DOCTORAL DISSERTATION

DOCTORAL PROGRAM IN PURE AND APPLIED MATHEMATICS (33^{th} CYCLE)

EQUILIBRIA AND SYSTEMIC RISK IN SATURATED NETWORKS

BY

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To my family.

DECLARATION

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

Torino, 2021

Leonardo Massai, January 10, 2022

This dissertation is presented in partial fulfillment of the requirements for the degree of **Philosophiae Diploma (PhD degree)** in **Pure and Applied Mathematics**.

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ABSTRACT

In this dissertation we undertake a fundamental study of network equilibria modeled as solutions of fixed point equations for monotone linear functions with saturation nonlinearities. The considered model extends one originally proposed to study systemic risk in networks of financial institutions interconnected by mutual obligations and it is one of the simplest continuous models accounting for shock propagation phenomena and cascading failure effects. This model also characterizes Nash equilibria of constrained quadratic network games with strategic complementarities. We first derive explicit expressions for network equilibria and prove necessary and sufficient conditions for their uniqueness, encompassing and generalizing results available in the literature. Then, we study jump discontinuities of the network equilibria when the exogenous flows cross certain regions of measure 0 representable as graphs of continuous functions. We discuss some implications of our results in the two main motivating applications. In financial networks, this bifurcation phenomenon is responsible for how small shocks in the assets of a few nodes can trigger major aggregate losses to the system and cause the default of several agents. In constrained quadratic network games, it induces a blow-up behavior of the sensitivity of Nash equilibria with respect to the individual benefits.

Finally, we study in details a relevant application by considering a deterministic continuous-time lossy dynamical flow networks with constant exogenous demands, fixed routing, and finite flow and buffer capacities. In the considered model, when the total net flow in a cell, consisting of the difference between the total flow directed towards it minus the outflow from it, exceeds a certain capacity constraint, then the exceeding part of it leaks out of the system. The ensuing network flow dynamics is a linear saturated system with compact state space that we analyze using tools from monotone systems and contraction theory. Specifically, we prove that there exists a set of equilibria that is globally asymptotically stable. Such equilibrium set reduces to a single globally asymptotically stable equilibrium for generic exogenous demand vectors. Moreover, we show that the critical exogenous demand vectors giving rise to non-unique equilibria correspond to phase transitions in the asymptotic behavior of the dynamical flow network.

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INTRODUCTION

1.1 INTRODUCTION AND MOTIVATION

A central aspect of complex socio-technical systems such as infrastructural, social, economic, and financial networks is the role played by interconnections in amplifying and propagating shocks through cascading mechanisms that may increase the fragility of a system [5, 18, 21]. The term *systemic risk* refers to the possibility that even small shocks localized in a limited part of the network can spill over, thus possibly achieving a significant global impact [29, 54, 2]. A key challenge is to find adequate models for network systems, that are sufficiently elaborate to incorporate such propagation phenomena, yet simple enough to allow for mathematical tractability. Whilst simple contagion models such as epidemic contact processes prove inadequate as they are based on purely pairwise interactions, more complex models taking into account cumulative neighborhood effects include the linear threshold model [4, 59, 51] whose applicability is however limited by the fact that states of the nodes are described by pure binary variables simply expressing whether the node has been affected by the shock.

In most of the applications where the network represents a physical infrastructure, a social or economic network, or an interconnected financial system, however, the cascading mechanism is rather triggered by a process naturally described in terms of continuous variables such as, e.g., power flows in electric grids, traffic volumes in transportation systems, the extent of individuals' involvement in a certain activity in social networks, prices or quantities of goods in an economic system, assets values and payments in financial networks. The most tractable continuous models of network interaction considered in the literature give rise to notions of equilibria that can be mathematically characterized as the solutions of a linear system of equations whose coefficients can be assembled in a (often sparse) matrix that describes the network of interconnections among the different nodes. Examples include competitive equilibria in production networks [1, 3] or Nash equilibria in network games with linear best replies including quadratic network games [9, 19, 27, 16, 32].

While the most basic formulations of such fundamental models consider no constraints on the involved variables, in several of the aforementioned applications it is natural to assume some a priori lower (e.g., non-negativity) and upper bounds (e.g., maximum available resource). E.g., in the financial context [25], where institutions are interconnected by mutual obligations and the payments are necessarily non-negative and upper bounded by the debt value. In the context of network games modeling peer effects on students' engagement, [19] suggests to "bound the strategy space in such a game rather naturally by simply acknowledging the fact that students have a time constraint and allocate their time between leisure and school work," and [17] acknowledge that "while in principle, a player's action could be any real number, all games in the literature place restrictions on players' actions which represent different real-world situations" and that "for peer effects in a classroom, there are natural lower and upper bounds: a student can study no less than zero hours and no more than twenty-four hours in a day." When a priori upper and lower bounds are taken into account in the network model, the related equilibrium notions end up being mathematically characterized as the solutions of linear systems of equations with saturation non-linearities. [19, 12, 16, 6, 17]. Such saturated network models exhibit a considerably richer behaviors than purely linear ones, including the possibility of cascading effects coded in terms of variable saturations and transition phases with respect to structure parameters. As we shall see in the following chapters, such cascade effects can have deep implications in terms of systemic risk, which is particularly crucial in financial networks. In this context, interconnections existing among financial entities originate additional channels for contagion and amplification of shocks to the financial system. In fact, among the several factors that amplified the global financial crisis of 2007-08, the role of the expanding interconnectedness of the financial network is perhaps the least understood. The unfolding of the crisis made clear the fact that both regulators and market participants had very bounded information about the network of obligations and liabilities connecting financial institutions. It also revealed that very little was known about the relationship between interconnectedness, financial stability and systemic risk. Since then, these kind of issues have attracted a considerably amount of attention and a growing body of literature began studying how connections can actually amplify or dampen shocks and the role of the network topology in such phenomena.

Motivated by all these applications, the aim of this thesis is to perform a systematic study of saturated network models. We will answer fundamental questions about uniqueness and continuity of the equilibria with results that also give new insights about the relation between shocks and network topology and how this interplay can originate cascading effects.

1.2 ORGANIZATION OF THE DISSERTATION

The thesis is organized as follows.

- In Chapter 2 we introduce the necessary mathematical tools and notation that will be used throughout the dissertation. In particular, in Section 2.2 we recall some basic notions of graph theory and non-negative matrices. In Section 2.3 we discuss some useful properties of monotone dynamical system, especially when the state space form a complete lattice. Then, in Section 2.4 we present the basics of game theory with a particular focus on supermodular games. Several examples are proposed to better explain some concepts.

- In Chapter 3 we present a fundamental analysis of a saturated equilibrium model, which is introduced in in Section 3.1. Here we discuss the relevant literature and previous work; we end this Section discussing the main contributions that we develop in the Chapter. In Section 3.2 we present the main applications of the model and in Section 3.3 we discuss important structural properties of the equilibria that are instrumental to prove the fundamental results presented in Section 3.4 and Section 3.5. Specifically in Section 3.4 we provide a necessary and sufficient condition for uniqueness of equilibria for a generic network and in Section 3.5 we study the continuity of the equilibria and the existence of critical shocks that trigger a jump discontinuity that has deep implications also for the concept of systemic risk. Finally, in Section 3.6 we draw some conclusions about this Chapter. The content of this Chapter is based on the accepted-for-publication paper [44] co-authored with my advisors Fabio Fagnani and Giacomo Como.

- In Chapter 4 we study in detail a relevant application of the saturation model in the form of a continuous-time flow dynamics on networks with finite capacities. In Section 4.1 we discuss the relevant literature and the main contributions presented in this Chapter. In Section 4.2 we introduce the model and in Section 4.3 we give important results about the geometry and stability of the equilibria. In Section 4.4 we study the continuity of the equilibria and the phase transition that occurs in the system for critical values of the exogenous shock. We end the Chapter with Section 4.5 where we draw some conclusions and discuss future research. The content of this Chapter is based on the published paper [43] co-authored with my advisors Fabio Fagnani and Giacomo Como.

- We wrap up the thesis with Chapter 5 where we summarize the main contributions of this work as a whole in Section 5.1 and discuss possible extensions and open problems that are object of current and future research in Section 5.2.

2

MATHEMATICAL CONCEPTS AND TOOLS

In this Chapter we are going to introduce the notation, theoretical concepts and tools that will be used throughout this dissertation. More in details, we present notions of graph theory, dynamical systems and game theory that will be used to build and study saturated network models in the following chapters.

2.1 BASIC NOTATION

To begin with, we explain the basic notation to be used throughout this work. Vectors are denoted with lower case, matrices and random variables with upper case, and sets (and set-valued functions) with calligraphic letters. A subscript associated to vectors, for instance $v_{\mathcal{A}}$, represents the sub-vector that is the restriction of a vector v in $\mathbb{R}^n$ on the set of indices $\mathcal{A} \subseteq \{1, 2, \ldots, n\}$. The same notation is used for matrices: $P_{\mathcal{A}\mathcal{B}}$ represents the sub-matrix of P obtained by considering rows and columns associated with the indices contained in sets $\mathcal{A}$ and $\mathcal{B}$, respectively. We view all vectors as column vectors and we use $x^\top$ to denote the transpose of a vector x; the same holds for matrices.

We indicate with $\mathbb{1}$ the all-1 vector, regardless of its dimension, and with I the identity matrix. Throughout the dissertation, the natural entry-wise partial order is considered on $\mathbb{R}^n$, so that, the inequality $x \leq y$ for two vectors x and y in $\mathbb{R}^n$ is to be understood as $x_i \leq y_i$ for every $i = 1, 2, \ldots, n$, whereas $x \lneq y$ means that $x \leq y$ with strict inequality in at least one entry. Analogously, the absolute value of a vector v in $\mathbb{C}^n$ is the vector $|v|$ in $\mathbb{R}^n_+$ with entries $(|v|)_i = |v_i|$ for $i = 1, \ldots, n$. A norm $\| \cdot \|$ on $\mathbb{C}^n$ is referred to as monotone if $\|v\| \leq \|w\|$ whenever $|v| \leq |w|$. Additional notations will be introduced throughout this work and explained when needed.

2.2 ELEMENTS OF GRAPH THEORY

In this section we present some basic notions of graph theory and we introduce the notation and the terminology used in the rest of this dissertation. Finally, we conclude presenting some relevant examples of graph topologies that we will use later on.

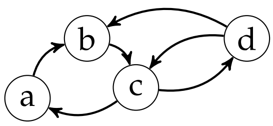
2.2.1 *Weighted directed graphs*

A finite *(directed weighted) graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is a mathematical entity identified by a triple:

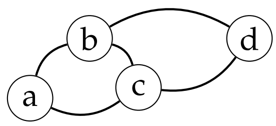
- a set of $n \in \mathbb{N}$ nodes, usually labeled by positive integer numbers, gathered in the node set $\mathcal{V} = \{1, \ldots, N\}$;
- a set of ordered pairs of nodes (i, j) with $i, j \in \mathcal{V}$, named links, which are collected in the link set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$;
- a weight matrix $W \in \mathbb{R}_+^{\mathcal{V} \times \mathcal{V}}$ that has the property that $W_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$, i.e., if (i, j) is a link. This also means that we can associate a weighted directed graph $\mathcal{G}_W = (\mathcal{V}, \mathcal{E})$ to any square matrix W in $\mathbb{R}^{n \times n}$ with node set $\mathcal{V} = \{1, 2, \ldots, n\}$, link set $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : W_{ij} \neq 0\}$ and weights given by W.

The presence of the edge (i, j) has to be interpreted as a connection between node i and node j and the associated weight W_{ij} quantifies the "strength" of the connection. Depending on the specific context, the link's weight may measure for instance the strength of a connection in terms of influence between two nodes, or, in the financial network applications, the nominal value of the debt that agent i has towards agent j. We shall refer to links (i, i) whose head node coincides with its tail node as *self-loops*.

In certain applications, links have an intrinsic bilateral meaning (e.g, symmetric interaction, friendship, partnership). This corresponds to a situation where two links (i, j) and (j, i) are either both present with the same weight $W_{ij} = W_{ji} > 0$, or both absent (so that $W_{ij} = W_{ji} = 0$). Graphs with this feature will be called undirected.



(a) Directed graph.



(b) Undirected graph.

Figure 1: Directed and undirected graphs.

When referring generically to a graph, we will implicitly intend it to be weighted and directed, unless it is otherwise specified or clear from the context.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, we introduce the following notions.

- The out-neighborhood and the in-neighborhood of a node $i \in \mathcal{V}$ are, respectively, the sets
$$\mathcal{N}_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}, \qquad \mathcal{N}_i^- = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$$
Nodes in $\mathcal{N}_i$ and $\mathcal{N}_i^-$ are referred to, respectively, as out-neighbors and in-neighbors of node i in $\mathcal{G}$.

- Nodes with no out-neighbors other than possibly themselves are called *sinks*, while nodes with no in-neighbors other than possibly themselves are called *sources*. E.g., the graph in Figure 2 contains a source (node 2) and a sink (node 4).

- The out-degree and in-degree of a node i are defined, respectively, as
$$w_i = \sum_{j \in \mathcal{V}} W_{ij}, \quad \text{and} \quad w_i^- = \sum_{j \in \mathcal{V}} W_{ji}.$$
Often we will use the shorter term degree for out-degree and the compact notation
$$w = W\mathbb{1}, \quad w^- = W^\top \mathbb{1}.$$

- $\mathcal{G}$ is called balanced if $w = w^-$.

- $\mathcal{G}$ is called regular if all its nodes have the same degree, i.e., if $w = w^- = \frac{1}{n}$.

Notice that in undirected graphs there is no distinction between out- and in-neighbors, out- and in-neighborhoods, and out- and in-degree.

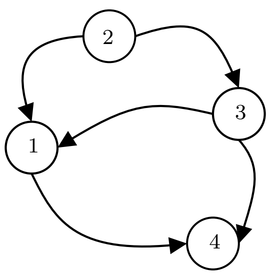


Figure 2: A directed graph with 4 nodes. Node 2 is a source and node 4 is a sink.

We end this subsection by introducing the notion of sub-graph. A sub-graph of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is any graph $\mathcal{H} = (\mathcal{U}, \mathcal{F}, Z)$ with node set $\mathcal{U} \subseteq \mathcal{V}$ link set $\mathcal{F} \subseteq \mathcal{E}$, and link weights $Z_{ij} \leq W_{ij}$ for every $i, j \in \mathcal{U}$.

2.2.2 *Reachability and connected components*

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. We introduce the following important definitions:

- A *walk* from node i to node j is a finite sequence of nodes $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ such that $\gamma_0 = i, \gamma_l = j$, and $(\gamma_{h-1}, \gamma_h) \in \mathcal{E}$ for all $h = 1, \ldots, l$, i.e., there is a link between every two consecutive nodes. Here, l is called the length of the walk. By convention, we consider walks of length 0 as going from a node to itself.

- A walk $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ such that $\gamma_h \neq \gamma_k$ for all $0 \leq h < k \leq l$, except for possibly $\gamma_0 = \gamma_l$, is called a *path*. In plain words, a path is a walk that does not pass through a previously visited node except possibly for ending in its start node;

- A node j is said to be *reachable* from a node i if there exists a walk from i to j;

- A graph $\mathcal{G}$ is called *strongly connected* if given any two nodes i and j, we have that i is reachable from j.

- Given a subset of nodes $\mathcal{U} \subseteq \mathcal{V}$, we say that $\mathcal{U}$ is *trapping* (in $\mathcal{G}$) if for every $i \in \mathcal{U}$ and every walk in $\mathcal{G}$ from i to some j, we have that $j \in \mathcal{U}$.

The analysis of the connectedness of a graph can be further refined by considering the so called *connected components* or *classes* of $\mathcal{G}$ that are the maximal subsets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k$ of the node set $\mathcal{V}$ such that, for every two nodes i and j in the same component $\mathcal{V}_h$, there exists a path from i to j. In other words, that means that the sub-graph associated to such a component is strongly connected. Note that the size of a connected component may range from 1 (in case there exists a node i such that there exists no other node $j \neq i$ such that both j is reachable from i and vice versa) to n (when the graph is strongly connected). The splitting in connected components constitutes a partition of the node set $\mathcal{V}$, i.e., one has that

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_k, \quad \mathcal{V}_h \cap \mathcal{V}_l = \emptyset, \quad h \neq l$$

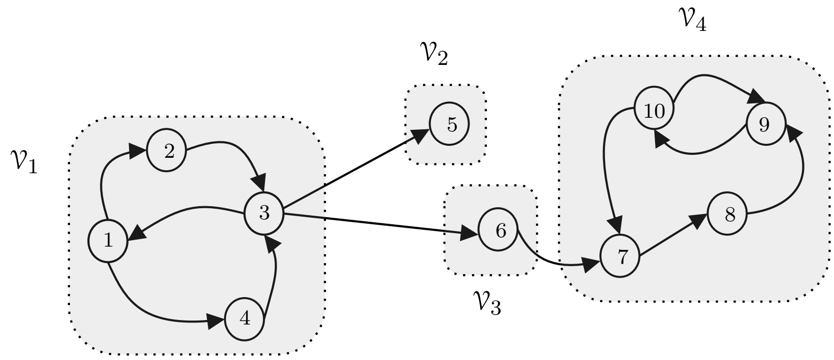


Figure 3: A graph with 10 nodes and 4 connected components. Notice that $\mathcal{V}_2$ and $\mathcal{V}_4$ are trapping sets. The sets $\mathcal{V}_1$ and $\mathcal{V}_3$ are also called *transient*.

In Figure 3 we show a graph consisting of 10 nodes, 4 connected components and 2 trapping sets, namely $\mathcal{V}_2$, which is also a sink, and $\mathcal{V}_4$. Intuitively, if we one starts moving at random from node to node according to the links present in the graph, it will eventually be "trapped" in either $\mathcal{V}_2$ or $\mathcal{V}_4$ and unable to get back to any of the other components.

2.2.3 *Elements of algebraic graph theory and non-negative matrices*

One of the key achievements of modern graph theory is the recognition that many graph properties admit an equivalent linear algebraic version. In this section, we introduce some of these notions that will be useful later on.

The most natural matrix associated to a weighted graph is of course the weighted matrix W. Typically, we will be working with a normalized version of such a matrix, called *normalized weighted matrix* and we will denote it with P.

P is defined element-wise by:

$$P_{ij} = \begin{cases} W_{ij}/w_i & \text{if } w_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Notice that all entries of P are non-negative: matrices with this property are simply referred to as non-negative. Moreover, by the definition of P it follows that

$$P\mathbb{1} \leq \mathbb{1}, \tag{2}$$

Non-negative square matrices satisfying property (2) are referred to as *sub-stochastic* matrices. In plain words, a non-negative matrix is sub-stochastic if the sum of the entries in each row never exceeds 1.

Notice that in the literature it is often assumed that sub-stochastic matrices have the additional property that for at least one row there is strict inequality. Here we prefer not to follow this convention and in this way our class of sub-stochastic matrices contains also matrices P for which $w_i > 0$ for all $i \in \mathcal{V}$ and, hence, satisfying:

$$P\mathbb{1} = \mathbb{1}, \tag{3}$$

Non-negative square matrices satisfying property (3) are also referred to as *stochastic matrices*.

In the following, we will denote the *spectral radius* of a matrix P, i.e., the largest absolute value of its eigenvalues, with the notation $\rho(P)$.

The structure of the normalized weight matrix P is also linked to the connectedness properties of the associated directed graph $\mathcal{G}_P$. In fact, a non-negative square matrix P is said to be *irreducible* if for every i and j, there exists $l \geq 1$ such that $\left(P^l\right)_{ij} > 0$. Equivalently, P is irreducible if and only if the associated graph $\mathcal{G}_P$ is strongly connected.

Finally, we present the following proposition gathering known important results about non-negative matrices that can be found, e.g., in the monograph [13].

Proposition 2.2.1 *Let P in $\mathbb{R}_+^{n\times n}$ be a non-negative square matrix. Then:*

(i) the spectral radius $\rho(P)$ is an eigenvalue of P and there exist vectors p and π in $\mathbb{R}_+^n \setminus \{0\}$ such that

$$Pp = \rho(P)p\,, \qquad \pi^\top P = \rho(P)\pi^\top\,.$$

Such vectors are called, respectively, a right and a left dominant eigenvector of P;

(ii) if Q is a principal square sub-matrix of P, then $\rho(Q) \leq \rho(P)$.

Moreover, if P is irreducible, then

(iii) the dominant eigenvectors p and π are unique up to normalization and have all positive entries. π is also referred as to the eigenvector centrality of the graph and also as to the invariant probability vector of the graph (when it is normalized such that $\mathbb{1}^\top\pi = 1$);

(iv) for every vector c in $\mathbb{R}^n$ such that $p^\top c = 0$, the equation $x = \rho(P)P^\top x + c$ admits infinite solutions x in $\mathbb{R}^n$;

(v) if Q is a principal proper square sub-matrix of P, then $\rho(Q) < \rho(P)$.

