Available online at www.sciencedirect.com
ScienceDirect
Journal of Differential Equations 366 (2023) 85–131
Journal of Differential Equations
www.elsevier.com/locate/jde

Boundary value problems with rough boundary data

Robert Denk [a], David Ploß [a], Sophia Rau [a], Jörg Seiler [b,*]

[a] *Universität Konstanz, Fachbereich für Mathematik und Statistik, Konstanz, Germany*
[b] *Università di Torino, Dipartimento di Matematica, V. Carlo Alberto 10, 10123 Torino, Italy*

Received 24 November 2022; revised 16 March 2023; accepted 1 April 2023

Abstract

We consider linear boundary value problems for higher-order parameter-elliptic equations, where the boundary data do not belong to the classical trace spaces. We employ a class of Sobolev spaces of mixed smoothness that admits a generalized boundary trace with values in Besov spaces of negative order. We prove unique solvability for rough boundary data in the half-space and in sufficiently smooth domains. As an application, we show that the operator related to the linearized Cahn–Hilliard equation with dynamic boundary conditions generates a holomorphic semigroup in $L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$.
© 2023 Elsevier Inc. All rights reserved.

MSC: primary 35J40; secondary 46E35, 47D06, 35K35

Keywords: Boundary value problem; Anisotropic Sobolev space; Generalized trace; Dynamic boundary condition; Holomorphic semigroup

1. Introduction

In the present paper, we study linear differential boundary value problems of the form

$$\begin{aligned} (\lambda - A)u &= f \quad \text{in } \Omega, \\ B_j u &= g_j \quad (j = 1, \dots, m) \text{ on } \Gamma, \end{aligned} \tag{1.1}$$

* Corresponding author.
E-mail addresses: robert.denk@uni-konstanz.de (R. Denk), david.ploss@uni-konstanz.de (D. Ploß), sophia.rau@uni-konstanz.de (S. Rau), joerg.seiler@unito.it (J. Seiler).

https://doi.org/10.1016/j.jde.2023.04.001
0022-0396/© 2023 Elsevier Inc. All rights reserved.

where Ω is either the half-space $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ or a domain in $\mathbb{R}^n$ with compact and sufficiently smooth boundary Γ. Moreover, A is a differential operator of order $2m$ and B_j is a boundary operator of order $m_j < 2m$ for $j = 1, \dots, m$. Whereas for sufficiently smooth f and g_j this problem can be solved by classical theory, we focus on rough boundary data $g_1, \dots, g_m$. In particular, we want to solve (1.1) for $f \in L^p(\Omega)$ but $g_j \in B^{s_j}_{pp}(\Gamma)$, where s_j may be zero or even negative. For such rough boundary data, even the formulation of the boundary conditions needs justification: It is known that the classical trace $u \mapsto u|_\Gamma$, first defined for smooth functions, has a continuous extension to an operator $\gamma_0 \colon H^s_p(\Omega) \to B^{s-1/p}_{pp}(\Gamma)$ *if and only if* $s > \frac{1}{p}$ ([23]). Nevertheless, it is possible to define a continuous trace on subspaces of $H^s_p(\Omega)$ for $s \le \frac{1}{p}$, see, e.g., Lions–Magenes ([24], [25]) and Roitberg ([29], [30]). In the present paper, we will introduce a class of Sobolev spaces $H^{s,\sigma}_p(\mathbb{R}^n)$ of anisotropic type, for which the trace exists as a continuous operator, following the ideas from Grubb ([17], [18]).

The motivation to study problem (1.1) with rough boundary data is two-fold: The first motivation arises in the study of stochastic partial differential equations (SPDEs) with boundary noise. Exemplarily, we mention here [33] and [27] for parabolic equations and reaction-diffusion systems with Neumann boundary conditions, [9] for the heat equation with Dirichlet boundary conditions, [3] and [8] for a free boundary value problem in fluid mechanics, and [10] for dynamical boundary conditions. The key step in the analysis of these problems is to understand the properties of the solution operator to the boundary value problem (formulated for Neumann boundary conditions)

$$\begin{aligned} \partial_t u - Au &= 0 \text{ in } (0,\infty) \times \Omega, \\ \partial_\nu u &= \xi \text{ on } (0,\infty) \times \Gamma, \end{aligned} \tag{1.2}$$

where ξ stands for the boundary noise and ∂_ν denotes the derivative in the direction of the outward pointing unit normal vector of the boundary Γ. As it is known that the paths of Gaussian white noise belong with probability one to some Besov space with negative regularity (see, e.g., [7], [20], [38]), this fits into the setting of (1.1) with $f = 0$. In the context of SPDEs, the solution operator is often denoted as the Neumann (or Dirichlet) map.

The second motivation for studying (1.1) arises from boundary value problems with Wentzell or dynamical boundary conditions. As a prototype example, we consider the heat equation with Wentzell boundary conditions

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in } (0,\infty) \times \Omega, \\ \Delta u + \partial_\nu u &= 0 \quad \text{on } (0,\infty) \times \Gamma, \\ u|_{t=0} &= u_0 \text{ in } \Omega. \end{aligned} \tag{1.3}$$

Replacing $\Delta u = \partial_t u$ in the boundary condition, we obtain the dynamic boundary condition $\partial_t u + \partial_\nu u = 0$. In a standard approach, one decouples $u =: u_1$ and $u|_\Gamma =: u_2$ and obtains a resolvent problem of the form

$$\begin{aligned} \lambda u_1 - \Delta u_1 &= f \text{ in } \Omega, \\ \lambda u_2 + \partial_\nu u_1 &= g \text{ on } \Gamma \end{aligned} \tag{1.4}$$

with the additional condition $u_1|_\Gamma = u_2$. The corresponding operator acts on the tuple $u = (u_1, u_2)$ as $Au = (\Delta u_1, -\partial_\nu u_1)$. From the point of view of maximal regularity for (1.3), the basic space for this operator would be $L^p(\Omega) \times B_{pp}^{1-1/p}(\Gamma)$, where the second component is the trace space of $H_p^2(\Omega)$ for the Neumann boundary operator. In fact, for boundary value problems with dynamic boundary conditions, the generation of a holomorphic semigroup in trace spaces was shown in [28] for the Cahn–Hilliard equation and in [12] for a general class of problems. However, a more natural basic space for the operator A is $L^p(\Omega) \times L^p(\Gamma)$. At least for $p = 2$, form methods can easily lead to the proof of the generation of a holomorphic semigroup. This was elaborated, e.g., for second-order equations in [6] and in [39], for the Bi-Laplacian in [11], and in an abstract setting in [14]. For the analysis in the basic space $L^p(\Omega) \times L^p(\Gamma)$, one has to deal with boundary values in L^p-spaces, which again is not covered by classical theory. In the present paper, we will apply our solution theory to the Cahn–Hilliard equation with dynamic boundary conditions.

Our analysis of (1.1) starts with the observation that (at least in the smooth situation) this problem fits into the framework of Boutet de Monvel's calculus of pseudodifferential boundary value problems. In this calculus, the solution operator for $f = 0$ is called a Poisson operator, and such operators have good mapping properties in the complete scale of Sobolev spaces. This follows, e.g., from the work of Grubb ([16], [17]) and Grubb and Kokholm [18]. However, the classical trace only exists for sufficiently smooth functions. Therefore, one has to define an appropriate generalization of the trace on the boundary. In the literature, one can find several approaches to generalized traces and corresponding boundary value problems with rough boundary data: By considering the dual boundary value problem as by Lions and Magenes ([24], [25]), one obtains unique solvability in some negative order spaces. However, these spaces depend on the boundary conditions, which is the reason for introducing the universal (but less natural) spaces $\Xi^s(\Omega)$ in [25], beginning of Section 6.3. By Roitberg ([29], [30]), generalized traces were defined using completion of smooth functions. Here, the solution of the boundary value problem is given as a tuple of the form $(u, g_0, \dots, g_{2m-1})$, where the first component u belongs to some dual space and $g_0, \dots, g_{2m-1}$ are generalized boundary traces. This concept leads to isomorphism results, but the considered spaces are non-standard and in general not even spaces of distributions on Ω. The Roitberg spaces are described in more detail in Remark 2.7 below. Another approach to rough boundary data was developed, e.g., by Hummel and Lindemulder ([19], [21]), where weighted Sobolev spaces (with respect to some distance function to the boundary) lead to a priori-estimates. The spaces are natural and do not depend on the operators, but the order at the boundary is still restricted to the non-negative scale (see [21], Theorem 6.2). Spaces of arbitrary negative (tangential) order can be obtained in combination of weighted spaces and spaces of dominating mixed smoothness, see [19], Theorem 6.1.

In this paper, we use another approach to a generalized trace by considering a new class of Sobolev spaces with mixed smoothness, which was introduced by Grubb ([17]) for $p = 2$. These spaces differ from anisotropic Sobolev spaces in the sense of [23] and [35] and from spaces with dominating mixed smoothness in the sense of [32] and [37]. For this class of Sobolev spaces, both the existence of a continuous trace and the unique solvability of parameter-elliptic model problems in the whole space and in the half-space follow immediately from known results. However, the passage from model problems (i.e., constant coefficients and no lower-order terms) to variable coefficients is not standard. It requires the application of an elaborate localization procedure, even for problems on the half-space. In domains, the definition of the Sobolev spaces with mixed smoothness is not canonical. Therefore, we work with classical Sobolev spaces in domains but employ local embeddings into our spaces of mixed smoothness. The necessity to estimate certain commutators leads to restrictions on the orders of the involved spaces (see Lemma 4.4); however these restrictions still allow to deal with boundary values in $L^p(\Gamma)$, for example.

The paper is structured as follows. In Section 2, we define and analyze Sobolev spaces of mixed smoothness, including parameter-dependent norms. We show trace results and typical embeddings. In Lemma 2.3, interpolation properties are shown which seem not to follow immediately from known results. Section 3 deals with boundary value problems in the half-space. The main result (Theorem 3.12) gives unique solvability of parameter-elliptic boundary value problems under appropriate smoothness assumptions on the coefficients. Note that we do not consider the infinitely smooth setting and thus pseudodifferential theory cannot be applied. Here, the boundary data may belong to Besov spaces with arbitrary low order. As a corollary, one obtains unique solvability in classical Sobolev spaces (Corollary 3.13). The situation in domains is studied in Section 4. The main result (Theorem 4.9) yields unique solvability in classical Sobolev spaces for rough boundary data. Finally, in Section 5, we apply the above results to the linearized Cahn–Hilliard equation with dynamic boundary conditions. We show that the related operator A generates a holomorphic semigroup in $L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$, see Theorem 5.6. In fact, we even show that, for every $\lambda_0 > 0$, the operator $A - \lambda_0$ generates a *bounded* holomorphic semigroup of angle $\frac{\pi}{2}$. In the proof, we use the bounded H^∞-calculus for the Laplacian with explicit symbol estimates, see Lemma 5.4. The same method can be applied to the (much easier) boundary value problem (1.4), and we obtain unique solvability of (1.4) and the generation of a holomorphic semigroup for the related operator.

2. Sobolev spaces of mixed smoothness and traces

Let us fix some notation used throughout the paper. We consider the Euclidean space $\mathbb{R}^n$ with variable $x = (x', x_n)$ and corresponding co-variable $\xi = (\xi', \xi_n)$. We fix $m \in \mathbb{N}$ and define

$$\langle \xi, \lambda \rangle := (1 + |\xi|^2 + |\lambda|^{1/m})^{1/2} \text{ and } \langle \xi', \lambda \rangle := (1 + |\xi'|^2 + |\lambda|^{1/m})^{1/2}$$

for $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$. Moreover, we write $\langle \xi \rangle := \langle \xi, 0 \rangle$, $\langle \xi' \rangle := \langle \xi', 0 \rangle$ and $\langle \lambda \rangle := \langle 0, \lambda \rangle$. For (suitable) functions $\varphi(\xi)$ defined on $\mathbb{R}^n$ we denote by $\varphi(D)$ its associated Fourier multiplier, which is defined by $\varphi(D)u := \mathscr{F}^{-1}(\varphi \mathscr{F} u)$, where $\mathscr{F}$ denotes the Fourier transform acting in the space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions. In particular, we have $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$ for $\alpha \in \mathbb{N}_0^n$. In case $\varphi(\xi) = \varphi(\xi')$ is independent of ξ_n, the associated Fourier multiplier will also be denoted by $\varphi(D')$.

For two Banach spaces X and Y let $L(X, Y)$ be the space of bounded linear operators $X \to Y$ and $L(X) := L(X, X)$. We shall write $X = Y$ if both spaces have the same elements and equivalent norms, and we write $X \subset Y$ if X is a subset of Y and the inclusion map $X \to Y$ is bounded.

2.1. *Some function spaces*

In the following, let $H_p^s(\mathbb{R}^n)$ and $B_{pp}^s(\mathbb{R}^n)$ denote the standard Bessel potential and Besov spaces for $s \in \mathbb{R}$. Throughout the paper, we assume $p \in (1, \infty)$. For $p = 2$, the following definition can also be found in [17], Appendix A.3.

Definition 2.1. For $s, \sigma \in \mathbb{R}$ and $p \in (1, \infty)$ we define the Bessel potential space of mixed smoothness $H_p^{s,\sigma}(\mathbb{R}^n)$ as

$$H_p^{s,\sigma}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D'\rangle^\sigma u \in H_p^s(\mathbb{R}^n)\}$$
$$= \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D\rangle^s \langle D'\rangle^\sigma u \in L^p(\mathbb{R}^n)\}$$

with canonical norm $\|u\|_{H_p^{s,\sigma}(\mathbb{R}^n)} := \|\langle D'\rangle^\sigma u\|_{H_p^s(\mathbb{R}^n)} = \|\langle D\rangle^s \langle D'\rangle^\sigma u\|_{L^p(\mathbb{R}^n)}$.

In the previous definition, $\langle D'\rangle^\sigma$ acts only on the x'-variable. Therefore, the above spaces have different smoothness in x'-direction and in x_n-direction, which is the reason of the notion of mixed smoothness.

Clearly, $H_p^{s,\sigma}(\mathbb{R}^n)$ is a Banach space. Since $\langle D'\rangle^\sigma$ and $\langle D\rangle^s$ leave $\mathscr{S}(\mathbb{R}^n)$ invariant, the rapidly decreasing functions are a dense subset of $H_p^{s,\sigma}(\mathbb{R}^n)$. For every $t, \tau \in \mathbb{R}$ the map

$$\langle D\rangle^t \langle D'\rangle^\tau : H_p^{s,\sigma}(\mathbb{R}^n) \to H_p^{s-t,\sigma-\tau}(\mathbb{R}^n)$$

is an isometric isomorphism with inverse $\langle D\rangle^{-t}\langle D'\rangle^{-\tau}$.

We remark that the scale $H_p^{s,\sigma}(\mathbb{R}^n)$ for $s, \sigma \in \mathbb{R}$ is different from the scale of anisotropic spaces $H_p^{s,\vec{a}}(\mathbb{R}^n)$ in the sense of [23], Proposition 2.10 (see also [35], Section 5.1.3). In particular, for $s > 0$ and $\sigma < -s$, we have positive smoothness with respect to x_n but negative smoothness in x', which is not allowed for the anisotropic spaces. $H_p^{s,\sigma}(\mathbb{R}^n)$ is also different from the space of dominating mixed smoothness (see, e.g., [37], Section 1.1.2). Spaces of dominating mixed smoothness are defined similarly as above, but with $\langle\xi\rangle$ being replaced by $\prod_{j=1}^n (1+\xi_j^2)^{1/2}$. We refer to [32], Section 1, and [20], Subsection 2.2, for further information on spaces of dominating mixed smoothness and applications to boundary value problems.

We state some elementary properties of the Bessel potential spaces with mixed smoothness which we shall use frequently later on.

Proposition 2.2. *Let $s, \sigma \in \mathbb{R}$.*

a) $H_p^{s,0}(\mathbb{R}^n) = H_p^s(\mathbb{R}^n)$ *and* $H_p^{0,\sigma}(\mathbb{R}^n) = L^p(\mathbb{R}, H_p^\sigma(\mathbb{R}^{n-1}))$.

b) $H_p^{t,\tau}(\mathbb{R}^n) \subset H_p^{s,\sigma}(\mathbb{R}^n)$ *whenever $s \le t$ and $\sigma \le \tau$.*

c) For $\sigma \ge 0$ we have

$$H_p^{s+\sigma}(\mathbb{R}^n) \subset H_p^{s,\sigma}(\mathbb{R}^n) \subset H_p^s(\mathbb{R}^n) \subset H_p^{s,-\sigma}(\mathbb{R}^n) \subset H_p^{s-\sigma}(\mathbb{R}^n).$$

d) If q is the dual coefficient to p, i.e. $\frac{1}{p}+\frac{1}{q}=1$, then the standard bilinear pairing $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to \mathbb{C}$ induces an identification of the dual space of $H_p^{s,\sigma}(\mathbb{R}^n)$ with $H_q^{-s,-\sigma}(\mathbb{R}^n)$.

e) In case of $s \ge 0$ we have

$$H_p^{s,\sigma}(\mathbb{R}^n) = L^p(\mathbb{R}, H_p^{s+\sigma}(\mathbb{R}^{n-1})) \cap H_p^s(\mathbb{R}, H_p^\sigma(\mathbb{R}^{n-1})).$$

f) For $\alpha \in \mathbb{N}_0^n$, the derivative $\partial^\alpha : H_p^{s,\sigma}(\mathbb{R}^n) \to H_p^{s-|\alpha_n|,\sigma-|\alpha'|}(\mathbb{R}^n)$ is continuous.

g) If $s \in \mathbb{N}_0$ is an integer, $u \in \mathscr{S}'(\mathbb{R}^n)$ belongs to $H_p^{s,\sigma}(\mathbb{R}^n)$ if and only if $\partial_n^j u \in H_p^{0,s+\sigma-j}(\mathbb{R}^n)$ for all $j = 0, \dots, s$. Moreover, $\|u\| := \sum_{j=0}^s \|\partial_n^j u\|_{H_p^{0,s+\sigma-j}(\mathbb{R}^n)}$ defines an equivalent norm on $H_p^{s,\sigma}(\mathbb{R}^n)$.

Proof. a) is clear. b) is true, since $\langle D\rangle^{s-t}\langle D'\rangle^{\sigma-\tau}$ is a bounded operator in $L^p(\mathbb{R}^n)$ due to Mikhlin's theorem.

c) By Mikhlin's theorem, both $\langle D\rangle^{-\sigma}\langle D'\rangle^{\sigma}$ and $\langle D'\rangle^{-\sigma}$ are bounded operators in $L^p(\mathbb{R}^n)$ for $\sigma \geq 0$. This yields the first and the second inclusion, respectively. The other two are verified analogously.

d) As $\langle D'\rangle^{\sigma}\colon H_p^{s,\sigma}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$ is an isometric isomorphism, the adjoint operator $(\langle D'\rangle^{\sigma})'\colon (H_p^s(\mathbb{R}^n))' \to (H_p^{s,\sigma}(\mathbb{R}^n))'$ is an isometric isomorphism in the dual spaces. In the bilinear pairing, this adjoint operator is again $\langle D'\rangle^{\sigma}$, and the dual space of $H_p^s(\mathbb{R}^n)$ is given by $H_q^{-s}(\mathbb{R}^n)$. Hence, the dual space of $H_p^{s,\sigma}(\mathbb{R}^n)$ is identified with $\langle D'\rangle^{\sigma}(H_q^{-s}(\mathbb{R}^n)) = H_q^{-s,-\sigma}(\mathbb{R}^n)$.

e) The equality is known to be true in case $\sigma = 0$. Applying $\langle D'\rangle^{-\sigma}$ to both sides of this equality yields the claim.

f) holds because $\partial^{\alpha}\langle D'\rangle^{-|\alpha'|}\langle D\rangle^{-\alpha_n}$ is bounded in $L^p(\mathbb{R}^n)$ due to Mikhlin's theorem.

g) Again this is known in case $\sigma = 0$. Then, the general case holds true because $\langle D'\rangle^{\sigma}$ commutes with ∂_n. □

2.2. Interpolation spaces

Let us briefly recall the complex interpolation method, following [22], Section C.2 (see also [34], Section 1.9). Let X_0 and X_1 be an interpolation couple of complex Banach spaces and $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Denote by $\mathcal{F}(X_0, X_1)$ the space of all continuous functions $f\colon \overline{S} \to X_0 + X_1$ such that $f|_S$ is holomorphic as an $(X_0 + X_1)$-valued function on S and, for $j \in \{0, 1\}$, the function $b \mapsto f(j+ib) : \mathbb{R} \to X_j$ is bounded and continuous. $\mathcal{F}(X_0, X_1)$ becomes a Banach space with the norm

$$\|f\|_{\mathcal{F}(X_0,X_1)} := \max_{j=0,1} \sup_{b\in\mathbb{R}} \|f(j+ib)\|_{X_j}.$$

For $\theta \in (0, 1)$, the complex interpolation space $[X_0, X_1]_\theta$ is defined as the space of all $x \in X_0 + X_1$ for which $x = f(\theta)$ for some $f \in \mathcal{F}(X_0, X_1)$, endowed with the norm

$$\|x\|_{[X_0,X_1]_\theta} := \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} \,:\, f(\theta) = x\}.$$

With this norm, the complex interpolation space becomes a Banach space satisfying $X_0 \cap X_1 \subset [X_0, X_1]_\theta \subset X_0 + X_1$. In the definition of the interpolation space and the norm, the space $\mathcal{F}(X_0, X_1)$ can be replaced by the subspace $\mathcal{F}_0(X_0, X_1)$ which consists of all $f \in \mathcal{F}(X_0, X_1)$ such that $b \mapsto \|f(j+ib)\|_{X_j}$ vanishes for $|b| \to \infty$ and $j = 0, 1$. We will also consider the space $\mathcal{F}_0(X_0, X_1; X_0 \cap X_1)$ consisting of all $f \in \mathcal{F}_0(X_0, X_1)$ for which $f(z) \in X_0 \cap X_1$ for all $z \in \overline{S}$ and where f is continuous on $\overline{S}$ and holomorphic in S as a function with values in $X_0 \cap X_1$ (note that the definition of this space differs from the one in [22]). By [34], Theorem 1.9.1, $\mathcal{F}_0(X_0, X_1; X_0 \cap X_1)$ is dense in $\mathcal{F}_0(X_0, X_1)$.

Lemma 2.3. *Let $s_0, \sigma_0, s_1, \sigma_1 \in \mathbb{R}$ and $\theta \in (0, 1)$. Then*

$$[H_p^{s_0,\sigma_0}(\mathbb{R}^n), H_p^{s_1,\sigma_1}(\mathbb{R}^n)]_\theta = H_p^{s_\theta,\sigma_\theta}(\mathbb{R}^n) \tag{2.1}$$

with $s_\theta = (1-\theta)s_0 + \theta s_1$ and $\sigma_\theta = (1-\theta)\sigma_0 + \theta\sigma_1$.

