Journal of Physics
Communications

TOPICAL REVIEW • OPEN ACCESS

A tale of analogies: a review on gravitomagnetic effects, rotating sources, observers and all that

To cite this article: Matteo Luca Ruggiero and Davide Astesiano 2023 *J. Phys. Commun.* **7** 112001

Journal of Physics Communications

OPEN ACCESS

RECEIVED
13 May 2023

REVISED
24 October 2023

ACCEPTED FOR PUBLICATION
2 November 2023

PUBLISHED
17 November 2023

TOPICAL REVIEW

A tale of analogies: a review on gravitomagnetic effects, rotating sources, observers and all that

Matteo Luca Ruggiero[1,2,*] and **Davide Astesiano**[3]

[1] Dipartimento di Matematica 'G.Peano', Università degli studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
[2] INFN—LNL, Viale dell'Università 2, 35020 Legnaro (PD), Italy
[3] Science Institute, University of Iceland, Dunhaga 3, 107 , Reykjavík, Iceland
[*] Author to whom any correspondence should be addressed.

E-mail: matteoluca.ruggiero@unito.it and dastesiano@hi.is

Keywords: gravitomagnetism, general relativity, rotation effects

Abstract

Gravitoelectromagnetic analogies are somewhat ubiquitous in General Relativity, and they are often used to explain peculiar effects of Einstein's theory of gravity in terms of familiar results from classical electromagnetism. Perhaps, the best known of these analogy pertains to the similarity between the equations of electromagnetism and those of the linearized theory of General Relativity. But the analogy is somewhat deeper and ultimately rooted in the splitting of spacetime, which is preliminary to the definition of the measurement process in General Relativity. In this paper we review the various approaches that lead to the introduction of a magnetic-like part of the gravitational interaction, briefly called *gravitomagnetic* and, then, we provide a survey of the recent developments both from the theoretical and experimental viewpoints.

1. Introduction

The close similarity between Newton's and Coulomb's laws prompted to investigate further analogies between electromagnetism and gravitation: as reported by McDonald [1], Maxwell himself considered the possibility that gravity could be described by a vector field, but he was puzzled by the (negative) sign of the energy of static gravitational configurations. A further step was the Maxwellian-like theory of gravity formulated by O. Heaviside [2], which has recently received interest for pedagogical purposes [3]: in this theory, the magnetic part of the gravitational field originates from mass currents. This analogy obtained a natural formulation in the framework of Relativity: in fact, since we know that magnetic interactions can be explained using electrostatics and Special Relativity (SR), it is expected that if we want to put toghether Newtonian gravity and Lorentz invariance, the presence of a magnetic-like component of the gravitational field or, for short, a *gravitomagnetic field*, is mandatory. As reported by Pfister [4], before completing General Relativity (GR), Einstein investigated the existence of a gravitational analogue of electromagnetic induction [5, 6] and, in addition, he suggested the presence of a Coriolis-like force inside a rotating spherical mass shell, which provokes *dragging* effects on test masses. Gravitomagnetic effects where subsequently investigated in the framework of GR. In particular, it is relevant to emphasise that the analogy between Einstein's equations in the weak-field and slow-motion approximation and Maxwell's equations was formulated for the first time by Thirring [7, 8]. Afterwards, Thirring calculated the dragging effects inside a rotating mass shell and, in collaboration with J. Lense, they solved perturbatively the equation of motion of a test particle in the field of a rotating mass, from which the so-called Lense-Thirring effect [9] originated (see Pfister [4] for a critical analysis of the content of these papers). However, it was soon clear that these effects are much smaller than the leading Newtonian gravitational ones, which can be called *gravitoelectric* in the spirit of the analogy: indeed, in his letter to Thirring [10], Einstein affirmed that these effects 'remain far below any observable quantity'. This is a consequence of the fact that both the speeds of the sources and that of the test particles (except for massless ones) must be compared to the speed of light, so that the overall effect is very small when we are in linearised GR; on the other hand, more favourable conditions can be met when the gravitational field is strong, which happens for instance for pulsars, black holes and other extreme astrophysical situations.

Nonetheless, since the very beginning of the space exploration era, proposals have been made to exploit space technologies to test these effects: in fact, the conceptual design of the Stanford Gravity Probe B (GPB) mission was proposed to NASA in 1961, but its launch took place only in 2004 and the final results were published in 2011 [11]. In particular, the gravitomagnetic precession of the gyroscope axis was measured by GPB with 19% precision. More recently, Ciufolini [12] suggested to use laser-ranging to measure the modification of the satellites orbits determined by the Earth rotation and obtained initial confirmation of the prevision of Einstein's theory with 10% precision [13], with subsequent improvements [14]. A comprehensive analysis of the attempt to measure the Lense-Thirring and related effects in the Solar System can be found in the review written by Iorio *et al* [15].

The purpose of this paper is to examine the recent developments in the study of the gravitomagnetic effects and in the search of observational tests. Actually, in the literature there are already several reviews and monographs on this subject, which we refer to for a comprehensive discussion (see e.g. Pfister [16], Ciufolini and Wheeler [17], Ruggiero and Tartaglia [18], Mashhoon [19], Iorio [4]); as for us, we will focus on the findings and proposals published during the last twenty years, with the aim to provide an up-to-date reference for all researchers working in this field. We specify that we have limited ourselves to considering gravitomagnetic effects in GR: there are, however, investigations also in gravitational theories that can be considered as generalizations of Einstein's theory. Just to mention few of them, we refer to the Lorentz-violating Standard-Model Extension [20, 21], $f(R)$ and scalar-tensor theories [22], non local gravity [23], Chern-Simons gravitational theory [24].

The paper is organised as follows: in section 2 we review the basic theoretical foundations of the gravitomagnetic analogy, and its different meanings, then in section 3 we resume recent theoretical progresses. Afterwords, we focus on the various proposals aimed at the measurements of gravitomagnetic effects: historically, the first attempts were made in the space around the Earth or in the Solar System, which is still a lively scenario for these purposes, as we discuss in section 4. The continuous technological improvements made it possibile to consider the feasibility of Earth-based experiments which are examined in section 5. The exploration of deep space lends itself to analyse various astrophysical events that can be used to verify the predictions of GR and, in particular, for testing gravitomagnetic effects: this is the topic of section 6. Another possibility is offered by analogue models, which arise in different contexts in physics: some of them, which are relevant for gravitomagnetic effects, are discussed in section 7. The interplay between quantum phenomena and GR is explored in section 8, while in section 9 we review recent interpretations of gravitmagnetic effects in view of the Mach principle.

2. Basic theoretical framework

The term gravitomagnetism is probably due to Thorne [25], even though the analogy with electromagnetism had already been used by Forward [26] to refer to a formalism useful to deal with GR 'in experimentally realisable conditions', that is to say when the gravitational field is weak and the speeds are much smaller than speed of light: actually, this is the well known *gravitoelectromagnetic* (GEM) analogy between the linearised version of GR and electromagnetism, which is ultimately rooted in the 'space plus time' splitting of electromagnetism in flat spacetime.

Actually, as keenly observed by Jantzen *et al* [27], gravitoelectromagnetism has 'many faces', since the description of GR effects in analogy with electromagnetism is somewhat ubiquitous. Here we briefly summarise the relevant features of these analogies, and refer to the works by Jantzen *et al* [27, 28], Lynden-Bell and Nouri-Zonoz [29], Bini and Jantzen [30], De Felice and Bini [31] for a thorough description of the approaches to spacetime splitting and the related gravitoelectromagnetic formalisms, and to the papers by Costa and Natário [32], Costa and Natário [33] for a comprehensive account of the various gravitelectromagnetic analogies. In addition, the role of rotating observers is thoroughly discussed in the monograph edited by Rizzi and Ruggiero [34].

In particular, we broadly follow the presentation of Costa and Natário [33] to emphasize the different 'levels' of the analogy which, starting from the potentials appearing in the spacetime metric, involves the fields, i.e. derivatives of the metric elements and, in the end, the tidal tensors, which are made of the derivatives of the fields.

Before starting, we believe that it is important to answer two important questions: why and to what extent gravitoelectromagnetism is useful? The power of analogies in science has been known since the time of Kepler, who wrote:

And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature, and they ought to be least neglected in Geometry.

In our case, analogies are useful and fruitful because they allow to get a better intuition of what happens in the 4-dimensional geometry of GR, with the possibility to understand new effects in terms of known ones: for instance, the celebrated Lense-Thirring precession is analogous to the precession of a magnetic dipole in the magnetic field [16]. On the other hand, it should be clear that there are no motivations to believe that gravitational interactions can be *completely* described in analogy with electromagnetism, since there are deep differences between the two theories: indeed, as we are going to discuss, there are limitations in the analogies that should be taken into account.

The plan of this short outlook on gravitoelectromagnetic analogies is as follows: in section 2.1 we resume some basic ideas of spacetime splitting, which enables to show that relativistic dynamics, once that a class of observers has been defined, can be naturally described using a gravitoelectromagnetic analogy in full GR, without approximation; to introduce the analogy, we use the Sagnac effect, which can be understood in terms of the action of a gravitomagnetic potential on the motion of test particles (section 2.2). The well known gravitoelectromagnetic analogy arising in linearized GR (section 2.3) can be seen as a particular case of the full theory approach , when a particular class of observers (those at rest at infinity) is considered. Eventually, in section 2.4 we focus on the gravitoelectromagnetic analogy that can be build using the components of the curvature tensor.

2.1. Splitting spacetime

The idea to split spacetime into *space plus time* is fundamental in GR to get a better understanding, on the basis of our 3-dimensional experience, of what happens in the 4-dimensional geometry. To do this, in a spacetime manifold $\mathcal{V}_4$ we consider a congruence Γ of timelike (future-oriented) worldlines that can be thought of as the four-velocity field of a set of test observers filling the spacetime and performing measurements: in this sense, it is possible to say that Γ constitute the physical reference frame. It seems therefore worthwhile examining the propertiers of the pair $(\mathcal{V}_4,\ \Gamma)$ on geometrical grounds.

There are different approaches to spacetime splitting in the literature (see Jantzen *et al* [27, 28], Bini and Jantzen [30], De Felice and Bini [31]); we use here Cattaneo's approach [35], which we appreciate for its clarity and physical-mathematical insight. In particular, our introduction to Cattaneo's splitting is largely taken from the paper written by Rizzi and Ruggiero [36] (see also references therein).

Given a time-like congruence Γ it is always possible to choose a system of admissible coordinates so that the lines $x^0 = var$ coincide with the lines of Γ ; in this case, such coordinates are said to be adapted to the physical frame defined by the congruence Γ. In physical terms, the observers are at rest in this system of coordinates. This splitting approach allows to describe kinematics and dynamics in GR in 'relative' terms, in the very spirit of Einsteinian approach, since they are expressed relatively to the reference frame Γ. Let us remark that the entire approach can be applied as well to flat spacetime, when we consider a metric adapted to a congruence of non inertial observers.

Let γ^μ be the components of the field of unit vectors tangent to the world-lines of the congruence Γ in adapted coordinates, parameterized by λ. The spatial coordinates[4] x^i are constant along the lines of Γ and $dx^i = 0$ along any line of Γ; the same holds for the components $\gamma^i = \frac{dx^i}{d\lambda}$. The component γ^0 directly comes from the condition $g_{\mu\nu}\gamma^\mu\gamma^\nu = -1$:

$$\gamma^0 = \frac{1}{\sqrt{-g_{00}}}. \tag{1}$$

The covariant components can be found in the usual way

$$\gamma_0 = g_{0\mu}\gamma^\mu = g_{00}\gamma^0 = -\sqrt{-g_{00}}, \tag{2}$$

$$\gamma_i = g_{i\mu}\gamma^\mu = g_{i0}\gamma^0. \tag{3}$$

Summarizing, we have

$$\begin{cases}\gamma^0 = \dfrac{1}{\sqrt{-g_{00}}},\\ \gamma^i = 0,\end{cases} \qquad \begin{cases}\gamma_0 = -\sqrt{-g_{00}},\\ \gamma_i = g_{i0}\gamma^0.\end{cases} \tag{4}$$

[4] Latin indices run from 1 to 3, and refer to space components, while Greek indices run from 0 to 3, and label spacetime components. The signature of spacetime is $(-1, 1, 1, 1)$.

The most general coordinates transformation which does not change the physical frame, i.e. the congruence Γ, has the form:

$$\begin{cases} x'^0 = x'^0(x^0, x^1, x^2, x^3), \\ x'^i = x'^i(x^1, x^2, x^3), \end{cases} \tag{5}$$

with the additional condition $\partial x'^0/\partial x^0 > 0$, which ensures that the change of time parameterization does not modify the arrow of time. The coordinates transformation (5) is said to be internal to the physical frame Γ.

At each point p in spacetime, the tangent space T_p can be split into the direct sum of two subspaces: Θ_p, spanned by γ^α, the local time direction, and Σ_p, the 3-dimensional subspace which is orthogonal to Θ_p; Σ_p is called local space platform. Accordingly, the tangent space can be written as the direct sum

$$T_p = \Theta_p \oplus \Sigma_p. \tag{6}$$

Let $\{e_\mu\}$ be a basis of T_p. A vector $v = v^\mu e_\mu \in T_p$ can be projected onto Θ_p and Σ_p using the time projector

$$P_\Theta \doteq -\gamma_\mu \gamma_\nu \tag{7}$$

and the space projector

$$P_\Sigma \doteq \gamma_{\mu\nu} \doteq g_{\mu\nu} + \gamma_\mu \gamma_\nu \tag{8}$$

in the following way:

$$\begin{cases} \bar{v}_\mu \equiv P_\Theta(v_\mu) \doteq -\gamma_\mu \gamma_\nu v^\nu, \\ \tilde{v}_\mu \equiv P_\Sigma(v_\mu) \doteq \gamma_{\mu\nu} v^\nu = (g_{\mu\nu} + \gamma_\mu \gamma_\nu) v^\nu = v_\mu + v^\nu \gamma_\nu \gamma_\mu. \end{cases} \tag{9}$$

The subspaces Θ_p and Σ_p are defined by

$$\begin{cases} \Theta_p \equiv \{v^\mu \in T_p \mid \quad v^\mu = \lambda \gamma^\mu \quad \forall \lambda \in \mathbb{R}\}, \\ \Sigma_p \equiv \{v^\mu \in T_p \mid \quad g_{\mu\nu} v^\mu \gamma^\nu = 0\}, \end{cases} \tag{10}$$

The projectors P_Θ, P_Σ define the mappings

$$P_\Theta: \quad T_p \longrightarrow \Theta_p \qquad P_\Sigma: \quad T_p \longrightarrow \Sigma_p. \tag{11}$$

From equation (9), $\forall v^\mu \in T_p$ we have

$$v_\mu = \bar{v}_\mu + \tilde{v}_\mu = P_\Theta(v_\mu) + P_\Sigma(v_\mu). \tag{12}$$

This defines the natural splitting of a vector v^μ. The superscripts $^-$,$^\sim$ denote respectively a time vector and a space vector, or more generally, a time tensorand a space tensor, since the above described procedure can be applied to each tensor index.

