Some results on time-varying fractional partial differential equations and birth-death processes

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Abstract. *Some results on specific types of branching processes are presented. Firstly, linear pure birth and birth-death processes governed by partial differential equations with time-varying coefficients are analysed. Such processes are constructed by inserting the fractional time derivative into the p.d.e. governing the law of fractional Brownian motion. We consider also pure birth processes stopped at first-passage time of Brownian motion and present the related distributions and the governing equations. Some explicit results on the mean values and low-order probabilities are obtained in terms of generalised Mittag-Leffler functions.*

Keywords. *Fractional derivatives, branching processes, partial differential equations with time-varying coefficients, processes with random time.*

1 Introduction

In this paper we consider compositions of different types of processes whose distribution is related either to fractional equations or to higher-order partial differential equations. In a previous paper of ours [4] we considered the fractional pure birth process $\mathsf{N}_\nu(t)$, $t > 0$ whose state probabilities

$$\hat{p}_k^\nu(t) = \Pr\left\{\mathsf{N}_\nu(t) = k\right\}, \tag{1}$$

where $k \geq 1$ (with one initial progenitor), $t > 0$, and $\nu \in (0, 1]$, satisfy the time-fractional difference-differential equations

$$\frac{d^\nu}{dt^\nu} p_k(t) = -\lambda_k p_k(t) + \lambda_{k-1} p_{k-1}(t), \tag{2}$$

subject to the initial conditions

$$p_k(0) = \begin{cases} 1, & k > 1 \\ 0, & k = 1 \end{cases}. \tag{3}$$

For $\lambda_k = \lambda k$, $k \geq 1$ we have the linear pure birth process. We have obtained the explicit distribution in the linear and non-linear cases with many related properties of the fractional birth process. We have also shown that $\mathsf{N}_\nu(t)$, $t > 0$ can be represented as

$$\mathsf{N}_\nu(t) = \mathsf{N}\left(T_{2\nu}(t)\right) \tag{4}$$

where $\mathsf{N}(t)$, $t > 0$, is the classical birth process and $T_{2\nu}(t)$ is a stochastic process independent from $\mathsf{N}(t)$ and possessing distribution related to the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial^{2\nu}}{\partial t^{2\nu}} f(x,t) = \dfrac{\partial^2}{\partial x^2} f(x,t), & \nu \in (0,1] \\ f(x,0) = \delta(x) \end{cases}. \tag{5}$$

The fractional derivative appearing in (2) and (5) must be understood as follows

$$\begin{cases} \dfrac{d^\nu h(t)}{dt^\nu} = \dfrac{\displaystyle\int_0^t \frac{\frac{d}{ds} h(s)}{(t-s)^\nu} ds}{\Gamma(1-\nu)}, & \nu \in (0,1) \\ h'(t), & \nu = 1 \end{cases}. \tag{6}$$

We consider here the equation

$$\frac{d^\nu}{dt^\nu} p_k(t) = \lambda t^{2H-1} \left\{(k-1) p_{k-1}(t) - k p_k(t)\right\} \tag{7}$$

which generalises (2) and produces for the probability generating function $\mathsf{G}(u,t) = \mathbb{E} u^{\mathsf{N}_\nu(t)}$ the p.d.e.

$$\frac{\partial^\nu}{\partial t^\nu} \mathsf{G}(u,t) = \lambda t^{2H-1} u(u-1) \frac{\partial}{\partial u} \mathsf{G}(u,t) \tag{8}$$

subject to the initial condition $\mathsf{G}(u,0) = u$. The time-dependent coefficient in (8) is inspired by the structure of the partial differential equation governing the distribution $g(t,x)$ of the fractional Brownian motion which reads

$$\frac{\partial}{\partial t} g(t,x) = H t^{2H-1} \frac{\partial^2}{\partial x^2} g(t,x) \tag{9}$$

with the initial condition $g(0,x) = \delta(x)$, and coincides with the classical heat equation for $H = 1/2$. From (4) the process $\mathsf{N}_{\frac{1}{2}}(t)$, $t > 0$, has the following representation

$$\mathsf{N}_{\frac{1}{2}}(t) = \mathsf{N}\left(|B(t)|\right) \tag{10}$$

where $B(t)$, $t>0$ is a standard Brownian motion independent from $\mathsf{N}(t)$. The process related to (7) and (8) has the representation

