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Truncation quantization in the edge calculus

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Abstract
Pseudodifferential operators on the half-space associated with classical symbols of order zero without transmission property are shown to belong to the so-called edge algebra.

1 Introduction

This paper concerns the analysis of pseudodifferential operators on smooth manifolds with boundary and their underlying symbols. Actually, the notion of pseudodifferential operators on manifolds with boundary is not uniquely defined and must be specified according to a particular problem or a specific context. Since, roughly speaking, the interesting effects occur near the boundary, we shall focus in this paper on the analysis of operators and symbols in the local model, i.e., the half-space $\mathbb{R}^{1+q}_+ = \{(r, y) \in \mathbb{R} \times \mathbb{R}^q \mid r > 0\}$ where q is a non negative integer.

In the context of classical boundary value problems (like the Dirichlet problem for the Laplacian), solution operators can be described in the framework of Boutet de Monvel's calculus, which was introduced in [1]. Here, to define pseudodifferential operators, one employs an embedding of the underlying manifold in a smooth surrounding space. In the local model, the half-space is embedded in the full space $\mathbb{R}^{1+q}$. One considers symbols which have the so-called transmission property (see [9] for a detailed discussion) with respect to the boundary $r = 0$. The associated operator on the half-space is then formed via the Fourier transform combined with the extension-by-zero operator and the operator of restriction from the full to the half-space; see Sect. 2 for more details. This passage from symbol to the operator we shall refer to

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Published online: 24 October 2023 Birkhäuser as *truncation quantization*. As it turns out, such operators preserve smoothness of functions up to the boundary. This is why the natural function spaces in this context are the usual Bessel potential spaces (restricted to the half-space).

Recently, in a series of papers [6–8], Grubb modified this approach in order to study the fractional Laplacian (and similar operators) on bounded domains. In this context, symbols have a so-called μ-transmission property; the before mentioned transmission property in Boutet de Monvel's calculus corresponds to the case $\mu = 0$. There arise function spaces not referring to smoothness up to the boundary anymore but to smoothness modulo a weight-factor r^{μ}.

In this work we study the truncation quantization for zero order symbols which do not possess any kind of transmission property. To this end we interpret the boundary as an edge, respectively the half-space as a wedge, and employ an alternative intrinsic quantization based on the Mellin transform in the direction normal to the boundary (i.e., in r-direction). This quantization does not make reference to a surrounding space and has its origin in the so-called edge calculus for operators on manifolds with edges with corresponding edge degenerate symbols, see for instance [20]. The main result is Theorem 6.6 where operators based on the truncation quantization are expressed in terms of the intrinsic quantization and thus are shown to be particular elements of the edge-calculus on the half-space. As a particular consequence one finds that truncated operators act in a natural scale of so-called edge Sobolev spaces; we refer the reader again to the introductory Sect. 2 for more details.

The present paper is closely related to our previous article [21] where the same kind of operators are discussed. Here we focus on a more detailed analysis of Mellin representations of so-called decoupled symbols which in turn is based on a Mellin representation of operators on the half-axis which is interpreted as a manifold with conical singularity in the origin. The latter result originates from the work of Eskin [5] which was refined by Rempel and Schulze in [16] and [19], respectively; see also Liu and Schulze [14]. Since these results are quite intricate, we make an effort to present a self-contained and relatively short proof for the case of the half-axis, see Sect. 4. In this connection we also make some useful observations on naturally occurring meromorphic Mellin symbols which are quotients of translated Gamma functions, see Theorem 3.8 and its proof in the Appendix. Though we shall not enter in details, let us remark that it is essentially this meromorphic structure which determines the mapping properties of truncated operators on functions with asymptotics at the boundary which, in general, will be more complex than smoothness or weighted smoothness up to the boundary. Let us mention in this context the work Liu and Witt [15] where the authors, based on the previously mentioned Mellin representation, construct algebras of operators on the half-axis that act between function spaces with prescribed asymptotic behaviour.

Boundary value problems for pseudodifferential operators have been studied in the past by means of different approaches. A basic intention has always been to understand the parametrices to elliptic problems for differential operators. In addition, there are numerous modifications of elliptic boundary value problems, such as transmission problems, Sobolev-type problems, problems with global projection conditions, problems in non-compact configurations with exit conditions at infinity, elliptic complexes, or combinations of such situations. In recent years increased the interest in ellipticity on spaces ("manifolds") with singularities, such as cones or polyhedral domains or domains with edges. The (inter)faces in such configurations may be regarded as generalisations of a boundary and, for obvious reasons, some intrinsic approach seems to be advisable, such as the method of using Mellin operators, if necessary with operator-valued symbols and with additional parameters in higher singular cases. The present investigation is also an attempt to draw attention to the singular analysis and pseudodifferential ideas, both of interest for applications and for problems of the above-mentioned kind.