Example:

Let us consider the following weighted directed graph:

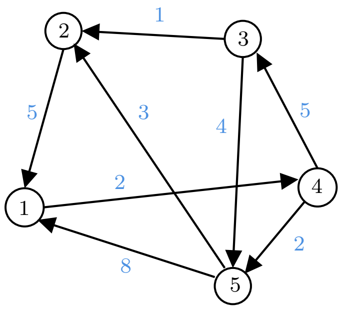


Figure 4: A weighted directed graph with 5 nodes and 8 links connecting them.

We can easily compute the weighted adjacency and normalized weighted matrices using their definitions:

$$W = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 5 & 0 & 2 \\ 8 & 3 & 0 & 0 & 0 \end{bmatrix} \implies P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & \frac{4}{5} \\ 0 & 0 & \frac{5}{7} & 0 & \frac{2}{7} \\ \frac{8}{11} & \frac{3}{11} & 0 & 0 & 0 \end{bmatrix}$$

Notice that the graph is strongly connected and hence $P \in \mathbb{R}_+^{5\times 5}$ is an irreducible matrix. According to Proposition 2.2.1(*iii*), we can compute the unique invariant probability vector π associated to the graph: $\pi \approx [0.25, 0.09, 0.18, 0.25, 0.22]^\top$. According to π, the most "central" nodes in the graph are 1 and 4.

2.2.4 *Spectral properties and non-expansive networks*

In this subsection, we introduce the notion of non-expansive network that will play a key role in the Chapter 3.

For a non-negative square matrix P in $\mathbb{R}_+^{n\times n}$, we shall consider the connected components $\mathcal{V}_1, \dots, \mathcal{V}_s$ of the associated digraph $\mathcal{G}_P$ and refer to them as the *classes* of P. Upon a possible permutation of the indices $i = 1, \ldots, n$, we can always assume that the matrix P admits the following block triangular structure

$$P = \begin{bmatrix} P^{(11)} & P^{(12)} & \ldots & P^{(1s)} \\ 0 & P^{(22)} & \ldots & P^{(2s)} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & P^{(ss)} \end{bmatrix}, \tag{4}$$

where, for $i, j = 1, \ldots, l$, $P^{(i,j)}$ in $\mathbb{R}_+^{\mathcal{V}_i \times \mathcal{V}_j}$ is the sub-matrix of P obtained by keeping only rows with index in $\mathcal{V}_i$ and columns with index in $\mathcal{V}_j$. Notice that this is equivalent to saying that the diagonal blocks $P^{(ii)}$ are irreducible and that in $\mathcal{G}_P$ there is no link from a node in a class $\mathcal{V}_l$ to any node in a class $\mathcal{V}_i$ with $i < l$. It then follows from Proposition 2.2.1 (ii) that $\rho(P^{(ii)}) \leq \rho(P)$. A class $\mathcal{V}_i$, for $1 \leq i \leq s$, will then be referred to (c.f. [13]) as:

- *basic* if $\rho(P^{(ii)}) = \rho(P)$;
- *final* if $P^{(ih)} = 0$ for every $h \neq i$.

We can state the following result, whose proof is presented in Appendix A.

Proposition 2.2.2 *Let P in $\mathbb{R}_+^{n \times n}$ be a non-negative square matrix. Then, there exists a positive vector v in $\mathbb{R}_+^n$ such that*

$$Pv \leq v\,, \tag{5}$$

if and only if $\rho(P) < 1$ or $\rho(P) = 1$ and every basic class of P is final.

Observe that to every positive vector v in $\mathbb{R}_+^n$ we may associate the weighted l_1-norm

$$\|x\| = \sum_{i=1}^{n} v_i |x_i|\,, \qquad x \in \mathbb{C}^n\,. \tag{6}$$

Clearly, the above is an absolute norm, hence a monotone norm [33]. Condition (5) implies that

$$\|P^\top x\| = v^\top P^\top |x| \leq v^\top |x| = \|x\|\,, \qquad \forall x \in \mathbb{C}^n\,. \tag{7}$$

We introduce the following definition.

Definition (Non expansive networks)
A network (P, w) is *non-expansive* if either

(i) $\rho(P) < 1$; or

(ii) $\rho(P) = 1$ and every basic class of P is final.

Remark A special class of non-expansive networks is provided by those networks (P, w) such that the matrix P is (row) sub-stochastic. Indeed, for a sub-stochastic matrix P, it can be easily checked that $\rho(P) \leq 1$ and that if $\rho(P) = 1$ then every basic class is necessarily final. We recall that in the literature it is often assumed that sub-stochastic matrices have the additional property that for at least one row there is strict inequality. Here we prefer not to follow this convention and in this way our class of sub-stochastic matrices contains also the stochastic matrices that are those for which $P\mathbb{1} = \mathbb{1}$.

2.3 ELEMENTS OF MONOTONE DYNAMICAL SYSTEMS

In this Section we present some notion of monotone dynamical system that will be exploited in the thesis. An in-depth study of this topic can be found in [30].

2.3.1 *Dynamical systems*

In very general terms, a dynamical system, also called a flow, is a tuple $(\mathcal{T}, \mathcal{X}, \phi)$ where $\mathcal{T}$ is a monoid, $\mathcal{X}$ is a non-empty set called the state space and $\phi(t, x)$ is a continuous function $\phi : \mathcal{T} \times \mathcal{X} \mapsto \mathcal{X}$ such that

- $\phi(0, x) = x$;
- $\phi\left(t_2, \phi\left(t_1, x\right)\right) = \phi\left(t_2 + t_1, x\right)$ for $t_1, t_2 \in \mathcal{T}$.

Usually $\phi(t, x)$ is called the evolution function and t the evolution parameter. In our case, the set $\mathcal{T}$ will be either the non-negative reals (continuous dynamical system) or the set of natural numbers (discrete dynamical system). In both cases, we shall also refer to the dynamical system as to a semi-flow.

We will also use the notation

$$\phi_x(t) \equiv \phi(t, x)$$
$$\phi^t(x) \equiv \phi(t, x)$$

whenever we take one of the variables as constant.

$$\phi_x : \mathcal{T} \mapsto \mathcal{X}$$

is called the flow through x.

Definition (Monotone dynamical system)
Consider a dynamical system $(\mathcal{T}, \mathcal{X}, \phi)$ where $\mathcal{X}$ is a metric space equipped with a partial order $\leq$. We say that the system is a *monotone dynamical system* when it preserves such a partial order, i.e.,

$$x \leq y \implies \phi^t(x) \leq \phi^t(y), \qquad t \in \mathcal{T}, \quad x, y \in \mathcal{X} \tag{8}$$

Monotonicity is a strong property of dynamical systems and it has been extensively studied due to the fact that the evolution of monotone systems is severely limited and much can be said about their asymptotic behavior.

2.3.2 *Monotone systems on lattices*

In this dissertation we will often consider monotone systems where the state space forms a complete lattice.

Definition (Lattice)
The partially ordered set $\mathcal{X}$ is a *lattice* $(\mathcal{X}, \leq)$ if for each two point set $\{x, y\} \subset \mathcal{X}$, there is a supremum (least upper bound) for $\{x, y\}$ (denoted $x \vee y$ and called the join of x and y) and an infimum (greatest lower bound) for $\{x, y\}$ (denoted $x \wedge y$ and called the meet of x and y) in $\mathcal{X}$. The lattice is complete if for all nonempty subsets $\mathcal{T} \subset \mathcal{X}, \inf(\mathcal{T}) \in \mathcal{X}$ and $\sup(\mathcal{T}) \in \mathcal{X}$. We denote the minimum and maximum element of a complete lattice $(\mathcal{X}, \leq)$ with $\underline{x}$ and $\bar{x}$ respectively.

Example:

- The real line (with the usual order) is a lattice as well as any compact subset of it.
- $\mathcal{X} = \prod_i \mathcal{A}_i$ where $\mathcal{A}_i \subseteq \mathbb{R}$ are compacts, with the component-wise order $x \leq y$ if and only if $x_i \leq y_i$ for all i. Moreover $(a \vee b)_i = \max\{a_i, b_i\}, (a \wedge b)_i = \min\{a_i, b_i\}$. This is also a complete lattice.

We first present an important fixed-point result that holds for monotone maps defined on complete lattices whose proof can be found in [55].

Proposition 2.3.1 *(Tarski's fixed point theorem)*
Let $(\mathcal{X}, \leq)$ be a complete lattice and $f : \mathcal{X} \mapsto \mathcal{X}$ an order-preserving function with respect to the partial order $\leq$. Then, the set of all fixed points of f is a complete lattice with respect to $\leq$. In particular, such set admits a minimum and a maximum element.

Tarski's fixed point theorem can be used to prove the following results about the asymptotic behavior of monotone discrete systems on lattices.

Proposition 2.3.2 *Consider the monotone discrete-time dynamical system defined by* $(\mathbb{N}, \mathcal{X}, \phi)$ *where* $(\mathcal{X}, \leq)$ *is a complete lattice. Then. the set of equilibria of the system also forms a complete lattice. In particular, such set admits a minimum element* $\underline{x}^*$ *and a maximum element* $\bar{x}^*$ *and we have* $\phi(t, \underline{x}) \to \underline{x}^*$ *and* $\phi(t, \bar{x}) \to \bar{x}^*$ *for* $t \to +\infty$*.*

Proof By Tarski's fixed point theorem 2.3.1 and by the monotonicity of the system, we have that $\phi^t(x) : \mathcal{X} \mapsto \mathcal{X}$ admits a complete lattice of fixed points. Moreover, since the system is monotone, for a fixed t, we have:

$$\bar{x} \leq \phi^t(\bar{x}) \leq \phi^t(\phi^t(\bar{x})) = \phi^{2t}(\bar{x}) \leq \phi^{3t}(\bar{x}) \leq \dots$$

This sequence is bounded below by $\underline{x}$ and hence it admits a limit: $\phi^t(\bar{x}) \to \bar{x}^*$. A completely analogous argument can be used to show that $\phi_t(\underline{x}) \to \underline{x}^*$.

∎

2.4 ELEMENTS OF GAME THEORY AND SUPERMODULAR GAMES

In this Section we introduce some elements of Game Theory, with particular focus on supermodular games. A wonderful introduction to the topic can be found in [36].

2.4.1 *Some basic notions*

We consider games in strategic form. There is a finite set of n players $\mathcal{V}$ and a set of actions $\mathcal{A}_i$ for each $i \in \mathcal{V}$. The assignment of an action to each player is described by a vector $x \in \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ that is called an action profile or configuration. Throughout, we shall denote by

$$\mathcal{X} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$$

the *configuration space*.

Each player $i \in \mathcal{V}$ is equipped with a *utility function* (a.k.a. reward or payoff function)

$$u_i : \mathcal{X} \to \mathbb{R}$$

that associates with every action profile x in $\mathcal{X}$ the utility $u_i(x)$ that player i gets when each player j is playing action $x_j \in \mathcal{A}$. We will often use the notation

$$x_{-i} = x_{|\mathcal{V} \setminus \{i\}}$$

for the vector obtained from the action profile x by removing its i -th entry, and, with a slight abuse of notation, write

$$u_i\left(x_i, x_{-i}\right) = u_i(x) \tag{9}$$

for the utility received by player i when she chooses to play action x_i, and the rest of the players choose to play x_{-i}. The triple $\Gamma = \left(\mathcal{V}, \{\mathcal{A}\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}}\right)$ will be referred to as a *(strategic form) game*.

Every player i is to be interpreted as a rational agent choosing her action x_i from the action set $\mathcal{A}$ so as to maximize her own utility $u_i\left(x_i, x_{-i}\right)$. In consideration of the fact that this utility depends not only on player i 's action x_i but also on the actions of the rest of the players' actions x_{-i}, it is natural to introduce the (set-valued) best response (BR) function

$$\mathcal{B}_i\left(x_{-i}\right) = \underset{x_i\in\mathcal{A}_i}{\operatorname{argmax}}\, u_i\left(x_i, x_{-i}\right)$$

Assuming that player i knows what the rest of the players' actions are and that these are not changing, choosing an action in $\mathcal{B}_i\left(x_{-i}\right)$ is for her the rational choice as it makes her utility as large as possible. Of course, when $\mathcal{A}_i$ is not finite, $\mathcal{B}_i\left(x_{-i}\right)$ could as well be an empty set.

Definition (Pure strategy Nash equilibrium)
A *(pure strategy) Nash equilibrium (NE)* for the game $\left(\mathcal{V}, \{\mathcal{A}\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}}\right)$ is an action configuration $x^* \in \mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i\left(x_{-i}^*\right), \quad i \in \mathcal{V}$$

The Nash equilibrium is called strict if, moreover, $\left|\mathcal{B}_i\left(x_{-i}^*\right)\right| = 1$ for every i.

The interpretation of a Nash equilibrium is the following: it is an action profile such that no player has any incentive to unilaterally deviate from her action, as the utility she is getting with that action is the best possible given the actions chosen by the other players. Note the emphasis on 'unilaterally': it is not at all guaranteed that coordinated deviations of more than one player from their actions in a Nash equilibrium could not lead to a higher utility for these players. As we shall see, there are games with multiple Nash equilibria and games which instead have none. We denote by $\mathcal{N}$ the set of Nash equilibria of a game.

Example:
(Cournot Competition)
This is a two-player game where the players are two firms producing a homogeneous good for the same market. The action of a firm i is a quantity, $x_i \in \mathcal{A} = [0, +\infty)$ representing the amount of good it produces. The utility for each firm is its total revenue minus its total cost,

$$u_i(x_1, x_2) = x_i p(x_1 + x_2) - cx_i, \quad i = 1, 2,$$

where $p(q)$ is the price of the good (as a function of the total quantity), and c is unit cost (same for both firms).

Assume for simplicity that $p(q) = \max\{0, 2-q\}$. Then, the best response is given by

$$\mathcal{B}_i(x_{-i}) = \begin{cases} 1 - c/2 - x_{-i}/2 & \text{if} \quad x_{-i} \leq 2 - c \\ 0 & \text{if} \quad x_{-i} > 2 - c \end{cases}$$

If $c < 2$ (the case when $c \geq 2$ is of no interest as 0 will be in this case a dominant strategy for both players), a simple computation shows that the only Nash equilibrium is given by the configuration

$$x_1 = x_2 = \frac{2-c}{3}$$

2.4.2 *Best response dynamics*

In this section, we introduce an important game-theoretic learning process, the *best response dynamics*. We start with the definition of the asynchronous best response dynamics, where players in a strategic form game get randomly activated one at a time and switch to a best response action.

Consider a strategic-form game $(\mathcal{V}, \{\mathcal{A}\}_{i \in \mathcal{V}}, \{u_i\}_{i \in \mathcal{V}})$. The continuous-time asynchronous best response dynamics is a Markov chain $X(t)$ with state space $\mathcal{X}$, where every player $i \in \mathcal{V}$ is equipped with an independent rate-1 Poisson clock. When her clock ticks at time t, player i updates her action to some y_i chosen from the action set $\mathcal{A}_i$ with conditional probability distribution that is uniform over the best response set (assuming that such a set is finite)

$$\mathcal{B}_i(X_{-i}(t)) = \underset{x_i \in \mathcal{A}_i}{\operatorname{argmax}} \{u_i(x_i, X_{-i}(t))\} \tag{10}$$

In particular, when the best response is unique, player i updates her action to such best response action. Hence, the continuous-time asynchronous best response dynamics is a continuous-time Markov chain $X(t)$ with state space coinciding with the configuration space $\mathcal{X}$ of the game and transition rate matrix Λ as follows: $\Lambda_{xy} = 0$ for every two configurations $x, y \in \mathcal{X}$ that differ in more than one entry, and

$$\Lambda_{xy} = \begin{cases} |\mathcal{B}_i(x_{-i})|^{-1} & \text{if} \quad y_i \in \mathcal{B}_i(x_{-i}) \\ 0 & \text{if} \quad y_i \notin \mathcal{B}_i(x_{-i}) \end{cases} \tag{11}$$

for every two configurations $x, y \in \mathcal{X}$ differing in entry i only, i.e., such that $x_i \neq y_i$ and $x_{-i} = y_{-i}$.

If the best response is unique, one could also consider the *discrete-time synchronous best response dynamics*, where all players update time to their unique best response at the same time. In such a case, the update rule (10) can be written as a discrete-time dynamics for each time $t \in \mathbb{N}$ as:

$$x_i(t+1) = \mathcal{B}_i(x_{-i}(t)) \tag{12}$$

Notice that, in contrast with the asynchronous best response, this is a deterministic dynamics.

A natural question that rises is under what assumptions the dynamics (10) (and (12)), starting from a certain initial state $x_0 \in \mathcal{X}$, converges to the set of Nash equilibria of Γ. It is well known that for certain classes of games, like the supermodular games that we will introduce in the next Section, this convergence is guaranteed.

2.4.3 *Supermodular games*

We introduce a particular family of games called *supermodular games* or *games with strategic complementarities*. These games enjoy very useful properties in terms of existence and structure of their Nash equilibria. A detailed study of supermodular games can be found in [45].

We start with some preliminaries:

Definition (Increasing Differences)
Given two lattices $\mathcal{A}_1$ and $\mathcal{A}_2$, a function $f : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}$ has increasing differences in its two arguments x and y if for all $x \geq x'$, the difference $f(x, y) - f(x', y)$ is non-decreasing in y.

In the game model that follows, if x is interpreted as one player's strategy, y as the other's, and u as the first player's utility, then the assumption of increasing differences is essentially the assumption of strategic complementarity: when the second player increases his choice variable(s), it becomes more profitable for the first to increase his as well.

Definition (Order Continuity and Semi-continuity) A chain $\mathcal{C} \subset \mathcal{A}$ is a totally ordered subset of the lattice $\mathcal{A}$, that is, for any $x \in \mathcal{C}$ and $y \in \mathcal{C}, x \geqslant y$ or $y \geqslant x$. Given a complete lattice $\mathcal{A}$, a function $f : \mathcal{A} \to \mathbb{R}$ is order continuous if it converges along every chain $\mathcal{C}$ (in both the increasing and decreasing directions), that is, if $\lim_{x\in\mathcal{C},x\downarrow\inf(\mathcal{C})} f(x) = f(\inf(\mathcal{C}))$ and $\lim_{x\in\mathcal{C},x\uparrow\sup(\mathcal{C})} f(x) = f(\sup(\mathcal{C}))$. It is order upper semi-continuous if $\limsup_{x\in\mathcal{C},x\downarrow\inf(\mathcal{C})} f(x) \leqslant f(\inf(\mathcal{C}))$ and $\limsup_{x\in\mathcal{C},x\uparrow\sup(\mathcal{C})} f(x) \leqslant f(\sup(\mathcal{C}))$.

We will now give the most general definition of a supermodular game.

Definition (Supermodular Game)
Consider now a game Γ in strategic form where each player $i \in \mathcal{V}$ has a strategy set $\mathcal{A}_i$ that comes with a partial order $\geq$. The game Γ is a supermodular game if, for each $i \in \mathcal{V}$:

- $\mathcal{A}_i$ is a complete lattice;
- $u_i : \mathcal{A}_i \to \mathbb{R} \cup \{-\infty\}$ is order upper semi-continuous at x_i (for fixed x_{-i}) and order continuous at x_{-i} (for fixed x_i) and has a finite upper bound;
- u_i has increasing differences in x_i and x_{-i}.

This definition is not always very practical. Luckily, for many games of interest, including the one that we will study in the second part of this dissertation, the conditions of supermodularity can be easily checked using the following Proposition.

Proposition 2.4.1 *Consider a game Γ in strategic form where each player $i \in \mathcal{V}$ has a strategy set $\mathcal{A}_i$ that comes with a partial order $\geq$. Then, Γ is supermodular if, for each $i \in \mathcal{V}$, the following conditions hold:*

- *$\mathcal{A}_i$ is an interval in $\mathbb{R}^{k_i}$, that is*

$$\mathcal{A}_i = \left[\underline{y}_i, \bar{y}_i\right] = \left\{x \mid \underline{y}_i \leqslant x \leqslant \bar{y}_i\right\}$$

- *u_i is twice continuously differentiable on $\mathcal{A}_i$;*
- $\dfrac{\partial^2 u_i}{\partial x_i \partial x_j} \geq 0$ *for all $i \neq j$.*

We point out that the differentiability condition can be replaced with a weaker condition on finite differences but this will suffice for our purposes.

Example:
(Bertrand Competition)
Suppose that $\mathcal{V}$ is a set of firms that simultaneously choose prices of a single good in order to maximize their profit. Assume that the demand of the good, given the prices $(p_i)_{i\in\mathcal{V}}$, is given by the linear function

$$d_i\left(p_i, p_{-i}\right) = a_i - b_i p_i + \sum_{j\neq i} d_{ij} p_j,$$

where $b_i, d_{ij} \geq 0$. The utility of each firm is given by its profit:

$$u_i\left(p_i, p_{-i}\right) = \left(p_i - c_i\right) D_i\left(p_i, p_{-i}\right),$$

where $c_i \geq 0$ is the marginal cost.

Notice that $\mathcal{A}_i = \mathbb{R}^+$ and $\dfrac{\partial^2 u_i}{\partial p_i \partial p_j} = d_{ij} \geq 0$. Hence, the game is supermodular by Proposition 2.4.1.