$$\mathfrak{N}_{\frac{1}{2}}(t)=\mathsf{N}\left(|B_H(t)|\right) \tag{11}$$

where $B_H(t)$, $t>0$ is a fractional Brownian motion independent from $\mathsf{N}(t)$, $t>0$. We also study in detail other properties and this involves a delicate analysis based on the generalised Mittag-Leffler functions of the form

$$E_{\alpha,m,l}(z)=1+\sum_{k=1}^{\infty} z^k \prod_{j=0}^{k-1} \frac{\Gamma\left(\alpha\left(jm+l\right)+1\right)}{\Gamma\left(\alpha\left(jm+l+1\right)+1\right)}. \tag{12}$$

2 Partial differential equations with time-varying coefficients and branching processes

It is well-known that the partial differential equation satisfied by the transition law of the fractional Brownian motion is a simple time-modification of the heat equation (see appendix 6). We are interested in analysing similar differential equations. An example of such equations is the analogue of that governing a classical linear birth process (see for example [2], page 87) but with time-varying coefficients i.e.

$$\begin{cases} \dfrac{\partial}{\partial t}G(u,t)=\lambda t^{2H-1}u(u-1)\dfrac{\partial}{\partial u}G(u,t) \\ G(u,0)=u \end{cases}, \tag{13}$$

where $t>0$, $H\in(0,1]$ and λ is the birth rate. By $G=G(u,t)$, $0<u\leq 1$, $t>0$, we denote the probability generating function of the pure birth process related to (7) for $\nu=1$. From (13) we infer that for the mean value $\mathbb{E}N(t)=G_u(u,t)|_{u=1}$ we arrive at the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\mathbb{E}N(t)=\lambda t^{2H-1}\mathbb{E}N(t) \\ \mathbb{E}N(0)=1 \end{cases}, \tag{14}$$

where $t>0$, $H\in(0,1]$ and λ is the rate of birth. Equation (14) is readily solved by standard methods and yields $\mathbb{E}N(t)=e^{\frac{\lambda}{2H}t^{2H}}$.

With similar arguments it is possible to solve also the following difference-differential equation

$$\begin{cases} \dfrac{d}{dt}p_1(t)=-\lambda t^{2H-1}p_1(t) \\ p_1(0)=1 \end{cases}, \tag{15}$$

with $t>0$, $H\in(0,1]$ and where $p_1(t)$ is the probability of having exactly one individual at time t given that at time $t=0$ we start with one individual, i.e. $p_1(t)=\Pr\{N(t)=1 \mid N(0)=1\}$. The solution to (15) reads $p_1(t)=e^{-\frac{\lambda}{2H}t^{2H}}$.

It can be checked by direct calculations that probability generating function has the form

$$G(u,t)=\frac{ue^{-\frac{\lambda}{2H}t^{2H}}}{1-u\left(1-e^{-\frac{\lambda}{2H}t^{2H}}\right)}, \qquad t>0. \tag{16}$$

This implies that the distribution of $N(t)$, $t>0$ is

$$p_k(t)=e^{-\lambda t^{2H}}\left(1-e^{-\lambda t^{2H}}\right)^{k-1}, \; k\geq 1. \tag{17}$$

Remark 2.1 *As in the case of* (44) *(see appendix 6) when $H=1/2$ we obtain, as a particular case, the classical linear pure birth process.*

A second example worth of being analysed is that of a linear birth-death process governed by partial differential equations with time-varying coefficients. For details about the classical case see [2], page 93. Here we modify the p.d.e. involving the probability generating function exactly in the same manner as in (13), thus obtaining

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{G}(u,t)=t^{2H-1}(\lambda u-\mu)(u-1)\dfrac{\partial}{\partial u}\mathcal{G}(u,t) \\ G(u,0)=u \end{cases}, \tag{18}$$

where $t>0$, $H\in(0,1]$ and where λ and μ are respectively the birth rate and the death rate. Again, by means of standard methods it can be shown that the function

$$\mathcal{G}(u,t)=\frac{(u-1)\,\mu e^{\frac{\lambda-\mu}{2H}t^{2H}}-\lambda u+\mu}{(u-1)\,\lambda e^{\frac{\lambda-\mu}{2H}t^{2H}}-\lambda u+\mu} \tag{19}$$

is the solution to (18).

