Windowed - Wigner representations in the Cohen class and uncertainty principles

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Abstract

For representations in the Cohen class, specific Cohen kernels depending only on one half of the variables are showed to produce two types of representations which can in a natural way be associated with time and frequency windows. This leads to the definition of representations with no interference for signals whose time-frequency content is confined in specific zones. We prove the main properties of these representations in the context of the Cohen class. We study then uncertainty principles at first in connection with support compactness and then in the framework of a general concept of duality among representations.

Keywords: Time-Frequency representations, Wigner sesquilinear and quadratic form, interference, uncertainty principle.

Mathematics Subject Classification: 42B10, 47A07.

1 Introduction

This paper deals with a subclass of time-frequency representations belonging to the Cohen class, namely, representations whose Cohen kernel has Fourier transform depending only on time or frequency variables. We start by giving some basic definitions and the motivations for studying such objects. A generic representation in the Cohen class is of the form

$$Q(f,g) = \sigma * \mathrm{Wig}(f,g), \tag{1.1}$$

where $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is the "kernel" and the Wigner transform is defined as

$$\mathrm{Wig}(f,g)(x,\omega) = \int e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\,dt,$$

$f, g \in \mathcal{S}(\mathbb{R}^d)$ (other functional settings can be considered as well, by choosing f, g, σ in such a way that (1.1) makes sense). Of course the Wigner transform itself belongs to the Cohen class, for $\sigma = \delta$, and was, in fact, one of the first time-frequency representations to be defined. Various subclasses of the Cohen class show specific interesting features, we recall here two of them. The first one is the "τ-Wigner transform", defined for each real number $\tau \in [0,1]$ as

$$\mathrm{Wig}^{(\tau)}(f,g) = \int e^{-2\pi i t\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\,dt, \tag{1.2}$$

cf. for example [15], where some problems concerning positivity are considered, and [20], [3], where Wig and $\mathrm{Wig}^{(\tau)}$ are studied in connection with pseudo-differential operators. We shall come back to these transforms in Section 2 for a comparison with the representations which we introduce in this paper later on.

Another relevant subclass of the Cohen class is the so-called "generalized spectrogram", defined by

$$\mathrm{Sp}_{\phi_1,\phi_2}(f,g)(x,\omega) = V_{\phi_1}f(x,\omega)\overline{V_{\phi_2}g(x,\omega)}, \tag{1.3}$$

where $f, g \in \mathcal{S}(\mathbb{R}^d)$, $V_{\phi_j}h(x,\omega) = \int e^{-2\pi it\omega}h(t)\overline{\phi_j(t-x)}\,dt$ are *Gabor transforms* with "windows" $\phi_j \in \mathcal{S}'(\mathbb{R}^d)$, $j = 1, 2$ (see e.g. [4], [8], and [13, Thm 11.2.3], for a justification of the functional setting we have mentioned).

Both the spectrograms and the Wigner transform are deeply connected with many aspects of pseudo-differential calculus, see for instance [3], [11], [17] and the references therein.

The generalized spectrogram does not preserve the supports, in fact it can be proved that the projections of its support in time and frequency are in general larger than the corresponding supports of the signal and of its Fourier transform respectively. The Wigner distribution on the other side satisfies the support properties both in time and in frequency, but as a counterpart it shows problems concerning interferences. Indeed in the time-frequency plane it displays a "ghost frequency" in the "middle" of any couple of "true" frequencies. Many attempts have been made in order to find representations with better behavior with respect to interferences. The Cohen class itself, cf. (1.1), is a way to filter the Wigner transform, and for some choices of the kernel σ it can reduce ghosts, see for example [8]. In the paper [2] we have introduced a modification of the Wigner transform, in order to find representations showing no (or at least reduced) interference; for one-variable signals we have defined

$$\mathrm{Wig}^M(f)(x,\omega) = \int_{-M}^{M} e^{-2\pi it\omega} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}\,dt, \tag{1.4}$$

for a fixed $M > 0$. We briefly recall the motivations that led us to the definition of Wig^M. Let us consider a signal f with two frequencies in two disjoint time intervals, say, $[k, k+\alpha]$ and $[h, h+\beta]$, with $k+\alpha < h$. The interference ("ghost") showed by the Wigner in the middle of the two frequencies can be understood geometrically considering that the function $f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}$ is supported in the set $D_1 \cup D_2 \cup D_3 \cup D_4$, cf. Figure 1. Through integration in the t variable, the x-projections of the sets D_1 and D_3 yield the true frequencies, while the ghosts originate from the x-projections of the sets D_2 and D_4, where the two different frequencies multiply each other. As proved in [2], if

$$h - k \geq \max\{2\alpha, \beta - \alpha\} \tag{1.5}$$

then there exists $M > 0$ such that the function $\chi_{[-M,M]}(t)f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}$ is supported in the set $D_1 \cup D_3$, where $\chi_{[-M,M]}$ is the characteristic function of the interval $[-M, M]$; so in this case (1.4) shows no interferences. The starting point of the present paper is the consideration

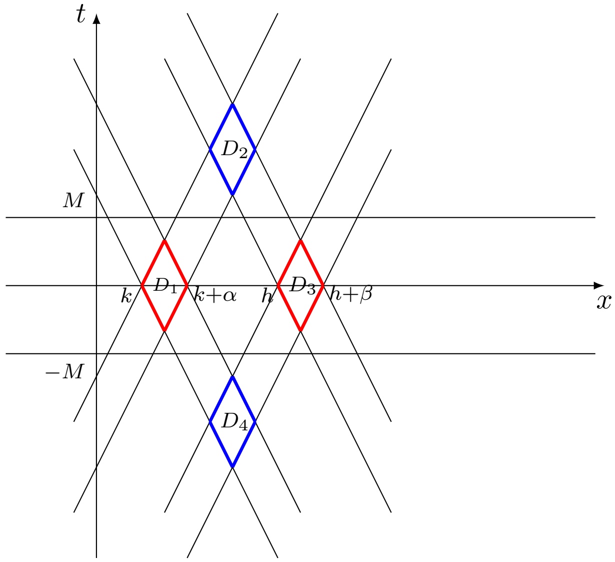


Figure 1:

that more generally we can substitute $\chi_{[-M,M]}(t)$ with a function (or distribution) $\psi(t)$, which leads to the following modification of the Wigner transform

$$\mathrm{Wig}_\psi(f,g)(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi it\omega}\psi(t) f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\,dt, \tag{1.6}$$

for $f, g, \psi \in \mathcal{S}(\mathbb{R}^d)$, see Section 2 for the definition in the distributions framework.

It is then natural to ask whether this method of eliminating interferences, which applies to disjoint time intervals, can be modified on the Fourier transform side to apply to disjoint frequency intervals. A corresponding modification of the Wigner transform in this direction is suggested by the following well-known formula for the classical Wigner:

$$\mathrm{Wig}(f,g)(x,\omega) = \mathrm{Wig}(\hat{f},\hat{g})(\omega,-x). \tag{1.7}$$

We are then led to define

$$\mathrm{Wig}^*_\psi(f,g)(x,\omega) := \mathrm{Wig}_\psi(\hat{f},\hat{g})(\omega,-x) = \int_{\mathbb{R}^d} e^{2\pi itx}\psi(t)\hat{f}\left(\omega+\frac{t}{2}\right)\overline{\hat{g}\left(\omega-\frac{t}{2}\right)}\,dt, \tag{1.8}$$

which is the natural counterpart of Wig_ψ on the Fourier transform side. The aim of this paper is to study Wig_ψ and Wig^*_ψ and is organized as follows. In Section 2 we specify some mapping properties, we prove that these representations belong to the Cohen class and analyze the marginal distributions and the support properties; moreover, we compare these representations with other relevant subclasses of the Cohen class, in particular we characterize for which ψ the representations (1.6) and (1.8) can be expressed as generalized spectrograms. In Section 3 we show on some examples the reduction of interference and we describe how Wig_ψ and Wig_ψ^* can be combined in order to have a better behavior both in time and frequency. Another relevant issue in time-frequency analysis is the presence of some forms of uncertainty principle associated with each representation. The literature on this subject is very vast, we follow in particular the lines of [10] and we prove in Section 4 different uncertainty principles for Wig_ψ and Wig_ψ^*, in the form of properties concerning the support of the representations. Finally in the same section we extend in a natural way the "duality" between Wig_ψ and Wig_ψ^* to a duality between general representations in the Cohen class. This leads to the formulation of a form of uncertainty principle involving couples of dual representations, which generalizes well-known forms of the principle, as well as yielding some new ones.

2 Wigner type representations associated with a window

We start by analyzing some mapping properties of Wig_ψ and Wig_ψ^*. We write in the following

$$\mathrm{Wig}_\psi(f) := \mathrm{Wig}_\psi(f,f) \text{ and } \mathrm{Wig}_\psi^*(f) := \mathrm{Wig}_\psi^*(f,f).$$

First of all we want to rewrite the definition of $\mathrm{Wig}_\psi(f,g)$ in such a way that it makes sense also for distributions. Let $\mathcal{T} : \mathcal{S}'(\mathbb{R}^{2d}) \to \mathcal{S}'(\mathbb{R}^{2d})$ be the extension to tempered distributions of the operator that acts on $F \in \mathcal{S}(\mathbb{R}^{2d})$ as

$$(\mathcal{T}F)(x,t) = F\left(x + t/2, x - t/2\right); \tag{2.1}$$

we then have

$$\mathrm{Wig}_\psi(f,g)(x,\omega) = \mathcal{F}_2[(1 \otimes \psi)(\mathcal{T}(f \otimes \overline{g}))], \tag{2.2}$$

where $\mathcal{F}_2$ is the partial Fourier transform with respect to the second variable. We have the following result.

Proposition 1. *The representations* $\mathrm{Wig}_\psi(f,g)$ *and* $\mathrm{Wig}_\psi^*(f,g)$ *define the following continuous maps:*

$$\begin{aligned}
(i) &\quad (\psi,f,g) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathrm{Wig}_\psi^{(*)}(f,g) \in \mathcal{S}(\mathbb{R}^{2d})\\
(ii) &\quad (\psi,f,g) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathrm{Wig}_\psi^{(*)}(f,g) \in \mathcal{S}'(\mathbb{R}^{2d})\\
(iii) &\quad (\psi,f,g) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \to \mathrm{Wig}_\psi^{(*)}(f,g) \in \mathcal{S}'(\mathbb{R}^{2d})\\
(iv) &\quad (\psi,f,g) \in L^\infty(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to \mathrm{Wig}_\psi^{(*)}(f,g) \in L^2(\mathbb{R}^{2d}),
\end{aligned}$$

where $\mathrm{Wig}_\psi^{(*)}(f,g)$ *stands for either* $\mathrm{Wig}_\psi(f,g)$ *or* $\mathrm{Wig}_\psi^*(f,g)$.

Proof. Concerning $\mathrm{Wig}_\psi(f,g)$ the conclusions follow from (2.2), since all the operators in which we have decomposed $\mathrm{Wig}_\psi(f,g)$ are continuous in the respective spaces. Let us analyze for example the first one; we have that

$$(f,g) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{T}(f \otimes \overline{g}) \in \mathcal{S}(\mathbb{R}^{2d})$$

is continuous; moreover, the multiplication by $1 \otimes \psi$ acts continuously as a map

$$(\psi, F) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^{2d}) \to (1 \otimes \psi)F \in \mathcal{S}(\mathbb{R}^{2d}),$$

and the partial Fourier transform is continuous from $\mathcal{S}(\mathbb{R}^{2d})$ to $\mathcal{S}(\mathbb{R}^{2d})$. The other map properties for $\mathrm{Wig}_\psi(f,g)$ can be proved in the same way. Regarding $\mathrm{Wig}^*_\psi(f,g)$, the continuity properties follow from (1.8) and the corresponding continuity of $\mathrm{Wig}_\psi(f,g)$, since the Fourier transform is bounded from $\mathcal{S} \to \mathcal{S}$, from $\mathcal{S}' \to \mathcal{S}'$ and from $L^2 \to L^2$. □

Furthermore the following map property on Lebesgue spaces holds.
Let $C^\infty_b(\mathbb{R}^d)$ be the space of $C^\infty(\mathbb{R}^d)$ functions with bounded derivatives.

