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THE DE FINETTI PROBLEM WITH UNCERTAIN COMPETITION*

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Abstract. We consider a resource extraction problem which extends the classical de Finetti problem for a Wiener process to include the case when a competitor, who is equipped with the ability to extract all the remaining resources in one piece, may exist. This situation is modeled as a nonzero-sum controller-and-stopper game with incomplete information. For this stochastic game we provide a Nash equilibrium with an explicit structure. In equilibrium, the agent and the competitor use singular strategies in such a way that a two-dimensional process, which represents available resources and the filtering estimate of active competition, reflects in a specific direction along a given boundary.

Key words. the de Finetti problem, uncertain competition, controller-and-stopper game

MSC codes. 91A27, 93E20, 60G40

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1. Introduction.

In the classical single-player de Finetti problem for a Wiener process, the value of a limited resource evolves, in the absence of extraction, as

$$Y_t = x + \mu t + \sigma W_t,$$

where μ and σ are positive constants and W is a standard Brownian motion. The de Finetti problem—also known as *the dividend problem*—then consists of maximizing

$$\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right]$$

over all adapted, nondecreasing, and right-continuous processes D with $D_{0-} = 0$, where $\tau_0 := \inf\{t \geq 0 : Y_t - D_t \leq 0\}$ is the extinction time (or bankruptcy time). It is well known (see, e.g., Asmussen and Taksar [1] and Jeanblanc and Shiryaev [12]) that the optimal strategy $\tilde{D}$ is given by $\tilde{D}_t = \sup_{0\leq s\leq t}(Y_s - B)^+$, where $(x)^+ := \max\{x, 0\}$ and B is a constant that can be calculated explicitly.

In the current article, we study the de Finetti problem under an additional threat of competition. One interpretation of this uncertain competition is that the agent, who exerts the control D to extract from the source Y, is subject to possible theft. For concreteness one may think of the owner of a fish farm who seeks to maximize the revenues from harvesting a stochastically fluctuating population but whose business possibly is monitored by another party who seeks to steal fish. Another possible interpretation is that Y represents the value of a *common* resource but where currently only one agent is extracting; unknown competition then corresponds to the possibility that

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another agent will decide to extract as well. We thus include the possibility that a competitor exists, and we assume that the competitor has the capacity to extract all the remaining resources at once at a random time γ of their choice (for a discussion of this feature, see Remark 2.6).

To model uncertain competition, we use a Bernoulli random variable θ indicating whether the competitor exists ($\theta=1$) or not ($\theta=0$), and we consider the maximization of

$$\mathbb{E}\left[\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t\right]$$

over singular controls D, where $\hat{\gamma} := \gamma 1_{\{\theta=1\}} + \infty 1_{\{\theta=0\}}$. At the same time, the competitor chooses γ to maximize the expected payoff

$$\mathbb{E}\left[e^{-r(\tau_0\wedge\gamma)}X^D_{\tau_0\wedge\gamma}\right],$$

where $X^D = Y - D$ represents the remaining resources after extraction.

The above problem setup models a game of control and stopping; see [3], [4], [7], [10], [13], [14], and [15] for various studies of such games. Notably, we formulate and solve a *nonzero-sum* game. Moreover, an important feature that distinguishes our game from the works mentioned above is *incomplete and asymmetric information,* which in our framework stems from the fact that the existence of the competitor is uncertain. We thus complement the existing literature by investigating the role of uncertain competition in stochastic games. This strand of research can be traced back to the nondynamic setting of an auction game with uncertain competition (see Hirshleifer and Riley [11, pp. 386–389]) and was more recently extended to a dynamic setting in [6] where a stopping game with uncertain competition was studied. In [6] the term "ghost" was also introduced to represent the players that may not exist. In Ekström, Lindensjö, and Olofsson [8], the authors proposed and studied a ghost game in a setting related to fraud detection and so-called "salami slicing" fraudulence. As in the current paper, a controller-and-stopper nonzero-sum game of ghost type is studied in [8], but with the "ghost" role inverted. More precisely, in [8] the controller is a ghost, whereas in the current paper the stopper is a ghost. Our aim is thus to investigate the role of uncertain competition in a singular stochastic control problem. To lay the groundwork for possible future studies, we have chosen the classical setting of the de Finetti problem, which is a well-studied problem in the singular control literature.

Since the competitor is equipped with a binary stopping control, inference about the existence of competition is based on observations of the events $\{\hat{\gamma}\le t\}$. Indeed, the strategies that we consider are based on observations/calculations of the two-dimensional process (X,Π): $X = X^D = Y - D$ is *observed* and represents the value of resources after extraction, whereas Π is *calculated* and represents the adjusted belief of active competition, i.e., the conditional probability that $\theta = 1$ given that stopping has not yet occurred (see section 3.2). Remarkably, our controller-and-stopper nonzero-sum game with incomplete information has an explicit equilibrium with an interesting and rich structure. In this equilibrium the controller extracts resources, and the competitor stops at a randomized stopping time, specified in terms of a generalized intensity, in such a way that the corresponding two-dimensional process (X,Π) reflects obliquely at a given monotone boundary $x = b(p)$ (see Figure 1 in section 6). To construct this two-dimensional reflected process, including a carefully specified reflection direction, we use the notion of perturbed Brownian motion, (see, e.g., Carmona, Petit, and Yor [5] and Perman and Werner [16]). To the best of our knowledge, this is the first time that a perturbed Brownian motion has been used as part of the solution in a stochastic control problem. We also note how the structure of the equilibrium strategy, determined by this two-dimensional reflection, differs completely from the equilibrium found in the controller-and-stopper "ghost" game in [8]. In fact, in equilibrium, the two players act simultaneously as follows when the sufficient statistic (X, Π) hits a certain boundary: the controller exerts control in the (negative) x-direction, and the ghost stopper stops with a generalized intensity that induces a decrease in Π in such a way that the two-dimensional process (X, Π) reflects obliquely along the boundary.

Admittedly, our problem formulation with Brownian dynamics and with the possibility of the competitor extracting all remaining resources in one piece is a bit stylized. However, we hope that the current study of an explicit setup will contribute towards the understanding of more involved problems, including more flexible diffusion models and more general payoff structures. We note, however, that the explicit construction of perturbed processes is not available for general diffusion processes, but one would then instead need to rely on appropriate existence and uniqueness results for such processes.

The paper is organized as follows. In section 2 we provide the precise game formulation of the de Finetti problem under uncertain competition. In section 3 we review the standard single-player de Finetti problem and use heuristic arguments to provide properties that should hold in the game version studied here. Section 4 uses the notion of perturbed Brownian motion to construct the candidate equilibrium. Our main result, Theorem 5.1, in which the candidate equilibrium is verified, is presented in section 5. Finally, section 6 illustrates our findings with a numerical study.

2. Problem setup. We begin by setting the mathematical stage necessary for our analysis. Throughout the paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard Brownian motion W, a Bernoulli random variable θ with $\mathbb{P}(\theta = 1) = 1 - \mathbb{P}(\theta = 0) = p \in [0, 1]$, and a Unif$(0, 1)$ random variable U are defined. Moreover, W, θ, and U are assumed to be independent.

We consider a stochastic game between player 1 and player 2 in which both players seek to maximize certain quantities to be specified. Let Y be a Brownian motion with drift given by

$$Y_t = x + \mu t + \sigma W_t,$$

where the initial condition satisfies $x \geq 0$, and μ and σ are given positive constants. Denote by $\mathbb{F}^W = (\mathcal{F}_t^W)_{0 \leq t < \infty}$ the augmentation of the filtration generated by the Brownian motion W; this filtration will represent the information that Player 1 (the "controller") is equipped with.

Definition 2.1 (admissible controls for player 1). *An admissible control for player* 1 *is a nondecreasing, right-continuous,* $\mathbb{F}^W$*-adapted process* $D = (D_t)_{t \geq 0}$ *satisfying* $D_{0-} = 0$ *and* $D_{\tau_0} \leq Y_{\tau_0}$ *on* $\{\tau_0 < \infty\}$*, where* $\tau_0 := \inf\{s \geq 0 : D_s \geq Y_s\}$*. We denote by* $\mathcal{A}_1$ *the set of admissible controls for player* 1*.*

For any strategy $D \in \mathcal{A}_1$, let $X = X^D := Y - D$, and define

$$\tau_0^X := \inf\{t \geq 0 : X_t \leq 0\}. \tag{2.1}$$

To simplify the notation, we will often omit the superscript and simply write X instead of X^D and τ_0 instead of τ_0^X.

In line with other studies of games with asymmetric information, we equip the more informed player 2 (the "competitor") with randomized stopping times. To define the strategies of player 2, we denote by $\mathcal{D}$ the Skorokhod space of càdlàg paths on $[0,\infty)$.