Proof. For simplicity, we write $H_p^{s_1,\sigma_1} := H_p^{s_1,\sigma_1}(\mathbb{R}^n)$ etc. in the proof. Due to $[X_0, X_1]_\theta = [X_1, X_0]_{1-\theta}$, we may assume $s_0 \le s_1$. Applying the operator $\langle D\rangle^{s_1}\langle D'\rangle^{\sigma_0}$ and setting $s := s_0 - s_1 \le 0$ and $\sigma := \sigma_1 - \sigma_0 \in \mathbb{R}$, by the retraction argument from [4], Proposition 2.3.2, it remains to show that

$$[H_p^{s,0}, H_p^{0,\sigma}]_\theta = H_p^{(1-\theta)s,\theta\sigma}. \tag{2.2}$$

Note that $H_p^{s,0} = H_p^s$ and $H_p^{0,\sigma} = L^p(\mathbb{R}, H_p^\sigma(\mathbb{R}^{n-1}))$ by Proposition 2.2 a).

We first show "$\subset$" in (2.2). For this, let $u \in [H_p^s, H_p^{0,\sigma}]_\theta$, and choose $g \in \mathcal{F}_0(H_p^s, H_p^{0,\sigma})$ with $g(\theta) = u$. By density, there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset \mathcal{F}_0(H_p^s, H_p^{0,\sigma}; H_p^s \cap H_p^{0,\sigma})$ such that $g_k \to g$ in $\mathcal{F}(H_p^s, H_p^{0,\sigma})$. It follows that $g_k(\theta) \to g(\theta)$ in $[H_p^s, H_p^{0,\sigma}]_\theta$. For $k \in \mathbb{N}$ let us define

$$f_k(z) := e^{z^2-\theta^2}\langle D'\rangle^{\sigma z} g_k(z) \quad (z \in \overline{S}).$$

First we show that $f_k(z) \in H_p^s$ with continuous and holomorphic dependence on $z \in \overline{S}$ and $z \in S$, respectively. This is equivalent to showing that

$$h_k(z) := \langle D\rangle^s \langle D'\rangle^{\sigma z} g_k(z) \quad (z \in \overline{S})$$

defines a function $h_k \colon \overline{S} \to L^p$ which depends on z as requested. In case of $\sigma \le 0$ note that $g_k \colon \overline{S} \to H_p^s$, hence $\langle D\rangle^s g_k \colon \overline{S} \to L^p$, with the requested dependence on z. Then the claim for h_k follows from Lemma 5.6.8 in [22]. By (5.53) of [22] we also find the estimate

$$\|h_k(z)\|_{L^p} \le C(1 + |\sigma \operatorname{Im} z|)\|g_k(z)\|_{H_p^s} \quad (z \in \overline{S}).$$

If $\sigma > 0$, note that $\langle D\rangle^s \in L(L^p)$ since $s \le 0$. Then write

$$\langle D'\rangle^{\sigma z} g_k(z) = \langle D'\rangle^{\sigma(z-1)}\langle D'\rangle^{\sigma} g_k(z).$$

Since $g_k \colon \overline{S} \to H_p^{0,\sigma}$, hence $\langle D'\rangle^\sigma g_k \colon \overline{S} \to L^p$, with the requested dependence on z, the claim again follows by Lemma 5.6.8 of [22]. Also

$$\|h_k(z)\|_{L^p} \le C(1 + |\sigma \operatorname{Im} z|)\|g_k(z)\|_{H_p^{0,\sigma}} \quad (z \in \overline{S}).$$

This yields

$$\|f_k(z)\|_{H_p^s} \le \widetilde{C}\|g_k(z)\|_{H_p^s \cap H_p^{0,\sigma}} \quad (z \in \overline{S})$$

with $\widetilde{C} := C \sup_{b\in\mathbb{R}}(1 + |\sigma b|)e^{1-\theta^2-b^2}$.

Arguing similarly, one finds $f_k(ib) \in H_p^s$ and $f_k(1 + ib) \in L^p$ with continuous dependence on $b \in \mathbb{R}$ and

$$\sup_{b\in\mathbb{R}} \|f_k(ib)\|_{H_p^s} \le C \sup_{b\in\mathbb{R}} \|g_k(ib)\|_{H_p^s},$$
$$\sup_{b\in\mathbb{R}} \|f_k(1+ib)\|_{L^p} \le C \sup_{b\in\mathbb{R}} \|g_k(1+ib)\|_{H_p^{0,\sigma}}.$$

Summing up, we have shown that $f_k \in \mathcal{F}(H_p^s, L^p)$ with

$$\|f_k\|_{\mathcal{F}(H_p^s,L^p)} \le C\|g_k\|_{\mathcal{F}(H_p^s,H_p^{0,\sigma})}.$$

As $(g_k)_{k\in\mathbb{N}}$ is convergent and therefore a Cauchy sequence, the above estimate, applied to $f_k - f_\ell$, yields that also $(f_k)_{k\in\mathbb{N}} \subset \mathcal{F}(H_p^s, L^p)$ is a Cauchy sequence. Again by the definition of the interpolation space $[H_p^s, L^p]_\theta$, we obtain that $(f_k(\theta))_{k\in\mathbb{N}}$ is a Cauchy sequence in $[H_p^s, L^p]_\theta = H_p^{(1-\theta)s}$ (for the last equality, see [22], Theorem 5.6.9). By completeness, there exists $v \in H_p^{(1-\theta)s}$ with $f_k(\theta) \to v$. On the other hand, $f_k(\theta) = \langle D'\rangle^{\theta\sigma} g_k(\theta)$ and $g_k(\theta) \to g(\theta) = u$ in $[H_p^s, H_p^{0,\sigma}]_\theta$, hence $f_k(\theta) \to \langle D'\rangle^{\theta\sigma} u$ in $\mathscr{S}'(\mathbb{R}^n)$. Therefore, $u = \langle D'\rangle^{-\theta\sigma} v \in H_p^{(1-\theta)s,\theta\sigma}$ with norm

$$\begin{aligned}\|u\|_{H_p^{(1-\theta)s,\theta\sigma}} &\le C\|v\|_{[H_p^s,L^p]_\theta} \le C \lim_{k\to\infty} \|f_k\|_{\mathcal{F}(H_p^s,L^p)} \le C \lim_{k\to\infty} \|g_k\|_{\mathcal{F}(H_p^s,H_p^{0,\sigma})}\\ &= C\|g\|_{\mathcal{F}(H_p^s,H_p^{0,\sigma})}.\end{aligned}$$

As $g \in \mathcal{F}_0(H_p^s, H_p^{0,\sigma})$ was arbitrary with $g(\theta) = u$, we obtain

$$\|u\|_{H_p^{(1-\theta)s,\theta\sigma}} \le C\|u\|_{[H_p^{s,0},H_p^{0,\sigma}]_\theta}$$

which finishes the proof of "$\subset$".

The proof of the embedding "$\supset$" follows in exactly the same way. Let $u \in H_p^{(1-\theta)s,\theta\sigma}$ and $v := \langle D'\rangle^{\theta\sigma} u \in H_p^{(1-\theta)s}$, and let $f \in \mathcal{F}_0(H^s, L^p)$ with $f(\theta) = v$. We approximate f by a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{F}_0(H_p^s, L^p; L^p)$ and set $g_k(z) := e^{z^2-\theta^2}\langle D'\rangle^{-\sigma z} f_k(z)$ for $z \in \overline{S}$ and $k \in \mathbb{N}$. If $\sigma \ge 0$, we see that $g_k(z) \in L^p \subset H_p^s$, and for $\sigma < 0$ we have $g_k(z) \in H_p^{0,\sigma}$, so we have $g_k(z) \in H_p^s + H_p^{0,\sigma}$ for all $z \in \overline{S}$. Therefore, we can argue as above to show the embedding "$\supset$" in (2.2). □

In addition to the spaces above, we will also consider the standard Besov spaces $B_{pp}^s(\mathbb{R}^n)$ for $p \in (1,\infty)$ and $s \in \mathbb{R}$. For $X \in \{\mathscr{S}, \mathscr{S}', H_p^s, B_{pp}^s, H_p^{s,\sigma}\}$ and a domain $\Omega \subset \mathbb{R}^n$, we define

$$X(\Omega) := \{u|_\Omega : u \in X(\mathbb{R}^n)\}$$

(where restriction is understood in the distributional sense) with the canonical norm

$$\|v\|_{X(\Omega)} := \inf\{\|u\|_{X(\mathbb{R}^n)} : u \in X(\mathbb{R}^n),\ u|_\Omega = v\}$$

(see, e.g., [35], Definition 4.1). Moreover, for a closed subset $A \subset \mathbb{R}^n$ we define

$$\dot{X}(A) := \{u \in X(\mathbb{R}^n) : \operatorname{supp} u \subset A\}.$$

Then by definition we see that $X(\Omega)$ can be identified with the quotient space $X(\mathbb{R}^n)/\dot{X}(\mathbb{R}^n \setminus \Omega)$. Note that we consider $H_p^{s,\sigma}(\Omega)$ only for $\Omega = \mathbb{R}^n_+$.

Remark 2.4. a) Let $\Omega = \mathbb{R}^n_+$. Then the restriction $r_{\mathbb{R}^n_+}: X(\mathbb{R}^n) \to X(\mathbb{R}^n_+)$, $u \mapsto u|_{\mathbb{R}^n_+}$ is a retraction. This follows from the fact that there exists a restriction-extension pair (R, E) for $(\mathscr{S}'(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n_+))$ in the sense of [5], Theorem VI.1.2.3, which yields the restriction-extension pair $(r_{\mathbb{R}^n_+}, e_{\mathbb{R}^n_+})$ on $(X(\mathbb{R}^n), X(\mathbb{R}^n_+))$ by [5], Lemma VII.2.8.1. In particular, the extension operator $e_{\mathbb{R}^n_+}$ is universal for all considered spaces. Later, we will also consider e^0_Ω, the canonical extension from Ω to $\mathbb{R}^n$ by zero.

b) Similarly, if $\Omega \subset \mathbb{R}^n$ has smooth boundary, the map $u \mapsto u|_\Omega$ is a retraction from $H^s_p(\mathbb{R}^n)$ to $H^s_p(\Omega)$ and from $B^s_{pp}(\mathbb{R}^n)$ to $B^s_{pp}(\Omega)$, and for all $N \in \mathbb{N}$ there exists a common co-retraction (i.e., a continuous right-inverse) for all $|s| < N$ (see [36], Theorem 3.3.4).

c) Due to a) and standard retraction-coretraction arguments (see [4], Section I.2.3), all statements of Proposition 2.2 and Lemma 2.3 remain valid if we replace $\mathbb{R}^n$ by $\mathbb{R}^n_+$, with the exception of Proposition 2.2 d) which has to be modified in the following form: For all $s, \sigma \in \mathbb{R}$ the dual space of $H_p^{s,\sigma}(\mathbb{R}^n_+)$ with respect to the standard pairing is given by $(H_p^{s,\sigma}(\mathbb{R}^n_+))' = \dot{H}_q^{-s,-\sigma}(\overline{\mathbb{R}^n_+})$. This follows from $(H^s_p(\mathbb{R}^n_+))' = \dot{H}^{-s}_q(\overline{\mathbb{R}^n_+})$ (see [5], Theorem VII.4.4.2) in the same way as in the proof of Proposition 2.2 d), noting that $\langle D'\rangle^\sigma u$ has support in $\overline{\mathbb{R}^n_+}$ if u does.

2.3. Boundary traces

To define the trace space of $H_p^{s,\sigma}(\mathbb{R}^n_+)$ on the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we first note that for the standard space $H^s_p(\mathbb{R}^n_+)$ the trace

$$\gamma_0 \colon H^s_p(\mathbb{R}^n_+) \to B^{s-1/p}_{pp}(\mathbb{R}^{n-1}),\ u \mapsto \gamma_0 u := u|_{\mathbb{R}^{n-1}} \tag{2.3}$$

exists and is continuous if and only if $s > \frac{1}{p}$. In fact, it was shown in [23], Theorem 2.4, that for $s \le \frac{1}{p}$ the map $u \mapsto \gamma_0 u$ is not even continuous from $H^s_p(\mathbb{R}^n_+)$ to $\mathscr{D}'(\mathbb{R}^{n-1})$. If $s > \frac{1}{p}$, then (2.3) is a retraction, and γ_0 is the unique extension of the classical boundary trace $u \mapsto u|_{x_n=0}$ for smooth functions $u \in \mathscr{S}(\mathbb{R}^n_+)$. We will also consider the higher-order traces $\gamma_j \colon H^s_p(\mathbb{R}^n_+) \to B^{s-j-1/p}_{pp}(\mathbb{R}^{n-1})$, $u \mapsto \gamma_0 \partial_n^j u$ for $j \in \mathbb{N}_0$ and $s > j + \frac{1}{p}$.

Definition 2.5. Let $j \in \mathbb{N}_0$, $s \in (j + \frac{1}{p}, \infty)$, and $\sigma \in \mathbb{R}$. Then we define the j-th order trace $\widetilde{\gamma}_j$ on $H_p^{s,\sigma}(\mathbb{R}^n_+)$ as

$$\widetilde{\gamma}_j \colon H_p^{s,\sigma}(\mathbb{R}^n_+) \to B^{s+\sigma-j-1/p}_{pp}(\mathbb{R}^{n-1}), \quad u \mapsto \langle D'\rangle^{-\sigma} \gamma_j \langle D'\rangle^\sigma u. \tag{2.4}$$

Remark 2.6. Note that $\widetilde{\gamma}_j$ is well-defined as $\langle D'\rangle^\sigma u \in H^s_p(\mathbb{R}^n_+)$ and the unique continuous extension of the classical trace which is defined on the dense subspace $\mathscr{S}(\mathbb{R}^n_+)$. The fact that γ_j is a retraction on the classical space $H^s_p(\mathbb{R}^n_+)$ immediately implies that (2.4) is a retraction, too. In fact, if e_j is a co-retraction to γ_j, then $\widetilde{e}_j := \langle D'\rangle^{-\sigma} e_j \langle D'\rangle^\sigma$ is a co-retraction to $\widetilde{\gamma}_j$. As $\widetilde{\gamma}_j$ is (for $\sigma \le 0$) the unique continuous extension of γ_j to $H_p^{s,\sigma}(\mathbb{R}^n_+)$, we will write γ_j instead of $\widetilde{\gamma}_j$ again.

Remark 2.7 *(Roitberg spaces).* There is a theory of generalized boundary value problems in spaces of negative regularity due to Roitberg [29]. In this theory, for $s \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$, the space $\widetilde{H}_p^{s,(\ell)}(\mathbb{R}^n_+)$ is defined as the set of all tuples $(u, g_0, \dots, g_{\ell-1})$ such that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n_+)$ satisfying $(u_k, \gamma_0 u_k, \dots, \gamma_{\ell-1}u_k) \to (u, g_0, \dots, g_{\ell-1})$, where the convergence takes place in the space

$$H_p^s(\mathbb{R}^n_+) \times \prod_{j=0}^{\ell-1} B_{pp}^{s-j-1/p}(\mathbb{R}^{n-1}) \text{ if } s \geq 0,$$

$$\dot{H}_p^s(\overline{\mathbb{R}^n_+}) \times \prod_{j=0}^{\ell-1} B_{pp}^{s-j-1/p}(\mathbb{R}^{n-1}) \text{ if } s < 0.$$

For $s > \ell - 1 + 1/p$, the space $\widetilde{H}_p^{s,(\ell)}(\mathbb{R}^n_+)$ can be identified with the standard Sobolev space $H_p^s(\mathbb{R}^n_+)$, and we have $g_j = \gamma_j u$ for $j = 0, \dots, \ell - 1$ in this case, see [29], Section 2.1.

Let $\ell \in \mathbb{N}_0$, $s > \ell - 1 + 1/p$, and $\sigma \leq 0$, and let $u \in H_p^{s,\sigma}(\mathbb{R}^n_+)$. By density, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n_+)$ with $\|u_k - u\|_{H_p^{s,\sigma}(\mathbb{R}^n_+)} \to 0$ $(k \to \infty)$. The continuity of (2.4) yields $\gamma_j u_k \to \gamma_j u \in B_{pp}^{s+\sigma-j-1/p}(\mathbb{R}^{n-1})$ for $j = 0, \dots, \ell - 1$. From this and Lemma 2.2 a), we obtain the continuous embeddings

$$\begin{aligned} &H_p^{s,\sigma}(\mathbb{R}^n_+) &&\subset \widetilde{H}_p^{s+\sigma,(\ell)}(\mathbb{R}^n_+) \quad \text{if } s+\sigma \geq 0,\\ &H_p^{s,\sigma}(\mathbb{R}^n_+) \cap \dot{H}_p^{s+\sigma}(\overline{\mathbb{R}^n_+}) &&\subset \widetilde{H}_p^{s+\sigma,(\ell)}(\mathbb{R}^n_+) \quad \text{if } s+\sigma < 0, \end{aligned}$$

where we identify $u \in H_p^{s,\sigma}(\mathbb{R}^n_+)$ with the tuple $(u, \gamma_0 u, \dots, \gamma_{\ell-1}u)$.

2.4. Parameter-dependent spaces

We will also need parameter-dependent versions of the above spaces. For this, we follow the approach of Grubb–Kokholm [18].

Definition 2.8. If X_λ and Y_λ are families of Banach spaces (parametrized by λ from some index set), a family of linear operators $T(\lambda)\colon X_\lambda \to Y_\lambda$ is said to be continuous if $T(\lambda) \in L(X_\lambda, Y_\lambda)$ for every fixed λ and the operator norm $\|T(\lambda)\|_{L(X_\lambda,Y_\lambda)}$ is uniformly bounded in λ. A continuous family is called an isomorphism if each $T(\lambda)$ is invertible and $T(\lambda)^{-1}\colon Y_\lambda \to X_\lambda$ is a continuous family, too.

In our context, the occurring families of spaces X_λ will consist of a fixed vector space X equipped with a norm depending on the parameter λ.

Definition 2.9. For $\lambda \in \mathbb{C}$ let κ_λ denote the homeomorphism of $\mathscr{S}'(\mathbb{R}^n)$ given on $\mathscr{S}(\mathbb{R}^n)$ by $(\kappa_\lambda u)(x) = u(\langle\lambda\rangle x)$. Then, we define the parameter-dependent norms by

$$\|u\|_{H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)} := \langle\lambda\rangle^{s+\sigma-n/p}\|\kappa_\lambda^{-1}u\|_{H_p^{s,\sigma}(\mathbb{R}^n)} \qquad (s,\sigma \in \mathbb{R}),$$

$$\|u\|_{B_{pp,\lambda}^{s}(\mathbb{R}^{n-1})} := \langle\lambda\rangle^{s-(n-1)/p}\|\kappa_\lambda^{-1}u\|_{B_{pp}^{s}(\mathbb{R}^{n-1})} \quad (s \in \mathbb{R}).$$

Additionally, we set $H^s_{p,\lambda}(\mathbb{R}^n) := H^{s,0}_{p,\lambda}(\mathbb{R}^n)$ for $s \in \mathbb{R}$. Analogously, we define $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)$ and $H^s_{p,\lambda}(\mathbb{R}^n_+)$.

Lemma 2.10.

a) The statements from Proposition 2.2 a)–f), Lemma 2.3, Remark 2.4 and Remark 2.6 remain valid in the spaces $H^{s,\sigma}_{p,\lambda}$ for $\lambda \in \mathbb{C}$ with respect to the parameter-dependent norms.

b) For all $s, \sigma \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we have

$$\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} = \|\langle D', \lambda\rangle^\sigma u\|_{H^s_{p,\lambda}(\mathbb{R}^n)} = \|\langle D, \lambda\rangle^s \langle D', \lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)}.$$

c) (Interpolation inequality) Let $s_0 < s < s_1$ and $\sigma \in \mathbb{R}$. For every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that, for every $\lambda \in \mathbb{C}$ and $u \in H^{s_1,\sigma}_{p,\lambda}(\mathbb{R}^n)$,

$$\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} \le \varepsilon \|u\|_{H^{s_1,\sigma}_{p,\lambda}(\mathbb{R}^n)} + C(\varepsilon)\langle\lambda\rangle^{s-s_0} \|u\|_{H^{s_0,\sigma}_{p,\lambda}(\mathbb{R}^n)}.$$

The analog statement holds for $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$.

Proof. a) We can apply the above results in the parameter-independent norms to the function $\kappa_\lambda^{-1} u$ and obtain constants independent of λ, noting also that κ_λ commutes with taking the trace on the boundary $\mathbb{R}^{n-1}$.

b) For $\sigma = 0$, the statement follows from [18], Formula (1.9). For general σ, we use the identity

$$\langle\lambda\rangle^\sigma \kappa_\lambda \langle D'\rangle^\sigma \kappa_\lambda^{-1} = \kappa_\lambda \big\langle \langle\lambda\rangle D', \lambda\big\rangle^\sigma \kappa_\lambda^{-1} = \langle D', \lambda\rangle^\sigma,$$

which is obtained by straightforward calculation. This yields

$$\begin{aligned}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} &= \langle\lambda\rangle^{s+\sigma-n/p} \big\|\langle D'\rangle^\sigma \kappa_\lambda^{-1} u\big\|_{H^s_p(\mathbb{R}^n)} = \langle\lambda\rangle^\sigma \big\|\kappa_\lambda \langle D'\rangle^\sigma \kappa_\lambda^{-1} u\big\|_{H^s_{p,\lambda}(\mathbb{R}^n)} \\ &= \big\|\langle D', \lambda\rangle^\sigma u\big\|_{H^s_{p,\lambda}(\mathbb{R}^n)} = \big\|\langle D, \lambda\rangle^s \langle D', \lambda\rangle^\sigma u\big\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

c) An application of the standard interpolation inequality gives

$$\begin{aligned}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} &= \langle\lambda\rangle^{s+\sigma-n/p} \big\|\langle D'\rangle^\sigma \kappa_\lambda^{-1} u\big\|_{H^s_p(\mathbb{R}^n)} \\ &\le \langle\lambda\rangle^{s+\sigma-n/p} \Big(\varepsilon \|\langle D'\rangle^\sigma \kappa_\lambda^{-1} u\|_{H^{s_1}_p(\mathbb{R}^n)} + C(\varepsilon) \|\langle D'\rangle^\sigma \kappa_\lambda^{-1} u\|_{H^{s_0}_p(\mathbb{R}^n)}\Big) \\ &\le \varepsilon \|u\|_{H^{s_1,\sigma}_{p,\lambda}(\mathbb{R}^n)} + C(\varepsilon)\langle\lambda\rangle^{s-s_0} \|u\|_{H^{s_0,\sigma}_{p,\lambda}(\mathbb{R}^n)}. \quad \square\end{aligned}$$

Let us remark that a statement analogous to Lemma 2.10 b) does not hold for the parameter-dependent Besov spaces (with $L^p(\mathbb{R}^n)$ being replaced by $B^0_{pp}(\mathbb{R}^{n-1})$). Although $\langle D', \lambda\rangle^s \colon B^s_{pp,\lambda}(\mathbb{R}^{n-1}) \to B^0_{pp,\lambda}(\mathbb{R}^{n-1})$ is an isomorphism, the norm in $B^0_{pp,\lambda}(\mathbb{R}^{n-1})$ still depends on λ, in contrast to $\|\cdot\|_{H^0_{p,\lambda}(\mathbb{R}^n)} = \|\cdot\|_{L^p(\mathbb{R}^n)}$. This was observed in [18], Section 1.1.

