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Engaging in covariational reasoning when modelling a real phenomenon: the case of the psychrometric chart

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Abstract
The mathematical modelling of a real-life phenomenon is an elaborated activity, and it often requires complex forms of covariational reasoning, such as second-order covariation. This study aims to characterize how students use several forms of covariational reasoning when modelling a real-life phenomenon. To achieve this research goal, it is proposed the analysis of a teaching experiment conducted in an 11th-grade classroom and focused on the mathematical modelling of the relationship between three quantities, temperature, absolute humidity, and relative humidity, which is mathematically represented in the psychrometric chart. The qualitative analysis was focused on covariational reasoning and the students' processes of mathematical modelling of the real-life phenomenon under investigation. Findings from five representative episodes showed an interlacing of several forms of covariational reasoning, the emergence of qualitative, quantitative, and global characterizations of covariational reasoning, and three different roles of covariation throughout the various steps of the modelling activities. From an educational point of view, the modelling activities described here offer practical insights for the design of activities aimed at promoting the modelling of real-life phenomena through a covariational approach.

Keywords Covariation · Second-order covariation · Multivariation · Modelling · Digital tools · Representations

1 Introduction

In Mathematics Education the mathematical modelling competence is intended as a lens onto the real world (Niss and Højgaard 2019). The PISA framework clearly remarks that "[b]eing more literate about change and relationships involves understanding fundamental types of change and recognizing when they occur in order to use suitable mathematical models to describe and predict change" (OECD-PISA 2022, p. 24). Recent studies have

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touched on how reasoning covariationally, namely envisioning how variables can change in tandem with each other (Thompson and Carlson 2017), can support the development of a conceptual understanding of scientific phenomena (Gonzalez 2019; Panorkou and Germia 2021; Rodriguez et al. 2019). Even if in the PISA framework there are no explicit references to covariational reasoning, it seems the suitable form of mathematical reasoning to model "the change and the relationships with appropriate functions and equations, as well as creating, interpreting and translating among symbolic and graphical representations of relationships" (OECD-PISA 2022, p. 24).

Numerous studies have investigated the potentialities of several digital tools to support covariational reasoning such as dynamic geometry environments (Ellis et al. 2016; Hegedus and Otálora 2023; Johnson et al. 2017) and augmented reality technology (Levy et al. 2020; Swidan et al. 2019). Such kind of environments may support the conceptualization of the dynamic aspects involved in tasks of mathematical modelling by enabling one to explore the continuous dynamic features of a real phenomenon and in some learning environments also by bringing such features together with the mathematical representations of the phenomenon.

However, the mathematical modelling of a real-life phenomenon is a complex activity: "The natural and designed worlds display a multitude of temporary and permanent relationships among objects and circumstances, where changes occur within systems of interrelated objects or in circumstances where the elements influence one another" (OECD-PISA 2022, p. 24). Therefore, reasoning about such real-life situations may require a cognitive effort more demanding than the one described in the covariational reasoning framework. On the one hand, such real-life contexts often include more than two quantities at stake: for instance, Panorkou and Germia (2021) analysed the case of Earth's gravitational force $F = G\frac{m_1 \times m_2}{d}$, and in this case, several researchers adopt the term *multivariation*, an extended framework whose conceptual and empirical validation is in progress (Jones 2022). On the other hand, other researchers have observed that in a process of mathematical modelling of classes of real-life phenomena, it may also be important to consider how the characteristic quantities affect the behaviour of the phenomenon itself. Arzarello (2019) started referring to this construct as *second-order covariation*, and it has been lately defined as the ability to envision a family of invariant relations and its characteristic parameters varying simultaneously (Bagossi 2022). The chosen term is in line with the terminology used by Bloedy-Vinner (2001) who calls "second order functions" those functions whose inputs are parameters and the outputs are corresponding parametric equations or functions. Referring to the gravitational force previously recalled, if we consider m_2 as fixed, an example of covariational reasoning at the second-order would be considering how the distance d between the two masses affects the gravitational force-mass relationship $F = f_d(m_1)$. Investigating students' covariational reasoning, especially the second-order one, in the modelling of real-life phenomena is a line of research that has already been initiated in the literature and some preliminary results have been discussed (Bagossi 2021, 2022).

The design principles underlying digital learning environment supporting second-order covariation have been explored only recently and in a few studies. For instance, Hoffkamp (2011) analysed the design features of some applets created with an interactive geometry software prompting what she calls metavariation, a form of variation leading to a qualitative and global view of functional dependency. Such task design allowed the visualization of the dynamic aspect of functional dependencies simultaneously in different representations. Bagossi and Swidan (accepted) instead have investigated the learning of second-order covariation in the conceptualization of the motion of a ball rolling along an inclined plane both using GeoGebra and augmented reality. In both the learning environments, students covaried the angle of inclination of the plane and the distance-time graph elaborating on a qualitative description and making explicit the direction of change of both the quantity and the graph.

The aim of this study is to characterize how students use several forms of mathematical reasoning, first-and second-order covariational reasoning, and multivariational reasoning, when they engage in modelling activities of a real-life phenomenon, and how these forms of mathematical reasoning interlace with and support the conceptualization process. To pursue this goal, we qualitatively analysed data from a teaching experiment on the conceptualization of the relationship between three quantities, temperature, absolute humidity, and relative humidity, which may allow learners to understand, for instance, the phenomena of condensation and evaporation. This study contributes to the literature investigating the relevance of covariational reasoning in activities of mathematical modelling with a deeper focus on the forms of covariation involved and the nature of the modelling processes. Indeed, findings highlight various characterizations and three different roles of covariational reasoning in the modelling processes; the role of real and mathematical representations, especially the ones supported by digital tools, in the various steps of the teaching experiment was also an element of discussion. Moreover, the activities described here can provide useful insights for the design of educational activities aimed at promoting modelling of real-life phenomena through a covariational approach.

2 Theoretical framework

2.1 Mathematical modelling

In Mathematics Education, modelling is intended as "the process of translating between the real world and mathematics in both directions" (Blum and Ferri 2009, p. 45). Mathematical models represent an ideal conceptualization of a real-life or scientific phenomenon; they are formulated in a mathematical language and make use of a wide variety of mathematical tools and results (Niss and Højgaard 2019). Hence, given the diversity of factors at play, mathematical modelling is a complex activity. As contained in the definition itself, the modelling cycle mainly consists of interactions (Fig. 1). The connections from the real world to the mathematical world can be described through two main processes: mathematising and applying (Yoon et al. 2010). *Mathematising* consists of interpreting a real context mathematically; it describes a process from the real world to the mathematical one. *Applying* instead consists of using mathematical knowledge for the creation of a mathematical model, having so the chance to experience mathematics can be practical. Naturally, there is also a process from the maths world to the real one which enables one to interpret the mathematical results in the real world: in this study, we will refer to this process as *interpreting*.

