Does DC imply $\mathsf{AC}_ω$, uniformly?

DOES DC IMPLY AC_ω, UNIFORMLY?

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Abstract. The Axiom of Dependent Choice DC and the Axiom of Countable Choice AC_ω are two weak forms of the Axiom of Choice that can be stated for a specific set: $\mathsf{DC}(X)$ asserts that any total binary relation on X has an infinite chain, while $\mathsf{AC}_\omega(X)$ asserts that any countable collection of nonempty subsets of X has a choice function. It is well-known that $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$. We study for which sets and under which hypotheses $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$, and then we show it is consistent with ZF that there is a set $A \subseteq \mathbb{R}$ for which $\mathsf{DC}(A)$ holds, but $\mathsf{AC}_\omega(A)$ fails.

1. Introduction

The **Axiom of Choice** AC is the statement $\forall X\, \mathsf{AC}(X)$, where

$$(\mathsf{AC}(X)) \quad X \neq \emptyset \Rightarrow \exists f \colon \mathscr{P}(X) \to X\ \forall A \subseteq X\, (A \neq \emptyset \Rightarrow F(A) \in A).$$

The function f is a choice function for X. Observe that $\mathsf{AC}(X)$ if and only if "X can be well-ordered".

By restricting the choice function we have that $\mathsf{AC}(X) \Rightarrow \mathsf{AC}_I(X)$, where

$(\mathsf{AC}_I(X))$ For any sequence $(A_i)_{i\in I}$ of nonempty subsets of X there is $(a_i)_{i\in I}$ such that $\forall i \in I\, (a_i \in A_i)$.

Of particular interest is the case when $I = \omega$: the **Axiom of Countable Choice** AC_ω is $\forall X\, \mathsf{AC}_\omega(X)$. (In the literature CC is another name for this axiom.)

Let R be a binary relation on a set X.

- An R**-chain** is a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_i \mathrel{R} x_{i+1}$ for all $i \in \omega$. The element x_0 is the starting point of the chain.
- An R**-cycle** is a finite string $x_0, \dots, x_n$ of elements of X such that $x_i \mathrel{R} x_{i+1}$ for all $i < n$ and $x_n \mathrel{R} x_0$.
- R is **total on** X if $\forall x \in X\, \exists y \in X\, x \mathrel{R} y$.

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Any R-cycle yields an R-chain.

The **Axiom of Dependent Choice** DC is $\forall X\, \mathsf{DC}(X)$, where

($\mathsf{DC}(X)$) For any nonempty, total $R \subseteq X^2$ there is $(x_n)_{n\in\omega}$ such that $\forall n \in \omega\, (x_n \mathrel{R} x_{n+1})$.

The axioms DC and AC_ω are ubiquitous in set theory and figure prominently in many areas of mathematics, including analysis and topology. They are probably the most popular weak-forms of the axiom of choice, since they are powerful enough to enable standard mathematical constructions, yet they are weak enough to avoid the pathologies given by AC.

It is well-known that $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$ (Theorem 2.4), so one may ask if this results holds uniformly, that is: does $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$ for all X? This implication holds for many Xs, but in order to prove it in general, $\mathsf{AC}_\omega(\mathbb{R})$ we must be assumed (Theorem 2.8). In Section 4 we will show that the assumption $\mathsf{AC}_\omega(\mathbb{R})$ cannot be dropped, as it is consistent with ZF that there is a set $A \subseteq \mathbb{R}$ for which $\mathsf{DC}(A)$ holds, but $\mathsf{AC}_\omega(A)$ fails (Theorem 4.1). In Section 5 we discuss some complementary results along with the question on the definability of the set constructed in Section 4.

Notation. Our notation is standard, see e.g. [Jec03]. We write $X \precsim Y$ to say that there is an injection from X into Y, and $X \approx Y$ to say that X and Y are in bijection. Ordered pairs are denoted by (a, b), finite sequences are denoted by $\langle a_0, \dots, a_n\rangle$ or by $(a_0, \dots, a_n)$, countable sequences are denoted by $\langle a_n \mid n \in \omega\rangle$ or by $(a_n)_{n\in\omega}$. The **concatenation** of a finite sequence s with a finite/countable sequence t is the finite/countable sequence $s^\frown t$ obtained by listing all elements of s and then all elements of t. The set of all finite (countable) sequences from X is ${}^{<\omega}X$ (respectively: ${}^{\omega}X$). The collection of all finite subsets of a set X is $[X]^{<\omega}$.

If Y is a subset of a topological space X, then $\mathrm{Cl}(Y)$ is its closure, and $\mathrm{Cl}_A(Y) \coloneqq \mathrm{Cl}(Y) \cap A$ is the closure of $Y \cap A$ with respect to $A \subseteq X$.

Following set-theoretic practice, we refer to members of ${}^{\omega}\omega$ or $\mathscr{P}(\omega)$ as "reals", and we effectively identify $\mathbb{R}$ with the Baire space ${}^{\omega}\omega$.

2. Basic constructions

For the reader's convenience let us recall a few notions and results that will be used throughout the paper.

A set X is **finite** if $X \approx n$ for some $n \in \omega$; otherwise it is **infinite**. A set X is **Dedekind-infinite** if $\omega \precsim X$; otherwise it is **Dedekind-finite** or simply **D-finite**. Every finite set is D-finite, and assuming AC_ω the converse holds.

It is consistent with ZF that infinite D-finite sets exist (see Section 3.1). By [Kar19] it is even consistent that every set is the surjective image of a D-finite set.

Let R be a binary relation. With abuse of notation we write

$$R(x) \coloneqq \{y \mid x \mathrel{R} y\}$$

for the set of all ys that are related to x, and

$$R \restriction A \coloneqq R \cap (A \times A)$$

for the restriction of R to the set A. The **transitive closure** of R

$$R^+ \coloneqq \{(x, y) \mid \exists \langle y_0, \dots, y_n \rangle \, (x \mathrel{R} y_0 \mathrel{R} y_1 \mathrel{R} \cdots \mathrel{R} y_n \mathrel{R} y)\}$$

is the smallest transitive relation containing R.

The next few results are folklore.

Proposition 2.1. *Let X be a set.*

(a) *If Y is the surjective image of X, then $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(Y)$.*
(b) *$\mathsf{DC}(X)$ is equivalent to the seemingly stronger statement: For any total $R \subseteq X \times X$ and for any $a \in X$, there is an R-chain starting from a.*
(c) *If $\emptyset \neq A_n \subseteq X$ and $A_n \cap A_m = \emptyset$, then $\mathsf{DC}(X)$ implies that there is a choice function for the A_n's.*
(d) *$\mathsf{DC}(X \times \omega) \Rightarrow \mathsf{AC}_\omega(X)$.*

Proof. (a) Assume $\mathsf{DC}(X)$ and let R be a total relation on Y and let $F \colon X \to Y$ be a surjection. The relation $S = \{(x, x') \in X^2 \mid (F(x), F(x')) \in R\}$ is total on X, so by assumption there is an S-chain $(x_n)_{n\in\omega}$. Then $(F(x_n))_{n\in\omega}$ is an R-chain.

(b) Suppose $R \subseteq X^2$ is total and let $a \in X$. Observe that $S = R \restriction R^+(a)$ is total on $R^+(a)$. By part (a) $\mathsf{DC}(R^+(a))$ holds, hence there is an S-chain $(y_n)_{n\in\omega}$. Let $(x_0, \dots, x_{k+1})$ witness that $y_0 \in R^+(a)$, i.e. $x_0 = a$, $x_{k+1} = y_0$ and $x_i \mathrel{R} x_{i+1}$ for all $i \leq k$: then $(x_0, \dots, x_k)^\frown (y_n)_{n\in\omega}$ is an R-chain starting from a.

(c) Let R be the relation on $\bigcup_n A_n \subseteq X$ defined by

$$x \mathrel{R} y \Leftrightarrow \exists n \in \omega \, (x \in A_n \wedge y \in A_{n+1})$$

By part (a) $\mathsf{DC}(\bigcup_n A_n)$ holds, hence by part (b) there is an R-chain $(x_n)_{n\in\omega}$ in $\bigcup_n A_n$ starting from any $a_0 \in A_0$. Observe that any R-chain $(a_n)_{n\in\omega}$ is such that $a_n \in A_n$ for all $n \in \omega$.

(d) Given $\emptyset \neq A_n \subseteq X$, let $\bar{A}_n = A_n \times \{n\} \subseteq X \times \omega$. By hypothesis and part (c), there is a sequence $(a_n, n)_{n\in\omega}$ such that $(a_n, n) \in \bar{A}_n$, hence $a_n \in A_n$. □

The gist of part (c) of Proposition 2.1 is that we can use dependent choice rather than countable choice whenever the set we choose from are disjoint. Here is an example of such application.

Lemma 2.2. *Suppose X is a first countable space and $a \in \mathrm{Cl}(A) \setminus A$ where $A \subseteq X$. Assume $\mathsf{DC}(A)$ holds. Then there are distinct $a_n \in A$ such that $a_n \to a$. In particular $\omega \precsim A$.*

Proof. Let $\{U_n \mid n \in \omega\}$ be a neighborhood base for a. Then choose $a_n \in (U_n \setminus U_{n+1}) \cap A$—these sets are pairwise disjoint, and by passing to a subsequence, if needed, we may assume they are nonempty. □

Lemma 2.3. *Let X be a set.*

(a) $X \times 2 \precsim X \Rightarrow X \times \omega \precsim X$.
(b) *If $X \neq \emptyset$ then* ${}^{<\omega}({}^{<\omega}X) \precsim {}^{<\omega}X$, *so* ${}^{<\omega}X \times 2 \precsim {}^{<\omega}X$.
(c) $\forall X \, \exists Y \, (X \subseteq Y \wedge {}^{<\omega}Y \precsim Y)$.

Proof. (a) If $f_0, f_1 \colon X \to X$ are injections with $\operatorname{ran}(f_0) \cap \operatorname{ran}(f_1) = \emptyset$, then define an injection $F \colon X \times \omega \to X$ as follows:

$$F(x,0) = f_0(x), \qquad F(x, n+1) = \underbrace{f_1 \circ \cdots \circ f_1}_{n+1 \text{ times}} \circ f_0(x).$$

(b) If X is a singleton, then ${}^{<\omega}X \approx \omega$, and the result follows at once. If X has at least two elements, the result follows from [AMR22, Proposition 2.1].