To formulate the physical equations relative to the frame Γ, we need the transverse partial derivative defined by

$$\tilde{\partial}_\mu \doteq \partial_\mu + \gamma_\mu \gamma^0 \partial_0. \tag{13}$$

By definition, it is a space vector: $\tilde{\partial}_0 = \partial_0 + \gamma_0 \gamma^0 \partial_0 = 0$, since $\gamma_0 \gamma^0 = -1$. Accordingly, this operator can be used to define the transverse gradient: in fact for a generic scalar field $\varphi(x)$ we obtain:

$$P_\Sigma(\partial_\mu \varphi) = \tilde{\partial}_\mu \varphi. \tag{14}$$

The metric g and the fields

$$\gamma = \gamma^\mu \partial_\mu, \qquad \omega^0 = \gamma_\mu dx^\mu, \tag{15}$$

are basic geometrical quantities associated to the pair $(\mathcal{V}_4, \Gamma)$. Using differential operators we can generate first order geometrical objects from these fields

$$C = -\mathcal{L}_\gamma(\omega^0), \qquad \Omega = -2d\omega^0, \qquad K = \mathcal{L}_\gamma(g), \tag{16}$$

where $\mathcal{L}_V$ is the Lie derivative with respect to the field V. These new objects are respectively called the curvature vector, the vortex tensor and the Killing tensor. Using the splitting procedure defined above we get

$$C_\mu = \tilde{C}_\mu, \tag{17}$$

$$\Omega_{\mu\nu} = \tilde{\Omega}_{\mu\nu} + C_\mu \gamma_\nu - C_\nu \gamma_\mu, \tag{18}$$

$$K_{\mu\nu} = \tilde{K}_{\mu\nu} - \gamma_\mu C_\nu - \gamma_\nu C_\mu. \tag{19}$$

Using the adapted coordinates as before, defined by the condition $\gamma(x^i) = 0$ where $i = 1, 2, 3$, we find

$$\tilde{C}_\mu = \gamma^\nu \nabla_\nu \gamma_\mu, \tag{20}$$

$$\widetilde{\Omega}_{\mu\nu} = \gamma_0\left[\widetilde{\partial}_\mu\left(\frac{\gamma_\nu}{\gamma_0}\right) - \widetilde{\partial}_\nu\left(\frac{\gamma_\mu}{\gamma_0}\right)\right], \tag{21}$$

$$\widetilde{K}_{\mu\nu} = \gamma^0\partial_0\gamma_{\mu\nu}. \tag{22}$$

$\widetilde{C}_\mu$ is the curvature vector[5] of the curve $x^0 = var$ of the congruence. When this vector is null, the curve is geodesic. If this is true for all curves of the congruence Γ, the frame is freely falling (and the congruence is said to be geodesic). $\widetilde{\Omega}_{\mu\nu}$ is the space vortex tensor which gives the local angular velocity of the reference fluid. When this tensor is null, the frame is said to be non rotating or *time-orthogonal*; for our purposes, it is relevant to point out that this tensor is not null when $g_{0i} \neq 0$. Actually, it is possible to show [37] that this tensor is simply related to velocity of rotation of the particle of the congruence relative to a Fermi-Walker frame, which is the standard for a non rotating frame in GR: consequently, when the vortex tensor is null, the coordinates are adapted to a non rotating frame. Eventually, $\widetilde{K}_{\mu\nu}$ is the Born space tensor, which gives the deformation rate of the reference fluid. This tensor provides the rate of deformation for the reference fluid: When this tensor equals zero, the frame is described as 'rigid'. Specifically, the Born tensor is zero when the metric components do not exhibit any time dependence. The natural decomposition of the covariant vector field γ becomes

$$\nabla_\mu\gamma_\nu = \frac{1}{2}(\tilde{\Omega}_{\mu\nu} + \tilde{K}_{\mu\nu}) - \gamma_\mu C_\nu. \tag{23}$$

We need some definitions to explain the relative formulation of kinematics and dynamics. To begin with, let us consider two infinitesimally close events in spacetime, whose coordinates are x^α and $x^\alpha + dx^\alpha$. We can introduce the definitions of the standard relative time interval

$$dT = -\frac{1}{c}\gamma_\mu dx^\mu, \tag{24}$$

and the standard relative space element

$$d\sigma^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta \equiv \gamma_{ij}dx^i dx^j. \tag{25}$$

These quantities are clearly reliant on the physical frame established by the vector field $\gamma^\mu(x)$ and establish the temporal and spatial intervals as perceived by an observer within the congruence. As we will demonstrate, they have a pivotal role in the standard relative formulation of a particle's kinematics and dynamics in either an inertial or gravitational field. By its very definition as indicated in equation (24), the standard relative time interval is found to be an invariant within spacetime. It represents the projection of the vector dx^μ along the vector of the congruence γ^μ. By using equations (24) and (25) the spacetime invariant ds^2 can be written in the form

$$ds^2 = d\sigma^2 - c^2dT^2. \tag{26}$$

Now, we are in position to define and describe the motion as seen by the observers in the congruence: as before, we follow the approach outlined by Rizzi and Ruggiero [36]. A point particle P is at rest in Γ if its world-line coincides with one of the lines of the congruence. On the contrary, when the world-line of P does not coincide with any of the lines of Γ, the particle is said to be in motion: in this case, since $dx^i \neq 0$, we can write the parametric equation of the world-line of P in terms of a parameter λ, $x^\alpha = x^\alpha(\lambda)$. Let dP denote the infinitesimal displacement of the particle P. It is either time-like or light-like and in both cases $dT \neq 0$, so we can express the world-coordinates of the moving particle using the standard relative time as a parameter: $x^\alpha = x^\alpha(T)$.

The expression

$$v^\alpha = \frac{dx^\alpha}{dT}$$

defines the relative 4-velocity of a particle in motion with respect to the physical frame Γ. Accordingly, we call standard relative velocity its space projection

$$\tilde{v}_\beta \doteq P_\Sigma(v_\beta) = \gamma_{\beta\alpha}\frac{dx^\alpha}{dT} = \gamma_{\beta i}\frac{dx^i}{dT}. \tag{27}$$

Since $\tilde{v}_\beta \in \Sigma_p$, then $\tilde{v}_0 = 0$. The contravariant components of the standard relative velocity are

$$\tilde{v}^i = \frac{dx^i}{dT} \quad \tilde{v}^0 = -\gamma_i\frac{\tilde{v}^i}{\gamma_0}, \tag{28}$$

[5] Where

$$\widetilde{C}_\mu = \gamma^\nu\nabla_\nu\gamma_\mu = \frac{dx^\nu}{ds}\frac{D\gamma_\mu}{dx^\nu} = \frac{D\gamma_\mu}{ds}.$$

(because $\gamma_\alpha \tilde{v}^\alpha = 0$). As a consequence, equation (27) can be written as

$$\tilde{v}_i = \gamma_{ij}\tilde{v}^j. \tag{29}$$

The (space) norm of the standard relative velocity is

$$\| v^\mu \|_\Sigma \doteq \tilde{v}^2 = \gamma_{ij}\tilde{v}^i\tilde{v}^j = \frac{d\sigma^2}{dT^2}. \tag{30}$$

For a photon, since $ds^2 = 0$, we get from equation (26) $\tilde{v}^2 = c^2$, which is the same result obtained SR: this is not surprising, since locally GR reduces to SR. When we consider material particles, we can introduce the proper time

$$d\tau^2 = -\frac{1}{c^2}ds^2, \tag{31}$$

and, using equations (24) and (30) we may write

$$\frac{dT}{d\tau} = \frac{1}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}} \tag{32}$$

Again, we notice that we obtain a result that is formally identical to the special relativistic one in terms of the Lorentz factor $\frac{1}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}}$, in which the relative velocity appears.

After introducing the quantities that enable to describe relative kinematics, we can pass to relative dynamics: in other words, we need to formulate the geodesic equations in relative terms To this end, let us consider the connection coefficients, in the coordinates $\{x^\mu\}$ adapted to the physical frame, $\Gamma^\alpha{}_{\beta\gamma}$. The geodesic equations are written as

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}U^\beta U^\gamma = 0, \tag{33}$$

in terms of the 4-velocity

$$U^\alpha = \frac{dx^\alpha}{d\tau}$$

and the proper time τ. Let m_0 be the proper mass of the particle: then the energy-momentum 4-vector is

$$P^\alpha = m_0 U^\alpha = m_0\frac{dx^\alpha}{d\tau}$$

Now we want to re-formulate the geodesic equations in their relative form, i.e. by means of the standard relative quantities that we have introduced so far. To this end, let us define the standard relative momentum

$$\tilde{p}_\alpha \doteq P_\Sigma(P_\alpha) = \gamma_{\alpha\beta}P^\beta = m_0\gamma_{\alpha i}\frac{dx^i}{dT}\frac{dT}{d\tau} = m\tilde{v}_\alpha, \tag{34}$$

where we introduced the standard relative mass

$$m \doteq \frac{m_0}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}} \tag{35}$$

in formal analogy with SR. Since $\tilde{p}_\alpha \in \Sigma_p$, then $\tilde{p}_0 = 0$. If we deal with massless particles, we can proceed as follows to formally define the momentum 4-vector: consider, for instance, a 'classical', i.e. perfectly localized monochromatic photon, we can set

$$\lim_{m_0\to 0}\left(m_0\frac{dT}{d\tau}\right) = \frac{h\nu}{c^2} \tag{36}$$

and obtain

$$P^\alpha = \frac{h\nu}{c^2}\frac{dx^\alpha}{dT} \tag{37}$$

where h is the Planck constant and, in terms of relative quantities, the relation that links the wavelength and the frequency of the photon to the speed of light is $\lambda\nu = \frac{d\sigma}{dT} = c$.

It is possible to show (see again, Rizzi and Ruggiero [36] and references therein) that the space projection of the geodesic equations can be written as

$$\frac{\hat{D}\tilde{p}_i}{dT} = m\tilde{G}_i \tag{38}$$

where $\frac{\hat{D}}{dT}$ is a suitably defined derivative operator. In other words, the variation of the space momentum vector is determined by the field $\widetilde{G}_i$:

$$\widetilde{G}_i = -c^2\widetilde{C}_i + c\widetilde{\Omega}_{ij}\widetilde{v}^j \tag{39}$$

It is often useful to split the field $\widetilde{G}_i$ into the sum of two fields $\widetilde{G'}_i$, $\widetilde{G''}_i$, defined as follows:

$$\widetilde{G'}_i \doteq -c^2\widetilde{C}_i = -c^2\left[\gamma_0\tilde{\partial}_i\gamma^0 - \partial_0\left(\frac{\gamma_i}{\gamma_0}\right)\right]$$
$$\widetilde{G''}_i \doteq c\widetilde{\Omega}_{ij}\widetilde{v}^j \tag{40}$$

The field $\widetilde{G'}_i$ can be interpreted as a dragging field (c^2C_α is the 4-acceleration a_α of the particles of the reference fluid) and the field $\widetilde{G''}_i$ can be interpreted as a Coriolis-like field.

The contravariant form of equation (38) (see also (39)) turns out to be:

$$\frac{\hat{D}\widetilde{p}^i}{dT} = -mc^2\widetilde{C}^i + mc(\widetilde{\Omega}^i_{\ j} - \widetilde{K}^i_{\ j})\widetilde{v}^j. \tag{41}$$

The above description of relative dynamics holds true in full GR and it can be explained in terms of $\widetilde{G}_i$ which, in turn, can be seen as the sum of the dragging field $\widetilde{G'}_i$, related to the curvature of the congruence, and the velocity dependent field $\widetilde{G''}_i$, related to vorticity of the congruence. It is relevant to point out that only the field $\widetilde{G}_i$ is invariant with respect to the internal transformations (5). This formalism lends itself to introduce an analogy with the electromagnetic dynamics: in other words, the velocity dependent field can be interpreted as a magnetic-like force, while the dragging term can be interpreted as an electric-like force, whence the gravitoelectromagnetic analogy. In order to see how this can be done, in next section we discuss the Sagnac effect [38] which, as we are going to show, can be interpreted as the consequence of the action of a gravitomagnetic potential, in analogy with the well known Aharonov–Bohm effect [39].

2.2. The Sagnac effect and the emerging gravitomagnetic analogy

We are in position to see how it is possible to describe some known relativistic effects in spacetime, on the basis of a gravitoelectromagnetic analogy deriving from the splitting approach described above. To this end, we consider the line-element in the form

$$ds^2 = g_{00}c^2dt^2 + 2g_{0i}cdtdx^i + g_{ij}dx^idx^j, \tag{42}$$

and we suppose that the metric elements do not depend on time. The above metric is quite general in its form and, in particular, it said to be non time-orthogonal, because $g_{0i} \neq 0$. We focus on the asymmetry in the propagation times of two signals in the spacetime described by the line-element (42); this asymmetry is the so-called Sagnac time delay, which we describe here with extensive reference to the papers by Rizzi and Ruggiero [40], Rizzi and Ruggiero [36], Ruggiero and Tartaglia [41], Tartaglia and Ruggiero [42].

Let us consider two massive or massless particles simultaneously emitted at a given location: they propagate in opposite directions along the same path and reach the emission point at different times. It is possible to demonstrate that for both massless particles and massive particles moving at equal speeds in opposite directions, the variation of their coordinate propagation time can be expressed as follows:

$$\Delta t = \frac{2}{c}\oint_\ell \frac{g_{0i}dx^i}{|g_{00}|} = -\frac{2}{c}\oint_\ell \frac{g_{0i}dx^i}{g_{00}}. \tag{43}$$

For instance, let us suppose that spacetime is axially symmetric: accordingly, the metric (42) can be written in adapted (cylindrical) coordinates in the form

$$ds^2 = g_{00}c^2dt^2 + 2g_{0\varphi}cdtd\varphi + g_{rr}dr^2 + g_{\varphi\varphi}d\varphi^2 + g_{zz}dz^2 \tag{44}$$

In particular, the line element in a uniformly rotating frame of reference in flat Minkowski spacetime is written in this form

$$ds^2 = -\left(1 - \frac{\omega^2r^2}{c^2}\right)c^2dt^2 + 2\frac{\omega r^2}{c}cdtd\varphi + dr^2 + r^2d\varphi^2 + dz^2, \tag{45}$$

where ω is the constant rotation rate. From the above metric, it is possible to derive the classical effect that was first pointed out by Sagnac and is now currently used in laser gyros [43]: in particular for a circular path of radius R we get the following proper time difference

$$\Delta\tau = \frac{4\pi R^2\omega}{c^2}\left(1 - \frac{\omega^2 R^2}{c^2}\right)^{-1/2} \tag{46}$$

(see e.g. the monograph edited by Rizzi and Ruggiero [34]).