2 Motivation: truncated operators in the edge algebra

Let us motivate here, on an informal level, some of the key ideas of this paper. We start with the reformulation of pseuso-differential operators on a (Euklidean) product-space as pseudo-differential operators with operator-valued symbols. We refer the reader to Sect. 7.2 for a short review of the theory of such operators.

Let $a(r, y, \rho, \eta)$ be a pseudo-differential symbol (detailed definitions of the relevant symbol classes will be provided in the subsequent sections) with variables $(r, y) \in \mathbb{R}^{1+q}$ and corresponding covariables (ρ, η). The associated pseudo-differential operator $A = \mathrm{op}(a)$ is given by

$$(Au)(r, y) = (2\pi)^{-(1+q)} \iint e^{i(r,y)(\rho,\eta)} a(r, y, \rho, \eta)\widehat{u}(\rho, \eta)\, d\rho d\eta,$$

where $\widehat{u}$ denotes the Fourier transform of the function $u \in \mathscr{S}(\mathbb{R}^{1+q})$. Now we identify functions of variables (r, y) as functions of the variable y taking values in functions of the variable r; for example, $\mathscr{S}(\mathbb{R}^{1+q})$ is identified with $\mathscr{S}(\mathbb{R}^q, \mathscr{S}(\mathbb{R}))$. Then

$$(Au)(\cdot, y) = [\mathrm{op}(\mathbf{a})\mathbf{u}](y) = (2\pi)^{-q} \int e^{iy\eta}\mathbf{a}(y, \eta)\widehat{\mathbf{u}}(\eta)\, d\eta,$$

where $\mathbf{u}(y) = u(\cdot, y)$ and $\mathbf{a}(y, \eta)$, for every (y, η), is the pseudo-differential operator on $\mathbb{R}$ given by

$$[\mathbf{a}(y, \eta)v](r) = [\mathrm{op}(a)(y, \eta)v](r) = (2\pi)^{-1} \int e^{ir\rho} a(r, y, \rho, \eta)\widehat{v}(\rho)\, d\rho.$$

In this sense, we have now $A = \mathrm{op}(\mathbf{a})$ with an operator-valued symbol $\mathbf{a}$. If a is a symbol of order zero (and has suitable behaviour in (r, y), for example, is compactly supported), it can be shown that

$$\mathbf{a} \in S^0(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}), H^s(\mathbb{R})), \qquad s \in \mathbb{R}. \tag{2.1}$$

In a similar way we can consider operators on a half-space: Let $\mathbf{r}^+$ denote the operator of restriction to $\mathbb{R}^n_+$ of distributions defined on $\mathbb{R}^n$, while $\mathbf{e}^+$ denotes the operator of extension by zero from the half-space to the full space (whenever such an operation makes sense, for example as operator $L^2(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n)$. We use the same notation $\mathbf{e}^+$ and $\mathbf{r}^+$ for different dimensions n. With a as above we associate the operator $A^+ = \mathbf{r}^+\mathrm{op}(a)\mathbf{e}^+$ (initially defined on $\mathscr{S}(\mathbb{R}^{1+q}_+) = \mathscr{S}(\mathbb{R}^{1+q})\big|_{\mathbb{R}^{1+q}_+}$) and then, similarly as before, we re-write A^+ as $\mathrm{op}(\mathbf{a}^+)$, where

$$\mathbf{a}^+(y,\eta) = \mathbf{r}^+\mathbf{a}(y,\eta)\mathbf{e}^+ = \mathbf{r}^+\mathrm{op}(a)(y,\eta)\mathbf{e}^+ =: \mathrm{op}^+(a)(y,\eta);$$

note that in the last formula the operators of extension and restriction are in dimension 1. As a matter of fact, for $\mathbf{a}^+$ there is no simple analogue of (2.1). In case a has order zero and satisfies the transmission property with respect to $r = 0$, it can be shown that

$$\mathbf{a}^+ \in S^0(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+), H^s(\mathbb{R}_+)), \qquad s > -1/2, \tag{2.2}$$

and that A^+ is an element of Boutet de Monvel's algebra of boundary value problems (see [18] for a short introduction). From (2.2) and the general theory of operators with operator-valued symbols one can derive the well-known fact that A^+ preserves smoothness up to the boundary, i.e., A^+ acts continuously in the Bessel potential space $H^s(\mathbb{R}^{1+q}_+)$ for every $s > -1/2$.

If a violates the transmission property the situation is more complicated. In this paper we analize the case of zero-order symbols and we will show, in particular, that

$$\mathbf{a}^+ \in S^0(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,0}(\mathbb{R}_+), \mathcal{K}^{s,0}(\mathbb{R}_+)), \qquad s \in \mathbb{R}, \tag{2.3}$$

with the cone Sobolev spaces $\mathcal{K}^{s,0}(\mathbb{R}_+)$, see Sect. 7.1.2. The abstract theory of operator-valued symbols then shows that A^+ acts continuously in the scale of so-called edge Sobolev space

$$\mathcal{W}^{s,0}(\mathbb{R}^{1+q}_+) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,0}(\mathbb{R}_+)), \qquad s \in \mathbb{R},$$

(see Sect. 7.2 again). In fact, in this paper we shall demonstrate a much stronger result than (2.3), namely that $\mathbf{a}^+(y,\eta)$ is a so-called edge symbol, see Theorem 6.6. Hence, for every zero-order symbol, A^+ is an element of the edge algebra on the half-space. This allows to obtain further mapping properties (and others) of A^+ from the theory of pseudo-differential operators on manifolds with edges; we refer the reader to existing literature like [13, 19], and [20].