(c) Given X take $Y = \mathrm{V}_\lambda$ with sufficiently large limit λ. □

From Lemma 2.3 and Proposition 2.1(d) we obtain at once:

Theorem 2.4. (a) *If $X \times 2 \precsim X$ then $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$. In particular: $\mathsf{DC}(\mathbb{R}) \Rightarrow \mathsf{AC}_\omega(\mathbb{R})$.*
(b) $\forall X \, \exists Y \, (X \subseteq Y \wedge (\mathsf{DC}(Y) \Rightarrow \mathsf{AC}_\omega(Y)))$.
(c) $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$.

Lemma 2.5. (a) *If $A \subseteq \mathbb{R}$ and $\mathsf{AC}_\omega(A)$ holds, then A is separable.*
(b) $\mathsf{AC}_\omega(\mathbb{R}) \Leftrightarrow \forall A \subseteq \mathbb{R} \, (A$ *is separable*$)$.
(c) *Suppose $A \subseteq \mathbb{R}$ contains a nonempty perfect set, and assume $\mathsf{DC}(A)$. Then $\mathsf{DC}(\mathbb{R})$ holds, and hence $\mathsf{AC}_\omega(A)$ holds.*

Proof. As A is second countable, part (a) of Lemma 2.5 follows.

(b) The direction $(\Rightarrow)$ is a direct consequence of part (a). For the other direction, fix a sequence $(A_n)_{n\in\omega}$ of nonempty subsets of $\mathbb{R}$ and consider the set $A = \{\langle n \rangle {}^\frown x \mid n \in \omega \text{ and } x \in A_n\}$. From an enumeration of a dense subset of A (which exists by assumption) we can extract a choice function for $(A_n)_{n\in\omega}$.

(c) If $P \subseteq A$ is perfect, then $P \approx \mathbb{R}$, and since A surjects onto P, then $\mathsf{DC}(\mathbb{R})$ holds, and hence $\mathsf{AC}_\omega(\mathbb{R})$ holds. □

Note that the implication in part (a) of Lemma 2.5 cannot be reversed: if $A \subseteq \mathbb{R}$ is a witness of the failure of countable choice, then the same is true of the separable set $A \cup \mathbb{Q}$.

2.1. $\mathsf{AC}_\omega(X)$ **follows from** $\mathsf{DC}(X)$ **together with** $\mathsf{AC}_\omega(\mathbb{R})$**.** Let us start with the following combinatorial result that might be of independent interest. It is stated for families of sets indexed by an arbitrary set I, but when $I = \omega$ the assumption $\mathsf{AC}_I(\mathscr{P}(I))$ becomes $\mathsf{AC}_\omega(\mathbb{R})$.

Lemma 2.6. *Let $(X_i)_{i\in I}$ be nonempty sets, and assume $\mathsf{AC}_I(\mathscr{P}(I))$. Then there are $(Y_i)_{i\in I}$ such that $\emptyset \neq Y_i \subseteq X_i$ and for all $i, j \in I$ either $Y_i = Y_j$ or else $Y_i \cap Y_j = \emptyset$.*

Proof. Let $F\colon \bigcup_{i\in I} X_i \to \mathscr{P}(I)$, $F(x) = \{i \in I \mid x \in X_i\}$ and let $A_i = \{a \in \operatorname{ran}(F) \mid i \in a\}$. Observe that for all $x \in X$ and all $i \in I$

$$x \in X_i \Leftrightarrow F(x) \in A_i. \tag{1}$$

In particular, $\emptyset \neq A_i \subseteq \mathscr{P}(I)$ for all $i \in I$. By $\mathsf{AC}_I(\mathscr{P}(I))$ pick $a_i \in A_i$, and let $Y_i = F^{-1}(\{a_i\}) \subseteq X$. Then

$$Y_i = \{x \mid F(x) = a_i\} = \{x \mid \{j \mid x \in X_j\} = a_i\}$$

and since $i \in a_i$, then $Y_i \subseteq X_i$. The sets Y_i need not be distinct as the a_is need not be distinct, but if $a_i \neq a_i$ then $Y_i \cap Y_j = \emptyset$. □

By (1) if the X_is are finite, then so are the A_is. If $\mathscr{P}(I)$ is linearly orderable (e.g. when I is well-orderable), the a_is can be chosen without appealing to any axiom. Therefore:

Corollary 2.7. *If $\mathscr{P}(I)$ is linearly orderable and $(X_i)_{i\in I}$ are finite, nonempty sets, then there are $\emptyset \neq Y_i \subseteq X_i$ such that for all $i, j \in I$ either $Y_i = Y_j$ or else $Y_i \cap Y_j = \emptyset$.*

Theorem 2.8. *Assume $\mathsf{AC}_\omega(\mathbb{R})$, then $\forall X\,(\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X))$.*

Proof. Assume $\mathsf{DC}(X)$ and let $\emptyset \neq X_n \subseteq X$ for $n \in \omega$. By Lemma 2.6 there are $\emptyset \neq Y_n \subseteq X_n$ such that for all $n, m \in \omega$ either $Y_n = Y_m$ or else $Y_n \cap Y_m = \emptyset$. Let $I \subseteq \omega$ be such that $\{Y_i \mid i \in I\} = \{Y_n \mid n \in \omega\}$ and $Y_i \cap Y_j = \emptyset$ for every distinct $i, j \in I$. If we can find $y_i \in Y_i$ for all $i \in I$, then we can extend this to a choice sequence $y_n \in Y_n \subseteq X_n$ for all $n \in \omega$ as required. If I is finite, the y_is can be found without any appeal to choice. If I is infinite, then $I \approx \omega$ so we can find the y_is by part (c) of Proposition 2.1. □

The following follows from the argument of Theorem 2.8 together with Corollary 2.7.

Corollary 2.9. $\forall X\,(\mathsf{DC}(X) \Rightarrow \mathsf{AC}^{<\omega}_\omega(X))$, *where $\mathsf{AC}^{<\omega}_\omega(X)$ asserts that every countable collection of nonempty finite subsets of X has a choice function.*

2.2. **Does $\mathsf{DC}(X)$ imply $\mathsf{AC}_\omega(X)$?** By Theorem 2.4 and Theorem 2.8

$$\forall X\,(\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)) \tag{2}$$

follows from either one of the following assumptions:

- $X \times 2 \precsim X$ for all infinite X,
- $\mathsf{AC}_\omega(\mathbb{R})$.

Sageev in [Sag75] proved that "$X \times 2 \precsim X$ for all infinite X" does not imply $\mathsf{AC}_\omega(\mathbb{R})$, while Monro in [Mon74] proved that DC (and hence the weaker $\mathsf{AC}_\omega(\mathbb{R})$) does not imply "$X \times 2 \precsim X$ for all infinite X". So neither assumption implies the other.

The obvious question is if (2) is a theorem of ZF. Suppose that there is a set X such that $\mathsf{DC}(X) \wedge \neg \mathsf{AC}_\omega(X)$. By the proof of Lemma 2.6 the set $A \coloneqq F[X] \subseteq \mathscr{P}(\omega)$ is such that $\mathsf{DC}(A)$ holds, as A is the surjective image of X, and $\mathsf{AC}_\omega(A)$ fails, as otherwise, arguing as in Theorem 2.8, $\mathsf{AC}_\omega(X)$ would hold. Therefore if (2) fails, then the witness of this failure can be taken to be a subset of $\mathbb{R}$. In Section 4 we construct a model of ZF in which

$$\exists A \subseteq \mathbb{R}\ (\mathsf{DC}(A) \wedge \neg \mathsf{AC}_\omega(A)) \tag{3}$$

showing that (2) is not a theorem of ZF. By Lemma 2.5 any A as in (3) is neither D-finite, nor it contains a perfect set. It can be shown that (3) fails both in Cohen's first model (Proposition 3.4) and in the Feferman-Levy model (Proposition 5.3), and hence in both these models (2) holds.

2.3. **An equivalent formulation of DC.** A **tree on** X is a $T \subseteq {}^{<\omega}X$ that is closed under initial segments, that is if $t \in T$ and $s \subseteq t$ then $s \in T$. A tree T on X is **pruned** if for every $t \in T$ there is $s \in T$ such that $t \subset s$. A **branch** of T is a $b \colon \omega \to X$ such that $\forall n \in \omega\, (b \restriction n \in T)$. A tree T it is **ill-founded** if it has a branch, otherwise it is **well-founded**. Let

($\mathsf{DC}_\omega(X)$) Any nonempty pruned tree on X is ill-founded

and let DC_ω be $\forall X\, \mathsf{DC}_\omega(X)$. As DC is equivalent to DC_ω (Corollary 2.11 below) the axiom of Dependent Choice is often is stated as DC_ω. The advantage of this formulation is that it can be generalized to ordinals larger than ω.

Proposition 2.10. $\mathsf{DC}_\omega(X) \Leftrightarrow \mathsf{DC}({}^{<\omega}X)$, *for every nonempty set* X.

Proof. ($\Rightarrow$) Suppose R is a binary relation on ${}^{<\omega}X$ such that $\forall s\, \exists t\, (s\ R\ t)$. If $\emptyset\ R\ \emptyset$, then $\langle \emptyset, \emptyset, \ldots \rangle$ is an R-chain as required, so we may assume otherwise. Let $R' \subseteq R$ be the sub-relation on ${}^{<\omega}X$ obtained by choosing the shortest possible t', that is

$$s\ R'\ t \Leftrightarrow s\ R\ t \wedge \forall t' \subset t\, \neg(s\ R\ t').$$

The relation R' is total and any R'-chain is an R-chain. Then

$$T = \{t \in {}^{<\omega}X \mid \exists s_0, \ldots, s_n\, (\emptyset\ R'\ s_0\ R' \ldots R'\ s_n\ \wedge\ t \subseteq s_1{}^\frown \ldots {}^\frown s_n)\}$$

is a pruned tree on X, so it has a branch. By the minimality assumption of R', given a branch b of T one can construct inductively an R'-chain $(s_n)_n$ such that $s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_n \subseteq b$ for all n.