Example:
(Cournot Duopoly)
Let us consider again the Cournot duopoly introduced in Example 2.4.1. Let us assume that the price function $p(q)$ is such that

$$p'(q) + x_i p''(q) \leq 0$$

which formalize the reasonable assumption that firm i's marginal revenue decreases in q_{-i}. Let us now re-parameterize the game by introducing the new variables $z_1 = x_1$ and $z_2 = -x_2$ so that $q = z_1 - z_2$. With this choice we have that $\mathcal{A}_i = \mathbb{R}^+$ and

$$\frac{\partial^2 u_1}{\partial z_1 \partial z_2} = -\left(p'(q) + z_1 p''(q)\right) \geq 0 \tag{13}$$

$$\frac{\partial^2 u_2}{\partial z_1 \partial z_2} = -p'(q) + z_2 p''(q) = -\left(p'(q) + q_2 p''(q)\right) \geq 0. \tag{14}$$

Hence, the game is supermodular by Proposition 2.4.1.

In general, submodular two player games can be made supermodular by reversing the order on one of the strategies so that they also exhibit the useful properties of supermodular games. This trick does not work, however, for more than two-player games, which may exhibit dramatically different properties than the supermodular ones.

Notice that supermodular games model the so called strategic complements effect: the increase of one player's action makes more profitable for the others also to increase theirs.

Most of the properties of supermodular games stem from the following key fact (see [45]):

Proposition 2.4.2 *For a supermodular game* $\left(\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}}\right)$*, the following facts hold:*

- *For every* $i \in \mathcal{V}$ *and* x_{-i}*, the best response set* $\mathcal{B}_i(x_{-i})$ *has a maximum and a minimum element denoted, respectively,* $\mathcal{B}_i^+(x_{-i})$ *and* $\mathcal{B}_i^-(x_{-i})$*;*
- $\mathcal{B}_i^+(x_{-i})$ *and* $\mathcal{B}_i^-(x_{-i})$ *are monotone non-decreasing in* x_{-i}*.*

The above proposition can be used to prove the following fundamental statement, whose proof can be found in [45], gathering key properties that these games feature.

Proposition 2.4.3 *Let* $\Gamma_c = \left(\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u(x_i, x_{-i}, c)\}_{i\in\mathcal{V}}\right)$ *be a family of supermodular games with utilities parameterized by* $c \in \mathbb{R}$*, then the following holds:*

- *For fixed* c*,* Γ_c *admits a non-empty complete lattice* $\mathcal{X}(c)$ *of Nash equilibria. In particular, there exist a minimal pure Nash equilibrium* $\underline{x}(c)$ *and a maximal pure Nash equilibrium* $\bar{x}(c)$ *in* $\mathcal{X}(c)$*;*
- *for fixed* c *the discrete-time synchronous best response dynamics, where each player* i *plays:*

$$\begin{cases} x_i(t+1) = \mathcal{B}_i^-(x_{-i}(t), c) \\ x_i(0) = \min \mathcal{A}_i \end{cases}, \qquad \begin{cases} x_i(t+1) = \mathcal{B}_i^+(x_{-i}(t), c) \\ x_i(0) = \max \mathcal{A}_i \end{cases}$$

converge to $\underline{x}(c)$ *and* $\overline{x}(c)$ *respectively for* $t \to \infty$*;*

- *Suppose that for all* $i, x_{-i}, u_i(x_i, x_{-i}, c)$ *is supermodular in* (x_i, c)*. Then* $\overline{x}(c), \underline{x}(c)$ *are monotone non-decreasing functions of* c*.*

Notice that the first two bullets of Proposition 2.4.3 follow from the fact that the discrete-time best response dynamics of a supermodular game is a monotone system on a complete lattice and hence Proposition 2.3.2 applies. Moreover, this result implies that, when the Nash equilibrium is unique ($\underline{x}(c) = \overline{x}(c)$), there is global convergence of the discrete-time synchronous best response dynamics to it.

3

THE SATURATED EQUILIBRIUM MODEL

3.1 INTRODUCTION

In this Chapter, we undertake a fundamental study of a saturated equilibrium models in networks with positive externalities. Precisely, we consider the following fixed point equation

$$x_i = \min\left\{\max\left\{\sum_{j=1}^{n} x_j P_{ji} + c_i, 0\right\}, w_i\right\}, \qquad i = 1, \dots, n, \tag{15}$$

where P in $\mathbb{R}_+^{n\times n}$ is a non-negative square matrix and w in $\mathbb{R}_+^n$ is a non-negative vector that jointly describe a *network*, while c in $\mathbb{R}^n$ is an *exogenous flow* vector. Equation (15) can be more compactly rewritten as

$$x = S_0^w\left(P^\top x + c\right). \tag{16}$$

where S_0^w denotes the vector saturation function

$$\left(S_0^w(x)\right)_i = \min\{\max\{x_i, 0\}, w_i\}, \qquad i = 1, \dots, n, \tag{17}$$

We shall refer to vectors x that are solutions of (16) as *equilibria* of the network (P, w) with exogenous flow c. Notice that the range of the vector saturation function S_0^w is contained in the complete lattice

$$\mathcal{L}_0^w = \{x \in \mathbb{R}^n : 0 \leq x \leq w\}. \tag{18}$$

As the lattice $\mathcal{L}_0^w$ is a nonempty, convex, and compact set, and $x \mapsto S_0^w(P^\top x + c)$ maps $\mathcal{L}_0^w$ in itself with continuity, existence of network equilibria directly follows from Brower's fixed point Theorem. Hence, the set $\mathcal{X} \subseteq \mathcal{L}_0^w$ of network equilibria is always nonempty. On the other hand, the structure of such network equilibria as well as their uniqueness and dependence on the exogenous flow prove to be more delicate issues. They will be the object of this Chapter.

In financial networks, starting with the seminal work of Eisenberg and Noe [25], the entries of the vector w represent the obligations of the various institutions, those of the exogenous flow c represent the balance between assets possessed by the entities and their obligations towards institutions outside the network, while

P is a row-stochastic or sub-stochastic matrix describing the way obligations of an entity are split among the others thus encoding the backbone of the financial system interconnections. An equilibrium x represents, in this context, a set of payments that clear the network in a consistent way. A key question is to understand the extent to which a shock hitting the value of the assets of a single node i (perturbation of c_i) reflects on the entire network and leads to possible cascade effects. In particular, a default node is defined as one for which the quantity $\sum_j P_{ji}x_j + c_i$ (representing the liquidity of the entity i) is below the value of its obligation w_i and the default is called partial or total if, respectively, $\sum_j P_{ji}x_j + c_i > 0$ or not. Despite its apparent simplicity, this framework has proved to be very useful for analyzing how losses propagate through the financial system. Previous works including [40, 2, 28, 50] have analyzed conditions for uniqueness of the clearing payment equilibrium x and studied its dependence on the exogenous flow vector c. In particular, Eisenberg and Noe themselves [25] find sufficient conditions for uniqueness of clearing payment equilibria x in the special case of non-negative exogenous flow vector c and prove monotonicity and concavity of x as a function of c. Glasserman and Young [28] also consider the case of non-negative exogenous flow c and extend the sufficient conditions for uniqueness of the clearing payment equilibrium x in [25] to cover the case where the matrix P has spectral radius $\rho(P) < 1$. They also estimate the extent to which interconnections increase expected losses and defaults under a wide range of shock distributions, providing bounds on the potential magnitude of network effects on contagion and loss amplification. [2] consider a particular case of the Eisenberg and Noe model where the network is regular and prove that the clearing payment equilibrium is generically unique with respect to values of the exogenous flow c in $\mathbb{R}^n$. Furthermore, they prove rigorous results about the resilience of different network topologies depending on the shock magnitude. Liu and Statum [40] use linear programming to provide a sensitivity analysis of Eisenberg and Noe model with respect to certain parameters. Ren it et al. [50] explore several sufficient conditions for uniqueness of the clearing payment equilibrium, in particular showing that this holds true in the case where at least one entry of the maximal equilibrium is saturated at its upper bound or at least one entry of its maximal equilibrium is saturated at 0.

It is worth mentioning that several papers have recently appeared in the theoretical computer science community about the strategic analysis of financial networks. Although this studies generally focus on a more computational-based approach, they propose interesting extensions of the standard Eisenberg and Noe model that could also be natural generalizations for our results.

In [14] the Eisenberg and Noe model is analyzed from a game-theoretic perspective where each institution is a rational agent in a directed graph that has an incentive to allocate payments in order to clear as much of its debt as possible. Differently from what will be considered in this dissertation, the authors study the properties of a priority-based clearing mechanisms for financial markets instead that the standard pro-rata allocation rule. A similar approach is investigated in [35] where institutions can chose how to prioritize their payments in order to maximize their utilities. As mentioned before, this strategic aspect results in a quite different model from the one studied here, where the standard fixed pro-rata allocation rule for payments is considered.

In [31] the Eisenberg and Noe model is enriched by considering financial derivatives and in particular CDSs (credit default swaps) and authors study the clearing problem with these new assets.

In [49] authors study whether two banks can gain more assets or mitigate the effects of external shocks by executing a debt swap in the Eisenberg and Noe model. It turns out that swapping debts can actually be positive for both banks involved in the swap when their goal is to mitigate their losses in the worst possible case, which is given by an exogenous shock that makes a subset of banks lose all their assets in such a way that the banks involved in the swap have the minimum amount of assets possible.

In quadratic network games, the entries of the vector x represent the actions strategically chosen by n players, each one seeking to maximize a utility function $u_i(x) = c_i x_i - x_i^2/2 + x_i \sum_j P_{ji} x_j$ given by the difference between a linear return and a quadratic cost depending only on her own action plus a bi-linear term coupling her action with those of her neighbors in the network. These quadratic utilities are motivated by the fact that they can model a wide variety of network games, like imitation and coordination games. Here, the entries of the exogenous flow c represent the constant marginal benefits of the individual players from their own actions and coincide with their optimal choices in the absence of network interaction, whereas the nonzero entries of the matrix P correspond to either strategic complementarities (if they are positive) of strategic substitutes (if they are negative) between neighbor players in the network. In the absence of any constraints on their actions, the players' best responses are linear functions and Nash equilibria are solutions of the linear system $x = P^\top x + c$ whose existence and uniqueness can be characterized in terms of the spectral properties of P. In particular, if P has spectral radius $\rho(P) < 1$, then there exists a unique Nash equilibrium and, in the case when all externalities are positive, [9] show how its aggregate performance can be evaluated in terms of the sum of the individual players' marginal benefits weighted by their so-called Bonacich network centrality [15]. When upper and lower bounds on the feasible players' actions are considered, the best responses prove to be described as the composition of linear functions with saturation non-linearities and Nash equilibria coincide with the solutions of the fixed point equation (16) [19, 17]. In this case, it is known that, while existence is ensured by convexity and compactness of the strategy profile space as argued before, uniqueness is lost in general. In this regard, [9] claim that "multiple equilibria will certainly emerge, which is a plausible outcome in the school setting", while [17] acknowledge that "our general knowledge of how unique versus multiple equilibria depend on parameters and the network is still very fragmented." For symmetric quadratic games of strategic substitutes (i.e., non-positive symmetric P), Bramoullé *e*t al [16] prove uniqueness of Nash equilibria when P has spectral radius $\rho(P) < 1$, building on the fact that in this case the quadratic game is potential [47] with strictly concave potential function. On the other hand, in the special case when the exogenous flow c is strictly positive, Belhaj *e*t al [12] provide sufficient conditions for uniqueness of Nash equilibria for a class of network games with strategic complements (non-negative P) that include quadratic games, generalizing a previous result for fixed points of monotone concave functions [37].

The present Chapter develops a systematic study of the network equilibria described by equation (16) in the general case of networks (P, w) where P is a non-negative square matrix with spectral radius $\rho(P) \leq 1$ and provides three fundamental contributions:

(i) We characterize a class of non-expansive networks (c.f. Definition 2.2.4) including as a special case networks where P is a row-stochastic or sub-stochastic matrix and we prove that, for this class, all network equilibria satisfy an invariance property (Theorem 3.3.2) with respect to a certain partition of the node set in surplus, exposed, and deficit nodes (c.f. Section 3.3.2);

(ii) We analyze the structure of the set of network equilibria with respect to topological properties of the network. In particular, we show how to effectively construct all network equilibria starting from anyone of them and prove necessary and sufficient conditions for uniqueness of the network equilibrium in the general case of spectral radius $\rho(P) \leq 1$ (Theorem 3.4.3). This result subsumes and extends the ones available in the previously surveyed literature on financial networks, as in this context P is always a stochastic or sub-stochastic matrix, hence with spectral radius $\rho(P) \leq 1$. It is worth emphasizing that uniqueness conditions we derive can be easily checked a priori without the need for computing the network equilibrium itself.

(iii) We show that network equilibria exhibit a jump discontinuity in their dependence on the exogenous flow vector c when this is crossing certain regions of measure 0 representable as graphs of continuous functions, where the uniqueness of equilibrium is lost (Theorem 3.5.1). We provide an analytical description of the discontinuity set and we quantify the size of the largest jump (Corollary 3.5.2). In the financial network application, this can be interpreted as a jump in the aggregate loss function (c.f. Section 3.5.1 and Example 3.5.1).

Notice that, in contrast to some of the previously reviewed literature, we do not make any symmetry or regularity assumptions on the matrix P describing the network (c.f. [9, 19, 27, 16, 17, 3, 2]), nor on the sign of the exogenous flow c (c.f. [25, 28, 12]). This creates several technical challenges as in particular, we cannot rely on the theory of potential games (which would require P to be symmetric) and we have to deal with possible effective saturations at both the upper and the lower bound (while, e.g., assuming non-negative c would have removed the impact of the lower saturation).

From a methodological viewpoint, it is worth pointing out that non-negativity of the matrix P allows one to interpret the considered network equilibria as the Nash equilibria of a particular class of games with strategic complementarities. This implies that some of the general results in the theory of supermodular games [56, 45, 58, 57] can be applied in order to guarantee, e.g., that the set of network equilibria is a complete lattice, as well as the validity of certain comparative statics [46], in particular that the minimal and maximal network equilibria are monotone functions of the exogenous flow vector c, of the upper saturation vector w, and of the matrix P itself (Proposition 3.3.1). However, we depart quite soon from the general theory of supermodular games and develop an approach to the study of such monotone linear saturated network systems that partly hinges on some of the theory of non-negative matrices [13] (c.f. Proposition 2.2.1). Key steps in our treatment include the derivation of some *a*d hoc technical results exploiting finer spectral and topological properties of the network (Propositions 2.2.2, 3.4.1, and 3.4.2) that then prove instrumental in the proof of our main results (Theorems 3.3.2, 3.4.3, and 3.5.1). We notice that our results for non-expansive networks are somewhat reminiscent of the Rural Hospitals Theorem [52, 53] in the matching literature which, under suitable assumptions, shows that the set of stable matchings (hence, the equilibria in that setting) is a distributive lattice and satisfies a fundamental invariance property.

The rest of this Chapter is organized as follows. The remainder of this Introduction is devoted to a brief explanation of the main notational conventions to be followed throughout the Chapter. Section 3.2 presents the two main motivating applications for the model considered, i.e., financial networks and network games with linear saturated best replies. Section 3.3 establishes a number of preliminary results on the structure of the equilibria. Uniqueness results as well a general expression describing all solutions in non-uniqueness cases is presented in Section 3.4. Section 3.5 is devoted to the analysis of jump discontinuities in the equilibrium with respect to the variation of the exogenous flow vector with a particular focus on financial networks. The Chapter ends with Section 3.6 dedicated to draw some conclusions and open problems.

3.2 APPLICATIONS

In this section, we describe two main motivating applications. We start in Section 3.2.1 by presenting a model of financial networks generalizing the one first considered in [25]. We then provide an interpretation of network equilibria as Nash equilibria for a class of network games with monotone linear saturated best responses, as explained in Section 3.2.2. Moreover, the considered notion of equilibrium in saturated networks and the results derived in the following sections may find application in other contexts, such as, e.g., in some dynamical flow network models with fixed routing, which will be the object of Chapter 4.

3.2.1 *Payment equilibria in financial networks*

We consider a set $\mathcal{V} = \{1, \dots, n\}$ of financial entities (e.g., banks, broke dealers,...) interconnected by internal and external obligations that are specified by a non-negative matrix W in $\mathbb{R}_+^{n\times n}$ and three non-negative vectors a, b, and u in $\mathbb{R}_+^n$ whose entries have the following interpretation:

- $W_{ij} \geq 0$ is the liability of node i to node j;
- a_i is the total value of assets and credits of i from external entities;
- b_i is the total liability of node i to external non-financial entities;
- u_i is the total liability of node i to external financial entities.

The quantity $v_i = \sum_j W_{ji} - \sum_j W_{ij} + a_i - b_i - u_i$ is the net worth of node i. If the condition $v_i \geq 0$ is verified for every i in $\mathcal{V}$, it means that each node is fully liable and in principle capable to pay back all its liabilities to the nodes in the network as well the external ones. In case when instead some nodes do not satisfy the condition $v_i \geq 0$, namely they are not fully liable, it is necessary to determine a consistent set of payments among the various nodes.

Put $w_i = \sum_j W_{ij} + u_i$ and

$$P_{ij} = \begin{cases} W_{ij}/w_i & \text{if } w_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

We define by X_{ij} the payment from node i to node j and by X_{io} the payment from node i to external financial entities. Assuming that liabilities to non-financial entities have a higher seniority and that all other payments (including those to external financial entities) should be proportional to the corresponding liabilities, a consistent set of payments among the nodes has to satisfy the relations

$$\begin{aligned} X_{ij} &= \min\left\{P_{ij}\max\left\{\sum_k X_{ki}+a_i-b_i,0\right\},W_{ij}\right\} \\ X_{io} &= \min\left\{\frac{u_i}{w_i}\max\left\{\sum_k X_{ki}+a_i-b_i,0\right\},u_i\right\} \end{aligned} \tag{19}$$

Let $x_i = \sum_j X_{ij} + X_{io}$ be the total payment of node i to the financial entities. Summing the relations in (19) and using the fact that $W_{ij} = w_i P_{ij}$, we obtain

$$x_i = \min\left\{\max\left\{\sum_k X_{ki}+a_i-b_i,0\right\},w_i\right\} \tag{20}$$

so that, $X_{ij} = x_i P_{ij}$. Relation (20) can thus be rewritten as

$$x_i = \min\left\{\max\left\{\sum_k x_k P_{ki}+a_i-b_i,0\right\},w_i\right\} \tag{21}$$

This set of relations is equivalent to (19). Indeed, if the vector x solves (21), then $X_{ij} = x_i P_{ij}$ solves (19). This coincides with (15) with exogenous flow $c = a - b$. It is worth noticing that, in the financial jargon, vectors x are called *clearing vectors*.

Notice that the matrix P is sub-stochastic in its strict sense (i.e., at least one row does not sum to 1) when either there exist nodes with a positive liability towards external financial entities, or nodes with no financial liabilities.

In this financial setting, it is often considered the case when we start from a fully liable configuration, that is $v_i \geq 0$ for all i, leading to a solution x of (21) such that $x_i \geq w_i$ for all i. We then assume that the outside assets suffer a shock $\epsilon \in \mathbb{R}^n_+$ so that their values reduce to $a - \epsilon$ possibly making some of the v_i's negative. The study of the number of nodes in default $x_i < w_i$ as a function of the shock ϵ is one of the key issues.

3.2.2 *Network games with monotone linear saturated best responses*

We consider games with player set $\mathcal{V} = \{1, \ldots, n\}$, whereby each player i in $\mathcal{V}$ chooses an action x_i from the compact interval $\mathcal{A}_i = [0, w_i]$, where $w_i > 0$. We gather all actions in a vector x to be referred to as the strategy profile. Following a standard notational convention in game theory, we indicate by x_{-i} in $\prod_{j\neq i}\mathcal{A}_j$ the strategy profile of all players other than player i.

First, we consider the case of quadratic utility functions

$$u_i\,(x) = u_i(x_i, x_{-i}) = c_i x_i - \frac{x_i^2}{2} + x_i \sum_j P_{ji} x_j\,, \tag{22}$$

for every player i in $\mathcal{V}$ and strategy profile x. In (22), c_i denotes the marginal benefit of individual i from its own action, while P is a non-negative matrix describing the strategic interactions among the various players. Notice that, absent network effects, i.e., in the special case $P = 0$, c_i is the optimal action of player i.

Such games are known in the literature as constrained quadratic network games. Notice that the quadratic utility function u_i in (22) implies that the best response of a player i in $\mathcal{V}$ is always unique and given by

$$B_i(x_{-i}) = \min\left\{\max\left\{\sum_{j=1}^{n} x_j P_{ji} + c_i, 0\right\}, w_i\right\}. \tag{23}$$

It follows that Nash equilibria for such constrained quadratic network games are exactly the solutions of the fixed point equation (16).