DEFINITION 2.2 (admissible controls for player 2). *An admissible control* $\Gamma = (\Gamma_t(X))_{t\geq 0}$ *for player* 2 *is a mapping* $(t,X)\mapsto\Gamma_t(X)$ *from* $[0-,\infty)\times\mathcal{D}$ *into* $[0,1]$ *which is progressively measurable for the canonical filtration on* $\mathcal{D}$*, nondecreasing and right-continuous in* t*, and satisfying* $\Gamma_{0-}(X)=0$*. We denote by* $\mathcal{A}_2$ *the set of admissible controls for player* 2*.*

Given a pair of admissible strategies $(D,\Gamma)\in\mathcal{A}_1\times\mathcal{A}_2$, we define a randomized stopping time γ as

$$\gamma := \gamma^{\Gamma} := \inf\{t\geq 0 : \Gamma_t(X^D) > U\}, \tag{2.2}$$

where we recall that U is a random variable which is Unif(0,1)-distributed and independent of θ and W. In accordance with the notation for $X=X^D$, we will often omit the superscript and simply write γ instead of γ^{Γ}.

Remark 2.3. We note that player 2 selects a universal map Γ and applies to any given path of $X=Y-D$ to generate randomized stopping time $\gamma=\gamma^{\Gamma}$ in (2.2). In this way, player 2 is equipped with feedback controls, and we will obtain a Markovian game structure.

Given a fixed discount rate $r>0$ and a pair $(D,\Gamma)\in\mathcal{A}_1\times\mathcal{A}_2$, we define the payoffs for player 1 and player 2 as

$$J_1(x,p,D,\Gamma) := \mathbb{E}\left[\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t\right] \tag{2.3}$$

and

$$J_2(x,p,D,\Gamma) := \mathbb{E}\left[e^{-r(\tau_0\wedge\gamma)}X_{\tau_0\wedge\gamma}\right], \tag{2.4}$$

respectively, where $\tau_0=\tau_0^X$ and $\gamma=\gamma^{\Gamma}$ are defined as in (2.1)–(2.2), and

$$\hat{\gamma} := \begin{cases}\gamma & \text{if } \theta = 1,\\ \infty & \text{if } \theta = 0.\end{cases}$$

The integral in (2.3) is interpreted in the Lebesgue–Stieltjes sense, with

$$\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t := \int_{[0,\tau_0\wedge\hat{\gamma}]} e^{-rt}dD_t.$$

The inclusion of the lower limit 0 of integration thus accounts for the contribution to player 1 from an initial push $dD_0=D_0>0$.

Each player seeks to maximize their respective profit, and we are looking for a Nash equilibrium to this nonzero-sum game in the sense of the following definition.

DEFINITION 2.4. *A pair* $(D^*,\Gamma^*)\in\mathcal{A}_1\times\mathcal{A}_2$ *is a Nash equilibrium (NE) if*

$$\begin{aligned}J_1(x,p,D^*,\Gamma^*) &\geq J_1(x,p,D,\Gamma^*),\\ J_2(x,p,D^*,\Gamma^*) &\geq J_2(x,p,D^*,\Gamma)\end{aligned}$$

for any pair $(D,\Gamma)\in\mathcal{A}_1\times\mathcal{A}_2$*.*

Remark 2.5. Note that it is a consequence of our setup that player 1 has precedence over player 2 in the sense that if a lump sum $dD_t > 0$ is paid out at the same time $t = \hat{\gamma}$ as player 2 stops, then player 1 receives the lump sum, whereas player 2 receives the reduced amount $Y_t - D_t$. Consequently, since player 1 may choose a strategy with $D_0 = x$, for any Nash equilibrium $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ we must have

$$J_1(x,p,D^*,\Gamma^*) = \sup_{D \in \mathcal{A}_1} J_1(x,p,D,\Gamma^*) \geq x.$$

Remark 2.6. We have equipped player 2 with (randomized) stopping times, corresponding to a situation in which this player is limited to extracting all remaining resources at once. This is not a restriction if one considers applications where the actions are *observable*. Indeed, if player 2 were to be equipped with observable increasing controls from $\mathcal{A}_1$ (or more generally, randomized ones), then their existence would be revealed immediately upon extracting, turning the game into a resource extraction game with *known* competition, for which a Nash equilibrium would be that both players try to extract all remaining resources immediately. Therefore, a situation with observable actions and unknown competition corresponds exactly to our setup. (If, on the other hand, actions are *not* directly observable, then allowing player 2 to extract at any rate would not degenerate into the game of the current article; for details on a related game of such type, see [8].)

Remark 2.7. Notice that for player 2 we have chosen to maximize their expected payoff when they are active, i.e., when $\theta = 1$. The case $p = 0$ thus corresponds to when player 2 is active, whereas player 1 is certain that player 2 is not.

Alternatively, one could set player 2 to maximize

$$\hat{J}_2(x,p,D,\Gamma) := \mathbb{E}\left[\theta e^{-r(\tau_0 \wedge \hat{\gamma})} X_{\tau_0 \wedge \hat{\gamma}}\right].$$

The formulations for J_2 and $\hat{J}_2$ have the following interpretations. Imagine that before the game starts, at time $t = 0-$, neither player knows θ and that the value of θ will be revealed to player 2 at time $t = 0$. Then, $\hat{J}_2$ is the expected payoff for player 2 at time $t = 0-$, whereas J_2 is the expected payoff at time $t = 0$ when $\theta = 1$. These games are referred to as the *ex-ante* version of the game and the *interim* version of the game, respectively (see [2], [9] for the classical theory of games under incomplete information). Also notice that the two formulations are equivalent as by independence one obtains $\hat{J}_2(x,p,D,\Gamma) = pJ_2(x,p,D,\Gamma)$, and so the second inequality in Definition 2.4 can be equivalently replaced by $\hat{J}_2(x,p,D^*,\Gamma^*) \geq \hat{J}_2(x,p,D^*,\Gamma)$ for $p > 0$.

PROPOSITION 2.8. *For a given pair $(D,\Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$, we have*

$$J_1(x,p,D,\Gamma) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1 - p\Gamma_{t-})dD_t\right],$$

where $\Gamma_t := \Gamma_t(X^D)$.

Proof. By conditioning and independence, we have

$$\begin{aligned} J_1(x,p,D,\Gamma) &= \mathbb{E}\left[\int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t\right] \\ &= p\mathbb{E}\left[\int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t \middle| \theta = 1\right] + (1-p)\mathbb{E}\left[\int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t \middle| \theta = 0\right] \\ &= p\mathbb{E}\left[\int_0^{\tau_0 \wedge \gamma} e^{-rt} dD_t\right] + (1-p)\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right]. \end{aligned} \tag{2.5}$$

For every $\rho \in [0,1)$, let $\gamma(\rho) := \inf\{t \geq 0 : \Gamma_t(X) > \rho\}$. Then, since the randomization device U is independent of θ and W, and using Fubini's theorem, we have that

$$\mathbb{E}\left[\int_0^{\tau_0 \wedge \gamma} e^{-rt} dD_t\right] = \mathbb{E}\left[\int_0^1 \int_0^{\tau_0 \wedge \gamma(\rho)} e^{-rt} dD_t d\rho\right]$$
$$= \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}\left\{\int_0^1 \mathbb{1}_{\{t \leq \gamma(\rho)\}} d\rho\right\} dD_t\right].$$

Note that

$$\{\Gamma_{t-} \leq \rho\} = \{t \leq \gamma(\rho)\},$$

so

$$\int_0^1 \mathbb{1}_{\{t \leq \gamma(\rho)\}} d\rho = \int_0^1 \mathbb{1}_{\{\Gamma_{t-} \leq \rho\}} d\rho = 1 - \Gamma_{t-}. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$J_1(x,p,D,\Gamma) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1 - p\Gamma_{t-}) dD_t\right]. \qquad \square$$

3. Background material and heuristics.

3.1. The single-player de Finetti problem. Note that if $p = 0$, then player 1 acts under no competition and thus faces the standard de Finetti problem for which the value function

$$V(x) := \sup_{D \in \mathcal{A}_1} \mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right] \tag{3.1}$$

and the optimal strategy $\tilde{D}$ are well known (see, e.g., [12]). To describe this solution in more detail, let ψ be the unique increasing solution of

$$\mathcal{L}\psi(x) = 0, \quad x \geq 0,$$

with $\psi(0) = 0$ and $\psi'(0) = 1$, where $\mathcal{L}$ denotes the differential operator

$$\mathcal{L} := \frac{\sigma^2}{2}\partial_x^2 + \mu\partial_x - r. \tag{3.2}$$

More explicitly,

$$\psi(x) = \frac{e^{\zeta_2 x} - e^{\zeta_1 x}}{\zeta_2 - \zeta_1}, \tag{3.3}$$

where ζ_i, $i = 1, 2$, are the solutions of the quadratic equation

$$\zeta^2 + \frac{2\mu}{\sigma^2}\zeta - \frac{2r}{\sigma^2} = 0$$

with $\zeta_1 < 0 < \zeta_2$. Setting

$$B := \frac{\ln(\zeta_1^2) - \ln(\zeta_2^2)}{\zeta_2 - \zeta_1}, \tag{3.4}$$

we have that ψ is concave on $[0,B]$ and convex on (B,∞), and

$$V(x)=\begin{cases}\frac{\psi(x)}{\psi'(B)}, & x\le B,\\ x-B+V(B), & x>B.\end{cases} \tag{3.5}$$

Moreover,

$$\tilde D_t=\sup_{s\in[0,t]}\big(Y_s-B\big)^+ \tag{3.6}$$

is an optimal strategy in (3.1), i.e.,

$$V(x)=\mathbb{E}\left[\int_0^{\tilde\tau_0}e^{-rt}d\tilde D_t\right],$$

where $\tilde X := X^{\tilde D}$ and $\tilde\tau_0 := \tau_0^{\tilde X}$. We also remark that $(\tilde X,\tilde D)$ is the solution of a Skorokhod reflection problem with reflection at the barrier B.