3 Symbols and pseudo-differential operators

We shall introduce various symbol classes which we will employ throughout this paper. Let E be a Fréchet space with topology give by a system of semi-norms p_j, $j \in \mathbb{N}$.

3.1 Fourier symbols

Definition 3.1 Denote by $S^\nu(\mathbb{R}^m, E)$, $\nu \in \mathbb{R}$, the space of all smooth functions $a(w) : \mathbb{R}^m \to E$ such that, for every $N \in \mathbb{N}$,

$$\|a\|_N := \sup\Big\{p_j(D_w^\alpha a(w))\langle w\rangle^{|\alpha|-\nu} \mid w \in \mathbb{R}^m,\ |\alpha| + j \le N\Big\} < +\infty.$$

Passing to the intersection over all $\nu \in \mathbb{R}$ we obtain the space of regularizing symbols that coincides with the space of rapidly decreasing, E-valued functions,

$$S^{-\infty}(\mathbb{R}^m, E) := \bigcap_{\nu\in\mathbb{R}} S^\nu(\mathbb{R}^m, E) = \mathscr{S}(\mathbb{R}^m, E).$$

A function $a(w) \in C^\infty(\mathbb{R}^m \setminus \{0\}, E)$ is called (positively) homogeneous of degree $\mu \in \mathbb{C}$ if

$$a(\lambda w) = \lambda^\mu a(w) \qquad \forall\, w \neq 0 \quad \forall\, \lambda > 0.$$

The space of all such symbols is denoted by $S^{(\mu)}(\mathbb{R}^m \setminus 0, E)$. By restriction in w to the unit sphere, we may identify $S^{(\mu)}(\mathbb{R}^m \setminus 0, E)$ with $C^\infty(S^{m-1}, E)$ and thus obtain a Fréchet topology. In the following definition, and throughout the paper, we let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be the set of non-negative integers.

Definition 3.2 A symbol $a \in S^{\mathrm{Re}\,\mu}(\mathbb{R}^m, E)$ is called classical of order $\mu \in \mathbb{C}$ if there exists a sequence of homogeneous symbols $a_{(\mu-k)} \in S^{(\mu-k)}(\mathbb{R}^m \setminus 0, E)$, $k \in \mathbb{N}_0$, such that

$$r_{a,N}(w) = a(w) - \chi(w) \sum_{k=0}^{N-1} a_{(\mu-k)}(w) \ \in\ S^{\mathrm{Re}\,\mu-N}(\mathbb{R}^m, E)$$

for every $N \in \mathbb{N}_0$ (in case $N = 0$ the sum is considered to be 0). Here, $\chi(w)$ is an arbitrary (fixed) zero excision function, i.e., $\chi \in C^\infty(\mathbb{R}^m)$ vanishes in a neighborhood of the origin and $1 - \chi$ has cpmpact support. The space of all such classical symbols is denoted by $S^\mu_{\mathrm{cl}}(\mathbb{R}^m, E)$.

The homogeneous components $a_{(\mu-k)}$ are uniquely determined by a. The projective topology with respect to the maps

$$\begin{aligned} a \mapsto a_{(\mu-k)} &: S^\mu_{\mathrm{cl}}(\mathbb{R}^m, E) \longrightarrow S^{(\mu-k)}(\mathbb{R}^m \setminus 0, E) \cong C^\infty(S^{m-1}, E), \qquad k \in \mathbb{N}_0,\\ a \mapsto r_{a,N} &: S^\mu_{\mathrm{cl}}(\mathbb{R}^m, E) \longrightarrow S^{\mu-N}(\mathbb{R}^m, E), \qquad N \in \mathbb{N}_0, \end{aligned}$$

turns $S^\mu_{\mathrm{cl}}(\mathbb{R}^m, E)$ into a Fréchet space.

If $V \subset \mathbb{R}^n$ is some open set, replacing in the above constructions E by the Fréchet space $C^\infty(V, E)$ yields the spaces

$$S^\mu_{(\mathrm{cl})}(V \times \mathbb{R}^m, E) := S^\mu_{(\mathrm{cl})}(\mathbb{R}^m, C^\infty(V, E)). \tag{3.1}$$

In the particular case $E=\mathbb{C}$ we shall simply write $S^{\mu}_{(\mathrm{cl})}(V\times\mathbb{R}^m)$.