Proposition 2. *For $1 < p < \infty$ the representation $\mathrm{Wig}_\psi(f,g)$ is well defined as bounded map $(\psi, f, g) \in L^\infty(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \times L^{p'}(\mathbb{R}^d) \to \mathrm{Wig}_\psi(f,g) \in L^\infty(\mathbb{R}^{2d})$. Moreover its range is a subset of the space of the continuous functions vanishing at infinity if $\psi \in C^\infty_b(\mathbb{R}^d)$.*

Proof. Let T_a and M_a denote the usual translations and modulations by $a \in \mathbb{R}^d$ respectively. Set for convenience $\psi_2(t) = \psi(2t)$ and $U_{\psi,f,g}(x,\omega) = (f, M_{2\omega} T_x \overline{\psi}_2 T_{2x} \widetilde{g})$, where $\widetilde{g}(t) := g(-t)$; then simple changes of variables yield

$$\mathrm{Wig}_\psi(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} U_{\psi,f,g}(x,\omega),$$

consequently we have the following estimate which proves the L^p- map property

$$\| \mathrm{Wig}_\psi(f,g)\|_{L^\infty} \le 2^d \|f\|_{L^p} \|T_x \overline{\psi}_2 T_{2x} \widetilde{g}\|_{L^{p'}} \le 2^d \|f\|_{L^p} \|\psi\|_{L^\infty} \|g\|_{L^{p'}}. \tag{2.3}$$

We prove next the continuity of $\mathrm{Wig}_\psi(f,g)$. For (x,ω) and (x',ω') in $\mathbb{R}^{2d}$ we have

$$\begin{aligned} &| \mathrm{Wig}_\psi(f,g)(x,\omega) - \mathrm{Wig}_\psi(f,g)(x',\omega')| \le \\ &\qquad 2^d |e^{4\pi i x\omega} (U_{\psi,f,g}(x,\omega) - U_{\psi,f,g}(x',\omega')| + 2^d |(e^{4\pi i x\omega} - e^{4\pi i x'\omega'}) U_{\psi,f,g}(x',\omega')|. \end{aligned} \tag{2.4}$$

Due to the boundedness of $U_{\psi,f,g}$ (from (2.3)) and the continuity and periodicity of the complex exponential we have that the second term on the right-hand side of (2.4) is arbitrarily small for suitably near (x,ω) and (x',ω'). We consider now the first term on the right-hand side of (2.4):

$$
\begin{aligned}
&|U_{\psi,f,g}(x,\omega)-U_{\psi,f,g}(x',\omega')|\le\\
&|(f,M_{2\omega}T_x\overline{\psi}_2T_{2x}\widetilde{g})-(f,M_{2\omega'}T_x\overline{\psi}_2T_{2x}\widetilde{g})|+|(f,M_{2\omega'}T_x\overline{\psi}_2T_{2x}\widetilde{g})-(f,M_{2\omega'}T_{x'}\overline{\psi}_2T_{2x'}\widetilde{g})|\le\\
&|(M_{-2\omega}f-M_{-2\omega'}f,T_x\overline{\psi}_2T_{2x}\widetilde{g}|+|(M_{-2\omega'}f,T_x\overline{\psi}_2T_{2x}\widetilde{g}-T_{x'}\overline{\psi}_2T_{2x'}\widetilde{g}))|\le\\
&\|M_{-2\omega}f-M_{-2\omega'}f\|_{L^p}\ \|\psi\|_{L^\infty}\ \|g\|_{L^{p'}}\ +\\
&\|f\|_{L^p}\ \left(\|T_x\overline{\psi}_2T_{2x}\widetilde{g}-T_{x'}\overline{\psi}_2T_{2x}\widetilde{g}\|_{L^{p'}}+\ \|T_{x'}\overline{\psi}_2T_{2x}\widetilde{g}-T_{x'}\overline{\psi}_2T_{2x'}\widetilde{g}\|_{L^{p'}}\right)\le\\
&\|M_{-2\omega}f-M_{-2\omega'}f\|_{L^p}\ \|\psi\|_{L^\infty}\ \|g\|_{L^{p'}}\ +\\
&\|f\|_{L^p}\ \left(\|T_x\overline{\psi}_2-T_{x'}\overline{\psi}_2\|_{L^\infty}\|g\|_{L^{p'}}+\ \|\psi_2\|_{L^\infty}\|T_{2x}\widetilde{g}-T_{2x'}\widetilde{g}\|_{L^{p'}}\right)
\end{aligned}
\tag{2.5}
$$

The terms $\|M_{-2\omega}f - M_{-2\omega'}f\|_{L^p}$ and $\|T_{2x}\widetilde{g} - T_{2x'}\widetilde{g}\|_{L^{p'}}$ are arbitrarily small due to the L^p-boundedness of translation and modulation, for $1 < p < \infty$, and the term $\|T_x\overline{\psi}_2 - T_{x'}\overline{\psi}_2\|_{L^\infty}$ is arbitrarily small from the uniform continuity of ψ. This proves the continuity of $\mathrm{Wig}_\psi(f,g)(x,\omega)$, we show next that it vanishes at infinity. Let $f_j, g_j \in S(\mathbb{R}^d)$ with $f_j \to f$ in $L^p(\mathbb{R}^d)$ and $g_j \to g$ in $L^{p'}$ for $j \to \infty$, then

$$
\begin{aligned}
&|\,\mathrm{Wig}_\psi(f,g)(x,\omega)|\le\\
&2^d|U_{\psi,f,g}(x,\omega)-U_{\psi,f_j,g}(x,\omega)|+2^d|U_{\psi,f_j,g}(x,\omega)-U_{\psi,f_j,g_j}(x,\omega)|+2^d|U_{\psi,f_j,g_j}(x,\omega)|\le\\
&2^d\|f-f_j\|_{L^p}\|\psi\|_{L^\infty}\|g\|_{L^{p'}}+2^d\|f_j\|_{L^p}\|\psi\|_{L^\infty}\|g-g_j\|_{L^{p'}}+2^d|U_{\psi,f_j,g_j}(x,\omega)|
\end{aligned}
$$

Clearly $\|f-f_j\|_{L^p}$ and $\|g-g_j\|_{L^{p'}}$ can be made arbitrarily small for large j, and $\|f_j\|_{L^p}$, $j \in \mathbb{N}$, are bounded. It remains to show that $U_{\psi,f_j,g_j}(x,\omega)$ vanish at infinity. For $f_j, g_j \in S(\mathbb{R}^d)$ and $\psi \in C_b^\infty(\mathbb{R}^d)$ we have easily that $f_j(t)\psi(2(t-x))\overline{\widetilde{g}_j(t-2x)} \in S(\mathbb{R}^{2d})$, then $U_{\psi,f_j,g_j}(x,\omega) = \mathcal{F}_{t\to 2\omega}[f_j(t)\psi(2(t-x))\overline{\widetilde{g}_j(t-2x)}] \in S(\mathbb{R}^{2d})$ vanishing therefore at infinity. □

For what concerns Wig* we have the following counterpart of Proposition 2:

Proposition 3. *For $1 < p < \infty$ the representation $\mathrm{Wig}^*_\psi(f,g)$ satisfies the mapping property: $(\psi, f, g) \in L^\infty(\mathbb{R}^d)\times\mathcal{F}^{-1}L^p(\mathbb{R}^d)\times\mathcal{F}^{-1}L^{p'}(\mathbb{R}^d) \to \mathrm{Wig}^*_\psi(f,g) \in L^\infty(\mathbb{R}^{2d})$. Its range is a subset of the space of the continuous functions vanishing at infinity in the case that $\psi \in C_b^\infty(\mathbb{R}^d)$.*

Proof. It is a straightforward consequence of the formula $\mathrm{Wig}^*_\psi(f,g)(x,\omega) = \mathrm{Wig}_\psi(\hat{f},\hat{g})(\omega,-x)$, Proposition 2, and some changes of variables. □

Remark 4. *From* (1.7) *we have that* $\mathrm{Wig}^*_{\mathbf{1}}(f,g)(x,\omega) = \mathrm{Wig}_{\mathbf{1}}(\hat{f},\hat{g})(\omega,-x) = \mathrm{Wig}(f,g)(x,\omega)$, *where* $\mathbf{1}$ *stands for the function identically* 1*; on the other hand we obviously have* $\mathrm{Wig}_{\mathbf{1}}(f,g) = \mathrm{Wig}(f,g)$*, so*

$$
\mathrm{Wig}_{\mathbf{1}}(f,g) = \mathrm{Wig}^*_{\mathbf{1}}(f,g) = \mathrm{Wig}(f,g). \tag{2.6}
$$

In the case $\psi = \delta$, writing (u, Φ) for the action of a (conjugate linear) tempered distribution $u \in S'(\mathbb{R}^{2d})$ on a Schwartz function $\Phi \in S(\mathbb{R}^{2d})$, we have from (2.2) that for every $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$,

$$\begin{aligned}(\mathrm{Wig}_\delta(f,g), \Phi) &= (1 \otimes \delta, \mathcal{T}(\overline{f} \otimes g) \cdot \mathcal{F}_2^{-1}\Phi) \\ &= \int f(x)\overline{g(x)} \left(\int \overline{\Phi(x,\omega)}\, d\omega \right) dx \\ &= \iint f(x)\overline{g(x)}\, \overline{\Phi(x,\omega)}\, dx\, d\omega = (f\overline{g} \otimes 1, \Phi),\end{aligned}$$

that is

$$\mathrm{Wig}_\delta(f,g)(x,\omega) = f(x)\overline{g(x)} \tag{2.7}$$

for every $f, g \in \mathcal{S}(\mathbb{R}^d)$. By (1.8) we then get

$$\mathrm{Wig}^*_\delta(f,g)(x,\omega) = \hat{f}(\omega)\overline{\hat{g}(\omega)}. \tag{2.8}$$

In particular,

$$\mathrm{Wig}_\delta(f)(x,\omega) = |f(x)|^2 \quad and \quad \mathrm{Wig}^*_\delta(f)(x,\omega) = |\hat{f}(\omega)|^2 \tag{2.9}$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. Then the representations Wig_ψ and Wig^*_ψ, when ψ runs from $\mathbf{1}$ to δ, constitute a "bridge" from the classical Wigner to $|f(x)|^2$ and to $|\hat{f}(\omega)|^2$, respectively. An explicit path is for example when ψ is a gaussian depending on $\lambda \in [0, \infty]$ of the kind

$$\psi_\lambda(x) = c(\lambda)e^{-\pi\lambda x^2}, \tag{2.10}$$

where $c(\lambda)$ is a (continuous or even more regular) function of λ that equals 1 for $\lambda = 0$ and equals $\lambda^{d/2}$ for λ sufficiently large; we then mean $\psi_\infty = \delta$.

From now on we shall fix our attention on $f, g \in \mathcal{S}(\mathbb{R}^d)$, allowing ψ to be a tempered distribution.

