T\left[(T^n \otimes (\mathfrak{c} - \lambda \mathrm{Id}_{T^2} - \mathfrak{b}) \otimes T^m) \otimes T^p\right] = \left[T^p \otimes (T^n \otimes (\mathfrak{c} - \lambda \mathrm{Id}_{T^2} - \mathfrak{b}) \otimes T^m)\right]\mathfrak{c}_T,$$

relation that shows us that $\mathfrak{c}_T$ maps $I_{\mathfrak{c},\mathfrak{b}} \otimes T$ into $T \otimes I_{\mathfrak{c},\mathfrak{b}}$. The other property can be proved similarly.

We claim that $X_{\mathfrak{c},\mathfrak{b}}$ is a coideal in T. In fact we will prove that $X_{\mathfrak{c},\mathfrak{b}}$ contains only primitive elements in T. Let $z \in T^2$. By Proposition 1.14 we have

$$\begin{aligned} \Delta_T(z) &= z \otimes 1 + z + \mathfrak{c}(z) + 1 \otimes z, \\ \Delta_T(\mathfrak{c}(z)) &= \mathfrak{c}(z) \otimes 1 + \mathfrak{c}(z) + \mathfrak{c}^2(z) + 1 \otimes \mathfrak{c}(z). \end{aligned}$$

Thus $\Delta_T(\mathfrak{c}(z) - \lambda z) = (\mathfrak{c}(z) - \lambda z) \otimes 1 + 1 \otimes (\mathfrak{c}(z) - \lambda z)$, as $\mathfrak{c}$ is a Hecke braiding of mark λ. It follows that $\mathfrak{c}(z) - \lambda z \in P(T)$. Since $\mathfrak{b}(z) \in V$ we deduce that $\mathfrak{c}(z) - \lambda z - \mathfrak{b}(z) \in P(T)$, so $X_{\mathfrak{c},\mathfrak{b}} \subseteq P(T)$.

Now, by (21), it easily follows that $I_{\mathfrak{c},\mathfrak{b}}$, the ideal generated by $X_{\mathfrak{c},\mathfrak{b}}$, is a coideal. It remains to show that $\mathfrak{c}$ factors through a braiding $\mathfrak{c}_U$ of $U := U(V, \mathfrak{c}, \mathfrak{b})$, that makes U a braided bialgebra. Let $\pi_{\mathfrak{c},\mathfrak{b}} : T \to U$ be the canonical projection. By the foregoing, $\mathfrak{c}_T$ maps the kernel of $\pi_{\mathfrak{c},\mathfrak{b}} \otimes \pi_{\mathfrak{c},b}$ into itself, so there is a K-linear morphism $\mathfrak{c}_U : U \otimes U \to U \otimes U$ such that

$$\mathfrak{c}_U(\pi_{\mathfrak{c},\mathfrak{b}} \otimes \pi_{\mathfrak{c},\mathfrak{b}}) = (\pi_{\mathfrak{c},\mathfrak{b}} \otimes \pi_{\mathfrak{c},\mathfrak{b}})\mathfrak{c}_T.$$

Since T is a $\mathfrak{c}_T$-bialgebra, this relation entails that U is a $\mathfrak{c}_U$-bialgebra and that the canonical projection $\pi_{\mathfrak{c},\mathfrak{b}}$ becomes a morphism of braided bialgebras. □

Remark 2.3. Let $(V, \mathfrak{c})$ be an arbitrary braided vector space. Let $K^n := \mathrm{Ker}(\mathfrak{S}_n)$ where $\mathfrak{S}_n$ denotes the quantum symmetrizer [AS, 2.3]. It is well-known that $\bigoplus_{n\mathbb{N}} K^n$ is an ideal and a coideal in $T(V, \mathfrak{c})$, see [AS, section 3]. Since

$$\mathfrak{c}_T^{n,m}(T^n \otimes K^m + K^n \otimes T^m) \subseteq K^m \otimes T^n + T^m \otimes K^n$$

it follows that $B(V, \mathfrak{c}) = T(V, \mathfrak{c}) / \left(\bigoplus_{n\in\mathbb{N}} K^n\right)$ is a quotient graded braided bialgebra of $T(V, \mathfrak{c})$, that is called the Nichols algebra of $(V, \mathfrak{c})$. Let us now assume that $\mathfrak{c}$is a braiding of Hecke-type of mark λ. By the definition of Hecke operators we have

$$\mathrm{Im}(\mathfrak{c} - \lambda \mathrm{Id}_{T^2}) \subseteq \mathrm{Ker}(\mathrm{Id}_{T^2} + \mathfrak{c}) = K^2 \subseteq \bigoplus_{n\in\mathbb{N}} K^n.$$

Therefore, there is a morphism of braided bialgebras $\varphi : S(V, \mathfrak{c}) \to B(V, \mathfrak{c})$ such that $\varphi|_V = \mathrm{Id}_V$. Obviously φ is surjective, since $B(V, \mathfrak{c})$ is generated by V. Later (see Theorem 2.17) we will see that the space of primitive elements in $S(V, \mathfrak{c})$ and the homogeneous component $S^1(V, \mathfrak{c}) = V$ are identical. By [Mo, Theorem 5.3.1], it follows that φ is injective too. Thus, $S(V,\mathfrak{c})$ and $B(V,\mathfrak{c})$ are isomorphic braided bialgebras.

We are going to investigate some basic properties of these objects.

Proposition 2.4. *Let $(A,\nabla,1,\Delta,\varepsilon,\mathfrak{c}_A)$ be a connected braided bialgebra. Let P be the space of primitive elements of A. Assume that there is $\lambda\in K^*$ such that $\mathfrak{c}_P:=\mathfrak{c}_A|_{P\otimes P}$ is a braiding of Hecke-type on P of mark λ. Then:*

a) $\nabla(\mathfrak{c}_P-\lambda\mathrm{Id}_{P^{\otimes 2}})(P\otimes P)\subseteq P$, so we can define $\mathfrak{b}_P:P\otimes P\to P$ by $\mathfrak{b}_P=\nabla(\mathfrak{c}_P-\lambda\mathrm{Id}_{P^{\otimes 2}})|_{P\otimes P}$.

b) The map $\mathfrak{b}_P$ is a braided bracket on the braided vector space $(P,\mathfrak{c}_P)$.

c) Let $f:(V,\mathfrak{c},\mathfrak{b})\to(P,\mathfrak{c}_P,\mathfrak{b}_P)$ be a morphism of braided brackets and assume that $\mathfrak{c}$ is a braiding of Hecke-type with mark λ. Then there is a unique morphism of braided bialgebras $\widetilde{f}:U(V,\mathfrak{c},\mathfrak{b})\to A$ that lifts f.