3.2. Adjusted beliefs. We now return to our version of the game including a ghost feature as described in section 2. At the beginning of the game, from the perspective of player 1 there is active competition (i.e., $\theta=1$) with probability p. As time passes, and if no stopping occurs, player 1's conditional probability of competition Π will decrease. More precisely, at time $t\ge 0$, assuming that the strategy pair $(D,\Gamma)\in\mathcal{A}_1\times\mathcal{A}_2$ is played, we have

$$\begin{aligned}\Pi_t=\Pi_t^\Gamma&:=\mathbb{P}(\theta=1|\mathcal{F}_t^W,\hat\gamma>t)=\frac{\mathbb{P}(\theta=1,\hat\gamma>t|\mathcal{F}_t^W)}{\mathbb{P}(\hat\gamma>t|\mathcal{F}_t^W)}\\ &=\frac{p\mathbb{P}(\gamma>t|\mathcal{F}_t^W)}{(1-p)+p\mathbb{P}(\gamma>t|\mathcal{F}_t^W)}=\frac{p(1-\Gamma_t(X^D))}{1-p\Gamma_t(X^D)}\end{aligned} \tag{3.7}$$

since $\mathbb{P}(\gamma>t|\mathcal{F}_t^W)=1-\mathbb{P}(U\le\Gamma_t|\mathcal{F}_t^W)=1-\Gamma_t$ for $\Gamma=\Gamma(X^D)$. Moreover, since the initial probability of the event $\{\theta=1\}$ is p, we also have $\Pi_{0-}:=p$. Also note that solving for Γ_t in the equation above gives

$$\Gamma_t=\Gamma_t^\Pi=\frac{p-\Pi_t}{p(1-\Pi_t)}, \tag{3.8}$$

so there is a bijection between Π and Γ.

3.3. Heuristics. This section illustrates the heuristic arguments which lead to the formulation of a Nash equilibrium for our problem. These heuristics will be rigorously supported in Theorem 5.1.

We start by providing some simple bounds for the payoff of player 1. Recalling the single-player value function V from (3.1), we have

$$J_1(x,p,D,\Gamma)=\mathbb{E}\left[\int_0^{\tau_0}e^{-rt}(1-p\Gamma_{t-})dD_t\right]\le\mathbb{E}\left[\int_0^{\tau_0}e^{-rt}dD_t\right]\le V(x)$$

for any strategy pair $(D,\Gamma)\in\mathcal{A}_1\times\mathcal{A}_2$, so it is clear that the risk of competition deflates the value from the perspective of player 1. On the other hand, if $\tilde D$ denotes the optimal control of the single-player de Finetti problem (see (3.6)), then

$$J_1(x,p,\tilde D,\Gamma)=\mathbb{E}\left[\int_0^{\tilde\tau_0}e^{-rt}(1-p\Gamma_{t-})d\tilde D_t\right]\ge(1-p)\mathbb{E}\left[\int_0^{\tilde\tau_0}e^{-rt}d\tilde D_t\right]=(1-p)V(x)$$

for any $\Gamma \in \mathcal{A}_2$. Thus it is clear that

$$(1-p)V(x) \leq J_1(x,p,D^*,\Gamma^*) \leq V(x) \tag{3.9}$$

if $(D^*,\Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium.

We make the ansatz that there exists a Nash equilibrium $(D^*,\Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

(a) there exists a nonincreasing continuous boundary $p = c(x)$ such that the overall effect of the equilibrium strategy $(D^*,\Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ amounts to reflection of the two-dimensional process $(X^*,\Pi^*) = (Y - D^*, \Pi^{\Gamma^*})$ along this boundary (see Figure 1);

(b) the corresponding equilibrium value v of player 1 satisfies

$$v(x,p) = (1-p)V(x) \quad \text{for } p \leq c(x). \tag{3.10}$$

Notice that the ansatz (3.10) coincides with the lower bound in (3.9) and that it thus bears some resemblance to the equilibrium obtained in the ghost Dynkin game studied in [6].

Given this ansatz, we need to determine

(i) the boundary $x \mapsto c(x)$;

(ii) the direction of reflection when the process (X^*,Π^*) is at the boundary;

(iii) the strategy pair (D^*,Γ^*) corresponding to the reflected process (X^*,Π^*).

We do this step by step, and then verify the constructed candidate Nash equilibrium (D^*,Γ^*) in section 5. We focus on starting points (x,p) with $p \leq c(x)$ here and treat the strategy for $p > c(x)$ directly in section 5.

First, note that by the bijection (3.8) between Γ and Π, we have that Π^* is of the form $\Pi^* = \Pi^*(X^D)$ for every $D \in \mathcal{A}_1$. Thus, to obtain the reflection of (X^*,Π^*) along the monotone boundary c, we need that

$$\Pi_t^* = \Pi_t^*(X^D) = p \wedge c\left(\sup_{0\leq s\leq t} X_s^D\right) \quad \text{for } t \geq 0. \tag{3.11}$$

With a slight abuse of notation, Π^* will be used to indicate both $\Pi^*(X^D)$ and $\Pi^*(X^*)$, but this will be clear from the context as it will depend on whether player 1 plays an arbitrary admissible strategy $D \in \mathcal{A}_1$ or the equilibrium strategy D^*.

Now, consider a starting point $(x,p) \in [0,\infty) \times (0,1)$ with $p \leq c(x)$. By (3.11), we expect in equilibrium that

$$(X_t^*,\Pi_t^*) = \left(Y_t - D_t^*, p \wedge c\left(\sup_{0\leq s\leq t}(Y_s - D_s^*)\right)\right)$$

for $D^* \in \mathcal{A}_1$ to be specified. By construction, Π^* is continuous, and from (3.7) and (3.8) we have

$$\Gamma_t^* = \frac{p - \Pi_t^*}{p(1-\Pi_t^*)}$$

and

$$d\Pi_t^* = -\frac{1}{1-\Gamma_t^*}\Pi_t^*(1-\Pi_t^*)d\Gamma_t^* \tag{3.12}$$

on $\{t \geq 0 : \Gamma_t^* < 1\}$.

Let $\hat{\gamma}^* := \gamma^* 1_{\{\theta=1\}} + \infty 1_{\{\theta=0\}}$. By the dynamic programming principle, one would expect that the process $M = M^D$ given by

$$M_t := \int_0^{t\wedge\hat{\gamma}^*} e^{-rs} dD_s + e^{-rt} v(X_t, \Pi_t^*) \mathbb{1}_{\{t<\hat{\gamma}^*\}}$$

is an $\mathbb{F}^{W,\hat{\gamma}^*}$-martingale if $D = D^* \in \mathcal{A}_1$ is an optimal response to $\Gamma^* \in \mathcal{A}_2$. Here, $\mathbb{F}^{W,\hat{\gamma}^*} = (\mathcal{F}^{W,\hat{\gamma}^*})_{0\le t<\infty}$ is the smallest right-continuous filtration to which W and $\mathbb{1}_{\{\cdot\ge\hat{\gamma}^*\}}$ are adapted, augmented with the $\mathbb{P}$-null sets of Ω. Moreover, by conditioning (cf. Proposition 2.8), M is an $\mathbb{F}^{W,\hat{\gamma}^*}$-martingale if and only if

$$\hat{M}_t := \int_0^t e^{-rs}(1 - p\Gamma_{s-}^*)\, dD_s + e^{-rt}(1 - p\Gamma_t^*) v(X_t, \Pi_t^*)$$

is an $\mathbb{F}^W$-martingale. Therefore, by an application of Ito's formula and (3.12), we see that when player 2 plays the equilibrium strategy Γ^*, if t is such that $\Pi_t^* = c(X_t^*)$ so that (X_t^*, Π_t^*) is at the boundary, then we need that

$$(1 - v_x)\, dD_t^* - \frac{\Pi_t^*}{1-\Gamma_t^*}\big((1-\Pi_t^*)v_p \;+ v\big)\, d\Gamma_t^* = 0 \tag{3.13}$$

for M to be martingale. Moreover, since c is assumed to be continuous and nonincreasing, we have for all times t that

$$\Pi_t^* = p \wedge c\left(\sup_{0\le s\le t}(Y_s - D_s^*)\right) \le c(Y_t - D_t^*) = c(X_t^*), \tag{3.14}$$

so we obtain from the ansatz (3.10) that

$$(1-p)v_p(x,p) + v(x,p) = 0$$

whenever $p \le c(x)$. Therefore, the second term of (3.13) is zero, and to satisfy the martingality condition we thus need to have $v_x(x,p) = 1$ at the boundary c. Consequently, the boundary $p = c(x)$ should be defined by

$$(1 - c(x))V'(x) = 1 \tag{3.15}$$

for $x \in [0, B]$, where B is the single-player boundary as specified in (3.4).