Proposition 5. *The representations Wig_ψ and Wig^*_ψ belong to the Cohen class. In particular, for every $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in \mathcal{S}'(\mathbb{R}^d)$ we have*

$$\mathrm{Wig}_\psi(f,g) = \left(\delta \otimes \hat{\psi}\right) * \mathrm{Wig}(f,g) \tag{2.11}$$

and

$$\mathrm{Wig}^*_\psi(f,g) = \left(\hat{\tilde{\psi}} \otimes \delta\right) * \mathrm{Wig}(f,g) \tag{2.12}$$

where $\tilde{\psi}(s) := \psi(-s)$.

Proof. We prove the result for $f, g, \psi \in \mathcal{S}(\mathbb{R}^d)$; the case of tempered distributions follows from standard density arguments. Using (2.2) and the fact that $\mathrm{Wig} = \mathrm{Wig}_{\mathbf{1}}$, in order to prove that $\mathrm{Wig}_\psi(f,g)$ can be written in the form $\sigma * \mathrm{Wig}(f,g)$ for suitable $\sigma \in S'(\mathbb{R}^{2d})$ we have to show

$$\mathcal{F}_2[(1\otimes\psi)(\mathcal{T}(f\otimes\overline{g}))] = \sigma * \mathcal{F}_2[\mathcal{T}(f\otimes\overline{g})].$$

As $\mathcal{F}_2[(1\otimes\psi)(\mathcal{T}(f\otimes\overline{g}))] = (\delta\otimes\hat{\psi}) * \mathrm{Wig}(f,g)$, we have that the equality is satisfied if and only if $\sigma = \delta\otimes\hat{\psi}$; so (2.11) holds. Consider now Wig^*_ψ; we indicate by $S : F \in S(\mathbb{R}^{2d}) \longrightarrow S(F) \in S(\mathbb{R}^{2d})$ the symplectic map defined as $S(F)(x,\omega) = F(\omega,-x)$ with obvious extension to $S'(\mathbb{R}^{2d})$. Then by (2.11) and (1.7) we have

$$\begin{aligned}
\mathrm{Wig}^*_\psi(f,g) &= S\left(\mathrm{Wig}_\psi(\hat{f},\hat{g})\right)\\
&= S\left((\delta\otimes\hat{\psi}) * \mathrm{Wig}(\hat{f},\hat{g})\right)\\
&= S\left[(\delta\otimes\hat{\psi}) * S^{-1}\big(\mathrm{Wig}(f,g)\big)\right]\\
&= S\left[(\delta\otimes\hat{\psi})\right] * \mathrm{Wig}(f,g),
\end{aligned}$$

which proves (2.12) since $S(\delta\otimes\hat{\psi}) = \tilde{\hat{\psi}}\otimes\delta = \hat{\tilde{\psi}}\otimes\delta$. $\square$

It is natural to ask now how Wig_ψ and Wig^*_ψ are related to other classes of time-frequency representations studied in the literature. One of these classes is the τ-Wigner transform, cf. (1.2). We recall from [3] that

$$\mathrm{Wig}^{(\tau)}(f,g) = \sigma * \mathrm{Wig}(f,g),$$

where

$$\sigma = \begin{cases} \frac{2^d}{|2\tau-1|^d} e^{2\pi i \frac{2}{2\tau-1} x\omega} & \text{for } \tau \neq \frac{1}{2} \\ \delta & \text{for } \tau = \frac{1}{2} \end{cases} \tag{2.13}$$

By comparing (2.13) with (2.11) and (2.12) we have immediately that Wig_ψ and Wig^*_ψ are not of the form (1.2), apart from the case $\psi \equiv 1$ and $\tau = 1/2$, in which $\mathrm{Wig}_{\mathbf{1}}$, $\mathrm{Wig}^*_{\mathbf{1}}$ and $\mathrm{Wig}^{(1/2)}$ coincide with the classical Wigner.

Now we want to compare Wig_ψ and Wig^*_ψ with the generalized spectrogram, cf. (1.3). To this aim we now prove a result on the image of the classical Wigner transform that has an interest in itself, and shall allow us to prove that the representations Wig_ψ and Wig^*_ψ can be written as generalized spectrograms only in some "limit" cases, for special choices of the windows in $\mathcal{S}'(\mathbb{R}^d)$.

Proposition 6. *The only distributions of the form $\delta_a\otimes v$ or $u\otimes\delta_a$ belonging to the image of the Wigner transform are multiples of $\delta_a\otimes e^{2\pi i b\omega}$ or $e^{2\pi i bx}\otimes\delta_a$, where $b\in\mathbb{R}^d$ and δ_a is the Dirac distribution centered in $a\in\mathbb{R}^d$. More precisely:*

(i) *There exist $f,g\in\mathcal{S}'(\mathbb{R}^d)$ such that $\mathrm{Wig}(f,g)(x,\omega) = u\otimes\delta_a$ if and only if we can find $c\in\mathbb{C}$ and $b\in\mathbb{R}^d$ such that $u = ce^{2\pi i bx}$;*

(ii) *There exist* $f, g \in \mathcal{S}'(\mathbb{R}^d)$ *such that* $\operatorname{Wig}(f,g)(x,\omega) = \delta_a \otimes v$ *if and only if we can find* $c \in \mathbb{C}$ *and* $b \in \mathbb{R}^d$ *such that* $v = ce^{2\pi i b\omega}$.

Proof. Fix $a, b \in \mathbb{R}^d$; we write T_a and M_b for the translation and modulation operators respectively, i.e. the extension to $\mathcal{S}'(\mathbb{R}^d)$ of the operators acting on $\mathcal{S}(\mathbb{R}^d)$ as $T_a\phi(t) = \phi(t-a)$ and $M_b\phi(t) = e^{2\pi i t b}\phi(t)$. We recall that the Wigner distribution satisfies the following property: for every $f, g \in \mathcal{S}'(\mathbb{R}^d)$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}^d$ we have

$$\begin{aligned}&\operatorname{Wig}(T_{a_1}M_{b_1}f, T_{a_2}M_{b_2}g)(x,\omega) = \\ &\quad = e^{-\pi i(a_1+a_2)(b_1-b_2)} M_{(b_1-b_2, a_2-a_1)} T_{\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)} \operatorname{Wig}(f,g)(x,\omega),\end{aligned} \tag{2.14}$$

cf. for example [13]. In particular the Wigner transform is covariant, in the sense that translations and modulations of the signals are reflected in translations in time and frequency on the corresponding Wigner:

$$\operatorname{Wig}(T_aM_bf, T_aM_bg)(x,\omega) = T_{(a,b)} \operatorname{Wig}(f,g)(x,\omega)$$

for every $a, b \in \mathbb{R}^d$.

We then have that the multiples of $\delta_a \otimes e^{2\pi i b\omega}$ and $e^{2\pi i bx} \otimes \delta_a$ belong to the image of the Wigner, indeed, since $\operatorname{Wig}(\delta,\delta) = \delta \otimes \mathbf{1}$ and $\operatorname{Wig}(\mathbf{1},\mathbf{1}) = \mathbf{1} \otimes \delta$ we have:

$$\operatorname{Wig}\left(\sqrt{c}(T_{a-b/2}\delta), \sqrt{c}(T_{a+b/2}\delta)\right) = c \cdot M_{(0,b)}T_{(a,0)}\left(\operatorname{Wig}(\delta,\delta)\right) = c(\delta_a \otimes e^{2\pi i b\omega}) \tag{2.15}$$

and

$$\operatorname{Wig}\left(\sqrt{c}(M_{a+b/2}\mathbf{1}), \sqrt{c}(M_{a-b/2}\mathbf{1})\right) = c \cdot M_{(b,0)}T_{(0,a)}\left(\operatorname{Wig}(\mathbf{1},\mathbf{1})\right) = c(e^{2\pi i bx} \otimes \delta_a). \tag{2.16}$$

We now prove the first point of the proposition.

(i) We start by considering the case $a = 0$. Suppose that $\operatorname{Wig}(f,g) = u \otimes \delta$, for a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$. Since $\operatorname{Wig}(f,g) = \mathcal{F}_2[\mathcal{T}(f \otimes \overline{g})]$, cf. (2.1), we have

$$f \otimes \overline{g} = \mathcal{T}^{-1}(u \otimes \mathbf{1}).$$

Now for every $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ we have

$$\begin{aligned}\left(\mathcal{T}^{-1}(u \otimes \mathbf{1}), \Psi\right) &= \left(u \otimes \mathbf{1}, \Psi\left(x + \frac{t}{2}, x - \frac{t}{2}\right)\right) \\ &= \left(u_x, \int \Psi\left(x + \frac{t}{2}, x - \frac{t}{2}\right) dt\right) \\ &= 2^d\left(u_x, \int \Psi(y, 2x-y)\, dy\right) \\ &= 2^d\left(u \otimes \mathbf{1}, \mathcal{P}\Psi\right),\end{aligned}$$

where $(\mathcal{P}\Psi)(x,t) = \Psi(t, 2x-t)$. So we can write $f \otimes \overline{g}$ as

$$f \otimes \overline{g} = \mathcal{P}^{-1}\left(u \otimes \mathbf{1}\right), \tag{2.17}$$

for $u \in \mathcal{S}'(\mathbb{R}^d)$. Let us fix now the test function Ψ as a tensor product $\Psi(x,t) = \varphi_1(x)\varphi_2(t)$; we then have

$$\begin{aligned}(f,\varphi_1)(\overline{g},\varphi_2) &= 2^d(u \otimes \mathbf{1}, \mathcal{P}(\varphi_1 \otimes \varphi_2)) \\ &= 2^d(u_x \otimes \mathbf{1}_t, \varphi_1(t)\varphi_2(2x-t)) \\ &= 2^d\big(\mathbf{1}_t, \varphi_1(t)\overline{(u_x, \varphi_2(2x-t))}\,\big).\end{aligned}$$

Now defining $\varphi_3(s) := \overline{\varphi_2(-2s)}$ we have that $(u_x, \varphi_2(2x-t)) = (u * \varphi_3)(t/2)$, and so we have

$$(f,\varphi_1)(\overline{g},\varphi_2) = 2^d \int \overline{\varphi_1(t)}(u * \varphi_3)(t/2)\,dt.$$