Proof. First, observe that, by Remark 1.12, $\mathfrak{c}_A(P\otimes P)\subseteq P\otimes P$ so that it makes sense to consider $\mathfrak{c}_P:P\otimes P\to P\otimes P$.

a) By assumption, $\mathfrak{c}_P$ is a Hecke operator on P of mark λ. For $z\in P\otimes P$, by (21) we get:

$$\begin{aligned}\Delta\nabla(z) &= \nabla(z)\otimes 1+z+\mathfrak{c}_P(z)+1\otimes\nabla(z),\\ \Delta\nabla\mathfrak{c}_P(z) &= \nabla\mathfrak{c}_P(z)\otimes 1+\mathfrak{c}_P(z)+\mathfrak{c}_P^2(z)+1\otimes\nabla\mathfrak{c}_P(z).\end{aligned}$$

Therefore $\nabla(\mathfrak{c}_P-\lambda\mathrm{Id}_{P^{\otimes 2}})(z)\in P$. This shows that $\nabla(\mathfrak{c}_P-\lambda\mathrm{Id}_{P^{\otimes 2}})(P\otimes P)\subseteq P$.

b) We have to prove the compatibility relation between $\mathfrak{c}$ and $\mathfrak{b}$, that is we have (23). But these relations follows immediately by the braid relation and the fact that A is a $\mathfrak{c}_A$-algebra.

c) Apply the universal property of $T(V,\mathfrak{c})$ (see Theorem 1.17) to get a morphism $f':T(V,\mathfrak{c})\to A$ of braided bialgebras that lifts f. Since f is a morphism of braided brackets, by the definition of $\mathfrak{b}_P$, it results that f' maps $\mathfrak{c}(z)-\lambda z-\mathfrak{b}(z)$ to 0. Therefore f' factors through a morphism $\widetilde{f}:U(V,\mathfrak{c},\mathfrak{b})\to A$, which lifts f and is compatible with the braidings (note that $U(V,\mathfrak{c},\mathfrak{b})$ is a braided bialgebra in view of Theorem 2.2). □

Proposition 2.5. *Let $(A,\nabla,1,\Delta,\varepsilon,\mathfrak{c}_A)$ be a connected braided bialgebra. Let P denote the primitive part of A. Assume that there is $\lambda\in K^*$ such that $\nabla\mathfrak{c}_A=\lambda\nabla$ on $P\otimes P$. If $\mathfrak{c}_P=\mathfrak{c}_A|_{P\otimes P}$ then $\mathfrak{c}_P$ is of Hecke-type on P of mark λ. Moreover, if $(V,\mathfrak{c})$ is a braided vector space such that $\mathfrak{c}$ is a Hecke operator of mark λ and $f:(V,\mathfrak{c})\to(P,\mathfrak{c}_P)$ is a morphism of braided vector spaces then there is a unique morphism of braided bialgebras $\widetilde{f}:S(V,\mathfrak{c})\to A$ that lifts f. If A is graded then $\widetilde{f}$ respects the gradings on $S(V,\mathfrak{c})$ and A.*

Proof. For $z\in P\otimes P$, by (21) we get:

$$\begin{aligned}\Delta\nabla(z) &= \nabla(z)\otimes 1+z+\mathfrak{c}_A(z)+1\otimes\nabla(z),\\ \Delta\nabla\mathfrak{c}_A(z) &= \nabla\mathfrak{c}_A(z)\otimes 1+\mathfrak{c}_A(z)+\mathfrak{c}_A^2(z)+1\otimes\nabla\mathfrak{c}_A(z).\end{aligned}$$

By assumption, $\nabla\mathfrak{c}_A(z)=\lambda\nabla(z)$ whence

$$0=\mathfrak{c}_A(z)+\mathfrak{c}_A^2(z)-\lambda z-\lambda\mathfrak{c}_A(z)=(\mathfrak{c}_A+\mathrm{Id}_{A^{\otimes 2}})(\mathfrak{c}_A-\lambda\mathrm{Id}_{A^{\otimes 2}})(z)$$

Thus $\mathfrak{c}_P$ is a Hecke operator of mark λ. By taking $\mathfrak{b}=0$ in Proposition 2.4, it results that there is $\widetilde{f}$ that lifts f. □

Remark 2.6. The above proposition still works under the slighter assumption $\nabla\mathfrak{c}_A=\lambda\nabla$ on $\mathrm{Im}\,f\otimes\mathrm{Im}\,f$.

2.7. Let $T^n:=T^n(V)$ and let $T^{\leq n}:=\bigoplus_{0\leq m\leq n}T^m$. By construction $\pi_{\mathfrak{c},\mathfrak{b}}$ is a morphism of algebras and coalgebras from $T(V,\mathfrak{c})$ to $U(V,\mathfrak{c},\mathfrak{b})$. Thus $U_n':=\pi_{\mathfrak{c},\mathfrak{b}}(T^{\leq n})$ defines a braided bialgebra filtration on $U(V,\mathfrak{c},\mathfrak{b})$, i.e. $(U_n')_{n\in\mathbb{N}}$ is an algebra and coalgebra filtration on $U(V,\mathfrak{c},\mathfrak{b})$ which is compatible with $\mathfrak{c}_U$. It will be called the *standard filtration* on $U(V,\mathfrak{c},\mathfrak{b})$. In general, this filtration and the coradical filtration $(U_n)_{n\in\mathbb{N}}$ are not identical, but we always have $U_n'\subseteq U_n$, for any $n\in\mathbb{N}$.

If $\mathfrak{b}=0$ then $S(V,\mathfrak{c}):=U(V,\mathfrak{c},0)$ is a graded $\mathfrak{c}_S$-bialgebra, $S(V,\mathfrak{c})=\bigoplus_{n\in\mathbb{N}}S^n(V,\mathfrak{c})$. The standard filtration on $S(V,\mathfrak{c})$ is the filtration associated to this grading.

PROPOSITION 2.8. *Let $\mathfrak{b}$ be a $\mathfrak{c}$-bracket on a braided vector space $(V, \mathfrak{c})$ of Hecke-type. Then $U(V, \mathfrak{c}, \mathfrak{b})$ is a connected coalgebra. Moreover, for every braided vector space $(V, \mathfrak{c})$, $S(V, \mathfrak{c})$ is a 0-connected graded braided coalgebra.*

Proof. We know that the tensor algebra of an arbitrary braided vector space is a 0-connected coalgebra. By definition, $U(V, \mathfrak{c}, \mathfrak{b})$ is a quotient coalgebra of $T(V, \mathfrak{c})$, where $\lambda = \mathfrak{m}(\mathfrak{c})$. Then, in view of [Mo, Corollary 5.3.5], $U(V, \mathfrak{c}, \mathfrak{b})$ is connected. In particular, braided symmetric algebras are connected coalgebras. They are also 0-connected since they are graded quotients of $T(V, \mathfrak{c})$. □

REMARK 2.9. Let $\mathfrak{b}$ be a $\mathfrak{c}$-bracket on a braided vector space $(V, \mathfrak{c})$ of Hecke-type. The composition of the inclusion $V \to T(V, \mathfrak{c})$ with the canonical projection $\pi_{\mathfrak{c},\mathfrak{b}}$ gives a map

$$\iota_{\mathfrak{c},\mathfrak{b}} : V \to U(V, \mathfrak{c}, \mathfrak{b}).$$

Its image is included in the space of primitive elements of $U(V, \mathfrak{c}, \mathfrak{b})$. In general $\iota_{\mathfrak{c},\mathfrak{b}}$ is neither injective nor onto. Our purpose now is to investigate when $\iota_{\mathfrak{c},\mathfrak{b}}$ is injective (see Theorem (4.3)).