If we start from the metric (44), perform the (local) coordinates transformation $\varphi' = \varphi - \Omega t$, and set, for given r_0, z_0, $\Omega = -\frac{cg_{0\varphi}}{g_{\varphi\varphi}}$, then in the new metric $g'_{\varphi t} = 0$ and the observers at $r = r_0, z = z_0$ do not experience any Sagnac effect. These observers are called Zero Angular Momentum Observers (ZAMO), so we see that they are not rotating with respect to the local spacetime geometry: differently speaking, their vorticity (21) is zero. It is easy to understand why they are called ZAMO: in fact, from the metric (44) we can write the Lagrangian

$$\mathcal{L} = \frac{1}{2}(g_{00}c^2\dot{t}^2 + 2g_{0\varphi}c\dot{t}\dot{\varphi} + g_{rr}\dot{r}^2 + g_{\varphi\varphi}\dot{\varphi}^2 + g_{zz}\dot{z}^2), \tag{47}$$

where dot means derivation with respect to an affine parameter λ. The angular momentum is given by

$$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = cg_{0\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}. \tag{48}$$

On setting $\Omega = \frac{d\varphi}{dt}$, we see that when $\Omega = -\frac{cg_{0\varphi}}{g_{\varphi\varphi}}$ the angular momentum is null.

If the metric (44) is inertial (i.e. Minkowski) at infinity, we see that the observers at rest have a non zero angular momentum and measure a Sagnac effect; on the other hand, the ZAMO are moving in the metric (44), but measure no Sagnac effect. This the so-called frame-dragging, which means that the ZAMO are dragged by the spacetime metric, and this effect depends on the non diagonal elements of the metric, hence on its vorticity (see also the discussion in section 9).

The spacetime metric (42) can be written, using the standard relative time element and the standard relative space element in the form (26); in particular, we may write

$$dT = -\frac{1}{c}\gamma_\alpha dx^\alpha = -\frac{1}{c}\gamma_0\left(dx^0 + \frac{\gamma_i}{\gamma_0}dx^i\right) = \frac{1}{c}\sqrt{-g_{00}}\left(dx^0 + \frac{g_{0i}}{g_{00}}dx^i\right). \tag{49}$$

So, if we set

$$A_i^G = c^2\frac{\gamma_i}{\gamma_0} = c^2\frac{g_{0i}}{g_{00}}, \tag{50}$$

the Sagnac effect (43) can be written in the form

$$\Delta t = -\frac{2}{c^3}\oint_\ell A_i^G dx^i. \tag{51}$$

As a consequence, the Sagnac effect can be interpreted as a gravitomagnetic Aharonov–Bohm [44, 45] effect, determined by the gravitomagnetic potential A_i^G. In addition, we may formally introduce the gravitoelectric potential

$$\phi^G = -c^2\gamma^0. \tag{52}$$

The use of this terminology, can be justified considering the equation of motion (38). In fact, it is possible to define the *gravitomagnetic field*[6]

$$\widetilde{B}_G^i \doteq (\widetilde{\boldsymbol{\nabla}} \times \tilde{\boldsymbol{A}}_G)^i \tag{53}$$

and the *gravitoelectric field*:

$$\widetilde{E}_i^G \doteq -(-\widetilde{\partial}_i\phi_G - \partial_0\widetilde{A}_i^G). \tag{54}$$

Then, the equation of motion (38) can be written in the form

$$\frac{\hat{D}\widetilde{p}_i}{dT} = m\widetilde{E}_i^G + m\gamma_0\left(\frac{\tilde{\boldsymbol{v}}}{c} \times \tilde{\boldsymbol{B}}_G\right)_i \tag{55}$$

which looks like the equation of motion of a particle acted upon by a generalized Lorentz force.

Consequently, a gravitoelectromagnetic analogy naturally emerges in full GR when we are dealing with a non time-orthogonal metric like (42). However, the great majority of experimental studies on GR is done in weak-field conditions, i.e. when the magnitude of the gravitational field allows a linearization of the relevant equations: this will be discussed in the next section.

[6] Here and henceforth boldface symbols refer to space vectors.

2.3. Gravitoelectromagnetic analogy in linearized general relativity

Einstein equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{56}$$

can be solved in the weak-field and slow-motion approximation: in this case the gravitational field can be considered as a perturbation of flat spacetime, described by the Minkowski tensor $\eta_{\mu\nu}$. Here and henceforth, we closely follow the approach given by Ruggiero [46] (see also references therein) to the solution of Einstein equations in this approximation.

As a consequence, the metric tensor can be written in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a perturbation: $|h_{\mu\nu}| \ll |\eta_{\mu\nu}|$. If we introduce $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$, where $h = h_\mu{}^\mu$ and perform a linear approximation, Einstein equations (56) become (see e.g. Straumann [47])

$$-\Box \bar{h}_{\mu\nu} - \eta_{\mu\nu} \bar{h}_{\alpha\beta}{}^{,\alpha\beta} + \bar{h}_{\mu\alpha,\nu}{}^{\alpha} + \bar{h}_{\nu\alpha,\mu}{}^{\alpha} = \frac{16\pi G}{c^4} T_{\mu\nu}. \tag{57}$$

The gauge freedom can be exploited setting the *Hilbert gauge condition*

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0. \tag{58}$$

Then, from (57) we get

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{59}$$

We remark that condition (58) can always be met through a gauge transformation. In fact, the Einstein equations remain invariant under infinitesimal transformations of this kind:

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}, \tag{60}$$

which, in terms of $\bar{h}_{\mu\nu}$ becomes

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu} \xi^\alpha_{,\alpha}. \tag{61}$$

As a consequence, when $\bar{h}^{\mu\nu}{}_{,\nu} \neq 0$, we may choose ξ^μ to be a solution of $\Box \xi^\mu = -\bar{h}^{\mu\nu}{}_{,\nu}$.

Equations (59) exhibit a manifest resemblance to Maxwell's equations for the electromagnetic four-potential. Consequently, they can be approached in a similar manner. Specifically, by disregarding the solution to the homogeneous wave equation, the general solution can be expressed using the concept of retarded potentials:

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4} \int_V \frac{T_{\mu\nu}(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{62}$$

In the above equation integration is extended to the volume V, containing the source. The components of the energy-momentum tensors are defined as $T^{00} = \rho c^2$ and $T^{0i} = cj^i$, in terms of the mass density ρ and mass current j^i of the source; as a consequence, $j^\mu = (c\rho, j_g^i) = (c\rho, \mathbf{j})$ is the mass-current four vector of the source. In linear approximation, $T^{\mu\nu}_{,\nu} = 0$, so we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{63}$$

To fix ideas, let us assume that the source consists of a finite distribution of slowly moving matter, with velocity $\mathbf{v}$ such that $|\mathbf{v}| \ll c$; consequently, we obtain that $T_{ij} \simeq \rho v_i v_j + p\delta_{ij}$, where p is the pressure. In particular, from equation (62) we see that $\bar{h}_{ij} = O(c^{-4})$: as a result, in this linear approach, we may neglect in the metric tensor terms that are $O(c^{-4})$.

In summary, the solution of equation (62) can be written in the form

$$\bar{h}_{00} = \frac{4G}{c^2} \int_V \frac{\rho(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{64}$$

$$\bar{h}_{0i} = -\frac{4G}{c^3} \int_V \frac{j^i(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \tag{65}$$

The other components of $\bar{h}_{\mu\nu}$ are zero at the given approximation level.

If we exploit the already mentioned analogy with electromagnetism, we may introduce the *gravitoelectromagnetic potentials*: namely, the gravitoelectric Φ and gravitomagnetic A^i potentials, defined by

$$\bar{h}_{00} \doteq 4\frac{\Phi}{c^2}, \quad \bar{h}_{0i} = -2\frac{A_i}{c^2}, \tag{66}$$

Taking into account equations (64) and (65), their expressions turn out to be

$$\Phi = G\int_V \frac{\rho(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\mathrm{d}^3x', \tag{67}$$

$$A_i = \frac{2G}{c}\int_V \frac{j^i(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\mathrm{d}^3x'. \tag{68}$$

In the end, the spacetime metric that characterizes the solutions of Einstein's equation in the weak-field approximation takes on the following form:

$$\mathrm{d}s^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}t^2 - \frac{4}{c}A_i\mathrm{d}x^i\mathrm{d}t + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{69}$$

It is useful to see the link between this *linear* gravitoelectromagnetic analogy and the one described before and valid in full theory. In the above metric, the weak-field approximation means that $\left|\frac{\Phi}{c^2}\right| \ll 1, \left|\frac{A_i}{c^2}\right| \ll 1$. In particular, we have $\frac{\Phi}{c^2} = \frac{1 + g_{00}}{2}$ and $\frac{A_i}{c^2} = -\frac{g_{0i}}{2}$. Now, if we consider the definition (52), in the weak-field approximation, we have

$$\phi^G = -c^2\gamma^0 = -c^2\frac{1}{\sqrt{-g_{00}}} = -c^2\frac{1}{\sqrt{1 - 2\frac{\Phi}{c^2}}} \simeq \Phi \tag{70}$$

Similarly, we have

$$A_i^G = c^2\frac{\gamma_i}{\gamma_0} = c^2\frac{g_{0i}}{g_{00}} \simeq -c^2 g_{0i} = 2A_i \tag{71}$$

As a consequence, apart from a 2 factor in the definition of gravitomagnetic effects, we obtain a correspondence between the two definitions.

Let us focus on how this analogy translates into the expression of the geodesic equations. Let us start from the line element (69) and calculate the geodesic equations up to linear order in $\boldsymbol{\beta} = \mathbf{v}/c$. From

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \tag{72}$$

we obtain for the space components [32, 48]:

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = \frac{\partial\Phi}{\partial x^i} - 2(\boldsymbol{\beta}\times\mathbf{B})_i + 2\frac{\partial A_i}{c\partial t} - 3\beta^i\frac{\partial\Phi}{c\partial t}. \tag{73}$$

Then, if we define the gravitoelectromagnetic fields as

$$\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}, \quad \mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{2}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{74}$$

the above equation (73) becomes

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -E^i - 2(\boldsymbol{\beta}\times\mathbf{B})_i - 3\beta^i\frac{\partial\Phi}{c\partial t}. \tag{75}$$

Accordingly, it is not warranted that the geodesic equations take a Lorentz-like form, due to the presence of the last term in equation (75), if the metric elements are time-depending [46]: this can be also seen from equation (41), where an additional term appears, depending on the Born tensor, which is not null for time-depending metrics. If no time-dependency is present, we have a Lorentz-like equation in the form

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -E^i - 2(\boldsymbol{\beta}\times\mathbf{B})_i. \tag{76}$$

In addition, starting from equation (59), it is possible to show [17, 18] that Einstein's equation can be formally written in analogy to Maxwell's equations for the gravitoelectromagnetic fields. However, as discussed by Costa and Natário [48], Ruggiero [32], Bini *et al* [46], when the metric is time-depending, it is not possible to obtain a one-to-one gravitoelectromagnetic analogy both for the geodesic equations and the field equations, since, in any case, non-Maxwellian terms appear (see also Bakopoulos and Kanti [49], Williams and Inan [50]).

Another limitation of this linear analogy is that there are no gravitoelectromagnetic waves: differently speaking, there is not a propagation of the metric components h_{0i} and h_{00}; in fact, the gauge condition (58) implies that $h^{0\alpha} = 0$ in the transverse and traceless (TT) frame (see e.g. Hobson *et al* [51], Chapter 18); hence, it becomes impossible to represent gravitoelectromagnetic waves within the TT frame, implying their nonexistence in any frame. This is because we can consistently cancel a stationary, uniform gravitational potential Φ and a gravitomagnetic vector potential A_i through a suitable coordinate transformation to a locally inertial frame.

The metric (69) is expressed in coordinates that are adapted to observers at rest at infinity: as we have seen, the gravitomagnetic effects derive from mass currents. To fix ideas, in the case of a localised source whose center of mass is at rest with respect to these observers, the gravitomagnetic potential is related to the angular momentum of the source. In particular, we get

$$\Phi = \frac{GM}{r}, \qquad A_i = \frac{G}{c}\frac{(\mathbf{J} \wedge \mathbf{x})_i}{r^3} \tag{77}$$

where $r = |\mathbf{x}|$, in terms of the mass M and angular momentum $\mathbf{J}$ of the source [17, 18]. On the other hand, as we have seen in section 2.2, gravitomagnetic effects arise also because of the rotation of the observers. To summarize, both the motion of the observers and that of the sources might contribute to the definition of the gravitomagnetic effects: a non null vortex tensor, in fact, is generally related to the rotation of the reference frame (see section 2.1).

2.4. Gravitoelectromagnetic analogy from the curvature tensor

We showed in the previous sections how to build a gravitoelectromagnetic analogy: in full GR this is possible using a splitting approach for a given congruence of observers; this approach reduces to the gravitoelectromagnetic formalism of linerized GR, when we consider inertial observers around a rotating localized source. Here, we focus on the gravitoelectromagnetic analogy that can be build using the components of the curvature tensor: in particular, we will show that, under suitable hypotheses, the geodesic equations take the Lorentz-like form when we use Fermi coordinates. The formulation of the spacetime element in Fermi coordinates depends on both the characteristics of the reference frame, such as the acceleration and rotation of the congruence, and the spacetime curvature, as influenced by the Riemann curvature tensor. Our focus is on the impacts of the curvature tensor, leading us to examine a geodesic and non-rotating frame. Nevertheless, it is important to note that, in general, there will be contributions stemming from the world-line acceleration and the tetrad rotation. (see e.g. Ruggiero and Ortolan [52]). The approach to gravitoelectromagnetism in Fermi coordinates which are describing here is largely taken from Ruggiero [53] (see also references therein).