In this work, we focus on the special case where the coefficients P_{ji} are all non-negative. In this way, we are considering games of pure strategic complements: for every player i, the higher the value of x_{-i}, the higher the rate of variation of the utility $u_i(x_i, x_{-i})$ of player i with respect to its own action x_i. As explained in Section 2.4.3, games such that actions belong to compact spaces and utilities u_i are twice differentiable functions with non-negative cross derivatives

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = P_{ji} \geq 0$$

for every i and j with $j \neq i$, are supermodular. It is known, as seen in Proposition 2.4.3, that supermodular games always admit a complete lattice of Nash equilibria and in our case they coincide with the solutions of (16). This fact will be exploited in the Section 3.3.1.

In fact, our analysis applies to the broader class of network games with linear saturated best response as in (23). This includes, e.g., games with player set $\mathcal{V}$, action space $\mathcal{A}_i = [0, w_i]$, for every player i in $\mathcal{V}$, and utility functions in the form

$$u_i(x) = \varphi_i\left(x_i - c_i + \sum_{j\neq i} P_{ji} x_j\right), \tag{24}$$

for a continuous function $\varphi_i : \mathbb{R} \to \mathbb{R}$ that is increasing on $(-\infty, 0]$ and decreasing in $[0, +\infty)$ [17]. Notice that (22) is a special case of (24) with $\varphi_i(y) = -y^2/2$.

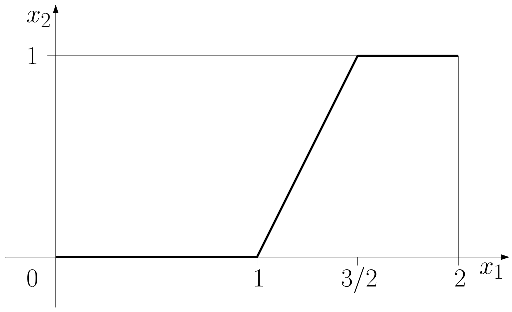


Figure 5: Set of network equilibria for the network in Example 3.3.

3.3 STRUCTURAL PROPERTIES OF NETWORK EQUILIBRIA

While existence of network equilibria is guaranteed for every network (P, w) and exogenous flow c, as discussed in Section 3.1, their uniqueness or multiplicity and more generally the structure of the network equilibrium set $\mathcal{X}$ remain more delicate issues, as also illustrated in the following simple example.

Example:
Consider a network (P, w) with $n = 2$,

$$P = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and the exogenous flow

$$c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

In this case, the fixed-point equation (16) reads

$$x_1 = S_0^2(x_1)\,, \qquad x_2 = S_0^1(x_1 + x_2/2 - 1)\,, \tag{25}$$

and the set of network equilibria is then

$$\mathcal{X} = \left\{ \left(t, S_0^1(2t-2)\right) : 0 \leq t \leq 2 \right\}, \tag{26}$$

as displayed in Figure 5.

In the rest of this section, we study structural properties of the set of network equilibria $\mathcal{X}$ for a network (P, w) with exogenous flow c, i.e., for the set of solutions of the fixed-point equation (16). Specifically, the contribution of this section is threefold. First, we exploit the fact that the network equilibrium set $\mathcal{X}$ can be interpreted as the set of Nash equilibria of the n-player supermodular game with utilities as in (22) and we establish a number of results concerning the lattice structure of $\mathcal{X}$ and its monotone dependence on the exogenous flow c. Second, we review some classical results on the spectral theory of non-negative matrices and derive some additional properties of the set of network equilibria $\mathcal{X}$ for a special class of non-expansive networks. Third, we introduce a fundamental partition of the node set into three subsets and prove that such partition is invariant with respect to the entire set of network equilibria for non-expansive networks. We wish to remark that, while the results concerning the lattice structure hold true in general for every network (P, w), the rest of the results are instead deeply connected to the finer spectral assumptions on the matrix P (c.f. Definition 2.2.4) and do not hold true for general networks. In particular, such results involve properties of the network equilibrium set that will play a crucial role in the following sections.

3.3.1 *Lattice properties of the set of network equilibria*

For a network (P, w) and an exogenous flow c, consider the following recursion on the complete lattice $\mathcal{L}_0^w$:

$$x(t+1) = S_0^w(P^\top x(t) + c)\,, \qquad t \geq 0\,. \tag{27}$$

Notice that equation (27) can be interpreted as the update rule of a synchronous best response dynamics for the supermodular game with utilities as in (22). The following proposition gathers a number of results on the network equilibria set $\mathcal{X}$ that follow immediately from Proposition 2.4.3 as a direct consequence of such game-theoretic interpretation.

Proposition 3.3.1 *Consider a network (P, w) and an exogenous flow c and let $\mathcal{X}$ be the corresponding set of network equilibria. Let $x(t)$, for $t = 0, 1, \dots,$ be the sequence generated by the recursion (27) with initial condition $x(0) = x_0$ in $\mathcal{L}_0^w$. Then:*

(i) *$\mathcal{X}$ is a complete lattice in $\mathbb{R}^n$. In particular, there exist a minimal network equilibrium $\underline{x}$ and a maximal network equilibrium $\overline{x}$ in $\mathcal{X}$;*

(ii) *if $x_0 = 0$, then $x(t)$ is non-decreasing and $\lim x(t) = \underline{x}$ as t grows large;*

(iii) *if $x_0 = w$, then $x(t)$ is non-increasing and $\lim x(t) = \overline{x}$ as t grows large;*

(iv) *both $\underline{x}$ and $\overline{x}$ are monotone non-decreasing functions of the exogenous flow c in $\mathbb{R}^n$, of the matrix P in $\mathbb{R}^n_+$, and of the upper saturation vector w in $\mathbb{R}^n_+$.*

Remark Observe that the recursion (27) can be implemented as a distributed iterative algorithm, whereby at every time $t = 0, 1, \ldots$, each node i in $\mathcal{V}$ updates in parallel its state $x_i(t)$ to

$$x_i(t+1) = S_0^{w_i}\left(\sum\nolimits_j P_{ji} x_i(t) + c_i\right) .$$

Notice that such update only requires each node i to observe the current states $x_j(t)$ of its in-neighbors $\{j \in \mathcal{V} : P_{ji} > 0\}$ and the total complexity of each iteration of (27) is of the order of the number of links in the network, i.e., the number of non-zero entries of P.

We now make some more refined considerations on the convergence time. Consider the recursion (27) with the initial condition $x(0) = 0$ and let $t_i^- = \inf\{t \geq 0 : x_i(t) = w_i\}$ for every $i = 1, \ldots, n$. By Proposition 3.3.1 (ii), whenever $t_i^- < +\infty$ we have $x_i(t) = w_i$ for every $t \geq t_i^-$. Analogously, by considering the recursion (27) this time with the initial condition $x(0) = w$ and letting $t_i^+ = \inf\{t \geq 0 : x_i(t) = 0\}$ for $i = 1, \ldots, n$, Proposition 3.3.1 (iii) guarantees that, whenever $t_i^+ < +\infty$ we have $x_i(t) = 0$ for every $t \geq t_i^+$. Observe that, since $\overline{x} \geq \underline{x}$, we necessarily have that at most one between (and possibly neither of) t_i^- and t_i^+ is finite. Let $t_i = \min\{t_i^-, t_i^+\}$ for all $i = 1, \ldots, n$. Then, when $t^* = \max\{t_i : 1 \leq i \leq n\} < +\infty$, we have a unique network equilibrium $x^* = \overline{x} = \underline{x}$ with every entry saturated from either below or above and convergence in finite time t^* is guaranteed to x^* from every initial condition $x(0)$ in $\mathcal{L}_0^w$. In contrast, when $t_i = +\infty$ for some i convergence typically occurs in infinite time, see Remark 3.3.2 for further considerations in this case.

In the following, we will make use of the definition of non-expansive network given in 2.2.4. A few comments are in order:

Remark Consider a non-expansive network (P, w) and let $\|\cdot\|$ be the monotone vector norm defined by (6) for a positive vector v satisfying (5). Then, for arbitrary vectors $x, \tilde{x}, c, \tilde{c}$ in $\mathbb{R}^n$, we have

$$\begin{aligned}
\|S_0^w(P^\top x + c) - S_0^w(P^\top \tilde{x} + \tilde{c})\| &= \sum_{i=1}^n v_i |S_0^{w_i}((P^\top x)_i + c_i) - S_0^{w_i}((P^\top \tilde{x})_i + \tilde{c}_i)| \\
&\leq \sum_{i=1}^n v_i |(P^\top x)_i + c_i - (P^\top \tilde{x})_i - \tilde{c}_i| \\
&\leq \sum_{i=1}^n v_i |(P^\top (x - \tilde{x}))_i| + \sum_{i=1}^n v_i |c_i - \tilde{c}_i| \\
&= \|P^\top (x - \tilde{x})\| + \|c - \tilde{c}\| \\
&\leq \|x - \tilde{x}\| + \|c - \tilde{c}\| ,
\end{aligned} \tag{28}$$

where the first inequality above follows from monotonicity of the weighted l_1-norm $\|\cdot\|$ and the last one from (7). This property justifies the terminology introduced in Definition 2.2.4.

Remark It is worth pointing out that existence of a (not necessarily monotone) vector norm $\|\cdot\|$ on $\mathbb{C}^n$ such that (7) holds true can be guaranteed under slightly weaker assumptions than those in Definition 2.2.4. Specifically [39] shows that this is equivalent to that either $\rho(P) < 1$ or $\rho(P) = 1$ and the geometric multiplicity of every eigenvalue λ of P with $|\lambda| = 1$ is equal to its algebraic multiplicity. In fact, notice that, when $\rho(P) = 1$, that every basic class of P is final implies that the geometric multiplicity of every eigenvalue λ of P with $|\lambda| = 1$ is equal to its algebraic multiplicity, but not vice versa. For a counterexample, take P as in Example 3.3: there P has unitary spectral radius and $\lambda = \rho(P) = 1$ is a simple eigenvalue, with algebraic and geometric multiplicities both equal to 1, however, there are two classes, $\mathcal{V}_1 = \{1\}$ and $\mathcal{V}_2 = \{2\}$, the first of which is basic but not final.

In fact, such stricter condition (ii) in Proposition 2.2.2 in the case when $\rho(P) = 1$ ensures not only existence of a vector norm $\|\cdot\|$ on $\mathbb{C}^n$ such that (7) holds true, but also that such a vector norm can be chosen as a weighted l_1-norm (6). It is exactly the monotonicity of such a norm that allows one to show that non-expansiveness is preserved when composing the affine map $P^\top x + c$ with the nonlinear saturation $S_0^w(\,\cdot\,)$, as in (28).

3.3.2 *Invariance property of network equilibria*

In this subsection, we show that the set of network equilibria $\mathcal{X}$ of every non-expansive network presents a relevant invariant property that will play a key role in the uniqueness results presented in the next section.

Consider an arbitrary network (P, w) with exogenous flow c. For a network equilibrium x in $\mathcal{X}$, we can always introduce the following partition of the node set $\mathcal{V} = \{1, 2, \dots, n\}$:

$$\mathcal{V} = \mathcal{V}_-^x \cup \mathcal{V}_+^x \cup \mathcal{V}_0^x\,, \tag{29}$$

where

- $\mathcal{V}_+^x = \left\{ i \in \mathcal{V} :\, c_i + \sum_{k \neq i} P_{ki} x_k > w_i \right\}$ is the set of *surplus* nodes;
- $\mathcal{V}_0^x = \left\{ i \in \mathcal{V} :\, 0 \leq c_i + \sum_{k \neq i} P_{ki} x_k \leq w_i \right\}$ is the set of *exposed* nodes;
- $\mathcal{V}_-^x = \left\{ i \in \mathcal{V} :\, c_i + \sum_{k \neq i} P_{ki} x_k < 0 \right\}$ is the set of *deficit* nodes.

Observe that, by the way these sets have been defined, it directly follows that

$$\begin{aligned} x_i &= 0\,, && \forall\, i \in \mathcal{V}_-^x\,, \\ x_i &= w_i\,, && \forall\, i \in \mathcal{V}_+^x\,, \\ x_i &= c_i + \textstyle\sum_{j\neq i} P_{ji} x_j\,, && \forall\, i \in \mathcal{V}_0^x\,. \end{aligned} \tag{30}$$

We now show that, if the network (P, w) is non-expansive, then partition (29) is invariant with respect to the chosen network equilibrium. This is stated in the following, which is the key result of this section and will be instrumental to all our future derivations.

Theorem 3.3.2 *For a non-expansive network (P, w) and any exogenous flow c in $\mathbb{R}^n$, the partition (29) is invariant over all equilibria x in $\mathcal{X}$.*

Proof We shall consider the maximal network equilibrium $\overline{x}$ and any another network equilibrium x in $\mathcal{X}$ and show that they share the same node partition (29). To begin with, notice that, since $\overline{x} \geq x$, we have $\mathcal{V}_+^{\overline{x}} \supseteq \mathcal{V}_+^x$ and $\mathcal{V}_-^{\overline{x}} \subseteq \mathcal{V}_-^x$. Let us split nodes in five different classes, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$, corresponding to the possible cases in which the entries of the network equilibria $\overline{x}$ and x can differ and are precisely defined as follows:

- $\mathcal{C}_1 = \mathcal{V}_+^x$ is the set of nodes that are surplus for both equilibria;
- $\mathcal{C}_2 = \mathcal{V}_+^{\overline{x}} \setminus \mathcal{V}_+^x$ is the set of nodes that are surplus for $\overline{x}$ but not for x;
- $\mathcal{C}_3 = \mathcal{V}_0^x \cap \mathcal{V}_0^{\overline{x}}$ is the set of nodes that are exposed for both equilibria;
- $\mathcal{C}_4 = \mathcal{V}_0^{\overline{x}} \setminus \mathcal{V}_0^x$ is the set of nodes that are exposed for $\overline{x}$ and deficit for x;
- $\mathcal{C}_5 = \mathcal{V}_-^{\overline{x}}$ is the set of nodes that are deficit for both equilibria.

We shall write any vector y in $\mathbb{R}^n$ in a block form $y = (y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)})$ and for simplicity of notation indicate $Q^{(ij)} := \left(P^\top\right)_{\mathcal{C}_i\mathcal{C}_j}$ for $i, j = 1, \dots, 5$. Notice that $\overline{x}^{(1)} = x^{(1)} = w^{(1)}$, $\overline{x}^{(5)} = x^{(5)} = 0$, and

$$w^{(2)} = \overline{x}^{(2)} < \sum_{k=1}^{4} Q^{(2k)} \overline{x}^{(k)} + c^{(2)}, \qquad x^{(2)} \geq \sum_{k=1}^{4} Q^{(2k)} x^{(k)} + c^{(2)}\,, \tag{31}$$

$$\overline{x}^{(3)} = \sum_{k=1}^{4} Q^{(3k)} \overline{x}^{(k)} + c^{(3)}, \qquad x^{(3)} = \sum_{k=1}^{4} Q^{(3k)} x^{(k)} + c^{(3)}\,, \tag{32}$$

$$\overline{x}^{(4)} = \sum_{k=1}^{4} Q^{(4k)} \overline{x}^{(k)} + c^{(4)}, \qquad 0 = x^{(4)} > \sum_{k=1}^{4} Q^{(4k)} x^{(k)} + c^{(4)}\,. \tag{33}$$

Put $z = \overline{x} - x \geq 0$ and notice that, for classes $\mathcal{C}_1$ and $\mathcal{C}_5$ we have that $z^{(1)} = z^{(5)} = 0$. For the remaining blocks, using (31), (32), and (33), we obtain

$$z^{(2)} < \sum_{k=2}^{4} Q^{(2k)} z^{(k)} , \qquad z^{(3)} = \sum_{k=2}^{4} Q^{(3k)} z^{(k)} , \qquad z^{(4)} < \sum_{k=2}^{4} Q^{(4k)} z^{(k)} . \tag{34}$$

Now, assume by contradiction that $\mathcal{C}_2 \cup \mathcal{C}_4 \neq \emptyset$, so that the above would imply that

$$z \lneq P^{\top} z . \tag{35}$$

Since the network is non-expansive, by Proposition 2.2.2 there exists a positive vector v such that (5) holds true. Together with (35), this would imply that

$$v^{\top} z < v^{\top} P^{\top} z \leq v^{\top} z ,$$

thus leading to a contradiction. This implies that necessarily $\mathcal{C}_2 = \mathcal{C}_4 = \emptyset$, so that $z = 0$, thus showing invariance of the node partition (29) with respect to the network equilibria x in $\mathcal{X}$. ∎

We gather some immediate consequences of Theorem 3.3.2 in the following result.

Corollary 3.3.3 *Let (P, w) be a non-expansive network. Then, for every exogenous flow c, there exists a partition of the node set*

$$\mathcal{V} = \mathcal{V}_{+} \cup \mathcal{V}_0 \cup \mathcal{V}_{-} , \tag{36}$$

such that, indicated with $z = (z^{(+)}, z^{(0)}, z^{(-)})$ the corresponding block decomposition of a vector z in $\mathbb{R}^n$ and with $P^{(\alpha\beta)} = P_{|\mathcal{V}_\alpha \times \mathcal{V}_\beta}$ for $\alpha, \beta = -, 0, +$,

(i) for every network equilibrium x in $\mathcal{X}$

$$x^{(-)} = 0 , \qquad x^{(0)} = P^{(00)\top} x^{(0)} + P^{(+0)\top} x^{(+)} + c^{(0)} , \qquad x^{(+)} = w^{(+)} ; \tag{37}$$

(ii) for every two network equilibria x and y in $\mathcal{X}$,

$$x^{(-)} = y^{(-)} , \quad x^{(+)} = y^{(+)} . \tag{38}$$

Corollary 3.3.3 implies that uniqueness can always be tested by simply looking at those entries of the network equilibria that belong to $\mathcal{V}_0$ and that such entries solve a linear system of equations. However, the outstanding difficulty in the analysis of the equilibrium set $\mathcal{X}$ stems from the fact that the partition (36) is not known a priori, a problem that will be dealt with in the next section.

Remark The necessity of the additional assumption that every basic class of P is final for networks (P, w) where P is non-stochastic and $\rho(P) = 1$ is illustrated by Example 3.3. In the network considered there, P has two classes: $\{1\}$ that is basic but not final and $\{2\}$ that is final but not basic. In fact, it is easily seen from (25) and (26) that, while node 1 is always exposed for every network equilibrium x in $\mathcal{X}$, node 2 is:

- a deficit node for every network equilibrium x in $\mathcal{X}_- = \{(t, 0) : 0 \le t < 1\}$;
- an exposed node for every network equilibrium x in $\mathcal{X}_0 = \{(t, 2t - 2) : 1 \le t \le 3/2\}$;
- a surplus node for every network equilibrium x in $\mathcal{X}_+ = \{(t, 1) : 3/2 < t \le 2\}$.

Therefore, partition (29) is clearly equilibrium-dependent in this case. As already pointed out in Remark 3.3 in this case the matrix P has unitary spectral radius and its eigenvalue $\lambda = \rho(P) = 1$ has algebraic and geometric multiplicities both equal to 1. This shows that, when $\rho(P) = 1$, the weaker condition that $\lambda = 1$ has algebraic multiplicity equal to its geometric multiplicity is not sufficient for the conclusions of Theorem 3.3.2 and Corollary 3.3.3 to hold true and the stricter assumption that every basic class be final is required.

Remark For a non-expansive network, consider once again the recursion (27) and, for $i = 1, \dots, n$, let t_i be defined as in Remark 3.3. Assume that $t_i = +\infty$ for some i and let $t^* = \max(\{0\} \cup \{t_i : t_i < +\infty\})$. Then, by combining the considerations in Remark 3.3 with Theorem 3.3.2 we get that the recursion (27) started in $x(0) = 0$ and $x(0) = w$ respectively determines partition (36) by time t^*. Indeed, the surplus, deficit, and exposed nodes are exactly those $i = 1, \dots, n$ such that $x_i(t^*) = w_i$, $x_i(t^*) = 0$, $0 < x_i(^*) < w_i$, respectively, for the sequence $x(t)$ generated by the recursion (27) started in an arbitrary initial condition $x(0)$ in $\mathcal{L}_0^w$. Notice that, once such partition has been determined, in other to find all network equilibria, one is simply left to solve the linear system

$$x_i = c_i + \sum_{j \neq i} P_{ji} x_j \,, \qquad \forall\, i \in \mathcal{V}_0 \,,$$

with boundary conditions $x_i = 0$ for all i in $\mathcal{V}_-$ and $x_i = w_i$ for all i in $\mathcal{V}_+$, something that can be performed in finite time using standard algorithms for linear systems, e.g., Gaussian elimination.

3.4 GEOMETRY AND UNIQUENESS OF NETWORK EQUILIBRIA

In this section, we undertake a fundamental geometric study of the set of network equilibria and, in particular, we derive necessary and sufficient conditions for their uniqueness. We shall first consider two relevant special cases:

- when the matrix P is asymptotically stable, i.e., such that $\rho(P) < 1$ (Proposition 3.4.1);
- when P is irreducible and such that $\rho(P) = 1$ (Proposition 3.4.2).

Then, we build on these two cases in order to prove a general result (Theorem 3.4.3) on the geometric structure of the network equilibrium set $\mathcal{X}$ for every network (P, w) such that P has spectrum contained in the closed unitary disk.