We suppose that both f and g are not identically zero, otherwise the corresponding Wigner transform is 0 and we are in a trivial case. So there is $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ such that $(\overline{g},\varphi_2) \neq 0$. For such φ_2 we then have

$$(f,\varphi_1) = 2^d \int \frac{(u * \varphi_3)(t/2)}{(\overline{g},\varphi_2)}\,\overline{\varphi_1(t)}\,dt$$

for every $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$. This implies that

$$f(t) = 2^d \frac{(u * \varphi_3)(t/2)}{(\overline{g},\varphi_2)},$$

that means in particular that f is a C^∞ function. We can reason in a similar way concerning g, since

$$\big(\mathcal{T}^{-1}(u \otimes 1), \Psi\big) = 2^d \left(u_x, \int \Psi(2x-y, y)\,dy\right) = 2^d\left(u \otimes 1, \mathcal{R}\Psi\right),$$

where $(\mathcal{R}\Psi)(x,t) = \Psi(2x-t, t)$. Taking as before a test function in the form of a tensor product we have

$$(f,\varphi_1)(\overline{g},\varphi_2) = 2^d\big(\mathbf{1}_t, \varphi_2(t)\overline{(u_x, \varphi_1(2x-t))}\,\big) = 2^d \int \overline{\varphi_2(t)}(u * \varphi_4)(t/2)\,dt,$$

where $\varphi_4(s) = \overline{\varphi_1(-2s)}$. We fix $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$ such that $(f,\varphi_1) \neq 0$ and we then get as before that

$$\overline{g(t)} = 2^d \frac{(u * \varphi_4)(t/2)}{(f,\varphi_1)},$$

obtaining in particular that also g is a C^∞ function. Since $f, g \in C^\infty(\mathbb{R}^d)$ we then have that also u in (2.17) is a C^∞ function, and f, g, u must satisfy

$$f(x)\overline{g(t)} = u\left(\frac{x+t}{2}\right) \tag{2.18}$$

for every $x, t \in \mathbb{R}^d$. We want now to prove that the only $f, g \in C^\infty(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ that satisfy (2.18) are modulations of a constant. We start by observing that from (2.18) we get

$$\partial_{x_j} f(x)\overline{g(t)} = f(x)\partial_{t_j}\overline{g(t)}, \tag{2.19}$$

for every $j = 1, \dots, d$. Recall that we are assuming that both f and g are not identically 0, otherwise $\mathrm{Wig}(f, g) \equiv 0$; then f and g cannot vanish at any point. Indeed suppose for example that there exist $\overline{x}, \tilde{x} \in \mathbb{R}^d$ such that $f(\overline{x}) = 0$ and $f(\tilde{x}) \neq 0$; from (2.18) we have that $u\left(\frac{\overline{x}+t}{2}\right) = 0$, that means that $u \equiv 0$; so $f(\tilde{x})\overline{g(t)} \equiv 0$, which implies that $g \equiv 0$. Then we can divide in (2.19) by $f(x)\overline{g(t)}$, obtaining

$$\frac{\partial_{x_j} f(x)}{f(x)} = \frac{\partial_{t_j}\overline{g(t)}}{\overline{g(t)}}.$$

The last equality is satisfied only if both sides are constants, and so for every $j = 1, \dots, d$ there exists $c_j \in \mathbb{C}$ such that

$$\partial_{x_j} f(x) = c_j f(x), \quad \partial_{t_j}\overline{g(t)} = c_j\overline{g(t)}. \tag{2.20}$$

Now (2.20) for $j = 1$ gives us that there exist $k_1, h_1 \in C^\infty(\mathbb{R}^{d-1})$ such that

$$f(x) = k_1(x_2, \dots, x_d)e^{c_1x_1}, \quad \overline{g(t)} = h_1(t_2, \dots, t_d)e^{c_1t_1}.$$

By substituting these expressions in (2.20) we obtain that for every $j = 2, \dots, d$

$$\partial_{x_j} k_1(x_2, \dots, x_d) = c_j k_1(x_2, \dots, x_d), \quad \partial_{t_j} h_1(t_2, \dots, t_d) = c_j h_1(t_2, \dots, t_d),$$

and then we can iterate the same procedure as before, obtaining finally that f and g are of the form

$$f(x) = Ae^{c_1x_1+\dots+c_dx_d}, \quad \overline{g(t)} = Be^{c_1t_1+\dots+c_dt_d},$$

for $A, B, c_1, \dots, c_d \in \mathbb{C}$. Since we assume that $f, g \in \mathcal{S}'(\mathbb{R}^d)$ the only possibility is that all the constants c_j are pure imaginary, that means that f and g are of the form $f(x) = Ae^{\pi ibx}$, $g(t) = Be^{-\pi ibt}$, for $b \in \mathbb{R}^d$. From (2.16) we finally obtain that $\mathrm{Wig}(f, g)$ is a multiple of $e^{2\pi ibx} \otimes \delta$, and so we have proved the thesis for $a = 0$. The conclusion for a general $a \in \mathbb{R}^d$ follows from the covariance property of the Wigner transform, since $\mathrm{Wig}(f, g) = u \otimes \delta_a$ if and only if $\mathrm{Wig}(M_{-a}f, M_{-a}g) = u \otimes \delta$. In particular we then have that $\mathrm{Wig}(f, g) = u \otimes \delta_a$ implies that $f = A \cdot (M_{a+b/2}\mathbf{1})$ and $g = B(M_{a-b/2}\mathbf{1})$.

(ii) Concerning the second part of the proposition we recall that for every $f, g \in \mathcal{S}'(\mathbb{R}^d)$

$$\mathrm{Wig}(\hat{f}, \hat{g}) = \mathcal{I}\left(\mathrm{Wig}(f, g)\right),$$

where $\mathcal{I}$ is the extension to $\mathcal{S}'(\mathbb{R}^{2d})$ of the operator that acts on $\mathcal{S}(\mathbb{R}^{2d})$ as $(\mathcal{I}\Psi)(x, \omega) = \Psi(-\omega, x)$. We then have that $\mathrm{Wig}(f, g) = \delta_a \otimes v$ if and only if $\mathrm{Wig}(\hat{f}, \hat{g}) = v \otimes \delta_{-a}$. From the point (i) of the proposition this implies that $\hat{f} = A \cdot (M_{-a+b/2}\mathbf{1})$ and $\hat{g} = B(M_{-a-b/2}\mathbf{1})$, and then $f = A\delta_{a-b/2}$ and $g = B\delta_{a+b/2}$. From (2.15) we then have that $\mathrm{Wig}(f, g)$ is a multiple of $\delta_a \otimes e^{2\pi ib\omega}$, and then the proof is complete. □

As a consequence of the previous result we obtain that the representations (1.6) and (1.8) can be written as generalized spectrograms only in some special cases. More precisely we have the following result.

Proposition 7. *The representations* (1.6) *and* (1.8) *can be written in the form* (1.3) *for every* $f, g \in \mathcal{S}(\mathbb{R}^d)$*, if and only if there exist* $c \in \mathbb{C}$ *and* $b \in \mathbb{R}^d$ *such that* $\psi = c\delta_b$*. In this case we have*

$$\mathrm{Wig}_{c\delta_b}(f,g) = \mathrm{Sp}_{\sqrt{c}\delta_{-b/2},\sqrt{c}\delta_{b/2}}(f,g) = ce^{-2\pi ib\omega} f\left(x+\frac{b}{2}\right)\overline{g\left(x-\frac{b}{2}\right)} \tag{2.21}$$

and

$$\mathrm{Wig}^*_{c\delta_b}(f,g) = \mathrm{Sp}_{\sqrt{c}e^{\pi ibt},\sqrt{c}e^{-\pi ibt}}(f,g) = ce^{2\pi ibx} \hat{f}\left(\omega+\frac{b}{2}\right)\overline{\hat{g}\left(\omega-\frac{b}{2}\right)} \tag{2.22}$$

Remark 8. *(i) In the particular case* $b = 0$, $c = 1$ *and* $f = g$ *we recover in* (2.21) *and* (2.22) *the limit cases* (2.9)*. On the other hand, all the* $\mathrm{Wig}_{\psi_\lambda}(f,g)$ *and* $\mathrm{Wig}^*_{\psi_\lambda}(f,g)$ *for* ψ_λ *of the form* (2.10) *are not generalized spectrograms.*

(ii) The way we can write Wig_ψ *as a generalized spectrogram is not unique, in fact we can arbitrarily split the constant* c *in the two windows, obtaining for example that* $\mathrm{Wig}_{c\delta_b}(f,g) = \mathrm{Sp}_{c_1\delta_{-b/2},c_2\delta_{b/2}}(f,g)$ *for every* c_1, c_2 *such that* $c_1c_2 = c$.

Proof of Proposition 7. Recall that for every $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\phi_1, \phi_2 \in \mathcal{S}'(\mathbb{R}^d)$ we have

$$\mathrm{Sp}_{\phi_1,\phi_2}(f,g) = \mathrm{Wig}(\tilde{\phi}_2, \tilde{\phi}_1) * \mathrm{Wig}(f,g), \tag{2.23}$$

cf. for example [3]. Then comparing (2.23) with (2.11) and (2.12) we have that Wig_ψ (respect. Wig^*_ψ) is a generalized spectrogram if and only if there exist $\phi_1, \phi_2 \in \mathcal{S}'(\mathbb{R}^d)$ such that $\mathrm{Wig}(\tilde{\phi}_2, \tilde{\phi}_1) = \delta \otimes \hat{\psi}$ (respect. $\mathrm{Wig}(\tilde{\phi}_2, \tilde{\phi}_1) = \hat{\hat{\psi}} \otimes \delta$). From Proposition 6 these equalities can be true if and only if $\psi = c\delta_b$ for some $c \in \mathbb{C}$ and $b \in \mathbb{R}^d$. Then (2.21) and (1.6) can be deduced directly from the definition of Wig_ψ and Wig^*_ψ. □

We want now to analyze the marginals and the support properties for Wig_ψ and Wig^*_ψ.

Proposition 9. *For every* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* $\psi \in L^1(\mathbb{R}^d)$ *such that* $\hat{\psi} \in L^1(\mathbb{R}^d)$ *we have*

$$(a) \int \mathrm{Wig}_\psi(f)(x,\omega)\,d\omega = |f(x)|^2\psi(0), \quad (a') \int \mathrm{Wig}_\psi(f)(x,\omega)\,dx = (\hat{\psi} * |\hat{f}|^2)(\omega),$$

$$(b) \int \mathrm{Wig}^*_\psi(f)(x,\omega)\,d\omega = (\hat{\hat{\psi}} * |f|^2)(x), \quad (b') \int \mathrm{Wig}^*_\psi(f)(x,\omega)\,dx = |\hat{f}(\omega)|^2\psi(0).$$

Proof. We start by proving properties (a) and (a'). We observe that for every $F, G \in S'(\mathbb{R}^d)$ such that $\hat{F}, \hat{G} \in L^1(\mathbb{R}^d)$ we have

$$\int \widehat{FG}(s)\, ds = \int \hat{F}(s-t)\hat{G}(t)\, dt\, ds = \int \hat{F}(s)\, ds \int \hat{G}(s)\, ds.$$

Since moreover $\int \hat{\Phi}(\omega)\, d\omega = \Phi(0)$ for every $\Phi \in \mathcal{S}(\mathbb{R}^d)$ we have that

$$\begin{aligned}\int \mathrm{Wig}_\psi(f)(x,\omega)\, d\omega &= \int \mathcal{F}_{t\to\omega}\left[\psi(t) f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}\right] d\omega \\ &= |f(x)|^2 \int \hat{\psi}(s)\, ds = |f(x)|^2 \psi(0).\end{aligned}$$

Let us now analyze the frequency marginal for Wig_ψ.

$$\begin{aligned}\int \mathrm{Wig}_\psi(f)(x,\omega)\, dx &= \int \left[\int e^{-2\pi i t\omega}\psi(t) f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)} dt\right] dx \\ &= \int e^{-2\pi i t\omega} e^{\pi i t(\eta+\xi)} e^{-2\pi i x(\eta-\xi)} \psi(t)\hat{f}(\xi)\overline{\hat{f}(\eta)}\, d\eta\, d\xi\, dt\, dx \\ &= \int e^{-2\pi i t\omega} e^{\pi i t(\eta+2\xi)} e^{-2\pi i x\eta} \psi(t)\hat{f}(\xi)\overline{\hat{f}(\eta+\xi)}\, d\xi\, dt\, d\eta\, dx \\ &= \int \mathcal{F}_{\eta\to x}\left[\int e^{-2\pi i t\omega} e^{\pi i t(\eta+2\xi)} \psi(t)\hat{f}(\xi)\overline{\hat{f}(\eta+\xi)}\, d\xi\, dt\right] dx \\ &= \int e^{-2\pi i t(\omega-\xi)}\psi(t)|\hat{f}(\xi)|^2\, d\xi\, dt \\ &= (\hat{\psi} * |\hat{f}|^2)(\omega).\end{aligned}$$

Formulas (b) and (b') are an immediate consequence of (1.8) and properties (a) and (a'). □

Remark 10. *When $\psi(0) = 1$ then Wig_ψ satisfies the time marginal and Wig^*_ψ satisfies the frequency marginal. Moreover observe that, even with $\psi \in \mathcal{S}'(\mathbb{R}^d)$, the only Wig_ψ or Wig^*_ψ enjoying both time and frequency marginals are the ones corresponding to $\psi \equiv 1$, i.e. the classical Wigner.*

As immediate consequence of Proposition 9 we have the following result concerning conservation of energy.