2.5. Multiplication operators

We finish this section with some considerations concerning multiplication operators. For a sufficiently smooth function $a\colon \mathbb{R}^n \to \mathbb{C}$, we define the multiplication operator M_a by $M_a u := au$ whenever the function u belongs to some Sobolev space of positive order. For negative order spaces, we define the multiplication operator by duality with respect to the canonical pairing $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In the following, $\mathrm{BUC}^r(\Omega)$ denotes the space of all functions which are r-times continuously differentiable in Ω and for which all derivatives up to order r are bounded and uniformly continuous.

Lemma 2.11. *Let $s, \sigma \in \mathbb{R}$, and define $r' = r'(s, \sigma) := \max\{|s|, |\sigma|, |s + \sigma|\}$ and*

$$r = r(s, \sigma) := \lfloor r' \rfloor + 1. \tag{2.5}$$

Let $a \in \mathrm{BUC}^r(\mathbb{R}^n)$ and let M_a denote the operator of multiplication by a.

a) There are constants $C = C(r) = C(s, \sigma) > 0$ and $\gamma = \gamma(s, \sigma) > 0$ such that for all $\lambda \in \mathbb{C}$

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} \leq C(r) \|a\|^{1-\gamma}_{\mathrm{BUC}^r(\mathbb{R}^n)} \|a\|^{\gamma}_{\infty}. \tag{2.6}$$

b) If we only have $a \in \mathrm{BUC}^{\lceil r' \rceil}(\mathbb{R}^n)$, M_a is still a multiplier in $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)$ and (2.6) holds with $\gamma = 0$.

c) For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, s, \sigma) > 0$ and a $\lambda_0 = \lambda_0\big(\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)}\big) > 0$ such that

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} < \varepsilon$$

whenever $\|a\|_\infty < \delta$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$.

d) The results in a), b) and c) hold analogously for $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$ with $a \in \mathrm{BUC}^r(\mathbb{R}^n_+)$.

e) The results in a), b) and c) also hold if we replace $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)$ by $B^s_{pp,\lambda}(\mathbb{R}^{n-1})$, taking $\sigma = 0$, i.e. $r' = |s|$, and $a \in \mathrm{BUC}^r(\mathbb{R}^{n-1})$ or $a \in \mathrm{BUC}^{\lceil r' \rceil}(\mathbb{R}^{n-1})$, respectively.

Proof. a) Consider the hexagon which is the convex hull of the vertex set

$$\mathcal{H} := \{(r, 0), (0, r), (-r, r), (-r, 0), (0, -r), (r, -r)\}$$

(see Fig. 2.1). In a first step we are going to show that for all $P \in \mathcal{H}$ we can deduce the bound

$$\|M_a\|_{L(H^P_p(\mathbb{R}^n))} \leq C(r) \|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)}.$$

For the first two vertices this follows from the fact that $H^{r,0}_p(\mathbb{R}^n) = H^r_p(\mathbb{R}^n)$ and $H^{0,r}_p(\mathbb{R}^n) = L^p(\mathbb{R}, H^r_p(\mathbb{R}^{n-1}))$ due to Lemma 2.2 a) as well as the product rule. Their counterparts $(0, -r)$ and $(-r, 0)$ can be treated by a duality argument. For the space $H^{r,-r}_p(\mathbb{R}^n)$ we use Lemma 2.2 g) to obtain

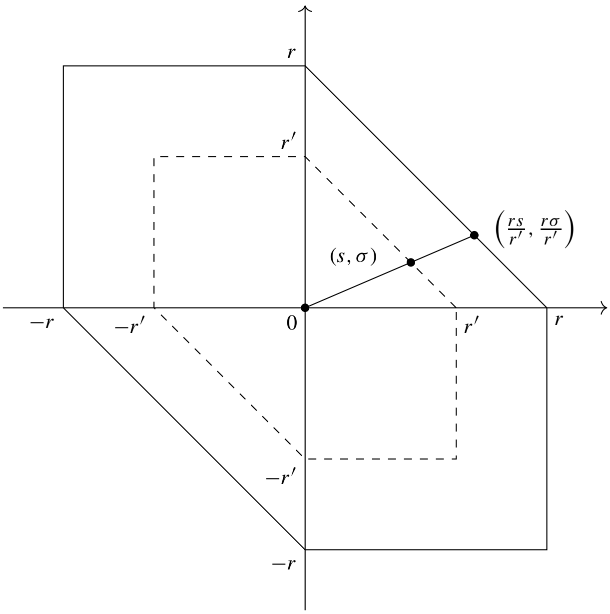

Fig. 2.1. In a first step, we see that the operator M_a is continuous on $H_p^P(\mathbb{R}^n)$ for every vertex $P \in \mathcal{H}$ of the outer hexagon and therefore by interpolation continuous on $H_p^{P_\theta}(\mathbb{R}^n)$ for every P_θ on its boundary. Finally, we interpolate between that boundary and the origin to get the continuity on $H_p^{s,\sigma}(\mathbb{R}^n)$ for every (s,σ) on the boundary of the dashed hexagon. In the origin, we have $\gamma = 1$, on the boundary of the outer hexagon, we have $\gamma = 0$.

$$\begin{aligned}
\|au\|_{H_p^{r,-r}(\mathbb{R}^n)} &\le C(r)\sum_{j=0}^{r} \|\partial_n^j(au)\|_{H_p^{0,-j}(\mathbb{R}^n)} \\
&\le C(r)\sum_{j=0}^{r}\sum_{l=0}^{j} \|(\partial_n^{j-l}a)(\partial_n^l u)\|_{H_p^{0,-j}(\mathbb{R}^n)} \\
&\le C(r)\sum_{j=0}^{r}\sum_{l=0}^{j} \|(\partial_n^{j-l}a)(\partial_n^l u)\|_{H_p^{0,-l}(\mathbb{R}^n)}.
\end{aligned}$$

Here we used $l \le j$ and hence $H_p^{0,-l}(\mathbb{R}^n)$ is continuously embedded in $H_p^{0,-j}(\mathbb{R}^n)$. Furthermore we have $\partial_n^{j-l}a \in \mathrm{BUC}^l(\mathbb{R}^n)$ as $j \le r$. So we may apply the boundedness of $M_{\partial_n^{j-l}a}$ on $H_p^{0,-l}(\mathbb{R}^n)$ and find

$$\begin{aligned}
\|au\|_{H_p^{r,-r}(\mathbb{R}^n)} &\le C(r)\sum_{j=0}^{r}\sum_{l=0}^{j} C(l)\|\partial_n^{j-l}a\|_{\mathrm{BUC}^l(\mathbb{R}^n)}\|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)} \\
&\le C(r)\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)}\sum_{j=0}^{r}\sum_{l=0}^{j}\|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&= C(r)\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)} \sum_{l=0}^{r} (r-l+1)\|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)} \\
&\le C(r)\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)} \sum_{l=0}^{r} \|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)} \\
&\le C(r)\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)} \|u\|_{H_p^{r,-r}(\mathbb{R}^n)}.
\end{aligned}
$$

The last vertex follows by duality again.

In a second step we interpolate along the edges of the hexagon, which is precisely the domain $\{(t,\tau) : \max\{|t|, |\tau|, |t+\tau|\} = r\}$. For any point $P_\theta = (1-\theta)P_0 + \theta P_1$, where $0 < \theta < 1$ and $P_0, P_1 \in \mathcal{H}$, we obtain by interpolation $H_p^{P_\theta}(\mathbb{R}^n) = [H_p^{P_0}(\mathbb{R}^n), H_p^{P_1}(\mathbb{R}^n)]_\theta$ and thus

$$
\|M_a\|_{L(H_p^{P_\theta}(\mathbb{R}^n))} \le C\|M_a\|^{1-\theta}_{L(H_p^{P_0}(\mathbb{R}^n))} \cdot \|M_a\|^{\theta}_{L(H_p^{P_1}(\mathbb{R}^n))} \le C(r)\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)}. \tag{2.7}
$$

Moreover, we observe that in $L^p(\mathbb{R}^n) = H_p^{0,0}(\mathbb{R}^n)$ we have $\|M_a\|_{L(L^p(\mathbb{R}^n))} = \|a\|_\infty$. Finally, we interpolate along a straight line that starts in the origin, passes through (s,σ) and hits the boundary of the hexagon in the point $\left(\frac{rs}{r'}, \frac{r\sigma}{r'}\right)$. More precisely, we use the interpolation

$$
H_p^{s,\sigma}(\mathbb{R}^n) = [L^p(\mathbb{R}^n), H_p^{rs/r', r\sigma/r'}(\mathbb{R}^n)]_{r'/r}
$$

to find that

$$
\|M_a\|_{L(H_p^{s,\sigma}(\mathbb{R}^n))} \le C\|M_a\|^{1-r'/r}_{L(L^p(\mathbb{R}^n))} \cdot \|M_a\|^{r'/r}_{L(H_p^{rs/r', r\sigma/r'}(\mathbb{R}^n))} \le C(r)\|a\|^{1-\gamma}_{\mathrm{BUC}^r(\mathbb{R}^n)} \|a\|^{\gamma}_{\infty}
$$

for $\gamma := 1 - \frac{r'}{r} = 1 - \frac{r'}{\lfloor r' \rfloor + 1} > 0$.

In order to carry the result over to the parameter-dependent norms, we observe the following for any bounded operator T in $H_p^{s,\sigma}(\mathbb{R}^n)$: By Definition 2.9 we have

$$
\begin{aligned}
\|Tu\|_{H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)} &= \langle\lambda\rangle^{s+\sigma-n/p} \|\kappa_\lambda^{-1}(Tu)\|_{H_p^{s,\sigma}(\mathbb{R}^n)} \\
&= \langle\lambda\rangle^{s+\sigma-n/p} \|(\kappa_\lambda^{-1} T \kappa_\lambda)(\kappa_\lambda^{-1} u)\|_{H_p^{s,\sigma}(\mathbb{R}^n)}.
\end{aligned}
$$

Dividing by $\|u\|_{H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)} = \langle\lambda\rangle^{s+\sigma-n/p} \|\kappa_\lambda^{-1} u\|_{L(H_p^{s,\sigma}(\mathbb{R}^n))}$ and passing to the supremum over all $0 \neq u \in H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)$ we conclude that

$$
\|T\|_{L(H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n))} = \|\kappa_\lambda^{-1} T \kappa_\lambda\|_{L(H_p^{s,\sigma}(\mathbb{R}^n))}.
$$

Since we have

$$
\kappa_\lambda^{-1} M_a \kappa_\lambda u(x) = \kappa_\lambda^{-1}[a(x)u(\langle\lambda\rangle x)] = a(\langle\lambda\rangle^{-1}x)u(x) = (\kappa_\lambda^{-1}a)(x)u(x)
$$

it holds

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} = \|M_{\kappa_\lambda^{-1}a}\|_{L(H^{s,\sigma}_p(\mathbb{R}^n))}.$$

Thus

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} \le C(r)\|\kappa_\lambda^{-1}a\|^{1-\gamma}_{\mathrm{BUC}^r(\mathbb{R}^n)}\|\kappa_\lambda^{-1}a\|^{\gamma}_{\infty} \le C(r)\|a\|^{1-\gamma}_{\mathrm{BUC}^r(\mathbb{R}^n)}\|a\|^{\gamma}_{\infty}, \tag{2.8}$$

since $\|\partial^\alpha(\kappa_\lambda^{-1}a)\|_\infty = \langle\lambda\rangle^{-|\alpha|}\|\partial^\alpha a\|_\infty \le \|\partial^\alpha a\|_\infty$ for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le r$.

b) Obviously $\lceil r'\rceil \ne r$ only for $r' \in \mathbb{N}_0$. So for $r' \notin \mathbb{N}_0$, the proof from a) remains unchanged. For $r' \in \mathbb{N}_0$, we proceed analogously as in a), just replacing r by $\lceil r'\rceil = r'$ but stop at (2.7). Carrying over this result to the parameter-dependent norms as before yields (2.6) with $\gamma = 0$.

c) Let $\varepsilon > 0$ and choose $\delta \in (0,1)$ with $\delta^\gamma < \frac{\varepsilon}{2C(r)}$. Let $a \in \mathrm{BUC}^r(\mathbb{R}^n)$ with $\|a\|_\infty < \delta$. As $\|\partial^\alpha(\kappa_\lambda^{-1}a)\|_\infty = \langle\lambda\rangle^{-|\alpha|}\|\partial^\alpha a\|_\infty$, there is a $\lambda_0 = \lambda_0(\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n)}) > 0$ such that

$$\sum_{|\alpha|=1}^{r} \|\partial^\alpha(\kappa_\lambda^{-1}a)\|_\infty \le 1$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| \ge \lambda_0$. For all such λ we obtain

$$\|\kappa_\lambda^{-1}a\|_{\mathrm{BUC}^r(\mathbb{R}^n)} \le \|\kappa_\lambda^{-1}a\|_\infty + 1 = \|a\|_\infty + 1 < \delta + 1 < 2$$

and therefore, using the analog of (2.8), $\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} \le C(r)2^{1-\gamma}\delta^\gamma < \varepsilon$.

d) There exists a bounded extension operator $E_{\mathbb{R}^n_+}\colon \mathrm{BUC}^r(\mathbb{R}^n_+) \to \mathrm{BUC}^r(\mathbb{R}^n)$ for any $r \in \mathbb{N}_0$ (see, e.g., the construction in [1], Theorem 5.19). Then for $\mathbb{R}^n_+$ part a) follows from

$$\begin{aligned}\|M_a u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)} &\le \|M_{E_{\mathbb{R}^n_+}a}(e_{\mathbb{R}^n_+}u)\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} \le C\|E_{\mathbb{R}^n_+}a\|_{\mathrm{BUC}^r(\mathbb{R}^n)}\|e_{\mathbb{R}^n_+}u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)}\\ &\le C\|a\|_{\mathrm{BUC}^r(\mathbb{R}^n_+)}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)}.\end{aligned}$$

e) Taking $\sigma = 0$, we use the result from a) and b) for the spaces $H^{s+\rho}_{p,\lambda}(\mathbb{R}^{n-1})$ and $H^{s-\rho}_{p,\lambda}(\mathbb{R}^{n-1})$ for a sufficiently small $\rho > 0$ such that $|s \pm \rho| \le \lfloor|s|\rfloor + 1$ still holds. Then the result follows by real interpolation of the λ-dependent, but classical Sobolev spaces, which was established in [18, (1.16)]. □

Remark 2.12. In the case of $r' \in \mathbb{N}_0$ in Lemma 2.11 b) and $\sigma = \lambda = 0$, we directly get back the classical results for the usual Sobolev spaces $H^s_p(\mathbb{R}^n)$. We remark that for $r' = s = \sigma = 0$, the assumption $a \in \mathrm{BUC}^1(\mathbb{R}^n)$ is not optimal for the statement in c), as in this case $\|a\|_{\mathrm{BUC}(\mathbb{R}^n)}$ coincides with $\|a\|_\infty$, and it would be sufficient to assume $a \in \mathrm{BUC}(\mathbb{R}^n)$. As we are mainly interested in the case $\sigma \ne 0$, we did not specify the smoothness for this specific case. Note that for positive $r' \in \mathbb{N}$ and for $a \in \mathrm{BUC}^{r'}(\mathbb{R}^n)$ the statement in c) seems to hold if λ_0 is allowed to depend also on ε. For non-integer r', the condition $a \in \mathrm{BUC}^r(\mathbb{R}^n)$ seems to be optimal, given that we only want to consider integer-valued smoothness parameters.

Furthermore, we would like to note that pointwise multipliers in Besov spaces with $\lambda = 0$ are described, e.g., in [26] and [40]. In particular, it is known that functions which are Hölder continuous with Hölder index larger than $|s|$ are multipliers in $B^s_{pp}(\mathbb{R}^{n-1})$ (see [31], Theorem 4.7.1 (ii)). For our purposes, however, Lemma 2.11 e) is sufficient.

3. Boundary value problems in the half-space

We now deal with boundary value problems in domains and in the half-space. In the following, let $\Omega \subset \mathbb{R}^n$ be a domain with compact and sufficiently smooth boundary Γ, or let $\Omega = \mathbb{R}^n_+$ with boundary $\Gamma = \mathbb{R}^{n-1}$. We consider the boundary value problem

$$\begin{aligned} (\lambda - A)u &= f \quad \text{in } \Omega, \\ B_j u &= g_j \quad (j = 1, \dots, m) \text{ on } \Gamma, \end{aligned} \tag{3.1}$$

where A and B_j are linear differential operators of order $2m$ and linear boundary operators of order $m_j < 2m$, respectively, of the form

$$A = A(x, D) = \sum\nolimits_{|\alpha| \le 2m} a_\alpha(x) D^\alpha, \tag{3.2}$$

$$B_j = B_j(x, D) = \sum\nolimits_{|\beta| \le m_j} b_{j\beta}(x) \gamma_0 D^\beta. \tag{3.3}$$

We also write $B = (B_1, \dots, B_m)$. Here, $a_\alpha \colon \overline{\Omega} \to \mathbb{C}$ and $b_{j\beta} \colon \Gamma \to \mathbb{C}$ are sufficiently smooth functions. More precisely, we will consider the following smoothness assumptions, depending on $(s, \sigma) \in \mathbb{R}^2$.

- **(S1)** Let $r' = r'(s - 2m, \sigma) := \max\{|s - 2m|, |\sigma|, |s + \sigma - 2m|\}$ and $r := \lfloor r' \rfloor + 1$. We assume $a_\alpha \in \mathrm{BUC}^r(\Omega)$ for all $|\alpha| = 2m$ and $a_\alpha \in \mathrm{BUC}^{\lceil r' \rceil}(\Omega)$ for all $|\alpha| < 2m$.
- **(S2)** If Ω is unbounded, then $a_\alpha(\infty) := \lim_{x \in \Omega, |x| \to \infty} a_\alpha(x)$ exists for all $|\alpha| \le 2m$. In addition, all derivatives of the function
$$x \mapsto a_\alpha\left(\frac{x}{|x|^2}\right) \quad (x \neq 0)$$
up to order r for $|\alpha| = 2m$ (and up to order $\lceil r' \rceil$ for $|\alpha| < 2m$) possess a continuous extension to $x = 0$.
- **(S3)** For each $j \in \{1, \dots, m\}$, let $k_j' := |s + \sigma - m_j - \frac{1}{p}|$ and $k_j := \lfloor k_j' \rfloor + 1$. We assume $b_{j\beta} \in \mathrm{BUC}^{k_j}(\Gamma)$ for all $|\beta| = m_j$ and $b_{j\beta} \in \mathrm{BUC}^{\lceil k_j' \rceil}(\Gamma)$ for all $|\beta| < m_j$.
- **(S4)** If $\Omega = \mathbb{R}^n_+$, then $b_{j\beta}(\infty) := \lim_{x \in \mathbb{R}^{n-1}, |x| \to \infty} b_{j\beta}(x)$ exists for all $j \in \{1, \dots, m\}$ and $|\beta| \le m_j$. In addition, all derivatives of the function
$$x \mapsto b_{j\beta}\left(\frac{x}{|x|^2}\right) \quad (x \neq 0)$$
up to order k_j for $|\beta| = m_j$ (and up to order $\lceil k_j' \rceil$ for $|\beta| < m_j$) possess a continuous extension to $x = 0$.
- **(S5)** The domain Ω is of class $C^{2m + \lceil r' \rceil}$.

In the following, let $\Lambda \subset \mathbb{C}$ be a closed sector in the complex plane with vertex at the origin. Then the family $\lambda - A(x, D)$ is called parameter-elliptic in Λ if the principal symbol $A_0(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha$ satisfies

$$|\lambda - A_0(x,\xi)| \geq C\big(|\lambda| + |\xi|^{2m}\big) \quad (x \in \overline{\Omega},\ \lambda \in \Lambda,\ \xi \in \mathbb{R}^n,\ (\xi,\lambda) \neq 0) \tag{3.4}$$

for some constant $C > 0$. Similarly, we define the principal symbols $B_{0,j}(x,\xi) := \sum_{|\beta|=m_j} b_{j\beta}(x)\xi^\beta$. The boundary value problem is called parameter-elliptic in Λ if $\lambda - A(x,D)$ is parameter-elliptic in Λ and if the following Shapiro–Lopatinskii condition holds:

Let $x_0 \in \partial\Omega$ be an arbitrary point of the boundary, and rewrite the boundary value problem $(\lambda - A_0(x_0,D), B_{0,1}(x_0,D), \dots, B_{0,m}(x_0,D))$ in the coordinate system associated with x_0, which is obtained from the original one by a rotation after which the positive x_n-axis has the direction of the interior normal to $\partial\Omega$ at x_0. Then the trivial solution $w = 0$ is the only stable solution of the ordinary differential equation on the half-line

$$\begin{aligned}
(\lambda - A_0(x_0,\xi',D_n))w(x_n) &= 0 \quad (x_n \in (0,\infty)),\\
B_{0,j}(x_0,\xi',D_n)w(0) &= 0 \quad (j = 1,\dots,m)
\end{aligned}$$

for all $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \Lambda$ with $(\xi',\lambda) \neq 0$.

In this section we show that parameter-elliptic problems induce an isomorphism between parameter-dependent spaces (in the sense of Definition 2.8). We focus on the case of the half-space.

3.1. Model problems and small perturbations

Lemma 3.1 *(Model problem in $\mathbb{R}^n$). Let $A_0(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ have constant coefficients $a_\alpha \in \mathbb{C}$, and let $\lambda - A_0(D)$ be parameter-elliptic in Λ. Then, for every $s,\sigma \in \mathbb{R}$ and every $\lambda_0 > 0$, the operator family*

$$\lambda - A_0\colon H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n) \to H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n) \tag{3.5}$$

is an isomorphism for $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$.

Proof. The result is well known in case $\sigma = 0$, see [16], Theorem 1.7, or can be obtained immediately from Mikhlin's theorem. Let us denote by $(\lambda - A_0)^{-1}_{(s,0)}$ the corresponding inverse operator. We use the description of the norm in $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)$ from Lemma 2.10 b).

Since A_0 commutes with $\langle D',\lambda\rangle^\sigma$ and $\langle D',\lambda\rangle^\sigma : H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n) \to H^{s,0}_{p,\lambda}(\mathbb{R}^n)$ is an isometric isomorphism, the inverse to (3.5) is

$$(\lambda - A_0)^{-1}_{(s,\sigma)} = \langle D',\lambda\rangle^{-\sigma}(\lambda - A_0)^{-1}_{(s,0)}\langle D',\lambda\rangle^\sigma,$$

which then has the same uniform bound as $(\lambda - A_0)^{-1}_{(s,0)}$. □

Let us now pass to the situation in the half-space, where we consider the following boundary problem.

Theorem 3.2 *(Model problem in $\mathbb{R}^n_+$). Let $(\lambda - A_0, B_0)$ be parameter-elliptic in Λ. Here again, we have $A_0(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ with constant coefficients $a_\alpha \in \mathbb{C}$, as well as $B_0 :=$*

$(B_{0,1}(D),\dots,B_{0,m}(D))$ where $B_{0,j}(D)=\sum_{|\beta|=m_j} b_{j\beta}\gamma_0 D^\beta$ with constant coefficients $b_{j\beta}\in\mathbb{C}$ for $j=1,\dots,m$. Then, for every $s>\max_j m_j+\frac{1}{p}$, $\sigma\in\mathbb{R}$, and $\lambda_0>0$, the family of operators

$$\begin{pmatrix}\lambda-A_0\\ B_0\end{pmatrix}: H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)\to H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n_+)\times\prod_{j=1}^m B^{s+\sigma-m_j-1/p}_{pp,\lambda}(\mathbb{R}^{n-1}) \tag{3.6}$$

is an isomorphism for $\lambda\in\Lambda$ with $|\lambda|\ge\lambda_0$.