($\Leftarrow$) If T is a pruned tree on X, let $R \subseteq T \times T$ be defined by

$$s\ R\ t \Leftrightarrow s \subset t \wedge \operatorname{lh}(s) + 1 = \operatorname{lh}(t).$$

As $T \subseteq {}^{<\omega}X$ then $\mathsf{DC}(T)$ holds, and since R is total, as T is pruned, there is an R-chain. Any such chain yields a branch of T. $\square$

Corollary 2.11. $\mathsf{DC} \Leftrightarrow \mathsf{DC}_\omega$.

Proposition 2.12. *Let X be a set.*

(a) $\mathsf{DC}_\omega(X) \Rightarrow \mathsf{DC}(X)$.
(b) $\mathsf{DC}_\omega(X) \Rightarrow \mathsf{AC}_\omega(X)$.

Proof. X injects into ${}^{<\omega}X$, so part (a) holds by Proposition 2.10.

For part (b) argue as follows. If $\emptyset \neq A_n \subseteq X$, then $\{\langle x_0, \dots, x_n\rangle \mid \forall i \leq n\,(x_i \in A_i)\}$ is a pruned tree on X, and any branch of it is a sequence $(a_n)_n$ such that $a_n \in A_n$ for all $n \in \omega$. $\square$

In light of Proposition 2.12, our main result, Theorem 4.1, tell us it is consistent with ZF that there is a set $A \subseteq \mathbb{R}$ for which $\mathsf{DC}(A)$ holds but $\mathsf{DC}_\omega(A)$ fails.

3. Symmetric extensions

The model we construct in Section 4 is an iterated symmetric extension. For the reader's convenience, lets us recall a few facts about forcing and symmetric extensions.

If $\mathbf{P}$ is a forcing notion, i.e. a preordered set with a maximum $1_\mathbf{P}$ we convene that $p \leq_\mathbf{P} q$ means that p is **stronger** than q. (When there is no danger of confusion we drop the subscript $\mathbf{P}$.) Dotted letters line $\dot{x}, \dot{y}, \dots$ vary over the class of $\mathbf{P}$-names, while $\check{x}$ is the canonical $\mathbf{P}$-name for x, while $\dot{G}$ is the $\mathbf{P}$-name for the generic filter. If F is a set of $\mathbf{P}$-names, then $F^\bullet$ is the $\mathbf{P}$-name $\{(\dot{x}, 1) \mid \dot{x} \in F\}$. If $G \subseteq \mathbf{P}$ is V-generic, then $\dot{x}_G$ is the object in $\mathrm{V}[G]$ obtained by evaluating $\dot{x}$ with G.

Let $\mathbf{P}$ be a forcing notion. Every automorphism $\pi \in \mathrm{Aut}(\mathbf{P})$ acts canonically on $\mathbf{P}$-names as follows: given $\dot{x}$ a $\mathbf{P}$-name,

$$\pi\dot{x} = \{(\pi\dot{y}, \pi p) \mid (\dot{y}, p) \in \dot{x}\}.$$

Lemma 3.1 (Symmetry Lemma, [Jec03, Lemma 14.37])**.** *Let $\mathbf{P}$ be a forcing notion, $\pi \in \mathrm{Aut}(\mathbf{P})$ and $\dot{x}_1, \dots, \dot{x}_n$ be $\mathbf{P}$-names. For every formula $\varphi(x_1, \dots, x_n)$*

$$p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \Leftrightarrow \pi p \Vdash \varphi(\pi\dot{x}_1, \dots, \pi\dot{x}_n).$$

Let $\mathcal{G}$ be a subgroup of $\mathrm{Aut}(\mathbf{P})$. A nonempty collection $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a **filter** on $\mathcal{G}$ if it is closed under supergroups and finite intersections. A filter $\mathcal{F}$ on $\mathcal{G}$ is said to be **normal** if for every $H \in \mathcal{F}$ and $\pi \in \mathcal{G}$, the conjugated subgroup $\pi H \pi^{-1}$ belongs to $\mathcal{F}$ as well.

We say that the triple $\langle \mathbf{P}, \mathcal{G}, \mathcal{F}\rangle$ is a **symmetric system** if $\mathbf{P}$ is a forcing notion, $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(\mathbf{P})$ and $\mathcal{F}$ is a normal filter on $\mathcal{G}$. Given a $\mathbf{P}$-name $\dot{x}$, we say that $\dot{x}$ is **$\mathcal{F}$-symmetric** if there exists $H \in \mathcal{F}$ such that for all $\pi \in H$, $\pi\dot{x} = \dot{x}$. This definition extends by recursion: $\dot{x}$ is **hereditarily $\mathcal{F}$-symmetric**, if $\dot{x}$ is $\mathcal{F}$-symmetric and every name $\dot{y} \in \operatorname{dom}(\dot{x})$ is hereditarily $\mathcal{F}$-symmetric. We denote by $\mathsf{HS}_{\mathcal{F}}$ the class of all hereditarily $\mathcal{F}$-symmetric names.

Theorem 3.2 ([Jec03, Lemma 15.51])**.** *Suppose that $\langle \mathbf{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system and $G \subseteq \mathbf{P}$ is a* V*-generic filter. Denote by $\mathcal{N}$ the class $\{\dot{x}_G \mid \dot{x} \in \mathsf{HS}_{\mathcal{F}}\}$, then $\mathcal{N}$ is a transitive model of* ZF*, and* $\mathrm{V} \subseteq \mathcal{N} \subseteq \mathrm{V}[G]$.

The class $\mathcal{N}$ is also known as a **symmetric extension** of V. Symmetric extensions are often used to produce models of ZF in which the axiom of choice fails. We next practise with this notion by discussing the construction due to Cohen of a symmetric extension in which there exists an infinite, D-finite set of reals. This model will be the first step of the iteration in our main construction (Theorem 4.1).

3.1. **The first Cohen model.** Let $\mathbf{P}$ be the forcing that adds countably many Cohen reals, i.e.

$$\mathbf{P} = \{p :\subset \omega \to {}^{<\omega}2 \mid \operatorname{dom}(p) \text{ is finite}\},$$

with $p \leq q$ if $\operatorname{dom}(p) \supseteq \operatorname{dom}(q)$ and $p(n) \supseteq q(n)$ for all $n \in \operatorname{dom}(q)$. Although this is not the standard presentation of such a forcing, this way of defining $\mathbf{P}$ will come useful in the Section 4. Let $\dot{a}_n$ be the canonical name for the n-th Cohen real, that is

$$\dot{a}_n = \{((\check{k}, i), p) \mid p \in \mathbf{P} \wedge n \in \operatorname{dom} p \wedge p(n)(k) = i\}.$$

Observe that $\dot{A} \coloneqq \{\dot{a}_n \mid n \in \omega\}^\bullet$ is forced to be a dense subset of ${}^{\omega}2$.

Every permutation π on ω induces an automorphism of $\mathbf{P}$ as follows: given $p \in \mathbf{P}$, we let $\pi p \in \mathbf{P}$ be defined by

$$\forall n \in \operatorname{dom}(p) \big(\pi p(\pi n) = p(n)\big).$$

We conflate the notation by using the same symbol π to denote both the permutation and the automorphism on $\mathbf{P}$ it induces. Let $\mathcal{G}$ be the group of all such automorphisms. For every finite $E \subset \omega$, let $\operatorname{Fix}(E)$ be the subgroup of $\mathcal{G}$ of all those automorphisms induced by permutations that pointwise fix the set E. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by $\{\operatorname{Fix}(E) \mid E \subset \omega \text{ finite}\}$. It is easy to check that $\mathcal{F}$ is actually a normal filter on $\mathcal{G}$, hence $\langle \mathbf{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system. Let G be a V-generic filter and let $\mathcal{N}_0$ be the corresponding symmetric extension, which we call **first Cohen model**.

Denote by A the realization of the name $\dot{A}$ in $\mathrm{V}[G]$, i.e. the set $\dot{A}_G$. Note that every $\dot{a}_n$ is in $\mathsf{HS}_{\mathcal{F}}$ and so is $\dot{A}$.

Proposition 3.3 ([Jec03, Example 15.52])**.** $\mathcal{N}_0 \vDash$ "*A is D-finite*".

The set A, being infinite and D-finite, it is certainly not separable as a subspace of $\mathbb{R}$ (indeed every infinite, separable T_1 space is Dedekind-infinite), and $\mathsf{DC}(A)$ also fails (see Lemma 2.2). The simultaneous local failure of both AC_ω and DC is not accidental, as the next proposition shows that the first Cohen model satisfies (2) and even more.

Proposition 3.4. $\mathcal{N}_0 \vDash \forall X\, (\mathsf{DC}(X) \Rightarrow \mathsf{AC}(X))$.

Lemma 3.5. *Let X be a linearly ordered set, and let $Y \subseteq [X]^{<\omega}$. If $\omega \precsim Y$, then $\omega \precsim \bigcup Y$.*

Proof. Let $\leq$ be a linear ordering of X, and let $(A_n)_{n\in\omega}$ be a sequence of distinct elements of Y. By passing to a subsequence we may assume that $A_{n+1} \nsubseteq A_0 \cup \cdots \cup A_n$, and that $A_0 \neq \emptyset$. Let x_0 be the least element of A_0, and x_{n+1} be the least element of $A_{n+1} \setminus (A_0 \cup \cdots \cup A_n)$. The x_ns are distinct, and belong to X, as required. □

Lemma 3.6. *If $\mathsf{DC}(Y)$ with $Y \subseteq [\mathbb{R}]^{<\omega}$ infinite, then $\omega \precsim \bigcup Y$.*

Proof. It is enough to show that $\omega \precsim Y$ and then apply Lemma 3.5 with $X = \mathbb{R}$. If $\bigcup Y$ has no limit points, then it is discrete, so $\omega \precsim Y$. Now suppose otherwise, and let $x \in \mathbb{R}$ be a limit point of $\bigcup Y$. Without loss of generality we may assume that $\{x\}, \emptyset \notin Y$. For all $A \in Y$ let $d(x, A) = \min\{|r - x| \mid r \in A \setminus \{x\}\}$ be the distance of x from the rest of A. Let $R \subseteq Y^2$ be the binary relation defined as follows: for every $A, B \in Y$,

$$R(A, B) \Leftrightarrow d(x, B) < d(x, A).$$

The relation R is acyclic and, by our hypothesis on x, it is total. If follows from $\mathsf{DC}(Y)$ that R has an infinite chain, and hence $\omega \precsim Y$. □

Proof of Proposition 3.4. In the first Cohen model, for every set X there is a map $s_X \colon X \to [A]^{<\omega}$, known as the least support map, such that $s^{-1}(\{B\})$ is well-orderable for every $B \in [A]^{<\omega}$ [Jec73, Theorem 5.21, Exercise 5.22].