Proposition 11. *For every $f \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in L^1(\mathbb{R}^d)$ such that $\hat{\psi} \in L^1(\mathbb{R}^d)$ we have*

$$\int \mathrm{Wig}_\psi(f)(x,\omega)\, dx\, d\omega = \int \mathrm{Wig}^*_\psi(f)(x,\omega)\, dx\, d\omega = \|f\|^2_{L^2}\psi(0).$$

In particular if $\psi(0) = 1$ we have conservation of energy for both Wig_ψ and Wig^*_ψ.

Concerning the support properties we have a similar situation as for the marginals, in the sense that the support property in time is satisfied only by Wig_ψ, and the one in frequency only by Wig^*_ψ. More precisely we have the following result.

Proposition 12. *For every $f \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in \mathcal{S}'(\mathbb{R}^d)$ we have:*

$$\Pi_x \mathrm{supp}\,(\mathrm{Wig}_\psi(f)) \subset \mathcal{C}(\mathrm{supp}\, f), \quad \Pi_\omega \mathrm{supp}\,(\mathrm{Wig}_\psi(f)) \subset \mathrm{supp}\,\hat{\psi} + \mathcal{C}\big(\mathrm{supp}\,\hat{f}\big),$$

$$\Pi_x \mathrm{supp}\,(\mathrm{Wig}^*_\psi(f)) \subset \mathrm{supp}\,\hat{\hat{\psi}} + \mathcal{C}(\mathrm{supp}\, f), \quad \Pi_\omega \mathrm{supp}\,(\mathrm{Wig}^*_\psi(f)) \subset \mathcal{C}\big(\mathrm{supp}\,\hat{f}\big),$$

where Π_x and Π_ω are the orthogonal projections of the corresponding subset of $\mathbb{R}^{2d}$ on x and ω respectively, and $\mathcal{C}$ indicates the convex hull.

Proof. The result is a consequence of Proposition 5 and the fact that the classical Wigner satisfies the support properties both in x and in ω. We have in fact from standard properties of the convolution product that

$$\begin{aligned} \mathrm{supp}\ \big(\mathrm{Wig}_\psi(f)\big) &= \mathrm{supp}\ \Big(\big(\delta \otimes \hat{\psi}\big) * \mathrm{Wig}(f)\Big) \\ &\subset \mathrm{supp}\ \Big(\delta \otimes \hat{\psi}\Big) + \mathrm{supp}\ (\mathrm{Wig}(f)) \\ &= \Big(\{0\} \times \mathrm{supp}\,\hat{\psi}\Big) + \mathrm{supp}\ (\mathrm{Wig}(f))\,. \end{aligned}$$

Then

$$\Pi_x \mathrm{supp}\,(\mathrm{Wig}_\psi(f)) \subset \{0\} + \Pi_x \mathrm{supp}\,(\mathrm{Wig}(f)) \subset \mathcal{C}(\mathrm{supp}\, f),$$

and

$$\Pi_\omega \mathrm{supp}\,(\mathrm{Wig}_\psi(f)) \subset \mathrm{supp}\,\hat{\psi} + \Pi_\omega \mathrm{supp}\,(\mathrm{Wig}(f)) \subset \mathrm{supp}\,\hat{\psi} + \mathcal{C}\big(\mathrm{supp}\,\hat{f}\big)$$

The proof for Wig^*_ψ can be done in the same way, using the corresponding expression (2.12) of Wig^*_ψ as an element of the Cohen class. ☐

Remark 13. *Proposition 12 contains as a particular case the well-known fact that the classical Wigner satisfies the support properties both in x and in ω; in fact, if $\psi \equiv 1$ we have that* $\mathrm{supp}\,\hat{\psi} = \mathrm{supp}\,\hat{\hat{\psi}} = \{0\}$*, and so by Proposition 12 we recover that* $\mathrm{Wig}_1 = \mathrm{Wig}^*_1 = \mathrm{Wig}$ *enjoy the supports.*

3 Reduction of interferences

This section is dedicated to the presentation of a method based on the representations (1.6) and (1.8) aimed at reducing the interferences appearing in the Wigner representation (which even yield a total cancelation in certain classes of signals). In [2] we have considered Wigner transforms of the form (1.4), i.e. a particular case of Wig_ψ, with $\psi(t) = \chi_{[-M,M]}(t)$, and we have provided a method of reduction of interferences for the class of signals consisting on pure frequencies appearing in different time slots. Using such re-defined Wigner distributions and limiting the integration on a suitable horizontal strip $\{(x,t) : t \in [-M, M]\}$ we have found a necessary condition to transmit signals without interferences: the "silence" between any couple of frequencies has to be longer or equal to the transmission time of each one of them. Now we are going to show that it is possible to avoid interferences also for signals characterized by different frequencies, even if they appear in the same time slot. Moreover, suitable combinations of Wig_ψ and Wig^*_ψ shall allow us to cancel out both interferences between separated time intervals and between separated frequency intervals.

The approach is based on the observation that Wig_ψ removes artifacts resulting from the interaction between frequencies defined in disjoint slots of time (*horizontal cuts*), whereas Wig^*_ψ deletes interferences among different frequencies independently from the time domains (*vertical cuts*). The combined application of these two filter has therefore the effect of removing interferences with "horizontal" and "vertical" cuts. Condition (1.5) which now applies both to time and frequency variables yields then a sort of a "grid" in the time-frequency plane. An ideal signal contained in this grid would show no interference (actually due to the Paley-Wiener Theorem this condition can be only approximatively satisfied). More precisely the representations that we consider are of the following type:

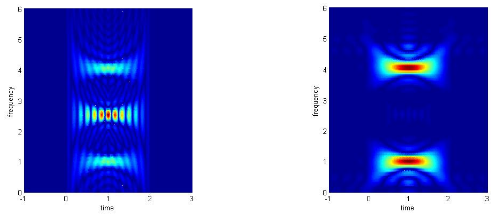
$$(\hat{\hat{\psi}}_2 \otimes \delta) * (\delta \otimes \hat{\psi}_1) * Wig(f) \tag{3.1}$$

In view of (2.11) and (2.12) we see that (3.1) amounts to the application of two successive filters on $Wig(f)$, one with the kernel defining Wig_{ψ_1} and the other one with the kernel defining $Wig^*_{\psi_2}$. Notice that this procedure differs from what happens in the engineering with the applications of low-pass and high-pass filters directly on the signal. In our situation we are actually filtering the Wigner transform of the signals in order to delete as much as possible false information from the images. In the first section and in [2] we have already seen the effect of Wig_ψ. We illustrate now the effect of Wig^*_ψ on the following particular case. Take $\psi(t) = \chi_{[-R,R]}(t)$ and consider the expression (1.8): we can interpret the multiplication of the Fourier transform of the signal with the Heaviside function in a similar way to the case of the cuts in the time-domain, with the difference that now the cut regards all artifacts between different frequencies. Example 1 shows this effect. Notice that, for the MATLAB implementation, we have used the following expression:

$$Wig^*_\chi(f)(x,\omega) = \int_{\mathbb{R}} e^{-2\pi i s\omega} \frac{\sin 2\pi(x-m)R}{\pi(x-m)} f\left(m+\frac{s}{2}\right) \overline{f\left(m-\frac{s}{2}\right)} dmds.$$

Example 1. *Let $f(t) = e^{2\pi it} \cdot \chi_{(0,2)}(t) + e^{8\pi it} \cdot \chi_{(0,2)}(t)$ be a signal with two frequencies, $\omega_1 = 1$ and $\omega_2 = 4$, with domain in $(0,2)$. The classical Wigner representation shows an interference between ω_1 and ω_2 (Figure 2(a)), but the application of Wig^*_ψ, where $\psi(t) = \chi_{[-R,R]}(t)$ with $R = 0.9$[1], improves the image showing essentially the true information contained in the original signal (Figure 2(b)).*

[1]The choise $R = 0.9$ is connected with the "essential" support of $\hat{f}(\omega)$, i.e. the set where $\hat{f}(\omega)$ is not "too" small; here R plays the role of M in (1.4), cf. Figure 1.

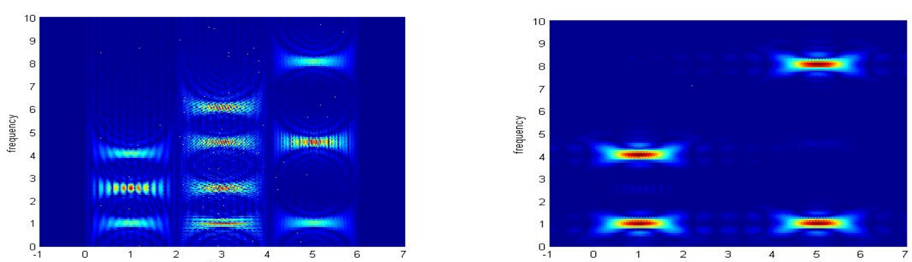


(a) $Wig(f)$ (b) $Wig^*_\psi(f)$

Figure 2:

The next two examples show the improvements given by the combined calculation of the windowed Wig_ψ and Wig^*_ψ. In particular, Example 2 sets a view of the actions of the two distributions separately, whereas Example 3 displays the efficacy of this method with signals with many different frequencies.

Example 2. *Let $f(t) = (e^{2\pi it} + e^{8\pi it}) \cdot \chi_{(0,2)} + (e^{16\pi it} + e^{2\pi it}) \cdot \chi_{(4,6)}$ be a signal with four frequencies: $\omega_1 = 1, \omega_2 = 4$ in $(0,2)$, and $\omega_3 = 8, \omega_4 = 1$ in $(4,6)$. As for the previous example, also in this case we first show the behavior of the classical Wigner distribution (Figure 3(a)) in such a way to compare it with the applications of (3.1) which does not present ghost frequencies (Figure 3(b)). In order to underline the effect of the two steps of the method we show in Figures 4(a) and 4(b) separately the actions of Wig_ψ and Wig^*_ψ. In particular note that the Wig^*_ψ works not only between frequencies defined in a same interval of time, but also between each couple with frequencies far enough from each other, according to the theoretical results (in this specific example the only artifact which remains is the one due to the interference between ω_1 and ω_4).*

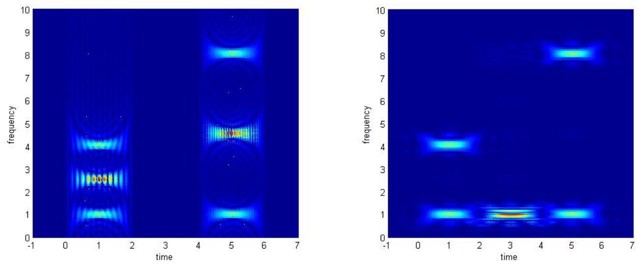


(a) $Wig(f)(x,\omega)$ (b) Filtered representation.