Proposition 3.4.1 *For a network (P, w) such that $\rho(P) < 1$ and every exogenous flow c in $\mathbb{R}^n$, there exists a unique network equilibrium x.*

Proof Let x and y in $\mathcal{X}$ be two network equilibria and put $\Delta = x - y$. We know from Corollary 3.3.3 (ii) that $\Delta_i = 0$ for every i in $\mathcal{V}_- \cup \mathcal{V}_+$. The proof is finished if $\mathcal{V}_0 = \emptyset$. Otherwise, let z in $\mathbb{R}^{\mathcal{V}_0}$ and Q in $\mathbb{R}^{\mathcal{V}_0 \times \mathcal{V}_0}$ be the restrictions of Δ to $\mathcal{V}_0$ and of P to $\mathcal{V}_0 \times \mathcal{V}_0$, respectively. It then follows from Corollary 3.3.3 (i) that z satisfies the equation $z = Q^\top z$. By Proposition 2.2.1 (ii), $\rho(Q) \leq \rho(P) < 1$, so that the matrix $(I - Q)$ is invertible and thus $z = 0$. Therefore, $x = y$. ∎

We now study the case of networks (P, w) with P irreducible and such that $\rho(P) = 1$. The following result gives an explicit characterization of the condition of non-uniqueness as well as a representation of the set of network equilibria.

Proposition 3.4.2 *Let (P, w) be a network such that P is irreducible and $\rho(P) = 1$. Let π and p be, respectively, left and right dominant eigenvectors of P, as in Proposition 2.2.1 (i). Then, for every exogenous flow c, there exists more than one network equilibrium in $\mathcal{X}$ if and only if*

$$p^\top c = 0, \quad \min_i \left\{ \frac{\nu_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} > 0\,, \tag{39}$$

where ν is any solution of the equation $\nu = P^\top \nu + c$ (c.f. Proposition 2.2.1 (iv)). Moreover, in this case, the set of network equilibria is given by

$$\mathcal{X} = \left\{ x = \nu + \alpha\pi : -\min_i \left\{ \frac{\nu_i}{\pi_i} \right\} \leq \alpha \leq \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} \right\}. \tag{40}$$

Proof We first analyze the solution on $\mathbb{R}^n$ of the non-saturated linear system

$$x = P^\top x + c\,. \tag{41}$$

Left multiplying by the vector p, we obtain

$$p^\top x = p^\top P^\top x + p^\top c = p^\top x + p^\top c$$

so that, for solutions of (41) to exist, it must hold true that $p^\top c = 0$. On the other hand, if condition $p^\top c = 0$ is satisfied, since P is irreducible, Proposition 2.2.1 (iii) and (iv) ensure that the set of solutions of (41) is an affine line

$$\mathcal{H} = \{x = \nu + t\pi : t \in \mathbb{R}\}\,. \tag{42}$$

where ν is any solution of (41). Notice that solutions of the linear system (41) that belong to the complete lattice $\mathcal{L}_0^w$ are necessarily network equilibria, i.e., $\mathcal{H} \cap \mathcal{L}_0^w \subseteq \mathcal{X}$. Moreover, observe that $\mathcal{H} \cap \mathcal{L}_0^w$ coincides with the right-hand side of (40) and that condition (39) is equivalent to saying that $\mathcal{H} \cap \mathcal{L}_0^w$ is a segment of strictly positive length.

We are now ready to prove the statements of the theorem. Suppose first that there are multiple equilibria, i.e., $|\mathcal{X}| > 1$. Since P is irreducible, the only class of $\mathcal{G}_P$ is basic and final, so that Theorem 3.3.2 implies that the node set partition (36) is common to all network equilibria. If $\mathcal{V}_- \cup \mathcal{V}_+ \neq \emptyset$, since $\mathcal{V}_0$ is a proper subset of $\mathcal{V}$, Proposition 2.2.1 (v) guarantees that the restriction Q of P to $\mathcal{V}_0 \times \mathcal{V}_0$ has spectral radius smaller than 1. Arguing exactly as in the proof of Proposition 3.4.1, we then deduce that $|\mathcal{X}| = 1$ thus reaching a contradiction. Therefore, necessarily $\mathcal{V} = \mathcal{V}_0$. In this case, it follows from Corollary 3.3.3 (i) that all network equilibria are solutions of (41), i.e., $\mathcal{H} \cap \mathcal{L}_0^w = \mathcal{X}$. By our previous considerations, since this set is nonempty, the condition $p^\top c = 0$ must hold true. Moreover, $|\mathcal{X}| > 1$ implies that $\mathcal{H} \cap \mathcal{L}_0^w$ must be a segment of positive length that, as previously observed, is equivalent to the second condition in (39).

Suppose now that the conditions in (39) hold true. Then previous considerations imply that $\mathcal{H} \cap \mathcal{L}_0^w \subseteq \mathcal{X}$ is a segment of positive length. Non-uniqueness of network equilibria is thus proven.

Finally, notice that, if any of the two equivalent conditions hold, then $\mathcal{H} \cap \mathcal{L}_0^w = \mathcal{X}$ and this is equivalent to representation (40). ∎

Remark The result above has a simple geometric interpretation in part already exploited in the proof. Assuming that $p^\top c = 0$, the line $\mathcal{H}$ defined in (42) is the set of solutions of the non-saturated linear system (41). The non-uniqueness condition (39) is simply the condition that this line intersects the interior part of the lattice $\mathcal{L}_0^w$ and the set of equilibria in this case is the segment obtained by this intersection.

The minimal and maximal equilibria are the boundary points of this interval. We notice that the arguments used in the proof also show that, in the case of non-uniqueness, necessarily all nodes must be exposed nodes, namely $\mathcal{V} = \mathcal{V}_0$.

Below, we report an explicit calculation of the network equilibria for a three-dimensional network and two possible exogenous flows, respectively yielding uniqueness and multiplicity of network equilibria.

Example:
Consider the network (P, w) where

$$P = \begin{bmatrix} 0 & 0.75 & 0.25 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}.$$

Notice that the matrix P is stochastic and irreducible, hence we can take $p = \mathbb{1}$. The associated graph $\mathcal{G}_P$ is depicted in Figure 6.

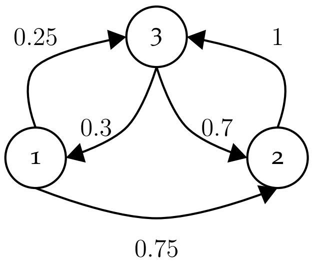


Figure 6: The network of Example 3.4.

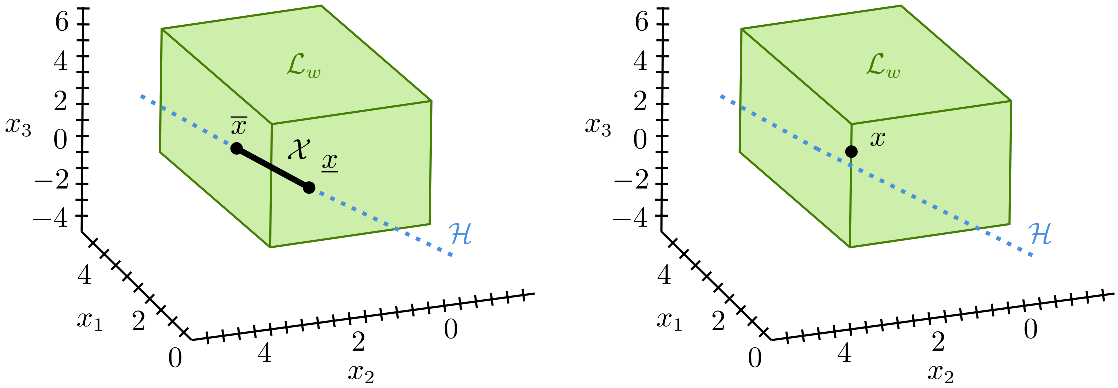
We analyze uniqueness for two possible exogenous flows

$$c^{(1)} = [-1, 1, 0]^\top \qquad c^{(2)} = [-2, 2, 0]^\top.$$

First of all, notice that $p^\top c^{(1)} = p^\top c^{(2)} = 0$. Moreover, a direct computation shows that

$$\begin{aligned} \min_i \left\{ \frac{\nu_i^1}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i^1}{\pi_i} \right\} &\approx 1.60 > 0 \\ \min_i \left\{ \frac{\nu_i^2}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i^2}{\pi_i} \right\} &\approx -6.41 < 0. \end{aligned} \tag{43}$$

By Proposition 3.4.2 we deduce that for the flow $c^{(1)}$ there are multiple equilibria, while for the flow $c^{(2)}$ the equilibrium is unique. The set of network equilibria $\mathcal{X}$ in the two cases is shown in Figure 7. Notice how in the first case the line $\mathcal{H}$ has a non-trivial intersection with the complete lattice $\mathcal{L}_0^w$ that is the segment of network equilibria. In contrast, in the second case, the line $\mathcal{H}$ does not intersect the complete lattice $\mathcal{L}_0^w$ and the unique network equilibrium is a single point lying on the boundary of the lattice as some of its entries x_i are necessarily saturated at either 0 or w_i.



(a) The network (P, w) with exogenous flow $c^{(1)}$ admits multiple equilibria (the black thick segment).

(b) The network (P, w) with exogenous flow $c^{(2)}$ admits a unique equilibrium (the black dot).

Figure 7: Sets of network equilibria for Example 3.4.

We now study the structure of network equilibria and give a full characterization of uniqueness in the general case of networks (P, w) where P is an arbitrary non-negative matrix with spectral radius $\rho(P) \leq 1$ and w is an arbitrary non-negative vector. Our analysis relies on the partition of the node set in the classes of P

$$\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_s \tag{44}$$

and on the corresponding triangular structure of P as described in (4).

Theorem 3.4.3 *Consider a network (P, w) such that $\rho(P) \leq 1$, and an exogenous flow c. Let (44) be the classes of P and assume that P is in the block triangular structure (4). Indicate the related split of a vector y in $\mathbb{R}^n$ as $y = [y^{(1)}, \ldots, y^{(s)}]^\top$. Then, the network equilibria x in $\mathcal{X}$ iteratively satisfy the following properties:*

(i) the projection $x^{(l)}$ on a class $\mathcal{V}_l$ such that $\rho(P^{(ll)}) < 1$ is unique;

(ii) given $(x^{(1)}, \ldots, x^{(l-1)})$, the projection $x^{(l)}$ on a class $\mathcal{V}_l$ such that $\rho(P^{(ll)}) = 1$ is non-unique if and only if

$$p^{(l)\top}\left(c^{(l)} + \sum_{1 \leq i < l} P^{(il)\top} x^{(i)}\right) = 0\,, \tag{45}$$

and

$$\min_{i\in\mathcal{V}_l}\left\{\frac{\nu_i^{(l)}}{\pi_i^{(l)}}\right\}+\min_{i\in\mathcal{V}_l}\left\{\frac{w_i-\nu_i^{(l)}}{\pi_i^{(l)}}\right\}>0\,, \tag{46}$$

where

- $p^{(l)}=P^{(ll)}p^{(l)}$ *is any right dominant eigenvector of the block* $P^{(ll)}$*;*
- $\pi^{(l)}=P^{(ll)\top}\pi^{(l)}$ *is any left dominant eigenvector of the block* $P^{(ll)}$*;*
- $\nu^{(l)}=P^{(ll)\top}\nu^{(l)}+\sum_{i=1}^{l}P^{(jl)\top}x^{(i)}+c^{(l)}$.

Moreover, in this case, given $[x^{(1)},\dots,x^{(l-1)}]^\top$*, the projection* $x^{(l)}$ *of any equilibrium satisfies*

$$x^{(l)}=\nu^{(l)}+\alpha\pi^{(l)}\,,\qquad -\min_{i\in\mathcal{V}_l}\left\{\frac{\nu_i^{(l)}}{\pi_i^{(l)}}\right\}\le\alpha\le\min_{i\in\mathcal{V}_l}\left\{\frac{w_i-\nu_i^{(l)}}{\pi_i^{(l)}}\right\}\,. \tag{47}$$

Proof It follows from (16) and the block triangular structure of P (4) that network equilibria satisfy the iterative relations

$$x^{(l)}=S_0^{w^{(l)}}\left(P^{(ll)\top}x^{(l)}+\sum_{0\le i<l}P^{(il)\top}x^{(i)}+c^{(l)}\right)\,,\qquad l=1,2,\dots,s\,. \tag{48}$$

The above says that the projection $x^{(l)}$ on the class $\mathcal{V}_l$ can be interpreted as a network equilibrium for the network $(P^{(ll)},w^{(l)})$ and exogenous flow $\sum_{i<l}P^{(il)\top}x^{(i)}+c^{(l)}$. The claim then follows from Propositions 3.4.1 and 3.4.2. ∎

Notice that, as Proposition 3.3.1 gives an efficient iterative way of computing the network equilibrium when this is unique, Theorem 3.4.3 provides an explicit way of computing, in an iterative way, the entire lattice of network equilibria $\mathcal{X}$ in the general case when $\rho(P)\le 1$.

Remark In the special case when the network is non-expansive (this includes the case when P is stochastic or sub-stochastic) Theorem 3.4.3 admits an important simplification. Indeed, in this case either $\rho(P)<1$, and then one can use Proposition 3.4.1 directly to compute the unique network equilibrium (e.g., by using (27) as a distributed iterative algorithm, c.f. Remark 3.3), or $\rho(P)=1$ and the basic classes are final so that we can always assume that in the partition (44) they are the last ones. Precisely, in the latter case, we can assume that

$$\rho(P^{(ll)})<1\quad for\quad l\le m,\qquad \rho(P^{(ll)})=1\quad for\quad m<l\le s\,. \tag{49}$$

The projection $(x^{(1)}, \dots, x^{(m)})$ of the network equilibria x on the first m classes is unique. For each basic class $\mathcal{V}_l$, with $m < l \leq s$, the non uniqueness condition of the projection $x^{(l)}$ is given by

$$p^{(l)\top} \left(c^{(l)} + \sum_{1 \leq i \leq m} P^{(il)\top} x^{(i)} \right) = 0\,, \tag{50}$$

together with (46). We notice that these conditions only depend on $(x^{(1)}, \dots, x^{(m)})$. In other words, once the solution on the non-basic classes is computed, the check of uniqueness and the parametrization of the solutions in case of non-uniqueness in the various basic classes are completely decoupled.

Remark Notice that our analysis has mostly focused on networks (P, w) with spectral radius $\rho(P) \leq 1$. In fact, Theorem 3.4.3 provides a complete description of the set of network equilibria $\mathcal{X}$ in this case. It is worth stressing out that, for networks with $\rho(P) > 1$, while $\mathcal{X}$ remains a nonempty complete lattice as per Proposition 3.3.1, its geometry can differ quite significantly in this case. In fact, consider a simple example with a single node, $P = 2$, and $w = 1$. Then, depending of the value of the exogenous flow c in $\mathbb{R}$ the set of network equilibria is

$$\mathcal{X} = \begin{cases} \{0\} & \text{if} \quad c < -1 \\ \{0, -c, 1\} & \text{if} \quad -1 \leq c \leq 0 \\ \{1\} & \text{if} \quad c > 0\,, \end{cases} \tag{51}$$

as illustrated in Figure 8. In particular, notice that for values of the exogenous flow c in $\mathcal{M} = [-1, 0]$, there are multiple isolated network equilibria, specifically $|\mathcal{X}| = 2$ for $c = -1$ and $|\mathcal{X}| = 3$ for $-1 < c < 0$. This is in stark contrast with the case $\rho(P) \leq 1$, where Theorem 3.4.3 in particular implies that, when the network equilibrium is not unique, there is in fact a continuum of network equilibria.

3.5 Continuity of network equilibria and the lack thereof

In this Section, we study the dependence of the network equilibria of a given network (P, w) on the exogenous flow c. This analysis is crucial to study the way exogenous shocks affect the payment equilibria in financial networks (c.f Section 3.2.1) or the individual marginal benefits affect the Nash equilibrium in quadratic network games (c.f. Section 3.2.2).

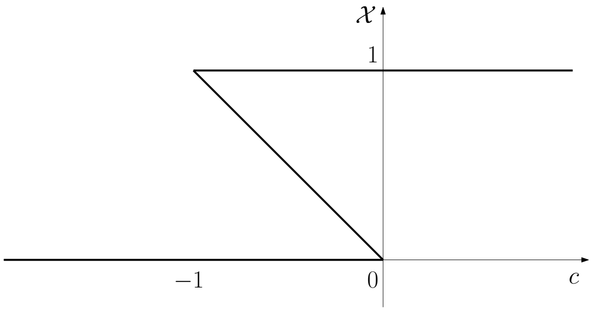


Figure 8: The set of network equilibria $\mathcal{X}$ for the network discussed in Remark 3.4 as a function of the exogenous flow c.

Let us consider a given network (P, w) and use the notation

$$\mathcal{X}(c), \quad \overline{x}(c), \quad \underline{x}(c)$$

to emphasize the dependence of, respectively, the set of network equilibria, and the maximal and minimal network equilibrium on the exogenous flow c. Moreover, let

$$\mathcal{U} = \{c \in \mathbb{R}^n : |\mathcal{X}(c)| = 1\} , \qquad \mathcal{M} = \mathbb{R}^n \setminus \mathcal{U} , \tag{52}$$

be the subsets of exogenous flows for which the network equilibrium is unique and, respectively, there are multiple network equilibria. For exogenous flows c in $\mathcal{U}$, we shall also use the notation

$$x(c) = \underline{x}(c) = \overline{x}(c)$$

for the unique equilibrium.

The following result gives a complete picture of the behavior of the set of network equilibria $\mathcal{X}(c)$ as a function of the exogenous flow c. It shows that the set of exogenous flows $\mathcal{M}$ for which the network equilibrium is not unique has Lebesgue measure 0 and is contained in the union of a finite number of graphs of continuous functions. Moreover, the network equilibrium $x(c)$ is a piece-wise continuous function of the exogenous flow c that undergoes jump discontinuities when c crosses the non-uniqueness set $\mathcal{M}$.

Theorem 3.5.1 *For a network (P, w) such that $\rho(P) \leq 1$, let m be number of basic classes of P and let $\mathcal{U}$ and $\mathcal{M}$ be defined as in (76). Then,*

(i) the non-uniqueness set $\mathcal{M}$ has Lebesgue measure 0 and is contained in the closed set consisting of the union of at most m graphs of scalar continuous functions;

(ii) the map $c \mapsto x(c)$ is continuous on the uniqueness set $\mathcal{U}$;

(iii) for every exogenous flow c^* in $\mathcal{M}$,

$$\liminf_{\substack{c\in\mathcal{U}\\ c\to c^*}} x(c) = \underline{x}(c^*)\,, \qquad \limsup_{\substack{c\in\mathcal{U}\\ c\to c^*}} x(c) = \overline{x}(c^*)\,.$$

Proof We start with a preliminary computation that will prove useful in the following derivations. Consider a sequence $c(1), c(2), \ldots$ of exogenous flows in $\mathbb{R}^n$ such that

$$c(t) \overset{t\to+\infty}{\longrightarrow} c^*\,, \qquad \underline{x}(c(t)) \overset{k\to+\infty}{\longrightarrow} x^*\,. \tag{53}$$

Since

$$\underline{x}(c(t)) = S_0^w\left(P^\top \underline{x}(c(t)) + c(t)\right)\,,$$

for all $t = 1, 2, \ldots$, passing to the limit in both sides of the above, by continuity we get that

$$x^* = S_0^w\left(P^\top x^* + c^*\right)\,,$$

thus showing that x^* belongs to $\mathcal{X}(c^*)$. In particular, this implies that

$$\underline{x}(c^*) \le x^* \le \overline{x}(c^*)\,. \tag{54}$$

Arbitrariness of the sequence satisfying (53) and (54) imply that

$$\underline{x}(c^*) \le \liminf_{\substack{c\in\mathcal{U}\\ c\to c^*}} x(c) \le \limsup_{\substack{c\in\mathcal{U}\\ c\to c^*}} x(c) \le \overline{x}(c^*) \tag{55}$$

In particular, for every exogenous flow c^* in $\mathcal{U}$, we have that $\underline{x}(c^*) = \overline{x}(c^*)$ and then relation (55) yields point (ii) of the claim.