The process of modelling is necessarily characterized by removing noise, meaning clearing all those disturbing elements that could make it challenging to move from the real situation to a mathematical model. Therefore, a mathematical model may be considered a conceptualization that is an approximation or an intentional simplification of the analysed phenomenon. Research suggests that digital tools can act as a bridge between the real world and the mathematics (Galbraith and Stillman 2006). Digital tools should not be intended only as means to complete computational tasks, but they can fulfil several purposes such as finding information, formulating equations, or visualizing solutions (Kaiser and Schukajlow 2022). Indeed, digital resources can provide different representations of the phenomenon itself which can introduce new learning opportunities (Doerr and Pratt 2008).

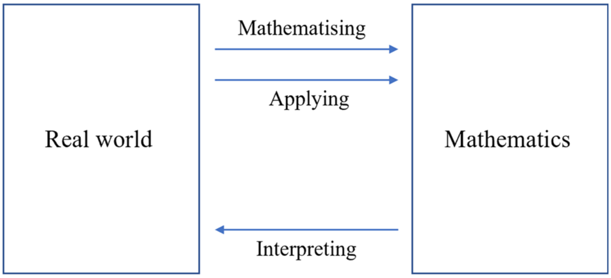


Fig. 1 A simplified version of the modelling cycle

Moreover, several studies discuss the potentialities of technological tools such as dynamic geometry software or graphic calculator to visualize the simultaneous change in quantities (Arzarello 2019; Ellis et al. 2016; Hoffkamp 2011; Johnson 2013; Swidan et al. 2022). They can "offer students opportunity for direct and dynamic manipulation of quantities and a prompt to explore their coordinated and smooth change" (Panorkou and Germia 2021, p. 323). Indeed, conceptualizing a mathematical situation requires elaborating on which quantities are varying, how they are varying simultaneously with the other quantities, i.e., how they are co-varying, and also how the involved quantities influence the situation itself. Hence, various forms of covariational reasoning can emerge in these modelling activities, and they should be suitably framed.

2.2 An extended framework for covariational reasoning

In this study, we will adopt the enlarged perspective on covariational reasoning preliminary introduced by Arzarello (2019) and then further elaborated by Bagossi (2022). Covariational reasoning should be intended as a form of mathematical reasoning with a larger epistemological and cognitive value, meaning as the ability to suitably envision relationships between two mathematical objects. This framework provides for the existence of various *orders* of covariation connoted by specific mathematical objects and their mutual relations and *levels* descriptive of a person's capacity to reason covariationally (Thompson and Carlson 2017). In the following, we will introduce in detail first-and second-order covariation and multivariation.

Covariational reasoning is defined as the ability to envision how two quantities' values vary simultaneously (Carlson et al. 2002). Elaborating on previous and well-consolidated research findings, Thompson and Carlson (2017) have proposed a taxonomy of five cognitive levels that should be intended as descriptive levels of a person's capacity to reason covariationally. These levels can be briefly summarized as follows: *precoordination of values* means envisioning a simultaneous change in the two quantities, *gross coordination* is envisioning the direction of change of the two quantities expressed as an increase/decrease relationship, *coordination of values* means establishing a relation between the two quantities' values, *chunky continuous covariation* means envisioning values changing in discrete chunks, and eventually, *smooth continuous covariation* refers to continuous and simultaneous changes in the two quantities. As the labels of the levels communicate, it is only at the last two levels of this framework that students really show forms of covariational reasoning. First-order covariational reasoning has its roots in quantitative reasoning (Thompson 2011), and empirical findings support its relevance for a solid understanding of many mathematical concepts such as functions, proportion, rate of change, and limits (Thompson et al. 2017). Moreover, covariation is crucial in activities of mathematical modelling because "the operations that compose covariational reasoning are the very operations that enable one to see invariant relationships among quantities in dynamic situations" (Thompson 2011, p. 46), and so it reveals essential for entering into the steps of the mathematical modelling process. In order to distinguish this form of covariational reasoning between two quantities from other covariational constructs, in this enlarged framework it is referred to as *first-order covariation* (COV 1).

A more recent theoretical construct is that of *second-order covariational reasoning* (COV 2), defined as the ability to envision a family of invariant relations and its characteristic parameters varying simultaneously (Arzarello 2019; Bagossi 2022). Only some findings about a characterization of COV 2 have been presented in the literature. Data from learning experiments with high school students revealed the emergence of both qualitative and quantitative forms of COV 2 (Bagossi 2021, 2022). Moreover, Swidan and colleagues (2022) identified the existence of a transitional phase from COV 1 and COV 2 in students' reasoning in which the idea of parameter appears, also in intuitive ways, introducing an idea of motion. Moreover, the conceptualization of COV 2, starting from that of COV 1, has also been characterized within the framework of category theory, interpreting the various epistemic steps through categorical tools (Asenova et al. accepted). Given its features and these initial results, second-order covariational reasoning seems to be relevant for the conceptualization of classes of real-life phenomena, parametric functions, and parametric equations.

Eventually, Jones (2022) elaborated on the construct of *multivariation* to better frame those forms of mathematical reasoning where more than two quantities are involved. Jones engaged in a conceptual analysis describing distinct theoretical multivariation structures and then investigated students' mental actions in reasoning about multivariation showing that they are connected to covariational reasoning but also new mental actions are used by the students. We will not go into the details of the rich findings presented by Jones (2022) because they go beyond the purpose of this paper. But concluding, we remark that the importance of studying multivariation was motivated by the several mathematical and scientific contexts that include more than two variables related to each other: just to mention some, physics laws, fractions, trigonometric relations, function composition, integral functions, multivariable functions, parametric equations, and differential equations.

Taking into account the importance of covariational reasoning for mathematical modelling, this study aims to explore the connection between the emerging forms of covariational reasoning, not only COV 1, and the various processes involved in an activity of mathematical modelling. Hence, the research question guiding this study can be formulated as: *How do students reason covariationally when engaging in real-life phenomena modelling activities?*

2.3 The psychrometric diagram

The real phenomenon investigated in this study is the relationship between temperature, absolute humidity, the quantity of water vapour contained in a unit volume of air, and relative humidity. We briefly recall that air cannot contain an unlimited amount of water vapour; once it has reached its maximum amount possible, it becomes saturated. However, the saturation limit varies with temperature: the higher the temperature, the more vapour can be contained in a given volume of air.

The relative humidity is exactly the ratio between absolute humidity and its saturation limit, at a given temperature, expressed as a percentage. For example, 1 kg of air at a temperature of 20 °C can contain at most 14.7 g of water vapour; therefore, the mixture consisting of 1 kg of dry air and 14.7 g of water vapour at a temperature of 20 °C has a relative humidity of 100 % . At the same temperature, if in 1 kg of dry air, there was 7.35 g of water vapour (i.e., half of the maximum possible quantity at that temperature), the mixture would have a relative humidity of 50 % . The thermodynamic conditions of condensation are called *dew point*.