Let $X \in \mathcal{N}_0$ be such that $\mathsf{DC}(X)$ holds. Then also $\mathsf{DC}(\operatorname{ran}(s_X))$ holds. If $\operatorname{ran}(s_X)$ were infinite then letting $Y = \operatorname{ran}(s_X)$ in Lemma 3.6 we have that $\omega \precsim \bigcup \operatorname{ran}(s_X) \subseteq A$, against the fact that A is D-finite. Hence $\operatorname{ran}(s_X)$ is finite, and X, being a finite union of well-orderable sets, is well-orderable. □

4. The main result

This section is devoted to proving the following:

Theorem 4.1. *It is consistent with ZF that there is a set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ and $\neg\mathsf{AC}_\omega(A)$.*

4.1. **Outline of the proof.** We prove the theorem via an iteration of symmetric extensions of length ω. We start the iteration with the first Cohen model $\mathcal{N}_0$, with $A \in \mathcal{N}_0$ being the generic D-finite set of reals (see Section 3.1). As already noted, in this model A is not separable (in particular $\mathsf{AC}_\omega(A)$ fails) and also $\mathsf{DC}(A)$ fails. Next, we define a chain of models $\mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_\omega$ such that, for each n, $\mathcal{N}_{n+1}$ is a symmetric extension of $\mathcal{N}_n$ that contains a generic set of chains for all binary relation in $\mathcal{N}_n$ that are total and acyclic on A. At the final stage, $\mathcal{N}_\omega$, which is our model, is going to be something resembling to "the model of sets definable from finitely many elements of $\bigcup_n \mathcal{N}_n$". If we do the construction properly, we can prove that in $\mathcal{N}_\omega$ we've added enough countable subsets of A (or, equivalently, enough sequences over A) to guarantee $\mathsf{DC}(A)$ (Theorem 4.10), but A is still not separable, in particular $\mathsf{AC}_\omega(A)$ fails (Corollary 4.8).

Actually, we don't only show that A is not separable in our model, but we give a topological characterization of its separable subsets: among the subsets of A, the separable ones are precisely those which are scattered with finite scattered height (Definition 4.5, Theorem 4.7).

4.2. **The symmetric system.** We start by defining recursively a sequence $\langle \mathbf{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle_{n\in\omega}$ of symmetric systems. Let $\langle \mathbf{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle$ be the symmetric system defined in Section 3.1, i.e. the one that induces the first Cohen model. For each n we denote by $\leq_n, \Vdash_n$ the ordering and the forcing relation of $\mathbf{P}_n$, respectively, and by HS_n the class $\mathsf{HS}_{\mathcal{F}_n}$, i.e. the class of all hereditarily $\mathcal{F}_n$-symmetric $\mathbf{P}_n$-names. We also let

$$\mathcal{R}_n = \{\dot{R} \in \mathsf{HS}_n \mid \forall \dot{x} \in \mathrm{dom}(\dot{R})\, \exists n, m \in \omega\, (\dot{x} = (\dot{a}_n, \dot{a}_m)^\bullet)\},$$

so that $\mathcal{R}_n$ is the set of all "good" hereditarily $\mathcal{F}_n$-symmetric $\mathbf{P}_n$-names for binary relations on $\dot{A}$.

Recursively on n, we define $\mathbf{P}_{n+1}$ to be the set of all the sequences $p = \langle p_k \mid k \leq n+1 \rangle$ such that

(1) $p \restriction n+1 \in \mathbf{P}_n$,
(2) $p_{n+1} \colon \mathrm{dom}(p_{n+1}) \to \mathcal{R}_n \times {}^{<\omega}\omega$ with $\mathrm{dom}(p_{n+1})$ a finite subset of ω,
(3) For each $k \in \mathrm{dom}(p_{n+1})$ with $p_{n+1}(k) = (\dot{R}, \langle n_0, \dots, n_l \rangle)$ we have $p \restriction n+1 \Vdash_n$ "$\dot{R}$ is total, acyclic and $\dot{a}_{n_0}\ \dot{R}\ \dot{a}_{n_1}\ \dot{R}\ \dots\ \dot{R}\ \dot{a}_{n_l}$",

where, at stage $n = 0$, we identify the conditions $p \in \mathbf{P}_0$ with their singleton sequence $\langle p \rangle$.

For each $p \in \mathbf{P}_{n+1}$ and $k \in \mathrm{dom}(p_{n+1})$ with $p_{n+1}(k) = (\dot{R}, s)$, we denote $\dot{R}$ and s by $p^R_{n+1}(k)$ and $p^s_{n+1}(k)$, respectively. Given $p, q \in \mathbf{P}_{n+1}$ we let $p \leq_{n+1} q$ if and only if

- $p \restriction n+1 \leq_n q \restriction n+1$,
- $\mathrm{dom}(p_{n+1}) \supseteq \mathrm{dom}(q_{n+1})$,
- $\forall k \in \mathrm{dom}(q_{n+1})\, (p^R_{n+1}(k) = q^R_{n+1}(k)$ and $p^s_{n+1}(k) \supseteq q^s_{n+1}(k))$.

This defines the forcing $\mathbf{P}_{n+1}$. Now we are left to define the subgroup $\mathcal{G}_{n+1}$ of $\mathrm{Aut}(\mathbf{P}_{n+1})$.

Consider a sequence $\vec{\pi} = \langle \pi_0, \dots, \pi_{n+1} \rangle$ with each π_i being a permutation of ω. By induction hypothesis[1] $\vec{\pi} \restriction n+1$ induces an automorphism $\vec{\pi} \restriction n+1 \in \mathcal{G}_n$. Note that, as in Section 3.1, we conflate the notation by using the same symbol to denote both sequences of permutations and the automorphisms they induce. Now, the sequence $\vec{\pi}$ induces an automorphism on $\mathbf{P}_{n+1}$ as follows: given $p \in \mathbf{P}_{n+1}$, we let $\vec{\pi}(p)$ be the condition in $\mathbf{P}_{n+1}$ such that $\vec{\pi}(p) \restriction n+1 = \vec{\pi} \restriction n+1(p \restriction n+1)$ and, for each $k \in \mathrm{dom}(p_{n+1})$ with $p^s_{n+1}(k) = \langle n_0, \dots, n_l \rangle$ and $p^R_{n+1}(k) = \dot{R}$,

$$\begin{aligned} \vec{\pi}(p)^R_{n+1}(\pi_{n+1}(k)) &= \vec{\pi} \restriction n+1(\dot{R}), \\ \vec{\pi}(p)^s_{n+1}(\pi_{n+1}(k)) &= \langle \pi_0(n_0), \dots, \pi_0(n_l) \rangle. \end{aligned}$$

Let $\mathcal{G}_{n+1}$ be the group of all such automorphisms on $\mathbf{P}_{n+1}$, i.e. the ones induced by sequences (of length $n+2$) of permutations of ω. For each sequence $\vec{H} = \langle H_0, \dots, H_{n+1} \rangle$ of subsets of ω, we let $\mathrm{Fix}(\vec{H})$ be the subgroup of all those $\vec{\pi} \in \mathcal{G}_{n+1}$ such that π_k pointwise fixes H_k for all $k \leq n+1$. We define $\mathcal{F}_{n+1}$ be the filter on $\mathcal{G}_{n+1}$ generated by $\{\mathrm{Fix}(\vec{H}) \mid H_k \text{ is finite for all } k \leq n+1\}$. From now on we use the symbol $\vec{H}$ to denote finite sequences of **finite** subsets of ω.

This ends the inductive definition of the sequence $\langle \mathbf{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle_{n \in \omega}$. Note that, for each $n < m$, there is a natural complete embedding $i_{n,m} \colon \mathbf{P}_n \to \mathbf{P}_m$ and a natural embedding $j_{n,m} \colon \mathcal{G}_n \to \mathcal{G}_m$. Thus we let $\mathbf{P}$ and $\mathcal{G}$ be the direct limits of the forcings $\mathbf{P}_n$ and of the groups $\mathcal{G}_n$, respectively. We now define the normal filter $\mathcal{F}$ on $\mathcal{G}$ in the expected way: we let $\mathcal{F}$ be the filter generated by

$$\{\mathrm{Fix}(\vec{H}) \mid H_k \text{ is finite for all } k < \mathrm{lh}(\vec{H})\},$$

where, given any $\vec{H}$ finite sequence of subsets of ω, $\mathrm{Fix}(\vec{H})$ is the subgroup of $\mathcal{G}$ made of all those $\vec{\pi}$ such that π_k pointwise fixes H_k for all $k < \mathrm{lh}(\vec{H})$.

Henceforth $\langle \mathbf{P}, \mathcal{G}, \mathcal{F} \rangle$ is our symmetric system, with HS being the class of all $\mathcal{F}$-symmetric $\mathbf{P}$-names and $\leq, \Vdash$ being the ordering and the forcing relation of $\mathbf{P}$, respectively.

Remark 4.2. Our iterative construction fits into the general framework developed by Asaf Karagila [Kar19] to deal with iterations of symmetric extensions.

For each $n, k \in \omega$, we let

$$\begin{aligned} \dot{f}_{n.k} &= \{((\check{l}, \dot{a}_m)^\bullet, p) \mid l, m \in \omega, p \in \mathbf{P}, p^s_{n+1}(k)(l) = m\}, \\ \dot{F}_n &= \{\dot{f}_{n,k} \mid k \in \omega\}^\bullet, \\ \dot{F} &= \{\dot{F}_n \mid n \in \omega\}^\bullet. \end{aligned}$$

Note that all names defined so far are all in HS. Given a $\dot{x} \in \mathsf{HS}$ we say that $\vec{H}$ is a **support** of $\dot{x}$ if $\vec{\pi}(\dot{x}) = \dot{x}$ for all $\vec{\pi} \in \mathrm{Fix}(\vec{H})$. Also, given

[1] At $n = 0$ we identify each $\pi \in \mathcal{G}_0$ with the singleton sequence $\langle \pi \rangle$.