Remark 2.2 *The probability generating function reduces to that of the classical case when $H=1/2$. From the probability generating function it is simple to extract the extinction probability $p_0(t)$ which reads*

$$p_0(t)=\mathcal{G}(0,t)=\frac{\mu-\mu e^{\frac{\lambda-\mu}{2H}t^{2H}}}{\mu-\lambda e^{\frac{\lambda-\mu}{2H}t^{2H}}}. \tag{20}$$

We also note that the behaviour of $p_0(t)$, when t tends to infinity, is the same as in the classical case and does not depend on the value of H.

3 Bifractional pure growth process

In order to furnish the system with a further type of fractionality, we substitute the integer time-derivative in (13) with a fractional derivative defined in (6). This kind of fractional derivative (usually called Caputo or Dzhrbashyan-Caputo fractional derivative) is closely related to the usual Riemann-Liouville fractional derivative (for details see [1], page 90). We therefore obtain the following fractional p.d.e.

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu}\mathfrak{G}(u,t) = \lambda t^{2H-1}u(u-1)\frac{\partial}{\partial u}\mathfrak{G}(u,t) \\ \mathfrak{G}(u,0) = u \end{cases}, \tag{21}$$

where $\nu \in (0,1]$, $t > 0$, $H \in (0,1]$, $\mathfrak{G}(u,t) = \mathbb{E}u^{\mathfrak{N}(t)}$, $0 < u \leq 1$ and the fractional derivative is defined in (6). From (21) we can extract some information on the expectation of the process $\mathfrak{N}(t)$, $t > 0$ which has probability generating function $\mathfrak{G}(u,t)$. By considering that

$$\left.\frac{\partial \mathfrak{G}}{\partial u}\right|_{u=1} = \mathbb{E}\mathfrak{N}(t), \qquad t > 0,\ \nu \in (0,1], \tag{22}$$

we obtain

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathbb{E}\mathfrak{N}(t) = \lambda t^{2H-1}\mathbb{E}\mathfrak{N}(t) \\ \mathbb{E}\mathfrak{N}(0) = 1 \end{cases}, \tag{23}$$

where $t > 0$, $\nu \in (0,1]$ and $H \in (0,1]$. The solution to (23) is directly obtained from [1], page 233 as

$$\mathfrak{N}(t) = E_{\nu,1+\frac{2H-1}{\nu},\frac{2H-1}{\nu}}\left(\lambda t^{\nu+2H-1}\right), \qquad t > 0, \tag{24}$$

where $E_{\alpha,m,l}(z)$ is the generalised Mittag-Leffler function defined as

$$E_{\alpha,m,l}(z) = 1+\sum_{k=1}^{\infty} z^k \prod_{j=0}^{k-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \tag{25}$$

and with $(1-\nu)/2 < H \leq 1$. The solution is then proved to be unique when $1/2 \leq H \leq 1$. For details see theorem 6.2.

We now address the problem of determining the probability law of the process $\mathfrak{N}(t)$, $t > 0$ i.e. $p_k^\nu(t) = \Pr\{\mathfrak{N}(t) = k \mid \mathfrak{N}(0) = 1\}$ for $k = 1$. We must solve the fractional difference-differential equation

$$\begin{cases} \frac{d^\nu}{dt^\nu}p_1(t) = -\lambda t^{2H-1}p_1(t), \\ p_1(0) = 1 \end{cases}, \tag{26}$$

where $t > 0$, $\nu \in (0,1]$ and $H \in (0,1]$. Similarly to the solution to (23), the solution to (26) is

$$p_k^\nu(t) = E_{\nu,1+\frac{2H-1}{\nu},\frac{2H-1}{\nu}}\left(-\lambda t^{\nu+2H-1}\right), \tag{27}$$

with again, $t > 0$, $(1-\nu)/2 < H \leq 1$ and where the solution is proved to be unique for $1/2 \leq H \leq 1$.

Remark 3.1 *From respectively* (27) *and* (24)*, when* $\nu = 1$*, we retrieve the results in section 2. When* $H = 1/2$*, they reduce to the corresponding expressions for the fractional linear pure birth process (see [4]).*

4 Bifractional diffusion

We consider here a generalised version of the so-called time-fractional diffusion equation (see e.g. [3]) as follows

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu}u(x,t) = Ht^{2H-1}\frac{\partial^2}{\partial x^2}u(x,t) \\ u(x,0) = \delta(x) \end{cases}, \tag{28}$$

with $\nu \in (0,2]$, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, together with the further condition $u_t(x,t)|_{t=0} = 0$ when $\nu \in (1,2]$ The fractional derivative appearing in (28) is the Dzhrbashyan-Caputo fractional derivative (6) but with $\nu \in (0,2]$ and the parameter $H \in (0,1]$.