If we use Fermi coordinates (cT, X, Y, Z), up to quadratic displacements $|X^i|$ from the reference world-line, the line element turns out to be (see e.g. Manasse and Misner [54], Misner *et al* [55])

$$ds^2 = -(1 + R_{0i0j}X^iX^j)c^2dT^2 - \frac{4}{3}R_{0jik}X^jX^kcdTdX^i + \left(\delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j. \tag{78}$$

Here, $R_{\alpha\beta\gamma\delta}(T)$ is the projection of the Riemann curvature tensor on the orthonormal tetrad $e^{\mu}_{(\alpha)}(\tau)$ of the reference observer, parameterized by the proper time[7] τ:
$R_{\alpha\beta\gamma\delta}(T) = R_{\alpha\beta\gamma\delta}(\tau) = R_{\mu\nu\rho\sigma}e^{\mu}_{(\alpha)}(\tau)e^{\nu}_{(\beta)}(\tau)e^{\rho}_{(\gamma)}(\tau)e^{\sigma}_{(\delta)}(\tau)$ and it is evaluated along the reference geodesic, where $T = \tau$ and $\mathbf{X} = 0$. If we set

$$\frac{\Phi}{c^2} = \frac{g_{00}+1}{2} \quad \frac{\Psi_{ij}}{c^2} = \frac{g_{ij}-\delta_{ij}}{2} \quad \frac{A_i}{c^2} = -\frac{g_{0i}}{2},$$

the spacetime metric (78) turns out to be

$$\mathrm{d}s^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}T^2 - \frac{4}{c}A_i\mathrm{d}X^i\mathrm{d}T + \left(\delta_{ij} + 2\frac{\Psi_{ij}}{c^2}\right)\mathrm{d}X^i\mathrm{d}X^j, \tag{79}$$

with the following definitions

$$\Phi(T, X^i) = -\frac{c^2}{2}R_{0i0j}(T)X^iX^j, \tag{80}$$

$$A_i(T, X^i) = \frac{c^2}{3}R_{0jik}(T)X^jX^k, \tag{81}$$

$$\Psi_{ij}(T, X^i) = -\frac{c^2}{6}R_{ikjl}(T)X^kX^l, \tag{82}$$

where Φ and A_i are, respectively, the gravitoelectric and gravitomagnetic potential, and Ψ_{ij} is the perturbation of the spatial metric. We point out that the line element (79) is a perturbation of flat Minkowski spacetime, that is to say $|\frac{\Phi}{c^2}| \ll 1, |\frac{\Psi_{ij}}{c^2}| \ll 1, |\frac{A_i}{c^2}| \ll 1$.

We see that even though the metric elements have a different meaning, the form of the spacetime interval in equation (79) is quite similar to equation (69): accordingly, the same consequences can be drawn for the geodesic equations, which we write up to linear order in $\boldsymbol{\beta} = \mathbf{V}/c$, where $V^i = \frac{\mathrm{d}X^i}{\mathrm{d}T}$. We define the gravitomagnetic field

[7] In $e^{\mu}_{(\alpha)}$ tetrad indices like (α) are within parentheses, while μ is a background spacetime index; however, for the sake of simplicity, we drop here and henceforth parentheses to refer to tetrad indices, which are the only ones used.

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}, \tag{83}$$

or, in terms of the curvature tensor

$$B_i(T, \mathbf{R}) = -\frac{c^2}{2}\epsilon_{ijk}R^{jk}{}_{0l}(T)X^l. \tag{84}$$

Accordingly, the space components of the geodesic equations are

$$\frac{d^2X^i}{dT^2} = \frac{\partial\Phi}{\partial X^i} - 2(\boldsymbol{\beta} \times \mathbf{B})_i + 2\frac{\partial A_i}{c\partial T} - 2\beta^j\frac{\partial\Psi_{ij}}{c\partial T} - \beta^i\frac{\partial\Phi}{c\partial T}. \tag{85}$$

In addition, exploiting once again the analogy with electromagnetism, we define the *gravitoelectric field*

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{2}{c}\frac{\partial\mathbf{A}}{\partial T}, \tag{86}$$

where, in terms of the curvature tensor, we have

$$E_i = c^2R_{0i0j}(T)X^j, \tag{87}$$

In summary, equation (85) becomes

$$\frac{d^2X^i}{dT^2} = -E^i - 2\left(\frac{\mathbf{V}}{c} \times \mathbf{B}\right)^i - 2\frac{V^j}{c}\frac{\partial\Psi_{ij}}{c\partial T} - \frac{V^i}{c}\frac{\partial\Phi}{c\partial T}. \tag{88}$$

Let us examine the meaning of equation (88) and relevance of the various terms First of all, it is important to stress that this equation defines the motion of a test mass with respect to the reference observer. Consequently, all quantities involved are *relative* to the reference observer at the origin of the frame. In addition, the geodesic equations do not take a Lorentz-like form if the fields are not static, due to the presence of the last terms which contain time-derivatives [46]. However, both terms—according to the definitions (80) and (82) - are quadratic in the displacements from the reference world-line. So, even if the fields are time-depending (such as in the case of a gravitational wave) we obtain the Lorentz-like force

$$\frac{d^2\mathbf{X}}{dT^2} = -\mathbf{E} - 2\left(\frac{\mathbf{V}}{c} \times \mathbf{B}\right), \tag{89}$$

if we confine ourselves to linear displacements from the reference world-line. In particular, in this case the gravitoelectric field turns out to be

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{90}$$

We see that a gravitoelectromagnetic analogy for the force equation holds true only if suitable hypotheses are assumed.

2.5. Summary

What we have shown in the previous sections is the possibility to describe gravitational dynamics in analogy with electromagnetism: in all cases, the dynamics of free particles is formally described by Lorentz-like equations (55), (76), (89), when suitable hypotheses are taken into account. The similarity with Maxwell's theory, then, allows to explain and investigate gravitational effects in terms of known electromagnetic ones, according to the reasonable principle for which similar equations lead to similar solutions. The presence of gravitomagnetic effects has no counterparts in Newtonian gravity and it is relevant not only from an experimental or observational point of view but, also, for fundamental reasons. For instance, the gravitational Larmor theorem [56] completes Einstein's formulation of the principle of equivalence, which can be rephrased in terms of the equivalence between the translational acceleration of the Einstein elevator and the Newtonian (i.e. gravitoelctric) field: however, a rotation of the elevator is needed to take into account the existence of the gravitomagnetic field.

The framework described so far is sufficient to encompass different approaches to the study of gravitomagnetic effects, both from a theoretical and experimental viewpoints. We refer to previous review works for a more detailed description of the mathematical and physical aspects of this formalism [16–18, 27, 28, 30, 32, 33].

3. Other theoretical developments

In the previous sections we have described the basic features of the gravitoelectromagnetic analogies; these theoretical foundations are the bases on which further progresses have been made, in different contexts, for the purpose to verify gravitomagnetic effects in experiments or observations.

The discovery of the first double pulsar [57] was an unprecedented opportunity to test GR effects; more generally speaking, pulsar astrophysics is an exciting laboratory to test relativistic gravity and, also, gravitomagnetic effects [58, 59]. In these systems we may evaluate the effect of the interaction of the spins of the sources and the orbital angular momentum: in particular O'Connell [60] focused on the possibility to measure spin-orbit effects in binary systems, which may produce not only a correction to the advance of periastron, but also a precession of the orbit about the spin direction. Surveys on general-relativistic spin effects in binary systems can be found in [61, 62]. Binary systems are relevant for gravitational waves physics: in this context Kaplan *et al* [63] showed that the Maxwell-like formalism can be useful to get physical insights into the numerical-relativity simulations of compact objects.

The analogy with electromagnetism can be applied to time-varying gravitomagnetic fields, as discussed by Mashhoon [64], who considered a linear temporal variation of the vector potential and analyzed its possible impact on some experimental tests, such as the gyroscope precession. Actually, it is possible to develop the analogy in order to introduce a gravitational induction law [48] which, for instance, can be used to understand higher order corrections in the interaction of a plane gravitational wave with a detector [53].

Costa *et al* [65] showed that the gravitomagnetic features can be used to distinguish (in the Weyl class of solutions describing the gravitational field of infinite cylinders) between the case of a static solution (the so-called Levi-Civita solution) and the one accounting for a rotating cylinder: in particular, in the latter case, a gravitomagnetic vector potential that cannot be eliminated by a global coordinate transformation is present. Herrera *et al* [66], Herrera [67] analysed the vorticity of the congruence of the observers world lines which, as we have seen above, is responsible for the dragging of inertial frames; their findings show that vorticity is related to the presence of a circular flow of super energy in the plane orthogonal to the vorticity vector, and this happens in stationary vacuum spacetimes and also in general Bondi-Sachs radiative ones.

The clock effect is another gravitomagnetic effect which refers to the difference in the proper time measurements of two clocks (freely) orbiting in opposite directions around a rotating source, and it was initially studied by Cohen and Mashhoon [68]. Lichtenegger *et al* [69] explained it in analogy with electromagnetism. Generalisations of this effect were discussed by Hackmann and Lämmerzahl [70], with possible applications to the GPS and geostationary satellites.

As we have seen in section 2.4, it is possible to obtain a gravitoelectromagnetic analogy based on the curvature tensor: a visualization tecnique of the electric-like and magnetic-like properties of the curvature tensor was developed by Owen *et al* [71], Nichols *et al* [72], Zhang *et al* [73], Nichols *et al* [74], with the aim to better understand the dynamics of compact objects merging.

Mashhoon [75] investigated the Stern–Gerlach force in a gravitomagnetic framework, in connection with spin-gravity coupling, and showed that it reduces to Mathisson's spin-curvature force. The dynamics of spinning particles in spacetime and the analogies with electromagnetism were investigated by Costa *et al* [76], and the case of the spacetime of a gravitational wave was considered by Bini *et al* [77]. A Stern–Gerlach force, together with a Faraday rotation, naturally arises in magnetized Kerr and Reissner-Nordström spacetimes [78], and this might have consequences in understanding the impact of both magnetic and gravitomagnetic fields on the propagation of electromagnetic signals in the strong field of compact objects. Magnetic helicity is related to the twist and braiding of the magnetic field lines: Bini *et al* [79] discussed gravitomagnetic helicity, both in the linear and spacetime curvature approaches to gravitoelectromagnetism.

The interplay between gravitational and electromagnetic field and its implications on gravitomagnetic effects was investigated by different authors. For instance, Ahmedov and Rakhmatov [80] considered the possibility that the interaction of the gravitomagnetic field with the electric field could lead to new measurements strategies. In [81] the interaction of the Kerr-Taub-NUT spacetime with a magnetic field is investigated. The relation between the vorticity tensor and the electromagnetic Poynting vector was studied by Herrera *et al* [82], to understand its role in producing gravitomagnetic effects, while in [83] the production of vorticity by electromagnetic radiation is focused on. The possibility that the Lense-Thirring effect could be produced by the simultaneous presence of electric and magnetic fields was studied by Gutiérrez-Ruiz and Pachón [84]. Eventually, gravitoelectromagnetic resonances were investigated by Tsagas [85] and the interaction between magnetic fields and gravitational waves, with emphasis on the gravitomagnetic effects, was studied by Tsagas [86].

There are analyses of gravitomagnetic effects in cosmology: they naturally arise in the so-called post-Friedmann formalism, which is an approach to the perturbation of the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime [87–90]. Instead, FLRW universes were studied with a gravitoelectromagnetic formalism in regions that are small if compared with the Hubble scale by Faraoni *et al* [91]. From a historical perspective, it is important to mention the rotating cosmological models of Kurt Gödel [92, 93].

A gravitomagnetic origin of friction for black-hole dynamics was investigated by Cashen *et al* [94], while Gutiérrez-Ruiz *et al* [95] focused on the possible impact of frame-dragging on chaotic dynamics of test particles around a family of stationary axially-symmetric solutions of Einstein's equation coupled with electromagnetic fields. The Aharonov–Bohm effect for the gravitational field of rotating cosmic string was studied by Barros *et al* [96].

4. Solar system tests

From a historical point of view, the first attempts were aimed at the measurements of the effects of the terrestrial gravitomagnetic field; in this case, a major difficulty is the coexistence, in Earth-based laboratories, of the Coriolis field due to the diurnal rotation, which is much greater than the terrestrial gravitomagnetic field but has a quite similar behaviour. Consequently, researchers turned their interest towards the space around the Earth first (where the Coriolis field is not present) and, then, in the Solar System.

A prototypical case is the already mentioned GPB mission, whose basic concept stemmed from Schiff's seminal paper [97] (even though the idea was independently considered by Pugh [98]), where it was shown that a gyroscope orbiting around the Earth undergoes a geodetic precession, due to its motion in curved spacetime, and a gravitomagnetic precession, entirely determined by the rotation of the Earth. The space mission was proposed at the beginning of the 60's, and launched in 2004 with the aim, in particular, to measure the gravitomagnetic precession to a precision of 1%. Actually, the gyroscope undergoes a precession

$$\boldsymbol{\Omega}_P = \frac{G}{c^2}\left[\frac{3(\mathbf{J}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{J}}{r^3}\right], \tag{91}$$

as determined by the Earth's angular momentum $\mathbf{J}$, where $\mathbf{r}$ is its position vector with respect to the center of the Earth; in the case of the GPB mission orbit, the integrated effect from (91) corresponds to 39 mas/year. However, due to experimental problems (electrostatic patches caused by the non-uniform coating of the gyroscopes), the effective accuracy was of 19% only (0.28 % for the geodetic precession), in any case in agreement with the predictions of GR [11].

Another approach to the measurement of the gravitomagnetic field of the Earth is the analysis of the orbits of satellites performed by laser-ranging. Actually, the line of the nodes of a test particle orbiting around a source of gravitational field is dragged by the angular momentum of the central body; however, for satellites around the Earth, this effect is much smaller than the one deriving from the non sphericity of the Earth. Ciufolini [12] originally suggested to use a couple of satellites, with supplementary inclinations, in order to get rid of the leading non relativistic effects. A detailed account account of the first approaches to this kind of measurements was given by Ciufolini [99]. Later results claimed for a confirmation of the GR predictions for the gravitomagnetic effect of the Earth within a 10% uncertainty [13], which were followed by a discussion on the error budget [100–105]. Eventually, the latest findings report for a confirmation of the gravitomagnetic effect within 2% uncertainty [14]. Furthermore, the LARES 2 satellite was recently successfully launched for improving the accuracy of previous tests of gravitomagnetism [106].

Lunar laser ranging (LLR) provided, over the years, several precision tests of GR [107]. Murphy *et al* [108] suggested that LLR provided accurate test of gravitomagnetic effects on the lunar orbit relative to Earth: in this case it is not the angular momentum of the Earth that determines this effect, rather it is due to the orbital motion of the Earth and the Moon in the Solar System, which can be seen as mass currents. According to Ciufolini [109], this kind of gravitomagnetic effect should thought of as *extrinsic* and opposed to the *intrinsic* gravitomagnetic effect determined by the spin angular momentum of a rotating object (see also Costa *et al* [110] for an analysis, based on curvature invariants, of the notions of intrinsic and exstrinsic gravitomagnetic effects) A subsequent debate [111–114] focused on the possibility that the extrinsic gravitomagnetic effect in the Earth-Moon system could be within the range of measurability with LLR; in addition, there are doubts about the possibility to measure the intrinsic gravitomagnetic effect of the Earth by LLR [115].