Proof. The proof is similar to that of Lemma 3.1. The result is known for $\sigma=0$, see [16], Theorem 1.9; let $L_{(s,0)}(\lambda)$ be the inverse. All involved operators commute with $\langle D',\lambda\rangle^\sigma$. Hence the inverse operator $L_{(s,\sigma)}(\lambda)$ for general σ is given by

$$\langle D',\lambda\rangle^{-\sigma}L_{(s,0)}(\lambda)\mathrm{diag}(\langle D',\lambda\rangle^\sigma,\dots,\langle D',\lambda\rangle^\sigma),$$

where the diagonal matrix acts as $\langle D',\lambda\rangle^\sigma$ on each space on the right-hand side of (3.6). Hence $L_{(s,\sigma)}(\lambda)$ has the same uniform norm-bound as $L_{(s,0)}(\lambda)$. □

Motivated by the last two results, we define the parameter-dependent spaces

$$\mathbb{E}^{s,\sigma}_\lambda(\Omega):=H^{s,\sigma}_{p,\lambda}(\Omega)\qquad(\text{with }\Omega=\mathbb{R}^n\text{ or }\Omega=\mathbb{R}^n_+), \tag{3.7}$$

as well as

$$\begin{aligned}\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n)&:=H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n),\\ \mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+)&:=H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n_+)\times\prod_{j=1}^m B^{s+\sigma-m_j-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})\end{aligned} \tag{3.8}$$

for $s,\sigma\in\mathbb{R}$.

$$\lambda-A_0\colon\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n)\to\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n)\text{ and }\begin{pmatrix}\lambda-A_0\\ B_0\end{pmatrix}\colon\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+)\to\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+)$$

are both isomorphisms. Below, we will consider the case of variable coefficients which are close to constant coefficients in an appropriate sense. As a preparation, we show some auxiliary continuity results.

Lemma 3.3. *Let $(s,\sigma)\in\mathbb{R}^2$.*

a) Let A be a differential operator in $\mathbb{R}^n$ as in (3.2) and assume (S1) to hold. Let $M_A:=\max_{|\alpha|=2m}\|a_\alpha\|_{BUC^r(\mathbb{R}^n)}+\max_{|\alpha|<2m}\|a_\alpha\|_{BUC^{\lceil r'\rceil}(\mathbb{R}^n)}$. Then for every $\varepsilon>0$ there exist constants $\delta=\delta(\varepsilon,s,\sigma)>0$ and $\lambda_0=\lambda_0(M_A)>0$ such that

$$\|A\|_{L(\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n),\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n))}<\varepsilon$$

holds for all $\lambda\in\mathbb{C}$ with $|\lambda|\ge\lambda_0$ provided $\max_{|\alpha|=2m}\|a_\alpha\|_\infty<\delta$.

b) Let (A, B) *be a boundary value problem of the form* (3.2)–(3.3) *in* $\mathbb{R}^n_+$ *and assume* (S1) *and* (S3) *to hold with* $s > \max_j m_j + \frac{1}{p}$. *Let*

$$M_{A,B} := \max_{|\alpha|=2m} \|a_\alpha\|_{\mathrm{BUC}^r(\mathbb{R}^n_+)} + \max_{|\alpha|<2m} \|a_\alpha\|_{\mathrm{BUC}^{\lceil r' \rceil}(\mathbb{R}^n_+)} + \\ + \max_{\substack{j=1,\dots,m \\ |\beta|=m_j}} \|b_{j\beta}\|_{\mathrm{BUC}^{k_j}(\mathbb{R}^{n-1})} + \max_{\substack{j=1,\dots,m \\ |\beta|<m_j}} \|b_{j\beta}\|_{\mathrm{BUC}^{\lceil k'_j \rceil}(\mathbb{R}^{n-1})}.$$

Then for every $\varepsilon > 0$ *there exist constants* $\delta = \delta(\varepsilon, s, \sigma) > 0$ *and* $\lambda_0 = \lambda_0(M_{A,B}) > 0$ *such that*

$$\left\| \begin{pmatrix} A \\ B \end{pmatrix} \right\|_{L(\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+), \mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+))} < \varepsilon$$

holds for all $\lambda \in \mathbb{C}$ *with* $|\lambda| \geq \lambda_0$ *provided*

$$\max_{|\alpha|=2m} \|a_\alpha\|_\infty + \max_{\substack{j=1,\dots,m \\ |\beta|=m_j}} \|b_{j\beta}\|_\infty < \delta.$$

Proof. a) Let $A_0 = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$ be the principal part of A and set $\widetilde{A} := A - A_0$. Let $\varepsilon > 0$ be fixed and $u \in H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)$ arbitrary. Then, due to Lemma 2.11 c), for appropriate $\varepsilon' > 0$ there exist $\delta(\varepsilon', s, \sigma) > 0$ and $\lambda_0(M_A) > 0$ such that for $|\lambda| \geq \lambda_0$ we have

$$\begin{aligned} \|A_0 u\|_{H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n)} &\leq \sum_{|\alpha|=2m} \|a_\alpha(\cdot) D^\alpha u\|_{H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n)} \\ &\leq \varepsilon' \sum_{|\alpha|=2m} \|D^\alpha u\|_{H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n)} \leq \frac{\varepsilon}{2} \|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)}, \end{aligned}$$

given $\max_{|\alpha|=2m} \|a_\alpha\|_\infty < \delta$. For $\widetilde{A}u$ we use Lemma 2.11 b), as we only need the fact that the coefficients are multipliers, which justifies the weaker regularity assumptions for the lower order terms. Thus, we obtain the estimate

$$\begin{aligned} \|\widetilde{A}u\|_{H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n)} &\leq \sum_{|\alpha|<2m} \|a_\alpha(\cdot) D^\alpha u\|_{H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n)} \\ &\leq C M_A \sum_{|\alpha|<2m} \|D^\alpha u\|_{H^{s-2m,\sigma}_{p,\lambda}(\mathbb{R}^n)} \\ &\leq C M_A \|u\|_{H^{s-1,\sigma}_{p,\lambda}(\mathbb{R}^n)} \leq C M_A \langle\lambda\rangle^{-1} \|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)}. \end{aligned} \tag{3.9}$$

The last inequality holds true because we have

$$\|u\|_{H^{s-1,\sigma}_{p,\lambda}(\mathbb{R}^n)} = \|\langle D, \lambda\rangle^{s-1} \langle D', \lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)} \leq C \langle\lambda\rangle^{-1} \|\langle D, \lambda\rangle^s \langle D', \lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)}$$

uniformly in λ, since $\langle\lambda\rangle\langle\xi,\lambda\rangle^{-1}$ is a Mikhlin multiplier with symbol estimates that are uniform in λ. As $\langle\lambda\rangle^{-1}$ vanishes for $|\lambda|\to\infty$, we can choose λ_0 so large that $CM_A\langle\lambda\rangle^{-1}<\frac{\varepsilon}{2}$ whenever $|\lambda|\geq\lambda_0$.

b) The calculations for A are similar to a), we just replace the whole space estimates by the half-space estimates. For the boundary operators B_j we use Lemma 2.11 e) instead, noting that (S3) yields the required smoothness. Hence for B_j we split off the lower order terms again. Then for a given $\varepsilon>0$, again, for appropriate $\varepsilon'>0$ there exist $\delta(\varepsilon',s,\sigma)>0$ and $\lambda_0(M_{A,B})>0$ such that for $|\lambda|\geq\lambda_0$ we obtain

$$\begin{aligned}\|B_{0,j}u\|_{B^{s+\sigma-m_j-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} &\leq \sum_{|\beta|=m_j}\|b_{j\beta}(\cdot)\gamma_0 D^\beta u\|_{B^{s+\sigma-m_j-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}\\ &\leq \varepsilon'\sum_{|\beta|=m_j}\|\gamma_0 D^\beta u\|_{B^{s+\sigma-m_j-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}\\ &\leq C\varepsilon'\sum_{|\beta|=m_j}\|D^\beta u\|_{H^{s-m_j,\sigma}_{p,\lambda}(\mathbb{R}^n_+)}\leq\frac{\varepsilon}{2}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)},\end{aligned}$$

given $\max_{|\alpha|=2m}\|a_\alpha\|_\infty+\max_{j=1,\dots,m}\max_{|\beta|=m_j}\|b_{j\beta}\|_\infty<\delta$. Here we also used the continuity of the trace from Definition 2.5. The lower order terms can be handled as in a), applying Lemma 2.11 e) once more. □

Lemma 3.4 *(Small perturbation in $\mathbb{R}^n$). Let $\lambda-A_0$ be as in Lemma 3.1, and let $A=A_0+\widetilde{A}$, where*

$$\widetilde{A}=\widetilde{A}(x,D)=\sum_{|\alpha|\leq 2m}\widetilde{a}_\alpha(x)D^\alpha.$$

Moreover, let $s,\sigma\in\mathbb{R}$ and assume (S1) to hold. Define $\delta=\delta(\frac{1}{2\rho},s,\sigma)$ and $\lambda_0=\max\{\lambda_0(M_{\widetilde{A}}),1\}$ as in Lemma 3.3 a), where

$$\rho:=\sup_{|\lambda|\geq 1}\|(\lambda-A_0)^{-1}\|_{L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n),\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n))}.$$

Then, if $\max_{|\alpha|=2m}\|\widetilde{a}_\alpha\|_\infty<\delta$, the operator family

$$\lambda-A\colon\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n)\to\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n)$$

is an isomorphism for $\lambda\in\Lambda$ with $|\lambda|\geq\lambda_0$.

Proof. Using Lemma 3.1, we can write

$$\lambda-A=(\lambda-A_0)\big(I-(\lambda-A_0)^{-1}\widetilde{A}\big)\quad(\lambda\neq 0).$$

Choosing δ and λ_0 as stated and applying Lemma 3.3 a) to $\widetilde{A}$, we obtain

$$\|\widetilde{A}\|_{L(\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n),\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n))}<\frac{1}{2\rho}$$

whenever $|\lambda| \geq \lambda_0$ and $\max_{|\alpha|=2m} \|\widetilde{a}_\alpha\|_\infty < \delta$. Therefore,

$$\|(\lambda - A_0)^{-1}\widetilde{A}\|_{L(\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n))} < \frac{1}{2}.$$

By the usual Neumann series argument, $I - (\lambda - A_0)^{-1}\widetilde{A}$ is invertible with

$$\|(I - (\lambda - A_0)^{-1}\widetilde{A})^{-1}\|_{L(\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n))} < 2$$

for every $|\lambda| \geq \lambda_0$. We conclude that, for such λ,

$$(\lambda - A)^{-1} = \big(I - (\lambda - A_0)^{-1}\widetilde{A}\big)^{-1}(\lambda - A_0)^{-1}$$

with

$$\|(\lambda - A)^{-1}\|_{L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n),\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n))} < 2\|(\lambda - A_0)^{-1}\|_{L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n),\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n))}.$$

Using Lemma 3.1 once more completes the proof. □

Theorem 3.5 *(Small perturbation in $\mathbb{R}^n_+$). Consider the boundary value problem $(\lambda - A, B)$ with $A = A_0 + \widetilde{A}$ and $B = B_0 + \widetilde{B}$, where $(\lambda - A_0, B_0)$ is as in Theorem 3.2,*

$$\widetilde{A} = \widetilde{A}(x, D) = \sum_{|\alpha|\leq 2m} \widetilde{a}_\alpha(x) D^\alpha$$

and $\widetilde{B} = (\widetilde{B}_1, \dots, \widetilde{B}_m)$ with

$$\widetilde{B}_j = \widetilde{B}_j(x, D) = \sum_{|\beta|\leq m_j} \widetilde{b}_{j\beta}(x)\gamma_0 D^\beta$$

and $m_j < 2m$. Moreover, let $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$, and assume (S1) and (S3) to hold. Define

$$\rho := \sup_{|\lambda|\geq 1} \|L(\lambda)\|_{L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+),\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+))},$$

where $L(\lambda)$ denotes the inverse of the map in (3.6), and choose $\delta = \delta(\frac{1}{2\rho}, s, \sigma)$ and $\lambda_0 = \max\{\lambda_0(M_{\widetilde{A},\widetilde{B}}), 1\}$ as in Lemma 3.3 b). If

$$\max_{|\alpha|=2m} \|\widetilde{a}_\alpha\|_\infty + \max_{\substack{j=1,\dots,m \\ |\beta|=m_j}} \|\widetilde{b}_{j\beta}\|_\infty < \delta,$$

then

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} : \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+) \to \mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+) \tag{3.10}$$

is an isomorphism for $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$.

Proof. We first note that by Theorem 3.2, the inverse $L(\lambda)$ of the map in (3.6) exists, so we can write

$$\begin{pmatrix}\lambda - A\\ B\end{pmatrix} = \begin{pmatrix}\lambda - A_0\\ B_0\end{pmatrix} + \begin{pmatrix}-\widetilde{A}\\ \widetilde{B}\end{pmatrix} = \begin{pmatrix}\lambda - A_0\\ B_0\end{pmatrix}\left(I + L(\lambda)\begin{pmatrix}-\widetilde{A}\\ \widetilde{B}\end{pmatrix}\right).$$

Choosing δ and λ_0 as stated and applying Lemma 3.3 b), we see that

$$\left\|\begin{pmatrix}-\widetilde{A}\\ \widetilde{B}\end{pmatrix}\right\|_{L(\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+),\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+))} < \frac{1}{2\rho}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$ provided

$$\max_{|\alpha|=2m} \|\widetilde{a}_\alpha\|_\infty + \max_{\substack{j=1,\dots,m\\ |\beta|=m_j}} \|\widetilde{b}_{j\beta}\|_\infty < \delta.$$

Therefore

$$\left\|L(\lambda)\begin{pmatrix}-\widetilde{A}\\ \widetilde{B}\end{pmatrix}\right\|_{L(\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+))} < \frac{1}{2},$$

which allows us to use the Neumann series just as above, yielding the desired isomorphism. □

3.2. General boundary value problems

The analysis of the general case of variable coefficients is based on the classical method of freezing the coefficients.

In the following, let $(\lambda - A, B)$ be a boundary value problem in $\mathbb{R}^n_+$ of the form (3.2)–(3.3) which is parameter-elliptic in Λ for all $x \in \overline{\mathbb{R}^n_+} \cup \{\infty\}$. Let $(s, \sigma) \in \mathbb{R}^2$, and assume the validity of (S1)–(S4).

For every $x_0 \in \mathbb{R}^n_+$, we consider the model problem $\lambda - A_0(x_0, D)$ with frozen coefficients $a_\alpha(x_0) \in \mathbb{C}$ and without lower-order terms. By the assumption of parameter-ellipticity, we can apply Lemma 3.1 and obtain the existence of the inverse operator

$$(\lambda - A_0(x_0, D))^{-1} \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n), \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n))$$

for $\lambda \in \Lambda$. In the same way, for $x_0 \in \mathbb{R}^{n-1} \cup \{\infty\}$ and $s > \max_j m_j + \frac{1}{p}$, we obtain from Theorem 3.2 the existence of the inverse operator

$$L_{x_0}(\lambda) := \begin{pmatrix}\lambda - A_0(x_0, D)\\ B_0(x_0, D)\end{pmatrix}^{-1} \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+)).$$

Lemma 3.6. *With the above notation, we have*

$$\rho_{A,B} := \sup_{\substack{x_0\in\mathbb{R}^n_+\\ \lambda\in\Lambda,\, |\lambda|\geq 1}} \|(\lambda - A_0(x_0, D))^{-1}\|_{L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n),\mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n))} +$$

$$+ \sup_{\substack{x_0\in\mathbb{R}^{n-1}\\ \lambda\in\Lambda,\,|\lambda|\geq 1}} \|L_{x_0}(\lambda)\|_{L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+),\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+))} < \infty.$$

Proof. Let us consider the first supremum and assume this supremum to be infinite. Then there exist sequences $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^n_+$ and $(\lambda_k)_{k\in\mathbb{N}} \subset \Lambda$ with $|\lambda_k| \geq 1$ such that $\|(\lambda_k - A_0(x_k, D))^{-1}\| \to \infty$ for $k \to \infty$. By passing to a subsequence we may assume that $x_k \to x^*$ for $k \to \infty$ where either $x^* \in \overline{\mathbb{R}^n_+}$ or $x^* = \infty$. Now write

$$\lambda_k - A_0(x_k, D) = \lambda_k - A_0(x^*, D) - \widetilde{A}^k(D), \qquad \widetilde{A}^k(D) := A_0(x_k, D) - A_0(x^*, D).$$

Since $A_0(x^*, D)$ satisfies the assumptions of Lemma 3.1, we get

$$\lambda_k - A_0(x_k, D) = (\lambda_k - A_0(x^*, D))\big[1 - (\lambda_k - A_0(x^*, D))^{-1}\widetilde{A}^k(D)\big].$$

Now let

$$\rho^* = \sup_{\lambda\in\Lambda,\,|\lambda|\geq 1} \|(\lambda - A_0(x^*, D))^{-1}\|_{L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n),\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n))}$$

which is finite due to Lemma 3.1. Moreover, observe that

$$\begin{aligned}\|\widetilde{A}^k(D)u\|_{\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n)} \leq{}& \|\langle D,\lambda\rangle^{s-2m}\langle D',\lambda\rangle^{\sigma}\widetilde{A}^k(D)\langle D,\lambda\rangle^{-s}\langle D',\lambda\rangle^{-\sigma}\|_{L(L^p(\mathbb{R}^n))}\times\\ &\times \|\langle D,\lambda\rangle^{s}\langle D',\lambda\rangle^{\sigma}u\|_{L^p(\mathbb{R}^n)}\\ ={}&\|\langle D,\lambda\rangle^{-2m}\widetilde{A}^k(D)\|_{L(L^p(\mathbb{R}^n))}\|u\|_{\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n)}.\end{aligned}$$

It is a straightforward consequence of Mikhlin's Theorem that

$$\sup_{\lambda\in\mathbb{C}} \|\langle D,\lambda\rangle^{-2m}\widetilde{A}^k(D)\|_{L(L^p(\mathbb{R}^n))} \xrightarrow{k\to\infty} 0,$$

since the (constant) coefficients of $\widetilde{A}^k(D)$ tend to zero with $k \to \infty$. It follows that

$$\sup_{\lambda\in\Lambda,\,|\lambda|\geq 1} \|(\lambda - A_0(x^*, D))^{-1}\widetilde{A}^k(D)\|_{L(\mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n))} \leq \frac{1}{2}$$

for all sufficiently large k. As above, using the Neumann series, we conclude that

$$\begin{aligned}&\|(\lambda_k - A_0(x_k, D))^{-1}\|_{L(\mathbb{F}^{s,\sigma}_{\lambda_k}(\mathbb{R}^n),\mathbb{E}^{s,\sigma}_{\lambda_k}(\mathbb{R}^n))}\\ &\qquad \leq 2\|(\lambda_k - A_0(x^*, D))^{-1}\|_{L(\mathbb{F}^{s,\sigma}_{\lambda_k}(\mathbb{R}^n),\mathbb{E}^{s,\sigma}_{\lambda_k}(\mathbb{R}^n))} \leq 2\rho^*\end{aligned}$$

for all sufficiently large k. This is a contradiction. □

Remark 3.7. In the following, we construct a finite covering of $\mathbb{R}^n_+$ consisting of balls and the complement of a ball centered in the origin. Afterwards, we need to extend the coefficients of the localized problems to $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^{n-1}$, respectively. To this end, we will use a general extension function. We fix $\chi \in C^\infty([0,\infty))$ with $0 \le \chi \le 1$, $\chi(z) = 1$ for $0 \le z \le 1$ and $\chi(z) = 0$ for $z \ge 2$ and define the function $\chi_U : \mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\}$ via

$$\chi_U(x) := \begin{cases} \frac{x}{|x|^2} + \chi\big(\frac{r'}{|x|}\big)(x - \frac{x}{|x|^2}) & \text{if there exists } r' > 0 : U = \mathbb{R}^n \backslash B(0, r'), \\ x' + \chi\big(\frac{|x-x'|}{r'}\big)(x - x') & \text{if there exist } r' > 0,\ x' \in \mathbb{R}^n : U = B(x', r'). \end{cases}$$

The function χ_U coincides with the identity on U and is later compatible with the parameter-ellipticity of the local operators. Since we use reflection techniques for the construction of χ_U, it is crucial that our covering consists of balls and the complement of a ball centered in the origin.

For the localization, we first apply Lemma 3.3 with $\varepsilon := \frac{1}{2\rho_{A,B}}$, where $\rho_{A,B}$ is taken from Lemma 3.6. We fix

$$\delta_0 := \delta\Big(\frac{1}{2\rho_{A,B}}, s, \sigma\Big) > 0 \tag{3.11}$$

as being defined in Lemma 3.3. Let $x_0 := \infty$ and $U_0 := \{x \in \mathbb{R}^n : |x| > r_0\}$ where r_0 is sufficiently large such that

$$\max_{|\alpha|=2m} \|a_\alpha(\cdot) - a_\alpha(x_0)\|_{L^\infty(\widetilde{U}_0 \cap \mathbb{R}^n_+)} + \\ + \max_{\substack{j=1,\dots,m \\ |\beta|=m_j}} \|b_{j\beta}(\cdot) - b_{j\beta}(x_0)\|_{L^\infty(\widetilde{U}_0 \cap \mathbb{R}^{n-1})} < \delta_0 \tag{3.12}$$

with $\widetilde{U}_0 := \big\{x \in \mathbb{R}^n : |x| > \frac{r_0}{2}\big\}$ (this is possible due to (S2) and (S4)). As the coefficients of A and B are continuous and $\overline{B(0, r_0)} \cap \mathbb{R}^{n-1}$ is compact, there exists a finite covering $\mathbb{R}^{n-1} \subset \bigcup_{k=0}^{K_0} U_k$ with $U_k := B(x_k, r_k) \subset \mathbb{R}^n$ for $k = 1, \dots, K_0$, where $x_k \in \mathbb{R}^{n-1}$ and $r_k > 0$ are chosen such that

$$\max_{|\alpha|=2m} \|a_\alpha(\cdot) - a_\alpha(x_k)\|_{L^\infty(\widetilde{U}_k \cap \mathbb{R}^n_+)} + \\ + \max_{\substack{j=1,\dots,m \\ |\beta|=m_j}} \|b_{j\beta}(\cdot) - b_{j\beta}(x_k)\|_{L^\infty(\widetilde{U}_k \cap \mathbb{R}^{n-1})} < \delta_0 \tag{3.13}$$

with $\widetilde{U}_k := B(x_k, 2r_k)$ for $k = 1, \dots, K_0$. We set

$$\delta_{\max} := \sup\Big\{\delta > 0 : \mathbb{R}^{n-1} \times [0, \delta] \subset \bigcup_{k=0}^{K_0} U_k\Big\}.$$

Similarly, as $\mathbb{R}^n_+ \setminus \bigcup_{k=0}^{K_0} U_k$ is compact, we can choose $x_k \in \mathbb{R}^n_+$ and $0 < r_k < \frac{\delta_{\max}}{2}$ for $k = K_0 + 1, \dots, K$ such that $U_k := B(x_k, r_k) \subset \Big\{z \in \mathbb{R}^n \mid z_n > \frac{\delta_{\max}}{2}\Big\}$,

$$\max_{|\alpha|=2m} \|a_\alpha(\cdot) - a_\alpha(x_k)\|_{L^\infty(\widetilde{U}_k)} < \delta_0 \tag{3.14}$$

with $\widetilde{U}_k := B(x_k, 2r_k)$ for $k = K_0 + 1, \dots, K$ and $\overline{\mathbb{R}^n_+} \subset \bigcup_{k=0}^K U_k$.