Taking (3.15) as a definition, we get

$$c(x) = \frac{V'(x) - 1}{V'(x)}, \qquad x \in [0, B], \tag{3.16}$$

from which it follows immediately by (3.5) that $c(B) = 0$, $c'(x) < 0$, and $c'(x) \to 0$ as $x \nearrow B$. Moreover, let

$$\hat{p} := (V'(0) - 1)/V'(0). \tag{3.17}$$

Then, $c : [0, B] \to [0, \hat{p}]$ defined by (3.15) is a continuous strictly decreasing bijection, and we denote its inverse by b; thus $b : [0, \hat{p}] \to [0, B]$ satisfies

$$b(c(x)) = x \quad \forall\, x \in [0, B]. \tag{3.18}$$

From here on, we will refer to b (instead of c) as the boundary when it is more convenient to do so. By convention, we also extend b and c by continuity and define $b(p) = 0$ for every $p \in (\hat{p}, 1]$, and $c(x) = 0$ for $x \in (B, \infty)$.

We now turn to (ii), i.e., to the question of how to specify the direction of reflection. Since player 2 in equilibrium only stops at time points when (X^*, Π^*) is at the boundary, we expect this player's equilibrium value u to be of the form $u(x,p) = g(p)\psi(x)$ for some function g and to satisfy the condition $u(b(p),p) = b(p)$. Consequently,

$$u(x,p) = b(p)\frac{\psi(x)}{\psi(b(p))} \tag{3.19}$$

for $x \le b(p)$. By another application of the dynamic programming principle, we expect the process

$$N_t = e^{-rt}u(X_t^*, \Pi_t^*)$$

to be a martingale when player 1 plays the equilibrium strategy D^*. After applying Ito's formula, this yields

$$-u_x\, dD_t^* + u_p\, d\Pi_t^* = 0 \tag{3.20}$$

on the boundary c, and the direction of reflection of (X^*, Π^*) thus needs to be $(u_p, -u_x)$ for this martingality condition to hold.

Having dealt with (i) and (ii) above, we are now in position to handle item (iii): construction of a candidate Nash equilibrium (D^*, Γ^*) so that the corresponding process (X^*, Π^*) reflects along the boundary c, defined by (3.15), in the direction $(u_p, -u_x)$. We first specify Γ^* by setting

$$\Gamma_t^*(X^D) = \frac{p - \Pi_t^*}{p(1-\Pi_t^*)} \quad \text{for } t \ge 0$$

(cf. (3.8)), where $\Pi_t^* = \Pi_t^*(X^D) = p \wedge c(\sup_{0\le s\le t}(X_s^D))$ for an arbitrary strategy $D \in \mathcal{A}_1$. The process (X^D, Π^*) then reflects at the boundary c by construction, but the direction of reflection is, for an arbitrary strategy $D \in \mathcal{A}_1$, not necessarily equal to $(u_p, -u_x)$. What remains is thus to construct the strategy D^* for player 1 which ensures this specific direction of reflection.

We expect player 1 to exert control only when at the boundary, which by the monotonicity of c translates to the process $X = Y - D$ being at its current maximum. Therefore, one expects D^* to be such that

$$dD_t^* = \lambda(\bar{X}_t^*)\, d\bar{X}_t^*$$

for some function λ to be determined, where $\bar{X}_t^* := b(p) \vee \sup_{0\le s\le t} X_s^*$ and $X^* = Y - D^*$. Moreover, from (3.11) we have, when player 1 plays the equilibrium strategy D^*, that $\Pi_t^* = c(\bar{X}_t^*)$, so (3.20) gives

$$\lambda(x) = \frac{c'(x)u_p(x,c(x))}{u_x(x,c(x))}. \tag{3.21}$$

Using (3.19), we then get

$$u_x(x,c(x)) = \frac{\psi'(x)}{\psi(x)}x$$

and

$$u_p(x,c(x)) = \frac{\psi(x) - x\psi'(x)}{\psi(x)c'(x)},$$

so that

$$\lambda(x)=\frac{\psi(x)-x\psi'(x)}{x\psi'(x)}. \tag{3.22}$$

Since $\psi(0)=0$ and ψ is concave on $[0,B]$, we have $\psi(x)\geq x\psi'(x)$ and thus also $\lambda\geq 0$ on $(0,B]$.

In the next section we study in detail the solvability of the equation

$$X^*_t=Y_t-\int_0^t\lambda(\bar{X}^*_s)\,d\bar{X}^*_s$$

using the notion of *perturbed Brownian motion,* which will allow us to obtain an explicit form of the equilibrium strategy D^* for player 1.

4. A perturbed Brownian motion with drift. To construct the equilibrium strategy D^* for player 1 we will use the notion of perturbed Brownian motion, which is a linear Brownian motion that gets an extra push when it hits its current maximum. Here we provide what is needed for the study of our problem, and we refer the reader to [5], [16], and the references therein for further details on such processes. First, define $\Lambda:[b(p),B]\to[0,\infty)$ by

$$\Lambda(x):=\int_{b(p)}^x\lambda(y)\,dy, \tag{4.1}$$

where

$$\lambda(x)=\frac{\psi(x)}{x\psi'(x)}-1$$

as in (3.22) and the boundary b is defined as in (3.18). Since $\lambda\geq 0$ on $(0,B]$, we note that Λ is increasing. Note also that $\lambda(x)$ is a bounded function for $x\in[0,B]$, so Λ is well-defined. For $x\leq b(p)$ we now consider the equation

$$X_t=Y_t-\Lambda(\bar{X}_t),\quad t\in[0,\tau_B], \tag{4.2}$$

where $Y_t=x+\mu t+\sigma W_t$, $\bar{X}_t:=b(p)\vee\sup_{0\leq s\leq t}X_s$, and $\tau_B=\tau_B^X:=\inf\{t\geq 0:X_t\geq B\}$. The process X is then a perturbed Brownian motion with drift.

To construct a solution of (4.2), let

$$\bar{Y}_t:=b(p)\vee\sup_{0\leq s\leq t}Y_s. \tag{4.3}$$

Define the function $f:[b(p),\infty)\to[b(p),B]$ by the relations

$$\begin{aligned}\Lambda(f(y))+f(y)&=y,\quad y\in[b(p),\Lambda(B)+B],\\ f(y)&=B,\quad y>\Lambda(B)+B,\end{aligned} \tag{4.4}$$

i.e., f is the inverse of the increasing function $x\mapsto y:=\Lambda(x)+x$ for $y\in[b(p),\Lambda(B)+B]$ and then extended constantly for $y>\Lambda(B)+B$. Now define

$$X_t:=Y_t-\bar{Y}_t+f(\bar{Y}_t). \tag{4.5}$$

Proposition 4.1. *Assume that $x\leq b(p)$. Then the process X in* (4.5) *solves* (4.2).

Proof. Let $t \in [0,\tau_B]$. Since $\bar{X}_t := b(p) \vee \sup_{s\in[0,t]} X_s$, we obtain from (4.5) that $\bar{X}_t = f(\bar{Y}_t)$ as $f(b(p)) = b(p)$. Consequently, $\tau_B = \inf\{t \geq 0 : Y_t \geq \Lambda(B) + B\}$, and so by (4.4) we have $f(\bar{Y}_t) - \bar{Y}_t = -\Lambda(f(\bar{Y}_t))$. This leads to

$$X_t = Y_t - \Lambda(\bar{X}_t),$$

which proves the claim. ∎

Remark 4.2. The setup in (4.2) of a perturbed Brownian motion is slightly more general than what is used in most literature on perturbed Brownian motions; in fact, the typical choice of perturbation used in the literature is linear, corresponding to a linear function Λ in (4.2). On the other hand, we only deal with one-sided perturbation, in which case the solution can be constructed explicitly as in (4.5) above. It is straightforward to check that the argument for pathwise uniqueness of solutions of (4.2) (cf. [5, Proposition 2.1]) carries over to our setting.

Remark 4.3. The function f defined in (4.4) is constructed in such a way that the process $X_t = Y_t - \bar{Y}_t + f(\bar{Y}_t)$ is a perturbed Brownian motion with drift for $t \in [0,\tau_B]$ (as proved in Proposition 4.1), and it is the Skorokhod reflection of the process Y_t at the barrier B for $t \in (\tau_B,\infty)$. Indeed, for $t \in (\tau_B,\infty)$, we have

$$\begin{aligned} X_t &= Y_t - \bar{Y}_t + f(\bar{Y}_t) = Y_t - \bar{Y}_t + B \\ &= Y_t - \sup_{s\in[0,t]}(Y_s - B) = Y_t - \sup_{s\in[0,t]}(Y_s - B)^+, \end{aligned} \tag{4.6}$$

i.e., we have $X_t = X_t^{\tilde{D}}$ for $t \in (\tau_B,\infty)$ where $\tilde{D}$ is defined as in (3.6).