Proposals and ideas to measure the gravitomagnetic effects in the Solar System by accurate determination of the orbits of planets and satellites were considered in various works: for instance, Iorio and Ruggiero [116] focused on Kerr-de Sitter solution, also to put constraint on the cosmological constant. Starting from the work by Bini *et al* [48], subsequent studies [117, 118] focused on the possibility to measure induction effects (due to a time-dependence of the gravitomagnetic field) on the motion of test particles. The analysis of the orbital plane of the Mars Global Surveyor (MGS) spacecraft suggested a possible evidence of the gravitomagnetic field of Mars [119], which raised a subsequent debate [120]; other gravitomagnetic effects modelled as perturbation of the dynamics of binary system were also considered by Iorio [121, 122]. Corrections to relativistic orbits due to higher order gravitomagnetic effects were studied by Capozziello *et al* [123].

Additional proposals to investigate gravitomagnetic effects in the space around the Earth were made by Ruggiero and Tartaglia [124], who suggested to use geostationary satellites to broadcast electromagnetic signals and measure their propagation times, in order to evidence the asymmetry—determined by the Earth rotation—for signals propagating in opposite directions (a sort of generalized Sagnac effect). Mirza [125] analysed anomalies in Earth flybys of several missions, and suggested that the interplay between the magnetic field and gravitational field can enhance the gravitomagnetic effects, which might have some relevance in explaining what is observed. Recently, Tartaglia *et al* [126] suggested to exploit the future LISA mission, which is designed as a detector of gravitational waves in space, to measure the gravitomagnetic field of the Milky Way, using the propagation times of electromagnetic signals. A model of accurate satellites tracking by means of electromagnetic signals that can be used to measure gravitomagnetic effects was considered by Schärer *et al* [127]. Eventually, Battista *et al* [128] studied the quantum corrections to the time delay in the gravitational field of a rotating object, and showed that these corrections are in any case too small to be detected in the Solar System.

5. Laboratory tests

Gravitomagnetic effects are in general much smaller than Newtonian ones, which makes it difficult to measure them, as we have already discussed. Nonetheless, several experimental concepts were proposed over the years. In 1977, in a seminal paper on the possibility to test relativistic gravity in terrestrial laboratory, Braginsky *et al* [129], asserted that 'Advancing technology will soon make possible a new class of gravitation experiments: pure laboratory experiments with laboratory sources of non-Newtonian gravity and laboratory detectors'. Even though none of the proposed experiments (which heavily exploited the gravitoelectromagnetic analogy) were performed up today, those proposals are still topical, and other ones were suggested.

For instance, Pascual-Sánchez [130], on the basis of a previous proposal by Braginsky *et al* [131], made a preliminary study on the possibility to use a Foucalult pendulum to measure the Lense-Thirring effect; the idea was to setup the experimental apparatus at the South pole, to get rid of the larger Coriolis effect due to Earth's rotation.

Other experimental proposals refer to the gravitational properties of coherent quantum systems, such as superconductors or superfluids; more generally speaking, also the quantum features of gravitation could be relevant [132–134] (see also section 8 below). Some recent reviews by Gallerati *et al* [135], Gallerati and Ummarino [136] carefully analyse these topics. For instance, rotating superconductors [137–139] showed apparently singular properties and it was suggested that they can be explained by gravitomagnetic fields much larger than those predicted by GR [140, 141]. However, the presence of such 'strange' gravitomagnetic fields would be at odds with known properties of compact objects, such as neutron stars [142]. In addition, a recent work by Tajmar *et al* [143] imposes narrower constraints on the impact of gravitomagnetic effects on the explanation of anomalous Cooper pair mass excess.

The results presented in that publication are imposing narrower constraints on a possible explanation of anomalous Cooper pair mass excess from gravitoelectromagnetism.

More recently, Ummarino and Gallerati [144], worked out the Maxwell and London equations taking into account the gravitational corrections in linear approximations, expressed in terms of gravitoelectric and gravitomagnetic fields, and investigated the modification of the gravitational field in a superconductor; the same formalism was used to suggest that a Josephson AC effect between two superconductors could be determined by the Earth's gravitational field [145] and to investigate the effects in a physical setup where the external electric and magnetic fields determine the presence of a vortex lattice [146].

The effects of rotations described as gravitomagnetic effects on topological superconductors and superfluids were studied by Nomura *et al* [147], Sekine [148], while the impact on Bose–Einstein condensates was considered by Camacho and Castellanos [149], with possible implications on the origin of dark matter [150].

The development of atom interferometry suggested to use this technique to perform precision tests of GR (see e.g. Dimopoulos *et al* [151, 152] and references therein). In principle, also the Lense-Thirring effect could be tested but, as we already stated, a major obstacle is the need to isolate this GR effect from the much larger Coriolis effect due to the rotation of the Earth; an alternative proposal was made by Angonin-Willaime *et al* [153] who suggested to use a satellite around the Earth to host an atomic interferometer.

Ring Laser Gyroscopes [43] (which are very precise rotation sensors whose operation is based on the Sagnac effect) are considered as very promising candidates to measure GR effects, such as the Lense-Thirring and de Sitter precessions, in a terrestrial laboratory: this is currently under investigation by the GINGER (Gyroscopes IN GEneral Relativity) collaboration [154–158].

6. Astrophysics

Astrophysics events are a natural arena to observe gravitomagnetic effects. After discussing in general terms their impact, we will focus on some specific phenomena, such as the propagation of electromagnetic signals in section 6.1, galactic dynamics in section 6.2 and gravitational waves in section 6.3.

The impact of gravitomagnetic effects is investigated for different astrophysical phenomena, such as the analysis of the stars motion in the Milky way. For instance, Kannan and Saha [159] studied the effects of a Kerr field in weak-field approximation on the stars orbiting near the center of the Milky way; Iorio [160] evaluated several relativistic effects on the radial velocity of a star orbiting the supermassive black hole in the galactic center. Various gravitomagnetic effects are evaluated in black holes environments, such as their role in the acceleration in accretion disks [161]; Rueda *et al* [162] considered the interaction of Kerr black hole with the magnetic field in order to understand the relation with gamma ray bursts; Ricarte *et al* [163] pointed out the effect of frame dragging on infalling gas. Moreover, the gravitomagnetic back reaction of a heavy object (not a test particle) on a black hole was discussed by Herdeiro *et al* [164], while the gravitomagnetic field of rotating rings was studied by Ruggiero [165, 166].

Pulsars are a natural laboratory to test relativistic gravity and, in particular, gravitomagnetic effects: Ruggiero and Tartaglia [167] studied the impact of the gravitomagnetic correction to the Shapiro time delay and its relevance in pulsar timing. The interplay between the gravitomagnetic field of a black hole and the spin of the pulsar companion can modify the rate at which pulses are received and this can give information on the black hole [168]. The observation of the orbital inclination of the pulsar PSR J1141-6545 in the binary system that contains a white dwarf can be explained as a combination of the effect of the Newtonian quadrupole and the Lense-Thirring effect, as suggested by Venkatraman Krishnan *et al* [169], even if Iorio [170] pointed out that care must be paid in the interpretation of these results. Gravitomagnetic tidal resonances [171–173] are important in the motion of a binary system, and they can impact on the emission of gravitational waves. Eventually, gravitomagnetic effects on circumbinary (i.e. orbiting two stars) exoplanets are studied by Iorio [174].

6.1. Effects on electromagnetic signals propagation

The bending effect on the propagation of light rays is one of the classical tests of GR; initially, only the effect of the mass was considered but it is clear that, at higher order, also mass currents contribute to this effect. The same was true for the so-called Shapiro effect, which is the time delay on the propagation of light signals in a gravitational field.

Kopeikin and Mashhoon [175], in weak-field approximation, focused on light propagation in the gravitational field determined by self-gravitating spinning bodies, that are moving with arbitrary velocity. Several effects are considered, such as the correction to the Shapiro time delay, the modification of the bending angle due to the spin of the gravitating bodies, the rotation of the plane of polarisation of electromagnetic waves.

The gravitomagnetic correction to the time delay of electromagnetic signals in gravitational lensing were calculated by Ciufolini *et al* [176], Ciufolini and Ricci [177], who considered both the case of propagation in a rotating mass shell and in the field of a rotating source, again in weak-field approximation.

The modification of the deflection angle and the time delay function due to gravitomagnetic effects were considered by Sereno [178], in a more general framework which can be applied to a post-Newtonian spacetime, and this formalism was used for spinning stars [179], spiral galaxies [180]; the time delay for extended rotating sources, modelled as isothermal spheres, was focused on by Sereno [181], and the effect due to moving lenses was considered by Sereno [182]. In addition, the Faraday rotation due to the gravitomagnetic field was considered by Sereno [183], and a possible application to binary pulsar systems by Ruggiero and Tartaglia [184].

In the above papers, gravitomagnetic effects on light rays propagation were studied in the weak-field approximation, which is undoubtedly sufficient in the Solar System. On the other hand, Kraniotis [185] exactly solved the geodesic equations for test particles and photons in Kerr spacetime and, also, in the spacetime of a rotating electrically charged black hole (Kerr-Newman) [186]; in addition, Kraniotis [187] calculated the frequency shift of light emitted by geodesics test particles in Kerr-Newman-de Sitter and Kerr-Newman spacetimes; in the latter paper also the pericentre shifts were calculated, which were already been considered by the same author [188].

Always in Kerr spacetime, the bending angle which takes into accounts also of the motion of the observer was considered by Arakida [189]: in particular, the case is considered where the observer is not located in a flat asymptotic region, so the effect of a finite distance from the lens is taken into account.

Iyer [190] obtained an exact analytical expression for the bending angle of light on the equatorial plane of a Kerr black-hole, and then expanded it in power series. In particular, the asymmetry between the direct and the retrograde propagation, due to the rotation of the sources, was evidenced (see also Iyer and Hansen [191]).

6.2. Galactic dynamics

The dark matter (DM) hypothesis, i.e. the hypothesis of the existence of a non-baryonic component of mass dominating the matter density budget in the Universe, is one of the foundations of the widely accepted ΛCDM cosmological model (e.g. [192]) and, at the same time, one of the greatest mysteries in physics. The DM hypothesis has been incredibly successful in interpreting different astrophysical observables, such as the velocity distribution of galaxies in galaxy clusters (GCs, since [193]), the rotation curves (RCs) of disc galaxies (e.g. [194]), the thermodynamical properties of x-ray emitting gas in GCs ([195]), the gravitational lensing produced by their mass distributions ([196]) and the observations of the two Bullet Clusters ([197, 198]); as well as cosmological ones, e.g., the anisotropies observed in the cosmic microwave background ([199, 200]) and the growth of cosmic structures from such anisotropies ([201, 202]).

For what concerns disc galaxies, the main evidence supporting the existence of DM are based on the observed rotation curves, whose flatness at large radii cannot be explained by the distribution of visible (baryonic) matter, if interpreted on the basis of Newtonian dynamics [203]. In this scenario, Newtonian gravity is employed instead of GR. This choice is made because, when we are far from the galactic center (where the flat behavior is observed), we can reasonably assume that the gravitational field is weak, and the stars within the Galaxy are not moving at speeds close to the speed of light. However, there has been a suggestion that GR might still have relevance in this context. Specifically, researchers have explored the issue of galactic rotation curves by examining it from two angles: one involving exact solutions of the equations of GR and the other using weak-field approximations.

It was proposed that considering gravitomagnetic effects might lead to a different interpretation of the role of dark matter in explaining the observed phenomena: even though there is no doubt that in galaxies the Newtonian approximation correctly applies locally, globally it may fail due to the overall rotation of the system. This may indicate the need of general relativistic corrections to the Newtonian approach.

This point of view was originally advocated in two pioneering works by Cooperstock and Tieu [204], Cooperstock and Tieu [205] and subsequently further developed by Carrick and Cooperstock[206]. Since some of these works were criticised by Cross [207], Menzies and Mathews [208], subsequently Balasin and Grumiller [209] proposed another solution of Einstein's equations which resolved some problematic features. The velocity profile derived from the BG model was first used by Crosta *et al* [210] as a basis for studing the Milky Way rotation curve: after recasting the BG model to make it consistent with the Gaia stellar data [211, 212], they showed that the GR rotation curve was in quite good agreement with the Gaia data, at a level statistically comparable to the state of the art CDM models they compared to in their article. It is relevant to point out that to obtain the astrometric accuracies needed in a mission like Gaia, it is important to correctly take into account relativistic effects, such as the gravitomagnetic ones (see the recent review by Crosta [213]).

A recent paper by Costa *et al* [214] analyses the solution obtained by Balasin and Grumiller (BG), and shows that it is not appropriate as a galactic model. In particular, the BG model is a rigid solution. The study of the generalization to non rigid rotation was done in Astesiano *et al* [215], Astesiano and Ruggiero [216].

Also Gupta and Lohiya [217] investigated the possibility of accounting for the observed anomalous velocities of stars in galaxies to be a result of a dynamic overall rotation of an inertial frame dragged by a galaxy. A somewhat different but related approach, which takes into account the gravitomagnetic effects originating from mass currents into the solution of Einstein equations in weak-field and slow-motion approximation was put forward by Ludwig [218], Ruggiero *et al* [219], Srivastava *et al* [220] and analyzed with detailed criticism by Ciotti [221]. In Astesiano and Ruggiero [222], the GR results of these papers are shown to be related to the existence of the homogeneous solutions of the gravitomagnetic field and not directly related to the mass currents. These homogeneous solutions can produce a 'strong gravitomagnetic limit' where these effects are of the same order as Newtonian ones. Criticisms toward the application of gravitomagnetic effects to study galactic dynamics were made by Lasenby *et al* [223]. Another different approach was proposed using Post-Newtonian corrections in galactic dynamics by Ramos-Caro *et al* [224] and Lobodzinski [225] who also proposed an explanation for spiral arms, without the presence of an exotic form of matter. The formalism is based on Boltzmann transport equation for the collisional matter and on the very-low-velocity gravitomagnetism.

The off-diagonal terms responsible for the gravitomagnetic effects contribute to the modified virial theorem as shown in [226]. They can also give rise to a consistent definition of a 'gravitational mass' in this very specific set up [227].