Figure 3:

Example 3. *Consider the signal f defined by*

$$\begin{aligned} f(t) = (e^{8\pi it} + e^{14\pi it}) \cdot \chi_{(-12,-10)} + e^{10\pi it} \cdot \chi_{(-8,-6)} + \\ (e^{2\pi it} + e^{12\pi it} + e^{8\pi it}) \cdot \chi_{(-4,-2)} + (e^{2\pi it} + e^{8\pi it}) \cdot \chi_{(0,2)} \\ (e^{16\pi it} + e^{2\pi it}) \cdot \chi_{(4,6)} + e^{10\pi it} \cdot \chi_{(9,11)}, \end{aligned}$$



(a) $Wig_\psi(f)$ (b) $Wig^*_\psi(f)$

Figure 4:

which has 11 *frequencies. In Figure 5 the huge amount of interferences creates serious problems in distinguishing the false from the true frequencies. On the contrary, Figure 6 shows only the true information contained in the signal, in fact it isolates the true frequencies.*

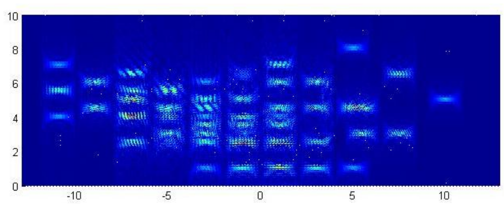


Figure 5: $Wig(f)(x, \omega)$

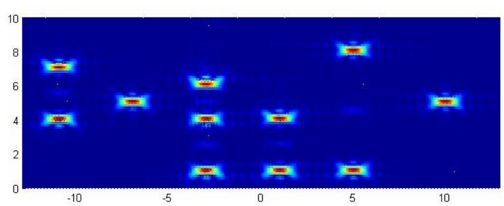


Figure 6: Filtered representaition.

Observe that the reason why in the application of the method it seems that there is a constriction of each time-interval in which the signal exists is linked to the fact that the two windowed distributions Wig_ψ and Wig^*_ψ do not satisfy separately the support property.

In this case this method eliminates in one step (i.e. Wig^*_ψ) the interaction between frequencies with same support and in the other step (i.e. Wig_ψ) the interaction between frequencies with same value.

4 Uncertainty principles

Two classical forms of the uncertainty principle concerning compactness of supports are the following:

Proposition 14. *For $f \in L^2(\mathbb{R}^d)$ the following results hold:*

a) *If* $\operatorname{supp} f$ *and* $\operatorname{supp} \widehat{f}$ *are compact, then* $f = 0$ *(Paley-Wiener Theorem);*

b) *If* $\operatorname{supp} \operatorname{Wig}(f)$ *is compact, then* $f = 0$ *(see e.g. [16], [19]).*

Both properties (a) and (b) can be seen as expressions of the uncertainty principles for Windowed-Wigner representations on $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$, as $|f(x)|^2$, $|\widehat{f}(\omega)|^2$ and $\operatorname{Wig}(f)(x,\omega)$ are all particular cases of this type of representations. Nevertheless they seem properties of a somewhat different nature: in (a) the two representations $|f(x)|^2$ and $|\widehat{f}(\omega)|^2$ (viewed as functions on $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$) are involved and it is supposed that the projections on $\mathbb{R}^d_x$ and $\mathbb{R}^d_\omega$ respectively of their supports are compact sets; (b) refers to only one representation on $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$ requiring compactness of its support with respect to all variables.

Many forms of the uncertainty principle have been considered in literature with the aim of constructing a unifying framework for (a) and (b). Most of them start from the observation that (a) can actually be reformulated as an uncertainty principle for the Rihaczek representation $R(f)(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\widehat{f}(\omega)}$ and they formulate therefore a simultaneous time-frequency condition on the representation taken into consideration (see e.g. [7], [9], [12], [14], [16]). We shall also follow a similar approach in Propositions 16 and 17 using the two types of windowed-Wigner representations as unifying framework.

In the second part of this section we propose a different point of view. Generalizing the couple $(|f|^2, |\widehat{f}|^2)$, we shall introduce a suitable duality for representations in the Cohen class and we shall express our results in terms of couples of dual representations (Propositions 25, 26, 28).

Let us remark that both (a) and (b) of Proposition 14 remain valid under the more general hypothesis of supports of finite measure instead of compact (see Benedicks [1] and Janssen [16]). A reasonable project, however outside of the aims of the present paper, would then be the research of a unifying extension of Proposition 14 under this more general hypothesis.

Our first form of the uncertainty principle relies on the following result for which we refer to [5].

Proposition 15. *Suppose that one of the following conditions is satisfied*

a) $\sigma \in S'(\mathbb{R}^{2d})$ *and* $\operatorname{supp} \widehat{\sigma} \neq \mathbb{R}^{2d}$*;*

b) $Q_\sigma = \sigma * \operatorname{Wig}$ *satisfies the Moyal equality i.e.*

$$(Q_\sigma(f_1, g_1), Q_\sigma(f_2, g_2))_2 = \overline{(g_1, g_2)}(f_1, f_2) \quad \text{for } f_j, g_j \in S(\mathbb{R}^d).$$

For $f \in S(\mathbb{R}^d)$ *we have then:* $\operatorname{supp} Q_\sigma f$ *compact* $\implies$ $f = 0$.

We have as consequence the following uncertainty principle for Windowed-Wigner representations:

Proposition 16. *For $\psi \in L^\infty(\mathbb{R}^d)$, suppose that one of the following conditions is satisfied*

a) $\operatorname{supp}\psi \neq \mathbb{R}^d$;

b) $|\psi(t)| = 1$ *for almost every* $t \in \mathbb{R}^d$.

For $f \in S(\mathbb{R}^d)$ then we have: $\operatorname{supp} \mathrm{Wig}_\psi(f)$ *or* $\operatorname{supp} \mathrm{Wig}^*_\psi(f)$ *compact* $\implies f = 0$.

Proof. Suppose that hypothesis a) is satisfied. As

$$\begin{aligned} \mathrm{Wig}_\psi &= Q_\sigma \quad \text{with} \quad \sigma = \delta \otimes \widehat{\psi}, \\ \mathrm{Wig}^*_\psi &= Q_{\sigma'} \quad \text{with} \quad \sigma' = \widehat{\widetilde{\psi}} \otimes \delta\ , \end{aligned} \tag{4.1}$$

(cf. Proposition 5) then $\operatorname{supp}\psi \neq \mathbb{R}^d$ implies both

$$\begin{aligned} &\operatorname{supp}\widehat{\sigma} = \operatorname{supp}(1 \otimes \tilde{\psi}) \neq \mathbb{R}^d \quad \text{and} \\ &\operatorname{supp}\widehat{\sigma}' = \operatorname{supp}(\psi \otimes 1) \neq \mathbb{R}^d. \end{aligned}$$

The thesis follows therefore from Proposition 15 (a).

Suppose now that hypothesis b) is satisfied. From (4.1) we have $|\tilde{\widehat{\sigma}}(\eta, y)| = |\psi(y)| = 1$ and $|\widehat{\sigma}'(\eta, y)| = |\psi(\eta)| = 1$ for every $(\eta, y) \in \mathbb{R}^{2d}$. It is well-known (see e.g. [13]) that this is equivalent to the fact that Q_σ and $Q_{\sigma'}$ satisfy Moyal's equality and the thesis follows then from Proposition 15 b). $\square$

A different form of uncertainty principles can be expressed in terms of projections:

Proposition 17. *Let $\psi \in S(\mathbb{R}^d)$, $\psi(0) \neq 0$.*

a) Suppose that at least one of the conditions $\operatorname{supp} f \neq \mathbb{R}^d$ *or* $\operatorname{supp}\psi \neq \mathbb{R}^d$ *is verified. Then* $\Pi_\omega \operatorname{supp} \mathrm{Wig}_\psi(f)(x_0, .)$ *compact in* $\mathbb{R}^d_\omega$ *for every fixed* $x_0 \implies f = 0$.

b) Suppose that at least one of the conditions $\operatorname{supp}\widehat{f} \neq \mathbb{R}^d$ *or* $\operatorname{supp}\psi \neq \mathbb{R}^d$ *is verified. Then* $\Pi_x \operatorname{supp} \mathrm{Wig}^*_\psi(f)(., \omega_0)$ *compact in* $\mathbb{R}^d_x$ *for every fixed* $\omega_0 \implies f = 0$.

Proof. a) For every fixed $x_0 \in \mathbb{R}^d$, from the Paley-Wiener theorem and the fact that

$$\mathrm{Wig}_\psi(f)(x_0, \omega) = \mathcal{F}_{t\to\omega}[\psi(t) f(x_0 + t/2)\overline{f(x_0 - t/2)}]$$

has compact support with respect to ω implies that $\psi(t) f(x_0 + t/2)\overline{f(x_0 - t/2)}$ is analytic. Under the hypothesis a) however $\psi(t) f(x_0 + t/2)\overline{f(x_0 - t/2)}$ can not have support $\mathbb{R}^d$ and therefore is identically 0 for every t. In particular for $t = 0$, as $\psi(0) \neq 0$, this means $|f(x_0)|^2 = 0$, i.e. $f = 0$.

b) From $\mathrm{Wig}^*_\psi(f)(x, \omega) = \mathrm{Wig}_\psi(\widehat{f})(\omega, -x)$ and from part a) we have $\widehat{f} = 0$, i.e. $f = 0$. $\square$

Remark 18. *If* $\operatorname{supp} \mathrm{Wig}_\psi(f)$ *is compact in* $\mathbb{R}^{2d}$ *then hypothesis a) is satisfied and therefore* $f = 0$. *Analogously if* $\operatorname{supp} \mathrm{Wig}^*_\psi(f)$ *is compact in* $\mathbb{R}^{2d}$ *then hypothesis b) is satisfied and* $f = 0$.

If we consider the previous two forms of uncertainty principle (Propositions 16 and 17) we observe that they generalize the non trivial fact that the Wigner representation can not be compactly supported on non zero signals $f \in S(\mathbb{R}^d)$ (Proposition 14 b)) to the cases of windowed-Wigner representations Wig_ψ and Wig^*_ψ, however in the case of the two "limits" representations $\mathrm{Wig}_\delta(f)(x,\omega) = |f(x)|^2$ and $\mathrm{Wig}^*_\delta(f)(x,\omega) = |\widehat{f}(\omega)|^2$ they do not reduce to Proposition 14 a), but to the trivial fact that, viewed as functions on $\mathbb{R}^{2d}$, $|f(x)|^2$ and $|\widehat{f}(\omega)|^2$ can not have compact support unless $f = 0$.

A formulation of the uncertainty principle which connects the two forms stated in Proposition 14 a) and b), is the following:

Proposition 19. *Let ψ_j $(j = 1, 2)$ be continuous, polynomially bounded functions such that $\psi_j(0) \neq 0$ and let $f \in S(\mathbb{R}^d)$. If $\Pi_x \mathrm{supp}\ \mathrm{Wig}_{\psi_1}(f)$ and $\Pi_\omega \mathrm{supp}\ \mathrm{Wig}^*_{\psi_2}(f)$ are compact then $f = 0$.*

Proof. From the condition that $\Pi_x \mathrm{supp}\ \mathrm{Wig}_{\psi_1}(f)$ is compact we have that there exist $M > 0$ such that for every $x_0 \in \mathbb{R}^d$, $|x_0| \geq M$ implies $\int e^{-2\pi i\omega t}\psi_1(t)f(x_0+t/2)\overline{f(x_0-t/2)}\,dt = 0$ for every $\omega \in \mathbb{R}^d$. This means $\psi_1(t)f(x_0+t/2)\overline{f(x_0-t/2)} = 0$ for every $t \in \mathbb{R}^d$. As $\psi_1(0) \neq 0$, taking $t = 0$, we have $|f(x_0)|^2 = 0$ for $|x_0| \geq M$, i.e. $\mathrm{supp}\, f$ is compact.