DEFINITION 2.10. Let $(B, \nabla_B, 1_B, \Delta_B, \varepsilon_B, \mathfrak{c}_B)$ be a graded braided bialgebra. For every $a, b, n \in \mathbb{N}$, set

$$\Gamma^B_{a,b} := \nabla^{a,b}_B \Delta^{a,b}_B.$$

LEMMA 2.11. *Let $(B, \nabla_B, 1_B, \Delta_B, \varepsilon_B, \mathfrak{c}_B)$ be a 0-connected graded braided bialgebra. Then*

$$\Delta^{n,0}_B(z) = z \otimes 1_B \qquad \text{and} \qquad \Delta^{0,n}_B = 1_B \otimes z, \text{ for every } z \in B^n.$$

Moreover

$$\Delta^{n,1}_B \nabla^{n,1}_B = \mathrm{Id}_{B^n \otimes B^1} + \left(\nabla^{n-1,1}_B \otimes B^1\right)\left(B^{n-1} \otimes \mathfrak{c}^{1,1}_B\right)\left(\Delta^{n-1,1}_B \otimes B^1\right). \tag{24}$$

Proof. The first assertion follows by Remark 1.16. Since B is a graded bialgebra we have

$$\begin{aligned}\Delta^{n,1}_B \nabla^{n,1}_B &= \left[\begin{array}{c}\left(\nabla^{n,0}_B \otimes \nabla^{0,1}_B\right)\left(B^n \otimes \mathfrak{c}^{0,0}_B \otimes B^1\right)\left(\Delta^{n,0}_B \otimes \Delta^{0,1}_B\right) + \\ + \left(\nabla^{n-1,1}_B \otimes \nabla^{1,0}_B\right)\left(B^{n-1} \otimes \mathfrak{c}^{1,1}_B \otimes B^0\right)\left(\Delta^{n-1,1}_B \otimes \Delta^{1,0}_B\right)\end{array}\right] \\ &= \mathrm{Id}_{B^n \otimes B^1} + \left(\nabla^{n-1,1}_B \otimes B^1\right)\left(B^{n-1} \otimes \mathfrak{c}^{1,1}_B\right)\left(\Delta^{n-1,1}_B \otimes B^1\right),\end{aligned}$$

so that (24) holds. □

DEFINITION 2.12. Let $(A, \nabla, 1_A)$ be a graded algebra. We say that A is a *strongly* $\mathbb{N}$*-graded algebra* whenever $\nabla^{i,j} : A_i \otimes A_j \to A_{i+j}$ is an epimorphism for every $i, j \in \mathbb{N}$ (equivalently $\nabla^{n,1} : A_n \otimes A_1 \to A_{n+1}$ is an epimorphism for every $n \in \mathbb{N}$).

Dually, let (C, Δ, ε) be a graded coalgebra. We say that C is a *strongly* $\mathbb{N}$*-graded coalgebra* whenever $\Delta^{i,j} : C_{i+j} \to C_i \otimes C_j$ is a monomorphism for every $i, j \in \mathbb{N}$ (equivalently $\Delta^{n,1} : C_{n+1} \to C_n \otimes C_1$ is a monomorphism for every $n \in \mathbb{N}$).

For more details on these (co)algebras see e.g. [AM1].

DEFINITION 2.13. An element $\lambda \in K^*$ is called n**-regular** whenever $(k)_\lambda \neq 0$, for any $1 \leq k \leq n$. If λ is n-regular for any $n > 0$, we will simply say that λ is **regular**.

REMARKS 2.14. 1) Note that $\lambda \neq 1$ is regular if and only if λ is not a root of one, while 1 is regular if and only if $char(K) = 0$.

2) If λ is n-regular (respectively regular) then λ^{-1} is also n-regular (respectively regular).

THEOREM 2.15. *Let $(B, \nabla_B, 1_B, \Delta_B, \varepsilon_B, \mathfrak{c}_B)$ be a 0-connected graded braided bialgebra and let $\lambda \in K^*$ be regular. The following are equivalent.*

(1) *B is a bialgebra of type one and $\mathfrak{c}^{1,1}_B$ is a braiding of Hecke-type of mark λ.*
(2) *B is strongly $\mathbb{N}$-graded as a coalgebra, $\nabla^{1,1}_B$ is surjective and $\mathfrak{c}^{1,1}_B$ is a braiding of Hecke-type of mark λ.*
(3) *B is strongly $\mathbb{N}$-graded as a coalgebra and $\left(\mathfrak{c}^{1,1}_B - \lambda \mathrm{Id}_{B^2}\right)\Delta^{1,1}_B = 0$.*

(4) *B is strongly $\mathbb{N}$-graded as an algebra, $\Delta_B^{1,1}$ is injective and $\mathfrak{c}_B^{1,1}$ is a braiding of Hecke-type of mark λ.*

(5) *B is strongly $\mathbb{N}$-graded as an algebra and $\nabla_B^{1,1}\left(\mathfrak{c}_B^{1,1}-\lambda \mathrm{Id}_{B^2}\right)=0$.*

Proof. (1) $\Rightarrow$ (2) By definition, B is strongly $\mathbb{N}$-graded both as a coalgebra and as an algebra.

(2) $\Rightarrow$ (3) We have

$$0=\left(\mathfrak{c}_B^{1,1}-\lambda \mathrm{Id}_{B^2}\right)\left(\mathfrak{c}_B^{1,1}+\mathrm{Id}_{B^2}\right)\overset{(24)}{=}\left(\mathfrak{c}_B^{1,1}-\lambda \mathrm{Id}_{B^2}\right)\Delta_B^{1,1}\nabla_B^{1,1}.$$

Since $\nabla_B^{1,1}$ is an epimorphism, we get $\left(\mathfrak{c}_B^{1,1}-\lambda \mathrm{Id}_{B^2}\right)\Delta_B^{1,1}=0$.

(3) $\Rightarrow$ (1) We have

$$\begin{aligned}
\Delta_B^{n,1}\Gamma_{n,1}^B &= \Delta_B^{n,1}\nabla_B^{n,1}\Delta_B^{n,1}\overset{(24)}{=}\Delta_B^{n,1}+\left(\nabla_B^{n-1,1}\otimes B^1\right)\left(B^{n-1}\otimes \mathfrak{c}_B^{1,1}\right)\left(\Delta_B^{n-1,1}\otimes B^1\right)\Delta_B^{n,1}\\
&= \Delta_B^{n,1}+\left(\nabla_B^{n-1,1}\otimes B^1\right)\left(B^{n-1}\otimes \mathfrak{c}_B^{1,1}\Delta_B^{1,1}\right)\Delta_B^{n-1,2}\\
&= \Delta_B^{n,1}+\left(\nabla_B^{n-1,1}\otimes B^1\right)\left(B^{n-1}\otimes \lambda\Delta_B^{1,1}\right)\Delta_B^{n-1,2}\\
&= \Delta_B^{n,1}+\lambda\left(\nabla_B^{n-1,1}\Delta_B^{n-1,1}\otimes B^1\right)\Delta_B^{n,1}=\Delta_B^{n,1}+\lambda\left(\Gamma_{n-1,1}^B\otimes B^1\right)\Delta_B^{n,1},
\end{aligned}$$

so that

$$\Delta_B^{n,1}\Gamma_{n,1}^B=\Delta_B^{n,1}+\lambda\left(\Gamma_{n-1,1}^B\otimes B^1\right)\Delta_B^{n,1}. \tag{25}$$

Let us prove by induction that

$$\Gamma_{n,1}^B=(n+1)_\lambda \mathrm{Id}_{B^{n+1}},\ \text{for every } n\geq 1. \tag{26}$$

$n=1$) We have

$$\Delta_B^{1,1}\Gamma_{1,1}^B\overset{(25)}{=}\Delta_B^{1,1}+\lambda\Delta_B^{1,1}=(2)_\lambda\Delta_B^{1,1}.$$

Since, by hypothesis, $\Delta_B^{1,1}$ is injective, we obtain $\Gamma_{1,1}^B=(2)_q \mathrm{Id}_{B^2}$.