6.3. Gravitational waves

The first direct detection of Gravitational Waves (GW), in 2015 [228], marked the beginning of gravitational waves astronomy and cosmology: in fact, apart from serving as a test for the theory, gravitational waves have become a potent instrument for exploring the Universe in the era of multi-messenger astronomy; technological developments and dedicated missions will help to greatly improve the information that can be obtained within this window. Accordingly, it is very important to properly model the measurement process[8] and, in this context, it is relevant to emphasise that the interaction of GW with a detector, modelled as a set of test masses, can be described in terms of gravitoelectromagnetic analogy [52, 229]. By adopting this approach, one can readily comprehend that while current devices detect the interaction between test masses and the gravitoelectric components of the wave, there are also gravitomagnetic interactions that offer the potential to observe the influence of gravitational waves on moving masses and spinning particles [77].

[8] Indeed, we focus here on the measurement process only, and neglect the impact of gravitomagnetic effects in the emission of gravitational waves, which is a very interesting topic, fully described in the literature on this field.

Notably, Ruggiero and Ortolan [230, 231] demonstrated the potential occurrence of a gravitomagnetic resonance phenomenon involving spinning particles within the influence of a gravitational wave. This observation opens up possibilities for devising innovative detectors capable of measuring collective spin excitations, such as spin waves within magnetized materials. In addition, Ramos and Mashhoon [232], using the gravitoelectromagnetic formalism for the components of the curvature tensor, studied the coupling of the helicity of the gravitational wave with the possible rotation of the detector.

Even though we consider spinless test masses, there are gravitomagnetic effects that need to be taken into account: Baskaran and Grishchuk [233] demonstrated that gravitomagnetic terms play a significant role in describing the displacements of interferometer test masses, particularly up to the second order in the distance parameter. As a result, they are important for precise measurements and the determination of gravitational waves source parameters. Specifically, while at the first order a detector's interaction with gravitational waves can be attributed to the influence of a gravitoelectric and gravitomagnetic field perpendicular to the propagation direction, at the second order, the gravitoelectric field exhibits a non-zero component along the propagation direction. These phenomena can be elucidated through the concept of gravitational induction [53].

Iorio and Corda [234] conducted a thorough examination of the interferometric response to these phenomena. Furthermore, it's noteworthy to underscore that investigating these types of effects could hold significance in testing alternative gravity theories beyond GR. Indeed, these theories often introduce longitudinal effects in gravitational radiation, which can arise from various sources such as massive modes, scalar fields, or a more complex geometric structure (see e.g. Capozziello *et al* [235], Capozziello and de Laurentis [236], Corda [237], Corda *et al* [238]).

7. Analogue models

Analogue gravity (see e.g. Barcelo *et al* [239] and references therein) investigates the possibility to describe physical systems in analogy with the formalism of curved spacetime, and this is typically fruitful for systems that can be experimentally tested in a laboratory and can give back new insights on the physics of general or special relativity. For example, it is possible to consider sound waves in a moving fluid in analogy with light waves in a curved spacetime: in particular, if the fluid flow is supersonic, it is possible to get a 'dumb hole', i.e. the acoustic analogue of a black hole [240, 241]. Accordingly, it is possible to mimic gravitomagnetic effects in these analogue models. For instance, Puthoff [242] studied the analogy between the equation of linearized turbulent fluid and those of General Relativity, in linear approximation, that leads to the linear analogy described in section 2.3. On the contrary, Kivotides [243] studies gravitomagnetic effects *on* turbulent fluids.

In addition, Chakraborty *et al* [244] focused on a model of the Lense-Thirring effect for a rotating acoustic analogue black hole, and suggested that Bose–Einstein condensate systems (BEC) could provide an important setup to test these effects, while Banerjee *et al* [245] addressed frame-dragging studying the hydrodynamics of nematic active fluids.

Another related interesting field of investigation pertains to the study of the gravitational field produced by light [246, 247]. In particular, it is possible to calculate in linear approximation of GR the gravitational field of electromagnetic beams called optical vortices, carrying orbital angular momentum [248]: the point is that every photon in a laser beam could carry angular momentum in addition to the angular momentum associated with its spin. Accordingly, these beams can generate a gravitational field that produces gravitomagnetic effects [249] which are very small to be detected, even if in principle present and important from the theoretical point of view. Similarly, it is possible to study the gravitomagnetic effects determined by the spin angular momentum of the light beams [250–252], which remains, however, too small to be detected with current technology.

8. Quantum effects

Despite the progress made in these fields, there are still many unanswered questions about the relationship between gravity and quantum mechanics. Additionally, there is still much to learn about the behavior of matter and energy at the Planck length, which is currently beyond the scope of experimental observation.

In the past years much attention was given to the theory of quantum fields in classical background gravitational fields, in particular regarding Hawking radiation by black holes (see Page [253] for a review on the subject). In this regard, the phenomenological thermodynamic properties of black holes are well understood, at least for quasi-stationary semiclassical black holes.

Parallel to this, some investigations have been done on quantum systems in classical background gravitational fields, for example for atomic beam interferometry [151] and on neutrons in the Earth's field [134]. A review on different aspects of the interaction of mesoscopic quantum systems with gravitational fields was written by Kiefer and Weber [254] (see also the discussion in section 5). In particular two proposed interaction mechanisms are considered:

- the use of quantum fluids as generator and/or detector of gravitational waves in the laboratory;
- the inclusion of gravitomagnetic fields in the study of the properties of rotating superconductors.

Experiments to detect such effects are expected to be quite difficult, but they would be of fundamental interest. In particular, in the gravitomagnetic limit, there are experimental setup for probing the 'diagonal part', or in other words, the so called 'inverse-square law' of gravity using quantum interference [255] and there are also quantum detection tests to observe the 'off-diagonal' terms, which defines what we call frame dragging [256].

A systematic treatment of the behaviour of a quantum system under the effects of a small and slowly rotating gravitational field was proposed by Adler and Chen [257] for spin zero particles and by Adler *et al* [258] for spin 1/2 particles. These works also take into account the possibile presence of an electromagnetic field, to which the Klein–Gordon and Dirac equations are minimally coupled.

A different approach is what in the literature is referred to as the Schrödinger-Newton model [259], which describes non-relativistic quantum objects under self-gravitation. In this model, in the Schrödinger equation also the Newtonian gravitational potential term appears and the source of this term is given by the square of the module of the wave function. The Schrödinger-Newton equation was first proposed to study self-gravitating bosonic stars by Ruffini and Bonazzola [260]. Since this model was completely non-relativistic, some authors recently considered a modification of the Schrödinger-Newton equation by taking into account certain relativistic corrections [261], with a particular focus on gravitomagnetic corrections [262]. Under the above approximation, Zhao *et al* [263] considered a short distance modification of the Schrödinger-Newton equation which also results in a short distance modification of the quantum mechanical virial theorem.

For what concerns quantum information, the dynamics of an orbiting qubits under the effect of gravitational frame dragging was studied by Lanzagorta [264]. In particular the author considered the Kerr spacetime geometry and a spin 1/2 qubits. Subsequently, another possible test for frame dragging effect using the same setup was proposed by Lanzagorta and Salgado [265].

We also note that the effect of a scattering process with gravitons as an intermediate state was investigated by Jesus *et al* [266]. In this setup, the gravitomagnetic limit is considered. This allows a Lagrangian formulation which includes interactions of gravitons with fermions and photons. On the other hand, the presence of an external gravitational field and how the frame dragging effects affect the scattering process was focused on by Kim [267].

9. Connections with the Mach's principle

The idea underlying the Mach's principle is that there is a relationshipe between the local and the global Universe and the nature of this link is mechanical. We are already familiar with connections of a different kind, for example the known 'Olber's paradox' [268], where the link is optical. Actually, according to Mach, the local inertia of a body is a consequence of the global distribution of matter in the Universe.

This view is a contrast to the old Newtonian paradigm, which states that inertia is an intrinsic property of matter and therefore it is completely independent from the rest of the Universe. Of course, until now, no such change in the value of inertia has been measured experimentally. From the Machian perspective, the fact that this change has not been detected is not a big deal, since nearby matter would make a little contribution to the inertia of a test body compared to the vastness of the observed Universe.

The Newtonian paradigm is based on absolute space and time: consequently, the notion of absolute motion comes naturally, as relative to the absolute space, and the absolute acceleration is the one appearing in the Newton second law. To prove the existence of absolute motion, in the *Principia* Newton describes a thought experiment to detect absolute rotation, which is the famous 'bucket experiment'. Newton's interpretation of this thought experiment is that if we measure the centrifugal forces responsible for the concavity of the water surface we are in fact measuring the absolute rotation of a body. In his interpretation, the centrifugal forces arise as a result of the rotation of water with respect to absolute space, since all possible inertial frames are tied to this fundamental 'entity'. The Focault's pendulum is a concrete realization of the above discussion, and it shows the absolute rotation of the Earth.

The main criticism toward the Newtonian interpretation were pointed out by Berkeley and, after many years, by Mach [269]. They claimed that all we can observe is that the centrifugal forces are due to the motion of the water with respect to the rest of the matter in the Universe, not only respect to the walls of the bucket. In particular Mach used the sentence 'respect to the fixed stars'. According to Mach, we do not know whether the result of the experiment would be the same if all the matter of the Universe were removed, nor whether such centrifugal forces could be produced by the rotation of the rest of the Universe while the walls of the bucket stay 'fixed'.

The theory of General Relativity presents Machian features, such as the dragging effects (see the monograph by Ciufolini and Wheeler [16] and the paper by Vassallo and Hoefer [270] for a recent review). To see how these effects arise in GR, let us consider the weak-field limit of a spacetime outside a slowly rotating stationary body in adapted coordinates; in particular, the line element (69) can be written as

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)dr^2 + r^2(d\theta^2 + r^2\sin^2\theta d\phi^2) - 4\frac{J}{r}\sin\theta dt d\phi, \tag{92}$$

where the non diagonal terms are given by equations (68) and (77). It is the field **B** in equation (74) which gives the Coriolis forces acting on test particles outside the body and the precession of gyroscopes. A similar argument applies for the forces arising inside a rotating shell of matter [9]. Let us consider a gyroscope at rest respect to the static observers. The static observers are defined by the four-velocity Z

$$Z = \frac{1}{\sqrt{-g_{tt}}}\partial_t \simeq \partial_t, \tag{93}$$

and are at rest in the coordinate system of (92). Due to the assumptions on the asymptotic flatness of the metric, the reference frame defined by the congruence of worldlines ∂_t corresponds to a rigid frame anchored to the asymptotic inertial frame at infinity. The key feature is that this reference frame is at rest compared to the distant stars, but it has non-zero angular momentum. The spin axis Ξ of the gyroscope is Fermi-Walker transported along its worldline [55]

$$\frac{D\Xi^\mu}{d\tau} = \Xi^\nu a_\nu V^\mu, \tag{94}$$

where τ is the proper time, $V = Z$ is the four velocity of the gyroscope and a is the four acceleration. We can read off the Christoffel symbols from equation (75) and after imposing the orthogonality condition $\Xi^\mu V_\mu = 0$ we find

$$\frac{D\Xi^i}{d\tau} = \frac{1}{2}(\Xi \times \mathbf{B})^i. \tag{95}$$

After using equation (77), which we report here for easier reference

$$A_i = \frac{G}{c}\frac{(\mathbf{J} \wedge \mathbf{x})_i}{r^3}, \tag{96}$$

we get the result in equation (91). This discussion shows, as stated before, that General Relativity possess some Machian properties: in fact, the gyroscope determines the axes of a local inertial frame and they are affected by the mass distribution and its motion.

For a more detailed application of gravitomagnetism to geodetic precession and frame dragging see Christillin and Barattini [271], where another quantitative confirmation of Mach's arguments can be found.

Such effects can arise also in setups which are different from the one that has been discussed. For example, in the context of cosmology, Schmid [272] showed that there is dragging of local inertial frames by a weighted average of the cosmological energy currents via gravitomagnetism, for all linear perturbations of all Friedmann-Robertson-Walker (FRW) universes.

10. Conclusions

Gravitoelectromagnetic analogies arise in different context in General Relativity and are often used to provide a better insight into complicated gravitational phenomena, even though it is clear that gravitational and electromagnetic interactions are essentially different. In this work, we reviewed these analogies and emphasized also the hypotheses on which they are based.

First, we showed that a splitting approach in full theory leads to a non-linear analogy with electromagnetism: to this end, we introduced the spatial tensor algebra formalism using Cattaneo's projection techniques. This led us to the form of the force equation for a test particle, as seen by an observer described by a timelike congruence Γ in the spacetime $\mathcal{V}_4$. This force contains a term which is proportional to the curvature of Γ and a term proportional to the spatial velocity of the test particle, which makes it possible to introduce an analogy with the electromagnetic dynamics: in particular, a gravitoelectric field is associated to the local linear acceleration, while a gravitomagnetic field is associated to the local angular acceleration.

Then, we pointed out that the well known gravitoelectromagnetic analogy that arises in linearized General Relativity can be seen as a limiting case of the previously discussed exact analogy. In particular, the linear formalism is very useful to deal with experiments and observations which are often performed in conditions where the gravitational field is weak and the speeds are small compared to the speed of light. Accordingly, the linear gravitoelectromagnetic analogy is a powerful tool to explain new gravitational effects in terms of known electromagnetic ones: we remark that gravitomagnetic effects are peculiar to General Relativity, since in Newtonian gravity there are no gravitational effects arising from mass currents. In addition, we pointed out some limitations of this linear analogy that arises when we are dealing with time-depending gravitational fields; in particular, the radiative regime is one of the limits of applicability of the gravitoelectromagnetic approximation to General Relativity

Eventually, we briefly sketched the gravitoelectromagnetic analogy that is based on the magnetic-like and electric-like parts of the Riemann tensor, using Fermi coordinates. This formal approach is useful, for instance, when we are directly dealing with tidal effects, such as in the case of gravitational waves physics.

After this basic introduction, we reviewed the recent theoretical developments which are aimed to suggest new possible tests of gravitomagnetic effects. The continuous improvements in technology and measurement techniques made it possible to imagine a pletora of possible consequences of these effects. Accordingly, we made a survey of the proposals to test gravitomagnetic effects emerged during the last twenty years, which refer to Solar System and Earth-based experiments, astrophysical observations, analogue models. In addition we reviewed the interplay between gravitomagnetic effects and other areas of physics, such as quantum effects and the Mach principle.

The presence of 'gravitomagnetic effects' in a somewhat wider sense is a natural consequence of the general relativistic approach to the description of the gravitational interaction and, consequently, even if these effects are generally very small and difficult to distinguish from other competing ones, there should be no doubts on their existence. On the other hand, their study is important to correctly model and understand complex gravitational phenomena on different scales, which range from the near space in the Solar System, to astrophysical events and, eventually, to galactic and cosmological dynamics.

We conjecture a further evolution of this useful formalism, and we hope that this review will be a helpful reference for researchers involved in the study of gravitational physics.