The relationship between the three quantities previously introduced is mathematically represented in a chart called *psychrometric* or *Carrier diagram*, a graph of the thermodynamic parameters of moist air at constant pressure, typically registered at sea level (Fig. 2). The chart may contain many thermodynamic parameters, but in the following, we will just focus on three main quantities: temperature (abscissa), absolute humidity (ordinate), and relative humidity (parameter). The mathematical relation between temperature and absolute humidity is given by an exponential-like function, and each of the green curves shown in Fig. 2 corresponds to a different value of the percentage of relative humidity.

3 Method

The teaching experiment (Steffe and Thompson 2000) described here is part of a wider research project investigating high school students' covariational reasoning. This teaching experiment was conducted in a hybrid modality at the beginning of the 2020–2021 school year in an 11th-grade class of a scientific-oriented high school in Italy. As already mentioned, the activities were focused on the conceptualization of the relationship between absolute humidity, relative humidity, and temperature described in the psychrometric diagram.

Reading and interpreting the psychrometric chart require several forms of covariational reasoning to appreciate the mutual relationships between the three quantities involved: reasoning at the first-order means being able to envision how absolute humidity and temperature are covarying, reasoning at the second-order means being able to elaborate on how relative humidity, i.e., the characteristic parameter, affects the family of exponential curves,

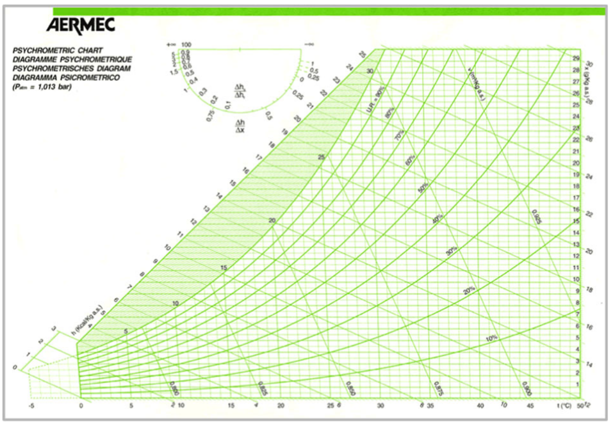


Fig. 2 An example of psychrometric diagram[1]

[1]Retrievable from: https:// www.edilportale.com/speciali/Climatizzazione/ARIA_UMIDA_01.asp?v = cl.

while multi-varying the three quantities enables to envision their simultaneous changes. The teaching experiment in its entirety had three main aims: investigating the relationship between humidity and temperature, reading and interpreting the psychrometric chart in order to explain a real-life phenomenon like condensation, and distinguishing the role of variables and parameters in reading charts.

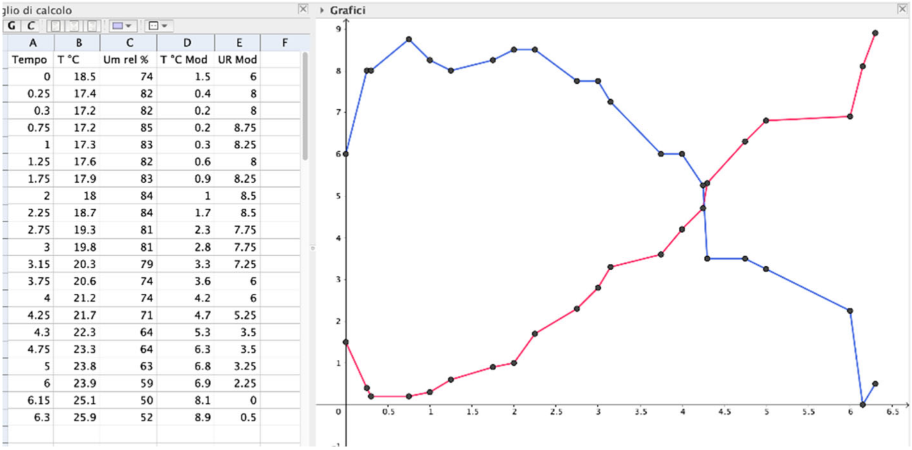
3.1 Participants

The 11th-grade classroom involved was made up of 22 students. Due to their previous studies in mathematics, students had mastery in identifying functions based on finite differences; students were used to working with digital tools and in particular with GeoGebra applets. By the time of this teaching experiment, students had not yet studied the exponential function; it was introduced formally after the teaching experiment. The notions of science required to investigate the relationship between temperature and humidity had already been studied by students during the previous school year, but a general review of these concepts was planned as an integral part of the activities during the design phase of the teaching experiment. Moreover, it is relevant to mention that in the previous school year, students were engaged in a teaching experiment on the conceptualization of the law of the motion of a ball rolling along an inclined plane (Bagossi 2021) increasing their mastery of working with functions and reasoning covariationally. Even during this activity, students used several representations, and in the end, they performed the real experiment in the physics laboratory.

3.2 Design of the teaching experiment

This teaching experiment was conducted during the period of the Coronavirus pandemic, hence it was held in a hybrid modality: partially in-presence and partially online through the Google Meet platform. The teaching experiment consisted of six main phases during which working group sessions were followed by classroom discussions. The activities proposed in each of the phases are described in detail in the following.

Phase 1 As homework, students were assigned the reading of a newspaper article dealing with the topic of hot temperatures in summer. After the reading, students were asked if they had ever heard of relative humidity and perceived temperature and wrote their answers on the Google Classroom platform of the Math course. After the such activity, the teacher initiated an in-presence classroom discussion on the answers provided: they reflected on the definitions of those terms, and some concrete examples referring to those notions were mentioned (e.g., condensation on the can, the steam on the mirror after the shower, sweating system of the human body). After that, the teacher displayed on the interactive whiteboard a GeoGebra file (Fig. 3) in which a table contained the values of temperature and relative humidity collected during a sunny day at regular intervals of time. Two graphs in a Cartesian plane represented those same sets of data with respect to time. The teacher explained how the data were collected and represented; in particular, since the magnitudes represented on the x- and y-axis have different units of measure, some suitable translations and dilatations were introduced to make them more readable and comparable. At the end of this phase, students were assigned as homework to elaborate on a possible relationship between temperature and relative humidity. Students uploaded their answers on the Classroom platform.