$n-1\Rightarrow n$) We have

$$\Delta_B^{n,1}\Gamma_{n,1}^B\overset{(25)}{=}\Delta_B^{n,1}+\lambda\left(\Gamma_{n-1,1}^B\otimes B^1\right)\Delta_B^{n,1}=\Delta_B^{n,1}+\lambda(n)_\lambda\Delta_B^{n,1}=(n+1)_\lambda\Delta_B^{n,1}$$

Since, by hypothesis, $\Delta_B^{n,1}$ is injective, we obtain $\Gamma_{n,1}^B=(n+1)_\lambda \mathrm{Id}_{B^{n+1}}$.

We have so proved (26).

Since λ is regular, we have $(n+1)_\lambda\neq 0$ so that

$$\nabla_B^{n,1}\Delta_B^{n,1}=\Gamma_{n,1}^B\overset{(26)}{=}(n+1)_\lambda \mathrm{Id}_{B^{n+1}}$$

is bijective for every $n\geq 1$. Therefore $\nabla_B^{n,1}$ is surjective for every $n\geq 1$. Equivalently B is strongly $\mathbb{N}$-graded as an algebra and hence of type one. Moreover

$$\left(\mathfrak{c}_B^{1,1}-\lambda \mathrm{Id}_{B^2}\right)\left(\mathfrak{c}_B^{1,1}+\mathrm{Id}_{B^2}\right)\overset{(24)}{=}\left(\mathfrak{c}_B^{1,1}-\lambda \mathrm{Id}_{B^2}\right)\Delta_B^{1,1}\nabla_B^{1,1}=0.$$

(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) It follows by dual arguments. □

Definition 2.16. A graded coalgebra $C=\bigoplus_{n\in\mathbb{N}}C^n$ is called **strict** if it is 0-connected and $P(C)=C^1$.

Theorem 2.17. *(cf.* [AS, Proposition 3.4]*) Let $(V,\mathfrak{c})$ be a braided vector space of Hecke-type with regular mark λ. Then $S(V,\mathfrak{c})$ is a bialgebra of type one. In particular $S(V,\mathfrak{c})$ is a strict coalgebra.*

Proof. By definition, we have

$$S:=S(V,\mathfrak{c})=U(V,\mathfrak{c},0)=\frac{T(V,\mathfrak{c})}{(\mathfrak{c}(z)-\lambda z\mid z\in V\otimes V)}.$$

Thus, since S is a graded quotient of the graded braided bialgebra $T(V,\mathfrak{c})$, we get that S is strongly $\mathbb{N}$-graded as an algebra. Moreover $\nabla_S^{1,1}\left(\mathfrak{c}_S^{1,1}-\lambda \mathrm{Id}_{S^2}\right)=0$. By Theorem 2.15, we conclude. □

3. Categorical subspaces

Definition 3.1. [Kh, 2.2] A subspace L of a braided vector space $(V,\mathfrak{c})$ is said to be **categorical** if

$$\mathfrak{c}\,(L\otimes V)\subseteq V\otimes L \qquad \text{and} \qquad \mathfrak{c}\,(V\otimes L)\subseteq L\otimes V. \tag{27}$$

Theorem 3.2. *Let $(V,\mathfrak{c})$ be a braided vector space of Hecke-type of mark λ. Assume $\lambda\neq 0,1$. Let L be a categorical subspace of V. Then $L=0$ or $L=V$.*

Proof. From $(\mathfrak{c}+\mathrm{Id}_{V\otimes V})\,(\mathfrak{c}-\lambda\mathrm{Id}_{V\otimes V})=0$ and $\lambda\neq 1$, we get $\mathfrak{c}=(\lambda-1)^{-1}\left(\mathfrak{c}^2-\lambda\mathrm{Id}_{V\otimes V}\right)$ so that

$$\mathfrak{c}\,(L\otimes V)=\frac{1}{\lambda-1}\left(\mathfrak{c}^2-\lambda\mathrm{Id}_{V\otimes V}\right)(L\otimes V)\overset{(27)}{\subseteq}L\otimes V.$$

Then

$$\mathfrak{c}\,(L\otimes V)\subseteq (V\otimes L)\cap(L\otimes V)=L\otimes L. \tag{28}$$

From $(\mathfrak{c}+\mathrm{Id}_{V\otimes V})\,(\mathfrak{c}-\lambda\mathrm{Id}_{V\otimes V})=0$ and $\lambda\neq 0$, we get $\mathrm{Id}_{V\otimes V}=\lambda^{-1}\left(\mathfrak{c}^2+(1-\lambda)\,\mathfrak{c}\right)$ so that

$$L\otimes V=\frac{1}{\lambda}\left(\mathfrak{c}^2+(1-\lambda)\,\mathfrak{c}\right)(L\otimes V)\overset{(28),(27)}{\subseteq}L\otimes L.$$

Since $L\subseteq V$, we deduce that $L=0$ or $L=V$. □

Proposition 3.3. *Let $\mathfrak{b}$ be a $\mathfrak{c}$-bracket on a braided vector space $(V,\mathfrak{c})$ of Hecke-type of mark λ. Assume $\lambda\neq 0,1$. Then $\mathfrak{b}$ is zero or surjective.*

Proof. Let $L=\mathrm{Im}\,(\mathfrak{b})\subseteq V$. We have that

$$\begin{aligned}\mathfrak{c}\mathfrak{b}_1 &= \mathfrak{b}_2\mathfrak{c}_1\mathfrak{c}_2\Rightarrow \mathfrak{c}\,(L\otimes V)\subseteq V\otimes L,\\ \mathfrak{c}\mathfrak{b}_2 &= \mathfrak{b}_1\mathfrak{c}_2\mathfrak{c}_1\Rightarrow \mathfrak{c}\,(V\otimes L)\subseteq L\otimes V.\end{aligned}$$

Thus L is a categorical subspace of V. By Theorem 3.2, we get that $L=0$ or $L=V$. □

Proposition 3.4. *Let V be an object in the monoidal category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over some Hopf algebra H. Assume that $\mathfrak{c}_{V,V}$ is a braiding of Hecke type of mark λ and that $\lambda\neq 0,1$. Then V is simple in ${}^H_H\mathcal{YD}$.*

Proof. Any subspace of V in ${}^H_H\mathcal{YD}$ is categorical. We conclude by Theorem 3.2. □

4. Trivial braided brackets

Let $\mathfrak{b}$ be a braided bracket on $(V,\mathfrak{c})$. Our aim now is to answer the following natural question: when is $\iota_{\mathfrak{c},\mathfrak{b}}:V\to U(V,\mathfrak{c},\mathfrak{b})$ injective?