Acknowledgments

MLR acknowledges the contribution of the local research project Modelli gravitazionali per lo studio dell'universo (2022) - Dipartimento di Matematica 'G.Peano', Università degli Studi di Torino, that supported the visit of DA to Torino, during which this work was started. This work is done within the activity of the Gruppo Nazionale per la Fisica Matematica (GNFM). The authors thank Filipe Costa, Mariateresa Crosta, Antonio Gallerati, José Natário for useful comments.

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Matteo Luca Ruggiero https://orcid.org/0000-0002-1844-5863
Davide Astesiano https://orcid.org/0000-0002-3934-3938

References

[1] McDonald K T 1997 *Am. J. Phys.* **65** 591
[2] Heaviside O 1894 *Electromagnetic Theory* (The Electrician Printing and Publishing Co)
[3] Herrmann F and Pohlig M 2022 *Am. J. Phys.* **90** 410
[4] Pfister H 2014 *Springer Proceedings in Physics* **157** 191
[5] Einstein A 1912 *Vierteljahrsschrift fur gerichtliche Medizin und offentliches Sanitatswesen* **44** 37
[6] Einstein A 2009 The collected papers of albert einstein *The Swiss Years: Writings, 1912-1914* vol 4 (Princeton, NJ: Princeton University Press) p 126
[7] Thirring H 2012 *Gen. Relativ. Gravitation* **44** 3225
[8] Pfister H 2012 *Gen. Relativ. Gravitation* **44** 3217
[9] Mashhoon B, Hehl F W and Theiss D S 1984 *Gen. Relativ. Gravitation* **16** 711
[10] Einstein A 2009 The collected papers of albert einstein *The Collected Papers of Albert Einstein.-vol 8: The Berlin Years: Correspondence 1914-1918* (Princeton University Press) p 365
[11] Everitt C W F *et al* 2011 *Phys. Rev. Lett.* **106** 221101
[12] Ciufolini I 1986 *PhRvL* **56** 278
[13] Ciufolini I and Pavlis E C 2004 *Natur* **431** 958
[14] Ciufolini I, Paolozzi A, Pavlis E C, Sindoni G, Ries J, Matzner R, Koenig R, Paris C, Gurzadyan V and Penrose R 2019 *Eur. Phys. J.* C **79** 872

[15] Iorio L, Lichtenegger H I M, Ruggiero M L and Corda C 2011 *Astrophys. Space Sci.* **331** 351
[16] Ciufolini I and Wheeler J A 1995 *Gravitation and Inertia* (Princeton University Press)
[17] Ruggiero M L and Tartaglia A 2002 *Nuovo Cim.* **B117** 743
[18] Mashhoon B 2007 *The Measurement of Gravitomagnetism: A Challenging Enterprise* ed L Iorio (NOVA Publishers)
[19] Iorio L (ed) 2007 *The Measurement of Gravitomagnetism: A Challenging Enterprise* (NOVA Publishers)
[20] Bailey Q G 2010 *Phys. Rev.* D **82** 065012
[21] Tasson J D 2012 *Phys. Rev.* D **86** 124021
[22] Dass A and Liberati S 2019 *Gen. Rel. Grav.* **51** 84
[23] Mashhoon B and Hehl F W 2019 *Universe* **5** 195
[24] Smith T L, Erickcek A L, Caldwell R R and Kamionkowski M 2008 *PhRvD* **77** 024015
[25] Thorne K S, Thorne K S, Price R H and MacDonald D A 1986 *Black Holes: The Membrane Paradigm* (Yale University Press)
[26] Forward R L 1961 *Proc. IRE* **49** 892
[27] Jantzen R T, Carini P and Bini D 1992 *Annals Phys.* **215** 1
[28] Jantzen R T, Carini P and Bini D 1996 *VII Marcel Grossmann Meeting on General Relativity (MG 7)* pp 133–52
[29] Lynden-Bell D and Nouri-Zonoz M 1998 *Rev. Mod. Phys.* **70** 427
[30] Bini D and Jantzen R T 2003 *Nuovo Cim.* B **117** 983
[31] De Felice F and Bini D 2010 *Classical Measurements in Curved Space-Times* (Cambridge University Press)
[32] Costa L F O and Natário J 2014 *Gen. Relativ. Gravitation* **46** 1792
[33] Costa L F O and Natário J 2021 *Universe* **7** 388
[34] Rizzi G and Ruggiero M L 2004 *Relativity in Rotating Frames: Relativistic Physics in Rotating Reference Frames* (Springer)
[35] Cattaneo C 1961 *Introduzione Alla Teoria Einsteiniana Della Gravitazione* (Libreria Eredi Virgilio Veschi)
[36] Rizzi G and Ruggiero M L 2004 The relativistic Sagnac Effect: two derivations *Relativity in Rotating Frames* **135** 179–220 gr-qc/0305084
[37] Rindler W and Perlick V 1990 *Gen. Relativ. Gravitation* **22** 1067
[38] Post E J 1967 *Rev. Mod. Phys.* **39** 475
[39] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[40] Rizzi G and Ruggiero M L 2003 *Gen. Relativ. Gravitation* **35** 2129
[41] Ruggiero M L and Tartaglia A 2015 *The European Physical Journal Plus* **130** 1
[42] Tartaglia A and Ruggiero M L 2021 From Kerr to Heinseberg *Entropy* **23** 1–9 arXiv:2103.03763
[43] Chow W W, Gea-Banacloche J, Pedrotti L M, Sanders V E, Schleich W and Scully M O 1985 *Rev. Mod. Phys.* **57** 61
[44] Ruggiero M L 2005 *Gen. Rel. Grav.* **37** 1845
[45] Ruggiero M L 2004 *Nuovo Cim.* B **119** 893
[46] Ruggiero M L 2021 *Universe* **7** 451
[47] Straumann N 2013 *General Relativity, With Applications to Astrophysics* (Springer)
[48] Bini D, Cherubini C, Chicone C and Mashhoon B 2008 *Class. Quant. Grav.* **25** 225014
[49] Bakopoulos A and Kanti P 2014 *Gen. Rel. Grav.* **46** 1742
[50] Williams L L and Inan N 2021 *New J. Phys.* **23** 053019
[51] Hobson M P, Efstathiou G P and Lasenby A N 2005 *General Relativity* (Cambridge: Cambridge University Press) 0521829518
[52] Ruggiero M L and Ortolan A 2020 *Journal of Physics Communications* **4** 055013
[53] Ruggiero M L 2022 *Gen. Rel. Grav.* **54** 97
[54] Manasse F and Misner C W 1963 *J. Math. Phys.* **4** 735
[55] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (WH Freeman and Co.)
[56] Mashhoon B 1993 *Phys. Lett.* A **173** 347
[57] Lyne A G *et al* 2004 *Science* **303** 1153
[58] Shao L *et al* 2015 *PoS* **AASKA14** 042
[59] Venkatraman Krishnan V *et al* 2020 *Science* **367** 577
[60] O'Connell R F 2004 *Phys. Rev. Lett.* **93** 081103
[61] O'Connell R F 2008 Gravito-Magnetism in one-body and two-body systems: Theory and Experiments *Proc. of Course CLXVIII of the International School of Physics ∀Enrico Fermi∀, Varenna, Italy, 2007,* arXiv:0804.3806 (https://doi.org/10.48550/arXiv.0804.3806)
[62] O'Connell R F 2010 *General Relativity and John Archibald Wheeler, Astrophysics and Space Science Library* ed I Ciufolini and R A A Matzner (Springer: Dordrecht) vol 367, p 325
[63] Kaplan J D, Nichols D A and Thorne K S 2009 *Phys. Rev.* D **80** 124014
[64] Mashhoon B 2008 *Class. Quant. Grav.* **25** 085014
[65] Costa L F O, Natário J and Santos N O 2021 *Class. Quant. Grav.* **38** 055003
[66] Herrera L, Carot J and Di Prisco A 2007 *Phys. Rev.* D **76** 044012
[67] Herrera L 2021 *Universe* **7** 27
[68] Cohen J M and Mashhoon B 1993 *Phys. Lett.* A **181** 353
[69] Lichtenegger H I M, Iorio L and Mashhoon B 2006 *Annalen Phys.* **15** 868
[70] Hackmann E and Lämmerzahl C 2014 *Phys. Rev.* D **90** 044059
[71] Owen R *et al* 2011 *Phys. Rev. Lett.* **106** 151101
[72] Nichols D A *et al* 2011 *Phys. Rev.* D **84** 124014
[73] Zhang F, Zimmerman A, Nichols D A, Chen Y, Lovelace G, Matthews K D, Owen R and Thorne K S 2012 *Phys. Rev.* D **86** 084049
[74] Nichols D A, Zimmerman A, Chen Y, Lovelace G, Matthews K D, Owen R, Zhang F and Thorne K S 2012 *Phys. Rev.* D **86** 104028
[75] Mashhoon B 2021 *Entropy* **23** 445
[76] Costa L F O, Natário J and Zilhão M 2016 *Phys. Rev.* D **93** 104006
[77] Bini D, Geralico A and Ortolan A 2017 *Phys. Rev.* D **95** 104044
[78] Chakraborty C 2022 *Phys. Rev.* D **105** 064072
[79] Bini D, Mashhoon B and Obukhov Y N 2022 *Phys. Rev.* D **105** 064028
[80] Ahmedov B J and Rakhmatov N I 2003 *Found. Phys.* **33** 625
[81] Abdujabbarov A A, Ahmedov B J and Kagramanova V G 2008 *Gen. Rel. Grav.* **40** 2515
[82] Herrera L, Gonzalez G A, Pachon L A and Rueda J A 2006 *Class. Quant. Grav.* **23** 2395
[83] Herrera L and Barreto W 2012 *Phys. Rev.* D **86** 064014
[84] Gutiérrez-Ruiz A F and Pachón L A 2015 *Phys. Rev.* D **91** 124047

[85] Tsagas C G 2011 *PhRvD* **84** 043524
[86] Tsagas C G 2010 *Phys. Rev.* D **81** 043501
[87] Bruni M, Thomas D B and Wands D 2014 *Phys. Rev.* D **89** 044010
[88] Thomas D B, Bruni M and Wands D 2015 *Mon. Not. Roy. Astron. Soc.* **452** 1727
[89] Barrera-Hinojosa C, Li B, Bruni M and He J-H 2021 *Mon. Not. Roy. Astron. Soc.* **501** 5697
[90] Barrera-Hinojosa C, Li B and Cai Y-C 2022 *Mon. Not. Roy. Astron. Soc.* **510** 3589
[91] Faraoni V, Vachon G, Vanderwee R and Jose S 2022 *Phys. Rev.* D **105** 083530
[92] Gödel K 2000 *Gen. Relativ. Gravitation* **32** 1419
[93] Gödel K 1949 *Rev. Mod. Phys.* **21** 447
[94] Cashen B, Aker A and Kesden M 2017 *Phys. Rev.* D **95** 064014
[95] Gutiérrez-Ruiz A F, Cárdenas-Avendaño A, Yunes N and Pachón L A 2021 *Class. Quant. Grav.* **38** 145013
[96] Barros A, Bezerra V B and Romero C 2003 *Mod. Phys. Lett.* A **18** 2673
[97] Schiff L I 1960 *PhRvL* **4** 215
[98] Pugh G E 2003 *Nonlinear Gravitodynamics: The Lense-Thirring Effect* ed R Remo and S Costantino (World Scientific Publishing Co. Pte. Ltd) pp 414–26
[99] Ciufolini I 2002 *eConf* **C020620** FRAT03
[100] Iorio L 2011 *EPL (Europhysics Letters)* **96** 30001
[101] Ries J C, Ciufolini I, Pavlis E C, Paolozzi A, Koenig R, Matzner R A, Sindoni G and Neumayer H 2011 *Eur. Phys. Lett.* **96** 1
[102] Iorio L, Ruggiero M L and Corda C 2013 *Acta Astronaut.* **91** 141
[103] Iorio L 2016 *Adv. Space Res.* **57** 493
[104] Iorio L 2018 *Universe* **4** 113
[105] Ciufolini I, Pavlis E C, Ries J, Matzner R, Koenig R, Paolozzi A, Sindoni G, Gurzadyan V, Penrose R and Paris C 2018 *European Physical Journal* C **78** 880
[106] Ciufolini I, Paolozzi A, Pavlis E C, Ries J C, Matzner R, Paris C, Ortore E, Gurzadyan V and Penrose R 2023 *European Physical Journal* C **83** 87
[107] Williams J G, Turyshev S G and Boggs D H 2004 *PhRvL* **93** 261101
[108] Murphy T W Jr., Nordtvedt K and Turyshev S G 2007 *Phys. Rev. Lett.* **98** 071102
[109] Ciufolini I 2008 arXiv:0809.3219
[110] Costa L F O, Wylleman L and Natário J 2021 *Phys. Rev.* D **104** 084081
[111] Kopeikin S M 2007 *Phys. Rev. Lett.* **98** 229001
[112] Murphy T W Jr+, Nordtvedt K and Turyshev S G 2007 *Phys. Rev. Lett.* **98** 229002
[113] Kopeikin S 2008 arXiv:0809.3392
[114] Xie Y and Kopeikin S 2009 *IAU Symposium 261: Relativity in Fundamental Astronomy: Dynamics, Reference Frames, and Data Analysis* arXiv:0905.2424
[115] Iorio L 2009 *Int. J. Mod. Phys.* D **18** 1319
[116] Iorio L and Ruggiero M L 2009 *Journal of Cosmology and Astro-Particle Physics* **2009** 024
[117] Ruggiero M L and Iorio L 2010 *Gen. Rel. Grav.* **42** 2393
[118] Bini D, Iorio L and Giordano D 2015 *Gen. Rel. Grav.* **47** 130
[119] Iorio L 2006 *Class. Quant. Grav.* **23** 5451
[120] Krogh K 2007 *Classical Quantum Gravity* **24** 5709
[121] Iorio L 2019 *Universe* **5** 87
[122] Iorio L 2019 *Mon. Not. Roy. Astron. Soc.* **484** 4811
[123] Capozziello S, De Laurentis M, Garufi F and Milano L 2009 *Phys. Scripta* **79** 025901
[124] Ruggiero M L and Tartaglia A 2019 *Eur. Phys. J. Plus* **134** 205
[125] Mirza B M 2019 *Mon. Not. Roy. Astron. Soc.* **489** 3232
[126] Tartaglia A, Bassan M, Pucacco G, Ferroni V and Vetrugno D 2021 Dark gravitomagnetism with LISA and gravitational waves space detectors *16th Marcel Grossmann meeting (2021)* arXiv:2110.05135 XVI marcel grossmann meeting onrecent developments in theoretical and experimental general relativity (https://doi.org/10.48550/arXiv.2110.05135)
[127] Schärer A, Bondarescu R, Saha P, Angélil R, Helled R and Jetzer P 2017 *Frontiers in Astronomy and Space Sciences* **4** 11 arXiv:1707.00319
[128] Battista E, Tartaglia A, Esposito G, Lucchesi D, Ruggiero M L, Valko P, Dell'Agnello S, Di Fiore L, Simo J and Grado A 2017 *Class. Quant. Grav.* **34** 165008
[129] Braginsky V B, Caves C M and Thorne K S 1977 *Phys. Rev.* D **15** 2047
[130] Pascual-Sánchez J F 2003 *Lect. Notes Phys.* **617** 330
[131] Braginsky V B, Polnarev A G and Thorne K S 1984 *Phys. Rev. Lett.* **53** 863
[132] Modanese G 1996 *EPL* **35** 413
[133] Modanese G 1996 *Phys. Rev.* D **54** 5002
[134] Colella R, Overhauser A W and Werner S A 1975 *Phys. Rev. Lett.* **34** 1472
[135] Gallerati A, Modanese G and Ummarino G 2022 *Front. in Phys.* **10** 941858
[136] Gallerati A and Ummarino G A 2022 *Symmetry* **14** 554
[137] Tate J, Cabrera B, Felch S B and Anderson J T 1989 *Phys. Rev. Lett.* **62** 845
[138] Podkletnov E and Nieminen R 1992 *Physica C* **203** 441
[139] Tajmar M, Plesescu F, Seifert B and Marhold K 2007 *AIP Conf. Proc.* **880** 1071
[140] de Matos C J 2008 *Int. J. Mod. Phys.* D **16** 2599
[141] de Matos C J and Tajmar M 2005 *Physica* C **432** 167
[142] Bambi C 2008 *Int. J. Mod. Phys.* D **17** 327
[143] Tajmar M, Neunzig O and Kößling M 2022 *Frontiers in Physics* **10** 892215
[144] Ummarino G A and Gallerati A 2017 *Eur. Phys. J.* C **77** 549
[145] Ummarino G A and Gallerati A 2020 *Class. Quant. Grav.* **37** 217001
[146] Ummarino G A and Gallerati A 2021 *Results Phys.* **30** 104838
[147] Nomura K, Ryu S, Furusaki A and Nagaosa N 2012 *PhRvL* **108** 026802
[148] Sekine A 2016 *Phys. Rev.* B **93** 094510
[149] Camacho A and Castellanos E 2012 *Mod. Phys. Lett.* A **27** 1250198