Remark 3.8 *(Local operators and extensions).* Let $x_0, \dots, x_K$ be chosen as above. Starting out from the coefficient functions a_α and $b_{j\beta}$ let us define

$$\begin{aligned} a^k_\alpha(x) &:= a_\alpha(\chi_{U_k}(x)) \qquad (x \in \overline{\mathbb{R}^n_+}), \\ b^k_{j\beta}(x) &:= b_{j\beta}(\chi_{U_k}(x)) \qquad (x \in \mathbb{R}^{n-1}) \end{aligned}$$

for $k = 0, \dots, K_0$ and, for $k = K_0 + 1, \dots, K$,

$$a^k_\alpha(x) := a_\alpha(\chi_{U_k}(x)) \qquad (x \in \mathbb{R}^n).$$

Here the function χ_{U_k} is defined as in Remark 3.7. These new coefficients have the same smoothness as before. a^k_α coincides with a_α on $U_k \cap \overline{\mathbb{R}^n_+}$ and $U_k \cap \mathbb{R}^n$, respectively, $b^k_{j\beta}$ coincides with $b_{j\beta}$ on $U_k \cap \mathbb{R}^{n-1}$. By (3.12)–(3.14), we have

$$\begin{aligned} \left\| a^k_\alpha(\cdot) - a_\alpha(x_k) \right\|_{L^\infty(\mathbb{R}^n_+)} &< \delta_0 \text{ for } k = 0, \dots, K_0, \\ \left\| a^k_\alpha(\cdot) - a_\alpha(x_k) \right\|_{L^\infty(\mathbb{R}^n)} &< \delta_0 \text{ for } k = K_0 + 1, \dots, K, \\ \left\| b^k_{j\beta}(\cdot) - b_{j\beta}(x_k) \right\|_{L^\infty(\mathbb{R}^{n-1})} &< \delta_0 \text{ for } k = 0, \dots, K_0. \end{aligned} \tag{3.15}$$

With the new coefficient functions we associate the operators A^k and $B^k = (B^k_1, \dots, B^k_m)$ via

$$A^k = A^k(x, D) := \sum_{|\alpha| \le 2m} a^k_\alpha(x) D^\alpha, \qquad B^k_j = B^k_j(x, D) := \sum_{|\beta| \le m_j} b^k_{j\beta}(x) \gamma_0 D^\beta.$$

We remark that the localization procedure contains a subtlety concerning the constants δ and λ_0 in Lemmas 3.3–3.4 and Theorem 3.5. We defined the neighborhoods U_k and the radii r_k in dependence of δ_0 which depends only on $\rho_{A,B}$, s, and σ, see (3.11). For the new coefficients a^k_α, $b^k_{j\beta}$, the $\| \cdot \|_\infty$-norm still satisfies the desired smallness conditions, as seen in (3.15). However, as χ_{U_k} appears in the definition of the new coefficients, the BUC^r-norm and BUC^{k_j}-norm of the new coefficients, respectively, depend on U_k and therefore on the radius r_k. Here, it is important that δ_0 does not depend on the BUC^r-norm (in contrast to λ_0, see Lemma 3.3). Due to this, the above modification of the coefficients might lead to a larger constant λ_0, but we do not have to redefine the radii r_k, which prevents a circular reasoning in the definition of U_k.

Lemma 3.9. *Let $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$, and assume* (S1)–(S4) *to hold. Then there exists a $\lambda_0 \ge 1$ such that the operators*

$$\begin{aligned} \begin{pmatrix} \lambda - A^k \\ B^k \end{pmatrix} &: \mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+) \to \mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+) \quad (k = 0, \dots, K_0), \\ \lambda - A^k &: \mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n) \to \mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n) \quad (k = K_0 + 1, \dots, K) \end{aligned}$$

defined in Remark 3.8 are isomorphisms for every $\lambda \in \Lambda$ with $|\lambda| \ge \lambda_0$. We denote the inverse operators by $L_k(\lambda)$ for $k = 0, \dots, K$.

Proof. We split the operators into a part with constant coefficients and a perturbation, i.e., $A^k = A_0^k + \widetilde{A}^k$ with

$$A_0^k = \sum_{|\alpha|=2m} a_\alpha(x_k) D^\alpha, \qquad \widetilde{A}^k = \sum_{|\alpha|=2m} \big(a_\alpha^k(\cdot) - a_\alpha(x_k)\big) D^\alpha + \sum_{|\alpha|<2m} a_\alpha^k(\cdot) D^\alpha.$$

The B^k can be decomposed in a similar way. Due to the smallness property (3.15), the considered operators thus fit into the setting of Lemma 3.4 and Theorem 3.5, respectively. This yields the assertion. □

In the following, we will fix a smooth partition of unity $\varphi_k \in C^\infty(\mathbb{R}^n)$, $k = 0, \dots, K$, with $\operatorname{supp} \varphi_k \subset U_k$, $0 \le \varphi_k \le 1$, and $\sum_{k=0}^K \varphi_k = 1$ on $\overline{\mathbb{R}^n_+}$. In addition, we fix functions $\psi_k \in C^\infty(\mathbb{R}^n)$ with $0 \le \psi_k \le 1$, $\operatorname{supp} \psi_k \subset U_k$ and $\psi_k = 1$ on $\operatorname{supp} \varphi_k$. We can solve (3.1) locally in U_k, using the extended local operators in the half-space and in the whole space and their inverses $L_k(\lambda)$. However, the solution operators $L_k(\lambda)$ are not local, so we have to multiply the half-space solution by ψ_k. In this way, commutators appear, which are estimated in the following lemma. We write $[\cdot, \cdot]$ for the standard commutator and use the notation ψ_k also for the operator of multiplication by ψ_k. For the boundary operators, the commutator $[B^k, \psi_k]$ is defined as

$$[B^k, \psi_k]u = B^k(\psi_k u) - (\gamma_0 \psi_k) B^k u.$$

Lemma 3.10. *Let $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$, and assume* (S1)–(S4) *to hold. Let $R_0(\lambda)$ be defined on $\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+)$ by*

$$R_0(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=0}^{K_0} \psi_k L_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} + \sum_{k=K_0+1}^{K} \psi_k L_k(\lambda)(\varphi_k f). \tag{3.16}$$

Then

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R_0(\lambda) = 1 + C(\lambda) \tag{3.17}$$

where $C(\lambda) \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathbb{F}_\lambda^{s+1,\sigma}(\mathbb{R}^n_+))$, and there exists a $\lambda_0 \ge 1$ such that $1 + C(\lambda) \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+))$ is invertible for all $\lambda \in \Lambda$ with $|\lambda| \ge \lambda_0$.

Proof. As first step of the proof we show the commutator estimates

$$C_k(\lambda) := \begin{pmatrix} -[A^k, \psi_k] \\ [B^k, \psi_k] \end{pmatrix} L_k(\lambda) \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathbb{F}_\lambda^{s+1,\sigma}(\mathbb{R}^n_+)) \qquad (k = 0, \dots, K_0),$$

$$C_k(\lambda) := -[A^k, \psi_k] L_k(\lambda) \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n), \mathbb{F}_\lambda^{s+1,\sigma}(\mathbb{R}^n)) \qquad (k = K_0 + 1, \dots, K).$$

We shall only consider the case $k = 0, \dots, K_0$, since the proof for $k = K_0 + 1, \dots, K$ is analogous (and simpler).

The operator $[A^k, \psi_k]$ is a differential operator of order not greater than $2m - 1$. Therefore, it is a bounded operator

$$[A^k, \psi_k]\colon \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+) = H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+) \to H^{s-2m+1,\sigma}_{p,\lambda}(\mathbb{R}^n_+).$$

For the boundary operators, we have for $j = 1, \dots, m$

$$\begin{aligned} B_j^k(\psi_k u) &= \sum_{|\beta|\le m_j} b^k_{j\beta}\gamma_0 D^\beta(\psi_k u) \\ &= \sum_{|\beta|\le m_j} b^k_{j\beta}\gamma_0\Big(\psi_k D^\beta u + \sum_{\gamma\le\beta,\gamma\ne\beta} c_{j,k,\beta,\gamma}(\cdot)D^\gamma u\Big) \\ &= (\gamma_0\psi_k)B_j^k u + \sum_{\beta,\gamma} b^k_{j\beta}(\gamma_0 c_{j,k,\beta,\gamma})\gamma_0 D^\gamma u, \end{aligned}$$

where the coefficients $c_{j,k,\beta,\gamma}$ depend on ψ_k. Consequently, the operator $[B_j^k, \psi_k]$ is a boundary operator of order not greater than $m_j - 1$. In the case $m_j = 0$, this operator is zero. Therefore, $[B_j^k, \psi_k]$ is continuous as an operator

$$[B_j^k, \psi_k]\colon \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+) \to B^{s+\sigma-m_j+1-1/p}_{pp,\lambda}(\mathbb{R}^{n-1}).$$

Hence the commutator estimates are true, since $L_k(\lambda) \in L(\mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+))$ by Lemma 3.9.

Now let $v := R_0(\lambda)(f, g)$. We write

$$\begin{pmatrix}\lambda - A\\ B\end{pmatrix} v = \sum_{k=1}^{K_0}\begin{pmatrix}\lambda - A\\ B\end{pmatrix}\psi_k L_k(\lambda)\begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix} + \sum_{k=K_0+1}^{K}\begin{pmatrix}\lambda - A\\ B\end{pmatrix}\psi_k L_k(\lambda)(\varphi_k f)$$

and treat each term separately. For $k = 1, \dots, K_0$, we obtain

$$\begin{aligned} \begin{pmatrix}\lambda - A\\ B\end{pmatrix}\psi_k L_k(\lambda)\begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix} &= \begin{pmatrix}\lambda - A^k\\ B^k\end{pmatrix}\psi_k L_k(\lambda)\begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix} \\ &= \left[\begin{pmatrix}\psi_k(\lambda - A^k)\\ (\gamma_0\psi_k)B^k\end{pmatrix}L_k(\lambda) + \begin{pmatrix}-[A^k,\psi_k]\\ [B^k,\psi_k]\end{pmatrix}L_k(\lambda)\right]\begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix} \\ &= \begin{pmatrix}\psi_k\varphi_k f\\ (\gamma_0\psi_k)(\gamma_0\varphi_k)g\end{pmatrix} + C_k(\lambda)\begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix} \\ &= \begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix} + C_k(\lambda)\begin{pmatrix}\varphi_k f\\ (\gamma_0\varphi_k)g\end{pmatrix}. \end{aligned}$$

For $k = K_0 + 1, \dots, K$, we obtain in the same way

$$\begin{pmatrix}\lambda - A\\ B\end{pmatrix}\psi_k L_k(\lambda)(\varphi_k f) = \begin{pmatrix}\varphi_k f + C_k(\lambda)(\varphi_k f)\\ 0\end{pmatrix}.$$

Summing up over k yields

$$\begin{pmatrix}\lambda - A\\ B\end{pmatrix} v = (1 + C(\lambda))\begin{pmatrix}f\\ g\end{pmatrix}$$

with

$$C(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=0}^{K_0} C_k(\lambda)\begin{pmatrix} \varphi_k f \\ (\gamma_0\varphi_k)g \end{pmatrix} + \sum_{k=K_0+1}^{K} \begin{pmatrix} C_k(\lambda)(\varphi_k f) \\ 0 \end{pmatrix}.$$

Note that for sake of readability we have dropped the extensions and restrictions from our notation, here. More precisely, the upper entry in the last term above would be $r_{\mathbb{R}^n_+} C_k(\lambda) e^0_{\mathbb{R}^n_+} \varphi_k f$.

From the above commutator estimates and the fact that multiplication by φ_k preserves the smoothness, we obtain $C(\lambda) \in L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+), \mathbb{F}^{s+1,\sigma}_\lambda(\mathbb{R}^n_+))$.

Proceeding as in the proof of Lemma 3.3 a) (see (3.9)) for the lower order terms and using the Neumann series as in Lemma 3.4, we obtain that for sufficiently large λ, the operator $1 + C(\lambda) \in L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+))$ is invertible, and the norm of the inverse is not greater than 2. □

The last result provides a solution operator for the boundary value problem (3.1). To show uniqueness, the following observation will be useful.

Lemma 3.11. *Let E, F be Banach spaces, and let $T \in L(E, F)$ be a retraction, i.e., there exists $R \in L(F, E)$ with $TR = \mathrm{id}_F$. Let E_0 be a dense subset of E. If $T|_{E_0} \colon E_0 \to F$ is injective, then T is injective.*

Proof. Let $f \in F$ and $u \in E$ with $Tu = f$. Choose a sequence $(u_n)_{n\in\mathbb{N}} \subset E_0$ with $u_n \to u$ $(n \to \infty)$ in E. As $T|_{E_0}$ is injective, we have $u_n = Rf_n$, where $f_n := Tu_n$. With the continuity of T, we see $f_n = Tu_n \to Tu = f$ in F, and from the continuity of R we get $u_n = Rf_n \to Rf$ in E. As the limit is unique, this yields $u = Rf$, which shows the injectivity of T. □

The following theorem is the key result of this section.

Theorem 3.12. *Let $p \in (1, \infty)$ and $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$. Let $(\lambda - A, B)$ be a boundary value problem in $\mathbb{R}^n_+$ of the form (3.2)–(3.3) which is parameter-elliptic in Λ for all $x \in \overline{\mathbb{R}^n_+} \cup \{\infty\}$, and assume (S1)–(S4) to hold. Then, there exists a $\lambda_0 \geq 1$ such that for every $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$, the operator*

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} : \mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+) \to \mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+) \tag{3.18}$$

is an isomorphism. Its inverse is given by

$$R(\lambda) = R_0(\lambda)(1 + C(\lambda))^{-1} \in L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+), \mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+)),$$

where $R_0(\lambda)$ and $C(\lambda)$ are defined in Lemma 3.10.

Proof. Let λ_0 be as in Lemma 3.10. For $R(\lambda) = R_0(\lambda)(1 + C(\lambda))^{-1}$, we have $R(\lambda) \in L(\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+), \mathbb{E}^{s,\sigma}_\lambda(\mathbb{R}^n_+))$ by Lemma 3.9 and Lemma 3.10. From (3.17) we obtain

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R(\lambda) = (1 + C(\lambda))(1 + C(\lambda))^{-1} = \mathrm{id}_{\mathbb{F}^{s,\sigma}_\lambda(\mathbb{R}^n_+)}.$$

In particular, the operator in (3.18) is surjective.

To show injectivity (i.e., uniqueness of the solution), we remark that $\mathbb{F}_\lambda^{2m,0}(\mathbb{R}^n_+)$ and $\mathbb{E}_\lambda^{2m,0}(\mathbb{R}^n_+)$ are classical spaces, and therefore we obtain unique solvability in these spaces (see, e.g., [2], Theorem 2.1). In particular, the restriction of the operator (3.18) to $\mathscr{S}(\mathbb{R}^n_+)$ is injective. Now we can apply Lemma 3.11 with $T = \binom{\lambda - A}{B}$ and $R = R(\lambda)$ in the spaces $E = \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+)$, $F = \mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+)$, and $E_0 = \mathscr{S}(\mathbb{R}^n_+)$. □

Corollary 3.13. *In the situation of Theorem 3.12, let additionally $\sigma \in (-\infty, 0]$. Then, there exists a $\lambda_0 \geq 1$ such that for every $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$ and*

$$(f, g) \in H_{p,\lambda}^{s-2m}(\mathbb{R}^n_+) \times \prod_{j=1}^m B_{pp,\lambda}^{s+\sigma-m_j-1/p}(\mathbb{R}^{n-1})$$

the boundary value problem (3.1) has a unique solution $u \in H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n_+)$. In particular, we have $u \in H_{p,\lambda}^{s+\sigma}(\mathbb{R}^n_+)$ and

$$\|u\|_{H_{p,\lambda}^{s+\sigma}(\mathbb{R}^n_+)} \leq C\Big(\|f\|_{H_{p,\lambda}^{s-2m}(\mathbb{R}^n_+)} + \sum_{j=1}^m \|g_j\|_{B_{pp,\lambda}^{s+\sigma-m_j-1/p}(\mathbb{R}^{n-1})}\Big)$$

with a constant C independent of λ.

Proof. This follows immediately from Theorem 3.12 and the continuous embeddings $H_{p,\lambda}^{s-2m}(\mathbb{R}^n_+) \subset H_{p,\lambda}^{s-2m,\sigma}(\mathbb{R}^n_+)$ and $H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n_+) \subset H_{p,\lambda}^{s+\sigma}(\mathbb{R}^n_+)$. □

In Theorem 3.12, we considered the half-space case. For an operator A acting in the whole space, the analog results hold, where the proofs are similar but much simpler, due to the absence of boundary operators. We obtain the following result.

Lemma 3.14. *Let $A = A(x, D)$ be an operator of the form (3.2) with coefficients $a_\alpha : \mathbb{R}^n \to \mathbb{C}$, and assume that $\lambda - A$ is parameter-elliptic in Λ. Let $s, \sigma \in \mathbb{R}$, and assume (S1) and (S2) to hold. Then, there exists a $\lambda_0 \geq 1$ such that for every $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$, the operator*

$$\lambda - A : \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n) \to \mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n)$$

is an isomorphism.

4. Boundary value problems in domains

We now consider (3.1) in a bounded or exterior domain. Throughout this section, we assume Ω to be a domain with compact boundary Γ, and $(\lambda - A, B)$ to be a boundary value problem which is parameter-elliptic in some sector $\Lambda \subset \mathbb{C}$. Moreover, we assume (S1)–(S3) and (S5) to hold.

We define $C^\infty(\overline{\Omega})$ as the restriction of all $u \in C_0^\infty(\mathbb{R}^n)$ to Ω. As the definition of the spaces $H_{p,\lambda}^{s,\sigma}$ is non-canonical in domains, we will only consider standard Sobolev spaces on Ω. For the construction of the solution operators, we will use local coordinates where the space $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)$ is available.

We start with some remarks concerning the localization technique: Let $x_0 \in \Gamma$. Since the domain Ω has a $C^{2m+\lceil r'\rceil}$-boundary, there is an open set $\widetilde{U}_{x_0}$ containing x_0, a radius $r_{x_0} > 0$ and a $C^{2m+\lceil r'\rceil}$-diffeomorphism $\vartheta_{x_0}\colon \widetilde{V}_{x_0} \to \widetilde{U}_{x_0}$, where $\widetilde{V}_{x_0} = B(0, 2r_{x_0})$, such that $\vartheta_{x_0}(\widetilde{V}_{x_0} \cap \mathbb{R}^n_+) = \widetilde{U}_{x_0} \cap \Omega$ and $\vartheta_{x_0}(0) = x_0$. We set $V_{x_0} := B(0, r_{x_0})$ and $U_{x_0} := \vartheta_{x_0}(V_{x_0})$. By compactness of Γ, there are $x_1, \ldots, x_{K_0} \in \Gamma$ and open sets $U_{x_1}, \ldots, U_{x_{K_0}}$ as above such that $\Gamma \subset \bigcup_{k=1}^{K_0} U_{x_k}$. For the sake of simplicity, we shall use k instead of x_k as index.

We proceed similarly as in the half-space case. Hence, we define

$$\delta_{\max} := \sup\left\{\delta > 0 \,\middle|\, \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) \le \delta\} \subset \bigcup_{k=1}^{K_0} U_k\right\}.$$

If Ω is bounded, $\Omega \setminus \bigcup_{k=1}^{K_0} U_k$ is compact, and we can choose x_k in Ω and $0 < r_k < \frac{\delta_{\max}}{2}$ such that

$$U_k := B(x_k, r_k) \subset \left\{x \in \Omega : \operatorname{dist}(x, \Gamma) > \frac{\delta_{\max}}{2}\right\} \tag{4.1}$$

for $k = K_0 + 1, \ldots, K$ and $\overline{\Omega} \subset \bigcup_{k=1}^{K} U_k$.

In the case of an exterior domain, this construction has to be slightly modified. We first define $U_{K_0+1} := \mathbb{R}^n \setminus \overline{B(0, r_{K_0+1})}$, where the radius r_{K_0+1} is chosen such that $\mathbb{R}^n \setminus \Omega \subset B(0, \frac{r_{K_0+1}}{2})$. Now $\Omega \setminus \bigcup_{k=1}^{K_0+1} U_k$ is compact, and we choose x_k and r_k with (4.1) for $k = K_0 + 2, \ldots, K$ such that again $\overline{\Omega} \subset \bigcup_{k=1}^{K} U_k$.

For formal reasons, we define $V_k := U_k$ and $\vartheta_k := \mathrm{id}_{V_k}$ for $k = K_0 + 1, \ldots, K$.

Remark 4.1 *(Local operators and extensions).* Let $x_1, \ldots, x_K$ be chosen as above. For $k \in \{1, \ldots, K_0\}$, we define the local operator $\widetilde{A}^k$ as the pullback of the operator A by ϑ_k. More precisely, for $v \in C^\infty(\widetilde{V}_k)$, we write

$$(\widetilde{A}^k v)(y) := A(v \circ \vartheta_k^{-1})(\vartheta_k(y)) =: \sum_{|\alpha| \le 2m} \widetilde{a}^k_\alpha(y) D^\alpha v(y) \quad (y \in \widetilde{V}_k \cap \overline{\mathbb{R}^n_+}).$$

The explicit description of the coefficients $\widetilde{a}^k_\alpha$ (Faà di Bruno-formula, see [15], Formula B) shows that $\widetilde{a}^k_\alpha$ contains the function $a_\alpha \circ \vartheta_k$ as well as derivatives of ϑ_k^{-1} up to order $2m + 1 - |\alpha|$ for $|\alpha| \ge 1$ (and no derivative for $|\alpha| = 0$), concatenated with ϑ_k. Hence we always need at most $2m$ derivatives of ϑ_k^{-1}, which ensures $\widetilde{a}^k_\alpha \in \mathrm{BUC}^{\lceil r'\rceil}$. For $|\alpha| = 2m$ at most one derivative of ϑ_k^{-1} appears and as $m \in \mathbb{N}$ we have $2m + \lceil r'\rceil - 1 \ge \lfloor r'\rfloor + 1$, which shows $\widetilde{a}^k_\alpha \in \mathrm{BUC}^r$ for $|\alpha| = 2m$. Consequently, condition (S5) implies that (S1) also holds for $\widetilde{a}^k_\alpha$. In the same way, we define the local operator $\widetilde{B}^k = (\widetilde{B}^k_1, \ldots, \widetilde{B}^k_m)$ via

$$(\widetilde{B}^k_j v)(y) := B_j(v \circ \vartheta_k^{-1})(\vartheta_k(y)) =: \sum_{|\beta| \le m_j} \widetilde{b}^k_{j\beta}(y) \gamma_0 D^\beta v(y) \quad (y \in \widetilde{V}_k \cap \mathbb{R}^{n-1}).$$

A simple calculation shows that $2m + \lceil r'\rceil \ge m_j + \lfloor k'_j\rfloor + 1 = m_j + k_j$ and thus (S5) also implies that the transformed operators $\widetilde{B}^k_j$ satisfy (S3) for all $|\beta| \le m_j$. For $k \in \{K_0 + 1, \ldots, K\}$, we set

$\widetilde{a}_\alpha^k(y) := a_\alpha(y)$ for $y \in B(x_k, 2r_k)$ with some obvious modifications in the case of an exterior domain for $k = K_0 + 1$.