5. Main result.

In this section, we state and prove our main result: an explicit Nash equilibrium for our game. To do that, let us fix $(x,p) \in [0,\infty) \times [0,1]$ and recall that Y is given by

$$Y_t = x + \mu t + \sigma W_t.$$

First, define a new process $Y^\wedge$ by

$$Y_t^\wedge := x \wedge b(p) + \mu t + \sigma W_t = Y_t - (x - b(p))^+,$$

so that $Y^\wedge$ starts below the boundary $b(p)$ (recall definition (3.18)). Then define $\bar{Y}^\wedge$ as in (4.3) but with $Y^\wedge$ instead of Y, i.e.,

$$\bar{Y}_t^\wedge := b(p) \vee \sup_{0\leq s\leq t} Y_s^\wedge.$$

Also, recall the definitions of $\Lambda : [b(p),B] \to [0,\infty)$ in (4.1) and $f : [b(p),\infty) \to [b(p),B]$ in (4.4), and define $D^* \in \mathcal{A}_1$ by $D_{0-}^* = 0$ and

$$D_t^* := (x - b(p))^+ + \bar{Y}_t^\wedge - f(\bar{Y}_t^\wedge), \quad t \geq 0. \tag{5.1}$$

Setting

$$X_t^* := Y_t - D_t^*,$$

Proposition 4.1 applied with $Y^\wedge$ in place of Y yields

$$X_t^* = Y_t^\wedge - \bar{Y}_t^\wedge + f(\bar{Y}_t^\wedge) = Y_t^\wedge - \Lambda(\bar{X}_t^*), \qquad t \in [0,\tau_B^*], \tag{5.2}$$

where $\tau_B^* = \tau_B^{X^*} := \inf\{t \geq 0 : X_t^* \geq B\}$. Note that by construction we have $dD_t^* = \lambda(X_t^*)dX_t^*$ for $t \in (0, \tau_B]$.

Moreover, for a given path $X = X^D \in \mathcal{D}$ (with $D \in \mathcal{A}_1$), define $Z^* = Z^*(X)$ by $Z_{0-}^* := p$ and

$$Z_t^* := p \wedge c\left(\sup_{0\leq s\leq t} X_s\right), \quad t \geq 0 \tag{5.3}$$

(the process Z will play the role of Π^*; cf. (3.11)), and define $\Gamma^* \in \mathcal{A}_2$ by

$$\Gamma_t^*(X) := \begin{cases} \mathbb{1}_{\{t\geq\tau_B\}}, & p = 0, \\ \frac{p-Z_t^*}{p(1-Z_t^*)}, & p > 0, \end{cases} \tag{5.4}$$

where we recall that $\tau_B := \inf\{t \geq 0 : X_t \geq B\}$.

THEOREM 5.1. *Let $(x,p) \in [0,\infty) \times [0,1]$. The pair (D^*,Γ^*) defined above is an NE for the stochastic game* (2.3)–(2.4)*, with equilibrium values*

$$J_1(x,p,D^*,\Gamma^*) = v(x,p) := \begin{cases} (1-p)V(x), & x \leq b(p), \\ (1-p)V(b(p)) + x - b(p), & x > b(p), \end{cases}$$

$$J_2(x,p,D^*,\Gamma^*) = u(x,p) := \begin{cases} b(p)\frac{\psi(x)}{\psi(b(p))}, & x \leq b(p), \\ b(p), & x > b(p) \end{cases}$$

(with the understanding that $b(p)\psi(x)/\psi(b(p)) = 0$ for $x = 0$ also when $b(p) = 0$). Here, V is the value of the single-player de Finetti problem given in (3.5)*, b is defined as in* (3.18)*, and ψ is given by* (3.3)*.*

Proof. Step 1. We first prove that D^* is an optimal response to Γ^*. Let $D \in \mathcal{A}_1$ be an arbitrary strategy for player 1 and set $X := Y - D$. Let Z^* be defined as in (5.3) and $\Gamma_t^* := \Gamma_t^*(X)$ as in (5.4) accordingly .

If $p = 0$, then $\theta = 0$ a.s., and so

$$J_1(x,0,D,\Gamma^*) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right].$$

Namely, the optimization problem for player 1 degenerates into the single-player de Finetti problem, and D^* coincides with its optimal solution $\tilde{D}$, as highlighted in Remark 4.3. Hence, also $v(x,0) = J_1(x,0,D^*,\Gamma^*) \geq J_1(x,0,D,\Gamma^*)$ for every $D \in \mathcal{A}_1$.

If $x = 0$, then $J_1(0,p,D,\Gamma^*) = 0$ for every $p \in [0,1]$, $D \in \mathcal{A}_1$, and so, in particular, $v(0,p) = J_1(0,p,D^*,\Gamma^*)$ for every $p \in [0,1]$.

Now let $p \in (0,1]$, and let us first consider $0 < x \leq b(p)$. (Note that this implies that $p \in (0,\hat{p})$ as $b(p) = 0$ for every $p \in [\hat{p},1]$, where $\hat{p}$ is as in (3.17).) By (5.4), we have

$$Z_t^* = \frac{p(1-\Gamma_t^*)}{1-p\Gamma_t^*}, \quad t \geq 0,$$

and as a consequence, $\Pi^{\Gamma^*} = Z^*$ (see (3.7)). Since Z^* and Γ^* are continuous and of finite variation, we obtain

$$dZ_t^* = -\frac{p(1-Z_t^*)}{1-p\Gamma_t^*}\, d\Gamma_t^*, \quad t \geq 0.$$

Now define

$$\tilde{v}(x,p) := (1-p)V(x) \in C^2([0,\infty)\times[0,1]).$$

By setting $\tau := \tau_0 \wedge T$ with $T \geq 0$ and applying Ito's formula to $e^{-rt}(1-p\Gamma_t^*)\tilde{v}(X_t, Z_t^*)$, we have that

$$\begin{aligned} e^{-r\tau}(1-p\Gamma_\tau^*)\tilde{v}(X_\tau, Z_\tau^*) = \tilde{v}(x,p) &+ \int_0^\tau e^{-rt}(1-p\Gamma_t^*)\mathcal{L}\tilde{v}(X_{t-}, Z_t^*)\,dt \\ &- \int_0^\tau e^{-rt}(1-p\Gamma_t^*)\tilde{v}_x(X_{t-}, Z_t^*)\,dD_t^c \\ &+ \int_0^\tau \sigma e^{-rt}(1-p\Gamma_t^*)\tilde{v}_x(X_{t-}, Z_t^*)\,dW_t \\ &- \int_0^\tau e^{-rt}p\big[(1-Z_t^*)\tilde{v}_p(X_{t-}, Z_t^*) + \tilde{v}(X_{t-}, Z_t^*)\big]d\Gamma_t^* \\ &+ \sum_{0\leq t\leq\tau} e^{-rt}(1-p\Gamma_t^*)\big(\tilde{v}(X_t, Z_t^*) - \tilde{v}(X_{t-}, Z_t^*)\big), \end{aligned} \tag{5.5}$$

where $\mathcal{L}$ is defined as in (3.2), and D^c denotes the continuous part of D. Notice that $\tilde{v}(x,p) = v(x,p)$ for $x \leq b(p)$ and that by definition of $\tilde{v}$, for every $t > 0$ we have

$$\mathcal{L}\tilde{v}(X_{t-}, Z_t^*) = 0 \quad \text{and} \quad (1-Z_t^*)\tilde{v}_p(X_{t-}, Z_t^*) + \tilde{v}(X_{t-}, Z_t^*) = 0.$$

Hence, (5.5) becomes

$$\begin{aligned} v(x,p) = e^{-r\tau}(1-p\Gamma_\tau^*)\tilde{v}(X_\tau, Z_\tau^*) &+ \int_0^\tau e^{-rt}(1-p\Gamma_t^*)\tilde{v}_x(X_{t-}, Z_t^*)\,dD_t^c \\ &- \int_0^\tau \sigma e^{-rt}(1-p\Gamma_t^*)\tilde{v}_x(X_{t-}, Z_t^*)\,dW_t \\ &- \sum_{0\leq t\leq\tau} e^{-rt}(1-p\Gamma_t^*)\big(\tilde{v}(X_t, Z_t^*) - \tilde{v}(X_{t-}, Z_t^*)\big). \end{aligned} \tag{5.6}$$

For the summation term, we have by the mean value theorem that

$$\sum_{0\leq t\leq\tau} e^{-rt}(1-p\Gamma_t^*)\big(\tilde{v}(X_t, Z_t^*) - \tilde{v}(X_{t-}, Z_t^*)\big) = -\sum_{0\leq t\leq\tau} e^{-rt}(1-p\Gamma_t^*)\tilde{v}_x(\xi_t, Z_t^*)\Delta D_t, \tag{5.7}$$

where $\xi_t \in (X_{t-}, X_t)$ and $\Delta D_t := D_t - D_{t-}$. By plugging (5.7) into (5.6) and using the fact that $\tilde{v} \geq 0$ and $\tilde{v}_x \geq 1$, we obtain

$$v(x,p) \geq \int_0^\tau e^{-rt}(1-p\Gamma_t^*)\,dD_t - \int_0^\tau \sigma e^{-rt}(1-p\Gamma_t^*)\tilde{v}_x(X_{t-}, Z_t^*)\,dW_t. \tag{5.8}$$

Let

$$\mathcal{O} := \{(x,p) \in [0,\infty)\times[0,1] : x \leq b(p)\} \cup ((B,\infty)\times\{0\}), \tag{5.9}$$

and note that $(X_{t-}, Z_t^*) \in \mathcal{O}$ for every $t \geq 0$ (by construction of Z_t) and that $\tilde{v}_x$ is bounded on $\mathcal{O}$ ($\tilde{v}_x(x,p) = 1$ for $(x,p) \in (B,\infty)\times\{0\}$). Thus, the stochastic integral above is a martingale, and by an application of the optional sampling theorem we have that

$$\tilde{v}(x,p) \geq \mathbb{E}\left[\int_0^{\tau_0\wedge T} e^{-rt}(1-p\Gamma_t^*)\,dD_t\right].$$

Letting $T \to \infty$ yields, by the monotone convergence theorem,

$$v(x,p) \geq \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1-p\Gamma^*_t)\,dD_t\right] = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1-p\Gamma^*_{t-})\,dD_t\right] = J_1(x,p,D,\Gamma^*)$$

for every $D \in \mathcal{D}$, where the last equality follows by Proposition 2.8.