Theorem 4.1 *Let* $u(x,t)$*,* $x \in \mathbb{R}$*,* $t \in \mathbb{R}^+$ *be the solution to the generalised time-fractional diffusion equation* (28)*. The function* $q(x,t)$ *can be written as*

$$q(x,t) = \tag{29}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\beta x}E_{\nu,1+\frac{2H-1}{\nu},\frac{2H-1}{\nu}}\left(-H\beta^2 t^{\nu+2H-1}\right),$$

where $t > 0$*,* $x \in \mathbb{R}$*,* $H \in (0,1]$ *and* $\nu \in (0,2]$*.*

Proof. By resorting to the Fourier transform of (28) we obtain the following differential equation

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu}U(\beta,t) = -Ht^{2H-1}\beta^2 U(\beta,t) \\ U(\beta,0) = 1 \end{cases}, \tag{30}$$

where $t > 0$, $H \in (0,1]$, $\nu \in (0,2]$ and with the additional condition $U_t(\beta,t)|_{t=0} = 0$ when $\nu \in (1,2]$. Equation (30) is solved by (see [1], page 233, examples 4.11 and 4.12)

$$U(\beta,t) = E_{\nu,1+\frac{2H-1}{\nu},\frac{2H-1}{\nu}}\left(-H\beta^2 t^{\nu+2H-1}\right), \tag{31}$$

with $(1-\nu)/2 < H \leq 1$. The solution is proved to be unique for $1/2 \leq H \leq 1$. Note that the function $E_{\alpha,m,l}(z)$ is the generalised Mittag-Leffler (25). From (31) it is straightforward to obtain (29).

5 Birth processes with randomly-varying time

Let $q(t,s)$ be the transition density of the first-passage time T_t at time t of a Brownian motion i.e.

$$q(t,s) = \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}ds, \qquad (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{32}$$

It is well-known that $q = q(t,s)$ is a solution to the partial differential equation

$$\frac{\partial^2}{\partial t^2} q(t,s) = 2\frac{\partial}{\partial s} q(t,s), \qquad (t,s) \in \mathbb{R} + \times \mathbb{R}^+. \tag{33}$$

We are interested in composing a classical pure birth process $\mathsf{N}(t)$, $t > 0$, with the first-passage time T_t, $t > 0$. We have that

$$\begin{aligned} \breve{p}_k(t) &= \Pr\{\mathcal{N}(T_t) = k | \, \mathsf{N}(0) = 1\} \\ &= \int_0^\infty p_k(t) q(t,s) ds. \end{aligned} \tag{34}$$

Theorem 5.1 *Let $\mathsf{N}(t)$, $t > 0$ be a classical pure birth process and T_t, $t > 0$ be the first-passage time at t of a standard Brownian motion. The state probabilities $\breve{p}_k(t) = Pr\{\mathsf{N}(T_t) = k | \, \mathsf{N}(0) = 1\}$ satisfy the following difference-differential equation*

$$\frac{d^2}{dt^2} \breve{p}_k(t) = 2\lambda_k \breve{p}_k(t) - 2\lambda_{k-1} \breve{p}_{k-1}(t), \tag{35}$$

where $t > 0$ and $k \geq 1$.

Proof. Result (35) is directly obtained by evaluating the second order derivative as follows

$$\begin{aligned} \frac{d^2}{dt^2} \breve{p}_k(t) &= \int_0^\infty p_k(s) \frac{\partial^2}{\partial t^2} q(t,s) ds \\ &= 2\int_0^\infty p_k(s) \frac{\partial}{\partial s} q(t,s) ds \\ &= 2q(t,s) p_k(s)|_{s=0}^\infty - 2\int_0^\infty \frac{d}{ds} p_k(s) q(t,s) ds \\ &= -2\int_0^\infty q(t,s) \left\{-\lambda_k p_k(s) + \lambda_{k-1} p_{k-1}(s)\right\} ds \\ &= 2\lambda_k \breve{p}_k(t) - 2\lambda_{k-1} \breve{p}_{k-1}(t) \end{aligned} \tag{36}$$

where $p_k(t)$, $t > 0$, $k \geq 1$, are the state probabilities of a classical pure birth process. Note that we applied equation (33) in the second step of the proof.