Again with the general extension function from Remark 3.7, we extend the coefficients $\widetilde{a}_\alpha^k$ and $\widetilde{b}_{j\beta}^k$ to $\mathbb{R}_+^n$, $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. We set

$$\begin{aligned} a_\alpha^k(y) &:= \widetilde{a}_\alpha^k(\chi_{V_k}(y)) \quad (y \in \overline{\mathbb{R}_+^n}) \quad \text{for } k = 1, \dots, K_0, \\ a_\alpha^k(y) &:= \widetilde{a}_\alpha^k(\chi_{V_k}(y)) \quad (y \in \mathbb{R}^n) \quad \text{for } k = K_0 + 1, \dots, K, \\ b_{j\beta}^k(y) &:= \widetilde{b}_{j\beta}^k(\chi_{V_k}(y)) \quad (y \in \mathbb{R}^{n-1}) \text{ for } k = 1, \dots, K_0. \end{aligned}$$

Finally, we define

$$\begin{aligned} A^k v(y) &:= \sum_{|\alpha| \le 2m} a_\alpha^k(y) D^\alpha v(y) \qquad (y \in \overline{\mathbb{R}_+^n}) \quad \text{for } k = 1, \dots, K_0, \\ A^k v(y) &:= \sum_{|\alpha| \le 2m} a_\alpha^k(y) D^\alpha v(y) \qquad (y \in \mathbb{R}^n) \quad \text{for } k = K_0 + 1, \dots, K, \\ B_j^k v(y) &:= \sum_{|\beta| \le m_j} b_{j\beta}^k(y) \gamma_0 D^\beta v(y) \quad (y \in \mathbb{R}^{n-1}) \text{ for } k = 1, \dots, K_0. \end{aligned}$$

The extended local operators A^k and B^k satisfy the above smoothness and ellipticity assumptions, so we can apply the results from Section 3. However, as we do not have the spaces $H_{p,\lambda}^{s,\sigma}$ in domains, we use the standard Sobolev spaces as in Corollary 3.13.

Therefore, we additionally consider the spaces

$$\mathcal{E}_\lambda^{s,\sigma}(\Omega) := H_{p,\lambda}^{s+\sigma}(\Omega), \tag{4.2}$$

$$\mathcal{F}_\lambda^{s,\sigma}(\Omega) := H_{p,\lambda}^{s-2m}(\Omega) \times \prod_{j=1}^m B_{pp,\lambda}^{s+\sigma-m_j-1/p}(\Gamma) \tag{4.3}$$

and the analog spaces with Ω being replaced by $\mathbb{R}_+^n$. We also set $\mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}^n) := H_{p,\lambda}^{s+\sigma}(\mathbb{R}^n)$ and $\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n) := H_{p,\lambda}^{s-2m}(\mathbb{R}^n)$. Note that for $\sigma \le 0$ we have the continuous embeddings (Proposition 2.2 c) and Remark 2.4 b))

$$\mathcal{F}_\lambda^{s,\sigma} \subset \mathbb{F}_\lambda^{s,\sigma} \quad \text{and} \quad \mathbb{E}_\lambda^{s,\sigma} \subset \mathcal{E}_\lambda^{s,\sigma}. \tag{4.4}$$

Lemma 4.2. *Let $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$, and let A^k, B^k denote the extended local operators. Then there exists a $\lambda_0 \ge 1$ such that the operator families*

$$\begin{aligned} \begin{pmatrix} \lambda - A^k \\ B^k \end{pmatrix} &: \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}_+^n) \to \mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}_+^n) \quad (k = 1, \dots, K_0), \\ \lambda - A^k &: \mathbb{E}_\lambda^{s,\sigma}(\mathbb{R}^n) \to \mathbb{F}_\lambda^{s,\sigma}(\mathbb{R}^n) \quad (k = K_0 + 1, \dots, K) \end{aligned} \tag{4.5}$$

for $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$ are isomorphisms. We denote the inverse operator by $L_k(\lambda)$. For $\sigma \leq 0$, the restrictions of $L_k(\lambda)$ to $\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+)$ and $\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n)$, respectively, yield bounded operator families

$$\begin{aligned} &L_k(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+)) \quad (k = 1, \dots, K_0), \\ &L_k(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n), \mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}^n)) \quad (k = K_0 + 1, \dots, K). \end{aligned} \tag{4.6}$$

Proof. We have seen above that A^k, B^k satisfy conditions (S1) and (S3). Conditions (S2) and (S4) follow directly from the fact that the extended coefficients are constant far away from the origin by construction. Hence the statement follows for $k \in \{1, \dots, K_0\}$ from Theorem 3.12 and Corollary 3.13 and for $k \in \{K_0 + 1, \dots, K\}$ from Lemma 3.14 and the embeddings (4.4). □

To solve (3.1) in Ω, we first construct an approximate solution operator $R_0(\lambda)$, using the local solution operators $L_k(\lambda)$ from Lemma 4.2 and the local coordinate maps ϑ_k for $k = 1, \dots, K$. Setting $\Theta_k v := v \circ \vartheta_k^{-1}$, the $C^{2m+\lceil r' \rceil}$-diffeomorphism ϑ_k induces isomorphisms

$$\begin{aligned} &\Theta_k \colon H^s_{p,\lambda}(V_k \cap \mathbb{R}^n_+) \to H^s_{p,\lambda}(U_k \cap \Omega) \quad (k = 1, \dots, K_0), \\ &\Theta_k \colon H^s_{p,\lambda}(V_k) \qquad\quad\; \to H^s_{p,\lambda}(U_k) \qquad\quad (k = K_0 + 1, \dots, K) \end{aligned} \tag{4.7}$$

for $s \in [0, 2m + \lceil r' \rceil]$. Since we have $\Theta_k(\dot{H}^s_{p,\lambda}(V_k \cap \mathbb{R}^n_+)) = \dot{H}^s_{p,\lambda}(U_k \cap \Omega)$, we even get (4.7) for all $|s| \leq 2m + \lceil r' \rceil$ via duality. Moreover, by the definition of the Besov space on the closed $C^{2m+\lceil r' \rceil}$-manifold Γ, the restriction $\vartheta_k|_{V_k \cap \mathbb{R}^{n-1}} \colon V_k \cap \mathbb{R}^{n-1} \to U_k \cap \Gamma$ also induces isomorphisms

$$\Theta_k \colon B^s_{pp,\lambda}(V_k \cap \mathbb{R}^{n-1}) \to B^s_{pp,\lambda}(U_k \cap \Gamma)$$

for $k = 1, \dots, K_0$ and all $|s| \leq 2m + \lceil r' \rceil$. We fix a smooth partition of unity $\varphi_k^\Omega \in C^\infty(\mathbb{R}^n)$, $k = 1, \dots, K$, with $\operatorname{supp} \varphi_k^\Omega \subset U_k$, $0 \leq \varphi_k^\Omega \leq 1$, and $\sum_{k=1}^K \varphi_k^\Omega = 1$ on $\overline{\Omega}$. Additionally, let $\psi_k^\Omega \in C^\infty(\mathbb{R}^n)$ with $0 \leq \psi_k^\Omega \leq 1$, $\operatorname{supp} \psi_k^\Omega \subset U_k$ and $\psi_k^\Omega = 1$ on $\operatorname{supp} \varphi_k^\Omega$. We set $\psi_k := \Theta_k^{-1} \psi_k^\Omega = \psi_k^\Omega \circ \vartheta_k$, where here and in the following, we identify functions with compact support and their trivial extensions for sake of readability. Without this identification, we have, e.g., $\psi_k = e^0_{V_k} \Theta_k^{-1}(r_{U_k} \psi_k^\Omega)$ for $k = K_0 + 1, \dots, K$, where again r_{U_k} stands for the restriction to U_k and $e^0_{V_k}$ for the trivial extension to $\mathbb{R}^n$ by zero.

In the following let $\lambda_0 \geq 1$ be given as in Lemma 4.2. The approximate solution operator $R_0(\lambda)$ is now for $\lambda \in \Lambda$, $|\lambda| \geq \lambda_0$ formally defined as

$$R_0(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=1}^{K_0} L_k^\Omega(\lambda) \begin{pmatrix} \varphi_k^\Omega f \\ (\gamma_0 \varphi_k^\Omega) g \end{pmatrix} + \sum_{k=K_0+1}^{K} L_k^\Omega(\lambda)(\varphi_k^\Omega f). \tag{4.8}$$

Here, $L_k^\Omega(\lambda)$ is defined by

$$\begin{aligned} L_k^\Omega(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} &:= \Theta_k \left(\psi_k L_k(\lambda) \Theta_k^{-1} \begin{pmatrix} f \\ g \end{pmatrix} \right) \quad (k = 1, \dots, K_0), \\ L_k^\Omega(\lambda) f &:= \Theta_k(\psi_k L_k(\lambda) \Theta_k^{-1} f) \qquad\qquad (k = K_0 + 1, \dots, K). \end{aligned} \tag{4.9}$$

Lemma 4.3. *Let $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$ and $\sigma \leq 0$. Then the operator $R_0(\lambda)$ in (4.8) is well-defined on $\mathcal{F}_\lambda^{s,\sigma}(\Omega)$ for $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$ and yields a bounded operator family*

$$R_0(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega), \mathcal{E}_\lambda^{s,\sigma}(\Omega)). \tag{4.10}$$

Proof. The continuity of $R_0(\lambda)$ in the corresponding spaces follows from (4.7) and Lemma 4.2. □

In the following, we will modify $R_0(\lambda)$ to get a solution. For this, we compute $(\lambda - A, B)R_0(\lambda)(f, g)$, where we may choose (f, g) sufficiently smooth such that the classical theory can be applied. Therefore, we introduce s', and assume from now on that $s, \sigma, s' \in \mathbb{R}$ satisfy

$$s > \max_{j=1,\ldots,m} m_j + \tfrac{1}{p}, \quad -1 < \sigma \leq 0, \quad s' \geq \max\{2m, s\}. \tag{4.11}$$

Moreover, we assume (S1)–(S3) and (S5) for (s, σ) (as before) and also for $(s', 0)$. The conditions with respect to $(s', 0)$ collapse to $r' = s' - 2m$ and $k'_j = s' - m_j - \frac{1}{p}$. In the end we take the maximum, respectively.

In contrast to the half-space situation, we have a restriction on σ in (4.11). This is essentially due to the commutator estimates and the fact that we only consider standard Sobolev spaces in Ω.

Lemma 4.4. *Let s, σ and s' satisfy (4.11). Let $0 < \varepsilon < \min\{1 + \sigma, \frac{1}{p}\}$.*

a) For $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_0$, define the operator $C_k(\lambda)$ by

$$C_k(\lambda) := \begin{pmatrix} -[A^k, \psi_k] \\ [B^k, \psi_k] \end{pmatrix} L_k(\lambda) \quad (k = 1, \ldots, K_0),$$
$$C_k(\lambda) := -[A^k, \psi_k] L_k(\lambda) \qquad (k = K_0 + 1, \ldots, K).$$

Then

$$C_k(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}_+^n), \mathcal{F}_\lambda^{s+\varepsilon,\sigma}(\mathbb{R}_+^n)) \qquad (k = 1, \ldots, K_0),$$
$$C_k(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n), \mathcal{F}_\lambda^{s+\varepsilon,\sigma}(\mathbb{R}^n)) \quad (k = K_0 + 1, \ldots, K).$$

b) Let $(f, g) \in \mathcal{F}_\lambda^{s',0}(\Omega)$, and set $v := R_0(\lambda)(f, g)$ with $R_0(\lambda)$ being defined in (4.8). Then

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} v = (1 + C(\lambda)) \begin{pmatrix} f \\ g \end{pmatrix} \tag{4.12}$$

holds with an operator $C(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega), \mathcal{F}_\lambda^{s+\varepsilon,\sigma}(\Omega))$ and there exists a $\lambda_1 \geq \lambda_0$ such that $1 + C(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega))$ is invertible for all $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_1$.

Proof. a) The operator $[A^k, \psi_k]$ is a differential operator of order not greater than $2m - 1$. Therefore, the mapping

$$[A^k, \psi_k]\colon \mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}_+^n) = H_{p,\lambda}^{s+\sigma}(\mathbb{R}_+^n) \to H_{p,\lambda}^{s-2m+1+\sigma}(\mathbb{R}_+^n) \subset H_{p,\lambda}^{s+\varepsilon-2m}(\mathbb{R}_+^n)$$

is bounded.

In analogy to the proof of Lemma 3.10 we obtain that the operator $[B_j^k, \psi_k]$ is a boundary operator of order not greater than $m_j - 1$. In the case $m_j = 0$, this operator is zero. As $\mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+) = H_{p,\lambda}^{s+\sigma}(\mathbb{R}^n_+)$ and $\sigma > -1$, the boundary operator $[B_j^k, \psi_k]$ is defined in the classical sense and is continuous as an operator

$$[B_j^k, \psi_k] \colon \mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+) \to B_{pp,\lambda}^{s+\sigma-m_j+1-1/p}(\mathbb{R}^{n-1}) \subset B_{pp,\lambda}^{s+\varepsilon+\sigma-m_j-1/p}(\mathbb{R}^{n-1}).$$

By Lemma 4.2, we have $L_k(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathcal{E}_\lambda^{s,\sigma}(\mathbb{R}^n_+))$. This and the above mapping properties for the commutators show $C_k(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\mathbb{R}^n_+), \mathcal{F}_\lambda^{s+\varepsilon,\sigma}(\mathbb{R}^n_+))$.

b) We first remark that $v \in \mathcal{E}_\lambda^{s',0}(\Omega)$ holds due to Lemma 4.3, and, as $s' \geq 2m$, the boundary operators B_j can be applied to v in the classical sense. Using calculations similar to the ones in the proof of Lemma 3.10 and the equality $\Theta_k(\psi_k \Theta_k^{-1}(\varphi_k^\Omega f)) = \psi_k^\Omega \varphi_k^\Omega f$, we obtain

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} v = (1 + C(\lambda)) \begin{pmatrix} f \\ g \end{pmatrix}$$

with

$$C(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=1}^{K_0} \Theta_k \left(C_k(\lambda)\Theta_k^{-1} \begin{pmatrix} \varphi_k^\Omega f \\ (\gamma_0 \varphi_k^\Omega) g \end{pmatrix} \right) + \sum_{k=K_0+1}^{K} \begin{pmatrix} \Theta_k(C_k(\lambda)\Theta_k^{-1}(\varphi_k^\Omega f)) \\ 0 \end{pmatrix}.$$

From a) and the fact that multiplication by φ_k^Ω and the coordinate transformations Θ_k, Θ_k^{-1} preserve the smoothness as $\varepsilon < \frac{1}{p}$, we obtain

$$C(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega), \mathcal{F}_\lambda^{s+\varepsilon,\sigma}(\Omega)).$$

Proceeding as in the proof of Lemma 3.3 a), and using the Neumann series as in Lemma 3.4, we obtain that for sufficiently large λ, the operator $1 + C(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega))$ is invertible, and the norm of the inverse is not greater than 2. □

The following theorem is the key result of this section and gives an a priori-estimate for the solution operator of (3.1) in spaces of rough regularity. Note that we first consider smooth functions, where the boundary operators are defined in a classical way and where we know unique solvability by classical results. However, the a priori-estimate gives a continuous extension of the solution operator to larger spaces.

Theorem 4.5. *Let $(\lambda - A, B)$ be parameter-elliptic in the sector Λ, and let $s, \sigma, s' \in \mathbb{R}$ satisfy (4.11). Assume (S1)–(S3), (S5) to hold for (s, σ) and $(s', 0)$. Then taking $\lambda_1 \geq \lambda_0$ as in Lemma 4.4 b), for all $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_1$, and every $(f, g) \in \mathcal{F}_\lambda^{s',0}(\Omega)$, the unique solution $u \in \mathcal{E}_\lambda^{s',0}(\Omega)$ of (3.1) is given by*

$$u = R(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} := R_0(\lambda)(1 + C(\lambda))^{-1}\begin{pmatrix} f \\ g \end{pmatrix},$$

and we have the a priori-estimate

$$\Big\| R(\lambda)\begin{pmatrix} f \\ g \end{pmatrix}\Big\|_{\mathcal{E}_\lambda^{s,\sigma}(\Omega)} \le C \Big\| \begin{pmatrix} f \\ g \end{pmatrix}\Big\|_{\mathcal{F}_\lambda^{s,\sigma}(\Omega)} \tag{4.13}$$

with a constant C not depending on f, g or λ. In particular, the solution operator $R(\lambda)$ extends uniquely to a continuous operator family

$$R(\lambda) \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega), \mathcal{E}_\lambda^{s,\sigma}(\Omega)).$$

Proof. First, we remark that $\mathcal{F}_\lambda^{s',0}(\Omega)$ and $\mathcal{E}_\lambda^{s',0}(\Omega)$ are classical spaces, and therefore we obtain unique solvability in these spaces (see, e.g., [2], Theorem 2.1). By Lemma 4.4 b) with $(s, \sigma) = (s', 0)$, the operator $1 + C(\lambda)$ is invertible in $L(\mathcal{F}_\lambda^{s',0}(\Omega))$, and from Lemma 4.3 we get $R_0(\lambda) \in L(\mathcal{F}_\lambda^{s',0}(\Omega), \mathcal{E}_\lambda^{s',0}(\Omega))$ because of $s' \ge s$. Therefore, $u := R(\lambda)\binom{f}{g} \in \mathcal{E}_\lambda^{s',0}(\Omega)$. As

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} u = (1 + C(\lambda))(1 + C(\lambda))^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

by Lemma 4.4 b), u is the unique solution of (3.1). Finally, the a priori-estimate (4.13) follows from Lemma 4.3 and $(1 + C(\lambda))^{-1} \in L(\mathcal{F}_\lambda^{s,\sigma}(\Omega))$. □

The existence of continuous solution operators given by Theorem 4.5 is the main part of the analysis of (3.1). To formulate the uniqueness of the solution, we have to consider a function space over Ω where the boundary operators are well-defined. For this, we apply the theory of Roitberg ([29], [30]), which leads to the space $H^s_{p,A,s_0}(\Omega)$ as defined below. Note that for the construction of the solution operator, the results by Roitberg were not used. We still assume (4.11) to hold.

Definition 4.6. Let $s_0 \in \mathbb{R}$ with $s_0 \ge s - 2m$. Then we define $H^s_{p,A,s_0}(\Omega)$ as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{H^s_{p,A,s_0}(\Omega)} := \|u\|_{H^s_p(\Omega)} + \|Au\|_{H^{s_0}_p(\Omega)}.$$

Remark 4.7. By the continuity of $A\colon H_p^{s_0+2m}(\Omega) \to H_p^{s_0}(\Omega)$ and the condition on s_0, we find that

$$\|u\|_{H^s_p(\Omega)} \le \|u\|_{H^s_{p,A,s_0}(\Omega)} \le C\|u\|_{H_p^{s_0+2m}(\Omega)}$$

for every $u \in C^\infty(\overline{\Omega})$. It follows that

$$H_p^{s_0+2m}(\Omega) \subset H^s_{p,A,s_0}(\Omega) \subset H^s_p(\Omega)$$

with dense embeddings.

Lemma 4.8. *Let* $s_0 \in \mathbb{R}$ *with* $s_0 > -1 + \frac{1}{p}$ *and* $s_0 \geq s - 2m$. *Then*

$$\begin{pmatrix} A \\ B \end{pmatrix} \colon H^{s+\sigma}_{p,A,s_0}(\Omega) \to H^{s_0}_p(\Omega) \times \prod_{j=1}^{m} B^{s+\sigma-m_j-1/p}_{pp}(\Gamma) \tag{4.14}$$

is well-defined and continuous.

Proof. For smooth u, we write $B_j u$ in the form

$$B_j u = \gamma_0 \sum_{|\beta| \leq m_j} b_{j\beta}(\cdot) D^\beta u = \sum_{l=0}^{m_j} M_{jl}(x, D')\gamma_l u, \tag{4.15}$$

where $\gamma_l \colon u \mapsto (\partial^l_\nu u)|_\Gamma$ is the classical trace and $M_{jl}(x, D')$ is a differential operator of order not greater than $m_j - l$ which contains only derivatives in tangential direction. We first show that for all $u \in C^\infty(\overline{\Omega})$ and $l = 0, \ldots, \max_j m_j$ we have

$$\|\gamma_l u\|_{B^{s+\sigma-l-1/p}_{pp}(\Gamma)} \leq C \|u\|_{H^{s+\sigma}_{p,A,s_0}(\Omega)}. \tag{4.16}$$

Indeed, if $s + \sigma > \max_j m_j + \frac{1}{p}$, this follows from classical trace results, where we can even replace the norm on the right-hand side of (4.16) by $\|u\|_{H^{s+\sigma}_p(\Omega)}$. If $s + \sigma \leq \max_j m_j + \frac{1}{p}$, we first note that we have $s + \sigma > -1 + \frac{1}{p}$ by (4.11). Therefore, we can apply [29], Theorem 6.1.1 and (6.1.29) (see also the text after [29], Definition 6.2.1) and obtain

$$\|\gamma_l u\|_{B^{s+\sigma-l-1/p}_{pp}(\Gamma)} \leq C\big(\|u\|_{H^{s+\sigma}_p(\Omega)} + \|Au\|_{\dot{H}^{s+\sigma-2m}_p(\overline{\Omega})}\big). \tag{4.17}$$

Now choose $0 < \varepsilon < 1$ such that

$$s + \sigma - 2m \leq \max_j m_j + \frac{1}{p} - 2m < -1 + \frac{1}{p} + \varepsilon < s_0.$$

Then

$$H^{s_0}_p(\Omega) \subset H_p^{-1+\frac{1}{p}+\varepsilon}(\Omega) = \dot{H}_p^{-1+\frac{1}{p}+\varepsilon}(\overline{\Omega}) \subset \dot{H}^{s+\sigma-2m}_p(\overline{\Omega}),$$

where the equality can be found, e.g., in [34], Theorem 4.3.2/1. Hence we may replace $\|Au\|_{\dot{H}^{s+\sigma-2m}_p(\overline{\Omega})}$ in (4.17) by $\|Au\|_{H^{s_0}_p(\Omega)}$ which yields (4.16).