Proposition 4.1. *Let $\mathfrak{b}$ be a $\mathfrak{c}$-bracket on a braided vector space $(V,\mathfrak{c})$ of Hecke type. If $\mathrm{gr}'U(V,\mathfrak{c},\mathfrak{b})$ is the graded associated to the standard filtration on $U(V,\mathfrak{c},\mathfrak{b})$, then $\mathrm{gr}'U(V,\mathfrak{c},\mathfrak{b})$ is a graded braided bialgebra and there is a canonical morphism $\theta:S(V,\mathfrak{c})\to\mathrm{gr}'U(V,\mathfrak{c},\mathfrak{b})$ of graded braided bialgebras. Moreover θ is surjective.*

Proof. Let $T^{\leq n}:=\bigoplus_{0\leq m\leq n}T^m$ and let $(U'_n)_{n\in\mathbb{N}}$ be the standard filtration on $U:=U(V,\mathfrak{c},\mathfrak{b})$. Let ∇_U and $\mathfrak{c}_U$ be the multiplication and the braiding of $U:=U(V,\mathfrak{c},\mathfrak{b})$, respectively. If $T:=T(V,\mathfrak{c})$ and $\mathfrak{c}_T$ is the braiding of T then the canonical projection $\pi_U:T\to U$ is a morphism of braided bialgebras. Since $\mathfrak{c}_T(T^{\leq n}\otimes T^{\leq m})\subseteq T^{\leq m}\otimes T^{\leq n}$ we deduce that $\mathfrak{c}_U(U'_n\otimes U'_m)\subseteq U'_m\otimes U'_n$, for any $n,m\in\mathbb{N}$. Hence $\mathfrak{c}_U$ induces a braiding $\mathfrak{c}_{\mathrm{gr}'U}:\mathrm{gr}'U\otimes\mathrm{gr}'U\to\mathrm{gr}'U\otimes\mathrm{gr}'U$. The standard filtration is a coalgebra filtration, as π_U is a morphism of coalgebras, so $\mathrm{gr}'U$ is a coalgebra. One can prove easily that, with respect to this coalgebra structure, $\mathrm{gr}'U$ becomes a graded braided bialgebra.

Let $\lambda=\mathfrak{m}(\mathfrak{c})$. We define $\theta^1:V\to U'_1/U'_0$ by $\theta^1=p\iota_{\mathfrak{c},\mathfrak{b}}$, where $p:U'_1\to U'_1/U'_0$ is the canonical projection. The image of θ^1 is included in the component of degree 1 of $\mathrm{gr}'U$, so $\mathrm{Im}\,\theta^1\subseteq P(\mathrm{gr}'U)$. Clearly θ^1 is a map of braided vector spaces. One can check that $\nabla_{\mathrm{gr}'U}\mathfrak{c}_{\mathrm{gr}'U}=\lambda\nabla_{\mathrm{gr}'U}$ on $U'_1/U'_0\otimes U'_1/U'_0$. By Proposition 2.5 (see also Remark 2.6) there is a unique morphism of graded braided bialgebras $\theta:S(V,\mathfrak{c})\to\mathrm{gr}'U$ that lifts θ^1. On the other hand, $\mathrm{gr}'U(V,\mathfrak{c},\mathfrak{b})$ is generated as an algebra by U'_1/U'_0. Since U'_1/U'_0 is included into the image of θ, we conclude that θ is surjective. □

Remark 4.2. The second statement in the following theorem is well known, see [Gu1]. Nevertheless we include it for sake of completeness.

Theorem 4.3. *Let K be a field with* $\operatorname{char} K \neq 2$. *Let $\mathfrak{b}$ be a $\mathfrak{c}$-bracket on a braided vector space $(V, \mathfrak{c})$ of Hecke-type of mark $\lambda \neq 0$ such that $(3)!_\lambda \neq 0$.*

Assume that the K-linear map $\iota_{\mathfrak{c},\mathfrak{b}} : V \to U(V, \mathfrak{c}, \mathfrak{b})$ is injective.

- *If $\lambda \neq 1$, then $\mathfrak{b} = 0$.*
- *If $\lambda = 1$, then $\mathfrak{b}$ fulfills*

$$\mathfrak{b}\mathfrak{c} = -\mathfrak{b} \qquad \text{and} \qquad \mathfrak{b}\mathfrak{b}_1(\mathrm{Id}_{V^{\otimes 3}} - \mathfrak{c}_2 + \mathfrak{c}_2\mathfrak{c}_1) = 0 \tag{29}$$

i.e. $(V, \mathfrak{c}, \mathfrak{b})$ is a generalized Lie algebra in the sense of [Gu1].

Proof. Denote by T^n the n-th graded component of $T = T(V, c)$ and set $T^{\leq n} := \bigoplus_{0 \leq m \leq n} T^m$ and $T^{\geq n} := \bigoplus_{m \geq n} T^m$.

Let $\gamma := \mathfrak{c} - \lambda \mathrm{Id}_{T^2} - \mathfrak{b}$, let $F := \mathrm{Im}(\gamma)$ and $R := \mathrm{Im}(\lambda \mathrm{Id}_{V^{\otimes 2}} - \mathfrak{c})$. We denote the component of degree 1 of the map θ of Proposition 4.1 by $\theta^1 : V \to U_1'/U_0'$.

Let $x \in \operatorname{Ker} \theta^1$. It follows that $\iota_{\mathfrak{c},\mathfrak{b}}(x) \in U_0'$, so there is $a \in K$ such that $x - a1 \in (F)$. Since $(F) \subseteq T^{\geq 1}$ we get $a = 0$. It results that $\iota_{\mathfrak{c},b}(x) = 0$, so $x = 0$. In conclusion, θ^1 is injective. But, in view of Theorem 2.17, $S(V, \mathfrak{c})$ is a strict coalgebra so that $P(S(V, \mathfrak{c})) = S^1(V, \mathfrak{c}) = V$. Thus, since θ^1, the restriction of θ to V, is injective, by [Mo, Lemma 5.3.3] it follows that θ is injective too. Since θ is always surjective, we conclude that θ is an isomorphism of braided bialgebras.

The algebras $S(V, \mathfrak{c})$ and $U(V, \mathfrak{c}, \mathfrak{b})$ are the quotients of $T(V, \mathfrak{c})$ through the two-sided ideals generated by R and F, respectively. Set

$$\zeta = (\lambda \mathrm{Id}_{V^{\otimes 3}} - \mathfrak{c}_1)(\lambda^2 \mathrm{Id}_{V^{\otimes 3}} - \lambda \mathfrak{c}_2 + \mathfrak{c}_2\mathfrak{c}_1) \overset{(2)}{=} (\lambda \mathrm{Id}_{V^{\otimes 3}} - \mathfrak{c}_2)(\lambda^2 \mathrm{Id}_{V^{\otimes 3}} - \lambda \mathfrak{c}_1 + \mathfrak{c}_1\mathfrak{c}_2).$$

Since $(2)_\lambda \neq 0$, one can easily see that $R = \{x \in T^2 \mid \mathfrak{c}(x) = -x\}$ (note that if x is in this set, then $(\lambda \mathrm{Id}_{V^{\otimes 2}} - c)(x) = (2)_\lambda x$) and since $(3)!_\lambda \neq 0$ one gets that

$$(R \otimes V) \cap (V \otimes R) = \{x \otimes T^3(V) \mid \mathfrak{c}_1(x) = \mathfrak{c}_2(x) = -x\} = \mathrm{Im}\, \zeta. \tag{30}$$