Consider now the partition (44) of the node set into the classes of P and assume without loss of generality that P is in the block triangular structure (4). As usual, we indicate the relative split of any vector y in $\mathbb{R}^n$ as $y = [y^{(1)}, \ldots, y^{(s)}]^\top$. Assume that $l_1 < \cdots < l_m$ are the indices among $\{1, \ldots, s\}$ corresponding to the basic classes $\mathcal{V}_{l_1}, \ldots, \mathcal{V}_{l_m}$. For a fixed j, we consider the projection of the set of equilibria on $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{l_j - 1}$. Notice that, because of the triangular structure of P, such projected set depends on $c = [c^{(1)}, \ldots c^{(s)}]^\top$ only through the sub-vector $[c^{(1)}, \ldots c^{(l_j-1)}]^\top$. Suppose that for a given c and for a given j, such projected set is a singleton and indicate the projected block components of such equilibrium as $x^{(i)}([c^{(1)}, \ldots c^{(l_j-1)}])$ for $i = 1, \ldots, l_j - 1$. It then follows from Theorem 3.4.3 that a necessary condition for the projection of the equilibria on $\mathcal{V}_{(l_j)}$ not to be unique, is that

$$p^{(l_j)\top}\left(c^{(l_j)} + \sum_{i<l_j} P^{(il_j)\top} x^{(i)}[c^{(1)}, \ldots c^{(l_j-1)}]\right) = 0 \tag{56}$$

Now, define the sets $\mathcal{U}_k, \mathcal{M}_k \subseteq \mathbb{R}^{\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{l_k}}$ as follows:

$$\begin{aligned} \mathcal{U}_k &= \{[c^{(1)}, \dots, c^{(l_k)}] : [c^{(1)}, \dots c^{(l_j)}] \text{ does not satisfy (56) } \forall j \leq k\}\,; \\ \mathcal{M}_k &= \{[c^{(1)}, \dots, c^{(l_k)}] : [c^{(1)}, \dots c^{(l_j)}] \in \mathcal{U}_j \;\forall j \leq k-1, \text{and (56) is satisfied for } j = k\}\,. \end{aligned} \tag{57}$$

Put $\tilde{\mathcal{M}}_k = \mathcal{M}_k \times \mathbb{R}^{\mathcal{V}_{l_k+1} \cup \cdots \cup \mathcal{V}_m}$ and notice that the considerations above imply that

$$\mathcal{M} \subseteq \bigcup_{k=1}^{m} \tilde{\mathcal{M}}_k \tag{58}$$

Applying item (ii) to the restricted network consisting of the nodes in $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{l_k}$ we deduce that, for every $i = 1, \dots, l_k$, the functions $x^{(i)}([c^{(1)}, \dots c^{(l_k)}])$ are continuous on the set $\mathcal{U}_k$. This fact, together with the definition of $\mathcal{M}_k$ and the form of condition (56), allows us to conclude that $\mathcal{M}_k$ is the graph of a continuous function defined on $\mathcal{U}_{k-1} \times \mathbb{R}^{\mathcal{V}_{l_k} \setminus \{s_k\}}$ where s_k is any element in $\mathcal{V}_{l_k}$. An analogous conclusion then holds true for $\tilde{\mathcal{M}}_k$. This proves (i).

We are now left with proving (iii). Let c^* in $\mathcal{M}$ be an exogenous flow giving rise to multiple equilibria and define the sequence of exogenous flows $c(t)$ as follows:

$$c^{(i)}(t) = c^{*(i)} - \frac{1}{t} p^{(i)} \quad \forall\, i = 1, \dots, s\,.$$

where $p^{(i)}$ is any right dominant eigenvector of the block $P^{(ii)}$ We claim that $c(t)$ necessarily belongs to $\mathcal{U}$ for sufficiently large t. Indeed, a simple iterative argument shows that, if t is sufficiently large, $[c^{(1)}(t), \dots c^{(l_k)}(t)] \in \mathcal{U}_k$ for every k and therefore $c(t) \notin \tilde{\mathcal{M}}_k$ for every k. The claim then follows from (58). Since $c(t) \leq c^*$ for every $t = 1, 2, \dots$, it follows from Proposition 3.3.1 (iv) that

$$x(c(t)) = \underline{x}(c(t)) \leq \underline{x}(c^*)\,.$$

Using relation (55), we deduce that

$$\liminf_{\substack{c \in \mathcal{U} \\ c \to c^*}} x(c) = \underline{x}(c^*)\,. \tag{59}$$

An analogous argument allows us to prove the other relation in (iii) concerning the $\limsup$. ∎

For the special case of non-expansive networks (P, w), we are able to characterize the maximum discontinuity jump of the network equilibrium as the exogenous flow c varies in $\mathbb{R}^n$, as stated in the following result.

Corollary 3.5.2 *For a non-expansive network (P, w), consider the partition (44) of the node set into the classes of P and let the block triangular structure of P be as in (4). Let $\pi^{(l)}$ be any left dominant eigenvalue relative to $P^{(ll)}$. Then,*

(i) for every exogenous flow c, indicated with

$$L_c = \{l = 1, \dots, s \,|\, \mathcal{V}_l \textit{ is basic and } (50) \textit{ is satisfied}\}$$

the norm of the jump discontinuity of the network equilibrium at c can be expressed as

$$\|\overline{x}(c) - \underline{x}(c)\|_p^p = \sum_{\substack{l=1,\dots,s:\\ l \in L_c}} \left(\left[\min_{i \in \mathcal{V}_l} \frac{w_i - \nu_i^{(l)}}{\pi_i^{(l)}} + \min_{i \in \mathcal{V}_l} \frac{\nu_i^{(l)}}{\pi_i^{(l)}} \right]^+ \right)^p \|\pi^{(l)}\|_p^p \,, \qquad (60)$$

where $\nu^{(l)}$ is defined in Theorem 3.4.3.

(ii) the maximum jump discontinuity norm is for $c = 0$ and is given by

$$\max_{c \in \mathbb{R}^n} \|\overline{x}(c) - \underline{x}(c)\|_p^p = \|\overline{x}(0) - \underline{x}(0)\|_p^p = \sum_{\substack{l=1,\dots,s:\\ \mathcal{V}_l \text{ basic}}} \left(\min_{i \in \mathcal{V}_l} \frac{w_i}{\pi_i^{(l)}} \right)^p \|\pi^{(l)}\|_p^p \,, \qquad (61)$$

Proof Formula (60) directly follows from Theorem 3.4.3 by virtue of the non-uniqueness condition (45) as modified in (50) and the structure of solutions as expressed in (47). From (60), we obtain that

$$\|\overline{x}(c) - \underline{x}(c)\|_p^p \leq \sum_{\substack{l=1,\dots,s:\\ l \in L_c}} \left(\min_{i \in \mathcal{V}_l} \frac{w_i}{\pi_i^{(l)}} \right)^p \|\pi^{(l)}\|_p^p \leq \sum_{\substack{l=1,\dots,s:\\ \mathcal{V}_l \text{ basic}}} \left(\min_{i \in \mathcal{V}_l} \frac{w_i}{\pi_i^{(l)}} \right)^p \|\pi^{(l)}\|_p^p$$

On the other hand, since for $c = 0$ every l for which $\mathcal{V}_l$ is a basic class belongs to L_c, and since we can choose $\nu^{(l)} = 0$, formula (60) yields (61).

∎

A few comments are in order. First, notice that, for networks such that $\rho(P) = 1$, Theorems 3.4.3 and 3.5.1 ensure that the network equilibrium is generically unique and at the same time characterize the set $\mathcal{M}$ of exogenous flows inducing multiple network equilibria. As a function of the exogenous flow c, the network equilibrium $x(c)$ is proven to be a piece-wise continuous function (it is also monotone in c thanks to Proposition 3.3.1) with jump discontinuities occurring exactly when crossing the non-uniqueness set $\mathcal{M}$. For the relevant family of non-expansive networks, Corollary 3.5.2 establishes an explicit formula for the value norm of these jumps. For networks with $\rho(P) < 1$, Proposition 3.4.2 guarantees that the network equilibrium $x(c)$ is unique for every value of the exogenous flow c and, in this case, it is a monotone continuous function of it.

Another relevant observation is that the multiplicity of network equilibria for networks (P, w) with spectral radius $\rho(P) = 1$ and particular exogenous flows c^* can also be interpreted as an indicator of high sensitivity in the dependence of the network equilibrium $\tilde{x}(c)$ of networks $(\tilde{P}, w)$ with spectral radius $\rho(\tilde{P}) < 1$ that are sufficiently close to the nominal network (P, w). This is first illustrated by the following simple example.

Example Consider the family of networks $(P^{(\varepsilon)}, w)$, indexed by $\varepsilon \in [0, 1)$, with $n = 2$ nodes and

$$P^{(\varepsilon)} = \begin{bmatrix} 1-\varepsilon & 1 \\ 0 & 1/2 \end{bmatrix}, \qquad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Notice that for $\varepsilon \in (0, 1)$ we have $\rho(P^{(\varepsilon)}) = \max\{1-\varepsilon, 1/2\}$ and for every exogenous flow c in $\mathbb{R}^2$ there exists a unique network equilibrium $x^{(\varepsilon)}(c)$ with entries

$$x_1^{(\varepsilon)}(c) = S_0^2(c_1/\varepsilon), \qquad x_2^{(\varepsilon)}(c) = S_0^1(2c_2 + 2S_0^2(c_1/\varepsilon)).$$

On the other hand, for $\varepsilon = 0$ we recover the same network as in Example 3.3, with $\rho(P^{(0)}) = 1$. For such network, the set of exogenous flows giving rise to multiple equilibria is the whole line $\mathcal{M} = \{(0, t) : t \in \mathbb{R}\}$. It is then clear as the sensitivity of the first entry of the network equilibrium satisfies

$$\frac{\partial x_1^\varepsilon}{\partial c_1}(0^+, c_2) = \frac{1}{\varepsilon} \xrightarrow{\varepsilon \downarrow 0} +\infty,$$

for every c_1 in $\mathbb{R}$.

We conclude this section by discussing implications of our results in the two main motivating applications presented in Section 3.2.

3.5.1 *Systemic risk in financial networks*

Consider the generalized Eisenberg and Noe financial network model introduced in Section 3.2.1. In order to measure the aggregated effect of a shock, it is useful to introduce a risk measure known as *systemic loss* [60]. Let c° be a nominal exogenous flow for which all nodes in the financial network are fully liable, i.e., such that $x(c^\circ) = w$. Then, let $c \leq c^\circ$ be the exogenous flow after a shock has negatively affected the assets and external credits of some of the financial entities in the network and let $x(c)$ be a corresponding network equilibrium. As in Section 3.2.1, let the net worth vectors before and after the shock be, respectively, $v^\circ = P^\top w + c^\circ - w$ and $v = P^\top x(c) + c - w$. Then, the systemic loss is defined as their aggregate difference

$$l\left(c^\circ, c\right) := \mathbb{1}^\top \left(v^\circ - v\right) = \mathbb{1}^\top \left(P^\top w + c^\circ - w - \left(P^\top x(c) + c - w\right)\right)$$

$$= \mathbb{1}^\top \left(c^\circ - c\right) + \mathbb{1}^\top (w - x(c))\,. \tag{62}$$

In the rightmost side of the expression above, the term $\mathbb{1}^\top(c^\circ - c)$ represents the direct loss inflicted by the shock, while $\mathbb{1}^\top(w - x(c))$ represents the indirect loss triggered by reduced payments and is also referred to as shortfall term. Then, we may apply (62) and Theorem 3.5.1 (iii) to obtain the following expression for the size of the jump discontinuity of the systemic loss at some point $c = c^*$:

$$\Delta l\left(c^*\right) := \limsup_{\substack{c \in \mathcal{U} \\ c \to c^*}} l\left(c^\circ, c\right) - \liminf_{\substack{c \in \mathcal{U} \\ c \to c^*}} l\left(c^\circ, c\right) = \|\overline{x}(c^*) - \underline{x}(c^*)\|_1\,. \tag{63}$$

Explicit estimates of the expression above can then be obtained using formula (60) in Corollary 3.5.2. Systemic loss jumps are expected to play a crucial role in the resilience analysis of the financial network as they will often be associated to important failure events where several nodes simultaneously lose their liability, as illustrated in the following example.

Example:
Consider the financial network of Example 3.4. (Figure 12). The set $\mathcal{M}$ of exogenous flows giving rise to multiple network equilibria is plotted in Figure 9.

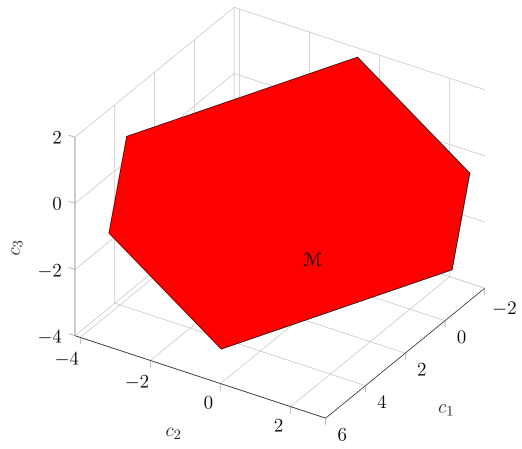
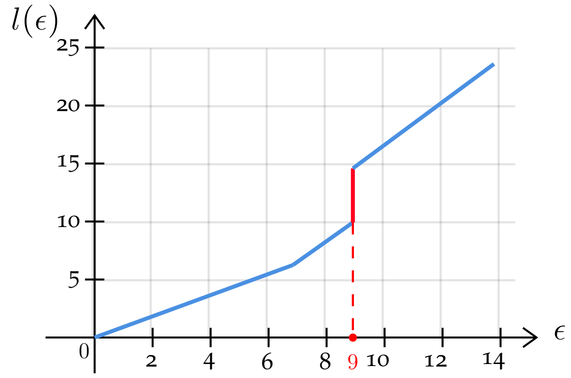


Figure 9: The set of critical shocks $\mathcal{M}$.

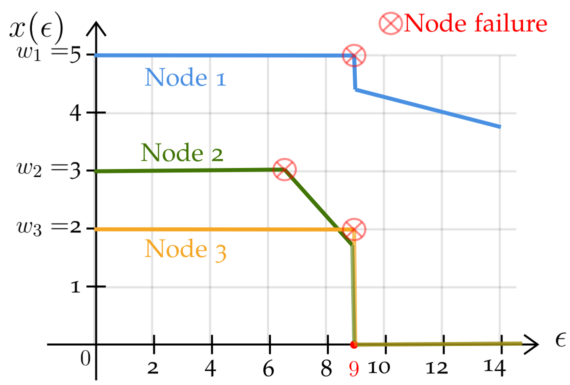
Consider an initial exogenous flow $c^\circ = [5, 2, 2]^\top$ and a perturbation of it $c = c^\circ - \epsilon q$, where $q = [0.07, 0.59, 0.34]^\top$, and $\epsilon \in [0, 14]$. A straightforward computation, using condition of Proposition 3.4.2, implies that the only case where we have multiple equilibria is for $\epsilon = 9$ corresponding to the exogenous flow $c^* = [4.4, -3.3, -1.1]^\top$ for which

$$\Delta l(c^*) = \min_i \{\nu_i / \pi_i\} + \min_i \{(w_i - \nu_i)/\pi_i\} \approx 4.44 > 0$$

The loss function and the equilibrium x as functions of ϵ are plotted in Figure 10.



(a) Loss as a function of ϵ. We can see the jump discontinuity at $\epsilon = 9$.



(b) Solution vector as a function of ϵ. We can see the jump discontinuity at $\epsilon = 9$, which brings nodes 1 and 3 to suddenly default.

Figure 10

In particular, Figure 10 (a) shows how the loss function varies piece-wise linearly until $\epsilon = 9$, where it undergoes the jump discontinuity of size $\Delta l(c^*)$. On the other hand, from Figure 10 (b) we can notice that all nodes are solvent for $\epsilon < 6.5$ while for $\epsilon \approx 6.5$ node 2 goes bankrupt as its outflow falls below $w_2 = 3$. As the shock magnitude increases, we reach the discontinuity point at $\epsilon = 9$ where the network suffers a dramatic crisis as nodes 1 and 3 suddenly default. Notice in particular how node 3 goes from fully solvent ($x_3 = w_3$) to completely insolvent ($x_3 = 0$) as the shock crosses the critical threshold $\epsilon = 9$.

Being able to compute critical shocks and the size of loss jumps around them can have a tremendous importance from a regulator perspective. A centralize authority should in fact try to keep the clearing system sufficiently far from the set of critical shocks to avoid sudden defaults. By computing the critical set, the regulator can estimate the distance of the current exogenous vector c from it and hence give a measure of robustness of the current clearing state. This can also suggest optimal aid policies aiming to maximize the distance from the critical set by injecting liquidity (i.e., by adjusting c) subject to a budget constraint.

3.5.2 *Sensitivity of Nash equilibria in constrained quadratic network games*

In the literature, the constrained quadratic games introduced in Section 3.2.2 are often studied [17] with the matrix P parameterized as $P^{(\delta)} = \delta G$ where G is some fixed matrix encoding the network interconnections and $\delta > 0$ is a parameter describing the strength of the network interaction among the agents. If we put $\delta^* = \rho(G)^{-1}$, we have that $\rho(\delta G) < 1$ for $\delta < \delta^*$. While Proposition 3.4.2 implies that, for every fixed $\delta < \delta^*$, the network equilibrium is unique and continuous in the exogenous flow c, its sensitivity to the variations of c may grow unbounded when δ approaches δ^*. As it turns out, this occurs when the limit network has multiple equilibria. Indeed, we have the following result showing that in this case, arbitrarily small variations in the exogenous flow c will determine, for δ close to δ^*, a variation in the equilibrium of the size of the set of equilibria for the limit case $\delta = \delta^*$.

Corollary 3.5.3 *For an irreducible matrix G in $\mathbb{R}_+^{n\times n}$ and a vector w in $\mathbb{R}_+^n$, and δ in $(0, \delta^*]$, where $\delta^* = 1/\rho(G)^{-1}$, let $P^{(\delta)} = \delta G$ and let $\overline{x}^{(\delta)}(c)$ and $\underline{x}^{(\delta)}(c)$ to be the minimal and maximal network equilibrium of the network $(P^{(\delta)}, w)$ with exogenous flow c in $\mathbb{R}^n$. Also, write $x^{(\delta)}$ for the network equilibrium when it is unique. Let c^* be an exogenous flow such that the (P^{δ^*}, w) has multiple network equilibria. Then,*

$$\sup_{\delta<\delta^*} \sup_{c:\,\|c-c^*\|\leq\epsilon} \|x^{(\delta)}(c) - x^{(\delta)}(c^*)\| \geq \|\overline{x}^{(\delta^*)}(c^*) - \underline{x}^{(\delta^*)}(c^*)\| > 0\,, \tag{64}$$

for every monotone norm $\|\cdot\|$ and every $\varepsilon > 0$.

Proof It follows from the comparative statics in Proposition 3.3.1 (iv) that, for $\delta < \delta^*$,

$$x^{(\delta)}(c^*) \leq \underline{x}^{(\delta^*)}(c^*) \lneq \overline{x}^{(\delta^*)}(c^*)\,. \tag{65}$$

Let p be any left dominant eigenvector of G and thus of all $P^{(\delta)}$. It then follows from Proposition 3.4.2 that,

$$\underline{x}^{(\delta^*)}(c^* + \epsilon p) = \overline{x}^{(\delta^*)}(c^* + \epsilon p)\,, \qquad \forall \epsilon > 0\,,$$

and thus, by Theorem 3.5.1 and Proposition 3.3.1 (v) again,

$$\lim_{\delta \downarrow \delta^*} x^{(\delta)}(c^* + \epsilon p) = \overline{x}^{(\delta^*)}(c^* + \epsilon p) \geq \overline{x}^{(\delta^*)}(c^*) \,. \tag{66}$$

For every monotone norm $\| \cdot \|$, (65) and (66) imply that

$$\lim_{\delta \downarrow \delta^*} \|\underline{x}^{(\delta)}(c^* + \epsilon p) - x^{(\delta)}(c^*)\| \geq \|\overline{x}^{(\delta^*)}(c^*) - \underline{x}^{(\delta^*)}(c^*)\| > 0 \,,$$

so that (64) holds true for every $\epsilon > 0$. ∎

3.6 CONCLUSION

This Chapter has analyzed network equilibria modeled as the solutions of a linear fixed point equation with saturation non-linearities. Necessary and sufficient conditions for uniqueness and a general expression describing all such equilibria for a general network with spectral radius not larger than 1 have been proved. Finally, the dependence of the network equilibria on the exogenous flows in the network has been studied highlighting the existence of jump discontinuities. This model was first considered to determine clearing payments in the context of networked financial institutions interconnected by obligations and it is one of the simplest continuous model where shock propagation phenomena and cascading failure effects may occur. It also describes the Nash equilibria of constrained quadratic network games with strategic complementarities. Our results contribute to an in-depth analysis of such applications.

4

A DYNAMICAL FLOW NETWORK MODEL WITH FINITE CAPACITIES

4.1 INTRODUCTION

In this Chapter, we study deterministic continuous-time models of dynamical flow networks where a saturated system such the one described by the fixed point equation (15) emerges naturally. The key point here is that we consider a dynamical model in contrast with the static one presented in Chapter 3 and this poses additional interesting problems that need to be addressed such that stability and convergence towards equilibria. Notice that in this case, the term "equilibrium" refers to the equilibrium point of a dynamical system, which will also satisfy the same fixed point equation studied in the previous Chapter.

More in details, we consider a finite number of cells exchanging some indistinguishable commodity among themselves and with the external environment. Cells possibly receive a constant exogenous inflow from outside the network and a constant flow is possibly drained out of them directly towards the external environment. We assume that the outflow from a cell is split among its immediately downstream cells in fixed proportions and that each cell has a finite flow and buffer capacity. When the total net flow in a cell, consisting of the difference between the total flow directed towards it minus the outflow from it, exceeds the cell's capacity, then the exceeding part of such net flow leaks out of the system. Also, when the difference between the total exogenous demand on a cell and the total inflow in it exceeds the cell's capacity, then the outflow towards the external environment is reduced by an amount equal to the exceeding part of this difference. The ensuing network flow dynamics turns out to be a linear saturated system with compact state space that we analyze using tools from monotone systems and contraction theory.