Tempo	T °C	Um rel %	T °C Mod	UR Mod
0	18.5	74	1.5	6
0.25	17.4	82	0.4	8
0.3	17.2	82	0.2	8
0.75	17.2	85	0.2	8.75
1	17.3	83	0.3	8.25
1.25	17.6	82	0.6	8
1.75	17.9	83	0.9	8.25
2	18	84	1	8.5
2.25	18.7	84	1.7	8.5
2.75	19.3	81	2.3	7.75
3	19.8	81	2.8	7.75
3.15	20.3	79	3.3	7.25
3.75	20.6	74	3.6	6
4	21.2	74	4.2	6
4.25	21.7	71	4.7	5.25
4.3	22.3	64	5.3	3.5
4.75	23.3	64	6.3	3.5
5	23.8	63	6.8	3.25
6	23.9	59	6.9	2.25
6.15	25.1	50	8.1	0
6.3	25.9	52	8.9	0.5

Fig. 3 GeoGebra applet interface showing the collected values of temperature versus time (red curve) and values of relative humidity versus time (blue curve) (color figure online)

Phase 2 The teacher led a classroom discussion commenting on students' answers to the homework assigned at the end of phase 1. After the discussion, the teacher made an experiment with her students (Fig. 4a): given a metal pot full of water at room temperature, they gradually added some cubes of ice; they regularly registered on the blackboard elapsed time and the getting lower temperature in the pot (Fig. 4b). When the outer surface of the pot started fogging, they registered that temperature corresponding to the dew point, i.e., the thermodynamic conditions of condensation. Then students replicated the experiment on their own as a homework activity.

Phase 3 Students faced a working group session of one-hour on Google Meet divided into five groups. They worked on a worksheet that, starting from the data collected during the pot experiment, asked for a possible relation between the initial room temperature and the dew point temperature. Then students were guided through the reading of a real psychrometric chart (Fig. 2) reported in their worksheets. Subsequently, using a GeoGebra applet reproducing the chart (Fig. 5), students were asked to find the coordinates of the point of intersections between the several green curves (corresponding to a different percentage of

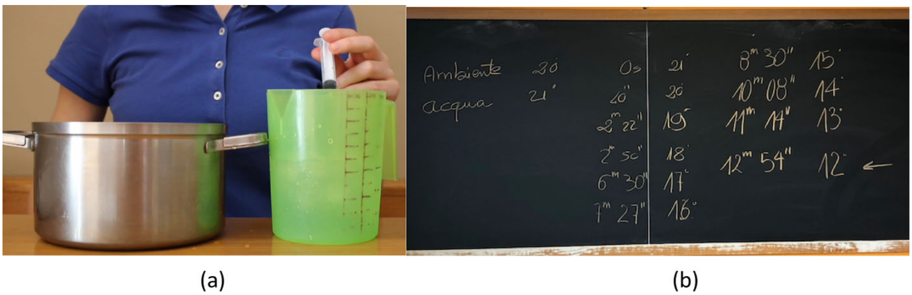


Fig. 4 **a** Reproduction of the pot experiment; **b** Data registered during the pot experiment and written on the blackboard

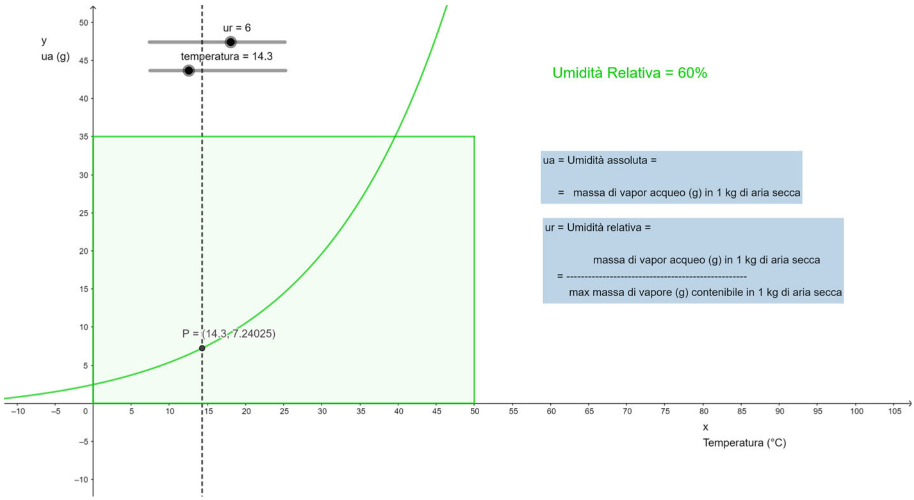


Fig. 5 GeoGebra applet simulating a psychrometric chart

relative humidity) and the horizontal line $y = y_{\text{DEW POINT}}$. The GeoGebra applet contained two sliders (top left in Fig. 5), one associated with relative humidity enabling one to move between the green curves, and the other associated with the value of temperature. Finally, students were asked again about a possible relationship between temperature and humidity. This task was followed by one-hour in-presence discussion led by the teacher during which students retraced the steps of the pot experiment on the psychrometric diagram.

Phase 4 Students were assigned the following task as homework: "What do you think will be the trend of the graph that represents the values of relative humidity as a function of temperature? Try to trace (freehand or with GeoGebra, you choose) a likely chart justifying your choices adequately." Students uploaded their works on the Classroom platform. Then the teacher devoted half an hour to an in-presence classroom discussion in which she commented on students' answers: she showed on the interactive whiteboard the various answers provided by the students underlining, in particular, the different approaches in drawing the graph and asking them to motivate their choices.

Phase 5 Students, divided into small groups, worked on a new GeoGebra applet (Fig. 6), showing the relationship between relative humidity, on the y-axis, and temperature, on the x-axis. A few questions guided students in observing which magnitudes were represented in the new reference system (Fig. 6) with respect to the old one (Fig. 5). Then students were provided with a worksheet with a table recalling each step of the pot experiment, which magnitudes varied and how, and how each step of the experiment could be represented on the Carrier diagram. The result was a cycle on the chart. Finally, work groups were asked to reproduce the same cycle in the new reference system. At the end of the phase, the groups gave their worksheets to the teacher.

Phase 6 This phase consisted of a one-hour classroom discussion during which the teacher first discussed students' answers to the previous task and then directed students' attention toward the idea that the two graphical representations, contained in the two GeoGebra applets

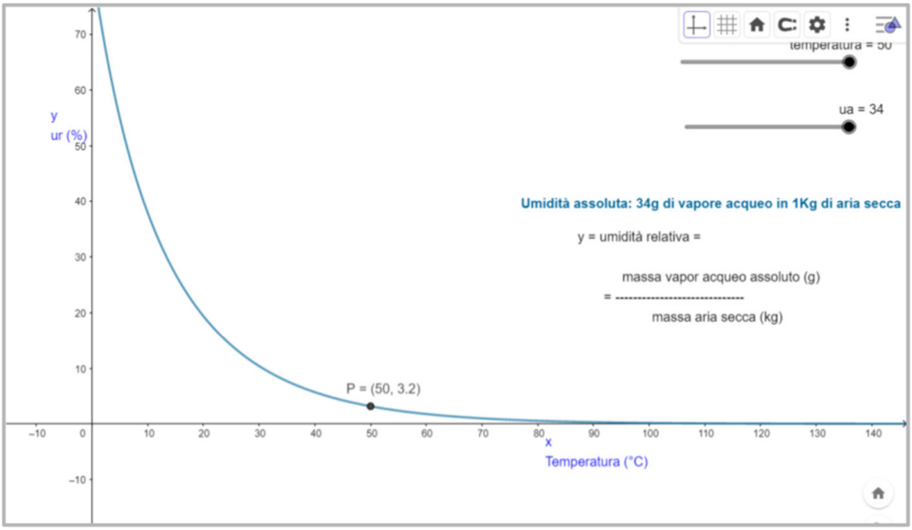


Fig. 6 The interface of the GeoGebra applet showing the relationship between relative humidity and temperature