Now notice that D^*_t defined in (5.1) is continuous for every $t \geq 0$, when $x \leq b(p)$, and that the same holds for $X^*_t := X^{D^*}_t$. Let $\tau^*_0 := \tau^{X^*}_0$; then (5.6) for $D = D^*$ and $\tau^* := \tau^*_0 \wedge T$ becomes

$$\begin{aligned} v(x,p) &= e^{-r\tau^*}(1-p\Gamma^*_{\tau^*})\tilde{v}(X^*_{\tau^*}, Z^*_{\tau^*}) + \int_0^{\tau^*} e^{-rt}(1-p\Gamma^*_t)\tilde{v}_x(X^*_t, Z^*_t)\,dD^*_t \\ &\quad - \int_0^{\tau^*} \sigma e^{-rt}(1-p\Gamma^*_t)\tilde{v}_x(X^*_t, Z^*_t)\,dW_t \\ &= e^{-r\tau^*}(1-p\Gamma^*_{\tau^*})\tilde{v}(X^*_{\tau^*}, Z^*_{\tau^*}) + \int_0^{\tau^*} e^{-rt}(1-p\Gamma^*_t)\,dD^*_t \\ &\quad - \int_0^{\tau^*} \sigma e^{-rt}(1-p\Gamma^*_t)v_x(X^*_t, Z^*_t)\,dW_t, \end{aligned}$$

where the last equality holds since $\tilde{v}_x(x,p) = 1$ if $x \geq b(p)$ and $dD^*_t = 0$ if $X^*_t < b(Z^*_t)$. Hence, again by taking expected values, we obtain

$$\begin{aligned} v(x,p) &= \mathbb{E}\left[e^{-r(\tau^*_0 \wedge T)}(1-p\Gamma^*_{\tau^*_0\wedge T})\tilde{v}(X^*_{\tau^*_0\wedge T}, Z^*_{\tau^*_0\wedge T}) + \int_0^{\tau^*_0\wedge T} e^{-rt}(1-p\Gamma^*_t)\,dD^*_t\right] \\ &\to \mathbb{E}\left[\int_0^{\tau^*_0} e^{-rt}(1-p\Gamma^*_t)\,dD^*_t\right] \end{aligned}$$

as $T \to \infty$ by dominated convergence (the first term tends to 0 since $\tilde{v}(X^*_{\tau^*_0}, Z^*_{\tau^*_0}) = 0$). Thus, we have proved that

$$J_1(x,p,D^*,\Gamma^*) = v(x,p) \geq \sup_{D\in\mathcal{A}_1} J_1(x,p,D,\Gamma^*) \quad \forall\,(x,p) \in \mathcal{O}.$$

Let us now consider $(x,p) \in ([0,\infty) \times [0,1]) \setminus \mathcal{O} =: \mathcal{O}^c$, i.e., $x > b(p)$ with $p \neq 0$. Then,

$$v(x,p) = v(b(p),p) + x - b(p) = J_1(b(p),p,D^*,\Gamma^*) + x - b(p) = J_1(x,p,D^*,\Gamma^*).$$

Thus, we are left to prove that also in this case,

$$J_1(x,p,D^*,\Gamma^*) \geq J_1(x,p,D,\Gamma^*) \quad \forall\, D \in \mathcal{A}_1.$$

For $(x,p) \in \mathcal{O}^c$, let the admissible strategy $D \in \mathcal{A}_1$ have an initial jump $\Delta D_0 = x - y$ with either $b(p) \leq y \leq x$ or $0 \leq y < b(p)$. In the former case, by definition (5.4) of Γ^*, we have that

$$J_1(x,p,D,\Gamma^*) = (1-\Gamma^*_0)J_1(b(q),q,D,\Gamma^*) + x - y = \frac{q(1-p)}{p}V(b(q)) + x - y,$$

where $q := c(y) \leq p$ (and hence $y = b(q)$). Since V is concave with $V'(b(p)) = 1/(1-p)$, then

$$\begin{aligned} J_1(x,p,D,\Gamma^*) &\leq \frac{q(1-p)}{p}\Big(V(b(p)) + \frac{y-b(p)}{1-p}\Big) + x - y \\ &= \frac{q}{p}\Big((1-p)V(b(p)) + y - b(p)\Big) + x - y \\ &\leq (1-p)V(b(p)) + x - b(p) = J_1(x,p,D^*,\Gamma^*). \end{aligned}$$

If instead $0 \le y < b(p)$, then by a similar argument,

$$\begin{aligned} J_1(x,p,D,\Gamma^*) &= J_1(y,p,D,\Gamma^*) + x - y = (1-p)V(y) + x - y \\ &\le (1-p)V(b(p)) + x - b(p) = J_1(x,p,D^*,\Gamma^*). \end{aligned}$$

This concludes Step 1, i.e., shows that the strategy D^* is an optimal response to Γ^*.

Step 2. We now prove that Γ^* is an optimal response to D^*. Recall that

$$u(x,p) := \begin{cases} b(p)\frac{\psi(x)}{\psi(b(p))}, & x \le b(p), \\ b(p), & x > b(p), \end{cases}$$

set $X^* := X^{D^*}$ with D^* as defined in (5.1), let $\tau_0^* := \tau_0^{X^*}$, and let

$$Z_t^* := p \wedge c\left(\sup_{0\le s\le t} X_s^*\right), \quad t \ge 0, \qquad Z_{0-}^* := p,$$

as in (5.3) with $D = D^*$.

Let $p \in [0,1]$ and assume $x \le b(p)$. If $p \in [\hat{p},1]$, then $b(p) = 0$ and so $x = 0$, and the strategy $\Gamma \in \mathcal{A}_2$ is irrelevant since the game stops immediately. It hence suffices to check $p \in [0,\hat{p})$. For notational convenience we treat the case $p = 0$ separately below and assume first $p \in (0,\hat{p})$. Note that $X_t^* \le b(Z_t^*)$ for every $t \ge 0$ and that Z_t^*, D_t^*, and X_t^* are continuous for every $t \ge 0$. Define

$$\tilde{u}(x,p) := b(p)\frac{\psi(x)}{\psi(b(p))} \in C^2([0,\infty) \times (0,\hat{p})),$$

and let τ be any $\mathbb{F}^W$-stopping time s.t. $\tau \le \tau_B^*$ a.s., where $\tau_B^* = \inf\{t \ge 0 : X_t^* \ge B\}$. Define $\tau^* = \tau_{\varepsilon,T}^* := \tau_0^* \wedge \tau_{B-\varepsilon}^* \wedge \tau \wedge T$ for $T, \varepsilon \ge 0$ arbitrary and note that $Z_t^* > 0$ for $t \in [0,\tau^*)$. By applying Ito's formula to $e^{-rt}\tilde{u}(X_t^*, Z_t^*)$, we obtain

$$\begin{aligned} e^{-r\tau^*}\tilde{u}(X_{\tau^*}^*, Z_{\tau^*}^*) &= \tilde{u}(x,p) + \int_0^{\tau^*} e^{-rs}\mathcal{L}\tilde{u}(X_s^*, Z_s^*)\,ds - \int_0^{\tau^*} e^{-rs}\tilde{u}_x(X_s^*, Z_t^*)\,dD_s^* \\ &\quad + \int_0^{\tau^*} \sigma e^{-rs}\tilde{u}_x(X_s^*, Z_s^*)\,dW_s + \int_0^{\tau^*} e^{-rs}\tilde{u}_p(X_s^*, Z_s^*)\,dZ_s^*. \end{aligned}$$

By definition of $\tilde{u}$, we have that $\mathcal{L}\tilde{u}(X_s^*, Z_s^*) = 0$ for every $0 \le s \le \tau^*$, and by construction of D^* and Z^* (recall (5.2)), we obtain

$$\begin{aligned} &\int_0^{\tau^*} e^{-rs}\tilde{u}_p(X_s^*, Z_s^*)\,dZ_s^* - \int_0^{\tau^*} e^{-rs}\tilde{u}_x(X_s^*, Z_s^*)\,dD_s^* \\ &\quad = \int_0^{\tau^*} e^{-rs}\Big(\tilde{u}_p(X_s^*, Z_s^*)c'(X_s^*) - \tilde{u}_x(X_s^*, Z_s^*)\lambda(X_s^*)\Big)\,d\bar{X}_s^* = 0, \end{aligned} \tag{5.10}$$

where the last equality holds by definition of λ in (3.21). Hence,

$$e^{-r\tau^*}\tilde{u}(X_{\tau^*}^*, Z_{\tau^*}^*) = \tilde{u}(x,p) + \int_0^{\tau^*} \sigma e^{-rs}\tilde{u}_x(X_s^*, Z_s^*)\,dW_s. \tag{5.11}$$