The study of dynamical flows in infrastructure networks has attracted a considerable amount of attention in recent years. In particular, there is a growing body of literature in the control systems field dealing with issues of stability, optimality, robustness, and resilience in dynamical flow networks. See, e.g., [48, 42, 26, 22, 11, 23, 20] and references therein. To the best of our knowledge, the study presented in this Chapter is the first one taking explicitly into account saturation constraints in a fixed routing network model.

Specifically, we give the following contributions:

- We prove that there exists a set of equilibria that is globally asymptotically stable. Such equilibrium set reduces to a single globally asymptotically stable equilibrium for generic exogenous demand vectors;
- we show the existence of critical exogenous demand vectors giving rise to non-unique equilibria correspond to phase transitions in the asymptotic behavior of the dynamical flow network.

Some of the results presented in this Chapter have a perfect correspondence with those presented in Chapter 3 and are proved to hold true also in this continuous-time setting. Moreover we exploit properties of monotone systems to prove that global convergence towards the equilibria is guaranteed.

The rest of this Chapter is organized as follows. The reminder of this section is devoted to the introduction of some notational conventions to be used. In Section 4.2 we present the class of dynamical flow network models that are the object of our study. Section 4.3 presents the main results concerning the equilibrium set characterization and its global asymptotic stability, as well as the dependence of such equilibria on the exogenous demand vector.

As usual, we shall consider the standard partial order on $\mathbb{R}^n$ whereby the inequality $a \leq b$ for two vectors $a, b \in \mathbb{R}^n$ is meant hold true entry-wise. A dynamical system with state space $\mathcal{X} \subseteq \mathbb{R}^n$ will be referred to as monotone if it preserves such partial order. Following the notation introduced in Chapter 3, for two vectors $a, b \in \mathbb{R}^n$ such that $a \leq b$, we shall denote by

$$\mathcal{L}_a^b = \{x \in \mathbb{R}^n : a \leq x \leq b\} = \Pi_{i=1}^n [a_i, b_i]$$

the complete lattice and let $S_a^b : \mathbb{R}^n \to \mathcal{L}_a^b$ be the vector saturation function defined by

$$\left(S_a^b(y)\right)_i = \max\{a_i, \min\{y_i, b_i\}\}, \qquad (67)$$

for $y \in \mathbb{R}^n$ and $i = 1, \ldots, n$.

4.2 A DYNAMICAL FLOW NETWORK MODEL WITH FINITE CAPACITY

We consider dynamical flow networks consisting of finitely many cells $i = 1, 2, \ldots, n$, exchanging an indistinguishable commodity both among themselves and with the external environment as described below. (See also Figure 11)

Let $x_i(t)$ be the quantity of commodity contained in cell $i = 1, 2, \ldots, n$ at time $t \geq 0$ and let $w_i > 0$ be its capacity. The state of the system is described by

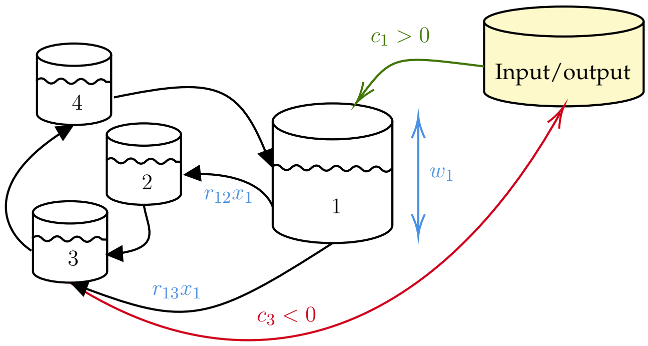


Figure 11: Illustration of a dynamical flow network with four cells.

the vector $x(t) = (x_i(t))_{1\leq i\leq n}$ and evolves in continuous time according to the following dynamical system

$$\dot{x} = f(x)\,, \tag{68}$$

where $f(x) = (f_i(x))_{1\leq i\leq n}$ is the vector of instantaneous net flows (inflows minus outflows) in the cells that will be assumed to satisfy the constraints

$$-x_i \leq f_i(x) \leq w_i - x_i\,, \qquad i = 0, \ldots, n\,, \tag{69}$$

throughout the evolution of the system. Notice that the leftmost inequality in (69) states that the outflow from cell i can never exceed the current inflow plus the total quantity of commodity in the cell, in particular implying the physically meaningful fact that the net flow $f_i(x)$ is non-negative when the cell is empty (i.e., when $x_i = 0$) so that $x_i(t)$ can never become negative. On the other hand, the rightmost inequality in (69) guarantees that the sum of the current total mass and the inflow in a cell i and can never exceed the difference between its capacity w_i and the current outflow, so that in particular, when the mass $x(t) = w_i$ has reached the capacity, the net flow $f_i(x)$ is non-positive, thus implying that the total mass will never exceed the capacity w_i if started below that. The complete lattice $\mathcal{L}_0^w$ is invariant for any dynamical flow network (68) satisfying (69).

Now, let each cell i possibly receive a constant exogenous inflow $\lambda_i \geq 0$ from outside the network and let a constant flow $\mu_i \geq 0$ possibly be drained directly from cell i towards the external environment, and let $c_i = \lambda_i - \mu_i$ be the exogenous net demand on cell i. Also, assume that constant fraction $R_{ij} \geq 0$ of the quantity of commodity x_i flows directly towards another cell $j \neq i$ in the network (fixed routing), while the remaining part $(1 - \sum_j R_{ij})x_i$ leaves the network directly. Notice that the routing matrix $R = (R_{ij}) \in \mathbb{R}^{n\times n}$ is necessarily sub-stochastic, i.e., with non-negative entries and such that its rows all have sum less than or equal to 1.

Conservation of mass and the constraint (69) imply that the net flow in each cell $i = 1, \ldots, n$ is given by

$$\begin{aligned} f_i(x) &= S_{-x_i}^{w_i - x_i}\left(\lambda_i - \mu_i + \textstyle\sum_j R_{ji} x_j - x_i\right) \\ &= S_0^{w_i}\left(\textstyle\sum_j R_{ji} x_j + c_i\right) - x_i \,. \end{aligned} \tag{70}$$

We may then rewrite the dynamical flow network (68)–(70) compactly as

$$\dot{x} = S_0^{w}\left(R^\top x + c\right) - x \,, \tag{71}$$

where $w \in \mathbb{R}^n$ is the vector of the cells' capacities. Observe that the function $f(x)$ as defined in (70) is Lipschitz continuous in $\mathbb{R}^n$, so that existence and uniqueness of a solution to the dynamical flow network (71) is ensured for every initial state $x(0) \in \mathcal{L}_0^w$.

Observe that in the dynamical network flow (71) it is understood that when the difference between the total flow $\lambda_i + \sum_j R_{ji} x_j$ directed towards a cell and the outflow $\mu_i + x_i$ from it exceeds the capacity w_i, then the exceeding part of it leaks out of the system. Moreover, the dynamical network flow (71) also assumes that, when the difference between the total exogenous demand μ_i on a cell i and the total inflow $\lambda_i + \sum_j R_{ji} x_j$ exceeds the cell's capacity w_i, then the outflow towards the external environment is reduced by an amount equal to the exceeding part of this difference.

In the following sections, we state the main results of this Chapter. These are concerned on the one hand with the geometry and global asymptotic stability of the dynamical flow network (71) and on the other hand on the dependence (in particular, continuity and the lack thereof) of the equilibria of (71) on the exogenous demand vector $c \in \mathbb{R}^n$.

4.3 GEOMETRY AND STABILITY OF EQUILIBRIA

In this Section we present important results about the geometry and stability of the equilibria. As far as the geometry is concerned, we notice that the equilibria of (71) are exactly the solutions of the saturated system (16) that we have thoroughly studied in Chapter 3 and this allows us to exploit several results already proved in that Chapter.

The dynamical nature of this model, however, imposes also questions about the stability and convergence towards such equilibria and in order to answer this, we will first present some technical results concerning properties of the system (71) that we will need to prove the main statement. Crucially, the routing the matrix R

considered in this Chapter is sub-stochastic, in contrast with the more general family of non-negative matrices considered in Chapter 3. This additional constraint will allow us to prove the monotonicity and non-expansiveness properties of the system (71) that are key to prove the stability results.

To improve the readability of this section, some technical proofs of the results presented here will be detailed in Appendix B.

We start with the following technical results that exploit monotonicity and, in part, results derived in Chapter 3.

Lemma 4.3.1 *The dynamical system (71) is monotone and non-expansive in l_1-distance on $\mathcal{L}_0^w$.*

Lemma 4.3.2 *The dynamical system (71) always admits a maximal equilibrium $\overline{x} \in \mathcal{L}_0^w$ and a minimal equilibrium $\underline{x} \in \mathcal{L}_0^w$. Moreover, the sets*

$$\mathcal{X}_\alpha = \left\{ x \in \mathcal{L}_{\underline{x}}^{\overline{x}} : \sum_i x_i = \alpha \sum_i \underline{x}_i + (1-\alpha) \sum_i \overline{x}_i \right\} \tag{72}$$

for $0 \leq \alpha \leq 1$ are all invariant for (71) and, for every initial condition $x(0) \in \mathcal{L}_0^w$, the solution of (71) is such that $x(t) \overset{t\to+\infty}{\longrightarrow} \mathcal{L}_{\underline{x}}^{\overline{x}}$.

We will also make use of the following result:

Lemma 4.3.3 *Let x^* be an equilibrium of the dynamical flow network (71) belonging to the interior of the lattice $\mathcal{L}_0^w$. Then, there exists an $\varepsilon > 0$ such that, every solution of (71) with initial condition $x(0) \in \mathcal{L}_0^w$ such that $\|x(0) - x^*\| < \varepsilon$, coincides with the solution of the linear dynamics*

$$\dot{x} = (R^\top - I)x + c\,. \tag{73}$$

We are now ready to present the main results assessing the geometry and stability of the equilibria.

Theorem 4.3.4 *Let $w \in \mathbb{R}^n$ be a positive vector and $R \in \mathbb{R}^{n\times n}$ a sub-stochastic matrix. Then,*

(i) if R is sub-stochastic and out-connected, then, for every exogenous demand vector $c \in \mathbb{R}^n$ the dynamical flow network (71) admits a globally asymptotically stable equilibrium $x^* \in \mathcal{L}_0^w$.

On the other hand, if R is stochastic and irreducible, then *(ii)* *for every exogenous demand vector $c \in \mathbb{R}^n$ the set of equilibria $\mathcal{X}(c)$ of the dynamical flow network (71) is a nonempty line segment joining two points $\underline{x} \leq \overline{x}$ on the boundary of the lattice $\mathcal{L}_0^w$;*

(iii) *for every initial state $x(0) \in \mathcal{L}_0^w$, the solution of (71) converges to the equilibrium set $\mathcal{X}(c)$ as t grows large;*

(iv) *the equilibrium set $\mathcal{X}(c)$ has positive length if and only if*

$$\min_i \left\{ \frac{\nu_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} > 0 \tag{74}$$

where ν is any solution of $\nu = R^\top \nu + c$.

Proof

(i) It immediately follows as a particular case of Theorem 3.4.3 that, when R is sub-stochastic out-connected, $\underline{x} = \bar{x} = x^*$ is the unique equilibrium. From Lemma 4.3.2 such an equilibrium is globally asymptotically stable.

(ii) It immediately follows from Theorem 3.4.3 that the equilibria of (71) form a line segment joining two points on the boundary of $\mathcal{L}_0^w$.

(iii) If $\underline{x} = \bar{x}$, then the global convergence follows from Lemma 4.3.2. Hence, we need to prove convergence in the case the system admits infinitely many equilibria. Notice that, for every $0 \leq \alpha \leq 1$, the set $\mathcal{X}_\alpha$ defined in (72) intersects the line segment $\mathcal{X}(c)$ in a single equilibrium point $x^*(\alpha) = \alpha \underline{x} + (1 - \alpha)\overline{x}$. Moreover, as discussed in the proof of point (ii) above, for every $0 < \alpha < 1$, such equilibrium $x^*(\alpha)$ belongs to the interior of the lattice $\mathcal{L}_0^w$, so that Lemma 4.3.3 implies that the dynamical flow network (71) reduces to the linear dynamical system (73) in a sufficiently small neighborhood of it. Now observe that all solutions of (73) with initial condition $x(0) \in \mathcal{X}_\alpha$ converge to x_α^* as t grows large. It then follows that, for every $0 \leq \alpha \leq 1$, there exists some $\varepsilon > 0$ such that for every solution $x(t)$ of the dynamical flow network with initial condition $x(0) \in \mathcal{X}_\alpha$ such that $\|x(0) - x^*(\alpha)\| < \varepsilon$ converges to $x^*(\alpha)$ as t grows large.
Now, let $\phi^t(x^\circ)$ be the solution of (71) started at $x(0) = x^\circ$. By Theorem 4.5 in [38] our last finding implies that, for every $0 \leq \alpha \leq 1$ there exists a $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that $\left\|\phi^t(x) - x^*(\alpha)\right\| \leq \beta\left(x - x^*(\alpha), t\right)$ for every $x \in \mathcal{X}_\alpha$ such that $\|x - x^*(\alpha)\|_1 \leq \varepsilon$. To prove global convergence to the set $\mathring{\mathcal{X}}(c)$ we need to show that for any $x^\circ \in \mathcal{X}_\alpha$ such that $\|x^\circ - x^*\|_1 > \varepsilon$, there exists a finite time $T \geq 0$ such that $\left\|\phi^T\left(x^\circ\right) - x^*(\alpha)\right\|_1 \leq \varepsilon$. For sake of notation, let us put $x^* = x^*(\alpha)$.

Now let $\hat{x} = x^* + \frac{\varepsilon}{\|x^\circ - x^*\|}\left(x^\circ - x^*\right)$, for which it is easily seen that $\|\hat{x} - x^*\|_1 = \varepsilon$, and

$$\begin{aligned}\|x^\circ - x^*\|_1 &= \|x^\circ - \hat{x}\|_1 + \|\hat{x} - x^*\|_1 \\ &= \|x^\circ - \hat{x}\|_1 + \varepsilon\end{aligned}$$

and consider the trajectories of the system starting from x° and $\hat{x}$. By the l_1-non expansive property ensured by Lemma 4.3.1 we have
$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi^t\left(x^\circ\right) - \phi^t(\hat{x})\right\|_1 \leq 0$, namely

$$\left\|\phi^t\left(x^\circ\right) - \phi^t(\hat{x})\right\|_1 \leq \|x^\circ - \hat{x}\|_1 .$$

By the triangle inequality,

$$\begin{aligned}\left\|\phi^t\left(x^\circ\right) - x^*\right\|_1 \leq& \left\|\phi^t\left(x^\circ\right) - \phi^t(\hat{x})\right\|_1 \\ &+ \left\|\phi^t(\hat{x}) - x^*\right\|_1 \\ &= \|x^\circ - \hat{x}\|_1 + \left\|\phi^t(\hat{x}) - x^*\right\|_1 \\ &= \|x^\circ - x^*\|_1 - \varepsilon \\ &+ \left\|\phi^t(\hat{x}) - x^*\right\|_1\end{aligned}$$

Due to the properties of the $\mathcal{KL}$ functions, there exists $T_{\frac{\varepsilon}{2}} \geq 0$ such that $\beta(x - y, t) \leq \frac{\varepsilon}{2}$ for all y such that $\|y - x^*\|_1 \leq \varepsilon$ and for all $t \geq T_{\frac{\varepsilon}{2}}$. Thus, we have

$$\begin{aligned}\left\|\phi^t\left(x^\circ\right) - x^*\right\|_1 &\leq \|x^\circ - x^*\|_1 - \varepsilon \\ &+ \left\|\phi^t(\tilde{x}) - x^*\right\|_1 \\ &\leq \|x^\circ - x^*\|_1 - \frac{\varepsilon}{2}\end{aligned} \tag{75}$$

for all $t \geq T_{\frac{\varepsilon}{2}}$. If $\left\|\phi^{T_{\frac{\varepsilon}{2}}}\left(x^\circ\right) - x^*(\alpha)\right\|_1 \leq \varepsilon$ the proof is complete with $T^- = T_{\frac{\varepsilon}{2}}$. Otherwise, the same argument can be reiterated. Since each step the ℓ_1 distance between $\phi^t(x)$ and x^* decreases by at least $\frac{\varepsilon}{2} > 0$, in no more than $\left\lceil \frac{2\|x^\circ - x^*\|_1}{\varepsilon} \right\rceil$ steps, i.e., for $T \leq \left\lceil \frac{2\|x^\circ - x^*\|_1}{\varepsilon} \right\rceil T_{\frac{\varepsilon}{2}}$, it holds $\|\phi^T\left(x^\circ\right) - x^*\|_1 \leq \varepsilon$.

(iv) It follows immediately from Theorem 3.4.3. ∎

4.4 CONTINUITY AND PHASE TRANSITIONS

Theorem 4.3.4 characterizes the set of equilibria $\mathcal{X}(c)$ and it is particularly relevant, in a given network, to study the behavior of such set with respect to possible variations of the exogenous net flow vector c. Indeed, this exogenous flow might be subject to shocks and variations that might affect the whole flow on the network. Thus, the resilience of the system with respect to shocks is in the end determined by the way solutions depend on the parameter vector c. Here we exploit what we proved in Chapter 3 showing that there exists a set of critical vector c such that the equilibria of (71) undergo a jump discontinuity, thus determining a phase transition in the asymptotic behavior of the system, and we will describe this critical set.

We follow the notation introduced in Chapter 3 where

$$\mathcal{U} = \{c \in \mathbb{R}^n : |\mathcal{X}(c)| = 1\} , \quad \mathcal{M} = \mathbb{R}^n \setminus \mathcal{U} , \tag{76}$$

are the subsets of exogenous flow vectors for which there is a unique equilibrium and, respectively, there are multiple equilibria. Moreover, we denote with $\underline{x}(c)$ and $\overline{x}(c)$ the smallest and largest equilibria for a given vector c. For exogenous flow vectors $c \in \mathcal{U}$, we shall also use the notation

$$x^*(c) = \underline{x}(c) = \overline{x}(c)$$

for the unique equilibrium.

We can now state the following result.

Theorem 4.4.1 *Let $w \in \mathbb{R}^n_+$ be a non-negative vector. Let $\mathcal{U}$ and $\mathcal{M}$ be defined as in (76). Then,*

(i) if R is sub-stochastic and out-connected, then, for every exogenous demand vector $c \in \mathbb{R}^n$ the map $c \mapsto x^*(c)$ is continuous.

On the other hand, if R is stochastic and irreducible, then

(ii) $\mathcal{M}$ is linear sub-manifold of co-dimension 1;

(iii) the map $c \mapsto x^*(c)$ is continuous on the set $\mathcal{U}$;

(iv) for every $c^* \in \mathcal{M}$,

$$\liminf_{\substack{c \in \mathcal{U} \\ c \to c^*}} x^*(c) = \underline{x}(c^*) , \qquad \limsup_{\substack{c \in \mathcal{U} \\ c \to c^*}} x^*(c) = \bar{x}(c^*) .$$

Proof Notice that (*i*), (*ii*), (*iii*) and (*iv*) all follow as a particular case of Theorem 3.5.1. In particular, since we are considering a matrix R stochastic and irreducible, the number of basic classes of R is $m = 1$. ∎

Theorem 4.4.1, and in particular the condition (iv), states that the equilibria of (71) undergo a jump discontinuity when the vector c crosses the set $\mathcal{M}$ for which the uniqueness condition for equilibria fails to hold. This in turn implies that even a slight change in the exogenous flow may trigger a phase transition in the system and a huge impact on the quantity of commodities exchanged at equilibrium in the network. We show this phenomenon in the following example.

Example:

Let us consider a flow model with an irreducible routing matrix R, in particular, we consider (71) with:

$$R = \begin{bmatrix} 0 & 0.75 & 0.25 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}, \; w = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}, \; c = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The corresponding flow network is shown in Fig. 12.

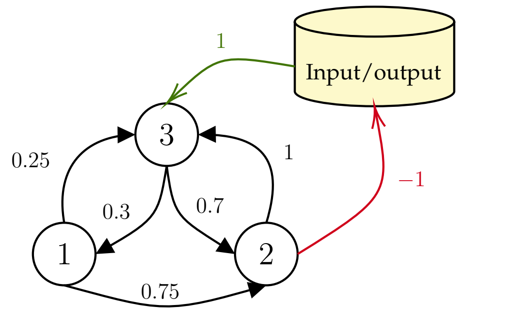


Figure 12: Flow network with three cells.