$p = \langle p_0, \dots, p_n \rangle \in \mathbf{P}$ and $\vec{H} = \langle H_0, \dots, H_n \rangle$ we write $p \restriction \vec{H}$ to denote the sequence $\langle p_0 \restriction H_0, \dots, p_n \restriction H_n \rangle$. Note that the latter sequence needs not, in general, belong to $\mathbf{P}$.

Lemma 4.3 (Restriction Lemma)**.** *Let $\varphi(x_1, \dots, x_n)$ be a formula in the forcing language, and let $\dot{x}_1, \dots, \dot{x}_n \in \mathsf{HS}$. For any $p \in \mathbf{P}$ and for any $\vec{H}$, if $\vec{H}$ is a support for each of the $\dot{x}_i$'s and, for all $m > 0$, for all $k \in H_m \cap \mathrm{dom}(p_m)$, $\vec{H} \restriction m$ is a support for $p^R_m(k)$ and $\mathrm{ran}(p^s_m(k)) \subseteq H_0$, then $p \restriction \vec{H} \in \mathbf{P}$ and*

$$p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \Leftrightarrow p \restriction \vec{H} \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n).$$

Proof. We prove the lemma by induction on the length of $\vec{H}$.

Let's first assume $\vec{H} = \langle H_0 \rangle$ for some finite $H_0 \subset \omega$, then $p \restriction \vec{H} \in \mathbf{P}_0$. Assume by contradiction that $p \restriction \vec{H} \nVdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$, then there is a $q \leq p \restriction \vec{H}$ such that $q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$. Let $\vec{\pi} \in \mathcal{G}$ such that π_0 fixes H_0 and such that $\pi_0[\mathrm{dom}(q_0)] \cap \mathrm{dom}(p_0) = H_0 \cap \mathrm{dom}(p_0)$ and $\pi_n[\mathrm{dom}(q_n)] \cap \mathrm{dom}(p_n) = \emptyset$ for all $n > 0$. Then $\vec{\pi}q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$ but p and $\vec{\pi}q$ are compatible, contradiction.

Now let's assume that $\vec{H} = \langle H_0, \dots, H_n \rangle$. The claim $p \restriction \vec{H} \in \mathbf{P}$ follows from the hypotheses of the lemma and the induction hypothesis—the latter being applied to the names of binary relations appearing in the range of $p \restriction \vec{H}$. Assume by contradiction that $p \restriction \vec{H} \nVdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$, then there is a $q \leq p \restriction \vec{H}$ such that $q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$. Let $\vec{\pi} \in \mathcal{G}$ such that π_k pointwise fixes H_k for each $k \leq n$ and such that $\pi_k[\mathrm{dom}(q_k)] \cap \mathrm{dom}(p_k) = H_k \cap \mathrm{dom}(p_k)$ for all $k \leq n$ and $\pi_k[\mathrm{dom}(q_k)] \cap \mathrm{dom}(p_k) = \emptyset$ for all $k > n$. Then $\vec{\pi}q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$ but p and $\vec{\pi}q$ are compatible, contradiction. □

4.3. **The model.** Fix a V-generic filter G for $\mathbf{P}$ and, for all n, let $\mathcal{N}_n$ be the symmetric extension obtained from $\langle \mathbf{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle$, and $\mathcal{N}$ be the symmetric extension, obtained from $\langle \mathbf{P}, \mathcal{G}, \mathcal{F} \rangle$. Clearly we have

$$\mathrm{V} \subseteq \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N} = \mathcal{N}_\omega.$$

For each $\mathbf{P}$-name defined so far (e.g. $\dot{A}$), we let its symbol without the dot (i.e. A) be its evaluation according to G (i.e. $\dot{A}_G$).

Lemma 4.4. *For every $n \in \omega$, for every total and acyclic binary relation $R \in \mathcal{N}_n$ on A, there is an an R-chain in $\mathcal{N}_{n+1}$.*

Proof. Let $p \in G$ and $\dot{R} \in \mathsf{HS}_n$ such that

$$p \Vdash \dot{R} \subseteq \dot{A} \times \dot{A} \text{ total and acyclic.}$$

We can suppose wlog that $p \in \mathbf{P}_n$. Now let

$$\dot{R}_g = \{((\dot{a}_n, \dot{a}_m)^\bullet, q) \mid n, m \in \omega, q \in \mathbf{P}_n, \text{ and } q \Vdash \dot{a}_n \; \dot{R} \; \dot{a}_m\}.$$

It readily follows that $\dot{R}_g$ is in $\mathcal{R}_n$ and $p \Vdash \dot{R} = \dot{R}_g$. Fix any $q \leq p$. Pick an $m \in \omega \setminus \mathrm{dom}(q_{n+1})$ and consider the finite sequence q' such that $q'_l = q_l$ for every $l \neq n+1$ and $q'_{n+1} = q_{n+1} \cup \{(m, (\dot{R}_g, \emptyset))\}$. Then $q' \in \mathbf{P}$, $q' \leq q$ and

$$q' \Vdash \dot{f}_{n,m} \text{ is an } \dot{R}_g\text{-chain, and } \dot{R}_g = \dot{R}.$$

By density,

$$p \Vdash \exists f \in \dot{F}_n \text{ which is an } \dot{R}\text{-chain.}$$

Since $F_n \in \mathcal{N}_{n+1}$ we are done. □

In order to get to the key result, we need to introduce the notion of scattered space.

Definition 4.5. Given a topological space X, we let by ordinal induction

$$\begin{aligned} X^{(0)} &= X, \\ X^{(\alpha+1)} &= \{x \in X^{(\alpha)} \mid x \text{ is a limit point of } X^{(\alpha)}\}, \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \qquad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

For every space X there is necessarily an ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$, and we call the least such ordinal the **scattered height** of the space. A topological space X is **scattered** if there is an α such that $X^{(\alpha)} = \emptyset$.

Every second countable scattered space is countable.

The next proposition tells us that, in $\mathcal{N}$, the closures with respect to A of the generic countable subsets of A we are iteratively adding are scattered with finite scattered height.

For each $t \in {}^{<\omega}2$ we denote by $\dot{\boldsymbol{N}}_t$ the canonical name for the basic open set $\boldsymbol{N}_t$, i.e. the set of all infinite binary sequences extending t.

Proposition 4.6. *For each $n, k \in \omega$, $\mathcal{N} \vDash (\mathrm{Cl}_A(\mathrm{ran}(f_{n,k})))^{(n+2)} = \emptyset$.*

Proof. We prove the proposition by induction on n. We first consider the case $n = 0$.

Let $k \in \omega$, $p \in \mathbf{P}_0$, $\dot{R} \in \mathcal{R}_0$ with support $\langle H_0 \rangle$ such that $p \Vdash$ "$\dot{R}$ is total and acyclic" and $\mathrm{dom}(p_0) = H_0$. For every $X \subseteq \omega$ we denote the name $\{\dot{a}_m \mid m \in X\}^\bullet$ by $\dot{A}_X$.

Claim 4.6.1. $\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle \Vdash \mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}$ *is discrete.*

Proof. Suppose by contradiction that there are $q \leq \langle p, \{(k, (\dot{R}, \emptyset))\}\rangle$ and $l \in \omega$ such that

$$q \Vdash \dot{f}_{0,k}(l) \notin \dot{A}_{H_0} \text{ and } \dot{f}_{0,k}(l) \text{ is a limit point of } \mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}.$$

Without loss of generality suppose that $\mathrm{lh}(q_1^s(k)) > l+1$ and let $m = q_1^s(k)(l)$, $t = q_0(m)$—in particular, $m \notin H_0$ and $q \Vdash \dot{f}_{0,k}(l) = \dot{a}_m \in \dot{\boldsymbol{N}}_t$. By assumption there must be a $z \leq q$ and an $h > l$ such that

$$z \Vdash \dot{f}_{0,k}(h) \in \dot{\boldsymbol{N}}_t \setminus \dot{A}_{H_0}.$$

Assume wlog $\mathrm{lh}(z_1^s(k)) > h$ and let $m' = z_1^s(k)(h)$, $t' = z_0(m')$—in particular, $m' \notin H_0$, $t' \supseteq t$ and $z_0 \Vdash \dot{a}_m \ \dot{R}^+ \ \dot{a}_{m'}$. By the Restriction Lemma,

$$p' = z_0 \restriction (\omega \setminus \{m\}) \cup \{(m, t')\} \Vdash \dot{a}_m \ \dot{R}^+ \ \dot{a}_{m'}.$$

Let $\pi_0 \colon \omega \to \omega$ be the permutation that swaps m and m' fixing everything else—in particular, $\pi_0 \in \mathrm{Fix}(H_0)$. Then

$$\pi_0 p' = p' \Vdash \dot{a}_{m'} \ \dot{R}^+ \ \dot{a}_m,$$

but then p' both extends p and forces $\dot{a}_m \ \dot{R}^+ \ \dot{a}_m$, which is a contradiction, since we assumed that p forces $\dot{R}$ to be acyclic. $\square$

Claim 4.6.2. $\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle \Vdash (\mathrm{Cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{0,k})))^{(1)} \subseteq \dot{A}_{H_0}$.

Proof. Suppose by contradiction that the claim is false, then there is a $q \le \langle p, \{(k, (\dot{R}, \emptyset))\}\rangle$ and an $m \notin H_0$ such that

$$q \Vdash \dot{a}_m \text{ is a limit point of } \mathrm{ran}(\dot{f}_{0,k}).$$

From Claim 4.6.1 it follows that q also forces $\dot{a}_m$ not to be in the range of $\dot{f}_{0,k}$. The condition $q' = \langle q_0, q_1 \restriction \{k\}\rangle$ extends p and, by the Restriction Lemma, forces the same statement. Let t be $q_0(m)$—in particular $q' \Vdash \dot{a}_m \in \dot{\boldsymbol{N}}_t$.