Since the canonical map $\theta : S(V, \mathfrak{b}) \to \mathrm{gr}' U(V, \mathfrak{c}, \mathfrak{b})$ is an isomorphism, by [BG, Lemma 0.4], it follows that the following conditions are satisfied:

$$F \cap T^{\leq 1} = 0 \tag{31}$$

$$\left(T^{\leq 1} \cdot F \cdot T^{\leq 1}\right) \cap T^{\leq 2} = F. \tag{32}$$

We claim (31) implies

$$\mathfrak{b}\mathfrak{c} = -\mathfrak{b}. \tag{33}$$

In fact we have

$$\gamma(\mathfrak{c} + \mathrm{Id}_{T^2}) = (\mathfrak{c} - \lambda \mathrm{Id}_{T^2} - \mathfrak{b})(\mathfrak{c} + \mathrm{Id}_{T^2}) = -\mathfrak{b}(\mathfrak{c} + \mathrm{Id}_{T^2}) \tag{34}$$

so that $\mathrm{Im}[\mathfrak{b}(\mathfrak{c} + \mathrm{Id}_{T^2})] = \mathrm{Im}[\gamma(\mathfrak{c} + \mathrm{Id}_{T^2})]$. We will prove that $\mathrm{Im}[\gamma(\mathfrak{c} + \mathrm{Id}_{T^2})] = F \cap T^{\leq 1}$, from which the conclusion will follow.

$\subseteq$) It follows by (34).

$\supseteq$) Let $y \in F \cap T^{\leq 1}$. Then there is x such that $y = \gamma(x) = \mathfrak{c}(x) - \lambda x - \mathfrak{b}(x)$. Since $y, \mathfrak{b}(x) \in T^{\leq 1}$ and $\mathfrak{c}(x), x \in V \otimes V$, it results $\mathfrak{c}(x) = \lambda x$ and $y = -\mathfrak{b}(x)$. Thus

$\gamma(\mathfrak{c} + \mathrm{Id}_{T^2})(x) = (2)_\lambda \gamma(x) = (2)_\lambda y$. Since $(2)_\lambda \neq 0$, we get $y \in \mathrm{Im}[\gamma(\mathfrak{c} + \mathrm{Id}_{T^2})]$.

Now, we define $\alpha := -(2)_\lambda^{-1} \mathfrak{b}|_R$. From (33), we deduce that

$$(\mathrm{Id}_{T^2} - \alpha)(\mathfrak{c} - \lambda \mathrm{Id}_{T^2}) = \mathfrak{c} - \lambda \mathrm{Id}_{T^2} + \frac{1}{(2)_\lambda} \mathfrak{b}(\mathfrak{c} - \lambda \mathrm{Id}_{T^2}) = \gamma$$

so that $F = \mathrm{Im}\gamma = \mathrm{Im}[(\mathrm{Id}_{T^2} - \alpha)(\mathfrak{c} - \lambda \mathrm{Id}_{T^2})] = \{x - \alpha(x) \mid x \in R\}$. By [BG, Lemma 3.3] (32) implies that α satisfies the following two conditions:

$$\begin{aligned} (\alpha \otimes V)(x) - (V \otimes \alpha)(x) &\in R \qquad \text{for all } x \in (R \otimes V) \cap (V \otimes R), \\ \alpha(\alpha \otimes V - V \otimes \alpha)(x) &= 0 \qquad \text{for all } x \in (R \otimes V) \cap (V \otimes R). \end{aligned}$$

The second property is equivalent to the fact that $\mathfrak{b}(\mathfrak{b}_1-\mathfrak{b}_2)=0$ on $\operatorname{Im}\zeta$, which at its turn is equivalent to

$$\mathfrak{b}\mathfrak{b}_1\zeta=\mathfrak{b}\mathfrak{b}_2\zeta. \tag{35}$$

Let us prove that $\mathfrak{b}$ satisfies the following conditions

$$\mathfrak{b}\mathfrak{b}_1(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_2+\mathfrak{c}_2\mathfrak{c}_1)=0, \tag{36}$$

$$\mathfrak{b}\mathfrak{b}_2(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_1+\mathfrak{c}_1\mathfrak{c}_2)=0. \tag{37}$$

We have

$$\mathfrak{b}\mathfrak{b}_2\zeta\overset{(30)}{=}\mathfrak{b}\mathfrak{b}_2\mathfrak{c}_1\mathfrak{c}_2\zeta\overset{(23)}{=}\mathfrak{b}\mathfrak{c}\mathfrak{b}_1\zeta\overset{(33)}{=}-\mathfrak{b}\mathfrak{b}_1\zeta$$

In view of (35), we obtain $\mathfrak{b}\mathfrak{b}_2\zeta=0=\mathfrak{b}\mathfrak{b}_1\zeta$ as $\operatorname{char}(K)\neq 2$. We have

$$0=\mathfrak{b}\mathfrak{b}_1\zeta=\mathfrak{b}\mathfrak{b}_1(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_1)(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_2+\mathfrak{c}_2\mathfrak{c}_1)\overset{(33)}{=}(2)_\lambda\mathfrak{b}\mathfrak{b}_1(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_2+\mathfrak{c}_2\mathfrak{c}_1).$$

and

$$0=\mathfrak{b}\mathfrak{b}_2\zeta=\mathfrak{b}\mathfrak{b}_2(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_2)(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_1+\mathfrak{c}_1\mathfrak{c}_2)\overset{(33)}{=}(2)_\lambda\mathfrak{b}\mathfrak{b}_2(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_1+\mathfrak{c}_1\mathfrak{c}_2).$$

so that $\mathfrak{b}$ satisfies (36) and (37). We have

$$0\overset{(23)}{=}\mathfrak{b}\left[\mathfrak{b}_1\mathfrak{c}_2\mathfrak{c}_1-\mathfrak{c}\mathfrak{b}_2\right]=\mathfrak{b}\mathfrak{b}_1\mathfrak{c}_2\mathfrak{c}_1-\mathfrak{b}\mathfrak{c}\mathfrak{b}_2\overset{(33)}{=}\mathfrak{b}\mathfrak{b}_1\mathfrak{c}_2\mathfrak{c}_1+\mathfrak{b}\mathfrak{b}_2\overset{(36)}{=}\mathfrak{b}\mathfrak{b}_1\left(-\lambda^2\mathrm{Id}_{V^{\otimes 3}}+\lambda\mathfrak{c}_2\right)+\mathfrak{b}\mathfrak{b}_2$$

so that $\mathfrak{b}\mathfrak{b}_2=\lambda\mathfrak{b}\mathfrak{b}_1\left(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_2\right)$. Similarly using $\mathfrak{c}\mathfrak{b}_1=\mathfrak{b}_2\mathfrak{c}_1\mathfrak{c}_2$ in (23), (33) and (37) we get $\mathfrak{b}\mathfrak{b}_1=\lambda\mathfrak{b}\mathfrak{b}_2\left(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_1\right)$. Using these formulas we obtain

$$\begin{aligned}\mathfrak{b}\mathfrak{b}_2 &= \lambda\mathfrak{b}\mathfrak{b}_1\left(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_2\right)=\lambda\lambda\mathfrak{b}\mathfrak{b}_2\left(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_1\right)\left(\lambda\mathrm{Id}_{V^{\otimes 3}}-\mathfrak{c}_2\right)\\ &= \lambda^2\mathfrak{b}\mathfrak{b}_2\left(\lambda^2\mathrm{Id}_{V^{\otimes 3}}-\lambda\mathfrak{c}_1+\mathfrak{c}_1\mathfrak{c}_2-\lambda\mathfrak{c}_2\right)\overset{(37)}{=}-\lambda^3\mathfrak{b}\mathfrak{b}_2\mathfrak{c}_2\overset{(33)}{=}\lambda^3\mathfrak{b}\mathfrak{b}_2\end{aligned}$$

so that $\left(\lambda^3-1\right)\mathfrak{b}\mathfrak{b}_2=0$. Assume $\mathfrak{b}$ is not zero. By Proposition 3.3, if $\lambda\neq 0,1$, we get that $\mathfrak{b}$ is surjective. Thus $\mathfrak{b}\mathfrak{b}_2=\mathfrak{b}\left(V\otimes\mathfrak{b}\right)$ is surjective too and hence $\lambda^3=1$, contradicting $(3)_\lambda\neq 0$. □