Remark 3.3 It is often very useful to use the fact that $S^{\mu}_{(\mathrm{cl})}(V\times\mathbb{R}^m,E)$ can be identified with the space $C^\infty(V)\,\widehat{\otimes}_\pi\,S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^m,E)$, where $E_1\,\widehat{\otimes}_\pi\,E_2$ denotes the completed projective tensor-product of two Fréchet spaces E_1 and E_2. In particular, by a result of Pietsch, given an arbitrary symbol $a\in S^{\mu}_{(\mathrm{cl})}(V\times\mathbb{R}^m,E)$ there exist bounded sequences $(p_j)\subset C^\infty(V)$, $(b_j)\subset S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^m,E)$, and an absolutely summable numerical sequence (λ_j) such that

$$a=\sum_{j=0}^{+\infty}\lambda_j p_j b_j,$$

see [17]. In many situations this result allows to reduce the analysis of general symbols to that of "product-symbols" of the form $a=pb$ with $b\in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^m,E)$ and $p\in C^\infty(V)$.

3.2 Holomorphic and meromorphic Mellin symbols

Definition 3.4 Denote by $M^{\mu}_{\mathcal{O}}(\mathbb{R}^m,E)$, $\mu\in\mathbb{R}$, the space of all holomorphic functions $h(z):\mathbb{C}\to S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m_w,E)$ (or, in case $m=0$, with values in E) such that

$$h_\delta(\tau,w):=h(\delta+i\tau,w)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{m+1}_{(\tau,w)},E)$$

locally uniformly in $\delta\in\mathbb{R}$. If $E=\mathbb{C}$, we simply write $M^{\mu}_{\mathcal{O}}(\mathbb{R}^m)$. In case $m=0$ we simply write $M^{\mu}_{\mathcal{O}}(E)$ and $M^{\mu}_{\mathcal{O}}$, respectively.

This is a Fréchet space in the obvious way. The space of regularizing symbols is denoted by

$$M^{-\infty}_{\mathcal{O}}(\mathbb{R}^m,E):=\bigcap_{\mu\in\mathbb{R}}M^{\mu}_{\mathcal{O}}(\mathbb{R}^m,E).$$

For the definition of meromorphic symbols we shall need the following definition; in it we shall use the projection $\pi_{\mathbb{C}}:\mathbb{C}\times\mathbb{N}_0\to\mathbb{C}$ defined by $\pi_{\mathbb{C}}(z,n)=z$.

Definition 3.5 A set $\mathcal{P}\subset\mathbb{C}\times\mathbb{N}_0$ is a called a (discrete) asymptotic type for Mellin symbols if $\pi_{\mathbb{C}}:\mathcal{P}\to\mathbb{C}$ is injective and $\pi_{\mathbb{C}}(\mathcal{P})\cap\{z\in\mathbb{C}\mid|\mathrm{Re}\,z|\le c\}$ is finite for any choice of the constant $c>0$.

In other words, an asymptotic type $\mathcal{P}$ is a set of the form

$$\mathcal{P}=\{(p_j,n_j)\in\mathbb{C}\times\mathbb{N}_0\mid j\in J\},\tag{3.2}$$

where either J is finite or $J=\mathbb{Z}$ and $\mathrm{Re}\,p_j\to\pm\infty$ if $j\to\pm\infty$; moreover, $p_j\neq p_k$ if $j\neq k$. Therefore, if h is a meromorphic function on $\mathbb{C}$ with poles in $\pi_{\mathbb{C}}(\mathcal{P})$ and $\chi\in C^\infty(\mathbb{C})$ is a zero excision function, it makes sense to define

$$\widehat{h}(z) := h(z) \prod_{p \in \pi_{\mathbb{C}}(\mathcal{P})} \chi(z-p), \tag{3.3}$$

since on any compact set only a finite number of factors is different from 1.

Definition 3.6 Let $\mathcal{P}$ be an asymptotic type as in (3.2). Denote by $M_{\mathcal{P}}^{-\infty}(\mathbb{R}^m, E)$ the space of all meromorphic functions $h : \mathbb{C} \to S^{-\infty}(\mathbb{R}^m_w, E)$, having poles at most in the points p_j of order at most $n_j + 1$ and such that

$$\widehat{h}_\delta(\tau, w) := \widehat{h}(\delta + i\tau, w) \in S^{-\infty}(\mathbb{R}^{m+1}_{(\tau,w)}, E)$$

locally uniformly in $\delta \in \mathbb{R}$, cf. (3.3). If $E = \mathbb{C}$, we simply write $M_{\mathcal{P}}^{\mu}(\mathbb{R}^m)$. In case $m = 0$ we simply write $M_{\mathcal{P}}^{\mu}(E)$ and $M_{\mathcal{P}}^{\mu}$, respectively.