The state probabilities $\breve{p}_k(t)$, $t > 0$, can also be written explicitly. This is shown for the linear case in the next theorem (λ is the birth rate).

Theorem 5.2 *The distribution of the linear pure birth process stopped at a random first-passage time T_t of a standard Brownian motion with a simple initial progenitor reads*

$$\breve{p}_k(t) = \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} e^{-t\sqrt{2\lambda m}}, \tag{37}$$

with $k \geq 1$ and $t > 0$.

Proof.

$$\begin{aligned} \breve{p}_k(t) &= \Pr\{\mathsf{N}(T_t) | \, \mathsf{N}(0) = 1\} \\ &= \int_0^\infty e^{-\lambda s} \left(1 - e^{-\lambda s}\right)^{k-1} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \sum_{m=0}^{k-1} \binom{k-1}{m} \int_0^\infty \left(-e^{-\lambda s}\right)^m e^{-\lambda s} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \int_0^\infty e^{-\lambda s(1+m)} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m e^{-t\sqrt{2\lambda(1+m)}} \\ &= \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} e^{-t\sqrt{2\lambda m}}. \end{aligned} \tag{38}$$

This concludes the proof.

In the next theorem we prove that the distribution (37) is actually a solution of the p.d.e. (35).

Theorem 5.3 *Let $\mathsf{N}(t)$, $t > 0$ be a linear pure birth process and T_t, $t > 0$ the first-passage time process of standard Brownian motion. The state probabilities of the composed process $\mathsf{N}(T_t)$, $t > 0$, i.e.*

$$\breve{p}_k(t) = \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} e^{-t\sqrt{2\lambda m}}, \tag{39}$$

where $k \geq 1$, $t > 0$, satisfy the following difference-differential equations

$$\frac{d^2}{dt^2} \breve{p}_k(t) = 2\lambda_k \breve{p}_k(t) - 2\lambda_{k-1} \breve{p}_{k-1}(t), \tag{40}$$

where $k \geq 1$, $t > 0$.

Proof. It is sufficient to calculate the left hand side and right hand side of (40) and show that they coincide. For the left hand side we have

$$\begin{aligned} &\frac{d^2}{dt^2} \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} e^{-t\sqrt{2\lambda m}} = \\ &= \sum_{m=1}^{k} \binom{k-1}{m-1} (-1)^{m-1} (2\lambda m) e^{-t\sqrt{2\lambda m}}. \end{aligned} \tag{41}$$

For the right hand side we have

$$
\begin{aligned}
&k\breve{p}_k(t) - (k-1)\breve{p}_{k-1}(t) = \qquad (42)\\
&= k\sum_{m=1}^{k}\binom{k-1}{m-1}(-1)^{m-1}e^{-t\sqrt{2\lambda m}}+\\
&\quad -(k-1)\sum_{m=1}^{k-1}\binom{k-2}{m-1}(-1)^{m-1}e^{-t\sqrt{2\lambda m}}\\
&= \sum_{m=1}^{k-1}(-1)^{m-1}e^{-t\sqrt{2\lambda m}}\left[\frac{(k-1)!}{(m-1)!\,(k-m-1)!}\times\right.\\
&\quad\left.\times\left(\frac{k}{k-m}-1\right)\right] + k(-1)^{k-1}e^{-t\sqrt{2\lambda k}}\\
&= \sum_{m=1}^{k-1}(-1)^{m-1}e^{-t\sqrt{2\lambda m}}\left(\frac{m\,(k-1)!}{(m-1)!\,(k-m)!}\right)+\\
&\quad + k(-1)^{k-1}e^{-t\sqrt{2\lambda k}}\\
&= \sum_{m=1}^{k}\binom{k-1}{m-1}(-1)^{m-1}e^{-t\sqrt{2\lambda m}}m,
\end{aligned}
$$

and this concludes the proof.