Remarks 4.4. Concerning the converse of Theorem 4.3, let us note that if $\mathfrak{b}=0$ then the map $\iota_{\mathfrak{c},\mathfrak{b}}:V\to U(V,\mathfrak{c},\mathfrak{b})=S(V,\mathfrak{c})$ is clearly injective. On the other hand, let K be a field of characteristic zero. Given a $\mathfrak{c}$-bracket $\mathfrak{b}$ on a braided vector space $(V,\mathfrak{c})$ where $\mathfrak{c}^2=\mathrm{Id}_{V\otimes V}$ (i.e. $(V,\mathfrak{c})$ of Hecke-type of regular mark 1), then the canonical map $\iota_{\mathfrak{c},\mathfrak{b}}:V\to U(V,\mathfrak{c},\mathfrak{b})$ is injective whenever $\mathfrak{b}$ fulfills (29) (see [Kh, Theorem 5.2]).

Example 4.5. [Mas] Let K be a field with $\operatorname{char}(K)=2$. Let $V=Kx$ and let $\mathfrak{c}:V\otimes V\to V\otimes V,\mathfrak{c}=\mathrm{Id}_{V\otimes V}$. Define $\mathfrak{b}:V\otimes V\to V$ by $\mathfrak{b}\left(x\otimes x\right)=ax$ for some $a\in K\backslash\left\{0\right\}$.

Assume there exists $\lambda\in K\backslash\left\{0,1\right\}$ such that $(3)!_\lambda\neq 0$ i.e. such that λ is not a primitive third root of unity. Clearly $\left(\mathfrak{c}+\mathrm{Id}_{V\otimes V}\right)\left(\mathfrak{c}-\lambda\mathrm{Id}_{V\otimes V}\right)=0$ so that $\mathfrak{c}$ is of Hecke-type of mark λ. Moreover $\mathfrak{b}$ is a $\mathfrak{c}$-bracket on the braided vector space $(V,\mathfrak{c})$. Thus we can consider the universal enveloping algebra

$$U=U(V,\mathfrak{c},\mathfrak{b}):=\frac{T(V,\mathfrak{c})}{\left(\mathfrak{c}(z)-\lambda z-\mathfrak{b}(z)\mid z\in V\otimes V\right)}\simeq\frac{K\left[X\right]}{\left(\left(1-\lambda\right)X^2-aX\right)}.$$

The canonical map $\iota_{\mathfrak{c},\mathfrak{b}}:V\to U(V,\mathfrak{c},\mathfrak{b})$ is clearly injective. Nevertheless $\mathfrak{b}\neq 0$.

Remark 4.6. Let $(V,I\oplus I^*=V^{\otimes 2},[,])$ be a braided Lie algebra in the sense of [Gu2, Definition 1] such that $I:=I_-=\operatorname{Im}(q\mathrm{Id}_{V^{\otimes 2}}-S)$ and $I^*:=I_+=\operatorname{Im}(\mathrm{Id}_{V^{\otimes 2}}+S)$ where $I_\pm\subseteq V^{\otimes 2}$ are as in [Gu2, Section 1]. Here $S:V^{\otimes 2}\to V^{\otimes 2}$ is a braiding of Hecke-type of mark q. Let $b:=-(2)_q[,]$. Then b is a morphism in the braided monoidal category $\mathfrak{A}$ is defined as before [Gu2, Definition 1]. Thus b is compatible with the braiding in $\mathfrak{A}$. This entails that $b:V\otimes V\to V$ is a S-bracket. In view of [Gu2, Proposition 5], we have that the map $\theta:S(V,\mathfrak{c})\to\mathrm{gr}'U(V,\mathfrak{c},\mathfrak{b})$ of Proposition 4.1 is an isomorphism. This implies that the canonical map $\iota_{S,\mathfrak{b}}:V\to U(V,S,b)$ is injective. In fact let $x\in V$ be in the kernel of $\iota_{S,\mathfrak{b}}$. Then $\pi_U(x)=\iota_{S,\mathfrak{b}}(x)=0$ so that $\theta(x)=\pi_U(x)+U_0'=0+U_0'$. Since θ is injective, we get that $x=0$. Therefore, in view of Theorem 4.3, if $q\neq -1$ is not a cubic root of one, we deduce that $b=0$ and hence $[,]=0$ (note that in [Gu2] the characteristic of K is assumed to be zero).

Anyway, we outline that the case considered in [Gu2] does not require, in general, neither that $I := I_- = \mathrm{Im}(q\mathrm{Id}_{V^{\otimes 2}} - S)$ nor that $I^* := I_+ = \mathrm{Im}(\mathrm{Id}_{V^{\otimes 2}} + S)$. In fact the aim of [Gu2] is to introduce a braided counterpart of the notion of S-Lie algebra (for S involutive) such that the corresponding enveloping algebra is a quadratic algebra. Here S needs not to be of Hecke-type.

5. A Milnor-Moore type theorem for braided bialgebras

In this section we prove the main result of this paper, Theorem 5.5, which represents a variant of Milnor-Moore Theorem for braided bialgebras. Then we deduce some consequences of this theorem, including applications to certain classes of bialgebras in braided categories.

Definition 5.1. Let $(A, \mathfrak{c}_A)$ be a connected braided bialgebra. Let $P := P(A)$. The braiding $\mathfrak{c}_P = \mathfrak{c}_A|_{P\otimes P}$ will be called the **infinitesimal braiding** of A.

Remarks 5.2. Let $(A, \mathfrak{c}_A)$ be a connected braided bialgebra. Let $P := P(A)$ and let $\mathfrak{c}_P$ be the infinitesimal braiding of A. If $\mathrm{gr}\, A$ denotes the graded associated with respect to the coradical filtration, then $\mathrm{gr}\, A$ is strictly graded. Thus

$$P(\mathrm{gr}\, A) = \mathrm{gr}^1 A \simeq P(A) = P.$$

Through this identification, $\mathfrak{c}^{1,1}_{\mathrm{gr}\, A}$ is equal to $\mathfrak{c}_P$. In conclusion the infinitesimal braiding of $\mathrm{gr}\, A$ is the infinitesimal braiding $\mathfrak{c}_P$ of A.