Since $\tilde{u}_x$ is bounded on $\{(x,p) : x \le b(p)\}$, the stochastic integral in (5.11) is a martingale. Since X^* and Z^* are continuous, applying the optional sampling theorem and using dominated convergence yields

$$\tilde{u}(x,p) = \mathbb{E}\left[e^{-r\tau^*}\tilde{u}(X_{\tau^*}^*, Z_{\tau^*}^*)\right] \to \mathbb{E}\left[e^{-r(\tau_0^*\wedge\tau)}\tilde{u}(X_{\tau_0^*\wedge\tau}^*, Z_{\tau_0^*\wedge\tau}^*)\right],$$

as $T\to\infty$ and $\varepsilon\to 0$, so

$$\tilde{u}(x,p)=\mathbb{E}\left[e^{-r(\tau_0^*\wedge\tau)}\tilde{u}(X^*_{\tau_0^*\wedge\tau},Z^*_{\tau_0^*\wedge\tau})\right] \tag{5.12}$$

for any $\mathbb{F}^W$-stopping time $\tau\le\tau_B$ a.s. Now, for any $\Gamma\in\mathcal{A}_2$, define the $\mathbb{F}^W$-stopping times

$$\gamma(\rho):=\inf\{t\ge 0:\Gamma_t(X^*)>\rho\},\qquad \rho\in[0,1),$$

and let $\gamma_B(\rho):=\gamma(\rho)\wedge\tau_B^*\le\tau_B^*$. Since $\tilde{u}=u$ on $\{(x,p):x\le b(p)\}$, equality (5.12) for $\tau=\gamma_B(\rho)$ reads

$$u(x,p)=\mathbb{E}\left[e^{-r(\tau_0^*\wedge\gamma_B(\rho))}u(X^*_{\tau_0^*\wedge\gamma_B(\rho)},Z^*_{\tau_0^*\wedge\gamma_B(\rho)})\right],\qquad \rho\in[0,1).$$

Thus,

$$\begin{aligned} u(x,p)&=\int_0^1\mathbb{E}\left[e^{-r(\tau_0^*\wedge\gamma_B(\rho))}u(X^*_{\tau_0^*\wedge\gamma_B(\rho)},Z^*_{\tau_0^*\wedge\gamma_B(\rho)})\right]d\rho\\ &\ge\int_0^1\mathbb{E}\left[e^{-r(\tau_0^*\wedge\gamma_B(\rho))}X^*_{\tau_0^*\wedge\gamma_B(\rho)}\right]d\rho, \end{aligned} \tag{5.13}$$

where the inequality holds because $\psi(x)$ is concave for $x\le B$ with $\psi(0)=0$.

Lastly, we note that

$$e^{-r(\tau_0^*\wedge\gamma_B(\rho))}X^*_{\tau_0^*\wedge\gamma_B(\rho)}\ge e^{-r(\tau_0^*\wedge\gamma(\rho))}X^*_{\tau_0^*\wedge\gamma(\rho)}\quad\text{a.s.} \tag{5.14}$$

since $X_t^*\le B$ for all $t>0$ and $r>0$, and thus

$$u(x,p)\ge\int_0^1\mathbb{E}\left[e^{-r(\tau_0^*\wedge\gamma(\rho))}X^*_{\tau_0^*\wedge\gamma(\rho)}\right]d\rho=J_2(x,p,D^*,\Gamma).$$

If $\Gamma=\Gamma^*$, then by (5.4) we have that $\gamma^*(\rho)\le\tau_B^*$ for every $\rho\in[0,1)$, where

$$\gamma^*(\rho):=\inf\{t\ge 0:\Gamma_t^*(X^*)>\rho\},\qquad \rho\in[0,1),$$

and thus the inequality in (5.14) is an equality in this case. Moreover, Γ_t^* only increases when Z_t^* increases and $Z^*=Z_t^*:=p\wedge c\Big(\sup_{0\le s\le t}X_s\Big)$, so

$$u(X^*_{\tau_0^*\wedge\gamma^*(\rho)},Z^*_{\tau_0^*\wedge\gamma^*(\rho)})=b(c(\bar{X}_{\tau_0^*\wedge\gamma^*(\rho)}))=X^*_{\tau_0^*\wedge\gamma^*(\rho)}$$

in (5.13). Thus all the inequalities above become equalities, and

$$u(x,p)=J_2(x,p,D^*,\Gamma^*). \tag{5.15}$$

If $p=0$, we have $u(x,0)=\tilde{u}(x,0)=b(0)\frac{\psi(x)}{\psi(b(0))}=B\frac{\psi(x)}{\psi(B)}$ and $Z_t^*=0$ for all $t\ge 0$. Applying Ito's formula to $e^{-rt}u(X_t^*,0)$ between 0 and $\tau_0\wedge\tau\le\tau_B^*$ and using the properties of $\psi(x)$ gives

$$\begin{aligned} e^{-r(\tau_0\wedge\tau)}\tilde{u}(X_{\tau_0\wedge\tau},0)&=\tilde{u}(x,0)-\int_0^{\tau_0\wedge\tau}e^{-rs}\tilde{u}_x(X_s^*,0)dD_s^*\\ &\quad+\int_0^{\tau_0\wedge\tau}e^{-rs}\sigma\tilde{u}_x(X_s^*,0)dW_s. \end{aligned}$$

Taking expected value and arguing as above thus gives

$$u(x,0) = \mathbb{E}\left[e^{-r(\tau_0\wedge\tau_B^*)}u(X^*_{\tau_0\wedge\tau_B^*},0)\right] = e^{-r(\tau_0\wedge\tau_B^*)}X_{\tau_0\wedge\tau_B^*} = J_2(x,0,D^*,\Gamma^*)$$

and

$$\begin{aligned} u(x,0) &= \int_0^1 \mathbb{E}\left[e^{-r(\tau_0\wedge\gamma_B(\rho))}u(X_{\tau_0\wedge\gamma_B(\rho)},0)\right]d\rho \\ &\geq \int_0^1 \mathbb{E}\left[e^{-r(\tau_0\wedge\gamma_B(\rho))}X_{\tau_0\wedge\gamma_B(\rho)}\right]d\rho \geq J_2(x,p,D^*,\Gamma), \end{aligned}$$

where we again have used convexity of ψ and the fact that any stopping time $\gamma(\rho) > \tau_B^*$ yields a lower payoff than τ_B^*.

The above treats the case $x \leq b(p)$, so let us finalize the proof by considering $x > b(p)$. We have, for every $\Gamma \in \mathcal{A}_2$, that

$$u(x,p) = u(b(p),p) \geq J_2(b(p),p,D^*,\Gamma) = J_2(x,p,D^*,\Gamma),$$

where the last equality holds by the precedence of player 1 over player 2 and since $D_0^* = x - b(p)$ for $x > b(p)$. Similarly, we obtain

$$u(x,p) = u(b(p),p) = J_2(b(p),p,D^*,\Gamma^*) = J_2(x,p,D^*,\Gamma^*).$$

Hence, Γ^* is an optimal response to D^*. Together with Step 1, this implies that (D^*,Γ^*) is an NE and that the equilibrium values are v and u, respectively. This concludes the proof. ∎

Remark 5.2. It is a remarkable feature of the equilibrium strategy (D^*,Γ^*) that it allows the process Π^* to reach 0 in finite time, thereby completely ruling out the possibility that a competitor exists. To the best of our knowledge, this fact has no counterpart in the literature on games with unknown competition (cf. [6], [8], [11]).

To see that this can happen, let $x \leq b(p)$. Then we have

$$X_t^* = Y_t - \bar{Y}_t + f(\bar{Y}_t),$$

and thus $\bar{X}_t^* = f(\bar{Y}_t)$ where f is an increasing bounded function such that $f(x) = B$ for all $x \geq \Lambda(B) + B$. Consequently, $\Pi_t^* = p \wedge c(\bar{X}_t^*) = p \wedge c(f(\bar{Y}_t)) = c(B) = 0$ for all

$$t \geq \tau_B = \inf\{s \geq 0 : Y_s \geq \Lambda(B) + B\}$$

the first time the drifted Brownian motion Y reaches $\Lambda(B) + B$ (which is finite a.s.).

Remark 5.3. It is not clear to us whether or not the Nash equilibrium obtained in Theorem 5.1 is unique. Also, note that the equilibrium obtained is specified in terms of a belief system described by Π, which is updated in a Bayesian way. As such, the obtained equilibrium bears much resemblance to the notion of *perfect Bayesian equilibrium,* which is the standard solution concept for games with asymmetric information in discrete time.

6. A numerical example. To provide the reader with further intuition, we conclude by looking at some numerical experiments. Throughout the section, we consider parameters $\mu = 0.03$, $\sigma = 0.12$, and $r = 0.01$. The optimal strategy $\tilde{D}$ in the single-player de Finetti problem given by (3.6) then amounts to reflection at $B \approx 1.12$.