Also this is a Fréchet space and thus it makes sense to define

$$M_{\mathcal{P}}^{\mu}(\mathbb{R}^m, E) := M_{\mathcal{O}}^{\mu}(\mathbb{R}^m, E) + M_{\mathcal{P}}^{-\infty}(\mathbb{R}^m, E) \tag{3.4}$$

as a non-direct sum of two Fréchet spaces. As above, replacing E by $C^\infty(V, E)$ yields the spaces

$$M_{\mathcal{P}}^{\mu}(V \times \mathbb{R}^m, E) := M_{\mathcal{P}}^{\mu}(\mathbb{R}^m, C^\infty(V, E)). \tag{3.5}$$

Of particular importance for us will be the cases $V = \mathbb{R}_+ \times U$ and $V = \overline{\mathbb{R}}_+ \times U$ with $U \subset \mathbb{R}^m$ an open set.

The following Example 3.7 and Theorem 3.8 introduce symbols which are fundamental in the subsequent considerations.

Example 3.7 Let $\mathcal{P} = \{(j, 0) \mid j \in \mathbb{Z}\}$. Then

$$g^{\pm}(z) := \frac{1}{1 - e^{\mp 2\pi i z}} \in M_{\mathcal{P}}^{0}, \qquad g(z) := \frac{e^{i\pi z}}{1 - e^{2\pi i z}} \in M_{\mathcal{P}}^{-\infty},$$

cf. [19, Example 1.1.52] for instance; see also [5]. Observe that $g^+(z) + g^-(z) \equiv 1$, $g^+(z)g^-(z) \equiv -g^2(z)$, and $g^{\pm}(z+1) \equiv g^{\pm}(z)$.

Theorem 3.8 *With $\Gamma(z)$ being the usual Gamma-function, let us define*

$$f_m(z) := \frac{\Gamma(1-z)}{\Gamma(1-z+m)}, \qquad m \in \mathbb{C}. \tag{3.6}$$

Then $f_m \in M_{\mathcal{P}_m}^{-m}$ where the asymptotic type $\mathcal{P}_m$ satisfies

(i) $\mathcal{P}_m = \emptyset$ if $m \in \mathbb{Z}$ and $m \leq 0$,
(ii) $\mathcal{P}_m = \{(j, 0) \mid j = 1, 2, \ldots, m\}$ if $m \in \mathbb{Z}$ and $m \geq 1$,
(iii) $\mathcal{P}_m = \{(j, 0) \mid j = 1, 2, 3 \ldots\}$ if $m \notin \mathbb{Z}$.

Moreover, f_m in case (i) has simple zeros in every integer $-m + 1 \leq k \leq 0$, in case (ii) it has no zeros at all, while in case (iii) it has simple zeros in $k + m$, $k \in \mathbb{N}$.

The previous theorem is frequently used in the literature; for convenience of the reader, we are going to give a self-contained proof in the appendix.

4 Mellin representation of truncated operators on $\mathbb{R}_+$

Given a symbol $a(r,\rho) \in S^0_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R})$, independent of r for large values of r, we consider the "truncated" operator

$$\mathrm{op}^+(a) : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+),$$

defined by

$$\mathrm{op}^+(a)u = \mathbf{r}^+(\mathrm{op}(a)\mathbf{e}^+u), \qquad u \in L^2(\mathbb{R}_+),$$

where $\mathbf{e}^+ : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes the operator of extension by zero, $\mathbf{r}^+ : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$ the operator of restriction; moreover, $\mathrm{op}(a)$ is the usual pseudo-differential operator on $\mathbb{R}$ associated with a. Note that $\mathrm{op}^+(a) = \mathrm{op}^+(b)$ for any other symbol $b(r,\rho) \in S^0_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R})$ satisfying $b(r,\rho) = a(r,\rho)$ whenever $r > 0$.

In the present section we will show that $\mathrm{op}^+(a)$ has a particular representation as a Mellin pseudo-differential operator, i.e., belongs to the so-called cone algebra. The result itself is known from [19, Theorem 2.1.26], however we shall give here a more concise proof which constitutes the base for the analysis of truncated symbols in relation with the edge algebra in Sect. 6.3.

4.1 Truncated operators and the cone algebra on $\mathbb{R}_+$

For a precise formulation of the representation theorem we need to recall the definition of the cone-algebra on the half-axis. Set

$$\Gamma_\delta := \{z \in \mathbb{C} \mid \mathrm{Re}\, z = \delta\}, \qquad \delta \in \mathbb{R}, \tag{4.1}$$

and recall that with a symbol $h(r,z) \in M^\mu_{\mathcal{P}}(\mathbb{R}_+)$ such that $\pi_{\mathbb{C}}\mathcal{P} \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$ we associate the Mellin pseudo-differential operator $\mathrm{op}^\gamma_M(h)$ given by

$$[\mathrm{op}^\gamma_M(h)u](r) := \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} h(r,z)\mathcal{M}u(z)\,dz,$$

where $\mathcal{M}$ denotes the Mellin transform defined by

$$(\mathcal{M}u)(z) = \int_0^{+\infty} r^z u(r)\,\frac{dr}{r}.$$

There exists an extensive literature on Mellin pseudo-differential operators; see for example [12]. In the sequel a function $\omega \in C^\infty_0(\overline{\mathbb{R}}_+)$ is called a *cut-off function* if it is constant 1 in some neighborhood of the origin. The notation $\omega \prec \omega'$ with two cut-off functions ω and ω' means that ω' is constant 1 in a neighborhood of the support of ω or, equivalently, that ω and $1-\omega'$ have disjoint supports in $\overline{\mathbb{R}}_+$.