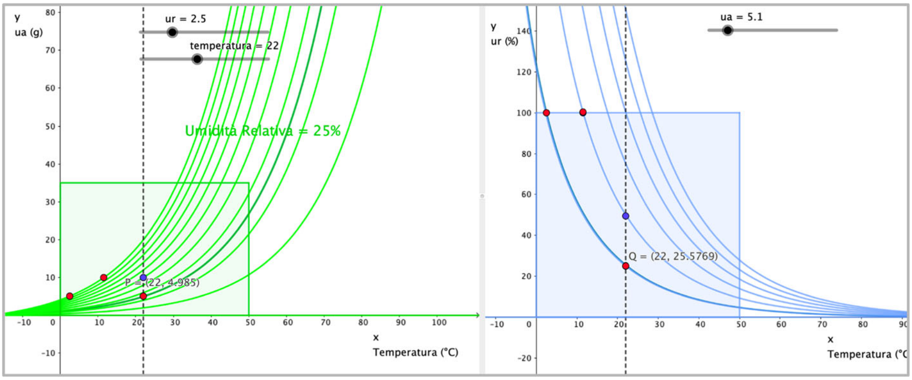


Fig. 7 The interface of the GeoGebra applet showing both diagrams simultaneously, side by side

(Figs. 5, 6), describe the same physical situation from two different mathematical points of view. Eventually, the teacher showed on the interactive whiteboard a new GeoGebra applet (Fig. 7) in which the two representations were displayed side by side.

3.3 Data collection and data analysis

All the activities were recorded through the recording function offered by the Meet platform and, the teacher also positioned by herself some devices within the classroom to record the classroom activities. All the written documents produced by the students were collected and shared on the Google Classroom platform.

The videos of all the activities were watched, and students' written protocols were read in order to identify episodes revealing forms of covariation. Then those episodes were transcribed, translated into English, and analysed with a three-level analysis: on the first level, emerging orders and levels of covariational reasoning were classified according to the enlarged theoretical framework of covariation by describing the mathematical objects involved and the specific features of the representations that prompted covariational reasoning. On the second level, the same episodes were analysed by identifying processes of mathematising, applying, and interpreting. For example, when students elaborated on a mathematical representation of the phenomenon that could be expressed in different registers, graphical, symbolic, or verbal, it was coded as mathematising. When students appealed to their mathematical knowledge for the elaboration of the model (well-known types of functions, their properties, reading and interpretation of graphs, or the use of digital tools), it was coded as applying. Students' ability to explain the mathematical model in light of the phenomenon itself was instead coded as interpreting the mathematical results. Eventually, in the third level of analysis, an overall interpretation of the episodes was elaborated by focusing on the various forms of covariational reasoning and their relevance within the modelling activities, which is the main theoretical contribution of this study.

4 Results

In this section, we will present the analysis of five representative episodes: Episode 1 refers to a selection of students' written protocols of the task administered during phase 1, episode 2 is an excerpt of the collective discussion during phase 3, episode 3 is from the discussion in phase 4, and the other two episodes are from the discussion during phase 6. The lines of the various episodes are progressively numbered. The teacher is denoted with T and the students as Si (e.g., S1 = Student 1).

4.1 Episode 1: COV 1 (gross coordination of values)

Eighteen students (over 22) uploaded on the Classroom platform their answers to the task administered at the end of phase 1: they were asked to describe the relationship between temperature and relative humidity presented in Fig. 3. All the students replied that there is a link or a relationship between temperature and relative humidity even if some of them specified that the relation "varies according to different factors" [1 - S11] or again that they "are not totally connected to each other because even with different temperatures the humidity is equal" [2 - S4]. In Table 1 some representative students' answers are collected. We can observe that despite the numerical values provided by the GeoGebra file, only in [5] we can see an explicit reference to them, while the students mainly elaborated on the temperature-relative humidity relationship in qualitative terms using increase/grows/decrease expressions to describe the direction of change of the two quantities [3-4-6-9]: hence, these forms of covariational reasoning were classified as COV 1 - gross coordination of values. Only a few students tried to describe this relationship globally speaking of inverse proportionality [6] or using a colloquial language, i.e., the two graphs "seem almost mirrored" [7]. Eventually, a few students motivated that relationship with a physical interpretation. Indeed, they refer to the process of evaporation of the water vapour, "when it is warmer the water tends to evaporate more and the air, consequently, to get drier" [8] or "[a]s the temperature increases, the water vapour particles will decrease" [9].

Table 1 Selection of students' answers to Task 1

Number line and Student	Answer to Task 1
[3 – S21]	When the temperature decreases, the relative humidity increases and vice versa
[4 – S4]	If the temperature increases, the humidity decreases, while if the temperature decreases, the humidity increases
[5 – S1]	The piecewise line representing the temperature [red] initially has rather low values, while the blue one [relative humidity] has higher values. Then the piecewise red line begins to grow while the other begins to decrease
[6 – S19]	The data, in the two open piecewise lines, are approximately inversely proportional, when one grows, the other decreases and vice versa
[7 – S15]	The two graphs, after they have been modified to facilitate the reading, seem almost mirrored, that is with the increase of the temperature the relative humidity decreases, and at 17. 2°, the minimum temperature recorded, the highest is relative humidity
[8 – S12]	When the temperature drops, the relative humidity tends to rise and vice versa. This fact has, in my opinion, a simple physical explanation: when it is warmer the water tends to evaporate more and the air, consequently, to get drier; when the temperature is lower, the water vapour present in the air tends not to rise, and the air is consequently wetter
[9 – S16]	As the temperature increases, the water vapour particles will decrease and, conversely, as the temperature decreases, they will be present in the air in greater quantity

Answers from 3 to 7 mainly reveal a mathematising approach, meaning that students made an effort to describe mathematically the trend of the blue and red graph provided in GeoGebra. Answers 8 and 9 show instead an interpreting approach: students tried to explain the trend of the two graphs in light of the physical phenomenon of evaporation.

4.2 Episode 2: Multivariation (qualitative)

This episode is an excerpt from the classroom discussion led by the teacher after the working group session during phase 3. Students, guided by the teacher, retraced the steps of the pot experiment on the psychrometric diagram reproduced in GeoGebra and shown on the interactive whiteboard (Fig. 5). This GeoGebra applet offers the possibility to visualize the whole family of exponential-like functions by moving the appropriate slider (Fig. 8). In the piece of transcript here reported, students were elaborating on the step during which they had already reached the dew point, that is a humidity of 100 %, but they kept on decreasing the temperature adding more ice in the pot and maintaining the percentage of relative humidity constant. The steps of the pot experiment are reproduced in Fig. 8, and the trait students are elaborating on is the oblique one.