[150] Sarkar S, Vaz C and Wijewardhana L C R 2018 *Phys. Rev.* D **97** 103022
[151] Dimopoulos S, Graham P W, Hogan J M and Kasevich M A 2007 *Phys. Rev. Lett.* **98** 111102
[152] Dimopoulos S, Graham P W, Hogan J M and Kasevich M A 2008 *Phys. Rev.* D **78** 042003
[153] Angonin-Willaime M C, Ovido X and Tourrenc P 2004 *Gen. Rel. Grav.* **36** 411
[154] Bosi F *et al* 2011 *Phys. Rev.* D **84** 122002
[155] Tartaglia A, Di Virgilio A, Belfi J, Beverini N and Ruggiero M L 2017 *Eur. Phys. J. Plus* **132** 73
[156] Di Virgilio A D V *et al* 2020 *Phys. Rev. Res.* **2** 032069
[157] Altucci C *et al* 2022 arXiv:2209.09328
[158] Di Virgilio A D V *et al* 2023 arXiv:2301.01386
[159] Kannan R and Saha P 2009 *Astrophys. J.* **690** 1553
[160] Iorio L 2011 *Mon. Not. Roy. Astron. Soc.* **411** 453
[161] Poirier J and Mathews G J 2015 arXiv:1504.02499
[162] Rueda J A, Ruffini R and Kerr R P 2022 *Astrophys. J.* **929** 56
[163] Ricarte A, Palumbo D C M, Narayan R, Roelofs F and Emami R 2022 *Astrophys. J. Lett.* **941** L12
[164] Herdeiro C A R, Rebelo C and Warnick C M 2009 *Phys. Rev.* D **80** 084037
[165] Ruggiero M L 2015 *Int. J. Mod. Phys.* D **24** 1550060
[166] Ruggiero M L 2016 *Astrophys. Space Sci.* **361** 140
[167] Ruggiero M L and Tartaglia A 2005 *Phys. Rev.* D **72** 084030
[168] Kocherlakota P, Biswas S, Joshi P S, Bhattacharyya S, Chakraborty C and Ray A 2019 *Mon. Not. Roy. Astron. Soc.* **490** 3262
[169] Venkatraman Krishnan V *et al* 2020 *Science* **367** 577
[170] Iorio L 2020 *Mon. Not. Roy. Astron. Soc.* **495** 2777
[171] Flanagan É É and Racine É 2007 *PhRvD* **75** 044001
[172] Favata M 2006 *Phys. Rev.* D **73** 104005
[173] Poisson E 2020 *Phys. Rev.* D **101** 104028
[174] Iorio L 2022 *Universe* **8** 546
[175] Kopeikin S and Mashhoon B 2002 *Phys. Rev.* D **65** 064025
[176] Ciufolini I, Kopeikin S, Mashhoon B and Ricci F 2003 *Phys. Lett.* A **308** 101
[177] Ciufolini I and Ricci F 2003 arXiv:gr-qc/0301030
[178] Sereno M 2003 *Phys. Rev.* D **67** 064007
[179] Sereno M 2003 *Mon. Not. Roy. Astron. Soc.* **344** 942
[180] Capozziello S, Cardone V F, Re V and Sereno M 2003 *Mon. Not. Roy. Astron. Soc.* **343** 360
[181] Sereno M 2005 *Mon. Not. Roy. Astron. Soc.* **357** 1205
[182] Sereno M 2005 *Mon. Not. Roy. Astron. Soc.* **359** 19
[183] Sereno M 2004 *Phys. Rev.* D **69** 087501
[184] Ruggiero M L and Tartaglia A 2007 *Mon. Not. Roy. Astron. Soc.* **374** 847
[185] Kraniotis G V 2005 *Class. Quant. Grav.* **22** 4391
[186] Kraniotis G V 2014 *Gen. Rel. Grav.* **46** 1818
[187] Kraniotis G V 2021 *Eur. Phys. J.* C **81** 147
[188] Kraniotis G V 2007 *Class. Quant. Grav.* **24** 1775
[189] Arakida H 2021 *Int. J. Mod. Phys.* D **30** 2150045
[190] Iyer S V 2018 arXiv:1808.06630
[191] Iyer S V and Hansen E C 2009 *Phys. Rev.* D **80** 124023
[192] Aghanim N *et al* (Planck) 2020 *Astron. Astrophys.* **641** A6 [Erratum: Astron.Astrophys. 652, C4 (2021)]
[193] Zwicky F 1933 *Helv. Phys. Acta* **6** 110
[194] Corbelli E and Salucci P 2000 *MNRAS* **311** 441
[195] Allen S W, Evrard A E and Mantz A B 2011 *Annual Review of Astronomy and Astrophysics* **49** 409
[196] Taylor A 1999 Evolution of Large Scale Structure : From Recombination to Garching *MPA- ESO cosmology conference, Garching, Germany, 2-7 August 1998* ed A J Banday, R K Sheth and L N da Costa p 226
[197] Clowe D, Bradac M, Gonzalez A H, Markevitch M, Randall S W, Jones C and Zaritsky D 2006 *Astrophys. J. Lett.* **648** L109
[198] Bradač M, Allen S W, Treu T, Ebeling H, Massey R, Morris R G, von der Linden A and Applegate D 2008 *Astrophys. J.* **687** 959
[199] Hinshaw G *et al* (WMAP) 2009 *Astrophys. J. Suppl.* **180** 225
[200] Ade P A R *et al* (Planck) 2016 *Astron. Astrophys.* **594** A13
[201] Conroy C, Wechsler R H and Kravtsov A V 2006 *ApJ* **647** 201
[202] Piattella O 2018 *Lecture Notes in Cosmology* (Berlin: Springer International Publishing)
[203] Strigari L E 2013 *Phys. Rep.* **531** 1
[204] Cooperstock F I and Tieu S 2005 arXiv: astro-ph/0507619
[205] Cooperstock F I and Tieu S 2007 *Int. J. Mod. Phys.* A **22** 2293
[206] Carrick J D and Cooperstock F I 2012 *Astrophysics and Space Science* **337** 321
[207] Cross D J 2006 arXiv:astro-ph/0601191
[208] Menzies D and Mathews G J 2007 arXiv:astro-ph/0701019
[209] Balasin H and Grumiller D 2008 *Int. J. Mod. Phys.* D **17** 475
[210] Crosta M, Giammaria M, Lattanzi M G and Poggio E 2020 *Mon. Not. R. Astron. Soc.* **496** 2107
[211] (Gaia Collaboration) 2016 *Astronomy & Astrophysics*(**595** A1
[212] (Gaia Collaboration) 2018 *Astronomy & Astrophysics*(**616** A1
[213] Crosta M 2019 *Nuovo Cimento Rivista Serie* **42** 443
[214] Costa L F O, Natário J, Frutos-Alfaro F and Soffel M 2023 arXiv:2303.17516
[215] Astesiano D, Cacciatori S L, Gorini V and Re F 2022 *Eur. Phys. J.* C **82** 554
[216] Astesiano D and Ruggiero M L 2022 *Phys. Rev.* D **106** 044061
[217] Gupta A and Lohiya D 2020 arXiv:2009.14553
[218] Ludwig G O 2021 *European Physical Journal* C **81** 186
[219] Ruggiero M L, Ortolan A and Speake C C 2022 *Class. Quant. Grav.* **39** 225015
[220] Srivastava Y, Immirzi G, Swain J, Panella O and Pacetti S 2023 *Eur. Phys. J.* C **83** 100
[221] Ciotti L 2022 *Astrophys. J.* **936** 180

[222] Astesiano D and Ruggiero M L 2022 *Phys. Rev.* D **106** L121501
[223] Lasenby A N, Hobson M P and Barker W E V 2023 arXiv:2303.06115
[224] Ramos-Caro J, Agon C A and Pedraza J F 2012 *Phys. Rev.* D **86** 043008
[225] Lobodzinski B 2014 arXiv:1406.6082
[226] Astesiano D 2022 *Gen. Rel. Grav.* **54** 63
[227] Sebens C T 2018 arXiv:1811.10602
[228] Abbott B P *et al* 2016 *Phys. Rev. Lett.* **116** 061102
[229] Ruggiero M L 2021 *Am. J. Phys.* **89** 639
[230] Ruggiero M L and Ortolan A 2020 *Phys. Rev.* D **102** 101501
[231] Ruggiero M L and Ortolan A 2021 *XVI Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics and Relativistic Field Theories* arXiv:2111.00502
[232] Ramos J and Mashhoon B 2006 *Phys. Rev.* D **73** 084003
[233] Baskaran D and Grishchuk L P 2004 *Classical Quantum Gravity* **21** 4041
[234] Iorio L and Corda C 2009 *AIP Conf. Proc.* **1168** 1072
[235] Capozziello S, Corda C and de Laurentis M F 2008 *Phys. Lett.* B **669** 255
[236] Capozziello S and de Laurentis M 2011 *Phys. Rep.* **509** 167
[237] Corda C 2011 *PhRvD* **83** 062002
[238] Corda C, Ali S A and Cafaro C 2010 *Int. J. Mod. Phys.* D **19** 2095
[239] Barcelo C, Liberati S and Visser M 2005 *Living Rev. Rel.* **8** 12
[240] Unruh W G 1981 *Phys. Rev. Lett.* **46** 1351
[241] Visser M 1998 *Advanced School on Cosmology and Particle Physics (ASCPP 98)* arXiv:gr-qc/9901047
[242] Puthoff H E 2008 arXiv:0808.3404
[243] Kivotides D 2021 *Int. J. Mod. Phys.* C **32** 2150129
[244] Chakraborty C, Ganguly O and Majumdar P 2018 *Annalen Phys.* **530** 1700231
[245] Banerjee A, Koley R and Majumdar P 2018 arXiv:1808.01828
[246] Ratzel D, Wilkens M and Menzel R 2016 *New J. Phys.* **18** 023009
[247] Spengler F, àö§tzel D R and Braun D 2022 *New J. Phys.* **24** 053021
[248] Padgett M J 2017 *Opt. Express* **25** 11265
[249] Strohaber J 2013 *Gen. Rel. Grav.* **45** 2457
[250] Schneiter F, Rätzel D and Braun D 2018 *Classical Quantum Gravity* **35** 195007
[251] Schneiter F, Rätzel D and Braun D 2019 *Classical Quantum Gravity* **36** 205007
[252] Schneiter F, Rätzel D and Braun D 2019 *Classical Quantum Gravity* **36** 119501
[253] Page D N 2005 *New J. Phys.* **7** 203
[254] Kiefer C and Weber C 2005 *Annalen Phys.* **14** 253
[255] Hammad F, Landry A and Mathieu K 2021 *Int. J. Mod. Phys.* D **30** 2150004
[256] Cong W, Bicak J, Kubiznak D and Mann R B 2021 *Phys. Rev.* D **103** 024027
[257] Adler R J and Chen P 2010 *Phys. Rev.* D **82** 025004
[258] Adler R J, Chen P and Varani E 2012 *Phys. Rev.* D **85** 025016
[259] Bahrami M, Großardt A, Donadi S and Bassi A 2014 *New J. Phys.* **16** 115007
[260] Ruffini R and Bonazzola S 1969 *Phys. Rev.* **187** 1767
[261] Brizuela D and Duran-Cabacés A 2022 *Phys. Rev.* D **106** 124038
[262] Manfredi G 2015 *Gen. Rel. Grav.* **47** 1
[263] Zhao Q, Faizal M and Zaz Z 2017 *Phys. Lett.* B **770** 564
[264] Lanzagorta M 2012 arXiv:1212.2200
[265] Lanzagorta M and Salgado M 2016 *Class. Quant. Grav.* **33** 105013
[266] Jesus W D R, Souza P R A, Santos A F and Khanna F C 2022 *Eur. Phys. J. Plus* **137** 260
[267] Kim J-W 2022 *Phys. Rev.* D **106** L081901
[268] Bonnor W B 1964 *MNRAS* **128** 33
[269] Mach E 1960 *The Science of Mechanics: A Critical and Historical Account of Its Development* (Open Court)
[270] Vassallo A and Hoefer C 2019 arXiv:1901.10766
[271] Christillin P and Barattini L 2012 arXiv:1206.4593
[272] Schmid C 2009 *Phys. Rev.* D **79** 064007