From the continuity of $M_{jl}(x, D') \colon B^{s+\sigma-l-1/p}_{pp}(\Gamma) \to B^{s+\sigma-m_j-1/p}_{pp}(\Gamma)$, the estimate in (4.16), and the definition of the norm in $H^{s+\sigma}_{p,A,s_0}(\Omega)$ we obtain

$$\|Au\|_{H^{s_0}_p(\Omega)} + \sum_{j=1}^{m} \|B_j u\|_{B^{s+\sigma-m_j-1/p}_{pp}(\Gamma)} \leq C \|u\|_{H^{s+\sigma}_{p,A,s_0}(\Omega)}$$

for all $u \in C^\infty(\overline{\Omega})$. As $C^\infty(\overline{\Omega})$ is dense in $H^{s+\sigma}_{p,A,s_0}(\Omega)$, the operator (4.14) is well-defined by unique extension and continuous. □

Now we are able to formulate our main result. At first we will cover the general situation and then, as a corollary, the simpler setting $f \in L^p(\Omega)$.

Theorem 4.9. *Let $(\lambda - A, B)$ be parameter-elliptic in the sector Λ, and let $s, \sigma \in \mathbb{R}$ with $s > \max_j m_j + \frac{1}{p}$ and $\sigma \in (-1, 0]$. Let $-1 + \frac{1}{p} < s_0 \le s + \sigma$, and $s_0 \ge s - 2m$. We fix $s' := \max\{2m, s_0 + 2m\}$. Assume conditions (S1)–(S3), (S5) to hold with respect to (s, σ) as well as $(s', 0)$. Then there exists a $\lambda_1 \ge 1$ such that for all $\lambda \in \Lambda$ with $|\lambda| \ge \lambda_1$ and all*

$$(f, g) \in H^{s_0}_p(\Omega) \times \prod_{j=1}^{m} B^{s+\sigma-m_j-1/p}_{pp}(\Gamma), \tag{4.18}$$

the boundary value problem (3.1) has a unique solution $u \in H^{s+\sigma}_{p,A,s_0}(\Omega)$. This solution is given by $u = R(\lambda)\binom{f}{g}$ and satisfies the a priori-estimate (4.13).

Proof. By the assumptions on s and σ, (4.11) holds, and for sufficiently large $|\lambda|$, we can define $u := R(\lambda)\binom{f}{g} \in \mathcal{E}^{s,\sigma}_\lambda(\Omega)$. By Theorem 4.5, u satisfies the a priori-estimate (4.13). For the rest of the proof let λ be arbitrary but fixed.

We want to show that $u \in H^{s+\sigma}_{p,A,s_0}(\Omega)$. For this we denote the space in (4.18) by $\mathcal{F}^{s,\sigma,s_0}_\lambda(\Omega)$ and first note that $\mathcal{F}^{s',0}_\lambda(\Omega)$ is dense in the space $\mathcal{F}^{s,\sigma,s_0}_\lambda(\Omega)$, as even smooth functions are dense. Therefore, we can choose a sequence $\binom{f_k}{g_k} \in \mathcal{F}^{s',0}_\lambda(\Omega)$ with $\|\binom{f_k}{g_k} - \binom{f}{g}\|_{\mathcal{F}^{s,\sigma,s_0}_\lambda(\Omega)} \to 0$ for $k \to \infty$ and set $u_k := R(\lambda)\binom{f_k}{g_k}$. By Theorem 4.5 and Remark 4.7, we know $u_k \in \mathcal{E}^{s',0}_\lambda(\Omega) = H^{s'}_p(\Omega) \subset H^{s+\sigma}_{p,A,s_0}(\Omega)$. Moreover, Bu_k is defined in the classical sense, and we have $\binom{\lambda-A}{B}u_k = \binom{f_k}{g_k}$ by Theorem 4.5. In particular, $Au_k = \lambda u_k - f_k$. This and (4.13) yield

$$\begin{aligned}
\|u_k - u_\ell\|_{H^{s+\sigma}_{p,A,s_0}(\Omega)} &= \|u_k - u_\ell\|_{H^{s+\sigma}_p(\Omega)} + \|Au_k - Au_\ell\|_{H^{s_0}_p(\Omega)} \\
&\le C_\lambda \left\| \begin{pmatrix} f_k \\ g_k \end{pmatrix} - \begin{pmatrix} f_\ell \\ g_\ell \end{pmatrix} \right\|_{\mathcal{F}^{s,\sigma}_\lambda(\Omega)} + \|f_k - f_\ell\|_{H^{s_0}_p(\Omega)} + |\lambda| \|u_k - u_\ell\|_{H^{s_0}_p(\Omega)} \\
&\le C_\lambda \left\| \begin{pmatrix} f_k \\ g_k \end{pmatrix} - \begin{pmatrix} f_\ell \\ g_\ell \end{pmatrix} \right\|_{\mathcal{F}^{s,\sigma,s_0}_\lambda(\Omega)} + |\lambda| \, \|u_k - u_\ell\|_{H^{s+\sigma}_p(\Omega)} \\
&\le C_\lambda \left\| \begin{pmatrix} f_k \\ g_k \end{pmatrix} - \begin{pmatrix} f_\ell \\ g_\ell \end{pmatrix} \right\|_{\mathcal{F}^{s,\sigma,s_0}_\lambda(\Omega)} \to 0 \quad (k, \ell \to \infty).
\end{aligned}$$

Recall that here we have used parameter-independent norms and the a priori-estimate (4.13) for fixed λ. We also use the condition $s_0 \le s + \sigma$. We have seen that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^{s+\sigma}_{p,A,s_0}(\Omega)$ and therefore convergent to some element $v \in H^{s+\sigma}_{p,A,s_0}(\Omega)$. By (4.13), we see $u_k \to u$ in $H^{s+\sigma}_p(\Omega)$, and therefore $u = v \in H^{s+\sigma}_{p,A,s_0}(\Omega)$.

As $u \in H^{s+\sigma}_{p,A,s_0}(\Omega)$ and $s_0 > -1 + \frac{1}{p}$, the expression Bu is well-defined in the sense of Lemma 4.8, which also yields the continuity of the operator in the respective spaces. Hence, the above approximation shows that $\binom{\lambda-A}{B}u = \binom{f}{g}$, therefore u is a solution of (3.1). For the uniqueness we make use of Lemma 3.11 once more, where we take $E = H^{s+\sigma}_{p,A,s_0}(\Omega)$, $F = \mathcal{F}^{s,\sigma,s_0}_{\lambda}(\Omega)$, $E_0 = \mathcal{E}^{s',0}_{\lambda}(\Omega)$, and $T = \binom{\lambda-A}{B}$. □

The conditions on the parameters s, σ and on the smoothness are much simpler in the case $f \in L^p(\Omega)$. We obtain the following corollary, which shows that boundary spaces of order close to -1 may appear.

Corollary 4.10. *Let $(\lambda - A, B)$ be parameter-elliptic in the sector Λ, and let $\tau \in \mathbb{R}$ with $\max_j m_j + \frac{1}{p} - 1 < \tau \leq 2m$ and $\tau \geq 0$ (the last condition is automatically satisfied except for $2m = 2$ and $m_1 = 0$). Assume (S1) and (S3) to hold for $r' := 2m - \tau$ and $k'_j := 2m - m_j - \frac{1}{p}$. Let Ω be of class $C^{2m+\lceil r' \rceil}$, and assume (S2) if Ω is unbounded.*

Then there exists a $\lambda_1 \geq 1$ such that for all $\lambda \in \Lambda$ with $|\lambda| \geq \lambda_1$ and all

$$(f, g) \in L^p(\Omega) \times \prod_{j=1}^{m} B^{\tau-m_j-1/p}_{pp}(\Gamma),$$

the boundary value problem (3.1) has a unique solution $u \in H^{\tau}_{p,A,0}(\Omega)$, which satisfies the a priori-estimate (4.13).

Proof. We apply Theorem 4.9 with $s_0 = 0$. If $\tau > \max_j m_j + \frac{1}{p}$, we choose $s := \tau$ and $\sigma := 0$. In the case $\tau \leq \max_j m_j + \frac{1}{p}$, we set $s := \max_j m_j + \frac{1}{p} + \varepsilon$ and $\sigma := \tau - s$ for $\varepsilon > 0$ sufficiently small. Note that for this choice of (s, σ) the conditions in Theorem 4.9 are fulfilled. □

5. Boundary value problems with dynamic boundary conditions

As an application of the above results, we consider a boundary value problem with dynamic boundary conditions, which is related to the linearized Cahn–Hilliard equation and was discussed in detail in [28]. We show that the corresponding operator generates a holomorphic semigroup in L^p. For simplicity, we restrict ourselves to the model problem situation and do not consider a general domain. The related model problem in the half-space has the form (see [28], Equation (2.1))

$$\begin{aligned} (\partial_t + \Delta^2)u &= f \text{ in } (0,\infty) \times \mathbb{R}^n_+, \\ \partial_t u + \partial_\nu u - \Delta' u &= g \text{ on } (0,\infty) \times \mathbb{R}^{n-1}, \\ \partial_\nu \Delta u &= 0 \text{ on } (0,\infty) \times \mathbb{R}^{n-1} \end{aligned}$$

(plus initial condition), where Δ' stands for the tangential Laplacian. Here we have set the constants to 1 and omitted lower-order terms. Following a standard approach for boundary value problems with dynamical boundary conditions, we decouple $u =: u_1$ and $\gamma_0 u =: u_2$ and consider the Cauchy problem in a product space, where now the compatibility condition $u_1 = \gamma_0 u_2$ has to be added. The corresponding resolvent problem is given by

$$
\begin{aligned}
(\lambda + \Delta^2)u_1 &= f \text{ in } \mathbb{R}^n_+, \\
\partial_\nu u_1 + (\lambda - \Delta')u_2 &= g \ \text{ on } \mathbb{R}^{n-1}, \\
\partial_\nu \Delta u_1 &= 0 \ \text{ on } \mathbb{R}^{n-1}, \\
\gamma_0 u_1 - u_2 &= 0 \ \text{ on } \mathbb{R}^{n-1}.
\end{aligned} \tag{5.1}
$$

It was shown in [28], Remark 2.2, that the related operator generates an analytic C_0-semigroup in the basic space $L^p(\mathbb{R}^n_+) \times B^r_{pp}(\mathbb{R}^{n-1})$ for all $r \in [2 - \frac{1}{p}, 3 - \frac{1}{p}]$. In the present paper, we consider the basic space $X := L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$. In order to define a suitable operator A representing (5.1), we have to verify the parameter-ellipticity of the auxiliary problem below.

In the following, let $\arg(\cdot)$ denote the argument of a complex number with values in $(-\pi, \pi]$. Furthermore let $\sqrt{\cdot}$ denote the principal branch of the complex square root which is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and for which we have $\operatorname{Re}\sqrt{z} > 0$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

Lemma 5.1. *Let $\theta \in (0, \pi)$. Then the boundary value problem $(\lambda + \Delta^2, \gamma_0, \partial_\nu \Delta)$ in $\mathbb{R}^n_+$ is parameter-elliptic in the sector $\Lambda := \overline{\Sigma_\theta}$, where*

$$\Sigma_\theta := \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta\}.$$

Proof. Obviously, the operator $\lambda + \Delta^2$ is parameter-elliptic in Λ. To see that the Shapiro–Lopatinskii condition holds, we first assume $\lambda \in \Lambda \setminus \{0\}$. Then every stable solution of the ODE

$$\big[\lambda + (\partial_n^2 - |\xi'|^2)^2\big] w(x_n) = 0 \quad (x_n > 0) \tag{5.2}$$

is of the form $w(x_n) = c_1 e^{-\tau_1 x_n} + c_2 e^{-\tau_2 x_n}$ with the roots $\tau_{1,2} = \tau_{1,2}(|\xi'|, \lambda)$, where

$$\tau_{1,2}(|\xi'|, \lambda) := \sqrt{|\xi'|^2 \pm i\sqrt{\lambda}}. \tag{5.3}$$

Note that we have $\operatorname{Re}\tau_j(|\xi'|, \lambda) > 0$ for $j = 1, 2$ and all $\xi' \in \mathbb{R}^{n-1}$.

The first boundary condition $w(0) = 0$ yields $c_1 = -c_2$, and from the second boundary condition we obtain

$$
\begin{aligned}
0 &= -\partial_n(\partial_n^2 - |\xi'|^2)w(0) = \tau_1(\tau_1^2 - |\xi'|^2)c_1 + \tau_2(\tau_2^2 - |\xi'|^2)c_2 \\
&= i\sqrt{\lambda}(\tau_1 c_1 - \tau_2 c_2).
\end{aligned} \tag{5.4}
$$

Therefore, $0 = \tau_1 c_1 - \tau_2 c_2 = (\tau_1 + \tau_2)c_1$. As $\operatorname{Re}(\tau_1 + \tau_2) > 0$, we obtain $c_1 = c_2 = 0$.

In the case $\lambda = 0$, every stable solution of (5.2) is of the form $w(x_n) = (c_1 + c_2 x_n)e^{-|\xi'|x_n}$. From $w(0) = 0$ we obtain $c_1 = 0$, and

$$0 = -\partial_n(\partial_n^2 - |\xi'|^2)w(0) = -2c_2|\xi'|^2$$

shows $c_1 = c_2 = 0$ for every $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. □

Definition 5.2. Now we are able to define the operator $A\colon X \supset D(A) \to X$ with $X := L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$ by

$$A(D)u := \begin{pmatrix} -\Delta^2 & 0 \\ -\partial_\nu & \Delta' \end{pmatrix} \quad (u \in D(A)),$$

where

$$D(A) := \{u = (u_1, u_2) \in X : \ A(D)u \in X, \ \partial_\nu \Delta u_1 = 0, \ \gamma_0 u_1 - u_2 = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

Remark 5.3. a) Due to the parameter-ellipticity of the boundary value problem in Lemma 5.1 we may use Theorem 3.2 to solve the system

$$\begin{aligned} (\lambda + \Delta^2)u_1 &= f \ \text{ in } \mathbb{R}^n_+, \\ \gamma_0 u_1 &= u_2 \text{ on } \mathbb{R}^{n-1}, \\ \partial_\nu \Delta u_1 &= 0 \ \text{ on } \mathbb{R}^{n-1}. \end{aligned} \tag{5.5}$$

Now, the existence of all traces is clear, as we are in the half-space situation and have the spaces $H_p^{s,\sigma}(\mathbb{R}^n_+)$ at our disposal. Using the embeddings $L^p(\mathbb{R}^n_+) \subset H_p^{0,-4}(\mathbb{R}^n_+)$ and $L^p(\mathbb{R}^{n-1}) \subset B_{pp}^{-1/p}(\mathbb{R}^{n-1})$ we obtain $u_1 \in H_p^{4,-4}(\mathbb{R}^n_+)$, which shows that A is well-defined, as $\partial_\nu u_1 \in B_{pp}^{-1-1/p}(\mathbb{R}^{n-1})$ and $\partial_\nu \Delta u_1 \in B_{pp}^{-3-1/p}(\mathbb{R}^{n-1})$.

b) Next, we observe that the operator A is densely defined. For this, let $(f, g) \in X = L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$, and let $\varepsilon > 0$. We first choose $\varphi_2 \in C_0^\infty(\mathbb{R}^{n-1})$ with $\|\varphi_2 - g\|_{L^p(\mathbb{R}^{n-1})} < \varepsilon$ and then define $\varphi_1 \in H_p^4(\mathbb{R}^n_+)$ as the unique solution of

$$\begin{aligned} (1 + \Delta^2)\varphi_1 &= 0 \ \text{ in } \mathbb{R}^n_+, \\ \gamma_0 \varphi_1 &= \varphi_2 \text{ on } \mathbb{R}^{n-1}, \\ \partial_\nu \Delta \varphi_1 &= 0 \ \text{ on } \mathbb{R}^{n-1}. \end{aligned}$$

By definition, we obtain $(\varphi_1, \varphi_2) \in D(A)$. In a second step, we choose $\varphi_1' \in C_0^\infty(\mathbb{R}^n_+)$ with $\|\varphi_1' + \varphi_1 - f\|_{L^p(\mathbb{R}^n_+)} < \varepsilon$. Then $(\varphi_1', 0) \in D(A)$, which implies that $u := (\varphi_1 + \varphi_1', \varphi_2) \in D(A)$. By construction, we know $\|u - (f, g)\|_X < 2\varepsilon$.

c) Finally, the operator A is closed. To see this, let $(u^k)_{k\in\mathbb{N}} \subset D(A)$ be a sequence with $u^k = (u_1^k, u_2^k) \to u = (u_1, u_2)$ in X and $Au^k \to v = (v_1, v_2)$ in X. Then we have $\Delta^2 u_1^k \to \Delta^2 u_1$ in $H_p^{-4}(\mathbb{R}^n_+)$ due to the continuity of the operator $\Delta^2 \colon L^p(\mathbb{R}^n_+) \to H_p^{-4}(\mathbb{R}^n_+)$ as well as $-\Delta^2 u_1^k \to v_1$ in $L^p(\mathbb{R}^n_+)$ and therefore also in $H_p^{-4}(\mathbb{R}^n_+)$. By uniqueness of the limit, we see that $-\Delta^2 u_1 = v_1 \in L^p(\mathbb{R}^n_+)$. Similarly, using the spaces from a), one shows $-\partial_\nu u_1 + \Delta' u_2 = v_2 \in L^p(\mathbb{R}^{n-1})$ and $\partial_\nu \Delta u_1 = \gamma_0 u_1 - u_2 = 0$. Therefore, $u \in D(A)$ and $Au = v$.

Now we want to show that the operator A generates a holomorphic semigroup in $L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$. The key step in the proof consists in the analysis of the solution operator of (5.1) with $f = 0$ and $\lambda \in \Sigma_\theta$. For this, we take the partial Fourier transform $(\mathscr{F}' u_1)(\xi', x_n) =: w(\xi', x_n) =: w(x_n)$ and obtain the ODE (5.2) as well as (5.4) from the boundary condition $\partial_\nu \Delta u_1 = 0$. From the proof of Lemma 5.1 we know that $w(x_n) = c_1 e^{-\tau_1 x_n} + c_2 e^{-\tau_2 x_n}$ and $\tau_1 c_1 = \tau_2 c_2$, where $\tau_{1,2} = \tau_{1,2}(|\xi'|, \lambda)$ are defined in (5.3). Inserting this into the second line of (5.1), we get

$$\tau_1 c_1 + \tau_2 c_2 + (\lambda + |\xi'|^2)(c_1 + c_2) = \hat{g} := \mathscr{F}' g.$$

With $c_2 = \frac{\tau_1}{\tau_2} c_1$, this yields

$$c_1 = \frac{\tau_2}{(\tau_1+\tau_2)(\lambda+|\xi'|^2)+2\tau_1\tau_2}\,\hat{g}.$$

For $\hat{u}_2(\xi') = (\mathscr{F}' u_1)(\xi', 0)$, we obtain

$$\hat{u}_2 = c_1 + c_2 = \frac{\tau_1+\tau_2}{(\tau_1+\tau_2)(\lambda+|\xi'|^2)+2\tau_1\tau_2}\hat{g} =: S(|\xi'|,\lambda)\hat{g}. \tag{5.6}$$

Therefore, we have to analyze the symbol $S(|\xi'|,\lambda)$. As we will use the bounded H^∞-calculus, we will extend this symbol with respect to the first variable to a small sector Σ_ε. We start with a technical result on the zeros τ_1 and τ_2.

Lemma 5.4. *Let $\theta \in (\frac{\pi}{2}, \pi)$ and $\varepsilon \in (0, \frac{\pi-\theta}{4})$.*

a) Let $\tau_{1,2}\colon \Sigma_\varepsilon \times \Sigma_\theta \to \mathbb{C}$ be defined by

$$\tau_{1,2}(z,\lambda) := \sqrt{z^2 \pm i\sqrt{\lambda}} \quad ((z,\lambda) \in \Sigma_\varepsilon \times \Sigma_\theta).$$

Then $\tau_{1,2}$ is holomorphic in $\Sigma_\varepsilon \times \Sigma_\theta$ and satisfies

$$C(|z|+|\lambda|^{1/4}) \le |\tau_j(z,\lambda)| \le C'(|z|+|\lambda|^{1/4}) \quad (j=1,2), \tag{5.7}$$

$$C(|z|+|\lambda|^{1/4}) \le |\tau_1(z,\lambda)+\tau_2(z,\lambda)| \le C'(|z|+|\lambda|^{1/4}) \tag{5.8}$$

for suitable constants $C, C' > 0$ and all $(z,\lambda) \in \Sigma_\varepsilon \times \Sigma_\theta$.

b) For all $(z,\lambda) \in \Sigma_\varepsilon \times \Sigma_\theta$ we have

$$\arg\Big(\frac{\tau_1(z,\lambda)\tau_2(z,\lambda)}{(\tau_1(z,\lambda)+\tau_2(z,\lambda))}\Big) \in \begin{cases} (-\varepsilon, \frac{\theta+\pi}{4}) & \text{if } \arg\lambda \in (\frac{\pi}{2}, \theta), \\ (-\frac{3\pi}{8}, \frac{3\pi}{8}) & \text{if } \arg\lambda \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ (-\frac{\theta+\pi}{4}, \varepsilon) & \text{if } \arg\lambda \in (-\theta, -\frac{\pi}{2}). \end{cases}$$

Proof. a) As $\pm i\sqrt{\lambda} \in \Sigma_{(\theta+\pi)/2}$, the condition on ε implies $z^2 \pm i\sqrt{\lambda} \in \Sigma_{(\theta+\pi)/2}$. This shows that τ_j is well-defined and holomorphic in $\Sigma_\varepsilon \times \Sigma_\theta$ with values in $\Sigma_{(\theta+\pi)/4}$. The function $\varphi(z,\lambda) := |\tau_j(z,\lambda)|(|z|+|\lambda|^{1/4})^{-1}$ is smooth and quasi-homogeneous of degree 0 in the sense that

$$\varphi(\rho z, \rho^4\lambda) = \varphi(z,\lambda) \quad (\rho > 0,\ z \in \Sigma_\varepsilon,\ \lambda \in \Sigma_\theta).$$

Therefore, its minimum and maximum are attained on the compact set

$$M := \big\{(z,\lambda) \in \overline{\Sigma_\varepsilon} \times \overline{\Sigma_\theta} : |z| + |\lambda|^{1/4} = 1\big\}$$

(here we note that τ_j can be extended continuously to M). As $\tau_j \neq 0$ for all $(z,\lambda) \in M$, we obtain $0 < C \le \varphi(z,\lambda) \le C' < \infty$, which yields (5.7).