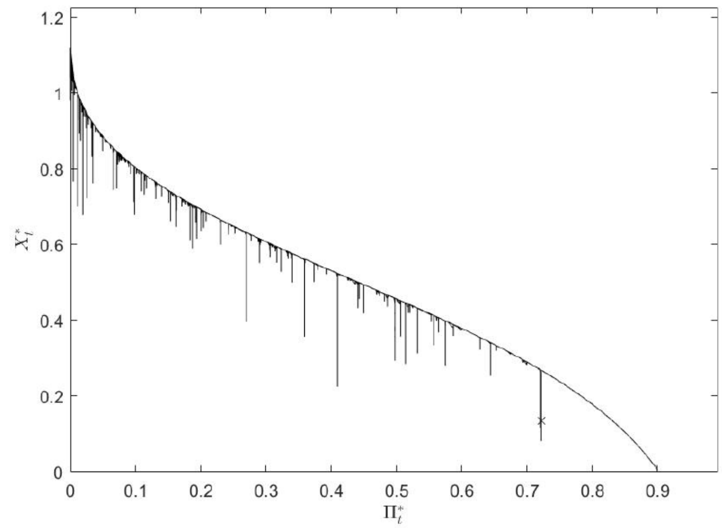

FIG. 1. *A simulated path of* (Π^*, X^*) *reflected along the boundary* $p \mapsto b(p)$*. The starting point* $(p,x) \approx (0.72, 0.13)$ *is illustrated by the cross* $\times$.

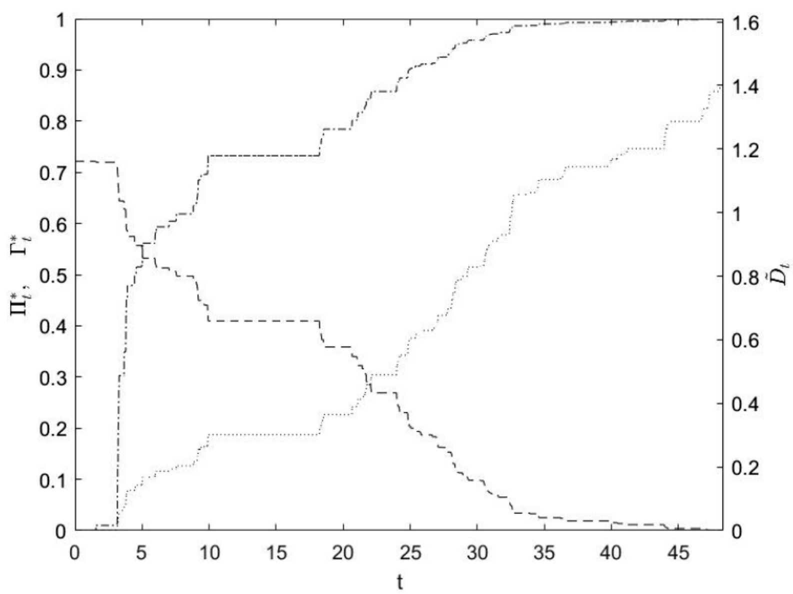

FIG. 2. *Auxiliary processes* Π^* *(dashed),* Γ^* *(dash-dot), and* D^* *(dotted).*

Note that whereas the qualitative form of the single-player strategy de Finetti problem is fixed, the nature of the NE strategy for player 1 varies depending on the value of $p \in [0,1]$. To be more precise, if player 1 is certain that no competitor exists, i.e., if $p = 0$, then the problem degenerates into the standard single-player de Finetti problem, and the optimal strategy is $\tilde{D}$ (and player 2 would stop as soon as X hits B). On the other hand, if player 1 has sufficient evidence of the existence of a competitor, i.e., if $p \in [\hat{p}, 1]$ where $\hat{p} = (V'(0) - 1)/V'(0)$ as in (3.17), then the agent extracts the whole resource immediately, and the game terminates at $t = 0$. The most interesting

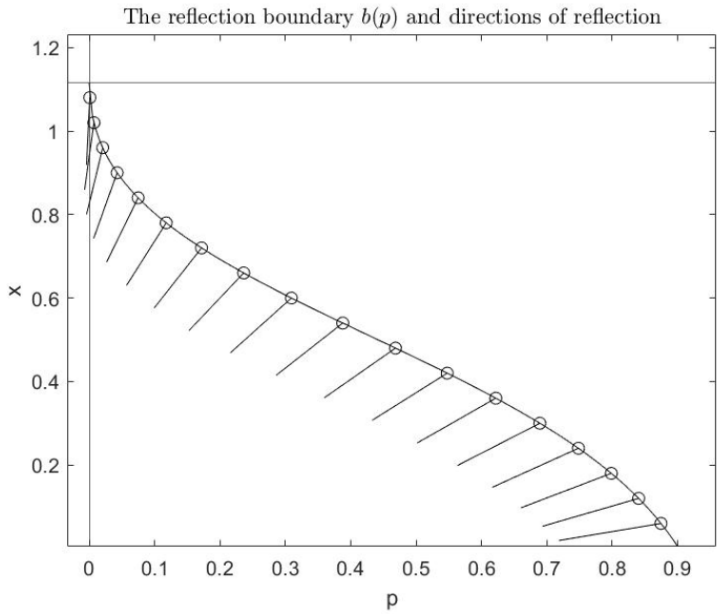

FIG. 3. *The boundary $p \mapsto b(p)$ and the direction of reflection for the equilibrium process (X^*, Π^*). The top horizontal line is $x = B \approx 1.1$ and represents the level at which to exert the control in the single-player de Finetti problem.*

scenario is when $p \in (0, \hat{p})$. In this case, the NE described in Theorem 5.1 amounts to a (possible) initial lump sum extraction of size $(x - b(p))^+$, and then continuous extraction so as to reflect the two-dimensional process (X^*, Π^*) along the boundary b, with reflection in the prescribed direction $(u_p, -u_x)$. Figures 1 and 2 are derived with initial values $p = 0.8 \cdot \hat{p} \approx 0.72$ and $x = \frac{b(p)}{2} \approx 0.13$, putting us in the last of the three cases above.

Figure 3 shows the boundary $p \mapsto b(p)$ (or, equivalently, $x \mapsto c(x)$) together with the direction of reflection of the equilibrium process (X^*, Π^*). Note that $b(0) = B$ and $b(\hat{p}) = 0$. Figures 1 and 2 show a simulated path of the equilibrium process (X^*, Π^*) and the corresponding processes Π^*, Γ^*, and D^*, respectively. Flat portions of Γ^*, Π^*, and D^* correspond to X^* being strictly below the boundary $b(\Pi^*)$. Note also that in Figure 1, the process Π^* reaches 0 in finite time, ruling out the existence of a competitor playing the equilibrium strategy if they did not stop yet; see Remark 5.2.

REFERENCES

[1] S. Asmussen and M. Taksar, *Controlled diffusion models for optimal dividend pay-out*, Insurance Math. Econom., 20 (1997), pp. 1–15.

[2] R. J. Aumann, M. Maschler, and R. E. Stearns, *Repeated Games with Incomplete Information*, MIT Press, 1995.

[3] E. Bayraktar and Y.-J. Huang, *On the multidimensional controller-and-stopper games*, SIAM J. Control Optim., 51 (2013), pp. 1263–1297, https://doi.org/10.1137/110847329.

[4] A. Bovo, T. De Angelis, and E. Issoglio, *Variational Inequalities on Unbounded Domains for Zero-Sum Singular-Controller vs. Stopper Games*, preprint, arXiv:2203.06247, 2022.

[5] P. Carmona, F. Petit, and M. Yor, *Beta variables as times spent in* $[0, \infty[$ *by certain perturbed Brownian motions*, J. Lond. Math. Soc. (2), 58 (1998), pp. 239–256.

[6] T. De Angelis and E. Ekström, *Playing with ghosts in a Dynkin game*, Stochastic Process. Appl., 130 (2020), pp. 6133–6156.

[7] T. De Angelis and G. Ferrari, *Stochastic nonzero-sum games: A new connection between singular control and optimal stopping*, Adv. Appl. Probab., 50 (2018), pp. 347–372.

[8] E. Ekström, K. Lindensjö, and M. Olofsson, *How to detect a salami slicer: A stochastic controller-and-stopper game with unknown competition*, SIAM J. Control Optim., 60 (2022), pp. 545–574, https://doi.org/10.1137/21M139044X.

[9] J. C. Harsanyi, *Games with incomplete information played by "Bayesian" players.* I. *The basic model*, Manag. Sci., 14 (1967), pp. 159–182.

[10] D. Hernandez-Hernandez, R. S. Simon, and M. Zervos, *A zero-sum game between a singular stochastic controller and a discretionary stopper*, Ann. Appl. Probab., 25 (2015), pp. 46–80.

[11] J. Hirshleifer and J. Riley, *The Analytics of Uncertainty and Information*, Cambridge University Press, 1992.

[12] M. Jeanblanc and A. N. Shiryaev, *Optimization of the flow of dividends*, Uspekhi Mat. Nauk, 50 (1995), pp. 25–46 (in Russian); translation in Russian Math. Surveys, 50 (1995), pp. 257–277.

[13] I. Karatzas and W. Sudderth, *Stochastic games of control and stopping for a linear diffusion*, in Random Walk, Sequential Analysis and Related Topics: A Festschrift in Honor of Yuan-Shih Chow, World Scientific, 2006, pp. 100–117.

[14] I. Karatzas and I.-M. Zamfirescu, *Martingale approach to stochastic differential games of control and stopping*, Ann. Probab., 36 (2008), pp. 1495–1527.

[15] H. D. Kwon and H. Zhang, *Game of singular stochastic control and strategic exit*, Math. Oper. Res., 40 (2015), pp. 869–887.

[16] M. Perman and W. Werner, *Perturbed Brownian motions*, Probab. Theory Related Fields, 108 (1997), pp. 357–383.