10. T: And then? What did we do after we reached the saturation of 100 %? Did we stop immediately? […]
11. S2: No, we waited until it condensed well, and, in the meanwhile, we continued to add ice.
12. T: So, what did you do?
13. S2: I continued to decrease the temperature.
14. T: Hence on the graph, where do you move?

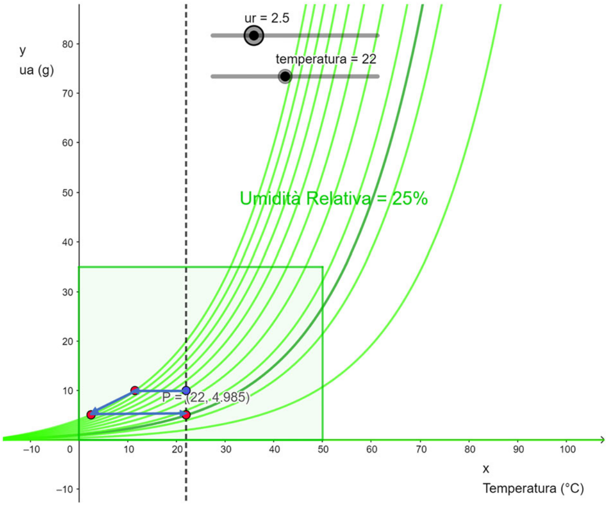


Fig. 8 The steps of the pot experiment on the psychrometric chart. Arrows are drawn for the convenience of the reader

15. S2: To the left.
16. T: To the left. Horizontally?
17. S2: No… [not really convinced]. If you have reached the dew point yes… only the temperature changes.
    [...]
18. S15: If we already have the dew point, [absolute] humidity is decreasing.
19. T: If we already have the dew point, humidity is decreasing. And so?
20. S9: It tends toward the *x*-axis.
21. T: Not only the temperature decreases, and it tends toward the *y*-axis but also toward the *x*-axis. In which way do we move on this graph?
22. S9: Following the curve.

During this discussion, students already had in front of them a graphical representation showing globally how relative humidity affects the trend of the family of functions. The whole episode "is centred around a game of displacement between the graphical representation and the experiment facilitated by the mediation of the teacher that constantly asks the students how they would move on the graph, inviting them to relate to the experiment with the GeoGebra applet" (Bagossi 2022, p. 4232). Specifically, students elaborated on the behaviour of the three quantities at stake, temperature, absolute, and relative humidity, when moving on the psychrometric chart which can be classified as a process of mathematising: first, students elaborated on which and how quantities were changing during the various steps of the pot experiment and then they succeeded in relating a multivariation between three quantities to a movement on the chart. In particular, in this episode, a decrease in temperature [13] and a decrease in absolute humidity [18] at a constant relative humidity, since the dew point has already been reached [18], corresponds to a movement "to the left" [15] and "following the curve" [22]. This episode was coded as revealing a multivariation between the three quantities at stake which is expressed in a qualitative form by making explicit the direction of change of each of the three quantities.

4.3 Episode 3: COV 2 (quantitative)

During phase 4, students were asked to sketch the trend of the graph of relative humidity with respect to temperature as homework. During the classroom discussion, the teacher commented on students' productions. In particular, in this episode, the teacher was showing on the interactive whiteboard the solution elaborated by S21 (Fig. 9a) and asked him to explain how he found it. This episode was selected because S21 was the only student who not only sketched the graph but also looked for a possible algebraic expression.

23. T: What did you do?
24. S21: First I tried to look for a function… I located all the points and then through a system I tried to look for a function passing through all the points, but it came a little higher or a little lower. Then I tried to play with the sliders and… [unclear].
25. T: What kind of function did you think of?
26. S21: I thought that relative humidity was a number *a* over *b* times the temperature plus *c*… […] but it came nothing good… *[The teacher writes the formula on the interactive whiteboard*, Fig. 9b*].*
27. T: In which sense nothing good?
28. S21: The function didn't touch all the points….
29. T: The function didn't touch all the points….
    [...]
30. T: And so, what can we conclude?
31. S21: [unclear]
32. T: The function is not that one or the data do not perfectly fit the function… but it can be that the function probably is not a hyperbole….

Phase 4 was designed to promote applying processes: students were asked to construct a graphical representation of the steps of the temperature versus relative humidity relationship as independent and dependent variable respectively starting from the data collected during the pot experiment. S21 explained that to identify the function shown on the interactive whiteboard (Fig. 9a), S21 located all the points [corresponding to the numerical values obtained during the pot experiment] on the Cartesian plane (by using GeoGebra), and then he tried to look for a function passing for all the points, but the function "came a little higher or a little lower" [24]. S21 also added that then he tried to play with the sliders, associated with the parameters S21 introduced in its analytical representation, but the final part of his sentence is not understandable [24]. Answering the teacher's question [25], S21 explained that he thought of a formula relating relative humidity as the dependent variable and temperature as the independent variable, and such formula contained three parameters named by the letters *a*, *b*, and *c* [26] (Fig. 9b).

In S21's claims, we can detect a form of second-order covariation that is condensed in the elaboration of a mathematical formula containing some non-well-identified parameters. The elaboration of such a parametric formula reveals a mental image in S21's understanding of the quantities' values continuously changing in a way that as the temperature increases, the relative humidity decreases and vice versa and by varying the parameters, the trend of such relationship changes: for this reason, we coded this episode as a form of quantitative COV 2.

To conclude, we remark that, as S21 himself realized, the function he elaborated on was not correct [28]. The teacher clarified the issue by presenting two possibilities for such a result: data are imprecise and do not perfectly fit the function, or the sought function is not a hyperbole [32]. Indeed, students had not yet studied the exponential function by the time of the teaching experiment.

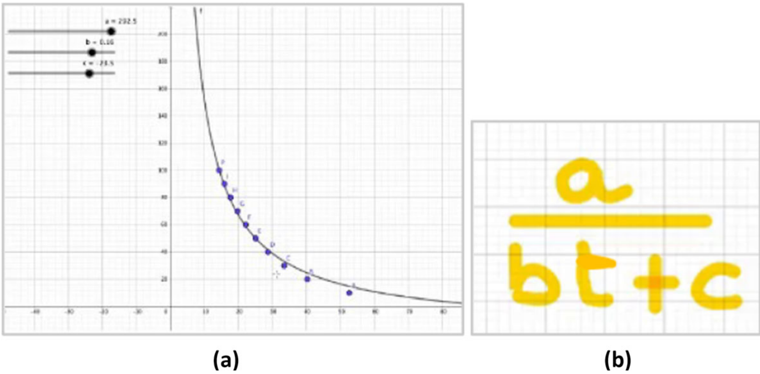


Fig. 9 **a** Tentative relative humidity-temperature graph proposed by S21; **b** Formula written by the teacher on the interactive whiteboard

4.4 Episode 4: Multivariation (qualitative) + COV 2 (qualitative)

The discussion during phase 6 was initially focused on reconstructing the cycle of the pot experiment on the new chart, the one describing the relationship between absolute humidity on the *y*-axis and temperature on the *x*- axis. The teacher showed on the interactive whiteboard one of the graphs made by the students as homework (Fig. 10). In this episode, they were collectively commenting on the second step of the experiment (represented by the yellow trait from P1 to the left), the one during which students continued to add ice cubes so as to decrease the temperature in the pot and then some drops of water formed on the outer wall of the pot.