Definition 4.1 Let $\gamma, \mu \in \mathbb{R}$ and $j, k \in \mathbb{N}$. Then

$$L^{\mu-j}(\mathbb{R}_+, \boldsymbol{g}), \qquad \boldsymbol{g} = (\gamma, \gamma-\mu, (-k, 0]),$$

is the space of all operators A of the form

$$A = \omega\, A_M\, \omega' \;+\; (1-\omega)\, A_\psi\, (1-\omega'') \;+\; M \;+\; G, \tag{4.2}$$

where $\omega, \omega', \omega'' \in C_0^\infty(\overline{\mathbb{R}}_+)$ are cut-off functions such that $\omega'' \prec \omega \prec \omega'$ and with

(i) a Mellin operator $A_M = r^{-\mu+j}\operatorname{op}_M^\gamma(h)$ with holomorphic Mellin symbol $h(r,z) \in M_{\mathcal{O}}^{\mu-j}(\overline{\mathbb{R}}_+)$,
(ii) a pseudo-differential operator $A_\psi = \operatorname{op}(a)$ with symbol $a(r,\rho) \in S_{\mathrm{cl}}^{\mu-j}(\mathbb{R}_\rho; S^0(\mathbb{R}_r))$,
(iii) a so-called *smoothing Mellin operator*

$$M = \omega \left\{ \sum_{\ell=0}^{k-j-1} r^{-\mu+j+\ell} \operatorname{op}_M^{\gamma_\ell}(h_\ell) \right\} \omega',$$

where $h_\ell \in M_{\mathcal{R}_\ell}^{-\infty}$ are meromorphic Mellin symbols and $\gamma - j - \ell \le \gamma_\ell \le \gamma$ with $\pi_{\mathbb{C}}\mathcal{R}_\ell \cap \Gamma_{\frac{1}{2}-\gamma_\ell} = \emptyset$,
(iv) a *Green operator* $G \in C_G(\mathbb{R}_+, \mathbf{g})$ (cf. Section 7.1.5 in the appendix).

With $A \in L^\mu(\mathbb{R}_+, \boldsymbol{g})$ as above one associates the so-called *conormal symbols*,

$$\sigma_M^{\mu-\ell}(A)(z) = \frac{1}{\ell!}(\partial_r^\ell h)(0,z) + h_\ell(z), \qquad 0 \le \ell \le k-1. \tag{4.3}$$

Note that $L^{\mu-j-1}(\mathbb{R}_+, \boldsymbol{g}) \subset L^{\mu-j}(\mathbb{R}_+, \boldsymbol{g})$ for every j. Moreover, one defines

$$L^{\mu-j}(\mathbb{R}_+, (\gamma, \gamma-\mu, (-\infty, 0])) := \bigcap_{k\in\mathbb{N}} L^{\mu-j}(\mathbb{R}_+, (\gamma, \gamma-\mu, (-k, 0])).$$

The terminology *cone algebra* originates from the fact that this class is closed under compositions, i.e., given $A \in L^{\mu-j}(\mathbb{R}_+, (\gamma, \gamma-\mu, (-k, 0]))$ and $B \in L^{\nu-\ell}(\mathbb{R}_+, (\gamma-\mu, \gamma-\mu-\nu, (-k, 0]))$ then $BA \in L^{\mu+\nu-j-\ell}(\mathbb{R}_+, (\gamma, \gamma-\mu-\nu, (-k, 0]))$. Moreover, there exists a notion of *ellipticity* in the cone algebra, which is defined in terms of the invertibility of associated principal and conormal symbols; the ellipticity of an element $A \in L^\mu(\mathbb{R}_+, (\gamma, \gamma-\mu, (-k, 0]))$ is then equivalent to the existence of a parametrix $B \in L^{-\mu}(\mathbb{R}_+, (\gamma-\mu, \gamma, (-k, 0]))$, i.e., B is an inverse of A modulo Green operators. For details let us refer the reader to [19].