We now show $q' \Vdash \dot{\boldsymbol{N}}_t \subseteq \mathrm{Cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0})$, which clearly contradicts Claim 4.6.1. Pick any $z \le q'$ and a $t' \supseteq t$. Fix an $m' \notin H_0 \cup \mathrm{dom}(z_0) \cup \mathrm{ran}(q_1^s(k))$. Define z' to be the condition such that $z'_0 = z_0 \cup \{(m', t')\}$ and $z'_i = z_i$ for every $i > 0$. Now, z' clearly extends z but, letting π_0 be the permutation of ω that swaps m and m', it extends also $\langle \pi_0 \rangle q'$, which means

$$z' \Vdash \dot{a}_{m'} \in \dot{\boldsymbol{N}}_{t'} \cap \mathrm{Cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}).$$

By density we have

$$q' \Vdash \dot{\boldsymbol{N}}_t \subseteq \mathrm{Cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}),$$

which, as said, is a contradiction. $\square$

Since H_0 is finite, it follows directly from Claim 4.6.2 that

$$\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle \Vdash (\mathrm{Cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{0,k})))^{(2)} = \emptyset.$$

For any fixed k, the set of conditions $\langle p, \{(k, (\dot{R}, \emptyset))\}$ we are considering is pre-dense in $\mathbf{P}$. It follows that for every $k \in \omega$

$$\Vdash (\mathrm{Cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{0,k})))^{(2)} = \emptyset.$$

Suppose now $n > 0$. Let $k \in \omega$, $p = \langle p_0, \dots p_n\rangle \in \mathbf{P}_n$, $\dot{R} \in \mathcal{R}_n$ with support $\vec{H} = \langle H_0, \dots, H_n\rangle$ such that $p \Vdash$ "$\dot{R}$ is total and acyclic". Assume also that, for each $i \le n$, $\mathrm{dom}(p_i) = H_i$, and, for all $0 < i \le n$, for all $j \in H_i$, $\vec{H} \restriction i$ is a support for $p_i^R(j)$.

Claim 4.6.3.

$$\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle \Vdash \operatorname{ran}(\dot{f}_{n,k}) \setminus \Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j\in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j})\Big) \textit{ is discrete.}$$

Proof. Suppose by contradiction that there are $q \leq \langle p, \{(k, (\dot{R}, \emptyset))\}\rangle$ and $l \in \omega$ such that

$$q \Vdash \dot{f}_{n,k}(l) \notin \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j\in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j}) \text{ and } \dot{f}_{n,k}(l) \text{ is a limit point of}$$
$$\operatorname{ran}(\dot{f}_{n,k}) \setminus \Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j\in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j})\Big).$$

Suppose wlog that $\operatorname{lh}(q^s_{n+1}(k)) > l+1$ and let $m = q^s_{n+1}(k)(l)$, $t = q_0(m)$—in particular, $q \Vdash \dot{f}_{n,k}(l) = \dot{a}_m \in \dot{\boldsymbol{N}}_t$. By assumption there must be a $z \leq q$ and an $h > l$ such that

$$z \Vdash \dot{f}_{n,k}(h) \in \dot{\boldsymbol{N}}_t \setminus \Bigg(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j\in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j})\Bigg).$$

Assume wlog $\operatorname{lh}(z^s_{n+1}(k)) > h$ and let $m' = z^s_{n+1}(k)(h)$, $t' = z_0(m')$—in particular $t' \supseteq t$ and $z \restriction n+1 \Vdash \dot{a}_m \ \dot{R}^+ \ \dot{a}_{m'}$. Let

$$p' = \langle z_0 \restriction (\omega \setminus \{m\}) \cup \{(m, t')\}, z_1 \restriction H_1, \dots, z_n \restriction H_n\rangle,$$

then, by the Restriction Lemma,

$$p' \Vdash \dot{a}_m \ \dot{R}^+ \ \dot{a}_{m'}.$$

Let $\pi_0 \colon \omega \to \omega$ be the permutation that swaps m and m' fixing everything else. Then

$$\langle \pi_0\rangle p' = p' \Vdash \dot{a}_{m'} \ \dot{R}^+ \ \dot{a}_m,$$

but then p' both extends p and forces $\dot{a}_m \ \dot{R}^+ \ \dot{a}_m$, which is a contradiction, since we assumed that p forces $\dot{R}$ to be acyclic. $\square$

Claim 4.6.4.

$$\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle \Vdash (\operatorname{Cl}_{\dot{A}}(\operatorname{ran}(\dot{f}_{n,k})))^{(1)} \subseteq \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j\in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j}).$$

Proof. Suppose by contradiction that this is not the case, then there is a $q \leq \langle p, \{(k, (\dot{R}, \emptyset))\}\rangle$ and an m such that

$$q \Vdash \dot{a}_m \text{ is a limit point of } \operatorname{ran}(\dot{f}_{n,k}) \text{ and}$$
$$\dot{a}_m \notin \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j\in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j}).$$

From Claim 4.6.3 it follows that q also forces $\dot{a}_m$ not to be in the range of $\dot{f}_{n,k}$. Let

$$q' = \langle q_0, q_1 \restriction H_1, \dots, q_n \restriction H_n, q_{n+1} \restriction \{k\}\rangle,$$

then q' extends p and, by the Restriction Lemma, forces the same statement. Let t be $q_0(m)$—in particular $q' \Vdash \dot{a}_m \in \dot{\boldsymbol{N}}_t$.

We now show that

$$q' \Vdash \dot{\boldsymbol{N}}_t \subseteq \mathrm{Cl}\left(\mathrm{ran}(\dot{f}_{n,k}) \setminus \left(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\right)\right),$$

which contradicts Claim 4.6.3. Pick any $z \leq q'$ and $t' \supseteq t$. Fix an $m' \in \omega$ such that

$$m' \notin H_0 \cup \mathrm{dom}(z_0) \cup \mathrm{ran}(q^s_{n+1}(k)) \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(q^s_{i+1}(j)).$$

Define z' to be the condition such that $z'_0 = z_0 \cup \{(m', t')\}$ and $z'_i = z_i$ for all $i > 0$. Now, z' clearly extends z but, letting π_0 be the permutation of ω that swaps m and m', it also extends $\langle \pi_0 \rangle q'$, which means that

$$z' \Vdash \dot{a}_{m'} \in \dot{\boldsymbol{N}}_{t'} \cap \mathrm{Cl}\left(\mathrm{ran}(\dot{f}_{n,k}) \setminus \left(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\right)\right).$$

By density,

$$q' \Vdash \dot{\boldsymbol{N}}_t \subseteq \mathrm{Cl}\left(\mathrm{ran}(\dot{f}_{n,k}) \setminus \left(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\right)\right),$$

which, as said, is a contradiction. □

It follows from Claim 4.6.4 and the induction hypothesis that

$$\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle \Vdash (\mathrm{Cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k})))^{(n+2)} = \emptyset.$$

For any fixed k, the set of conditions $\langle p, \{(k, (\dot{R}, \emptyset))\}\rangle$ we are considering is pre-dense in $\mathbf{P}$. Hence, for every $k \in \omega$,

$$\Vdash (\mathrm{Cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k})))^{(n+2)} = \emptyset. \qquad \square$$

In light of Proposition 4.6, we can prove that in $\mathcal{N}$ every separable subset of A is scattered with finite scattered height.

Theorem 4.7. *In the model $\mathcal{N}$ the following holds: for every separable $S \subseteq A$ there is an $n \in \omega$ such that $S^{(n)} = \emptyset$.*

Proof. Let $S \in \mathcal{N}$ be a separable subset of A and fix a function $f \colon \omega \to A$ such that $S \subseteq \mathrm{Cl}_A(\mathrm{ran}(f))$. Then there must be a $p \in G$ such that

$$p \Vdash \dot{f} \colon \check{\omega} \to \dot{A},$$

where $\dot{f} \in \mathsf{HS}$ is a symmetric name for f, with support $\vec{H} = \langle H_0, \dots, H_n\rangle$. We can assume wlog that $\mathrm{dom}(p_i) = H_i$ for each i, and that for all $i > 0$, for all $j \in H_i$, $H \restriction i$ is a support for $p^R_i(j)$. We claim that

$$p \Vdash \mathrm{ran}(\dot{f}) \subseteq \dot{A}_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j}).$$

If we manage to do so, then Proposition 4.6 ensures that $(\mathrm{Cl}_A(\mathrm{ran}(f))^{(n+2)} = \emptyset$, and we would be done.

Suppose that the claim is false, then there exist a $q \leq p$, $l, m \in \omega$ such that

$$q \Vdash \dot{f}(l) = \dot{a}_m \notin \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j}).$$

Let $q' = \langle q_0, q_1 \restriction H_1, \ldots, q_n \restriction H_n \rangle$, then, by the Restriction Lemma, q' forces the same statement. Fix an $m' \in \omega$ such that

$$m' \notin H_0 \cup \mathrm{dom}(q_0) \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(q^s_{i+1}(j)).$$

Let π_0 be the permutation of ω that swaps m and m', then $\langle \pi_0 \rangle q'$ and q' are compatible, but they both extend p and

$$\langle \pi_0 \rangle q' \Vdash \dot{f}(l) = \dot{a}_{m'} \neq \dot{a}_m,$$

which is a contradiction. □

Corollary 4.8. *$\mathcal{N} \vDash \neg \mathsf{AC}_\omega(A)$.*

Proof. Assume by contradiction that $\mathsf{AC}_\omega(A)$ holds, then A is certainly separable. By Theorem 4.7, A would be scattered of finite scattered height. But actually A has no isolated points, contradiction. □

Now we are left to prove that $\mathsf{DC}(A)$ holds in $\mathcal{N}$. Let $\dot{\mathcal{N}}_n$ be the canonical name for the intermediate model $\mathcal{N}_n$.