Analogously, from the condition that $\Pi_\omega \mathrm{supp}\ \mathrm{Wig}^*_{\psi_2}(f)$ is compact there exist $N > 0$ such that $|\omega_0| \geq M$ implies $\int e^{-2\pi i\omega t}\psi_2(t)\widehat{f}(\omega_0+t/2)\overline{\widehat{f}(\omega_0-t/2)}\,dt = 0$ for every $x \in \mathbb{R}^d$. This means $\psi_2(t)\widehat{f}(\omega_0+t/2)\overline{\widehat{f}(\omega_0-t/2)} = 0$ for every $t \in \mathbb{R}^d$, which, for $t = 0$, yields $|\widehat{f}(\omega_0)|^2 = 0$, i.e. $\mathrm{supp}\,\widehat{f}$ is compact. Therefore $f = 0$, according to Proposition 14, (a). □

In the case $\psi_1 = \psi_2 = 1$ we recapture the fact that the compatcness of $\mathrm{supp}\ \mathrm{Wig}(f)$ implies $f = 0$ (Proposition 14 (b)). Proposition 14 (a) on the other side is recaptured as limit case when $\psi_j \longrightarrow \delta$.

Remark that Proposition 19 is an example where two separate conditions with respect to time and frequency are imposed on two (a priori) different representations Wig_{ψ_1} and $\mathrm{Wig}^*_{\psi_2}$ to obtain $f = 0$. An intermediate way is pursued in [6] where two separate conditions are imposed on one representation, namely the ambiguity function.

We change now slightly our point of view and, regarding a signal as a phenomenon of which the couple $(f, \widehat{f})$ are two different "expressions", we propose a generalization of the couple $(|f|^2, |\widehat{f}|^2)$ by the introduction of a *duality* among representations in the Cohen class. We show then how this allows a unified view on some known uncertainty principle, as well as the formulation of some new ones.

Definiton 20. *Let $Q_\sigma = \sigma * \mathrm{Wig}$ be the general representation with Cohen kernel $\sigma \in S'(\mathbb{R}^{2d})$ defined (for simplicity) on signals in $S(\mathbb{R}^d)$. Let $S : F \in S(\mathbb{R}^{2d}) \longrightarrow S(F) \in S(\mathbb{R}^{2d})$ be the symplectic map defined as $S(F)(x,\omega) = F(\omega,-x)$ with obvious extension to $S'(\mathbb{R}^{2d})$. We set $\sigma^*(x,\omega) = S(\sigma)(x,\omega) = \sigma(\omega,-x)$ and we call σ^** **dual kernel** *of σ and $Q^*_\sigma = \sigma^* * \mathrm{Wig}$* **dual representation** *of Q_σ.*

Remark 21. *Clearly as* $\delta^* = \delta$ *we have* $\mathrm{Wig}^* = \mathrm{Wig}$. *More generally* $\sigma^{**} = \widetilde{\sigma}$, *where* $\widetilde{\sigma}(x,\omega) = \sigma(-x,-\omega)$, *and therefore for every* $\sigma \in S'(\mathbb{R}^d)$ *the kernel* $\sigma + \sigma^* + \widetilde{\sigma} + \widetilde{\sigma}^*$ *is self-dual;*
Furthermore simple computations yield the following expression for the dual representaion

$$\begin{aligned} Q^*_\sigma(f,g)(x,\omega) = \; & \big(\sigma * S\big(\mathrm{Wig}(f,g)\big)\big)(\omega,-x) = \\ & \big(\sigma * \mathrm{Wig}(\widehat{f},\widehat{g})\big)(\omega,-x) \;=\; Q_\sigma(\widehat{f},\widehat{g})(\omega,-x) \end{aligned} \tag{4.2}$$

which generalizes the well-known fact that $\mathrm{Wig}(f,g)(x,\omega) = \mathrm{Wig}(\widehat{f},\widehat{g})(\omega,-x)$.

On the other hand the duality between the representations Wig_ψ and Wig^*_ψ is a particular case of Definition 20 and we show next that the same applies to some other well-known subclasses of the Cohen class. More precisely the following property shows in which sense this holds for τ-Wigner representations (1.2), generalized spectrograms (1.3) and the Rihaczek and the conjugate Rihaczek representations defined as $R(f,g)(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\widehat{g}(\omega)}$ and $\widetilde{R}(f,g)(x,\omega) = e^{2\pi i x\omega}\overline{g(x)}\widehat{f}(\omega)$ respectively (see [3]).

Proposition 22. *The following dualities for subclasses of the Cohen class hold:*

(i) If $Q_\sigma = |f(x)|^2$ *then* $Q^*_\sigma = |\widehat{f}(\omega)|^2$;

(ii) If $Q_\sigma = \mathrm{Wig}_\psi$ *then* $Q^*_\sigma = \mathrm{Wig}^*_\psi$;

(iii) If $Q_\sigma = \mathrm{Wig}^{(\tau)}$ *then* $Q^*_\sigma = \mathrm{Wig}^{(1-\tau)}$;

(iv) If $Q_\sigma = \mathrm{Sp}_{\phi,\psi}$ *then* $Q^*_\sigma = \mathrm{Sp}_{\widehat{\phi},\widehat{\psi}}$;

(v) If $Q_\sigma = R$ *then* $Q^*_\sigma = \widetilde{R}$.

Proof. *(i)* is a particular case of *(ii)*. *(ii)* follows directly from Definition 20 and Remark 21. For *(iii)* we recall that, from (2.13), the expression of the Cohen kernel of $\mathrm{Wig}^{(\tau)}$ is $\frac{2^d}{|2\tau-1|^d} e^{\frac{4\pi i x\omega}{2\tau-1}}$, an easy computation yields then the conclusion. Analogously for *(iv)* we have from (2.23) that the Cohen kernel of $\mathrm{Sp}_{\phi,\psi}$ is $\mathrm{Wig}(\tilde{\phi},\tilde{\psi})(x,\omega)$, then for the dual kernel we have $\mathrm{Wig}(\tilde{\phi},\tilde{\psi})(\omega,-x) = \mathrm{Wig}(\widehat{\phi},\widehat{\psi})(x,\omega)$ which proves the assertion. Finally case *(v)* is a particular case of both *(iii)* and *(iv)* as $R(f,g) = e^{-2\pi i x\omega} * \mathrm{Wig}(f,g)$ and $\tilde{R}(f,g) = e^{2\pi i x\omega} * \mathrm{Wig}(f,g)$ (see [3]). $\square$

We show now that dual representations have symmetrical behavior with respect to the marginal distribution conditions.

Lemma 23. Q_σ *satisfies the marginal condition with respect to the* x*-variables (resp. the* ω*-variables) if and only if* Q^*_σ *satisfies the marginal condition with respect to the* ω*-variables (resp. the* x*-variables).*

Proof. Suppose that $\int_{\mathbb{R}^d} Q_\sigma(f)(x,\omega)\,dx = |\widehat{f}(\omega)|^2$ for all $f \in S(\mathbb{R}^d)$, then, using (4.2):

$$\int_{\mathbb{R}^d} Q^*_\sigma(f)(x,\omega)\,d\omega = \int_{\mathbb{R}^d} Q_\sigma(\widehat{f})(\omega,-x)\,d\omega = |\widehat{\widehat{f}}(-x)|^2 = |f(x)|^2$$

Viceversa suppose that $\int_{\mathbb{R}^d} Q^*(f)(x,\omega)\,d\omega = |f(x)|^2$, then

$$\int_{\mathbb{R}^d} Q_\sigma(f)(x,\omega)\,dx = \int_{\mathbb{R}^d} Q^*_\sigma(\mathcal{F}^{-1}f)(-\omega,x)\,dx = |(\mathcal{F}^{-1}f)(-\omega)|^2 = |\widehat{f}(\omega)|^2$$

The case of the other marginal conditions is analogous. □

Before we proceed with the study of dual representations, we summarize in a general form four equivalent formulations of the uncertainty principle. We consider from now on $d = 1$, as the generalization to the multi-dimensional case is a trivial exercise of indices.

Proposition 24. *Let Q_1 and Q_2 be two representations in the Cohen class satisfying a.e. the marginal conditions:*

$$\int_{\mathbb{R}} Q_1 f(x,\omega)\,d\omega = |f(x)|^2, \qquad \int_{\mathbb{R}} Q_2 f(x,\omega)\,dx = |\widehat{f}(\omega)|^2.$$

For $f \in S(\mathbb{R})$, the following hold and are equivalent:

(a) $\left(\int_{\mathbb{R}} x^2 Q_1 f(x,\omega)\,dxd\omega\right)^{1/2} \left(\int_{\mathbb{R}} \omega^2 Q_2 f(x,\omega)\,dxd\omega\right)^{1/2} \geq \frac{1}{4\pi}\|f\|^2_{L^2}$*;*

(b) $\left(\int_{\mathbb{R}^2} (x-a)^2 Q_1 f(x,\omega)\,dxd\omega\right)^{1/2} \left(\int_{\mathbb{R}^2} (\omega-b)^2 Q_2 f(x,\omega)\,dxd\omega\right)^{1/2} \geq \frac{1}{4\pi}\|f\|^2_{L^2}$*, for all $a, b \in \mathbb{R}$;*

(c) $\int_{\mathbb{R}^2} x^2 Q_1 f(x,\omega) + \omega^2 Q_2 f(x,\omega)\,dxd\omega \geq \frac{1}{2\pi}\|f\|^2_2$*.*

(d) $\int_{\mathbb{R}^2} (x-a)^2 Q_1 f(x,\omega) + (\omega-b)^2 Q_2 f(x,\omega)\,dxd\omega \geq \frac{1}{2\pi}\|f\|^2_2$*, for all $a, b \in \mathbb{R}$.*

Furthermore equalities are attained if and only if $f(x) = ce^{-\pi x^2}$, with $c \in \mathbb{C}$, in cases (a) and (c), and if and only if $f(x) = ce^{2\pi i b(x-a)}e^{-\pi k(x-a)^2}$ with $k > 0$, in cases (b) and (d).

Proof. (It is a generalization of the classical case see [11, Cor. 1.35, 1.37]). The classical uncertainty principle for $f \in L^2(\mathbb{R})$ asserts that

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}} \omega^2 |\widehat{f}(\omega)|^2\,d\omega\right)^{1/2} \geq \frac{1}{4\pi}\|f\|^2_{L^2}.$$

From the hypothesis on the marginal conditions of Q_1 and Q_2 this is equivalent to

$$\left(\int_{\mathbb{R}} x^2 Q_2 f(x,\omega)\,dxd\omega\right)^{1/2} \left(\int_{\mathbb{R}} \omega^2 Q_1 f(x,\omega)\,dxd\omega\right)^{1/2} \geq \frac{1}{4\pi}\|f\|^2_{L^2}$$

which proves (a).