Now the announced theorem reads as follows:

Theorem 4.2 *Let $a(r,\rho) \in S_{\mathrm{cl}}^0(\mathbb{R}\times\mathbb{R})$ be independent of r for large r. Then*

$$\operatorname{op}^+(a) \in L^0(\mathbb{R}_+, \boldsymbol{g}) \quad \text{for} \quad \boldsymbol{g} = (0, 0, (-\infty, 0]). \tag{4.4}$$

If the homogeneous components $a_{(-j)}(r,\rho)$ of $a(r,\rho)$ of order $-j$ have the form

$$a_{(-j)}(r,\rho) = \{a_j^+(r)\theta^+(\rho) + a_j^-(r)\theta^-(\rho)\}(i\rho)^{-j}, \qquad a_j^\pm(r) \in C^\infty(\mathbb{R}), \tag{4.5}$$

where $\theta^\pm(\rho)$ denotes the characteristic function of $\mathbb{R}_\pm \subset \mathbb{R}$, the sequence of conormal symbols of $\mathrm{op}^+(a)$ is given by

$$\sigma_M^{-\ell}(\mathrm{op}^+(a))(z) = \sum_{j+k=\ell} \frac{1}{j!}\Big(\partial_r^j a_k^+(0)g^+(z) + \partial_r^j a_k^-(0)g^-(z)\Big) f_k(z). \tag{4.6}$$

For the definition of the functions $g^\pm$ and f_k recall Example 3.7 and Theorem 3.8, respectively.

The proof shall be given in the following Sect. 4.2.

4.2 Proof of Theorem 4.2

For cut-off functions $\omega'' \prec \omega \prec \omega'$ we have

$$\mathrm{op}^+(a) = \omega\mathrm{op}^+(a)\omega' + (1-\omega)\mathrm{op}(a)(1-\omega'') + G$$

where G has an integral kernel belonging to $\mathscr{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$, hence both G and G^* map $L^2(\mathbb{R}_+)$ into $\mathscr{S}(\overline{\mathbb{R}}_+) = \mathscr{S}_T^0(\mathbb{R}_+)$ where $T = \{(-j,0) \mid j \in \mathbb{N}\} \in \mathrm{As}(0,(-\infty,0])$. Thus G is a Green operator. Therefore we can concentrate on the operator $\omega\mathrm{op}^+(a)\omega'$.

The symbol $a(r,\rho)$ has an asymptotic expansion

$$a(r,\rho) \sim \sum_{k=0}^{\infty} \chi(\rho)\Big\{a_k^+(r)\theta^+(\rho) + a_k^-(r)\theta^-(\rho)\Big\}(i\rho)^{-k}, \tag{4.7}$$

where $\chi(\rho)$ is an arbitrary excision function.

Definition 4.3 With $\mu \in \mathbb{C}$ and $\delta > 0$ define

$$l_\pm^\mu(\rho) = l_\pm^\mu(\rho,\delta) = (\delta \pm i\rho)^\mu, \qquad \rho \in \mathbb{R}.$$

In the expansion (4.7) we avoid the singularity of $(i\rho)^{-k}$ at $\rho = 0$ by inserting the asymptotic expansion

$$\chi(\rho)(i\rho)^{-k} \sim \sum_{j=k}^{\infty} \binom{-k}{j-k}(-\delta)^{j-k} l_+^{-j}(\rho,\delta),$$

see (2.1.61) in (the proof of) [19, Lemma 2.1.12] for further details. Hence, if we set

$$c_j^\pm(r,\delta) = \sum_{k=0}^{j} \binom{-k}{j-k}(-\delta)^{j-k} a_k^\pm(r), \tag{4.8}$$

the symbol

$$q_{(M)}(r,\rho) := a(r,\rho) - \sum_{j=0}^{M} \chi(\rho)\Big\{\big[c_j^+(r,\delta)\theta^+(\rho) + c_j^-(r,\delta)\theta^-(\rho)\big] l_+^{-j}(\rho)\Big\} \tag{4.9}$$

belongs to $S^{-(M+1)}(\overline{\mathbb{R}}_+ \times \mathbb{R})$.

Remark 4.4 For purposes below let us remark that $c_0^\pm = a_0^\pm$ and

$$a_k^\pm(r) = \sum_{j=1}^{k} \binom{k-1}{k-j} (-\delta)^{k-j} c_j^\pm(r,\delta), \qquad k \geq 1.$$

This can be obtained for example by inserting above the expansions

$$l_+^{-j}(\rho,\delta) = \sum_{\ell=0}^{\infty} \binom{\ell+j-1}{\ell} (-\delta)^\ell \chi(\rho)(i\rho)^{-j-\ell}, \qquad j \geq 1,$$

and then comparing coefficients with the expansion (4.7).

For simplicity of presentation, in the sequel we shall often suppress δ from the notation.