6 Appendix

It is possible to obtain the p.d.e. satisfied by the probability density function of the fractional Brownian motion $B^H(t)$, $z \in \mathbb{R}^+$ by a simple time-scaling procedure as follows

Theorem 6.1 *Let $u(z,x)$ be the transition density of a standard Brownian motion satisfying the heat equation*

$$\frac{\partial}{\partial z}u(z,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(z,x) \qquad (43)$$

together with the usual initial condition $u(0,x) = \delta(x)$. After the substitution $z = t^{2H}$, $H \in (0,1]$, $u(z,x) = g(t,x)$ satisfies the following partial differential equation with time-varying coefficients

$$\frac{\partial}{\partial t}g(t,x) = Ht^{2H-1}\frac{\partial^2}{\partial x^2}g(t,x) \qquad (44)$$

with the initial condition $g(0,x) = \delta(x)$.

Proof. By evaluating the partial derivative of $g(t,x)$ with respect to time t we have

$$
\begin{aligned}
\frac{\partial}{\partial t}g(t,x) &= \frac{\partial}{\partial t}u(z,x) = \frac{\partial}{\partial z}u(z,x)2Ht^{2H-1} \qquad (45)\\
&= Ht^{2H-1}\frac{\partial^2}{\partial x^2}u(z,x)\\
&= Ht^{2H-1}\frac{\partial^2}{\partial x^2}g(t,x)
\end{aligned}
$$

thus obtaining (44).

In the following theorem we prove that the generalised Mittag-Leffler function (25) is the solution to the differential equation considered in section 3.

Theorem 6.2 *The generalised Mittag-Leffler function*

$$
\begin{aligned}
E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(\lambda t^{\alpha+\beta}\right) &= 1+\sum_{k=1}^{\infty}\lambda^k t^{k(\alpha+\beta)}\times \qquad (46)\\
&\times\prod_{j=0}^{k-1}\frac{\Gamma\left(\alpha\left(j+j\frac{\beta}{\alpha}+\frac{\beta}{\alpha}\right)+1\right)}{\Gamma\left(\alpha\left(j+j\frac{\beta}{\alpha}+\frac{\beta}{\alpha}+1\right)+1\right)}
\end{aligned}
$$

solves the ordinary differential equation with variable coefficients

$$\begin{cases} \dfrac{d^\alpha}{dt^\alpha}y(t) = \lambda t^\beta y(t) \\ y(0) = 1 \end{cases}, \qquad (47)$$

where $t \in \mathbb{R}^+$, $\alpha \in (0,1]$, $\beta > 0$ and where d^α/dt^α is the Dzhrbashyan-Caputo fractional derivative (6).

Proof. The theorem is simply proved by evaluating the fractional derivative in the left hand side of (47) as follows

$$
\begin{aligned}
&\frac{d^\alpha}{dt^\alpha}y(t) = \qquad (48)\\
&= \sum_{k=1}^{\infty}\lambda^k\prod_{j=0}^{k-1}\frac{\Gamma(\alpha j+\beta j+\beta+1)}{\Gamma(\alpha j+\beta j+\beta+\alpha+1)}\frac{d^\alpha}{dt^\alpha}\left[t^{k(\alpha+\beta)}\right]\\
&= \sum_{k=1}^{\infty}\lambda^k\frac{\Gamma(k\alpha+k\beta+1)}{\Gamma(k\alpha+k\beta+1-\alpha)}t^{k(\alpha+\beta)-\alpha}\times\\
&\quad\times\prod_{j=0}^{k-1}\frac{\Gamma(\alpha j+j\beta+\beta+1)}{\Gamma(\alpha j+\beta j+\beta+\alpha+1)}\\
&= \lambda t^\beta\sum_{r=0}^{\infty}\lambda^r t^{r(\alpha+\beta)}\frac{\Gamma(r(\alpha+\beta)+\alpha+\beta+1)}{\Gamma(r(\alpha+\beta)+\beta+1)}\times\\
&\quad\times\prod_{j=0^r}^{r}\frac{\Gamma(\alpha j+j\beta+\beta+1)}{\Gamma(\alpha j+\beta j+\beta+\alpha+1)}\\
&= \lambda t^\beta\left[1+\sum_{r=1}^{\infty}\lambda^r t^{r(\alpha+\beta)}\prod_{j=0}^{r-1}\frac{\Gamma(\alpha j+\beta j+\beta+1)}{\Gamma(\alpha j+\beta j+\beta+\alpha+1)}\right].
\end{aligned}
$$

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