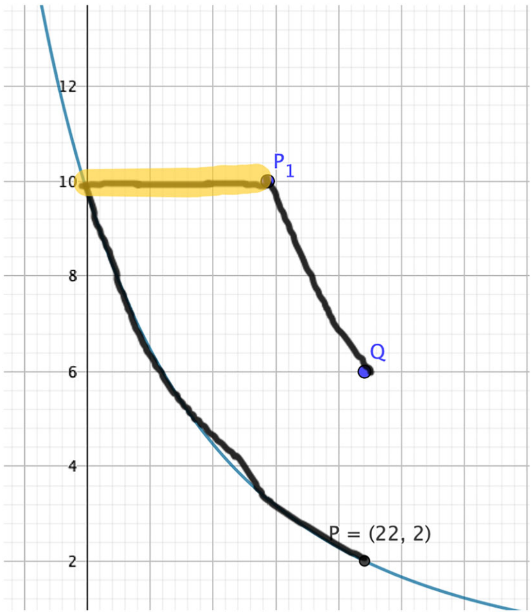


Fig. 10 Relative humidity – temperature graph made by one of the students

33. T: What is happening instead on the horizontal trait [of the graph]?
34. S8: The relative humidity maintains constant.
35. T: The relative humidity maintains constant.
36. S8: And the temperature decreases.
37. T: The temperature decreases. The absolute humidity? Does it decrease or remain constant?
38. S19: Decreases.
39. T: Decreases. Why?
40. S15: You have the condensation.
41. T: Ok, you have the condensation, and this is what happens practically. But on the graph why? […]
42. S19: The curve changes.

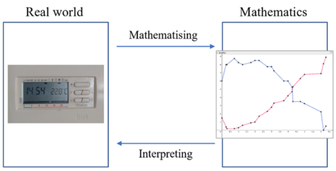
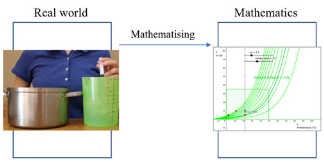
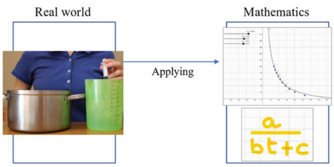
In this episode, students interpreted the representation in the new reference system in light of the pot experiment. In this new representation, what is not represented on the two axes is the absolute humidity which becomes the new parameter. Initially, students conceptualized the variation and the direction of change of each of the three quantities: "relative humidity remains constant" [34], "the temperature decreases" [36] while the absolute humidity "decreases" [38]. This reasoning was coded as a multivariation between three quantities expressed in a qualitative form, analogous to the one identified in episode 2. Then, S19 elaborated on a different form of reasoning: stimulated by the teacher, S19 claimed that a decrease in absolute humidity [38] relates to a change in the curve [42], analogous to the exponential-like curves in the previous psychrometric chart (Fig. 8). This form of reasoning can be classified as a covariation between a quantity (absolute humidity) and a family of functions (relative humidity-temperature curves). The direction of change of the objects involved is outlined: indeed, the relative humidity (parameter) is decreasing while the curve is changing (it can be seen in the graphical representation in Fig. 10) from the one passing through P1 to the other one. Summing up, this form of reasoning was coded as a qualitative form of COV 2 in which the direction of change of two objects was made explicit.

4.5 Episode 5: COV 2 (global)

This last episode is again from the classroom discussion during phase 6, but towards the end of it: after having introduced the new GeoGebra applet shown in Fig. 7, the teacher guided students to reflect on the similarities and differences between the two psychrometric charts.

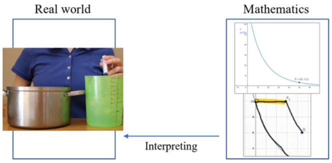
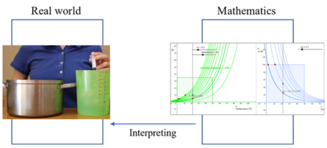
43. T: Are they two different situations/scenarios?
44. S21: No.
45. T: No. Why do the two graphs are different if they are not two different situations?
46. S21: The value represented on the *y*-axis is different.
47. T: The value represented on the *y*-axis is different. If you should make a comparison with something that is not mathematical but concerns real life… We have the same situation/scenario, but the value represented on the *y*-axis is different… If you should make an analogy…?
48. S2: From the physical point of view, they represent the same thing but from the graphical point of view no… because they are two different values.
49. T: Oh! From the physical point of view, they represent the same thing but from the graphical point of view no because they are two different values. […] Two different situations depending on what?
50. S21: A different point of view.

Table 2 Collection of the findings from the five episodes concerning the representations involved, the steps of the modelling process and emerging covariational reasoning

	Representations	Modelling process	Covariational reasoning
Episode 1 (Phase 1)		**Mathematising:** when describing the trend of the blue/red curve. **Interpreting:** when explaining the trend of the two curves by referring to the physical phenomenon of evaporation.	Qualitative coordination of the two quantities' values (**COV 1**) using increase/grow/decrease expressions to describe their direction of change (**gross coordination of values**).
Episode 2 (Phase 3)		**Mathematising** the real experiment: first, students elaborated on which and how quantities were changing during the various steps of the pot experiment. Then they succeeded in relating a multivariation between three quantities to a movement on the chart.	**Multivariation** between the three quantities at stake: it is expressed in a **qualitative** form by making explicit the direction of change of each of the three quantities.
Episode 3 (Phase 4)		**Applying** when elaborating on a mathematical (graphical or algebraic) representation starting from the data collected during the pot experiment.	Second-order covariation (**COV 2**) condensed in the elaboration of a mathematical formula containing some non-well-identified parameters (**quantitative**): such formula, jointly with the graphical representation, reveals an understanding of the quantities as changing continuously.