Lemma 4.9. *Let $n \in \omega$ and $\dot{x} \in \mathsf{HS}$ with support $\vec{H} = \langle H_0, \ldots, H_n \rangle$, then*

$$\Vdash \dot{x} \subseteq \dot{\mathcal{N}}_n \Rightarrow \dot{x} \in \dot{\mathcal{N}}_n$$

Proof. For each $(\dot{y}, p) \in \dot{x}$ fix a maximal antichain $A_{(\dot{y},p)}$ below p and a map $f_{(\dot{y},p)} \colon A_{(\dot{y},p)} \to \mathsf{HS}_n$ such that, for each $q \in A_{(\dot{y},p)}$, either $q \Vdash \dot{y} = f_{(\dot{y},p)}(q)$ or $q \Vdash \dot{y} \notin \dot{\mathcal{N}}_n$. Let $A'_{(\dot{y},p)} = \{q \in A_{(\dot{y},p)} \mid q \Vdash \dot{y} \in \dot{\mathcal{N}}_n\}$ and

$$C = \{\vec{\pi}(f_{(\dot{y},p)}(q)) \mid (\dot{y}, p) \in \dot{x},\ q \in A'_{(\dot{y},p)},\ \vec{\pi} \in \mathrm{Fix}(\vec{H})\}.$$

Consider the following name:

$$\dot{x}' = \{(\dot{y}, q) \mid \dot{y} \in C,\ q \in \mathbf{P}_n \text{ and } q \Vdash \dot{y} \in \dot{x}\}.$$

Claim 4.9.1. *$\dot{x}' \in \mathsf{HS}_n$ with support $\vec{H}$.*

Proof. Let $\vec{\pi} \in \mathrm{Fix}(\vec{H})$ and $(q, \dot{y}) \in \dot{x}'$. By definition, $q \Vdash \dot{y} \in \dot{x}$, hence $\vec{\pi} q \Vdash \vec{\pi}\dot{y} \in \dot{x}$. Since $\vec{\pi}\dot{y} \in C$, this means that $(\vec{\pi} q, \vec{\pi}\dot{y}) \in \dot{x}'$. Hence $\vec{\pi}\dot{x}' = \dot{x}'$. □

Suppose there is a $p \in \mathbf{P}$ such that $p \Vdash \dot{x} \subseteq \dot{\mathcal{N}}_n$.

Claim 4.9.2. *$p \Vdash \dot{x}' = \dot{x}$.*

Proof. Let $q \leq p$ and $\dot{z} \in \mathsf{HS}$ such that $q \Vdash \dot{z} \in \dot{x}$. By definition of C and our hypothesis on p, there is an $r \leq q$ and a $\dot{y} \in C$ such that $r \Vdash \dot{z} = \dot{y} \in \dot{x}$. By the Restriction Lemma, $r \restriction n+1 \Vdash \dot{y} \in \dot{x}$, hence $(r \restriction n+1, \dot{y}) \in \dot{x}'$ and, in particular, $r \Vdash \dot{z} = \dot{y} \in \dot{x}'$. By density, $p \Vdash \dot{x} \subseteq \dot{x}'$.

The other inclusion is immediate, as it follows directly from the definition of $\dot{x}'$. □

By density, $\Vdash \dot{x} \subseteq \dot{\mathcal{N}}_n \to \dot{x} \in \dot{\mathcal{N}}_n$. □

Theorem 4.10. $\mathcal{N} \vDash \mathsf{DC}(A)$.

Proof. Since every binary relation $R \in \mathcal{N}$ on A is a subset of $A \times A \in \mathcal{N}_0$, it follows from Lemma 4.9 that $R \in \mathcal{N}_n$ for some n. Now, either R is cyclic, but then it surely has a chain, or it is acyclic, but then Lemma 4.4 says that in $\mathcal{N}_{n+1} \subseteq \mathcal{N}$ there is a chain for this relation. □

This finishes the proof of Theorem 4.1.

5. Some complementary results

We collect some facts related to our main results.

5.1. **Dependent Choice propagates under finite unions.** By Proposition 2.1 the axiom $\mathsf{DC}(X)$ is closed under surjective images, and hence under subsets. The next result shows that it is also closed under finite unions.

Theorem 5.1. $\mathsf{DC}(X) \wedge \mathsf{DC}(Y) \Rightarrow \mathsf{DC}(X \cup Y)$.

Corollary 5.2. $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(X \times n)$*, for all sets* X *and all* $n \in \omega$.

The natural progression from Corollary 5.2 would be to prove that $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(X \times \omega)$, but this cannot be established in ZF, since $\mathsf{DC}(X \times \omega)$ implies $\mathsf{AC}_\omega(X)$ (part (d) of Proposition 2.1) and we know from Theorem 4.1 that $\mathsf{DC}(X)$ does not necessarily imply $\mathsf{AC}_\omega(X)$.

If a binary relation R is such that $\operatorname{ran}(R) \subseteq \operatorname{dom}(R)$, then it is total on its domain. The largest $R' \subseteq R$ such that $\operatorname{ran}(R') \subseteq \operatorname{dom}(R')$ is

$$\mathcal{D}(R) = \bigcup \{S \subseteq R \mid \operatorname{ran}(S) \subseteq \operatorname{dom}(S)\}.$$

By part (a) of Proposition 2.1 it is easy to see that

$$\mathsf{DC}(X) \Leftrightarrow \forall R \subseteq X^2\,(\mathcal{D}(R) \neq \emptyset \Rightarrow \text{there is a } \mathcal{D}(R)\text{-chain}). \tag{4}$$

Proof of Theorem 5.1. Suppose $\mathsf{DC}(X)$ and $\mathsf{DC}(Y)$, and let $R \subseteq (X \cup Y)^2$ be total, towards proving that there is an R-chain. Without loss of generality, we may assume that X and Y are nonempty and disjoint. If $\mathcal{D}(R \restriction X) \neq \emptyset$, then by $\mathsf{DC}(X)$ and (4) there is a $\mathcal{D}(R \restriction X)$-chain, which is, in particular an R-chain. Similarly, if $\mathcal{D}(R \restriction Y) \neq \emptyset$, then there is an R-chain. Therefore, without loss of generality, we may assume that R is acyclic, and that

$$\mathcal{D}(R \restriction X) = \mathcal{D}(R \restriction Y) = \emptyset. \tag{5}$$

Recall that R^+ is the smallest transitive relation containing R. If $x \in X \cup Y$ and $R^+(x) \subseteq X$, then $R \restriction R^+(x)$ would witness that $\mathcal{D}(R \restriction X) \neq \emptyset$, against (5). Similarly $R^+(x)$ cannot be included in Y. Therefore

$$\forall x \in X \cup Y \, (R^+(x) \nsubseteq X \wedge R^+(x) \nsubseteq Y). \tag{6}$$

Here is the idea of the proof. By (6) any R-chain $(z_n)_{n\in\omega}$ must visit both X and Y infinitely often, so $(z_n)_{n\in\omega}$ can be seen as the careful merging of two sequence $(x_n)_{n\in\omega}$ in X and $(y_n)_{n\in\omega}$ in Y. The sequence $(x_n)_{n\in\omega}$ is obtained by applying $\mathsf{DC}(X)$ to a total relation R_X on X such that $R \restriction X \subseteq R_X \subseteq R^+$. Using $(x_n)_{n\in\omega}$ a suitable total relation R_Y on some $Y' \subseteq Y$ is defined, and by $\mathsf{DC}(Y)$ the required sequence $(y_n)_{n\in\omega}$ is obtained. Here come the details.

Let R_X be the relation on X given by $R \restriction X$, together with all pairs (x, x') such that $x \mathrel{R} y_0 \mathrel{R} y_1 \mathrel{R} \cdots \mathrel{R} y_n \mathrel{R} x'$ for some finite sequence of elements of Y

$$\begin{aligned} R_X = (R \restriction X) \cup \{(x,x') \in X^2 \mid \exists m \geq 1 \, \exists s \in {}^m Y \\ (x \mathrel{R} s(0) \wedge s(m-1) \mathrel{R} x' \wedge \forall i < m \, (s(i) \mathrel{R} s(i+1)))\}. \end{aligned}$$

It is immediate that $R_X \subseteq R^+$.

Claim 5.2.1. *R_X is total on X.*

Proof. We must show that $\mathrm{dom}(R_X) = X$. Let $x \in X$. As R is total on $X \cup Y$, it follows that $\emptyset \neq R(x) \subseteq R^+(x)$. By (6) $R^+(x) \nsubseteq Y$ so there are $y_0, \ldots, y_n \in Y$ and $x' \in X$ such that $x \mathrel{R} y_0 \mathrel{R} \ldots \mathrel{R} y_n \mathrel{R} x'$. Thus $(x, x') \in R_X$, so $x \in \mathrm{dom}(R_X)$. $\square$

By $\mathsf{DC}(X)$ there is an R_X-chain $(x_n)_{n\in\omega}$.

Claim 5.2.2. $\forall n \, \exists m > n \, \neg(x_m \mathrel{R} x_{m+1})$.

Proof. Towards a contradiction suppose that there is $\bar{n} \in \omega$ such that $x_m \mathrel{R} x_{m+1}$ for every $m \geq \bar{n}$. Then $R \restriction \{x_m \mid m \geq \bar{n}\}$ is total on $\{x_m \mid m \geq \bar{n}\}$ and contained in $R \restriction X$, against (5). $\square$

Let $(n_k)_{k\in\omega}$ be the sequence enumerating the set of ms such that $\neg(x_m \mathrel{R} x_{m+1})$. By the definition of R_X, each x_{n_k} is linked to x_{n_k+1} via R through some finite path in Y, and let Y_k be the collection of all places visited by these paths:

$$\begin{aligned} Y_k \coloneqq \bigcup \Big\{ \mathrm{ran}(s) \mid \exists s \, \exists m \, \big(s \in {}^{m+1}Y \wedge x_{n_k} \mathrel{R} s(0) \wedge s(m) \mathrel{R} x_{n_k+1} \wedge \\ \forall i < m \, (s(i) \mathrel{R} s(i+1)) \big) \Big\}. \end{aligned}$$

Claim 5.2.3. *The Y_ks are nonempty, pairwise disjoint subsets of Y.*

Proof. For each k we have $(x_{n_k}, x_{n_k+1}) \in R_X \setminus R$. This means that there is some $\langle y_0, \ldots, y_m \rangle \in {}^{<\omega}Y$ such that $x_{n_k} \; R \; y_0 \; R \ldots R \; y_m \; R \; x_{n_k+1}$. In particular, $Y_k \neq \emptyset$.