Because of $\tau_j \in \Sigma_{(\theta+\pi)/4}$, there exists a constant $C_\theta > 0$ with $\operatorname{Re}\tau_j \geq C_\theta|\tau_j|$. Consequently,

$$|\tau_1+\tau_2| \geq \operatorname{Re}(\tau_1+\tau_2) \geq \operatorname{Re}\tau_1 \geq C_\theta|\tau_1| \geq CC_\theta(|z|+|\lambda|^{1/4}).$$

As the other inequality in (5.8) is obvious, we obtain a).

b) For $\operatorname{Re}\lambda \geq 0$ we have $\tau_1, \tau_2 \in \Sigma_{3\pi/8}$. Consequently, the same holds for $\frac{\tau_1\tau_2}{\tau_1+\tau_2} = \frac{1}{\tau_1^{-1}+\tau_2^{-1}}$. Now, let $\arg\lambda \in (\frac{\pi}{2}, \theta)$. Analogously, we get $\frac{\tau_1\tau_2}{\tau_1+\tau_2} \in \Sigma_{(\theta+\pi)/4}$. To see $\arg\left(\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right) > -\varepsilon$, it is sufficient to prove $\operatorname{Im}\left(\frac{|\tau_1+\tau_2|^2\tau_1\tau_2}{z|z|^2(\tau_1+\tau_2)}\right) \geq 0$. We set $c := \sqrt{\lambda/z^4}$ and obtain

$$\frac{|\tau_1+\tau_2|^2\tau_1\tau_2}{z|z|^2(\tau_1+\tau_2)} = \frac{|\tau_1|^2\tau_2+|\tau_2|^2\tau_1}{z|z|^2} = |1+ic|\sqrt{1-ic}+|1-ic|\sqrt{1+ic}.$$

By the condition on λ and z, we know that $c = a+ib$ for some $a, b > 0$. In a first step, we show

$$\operatorname{Im}(\sqrt{1+ic}+\sqrt{1-ic}) \geq 0. \tag{5.9}$$

Using the formula $\operatorname{Im}\sqrt{x \pm iy} = \pm\sqrt{\frac{|x\pm iy|-x}{2}}$ for all $x \in \mathbb{R}$ and $y > 0$, we get

$$\operatorname{Im}(\sqrt{1+ic}+\sqrt{1-ic}) = \sqrt{\frac{|1-b+ia|-(1-b)}{2}} - \sqrt{\frac{|1+b-ia|-(1+b)}{2}},$$

such that (5.9) is equivalent to

$$\sqrt{(1+b)^2+a^2} - \sqrt{(1-b)^2+a^2} \leq 2b,$$

which holds by the reverse triangle inequality in $\mathbb{R}^2$ applied to the points $(1+b, a)$ and $(1-b, a)$. With the inequalities $|1-ic| \geq |1+ic|$ and $\operatorname{Im}\sqrt{1+ic} \geq 0$ as well as (5.9), we finally get

$$\operatorname{Im}(|1+ic|\sqrt{1-ic}+|1-ic|\sqrt{1+ic}) \geq |1+ic|\operatorname{Im}(\sqrt{1-ic}+\sqrt{1+ic}) \geq 0.$$

Consequently, the statement in b) holds for $\arg\lambda \in (\frac{\pi}{2}, \theta)$. The statement for $\arg\lambda \in (-\theta, -\frac{\pi}{2})$ follows from $\overline{\tau_{1,2}(z,\lambda)} = \tau_{2,1}(\overline{z}, \overline{\lambda})$. □

Lemma 5.5. *Let $\theta \in (\frac{\pi}{2}, \pi)$ and $\varepsilon \in (0, \frac{\pi-\theta}{4})$. For $\lambda \in \Sigma_\theta$ and $z \in \Sigma_\varepsilon$, define*

$$m(z,\lambda) := (\lambda+z^2)S(z,\lambda) = \frac{(\lambda+z^2)(\tau_1+\tau_2)}{(\lambda+z^2)(\tau_1+\tau_2)+2\tau_1\tau_2}. \tag{5.10}$$

Then $m\colon \Sigma_\varepsilon \times \Sigma_\theta \to \mathbb{C}$ is holomorphic and bounded.

Proof. For the boundedness we notice that

$$m(z,\lambda) = \frac{1}{1+\frac{2\tau_1\tau_2}{(\lambda+z^2)(\tau_1+\tau_2)}}$$

and show $\frac{\tau_1\tau_2}{(\lambda+z^2)(\tau_1+\tau_2)} \in \Sigma_\varphi$ for some $\varphi \in (0,\pi)$. In the case $\operatorname{Re}\lambda \geq 0$ we have $\frac{\tau_1\tau_2}{\tau_1+\tau_2} \in \Sigma_{3\pi/8}$ due to Lemma 5.4 b), which yields

$$\frac{\tau_1\tau_2}{(\lambda+z^2)(\tau_1+\tau_2)} \in \Sigma_{7\pi/8}.$$

If $\arg\lambda \in (\frac{\pi}{2},\theta)$ we use again Lemma 5.4 b) and obtain $\arg\left(\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right) \in (-\varepsilon, \frac{\theta+\pi}{4})$. By the condition on λ and z, we get $\arg(\lambda+z^2)^{-1} \in (-\theta, 2\varepsilon)$ and therefore

$$\frac{\tau_1\tau_2}{(\lambda+z^2)(\tau_1+\tau_2)} \in \Sigma_{\theta+\varepsilon}.$$

For $\arg\lambda \in (-\theta, -\frac{\pi}{2})$ we argue in the same way to see $\frac{\tau_1\tau_2}{(\lambda+z^2)(\tau_1+\tau_2)} \in \Sigma_{\theta+\varepsilon}$. Obviously, m is holomorphic in $\Sigma_\varepsilon \times \Sigma_\theta$. □

The last two lemmas allow us to prove the main result of this section. We recall that the operator A is described in Definition 5.2.

Theorem 5.6. *For every $\lambda_0 > 0$, the operator $A - \lambda_0$ generates a bounded holomorphic C_0-semigroup of angle $\frac{\pi}{2}$ in $X := L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$. In particular, A generates a holomorphic C_0-semigroup of angle $\frac{\pi}{2}$ in X. Furthermore we obtain H^2-regularity of the solution. More precisely for any $\varepsilon > 0$ we have*

$$D(A) \subset H_p^{2+1/p-\varepsilon}(\mathbb{R}^n_+) \times H_p^2(\mathbb{R}^{n-1}).$$

We may choose $\varepsilon = 0$ if $p \geq 2$.

Proof. Let $\theta \in (\frac{\pi}{2}, \pi)$, and let $\lambda_0 > 0$. Then there is some $\lambda_0' > 0$ with

$$\lambda_0 + \Sigma_\theta \subset \{\lambda \in \Sigma_\theta : |\lambda| \geq \lambda_0'\}. \tag{5.11}$$

We show that for every $\lambda \in \Sigma_\theta$ with $|\lambda| \geq \lambda_0'$ and every $(f,g) \in X$, equation (5.1) has a unique solution $u = (u_1, u_2) \in D(A)$ and $|\lambda|\,\|u\|_X \leq C\|(f,g)\|_X$ with a constant not depending on λ.

(i) Let $(f,g) \in X$. We construct the unique solution $u = (u_1,u_2) \in D(A)$ of equation (5.1) by solving two different boundary value problems. First, we consider the boundary value problem

$$\begin{aligned} (\lambda+\Delta^2)u_1^0 &= f \text{ in } \mathbb{R}^n_+,\\ \gamma_0 u_1^0 &= 0 \text{ on } \mathbb{R}^{n-1},\\ \partial_\nu \Delta u_1^0 &= 0 \text{ on } \mathbb{R}^{n-1}. \end{aligned} \tag{5.12}$$

By Lemma 5.1, this problem is parameter-elliptic, and by classical results (see [16], Theorem 1.9, or apply Theorem 3.2 with $s=4$ and $\sigma=0$), there exists a unique solution $u_1^0 \in H_p^4(\mathbb{R}^n_+)$ of (5.12) and

$$|\lambda|\|u_1^0\|_{L^p(\mathbb{R}^n_+)} \leq C\|f\|_{L^p(\mathbb{R}^n_+)},$$

where the constant C depends on θ and λ_0' but not on f or λ.

(ii) Next, we solve

$$\begin{aligned}(\lambda+\Delta^2)u_1' &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \partial_\nu u_1' + (\lambda-\Delta')u_2 &= g' \quad \text{on } \mathbb{R}^{n-1},\\ \partial_\nu \Delta u_1' &= 0 \quad \text{on } \mathbb{R}^{n-1},\\ \gamma_0 u_1' &= u_2 \text{ on } \mathbb{R}^{n-1}\end{aligned} \tag{5.13}$$

such that the solution of (5.1) is given by $u=(u_1,u_2)$ with $u_1 = u_1' + u_1^0$. Here, we have set $g' := g - \partial_\nu u_1^0$. Note that

$$\|\partial_\nu u_1^0\|_{L^p(\mathbb{R}^{n-1})} \le C\|\partial_\nu u_1^0\|_{B^{3-1/p}_{pp}(\mathbb{R}^{n-1})} \le C\|u_1^0\|_{H^4_p(\mathbb{R}^n_+)} \le C\|f\|_{L^p(\mathbb{R}^n_+)}$$

and therefore

$$\|g'\|_{L^p(\mathbb{R}^{n-1})} \le C\big(\|g\|_{L^p(\mathbb{R}^{n-1})} + \|f\|_{L^p(\mathbb{R}^n_+)}\big) \le C\|(f,g)\|_X.$$

With the same calculations as those leading up to (5.6), we observe that the boundary value problem (5.13) possesses a unique solution (u_1', u_2) satisfying $\hat{u}_2 = S(|\xi'|,\lambda)\hat{g}'$ and therefore $u_2 = S(|D'|,\lambda)g'$. Since m is bounded due to Lemma 5.5 and

$$(-\Delta')^{1/2}\colon L^p(\mathbb{R}^{n-1}) \supset D((-\Delta')^{1/2}) = W^1_p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$$

has a bounded H^∞-calculus (see, e.g., [13], Corollary 2.10), the operator

$$m((-\Delta')^{1/2},\lambda) = (\lambda+\Delta')S(|D'|,\lambda)$$

is well-defined and a bounded operator in $L^p(\mathbb{R}^{n-1})$. The operator norm can be estimated by a constant independent of $\lambda \in \Sigma_\theta$. This shows that

$$S(|D'|,\lambda)\colon L^p(\mathbb{R}^{n-1}) \to H^2_p(\mathbb{R}^{n-1}) \tag{5.14}$$

is continuous, and as $\frac{\lambda}{\lambda+z^2} = \frac{1}{1+\frac{z^2}{\lambda}}$ is a bounded holomorphic function as well, we also obtain the boundedness of $\lambda S(|D'|,\lambda)$ on $L^p(\mathbb{R}^{n-1})$.

(iii) With (ii) and $u_2 = S(|D'|,\lambda)g'$ we get

$$|\lambda|\,\|u_2\|_{L^p(\mathbb{R}^{n-1})} \le C\|g'\|_{L^p(\mathbb{R}^{n-1})} \le C\|(f,g)\|_X.$$

The function u_1' in particular solves the problem

$$\begin{aligned}(\lambda+\Delta^2)u_1' &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \gamma_0 u_1' &= u_2 \text{ on } \mathbb{R}^{n-1},\\ \partial_\nu \Delta u_1' &= 0 \quad \text{on } \mathbb{R}^{n-1}.\end{aligned} \tag{5.15}$$

As this boundary value problem is parameter-elliptic due to Lemma 5.1, we can apply Theorem 3.2 with $s := 4$, $\sigma := -4 + 1/(2p)$. We use the embeddings $L^p(\mathbb{R}^{n-1}) = H^0_{p,\lambda}(\mathbb{R}^{n-1}) \subset B^{-1/(2p)}_{pp,\lambda}(\mathbb{R}^{n-1})$ and $H^{4,-4+1/(2p)}_{p,\lambda}(\mathbb{R}^n_+) \subset H^{1/(2p)}_{p,\lambda}(\mathbb{R}^n_+)$ (see Proposition 2.2 c)) and obtain from Theorem 3.2 that $u_1' \in H^{1/(2p)}_{p,\lambda}(\mathbb{R}^n_+)$ satisfies the estimate

$$\begin{aligned}|\lambda|^{1/(8p)}\|u_1'\|_{L^p(\mathbb{R}^n_+)} &\le C\|u_1'\|_{H^{1/(2p)}_{p,\lambda}(\mathbb{R}^n_+)} \le C\|u_1'\|_{H^{4,-4+1/(2p)}_{p,\lambda}(\mathbb{R}^n_+)}\\ &\le C\|u_2\|_{B^{-1/(2p)}_{pp,\lambda}(\mathbb{R}^{n-1})} \le C\|u_2\|_{L^p(\mathbb{R}^{n-1})}\end{aligned}$$

for $\lambda \in \Sigma_\theta$ with $|\lambda| \ge \lambda_0'$. Altogether, $u = (u_1, u_2)$ with $u_1 = u_1' + u_1^0$ is the unique solution of (5.1) and fulfills the uniform estimate $|\lambda|\,\|u\|_X \le C\|(f, g)\|_X$ for $\lambda \in \Sigma_\theta$ with $|\lambda| \ge \lambda_0'$. Writing $\lambda - A = (\lambda - \lambda_0) - (A - \lambda_0)$ and recalling (5.11), we see that $A - \lambda_0$ generates a bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$ in X, and therefore A generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$ in X.

(iv) From (5.14) we even know that $u_2 = S(|D'|, \lambda)g'$ lies in $H^2_p(\mathbb{R}^{n-1})$. Consequently, we can also apply Theorem 3.2 to (5.15) with $s = 4$ and $\sigma = -2 + \frac{1}{p} - \varepsilon$. Hence, taking a fixed $\lambda \in \Sigma_\theta$, we obtain the desired higher regularity due to

$$\begin{aligned}\|u_1\|_{H^{2+1/p-\varepsilon}_p(\mathbb{R}^n_+)} &\le C_\lambda\|u_1\|_{H^{4,-2+1/p-\varepsilon}_{p,\lambda}(\mathbb{R}^n_+)}\\ &\le C_\lambda\|u_2\|_{B^{2-\varepsilon}_{pp,\lambda}(\mathbb{R}^{n-1})} \le C_\lambda\|u_2\|_{H^2_p(\mathbb{R}^{n-1})}.\end{aligned}$$

For $p \ge 2$, the last embedding also holds for $\varepsilon = 0$. □

Remark 5.7. a) In the above estimates we could show that

$$|\lambda|\,\|u_2\|_{L^p(\mathbb{R}^{n-1})} \le C\|g'\|_{L^p(\mathbb{R}^{n-1})}$$

holds for all $\lambda \in \Sigma_\theta$. The condition $|\lambda| \ge \lambda_0$ with arbitrary small $\lambda_0 > 0$ was only used for the uniform estimate of $\|u_1\|_{L^p(\mathbb{R}^n_+)}$.

b) The proof of Theorem 5.6 is essentially based on the estimate from Lemma 5.5 and an application of the general result from Theorem 3.2. We expect generation of an analytic semigroup in some general setting, starting from Theorem 3.2 in $\mathbb{R}^n_+$ or Theorem 4.9 in domains. However, it is not so obvious to obtain an analogue of Lemma 5.5, which can be seen as an estimate on (a part of) the Lopatinskii matrix related to the dynamic boundary value problem. Analog estimates might heavily depend on the mixed-order structure of the Lopatinskii matrix and on the orders of the boundary operators. We plan to address this question in future research.

Example 5.8. With exactly the same methods as for (5.1), one can also treat the more simple boundary value problem with dynamics boundary condition given as

$$\begin{aligned}\lambda u_1 - \Delta u_1 &= f \quad \text{in } \mathbb{R}^n_+,\\ \lambda u_2 + \partial_\nu u_1 &= g \quad \text{on } \mathbb{R}^{n-1},\\ \gamma_0 u_1 - u_2 &= 0 \quad \text{on } \mathbb{R}^{n-1}.\end{aligned} \tag{5.16}$$

The operator A related to (5.16) acts in the space $X := L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$ and is defined by $D(A) := \{u = (u_1, u_2) \in X : A(D)u \in X,\ \gamma_0 u_1 - u_2 = 0\}$, where

$$A(D)u := \begin{pmatrix} \Delta & 0 \\ -\partial_\nu & 0 \end{pmatrix} u \quad (u \in D(A)).$$

In the same way as above, but with much simpler resolvent estimates, one sees that $A - \lambda_0$ generates for every $\lambda_0 > 0$ a bounded holomorphic C_0-semigroup in X. The symbol which we have to estimate now has the form

$$m(z,\lambda) := \frac{\lambda}{\lambda + \sqrt{\lambda + z^2}}$$

for $(z, \lambda) \in \Sigma_\varepsilon \times \Sigma_\theta$.

Data availability

No data was used for the research described in the article.

References

[1] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, second edition, Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] M.S. Agranovich, R. Denk, M. Faierman, Weakly smooth nonselfadjoint spectral elliptic boundary problems, in: Spectral Theory, Microlocal Analysis, Singular Manifolds, in: Math. Top., vol. 14, Akademie Verlag, Berlin, 1997, pp. 138–199.

[3] A. Agresti, M. Hieber, A. Hussein, M. Saal, The stochastic primitive equations with transport noise and turbulent pressure, Stoch. Partial Differ. Equ., Anal. Computat. (2022), https://doi.org/10.1007/s40072-022-00277-3.

[4] H. Amann, Linear and Quasilinear Parabolic Problems. Vol. I. Abstract Linear Theory, Monographs in Mathematics, vol. 89, Birkhäuser Boston, Inc., Boston, MA, 1995.

[5] H. Amann, Linear and Quasilinear Parabolic Problems. Vol. II. Function Spaces, Monographs in Mathematics, vol. 106, Birkhäuser/Springer, Cham, 2019.

[6] W. Arendt, G. Metafune, D. Pallara, S. Romanelli, The Laplacian with Wentzell-Robin boundary conditions on spaces of continuous functions, Semigroup Forum 67 (2) (2003) 247–261.

[7] S. Aziznejad, J. Fageot, Wavelet analysis of the Besov regularity of Lévy white noise, Electron. J. Probab. 25 (2020) 158.

[8] T. Binz, M. Hieber, A. Hussein, M. Saal, The primitive equations with stochastic wind driven boundary conditions, Preprint, arXiv:2009.09449, 2020, 2020.

[9] Z. Brzeźniak, B. Goldys, S. Peszat, F. Russo, Second order PDEs with Dirichlet white noise boundary conditions, J. Evol. Equ. 15 (1) (2015) 1–26.

[10] I. Chueshov, B. Schmalfuss, Qualitative behavior of a class of stochastic parabolic PDEs with dynamical boundary conditions, Discrete Contin. Dyn. Syst. 18 (2–3) (2007) 315–338.

[11] R. Denk, M. Kunze, D. Ploß, The Bi-Laplacian with Wentzell boundary conditions on Lipschitz domains, Integral Equ. Oper. Theory 93 (2021) 13.

[12] R. Denk, J. Prüss, R. Zacher, Maximal L_p-regularity of parabolic problems with boundary dynamics of relaxation type, J. Funct. Anal. 255 (11) (2008) 3149–3187.

[13] R. Denk, J. Saal, J. Seiler, Inhomogeneous symbols, the Newton polygon, and maximal L^p-regularity, Russ. J. Math. Phys. 15 (2) (2008) 171–191.

[14] K.-J. Engel, G. Fragnelli, Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions, Adv. Differ. Equ. 10 (11) (2005) 1301–1320.

[15] L.E. Fraenkel, Formulae for high derivatives of composite functions, Math. Proc. Camb. Philos. Soc. 83 (2) (1978) 159–165.

[16] G. Grubb, Parameter-elliptic and parabolic pseudodifferential boundary problems in global L_p Sobolev spaces, Math. Z. 218 (1) (1995) 43–90.

[17] G. Grubb, Functional Calculus of Pseudodifferential Boundary Problems, second edition, Progress in Mathematics, vol. 65, Birkhäuser Boston, Inc., Boston, MA, 1996.

[18] G. Grubb, N.J. Kokholm, A global calculus of parameter-dependent pseudodifferential boundary problems in L_p Sobolev spaces, Acta Math. 171 (2) (1993) 165–229.

[19] F. Hummel, Boundary value problems of elliptic and parabolic type with boundary data of negative regularity, J. Evol. Equ. 21 (2) (2021) 1945–2007.

[20] F. Hummel, Sample paths of white noise in spaces with dominating mixed smoothness, Banach J. Math. Anal. 15 (3) (2021) 54.

[21] F. Hummel, N. Lindemulder, Elliptic and parabolic boundary value problems in weighted function spaces, Potential Anal. 57 (4) (2022) 601–669.

[22] T. Hytönen, J. van Neerven, M. Veraar, L. Weis, Analysis in Banach Spaces. Vol. I. Martingales and Littlewood-Paley Theory, Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 63, Springer, Cham, 2016.

[23] J. Johnsen, W. Sickel, On the trace problem for Lizorkin-Triebel spaces with mixed norms, Math. Nachr. 281 (5) (2008) 669–696.

[24] J.-L. Lions, E. Magenes, Problemi ai limiti non omogenei (V), Ann. Sc. Norm. Super. Pisa, Cl. Sci., Ser. 3 16 (1) (1962) 1–44.

[25] J.-L. Lions, E. Magenes, Non-homogeneous Boundary Value Problems and Applications. Vol. I, Die Grundlehren der mathematischen Wissenschaften, vol. 181, Springer-Verlag, New York-Heidelberg, 1972.

[26] V.G. Maz'ya, T.O. Shaposhnikova, Theory of Sobolev Multipliers, Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 337, Springer-Verlag, Berlin, 2009, With applications to differential and integral operators.

[27] W.W. Mohammed, D. Blömker, Fast diffusion limit for reaction-diffusion systems with stochastic Neumann boundary conditions, SIAM J. Math. Anal. 48 (5) (2016) 3547–3578.

[28] J. Prüss, R. Racke, S. Zheng, Maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions, Ann. Mat. Pura Appl. (4) 185 (4) (2006) 627–648.

[29] Y. Roitberg, Elliptic Boundary Value Problems in the Spaces of Distributions, Mathematics and Its Applications., vol. 384, Kluwer Academic Publishers Group, Dordrecht, 1996, Translated from the Russian by Peter Malyshev and Dmitry Malyshev.

[30] Y. Roitberg, Boundary Value Problems in the Spaces of Distributions, Mathematics and Its Applications, vol. 498, Kluwer Academic Publishers, Dordrecht, 1999, Translated from the Russian manuscript by Inna Roitberg.

[31] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, De Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.

[32] H.-J. Schmeisser, Recent Developments in the Theory of Function Spaces with Dominating Mixed Smoothness, NAFSA 8—Nonlinear Analysis, Function Spaces and Applications, vol. 8, Czech. Acad. Sci., Prague, 2007, pp. 144–204.

[33] R. Schnaubelt, M. Veraar, Stochastic equations with boundary noise, in: Parabolic Problems, in: Progr. Nonlinear Differential Equations Appl., vol. 80, Birkhäuser/Springer, Basel AG, Basel, 2011, pp. 609–629.

[34] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[35] H. Triebel, Theory of Function Spaces. III, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.

[36] H. Triebel, Theory of Function Spaces, Modern Birkhäuser Classics, Birkhäuser/Springer, Basel AG, Basel, 2010, Reprint of 1983 edition [MR0730762], also published in 1983 by Birkhäuser Verlag, MR0781540.

[37] H. Triebel, Function Spaces with Dominating Mixed Smoothness, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2019.

[38] M.C. Veraar, Regularity of Gaussian white noise on the d-dimensional torus, in: Marcinkiewicz Centenary Volume, in: Banach Center Publ., vol. 95, Polish Acad. Sci. Inst. Math., Warsaw, 2011, pp. 385–398.

[39] M. Warma, Parabolic and elliptic problems with general Wentzell boundary condition on Lipschitz domains, Commun. Pure Appl. Anal. 12 (5) (2013) 1881–1905.

[40] W. Yuan, W. Sickel, D. Yang, Morrey and Campanato Meet Besov, Lizorkin and Triebel, Lecture Notes in Mathematics, vol. 2005, Springer-Verlag, Berlin, 2010.