At this point of the discussion, the teacher projected on the interactive whiteboard the new applet simultaneously showing the relationships absolute humidity versus temperature and relative humidity versus temperature (Fig. 7). At the teacher's question asking if the graphs were two different situations or scenarios [43], first S21 observed that "the value represented on the *y*-axis is different" [46], and then S2 claimed that from the physical standpoint the situation is the same: what differs is the graphical representation [48]. S21 concluded by saying that the difference between the two situations depends on "a different point of view" [50]. This interpreting process of the two mathematical representations suggests that the different role assumed by variables and parameters does not determine a different physical situation but a change of standpoint resulting in a different graphical representation. The

Table 2 (continued)

Episode 4 (Phase 6)		**Interpreting** the graphical representation in the new reference system in light of the pot experiment so as to reconstruct the steps of the cycle of the experiment.	Students conceptualized the simultaneous variation and the direction of change of each of the three quantities: **multivariation** in a **qualitative** form. **COV 2** between a quantity (absolute humidity) and a family of functions (relative humidity-temperature curves). The direction of change of the objects involved is **qualitatively** outlined: indeed, the relative humidity (parameter) is decreasing while the curve is changing.
Episode 5 (Phase 6)		**Interpreting** process of the two mathematical representations suggesting that the different role assumed by variables and parameters does not determine a different physical situation but a change of standpoint when describing it resulting in a different graphical representation.	The mathematical objects involved are two families of functions, both representing the same phenomenon: **COV 2**. Students succeeded in interpreting and relating the two families of functions **globally**.

approach that emerged in this episode can also be intended as a form of conceptualization of the different roles of variables and parameters within mathematical representations. The two psychrometric charts display a form of COV 2: indeed, the mathematical objects involved are two families of functions, produced by the same three quantities, both representing the same phenomenon. Students succeeded in interpreting and relating the two families of functions globally: hence, this form of reasoning was coded as COV 2 global.

The qualitative analysis of the five episodes presented previously is now summed up in Table 2 in order to make more evident students' emerging forms of covariational reasoning in relation to the modelling processes, with particular attention to the real and mathematical representations involved in each episode.

5 Discussion

The research question guiding this study is: *How do students reason covariationally when engaging in real-life phenomena modelling activities?* In order to elaborate on an answer, we qualitatively analysed five episodes of classroom activities from a teaching experiment whose goal was to conceptualize the relationship between three quantities: temperature, absolute humidity, and relative humidity.

First of all, the results summarized in Table 2 make evident that the modelling of such a real-life phenomenon required a continuous moving back and forward from the real to the math world (Blum and Ferri 2009): if the movement from the real to the math world was predominant in the first episodes, then movements in the other direction were predominant in the last phases. These processes were prompted by the design of the tasks themselves, but the students made them concrete and shaped them according to their background knowledge, concerning both mathematics and science. One interesting achievement in students' learning process was the awareness that the connection between the real and the math world is not unique: indeed, as episode 5 showed, several mathematical representations can be adopted to describe the same phenomenon.

All the modelling processes were constantly interlaced with various forms of covariational reasoning, not only of first-order, and the role played by such kind of reasoning differed throughout the teaching experiment. In episode 1, covariational reasoning turned out to be the arrival point in students' mathematising process which enabled them to read and describe the graphs in Fig. 2 conceived as the simultaneous variation of two quantities' values; in episodes 2 and 4, covariational reasoning was used to mediate between the steps of the pot experiment and their representation as a movement on the psychrometric diagram; eventually in episodes 3 and 6, covariational reasoning between quantities worked as a springboard making possible the creation and interpretation of function or family of functions graphically representing the phenomenon. Moreover, covariational reasoning was detected in all three types of modelling processes, mathematising, applying and interpreting, so it seems to be relevant in the overall process and not just in some specific phases.

The variety of the forms of covariational reasoning detected is much more complex than the first-order one which confirms the research assumption that modelling real-life phenomena requires several forms of mathematical reasoning. When mathematising or interpreting the cycle of the pot experiment on the psychrometric chart, students used multivariational reasoning to conceptualize how the three quantities are varying simultaneously (episodes 2 and 4). Then, moving to the conceptualization of the relative humidity-temperature relationship, second-order covariation became more relevant. In episode 3, students applied their mathematical knowledge concerning functional relationship to elaborate on a formula to sketch in GeoGebra the graph touching all points: S21 translated a suitable symbolic relationship into a graphical one and used the sliders to control the parameters introduced in its symbolic representation and to adjust the trend of the function to the data. In episode 4 COV 2 was used to read the cycle of the experiment in the new graphical representation, and in the last episode, it enabled to interpret globally the two graphical representations.

Another aspect of the analysed teaching experiment that deserves to be discussed is the role of the various representations involved in the different phases. First, all the modelling activities were supported by a strong connection with the real world through both the collection of real data presented in phase 1 and the performing of the pot experiment during phase 2. The latter enabled students not only to experience one important step of the experimental scientific method but also to create a connection with the experimentation conducted in the previous school year as mentioned in the Method section. Specifically, the classroom experiment constituted the starting point of all the activities, and it became a solid reference point throughout the whole teaching experiment, both in reading the real psychrometric chart and the graphical representations shown in GeoGebra; it also enabled students to interpret the mathematical representations from a physical standpoint. The representations provided by the digital tools instead, i.e., the GeoGebra applets reproducing the psychrometric diagrams (Figs. 5, 6, 7), facilitated the students' visualization of the whole family of functions at once supporting forms of second-order covariational and multivariational reasoning. Generally, what we observed is that all the chosen representations, both real and digital ones, truly acted as a bridge in the students' conceptualization process between the real and the math world (Galbraith and Stillman 2006).

6 Conclusion

This paper contributes to the literature in Mathematics Education showing the relevance of covariational reasoning to various grade levels and to several topics in all STEM disciplines (Gantt et al. 2023). In particular, this study involved 11th-grade students and focused on the mathematical modelling of the real-life phenomenon of condensation. The findings of the study remark the relevance of forms of covariational reasoning more complex than the one involving two varying quantities. The limitations of the study concern the generalizability of the findings given the small number of participants in the teaching experiment and their previous background about covariation. The results discussed here should be intended as some seeds for further research involving other samples of students coming from different backgrounds.

Even if the findings are discussed from a mathematical point of view, as Moutsios-Rentzos and Kalavasis (2016) observed, "[l]earning mathematics is an inherent interdisciplinary phenomenon that emerges through the collective mind's ability to continuously reflect upon experience, with the purpose for the experience to disappear with a trace that is the initial sketch of the 'mathematical idea'" (p. 114). In the episodes discussed here and offering some highlights into students' learning process, it is possible to grasp how students continuously reflected on the concrete experience provided by the pot experiment to keep in mind which quantities were covarying. But while progressing in the modelling activities, covariation became increasingly detached from the experience and turned into a mathematical idea shaped into related mathematical representations. Research highlights that there are only a few recent studies examining "students' covariational reasoning within the context of science in ways that illustrate the reciprocal relationship between the two disciplines [math and science]" (Panorkou and Germia 2021, p. 320). Hence, considering such kind of modelling activities from an interdisciplinary perspective could be an added educational value.

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Declarations

Conflict of interest The author declares no conflict of interest.

Ethics statement This research was conducted in accordance with the Italian statutory guidelines and requirements concerning scientific research and investigation involving human participants. The school granted approval for this research with approval number 0004849. All participants' parents or legal guardians gave written informed consent to participate in the study and publish its results.

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