Towards a contradiction suppose there are indeces $k < j$ such that $Y_k \cap Y_j \neq \emptyset$. Pick $y \in Y_k \cap Y_j$. Then $y \; R^+ \; x_{n_k+1} \; R^+ \; x_{n_j} \; R^+ \; y$, if $x_{n_k+1} \neq x_{n_j}$, or $y \; R^+ \; x_{n_k+1} = x_{n_j} \; R^+ \; y$ otherwise. Either way, this contradicts our assumption that R is acyclic. □

Now we let R_Y be the relation on $\bigcup_{k \in \omega} Y_k$

$$\bigcup_{k \in \omega} (R \restriction Y_k) \cup \bigcup_{k \in \omega} \{(y, y') \in Y_k \times Y_{k+1} \mid y \; R \; x_{n_k+1} \text{ and } x_{n_{k+1}} \; R \; y'\}.$$

It readily follows from the definition that $R_Y \subseteq R^+$.

Claim 5.2.4. *R_Y is total on $\bigcup_{k \in \omega} Y_k$.*

Proof. Pick $k \in \omega$ and $y \in Y_k$, towards proving that $y \in \operatorname{dom}(R_Y)$. Then there is a finite sequence $\langle y_0, \ldots, y_m \rangle$ of elements of Y_k such that $x_{n_k} \; R \; y_0 \; R \cdots R \; y_m \; R \; x_{n_k+1}$, and $y = y_i$ for some $0 \leq i \leq m$. If $i < m$, then $y \; R \; y_{i+1}$. If $i = m$ then $y \; R_Y \; y'$ for any $y' \in Y_{k+1}$ such that $x_{n_{k+1}} \; R \; y'$. In either case $y \in \operatorname{dom}(R_Y)$. □

By $\mathsf{DC}(Y)$, there is an R_Y-chain $(y_n)_{n \in \omega}$. By part (b) of Proposition 2.1 we can suppose that $y_0 \in Y_0$ and that $x_{n_0} \; R \; y_0$. As the Y_ks are disjoint, for every n there is a unique k such that $y_n \in Y_k$, and let $i(n)$ be this k.

Claim 5.2.5. *The set $I_k = \{n \in \omega \mid i(n) = k\}$ is a finite interval of natural numbers.*

Proof. By definition of R_Y it follows that either $i(n+1) = i(n)$ or else $i(n+1) = i(n) + 1$, so it is enough to show that I_k is finite. Towards a contradiction, suppose $I_{\bar{k}}$ is infinite, for some $\bar{k} \in \omega$. This means that there is $\bar{n}$ such that $i(n) = i(\bar{n})$ for all $n \geq \bar{n}$, that is $\{y_n \mid n \geq \bar{n}\} \subseteq Y_{\bar{k}}$. But then $R \restriction \{y_n \mid n \geq \bar{n}\}$ would be a total on $\{y_n \mid n \geq \bar{n}\}$ and contained in $R \restriction Y$, against (5). □

Let $m_k = \max(I_k)$ so that $I_0 = [0; m_0]$ and $I_{k+1} = [m_k + 1; m_{k+1}]$. Then

$$\langle x_0, \ldots, x_{n_0} \rangle^\frown \langle y_0, \ldots, y_{m_0} \rangle^\frown \langle x_{n_0+1}, \ldots, x_{n_1} \rangle^\frown \langle y_{m_0+1}, \ldots, y_{m_1} \rangle^\frown \cdots$$
$$\cdots^\frown \langle x_{n_k+1}, \ldots, x_{n_{k+1}} \rangle^\frown \langle y_{m_k+1}, \ldots, y_{m_{k+1}} \rangle^\frown \cdots$$

is the required R-chain. □

5.2. **The Feferman-Levy model.** Feferman and Levy showed that the following is consistent relative to ZF:

(FL) $\qquad \mathbb{R}$ is the countable union of countable sets.

(See [Jec73, p. 142] for an exposition of the Feferman-Levy model.)

The next result shows that in the Feferman-Levy model the statement of Theorem 4.1 fails, that is there is no set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ and $\neg\mathsf{AC}_\omega(A)$.

Proposition 5.3. FL *implies that if* $\mathsf{DC}(A)$ *holds with* $A \subseteq \mathbb{R}$*, then* A *is countable.*

We need a preliminary result.

Lemma 5.4. *Assume* FL*. Then there is a sequence of pairwise disjoint, nonempty, countable sets* $(X_n)_{n\in\omega}$ *such that* $\mathbb{R} = \bigcup_n X_n$*, and such no infinite subsequence of* $(X_n)_{n\in\omega}$ *has a choice function.*

Proof. Fix a bijection $\pi\colon \mathbb{R} \to \mathbb{R}^\omega$, and for each $m \in \omega$ let $\pi_m\colon \mathbb{R} \to \mathbb{R}$ be defined as $\pi_m(x) = \pi(x)_m$. If $Y \subseteq \mathbb{R}$ and $f\colon \omega \to Y$ is surjective, then $\tilde{Y}$, the closure of Y under the π_ms, is also countable, as

$$\tilde{f}\colon {}^{<\omega}\omega \times \omega \to \tilde{Y} \qquad (\langle n_0, \dots, n_k\rangle, m) \mapsto \pi_{n_k} \circ \dots \circ \pi_{n_0} \circ f(m)$$

is surjective. By FL let $(Y_n)_{n\in\omega}$ be a sequence of countable sets such that $\mathbb{R} = \bigcup_n Y_n$, and without loss of generality we may assume that each Y_n is closed under every π_m. Then let $X_n = Y_n \setminus \bigcup_{m<n} X_m$ for each $n \in \omega$. If necessary, we can pass to a subsequence to get them to be nonempty.

We claim that no infinite subsequence of $(X_n)_{n\in\omega}$ has a choice function. Otherwise there would be an infinite sequence $(x_n)_{n\in\omega} \in \mathbb{R}^\omega$ whose range intersects infinitely many X_ns. Let $x \in \mathbb{R}$ be such that $\pi(x) = (x_n)_{n\in\omega}$. Then there must be an $k \in \omega$ with $x \in X_k \subseteq Y_k$, and hence

$$\forall n \in \omega\ (x_n = \pi_n(x) \in Y_k \subseteq X_0 \cup \dots \cup X_k)$$

as Y_k is closed under the π_ns. But this contradicts the assumption that $\{x_n \mid n \in \omega\}$ intersects infinitely many X_ns. $\square$

Proof of Proposition 5.3. Fix $(X_n)_{n\in\omega}$ as in Lemma 5.4. Let $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds, and let $I = \{n \in \omega \mid A \cap X_n \neq \emptyset\}$. If I is infinite then (modulo a trivial reindexing) $\mathsf{DC}(A)$ would imply the existence of a choice function for the family $\{A \cap X_n \mid n \in I\}$, which is, in particular, a choice function for $\{X_n \mid n \in I\}$, against Lemma 5.4. So I must be finite, that is $A \subseteq X_0 \cup \dots \cup X_k$ for some k. But the finite union of countable sets is countable, so A is countable. $\square$

5.3. **Definability of the counterexample.** Theorem 4.1 shows that the statement (3) is consistent with ZF, that is to say: it is consistent that there is a set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ and $\neg\mathsf{AC}_\omega(A)$. The set A constructed in the proof of Theorem 4.1 is a set of Cohen reals, so it is not ordinal definable. But what is the possible descriptive-complexity of a set A as above?

By part (c) of Proposition 2.5 the set A cannot contain a perfect set. Recall that a set has the perfect set property if it is either countable, or else it contains a perfect subset. Assuming $\mathsf{AC}_\omega(\mathbb{R})$ every Borel set has the perfect-set property. In a choice-less context the situation becomes murky. Assuming FL, every set of reals is $\mathbf{F}_{\sigma\sigma}$ (i.e. countable union of $\mathbf{F}_\sigma$ sets), and by taking complements it is also $\mathbf{G}_{\delta\delta}$ (i.e. countable intersection of $\mathbf{G}_\delta$ sets), so every set is $\mathbf{\Delta}^0_4$, as $\mathbf{F}_\sigma = \mathbf{\Sigma}^0_2 \subset \mathbf{\Pi}^0_3$, and hence $\mathbf{F}_{\sigma\sigma} \subseteq \mathbf{\Sigma}^0_4$. Therefore FL collapses the Borel hierarchy at level 4. Moreover FL implies that there is an uncountable set in $\mathbf{\Pi}^0_3$ without a perfect subset [Mil11, Theorem 1.3].

On the other hand A. Miller has shown in ZF that $\mathbf{\Sigma}^0_3 \neq \mathbf{\Pi}^0_3$ [Mil08, Theorem 2.1], and that every set in $\mathbf{\Sigma}^0_3$ has the perfect-set property [Mil11, Theorem 1.2].

Recall that a subset of $\mathbb{R}$ is $\mathbf{\Pi}^1_n$ if it is the complement of a $\mathbf{\Sigma}^1_n$, and it is $\mathbf{\Sigma}^1_n$ if it is the projection of a $\mathbf{\Pi}^1_{n-1}$ set $C \subseteq \mathbb{R} \times \mathbb{R}$, where $\mathbf{\Pi}^1_0$ is the collection of closed sets. The lightface hierarchy Σ^1_n, Π^1_n is obtained by replacing $\mathbf{\Pi}^1_0$ with Π^1_0, the collection of recursively-closed sets, see [Kan09, Ch. 3, §12]. Working in ZF, every $\mathbf{\Sigma}^1_1$ set has the perfect set property, and by Mansfield-Solovay theorem (see [Kan09, Ch. 3, Corollary 14.9]) every $\mathbf{\Sigma}^1_2$ set is either well-orderable, being included in $\mathrm{L}[a]$ for some real a, or else it contains a perfect set.

By part (c) of Lemma 2.5 we obtain:

Corollary 5.5. *If $A \subseteq \mathbb{R}$ is $\mathbf{\Sigma}^0_3$ or $\mathbf{\Sigma}^1_2$ and $\mathsf{DC}(A)$ holds, then $\mathsf{AC}_\omega(A)$.*

We conclude with a question.

Question 5.6. *Is it consistent with ZF that there is a Π^1_2 set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ and $\neg\mathsf{AC}_\omega(A)$?*

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