Definition 5.3. Let $(A, \mathfrak{c}_A)$ be a connected braided bialgebra and let $P := P(A)$. The component $\Delta^{1,1}_{\mathrm{gr}\, A} : A_2/A_1 \to A_1/A_0 \otimes A_1/A_0 = P \otimes P$ is called the *infinitesimal comultiplication of* A.

Let $\mathfrak{c}_P = \mathfrak{c}_A|_{P\otimes P}$ and let $\lambda \in K^*$. We will say that $\Delta^{1,1}_{\mathrm{gr}\, A}$ is λ-*cocommutative* (or that $(A, \mathfrak{c}_A)$ is *infinitesimally* λ-*cocommutative*) if $\mathfrak{c}_P \circ \Delta^{1,1}_{\mathrm{gr}\, A} = \lambda\Delta^{1,1}_{\mathrm{gr}\, A}$, that is we have:

$$\mathfrak{c}^{1,1}_{\mathrm{gr}\, A} \circ \Delta^{1,1}_{\mathrm{gr}\, A} = \lambda\Delta^{1,1}_{\mathrm{gr}\, A}. \tag{38}$$

Proposition 5.4. *Let K be a field with* $\mathrm{char}\, K \neq 2$. *Let A be a connected braided bialgebra and assume that its infinitesimal braiding is of Hecke-type of mark $\lambda \neq 0, 1$ such that* $(3)!_\lambda \neq 0$. *Let P be the space of primitive elements of A and let $\mathfrak{b}_P = \nabla(\mathfrak{c}_P - \lambda\mathrm{Id}_{P^{\otimes 2}})|_{P\otimes P}$ be the $\mathfrak{c}_P$-bracket on the braided vector space $(P, \mathfrak{c}_P)$ defined in Proposition 2.4. Then $\mathfrak{b}_P = 0$.*

Let $f : (V, \mathfrak{c}, \mathfrak{b}) \to (P, \mathfrak{c}_P, 0)$ be a morphism of braided brackets and assume that $\mathfrak{c}$ is a braiding of Hecke-type with mark λ. Then there is a unique morphism of braided bialgebras $\widetilde{f} : U(V, \mathfrak{c}, \mathfrak{b}) \to A$ that lifts f.

Proof. By Proposition 2.4(b) it follows that $\mathfrak{b}_P$ is a $\mathfrak{c}_P$-bracket on $(P, \mathfrak{c}_P)$, hence we can apply the universal property of $U := U(P, \mathfrak{c}_P, \mathfrak{b}_P)$. There is a unique morphism of braided bialgebras $\phi_A : U \to A$ that lifts Id_P. Observe that the canonical map $\iota_{\mathfrak{c},\mathfrak{b}} : P \to U$ is injective, as $\phi_A\iota_{\mathfrak{c},\mathfrak{b}}$ is the inclusion of P into A. Now apply Theorem 4.3 to obtain that $\mathfrak{b}_P = 0$. The last part follows by 2.4. □

Theorem 5.5. *Let K be a field with* $\mathrm{char}\, K \neq 2$. *Let $(A, \mathfrak{c}_A)$ be a connected braided bialgebra which is infinitesimally λ-cocommutative for some regular element $\lambda \neq 0$ in K. Let $P = P(A)$. Then*

- *the infinitesimal braiding $\mathfrak{c}_P$ of A is of Hecke-type of mark λ and*
- *A is isomorphic as a braided bialgebra to the symmetric algebra $S(P, \mathfrak{c}_P)$ of $(P, \mathfrak{c}_P)$ whenever $\lambda \neq 1$.*

Proof. Let $B := \mathrm{gr}A$. Clearly B is strongly $\mathbb{N}$-graded as a coalgebra (see e.g. [AM2, Theorem 2.10]). By assumption $(A, \mathfrak{c}_A)$ is infinitesimally λ-cocommutative and hence the same holds for $(B, \mathfrak{c}_B)$ i.e. $\left(\mathfrak{c}^{1,1}_B - \lambda\mathrm{Id}_{B^2}\right)\Delta^{1,1}_B = 0$. Since B is also 0-connected, by Theorem 2.15, B is a bialgebra of type one and $\mathfrak{c}^{1,1}_B$ is a braiding of Hecke-type of mark λ. In particular the infinitesimal braiding of A is of Hecke-type of mark λ and B is generated as an algebra by B^1 so that A is generated as a K-algebra by $P = P(A) = B^1$. Therefore, the canonical braided bialgebra homomorphism $f : U(P, \mathfrak{c}_P, \mathfrak{b}_P) \to A$, arising by the universal property of the universal enveloping algebra, is surjective. Assume $\lambda \neq 1$. By Proposition 5.4, $\mathfrak{b}_P = 0$ hence $U(P, \mathfrak{c}_P, \mathfrak{b}_P) = S(P, \mathfrak{c}_P)$. On the other hand, by Theorem 2.17, P is the primitive part of $S(P, \mathfrak{c}_P)$ and the restriction of f to P is injective so that f is injective by [Mo, Lemma 5.3.3]. In conclusion f is an isomorphism. $\square$

Remark 5.6. Let (H, c_H) be a connected braided Hopf algebra and let $B := grH$ be the graded coalgebra associated to the coradical filtration of H. Let $\lambda \in K^*$. Since B is always strongly $\mathbb{N}$-graded as a coalgebra (see e.g. [AM2, Theorem 2.10]), in view of Theorem 2.15, the following assertions are equivalent:

- H is cosymmetric in the sense of [Kh, Definition 3.1] (see also [Kh, Theorem 3.5]) and $\mathfrak{c}_B^{1,1}$ is a braiding of Hecke-type of mark λ,
- $\mathfrak{c}_B^{1,1} \circ \Delta_B^{1,1} = \lambda \Delta_B^{1,1}$ i.e. H is infinitesimally λ-cocommutative.

Remark 5.7. With hypothesis of Theorem 5.5, if $\lambda = 1$ then A is isomorphic as a braided bialgebra to the universal enveloping algebra $U(P, \mathfrak{c}_P, \mathfrak{b}_P)$ of $(P, \mathfrak{c}_P, \mathfrak{b}_P)$. In fact regularity of λ in this case means char $(K) = 0$. By Theorem 4.3 $\mathfrak{b}\mathfrak{c} = -\mathfrak{b}$ and $\mathfrak{b}\mathfrak{b}_1(\mathrm{Id}_{V^{\otimes 3}} - \mathfrak{c}_2 + \mathfrak{c}_2\mathfrak{c}_1) = 0$ so that [Kh, Theorem 6.1] applies.

Corollary 5.8. *Let K be a field with* char $K \neq 2$. *Let $(V, \mathfrak{c})$ be a braided vector space such that $\mathfrak{c}$ is a braiding of Hecke-type of regular mark $\lambda \neq 0, 1$. Let A be a braided bialgebra such that* gr A *is isomorphic as a braided bialgebra to $S(V, \mathfrak{c})$ then A is isomorphic to the symmetric algebra $S(V, \mathfrak{c})$ of $(V, \mathfrak{c})$.*

Proof. Obviously the infinitesimal comultiplication of $S(V, \mathfrak{c})$ is λ-cocommutative and, by Proposition 2.8, $S(V, \mathfrak{c})$ is connected. Thus gr A has the same properties. Hence A itself is connected and by Remark 5.2, $(A, \mathfrak{c}_A)$ is infinitesimally λ-cocommutative. We conclude by applying Theorem 5.5. $\square$

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