Since $\mathbb{1}^\top c = 0$ and $\min_i \left\{ \frac{(Hc)_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - (Hc)_i}{\pi_i} \right\} \approx 9.62 > 0$ then (71) admits multiple equilibria because of Theorem 4.3.4(iv). Indeed one can compute $\bar{x}(c) \approx [1.62, 4, 5.41]^\top$ and $\underline{x}(c) \approx [0.32, 0, 1.08]^\top$. We highlight the big jump that occurs for this particular vector c; notice how in the largest solution $\bar{x}$, cell 2 can deliver its total outflow capacity 4 while in the smallest solution $\underline{x}$ it outputs 0. A slight change of the exogenous flow around c could then have a huge impact on the network. In Fig. 13 we show some trajectories (in red) for different initial conditions in the phase space; we also plot the two lattices $\mathcal{L}_0^w$ and $\mathcal{L}_{\underline{x}}^{\bar{x}}$ (in green and light blue respectively); finally, the segment of equilibria $\mathcal{X}$ is plot in orange.

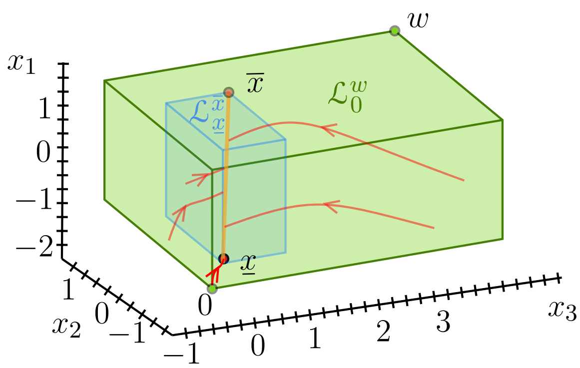


Figure 13: Trajectories in the phase space in case of multiple equilibria.

We can notice how all trajectories (red curves) converge to the set of equilibria (orange segment).

Let us now change slightly the vector c by setting: $c = [\frac{\alpha}{3}, -1, \frac{2\alpha}{3}]^\top$ with $\alpha \in [0, 9]$. Notice that we have multiple equilibria when $\alpha = 1 \implies c^* = [\frac{1}{3}, -1, \frac{2}{3}]^\top$ as in that case one can check that condition of Theorem 4.3.4(iv) holds. In Figure 14 we show the set of equilibria $\mathcal{X}(c)$ in the phase space as c varies as a function of α.

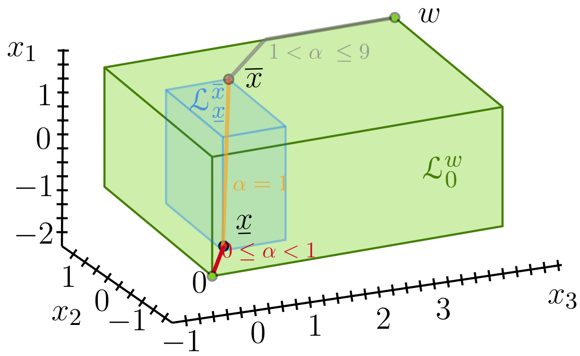


Figure 14: Set of equilibria in the phase space as α varies.

Notice that $x^*(c)$ is a piece-wise linear function. We can see that for $0 \leq \alpha < 1$ the equilibria (red segment) start from 0, they are unique and located on $\partial\mathcal{L}_0^w$, then when $\alpha = 1$ (and $c = c^*$) we have multiple equilibria (orange segment) and finally when $\alpha > 1$ the unique equilibria (gray segment) are located on $\partial\mathcal{L}_0^w$ until they eventually reach w, which means that all cells output their maximal flow.

We appreciate a phase transition of the dynamical system as the parameter α crosses the value $\alpha = 1$. In this case in fact, the equilibria undergo a jump discontinuity going from $\underline{x}(c^*)$ to $\bar{x}(c^*)$

4.5 CONCLUSIONS

In this Chapter we have introduced a nonlinear dynamical system that models a flow dynamic between cells with finite flow capacity. We have completely characterized the set of equilibria of the system and proved the global convergence of the solutions toward this set. Moreover, we have shown how the model exhibits critical phase transitions as the exogenous flow approaches a set of critical values. Future work includes a more in-depth analysis of the discontinuities and their relationship to the network structure and extending the dynamical flow model to allow for non-linearities in the dependence of the outflow from a cell on the mass of commodity in it.

5

CONCLUSIONS AND FUTURE RESEARCH

5.1 CONCLUSION

In this dissertation, we have undertaken a fundamental study of a saturated network model that has found several applications in literature. Specifically, in Chapter 3 we have considered a generalization of a saturated model first introduced with the key work of Eisenberg and Noe [25] in the context of financial networks. We have used novel approaches, leveraging concepts in the theory of supermodular games and non-expansive networks to obtain key results assessing existence and uniqueness of network equilibria that generalize previous results present in the literature. Within this framework, we have also given new insights on how the network structure can affect the propagation of shocks and on the important concept of systemic risk that these models try to capture. More in details, we have shown the equilibria can experience a jump discontinuity around certain critical shocks; this has profound implications for systemic risk, in particular in the context of financial networks, as arbitrarily small variants of the shock around such critical thresholds can trigger default cascades by causing multiple nodes to suddenly default. Moreover, this fundamental analysis of the dependence between the network equilibria allowed us to quantify the sensitivity of Nash equilibria with respect to exogenous inputs in a certain family of constrained quadratic games.

Our main contributions are the followings.

- We have completely characterized the network equilibria by introducing a class of non-expansive networks and proved that all such equilibria satisfy a fundamental invariant property with respect to a specific partition of the node set.

- We have characterized the structure of the set of network equilibria with respect to the structure of the underlying network itself. We have shown how to construct all equilibria given one of them and proved a Theorem that gives necessary and sufficient conditions for the uniqueness of the network equilibria in a very general case where we only assume that the spectral radius of the weighted adjacency matrix of the underlying network is less or equal than 1. Not only this results generalizes all previous known results but the condition for uniqueness can be easily checked a priori without the need for computing any network equilibrium.

- We have proved the existence of critical shocks for which the network equilibria exhibit a jump discontinuity. Such thresholds appear when the uniqueness conditions for network equilibria fail to hold true. The discontinuity set is described analytically as well as the largest jump that may occur in the network.

- We have provided a sensitivity analysis of the equilibria of certain constrained quadratic games with respect to the exogenous input. Moreover, we have quantified the effect of the jump discontinuity in the context of financial networks, highlighting the potential dramatic implications that such phenomenon can have in terms of systemic risk.

In Chapter 4 we have considered in details a relevant application of the saturated model studied in the previous Chapter. We have studied a dynamical flow model on networks with capacity constraints. In particular, the model represents a number of cells with finite capacity exchanging a common commodity around the network and with the external environment. Implementing these capacity constraints on the amount of commodities that a certain cell can contain is extremely relevant in a number of applications such as in the context of infrastructure and transportation networks. We have studied this model leveraging a number of results from the theory of monotone system and contraction theory while also exploiting the results derived in the previous Chapter to study the structure of the equilibria and their asymptotic stability. After completely characterizing the set of equilibria of the dynamical model, we have proved the global convergence of all solutions toward this set. Moreover, we have shown the existence of critical thresholds for the flow coming from the external environment that triggers phase transitions in the system by generating jump discontinuities in the set of equilibria.

Our main contributions are the followings.

- We have introduced a novel dynamical continuous-time flow model with finite capacities that is described by means of a differential equation involving saturation functions.

- We have proved that the system is monotone a non-expansive, this was used to prove the global convergence of the solutions toward the equilibria. Crucially, the fact that in this model the routing matrix is necessarily a substochastic matrix was key to prove the monotonicity and non-expansiveness properties that we needed to establish the convergence toward the equilibria.

- Exploiting the theory developed in the previous Chapter, we proved the existence of critical thresholds for the flow coming from the external environment that triggers phase transitions in the system.

To sum up, in this dissertation we have provided results that completely characterize the structure and the stability properties of the equilibria of a certain saturated network model that arises in several applications and attracted a growing attention in literature over the past few years. Our results about uniqueness and continuity of the network equilibria shed new light also on how this model can capture the systemic effect of exogenous shocks that hit certain nodes of the network.

5.2 CURRENT AND FUTURE RESEARCH

There are several interesting research directions to explore for future developments. One of the most interesting one that we are currently working on is the studying of the saturated network model in random networks with prescribed degree distributions.

A growing body of literature has started to study financial contagion models on random graphs. In [7] a framework for testing the possibility of large cascades in financial networks is studied on inhomogeneous random graphs. The proposed model is however simpler than the saturated one and it considers different probabilities of emergence of "contagious links" conditional on a shock, where a contagious link triggers the default of a bank following the default of its counterparty. The authors give bounds on the size of the cascade through contagious links and check under which conditions such cascades are "small". In [24] a threshold model is studied on a random directed network and it is shown that when the network has a degree distribution without second moment, a small number of initially defaulted banks can trigger a large default cascade. In [8] a model of financial contagion is studied on large random financial networks with prescribed degree distributions and authors provide analytical expressions for the asymptotic fraction of defaults, in terms of network characteristics.

Despite these results, an analytic study of saturation models such as the Eisenberg and Noe and its generalizations on random structures is still a largely uncharted territory of research. We are currently studying the saturated equilibrium model on a a particular family of infinite random acyclic graphs that have a tree-like structures with prescribed in and out degree distributions. As mentioned earlier, this is particularly relevant as many real-world networks proved to feature local tree-like structures. Shifting the setting towards random networks instead that deterministic ones poses several challenges from a mathematical point of view. In fact the equation describing the model defines a stochastic process that, in general, is non Markovian. Our focus is currently devoted to studying the existence and uniqueness of the invariant distribution of such a process.

We have already obtained promising preliminary results in this area. We were able to prove the existence and uniqueness of the invariant distribution on certain tree-like structures under some conditions on the in and out degree distribution of nodes. Moreover, we have proved the uniqueness of the invariant distribution for the particular case of an infinite line graph leveraging the theory of Markov chains on general state spaces and we are currently working on refining these results to address more general cases.

Another interesting line of research would be to develop novel notions of node centrality measures that could somehow quantify the importance of each node in a network in terms of its ability to spread shocks across the network and to "infect" other nodes, possibly triggering cascade effects. This is extremely relevant for all the institutions that play a role in the financial system, especially banks. Current frameworks on bank capital adequacy, stress testing, and market liquidity risk, such as the Basel III voluntary agreement, make use of risk measures that still do not properly take into account the underlying network structure linking the financial institutions, while common measures of importance like eigenvector centrality proved to be inadequate. While some numerical stress testing of Eisenberg and Noe-like models have been proposed to highlight the most fragile nodes, this approach does not allow to fully understand the role of the network in the contagion process. Promising results in this direction have been obtained in [10] where a framework to monitor systemic risk is proposed allowing to estimate and disentangle not only first-round effects (i.e., direct defaults) and second-round effects (i.e. distress induced in the inter-bank network), but also third-round effects induced by possible fire sales. In [28], a centrality measure known as node depth is introduced (notably, the node depth is the dual to the concept of eigenvector centrality in the networks literature). The authors show that the node depth measures the amplification of losses due to interconnections among nodes in the default set but this concept breaks down in presence of deficit nodes and it cannot be used in the more general model that we have studied in this thesis. The studying of measures of vulnerability and contagion is still lacking rigorous analytical results that can help to clarify the role of the topology and we would like to develop this theory by also leveraging the results that we have obtained regarding the existence of critical shocks. Developments in this topic could have important implications also in designing optimal aid policies to improve the robustness of the financial system.

Finally, encouraged by our results presented in this dissertation, we are also planning to extend them to more general versions of the saturated network model. The model can be enriched in a number of different ways, especially in the context of financial networks, for example by considering fire sales, bankruptcy costs and allow for asynchronous clearing processes that are more suitable to describe real financial systems.

A

APPENDIX A: TECHNICAL RESULTS ON NON-NEGATIVE MATRICES

Proof of Proposition 2.2.2 We start with the following result.

Lemma A.0.1 *Let P in $\mathbb{R}_+^{n\times n}$ be a non-negative square matrix such that*

- *there exists a non-negative vector $v \neq 0$ such that $Pv \leq v$;*
- *for every $i = 1, \dots, n$, there exists a path in $\mathcal{G}_P$ connecting i to some j such that $(Pv)_j < v_j$.*

Then, $\rho(P) < 1$.

Proof Notice that, for every $h \geq 0$, $P^h v \leq v$, so that, for $t \geq h$, non-negativity of P^{t-h} implies that $(P^t v) = P^{t-h}P^h v \leq P^h v$. On the other hand, existence of a length-l_i path from i to j in $\mathcal{G}_P$ is equivalent to that $(P^{l_i})_{ij} > 0$. Therefore, if there exists a length-l_i path in $\mathcal{G}_P$ from i to some j such that $(Pv)_j < v_j$, then, for every $t > l_i$,

$$(P^t v)_i \leq (P^{l_i+1}v)_i = \sum_{k=1}^{n}(P^{l_i})_{ik}(Pv)_k = \sum_{k=1}^{n}(P^{l_i})_{ik}v_k < (P^{l_i}v)_i \leq v_i\,.$$

Therefore, with $t = 1 + \max_i l_i$, we have $(P^t v)_i < x_i$ for every i. Since $x_i > 0$ for every i, we can find $\epsilon > 0$ such that $P^t v \leq (1-\epsilon)v$. This implies that $\lim P^{tm} = 0$ as m grows large and thus $\rho(P^t) < 1$. This yields $\rho(P) < 1$. ∎

We can now proceed to the proof of Proposition 2.2.2.

First, we prove existence of a positive vector v satisfying (5) for every non-expansive network. We proceed by induction on the number s of classes of P. If $s = 1$, i.e., P if is irreducible, the result follows from Proposition 2.2.1 (iii). Now, assume that the result holds true for $s-1$ and let us prove it for s. Consider the block structure (4) and notice that by the inductive hypothesis we can find vectors $x^{(l)}$ of dimension $|\mathcal{V}_l|$ for $l = 2, \dots, s$ with all positive entries such that

$$\sum_{h=l}^{s} P^{(lh)}v^{(h)} \leq v^{(l)}\,.$$

We now show that we can find $\alpha > 0$ and $x^{(1)}$ of dimension $|\mathcal{V}_1|$ with all positive entries, such that

$$P^{(11)}v^{(1)} + \alpha \sum_{j=2}^{s} P^{(1j)}v^{(j)} \leq v^{(1)} . \tag{77}$$

Indeed, if $\rho(P^{(11)}) < 1$ this simply follows from a continuity argument. Instead, if $\rho(P^{(11)}) = 1$, then since $P^{(11)}$ is irreducible, it admits a positive right dominant eigenvalue $v^{(1)} = P^{(11)}v^{(1)}$ by Proposition 2.2.1 (iii). On the other hand, since $\mathcal{V}_1$ is final, we have that $P^{(1h)} = 0$ for every $h = 1, \dots, s$, so that (77) is satisfied as an equality for all possible values of $\alpha > 0$. This implies that the vector $v = (v^{(1)}, \dots, v^{(s-1)}, v^{(s)})$ has all positive entries and satisfies $Pv \leq v$.

Finally, we prove that, existence of a positive vector v satisfying (5) implies that the network is non-expansive. From (5), using the fact that all entries of v are strictly positive, we deduce that P^t is a bounded sequence, so that $\rho(P) \leq 1$. Now, assume that $\mathcal{V}_l$ is a non final class such that $\rho(P^{(ll)}) = 1$. Indicating as usual with $v^{(l)}$ the restriction of v to $\mathcal{V}_l$, we obtain the relation

$$P^{(ll)}v^{(l)} + \sum_{h=l+1}^{s} P^{(lh)}v^{(h)} \leq v^{(l)}$$

from which we deduce that $P^{(ll)}v^{(l)} \lneq v^{(l)}$. Since $P^{(ll)}$ is irreducible, we can apply Lemma A.0.1 and conclude that $\rho(P^{(ll)}) < 1$. ∎

B

APPENDIX B: TECHNICAL RESULTS ON MONOTONE SYSTEMS

Proof of Lemma 4.3.1 We first prove that $\mathcal{L}_0^w$ is invariant. It is enough to show that when a component x_i reaches the boundary of $\mathcal{L}_0^w$, i.e. $x_i = w_i$ or $x_i = 0$, then the derivative is non positive or non negative respectively. For $x_i = w_i$, since obviously $S_0^{w_i}\left(\sum_j R_{ji}x_j + c_i\right) \leq w_i$ we have that:

$$\dot{x}_i = S_0^{w_i}\left(\sum_j R_{ji}x_j + c_i\right) - w_i \leq 0 \tag{78}$$

When $x_i = 0$, since $S_0^{w_i}\left(\sum_j R_{ji}x_j + c_i\right) \geq 0$ we have that

$$\dot{x}_i = S_0^{w_i}\left(\sum_j R_{ji}x_j + c_i\right) \geq 0 \tag{79}$$

and this completes the proof.

We now prove that (71) is a monotone system. Set $f_i(x) = S_0^{w_i}\left(\sum_j R_{ji}x_j + c_i\right) - x_i$. It is enough to show that $\dfrac{\partial f_i}{\partial x_k} \geq 0,\ \forall\, k \neq i$ almost everywhere (i.e. excluding 0-measure set of points where f_i is not differentiable).

It is immediate to see that:

$$\frac{\partial f_i}{\partial x_k} = \begin{cases} 0 & \text{if } \sum_j R_{ji}x_j + c_i < 0 \\ r_{ki} & \text{if } 0 < \sum_j R_{ji}x_j + c_i < w_i \\ 0 & \text{if } \sum_j R_{ji}x_j + c_i > w_i \end{cases} \tag{80}$$

Since (80) is non negative, therefore Theorem 1.2 in [34] implies that (71) is a monotone system.

Finally, we show that (71) is non expansive in l_1 distance on $\mathcal{L}_0^w$. By monotonicity and using the fact that

$$\sum_i \frac{\partial f_i}{\partial x_k} \leq \sum_i r_{ki} - 1 \leq 0 \tag{81}$$

the result follows by using Lemma 5 in [41]. ∎

Proof of Lemma 4.3.2 From monotonicity and the fact that $\mathcal{L}_0^w$ is invariant, the two Cauchy problems

$$\begin{cases} \dot{x} = S_0^w\left(R^\top x + c\right) - x \\ x_0 = 0 \end{cases} \qquad \begin{cases} \dot{x} = S_0^w\left(R^\top x + c\right) - x \\ x_0 = w \end{cases} \tag{82}$$

admit unique solutions that converge to a lower equilibrium $\underline{x}$ and largest equilibrium $\bar{x}$ respectively, i.e. $\underline{x} \leq \bar{x}$;

Now, let $\underline{y} = \sum_i \underline{x}_i$ and $\overline{y} = \sum_i \overline{x}_i$. Consider an initial state $x(0) \in \mathcal{L}_{\underline{x}}^{\overline{x}}$ for $0 \leq \alpha \leq 1$. Since the system is non-expansive in l_1, both $\|x(t) - \underline{x}\|_1$ and $\|x(t) - \overline{x}\|_1$ cannot increase in time, which implies that $\sum_i x_i(t)$ remains constant. It follows that the sets $\mathcal{X}_\alpha = \{x \in \mathcal{L}_{\underline{x}}^{\overline{x}} : \sum_i x_i = \alpha \underline{y} + (1-\alpha)\overline{y}\}$ are all invariant.

The last claims of the Lemma follow directly from monotonicity. Indeed, for any $x^\circ \in \mathcal{L}_0^w$, let $\phi^t(x^\circ)$ be the solution of (71) at time $t \geq 0$. Since $\phi^t(0) \overset{t\to+\infty}{\longrightarrow} \underline{x}$ and $\phi^t(w) \overset{t\to+\infty}{\longrightarrow} \bar{x}$, then it must be be $\phi^t(x^\circ) \overset{t\to+\infty}{\longrightarrow} \mathcal{L}_{\underline{x}}^{\bar{x}}$ $\forall\, x^\circ \in \mathcal{L}_0^w$ and in particular, $\forall\, x^\circ \in \mathcal{L}_{\underline{x}}^{\bar{x}}$. ∎

Proof of Lemma 4.3.3 Observe that an equilibrium $x^* \in$ the interior of $\mathcal{L}_0^w$ is such that $S_0^w(R^\top x^* + c) = x^*$ belongs to the interior of $\mathcal{L}_0^w$ which in turn implies that

$$f(x^*) = S_0^w(R^\top x^* + c) - x^* = (R^\top - I)x^* + c\,.$$

Since the map $f(x)$ is continuous, there necessarily exists an $\varepsilon > 0$ such that for all $\|x - x^*\| < \varepsilon$ we have that $f(x) = (R^\top - I)x + c$. Since R is stochastic, it has spectral radius in the unitary disk centered in zero so that $R^\top - I$ has all eigenvalues with non-positive real part. Hence x^* is locally stable (both for the linear dynamical system (73) and the nonlinear dynamical flow network (71), as they locally coincide), so that we can always find a number $\delta \leq \varepsilon$ such that if $\|x(0) - x^*\| < \delta$ then $\|x(t) - x^*\| < \varepsilon$ for all $t \geq 0$. This ensures that the trajectories of the system remain in the region where the dynamics is linear and hence the claim follows. ∎

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