(b) is obtained applying (a) to the function $g(x) = T_{-a}M_{-b}f(x) = e^{-2\pi ib(x+a)}f(x+a)$ and using the covariance property, i.e. $Q_jg(x,\omega) = Q_jf(x+a,\omega+b)$, (j=1,2), which holds for all members of the Cohen class.
(c) and (d) are obtained from (a) and (b) respectively using the elementary inequality $\frac{\alpha^2+\beta^2}{2} \geq \alpha\beta$, which holds for every $\alpha, \beta \geq 0$.
Then (a) implies (b), (c) and (d). Of course (b) implies (a) and (d) implies (c). We show now that (c) implies (a).
For $\alpha > 0$ and $f \in S(\mathbb{R})$, let $f_\alpha(x) = \alpha^{1/2}f(\alpha x)$, then using the marginal conditions we have

$$\int_{\mathbb{R}} x^2 Q_1 f_\alpha(x,\omega)\, dxd\omega = \alpha^{-2} \int_{\mathbb{R}} x^2 Q_1 f(x,\omega)\, dxd\omega$$

and

$$\int_{\mathbb{R}} \omega^2 Q_2 f_\alpha(x,\omega)\, dxd\omega = \alpha^{2} \int_{\mathbb{R}} \omega^2 Q_2 f(x,\omega)\, dxd\omega.$$

Applying (c) to f_α we have then

$$\frac{1}{\alpha^2}\int_{\mathbb{R}^2} x^2 Q_1 f(x,\omega)\, dxd\omega + \alpha^2 \int_{\mathbb{R}^2} \omega^2 Q_2 f(x,\omega)\, dxd\omega \geq \frac{1}{2\pi}\|f\|_2^2,$$

for every $f \in S(\mathbb{R})$. Minimizing the left side over $\alpha > 0$ we get (a).
Finally we remark that

$$\int_{\mathbb{R}} x^2 |f(x)|^2\, dx = \int_{\mathbb{R}} x^2 Q_2 f(x,\omega)\, dxd\omega \text{ and } \int_{\mathbb{R}} \omega^2 |\widehat{f}(\omega)|^2\, d\omega = \int_{\mathbb{R}} \omega^2 Q_1 f(x,\omega)\, dxd\omega$$

implies that equalities are obtained exactly in the same cases as the classical ones i.e. when $Q_1f(x,\omega) = |f(x)|^2$ and $Q_2(f)(x,\omega) = |\widehat{f}(\omega)|^2$. □

In the following we shall restrict our considerations to uncertainty principles of the form (c); In view of Proposition 24 every result could be equivalently reformulated in each of the forms (a), (b) or (d).

Lemma 23 and Proposition 24 yield the following general uncertainty principle for couples of dual representations defined by 20:

Proposition 25. *for $\sigma \in S'(\mathbb{R}^2)$ and $f \in S(\mathbb{R})$ suppose that $\int_{\mathbb{R}} Q_\sigma f(x,\omega)\, d\omega = |f(x)|^2$ (equivalently $\int_{\mathbb{R}} Q_\sigma^* f(x,\omega)\, dx = |\widehat{f}(\omega)|^2$), then:*

$$\int_{\mathbb{R}^2} x^2 Q_\sigma f(x,\omega) + \omega^2 Q_\sigma^* f(x,\omega)\, dxd\omega \geq \frac{1}{2\pi}\|f\|_2^2 \tag{4.3}$$

Equality holds if and only if $f(x) = ce^{-\pi x^2}$, with $c \in \mathbb{C}$.

We examine now some consequences of (4.3) starting with the case of τ-Wigner representations.

Proposition 26. *For $\tau \in [0,1]$ the following uncertainty principle holds:*

$$\int_{\mathbb{R}^2} (x^2+\omega^2)\, \Re \mathrm{Wig}^{(\tau)} f(x,\omega)\, dx d\omega \geq \frac{1}{2\pi}\|f\|_2^2 \tag{4.4}$$

for all $f \in S(\mathbb{R})$.

Proof. From the fact that $(\mathrm{Wig}^{(\tau)})^* = \mathrm{Wig}^{(1-\tau)}$ (Proposition 22, (iii)), from (4.3) and the fact that $\overline{\mathrm{Wig}^{(\tau)} f} = \mathrm{Wig}^{(1-\tau)} f$, we have

$$\int_{\mathbb{R}^2} x^2\, \mathrm{Wig}^{(\tau)} f(x,\omega) + \omega^2 \overline{\mathrm{Wig}^{(\tau)} f(x,\omega)}\, dx d\omega \geq \frac{1}{2\pi}\|f\|_2^2. \tag{4.5}$$

Substituting τ with $1-\tau$ we also have

$$\begin{aligned} \textstyle\int_{\mathbb{R}^2} x^2\, \mathrm{Wig}^{(1-\tau)} f(x,\omega) + \omega^2 \overline{\mathrm{Wig}^{(1-\tau)} f(x,\omega)}\, dx d\omega = \\ \textstyle\int_{\mathbb{R}^2} x^2 \overline{\mathrm{Wig}^{(\tau)} f(x,\omega)} + \omega^2\, \mathrm{Wig}^{(\tau)} f(x,\omega)\, dx d\omega \geq \quad \frac{1}{2\pi}\|f\|_2^2. \end{aligned} \tag{4.6}$$

Summing up (4.5) and (4.6) we get

$$\int_{\mathbb{R}^2} (x^2+\omega^2)\ \Re\, \mathrm{Wig}^{(\tau)} f(x,\omega)\, dx d\omega \geq \frac{1}{2\pi}\|f\|_2^2.$$

□

Inequality (4.4) represents a natural extension to the representations $\mathrm{Wig}^{(\tau)}$ of the well-known form of the uncertainty principle of the Wigner transform which recaptured for $\tau = 1/2$.

In [3] it the integrated representation $Q(f,g) = \int_{[0,1]} \mathrm{Wig}^{(\tau)}(f,g)\, d\tau$, actually the Born-Jordan representation, is studied in connection with quantization and interferences. Now, using the equality $\overline{\mathrm{Wig}^{(\tau)} f} = \mathrm{Wig}^{(1-\tau)} f$, simple integration and changes of variable yield the following corollary of Proposition 26.

Corollary 27. *For the Born-Jordan representation $Q(f,g) = \int_{[0,1]} \mathrm{Wig}^{(\tau)}(f,g)\, d\tau$ we have*

$$\int_{\mathbb{R}^2} (x^2+\omega^2)\, Q(f)(x,\omega)\, dx d\omega \geq \frac{1}{2\pi}\|f\|_2^2 \tag{4.7}$$

for all $f \in S(\mathbb{R})$.

We consider next the form of uncertainty principle obtained as consequence of the duality between Windowed-Wigner representations.

Proposition 28. *Let $\phi, \psi \in L^1(\mathbb{R})$ such that $\widehat{\phi}, \widehat{\psi} \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}}^2 \widehat{\phi}(\omega)\, d\omega = \int_{\mathbb{R}}^2 \widehat{\psi}(\omega) = 1$, then:*

$$\int_{\mathbb{R}^2} x^2\ \mathrm{Wig}_\phi f(x,\omega) + \omega^2\ \mathrm{Wig}^*_\psi f(x,\omega)\, dx d\omega \geq (2\pi)^{-1}\|f\|_{L^2}^2 \tag{4.8}$$

where the left-hand side is independent of the window functions ϕ and ψ (satisfying the hypothesis) and equality holds if and only if $f(x) = ce^{-\pi x^2}$ with $c \in \mathbb{C}$.

Proof. It is an immediate consequence of (4.3), which holds if and only if $f(x) = ce^{-\pi x^2}$, for $c \in \mathbb{C}$, and the fact that Wig_ψ and Wig^*_ψ satisfy the marginal densities $\int_{\mathbb{R}} \mathrm{Wig}_\phi f(x,\omega)\, d\omega = |f(x)|^2$ and $\int_{\mathbb{R}} \mathrm{Wig}^*_\psi f(x,\omega)\, dx = |\widehat{f}(\omega)|^2$ under the hypothesis $\int_{\mathbb{R}^2} \widehat{\phi}(\omega)\, d\omega = \int_{\mathbb{R}^2} \widehat{\psi}(\omega) = 1$. □

Remark 29. *Finally we remark that* (4.3) *applied to the Rihaczek and conjugate Rihaczek representations yields the classical uncertainty principle:*

$$\begin{array}{rl} \int_{\mathbb{R}^2} x^2 Rf(x,\omega) + \omega^2 R^* f(x,\omega)\, dx d\omega & = \\ \int_{\mathbb{R}} x^2 |f(x)|^2 dx + \int_{\mathbb{R}} \omega^2 |\widehat{f}(\omega)|^2\, d\omega & \geq \frac{1}{2\pi} \|f\|_2^2 \end{array} \tag{4.9}$$

for all $f \in S(\mathbb{R})$.

References

[1] M. Benedicks. On Fourier transforms of functions supported on sets of finite Lebesgue measure, *J. Math. Anal. Appl.* **106** (1), 180-183, (1985).

[2] P. Boggiatto, E. Carypis, and A. Oliaro. Wigner representations associated with linear transformations of the time-frequency plane, Operator Theory: Advances and Applications Vol. **213**, "Pseudo-Differential Operators: Analysis, Applications and Computations", Springer Basel (2011), 275–288

[3] P. Boggiatto, G. De Donno and A. Oliaro, Time-Frequency representations of Wigner type and Pseudo-differential Operators, *Trans. Amer. Math. Soc.* **362**, no. 9, 4955-4981, (2010).

[4] P. Boggiatto, G. De Donno, A. Oliaro, A class of quadratic time-frequency representations based on the short-time Fourier transform, Operator Theory: Advances and Appl., Vol. 172, 235–249, (2006).

[5] P. Boggiatto, C. Fernández, A. Galbis. Supports of representations in the Cohen class, J. Fourier Anal. Appl., **17** (6), 1180–1197, (2011).

[6] A. Bonami, Demange B., and P. Jaming. Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transform, *Revista Matematica Iberoamericana*, **19**, 23–25, (2003).

[7] L. Cohen. Time-Frequency Distributions - A Review. *Proc. of IEEE,* **77**(7), 941–981, (1989).

[8] L. Cohen. *Time-Frequency Analysis.* Prentice Hall Signal Proc. series, New Jersey, 1995.

[9] L. Cohen. *The uncertainty principle for the short-time Fourier transform Proc. Int. Soc. Opt. Eng.,* **22563**, 80-90, (1995).

[10] C. Fernández, A. Galbis. Annihilating sets for the short-time Fourier transform. *Adv. Math.* **224**, no. 5, 1904-1926, (2010)

[11] G. Folland. *Harmonic Analysis in Phase Space.* Princeton University Press, Princeton New Jersey 1989.

[12] G. B. Folland, A. Sitaram. The uncertainty principle: a mathematical survey. *J. Fourier Anal. Appl.*, **3**(3), 207–238, (1989).

[13] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Birkhäuser, Boston, 2001.

[14] K. Gröchenig. Uncertainty principles for time-frequency representations. *Advances in Gabor Analysis*, H. G. Feichtinger, T. Strohmer, eds., Birkhauser, Boston, 11–30, (2003).

[15] A.J.E.M. Janssen. Bilinear phase-plane distribution functions and positivity. *J. Math. Phys.* **26**, no. 8, 1986-1994 (1985).

[16] A.J.A.M. Janssen. Proof of a conjecture on the supports of Wigner distributions. *J. Fourier Anal. Appl.*, **4**(6), 723–726, (1998).

[17] Y. Liu, M.-W. Wong. Polar wavelet transforms and localization operators. *Integral Equations Operator Theory* **58**, no. 1, 99–110, (2007).

[18] F. Treves. *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York-London (1967).

[19] E. Wilczok. *New Uncertainty Principles for the Continuous Gabor Transform and the Continuous Wavelet Transform.* Documenta Math., (5) 201–226, 2000.

[20] M.-W. Wong *Weyl Transforms.* Universitext, Springer-Verlag, New York, (1998).