Let us now analize the operators $\omega \mathrm{op}^+(\chi\theta^\pm l_+^{-j})\omega'$. Choose a cut-off function ω_0 with $\omega_0 \equiv 1$ on the supports of ω and ω'. Then

$$\omega \mathrm{op}^+(\chi\theta^\pm l_+^{-j})\omega' = \omega \mathrm{op}^+(\theta^\pm)\omega_0\, \omega_0' \mathrm{op}^+(l_+^{-j})\omega' + R \tag{4.10}$$

with

$$R = \omega \mathrm{op}^+(\chi\theta^\pm)(1-\omega_0)\mathrm{op}^+(l_+^{-j})\omega' + \omega \mathrm{op}^+((1-\chi)\theta^\pm)(1-\omega_0)\mathrm{op}^+(l_+^{-j})\omega';$$

here we have employed the relation

$$\mathrm{op}^+(\chi\theta^\pm l_+^{-j}) = \mathrm{op}^+(\chi\theta^\pm)\mathrm{op}^+(l_+^{-j}) \tag{4.11}$$

as operators $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, $j \in \mathbb{N}$; this relation holds true, since the holomorphicity of l_+^{-j} in the lower complex half-plane yields that $\mathrm{op}(l_+^{-j})$ leaves the space $e^+L^2(\mathbb{R}_+) \subset L^2(\mathbb{R})$ invariant and therefiore $(1-\mathbf{r}^+)\mathrm{op}(l_+^{-j})\mathbf{e}^+ = 0$ on $L^2(\mathbb{R}_+)$. Since ω and $1-\omega_0$ have disjoint supports, $1-\chi$ is compactly supported, and using that $\mathrm{op}^+(l_+^{-j})$ and its adjoint map $\mathscr{S}(\overline{\mathbb{R}}_+)$ into itself, it follows that R has an integral kernel in $\mathscr{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$.

The following result is due to [5]; a proof can also be found in [19], Proposition 2.1.4.

Lemma 4.5 *With the meromorphic Mellin symbols $g^{\pm}(z)$ from Example 3.7,*

$$\mathrm{op}^+(\theta^{\pm}) = \mathrm{op}_M^0(g^{\pm}).$$

Combining (4.9), (4.10), and Lemma 4.5 we find that

$$\omega \mathrm{op}^+(a)\omega' = \sum_{j=0}^{M} \omega\, \mathrm{op}_M^0(h_j)\, \omega_0\, \mathrm{op}^+(l_+^{-j})\, \omega' + R_M, \tag{4.12}$$

$$h_j(r,z) := c_j^+(r)g^+(z) + c_j^-(r)g^-(z), \tag{4.13}$$

with a remainder R_M that has a compactly supported integral kernel in $C^m(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$, where $m = m(M) \to +\infty$ as $M \to +\infty$.

For the following result in case of real order μ see Proposition 4.1.15 in [19].

Proposition 4.6 *Let $\mu \in \mathbb{C}$, $\gamma > -1/2$, $m \in \mathbb{N}$, and ω, ω' be arbitrary cut-off functions. Then, if $N = N(\mu, m) \in \mathbb{N}$ is taken sufficiently large,*

$$[\omega \mathrm{op}^+(l_+^{\mu})\omega' u](r) = \omega r^{-\mu} \sum_{k=0}^{N-1} \binom{\mu}{k} (r\delta)^k [\mathrm{op}_M^{\gamma}(f_{k-\mu})\omega' u](r) + (G_N u)(r) \tag{4.14}$$

for every $u \in C_0^{\infty}(\mathbb{R}_+)$, where G_N is an integral operator with integral kernel from $C_0^{2m}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ inducing continuous maps

$$G_N : \mathcal{K}^{0,\gamma}(\mathbb{R}_+) \to r^m C_0^m(\overline{\mathbb{R}}_+) \tag{4.15}$$

(for the definition of the functions $f_{k-\mu}$ recall Definition 3.6).

Proof Let us first consider μ with $\mathrm{Re}\,\mu < -1/2$. Fix $u \in C_0^{\infty}(\mathbb{R}_+)$ and let $r > 0$. For simplicity of notation set $u' = \omega' u$. Then

$$[\omega \mathrm{op}^+(l_+^{\mu})\omega' u](r) = \omega(r)[(\mathscr{F}^{-1} l_+^{\mu}) * \mathbf{e}^+ u'](r) = \omega(r)(\mathbf{e}^+ g * \mathbf{e}^+ u')(r),$$

where we have used the fact that the inverse Fourier transform of l_+^{μ} is the regular distribution with L^1-density $\mathbf{e}^+ g$ where $g(r) = r^{-\mu-1}e^{-\delta r}/\Gamma(-\mu)$. Using the Taylor expansion of $e^{-\delta r}$ we can write

$$g(r) = \frac{1}{\Gamma(-\mu)} \sum_{k=0}^{N-1} \frac{(-\delta)^k}{k!} r^{-\mu-1+k} + g_N(r), \tag{4.16}$$

where $g_N \in C^{\infty}(\mathbb{R}_+)$ and $\mathbf{e}^+ g_N$ belongs to $C^{2m}(\mathbb{R})$ with holomorphic dependence on $\mu \in \{z \mid \mathrm{Re}\, z < M\}$, $M \in \mathbb{N}$, provided $N > 2m + M + 1$.

Let $\widetilde{\omega}$ such that $\widetilde{\omega} \equiv 1$ in a neighborhood of the support of ω and write $g_N = g_N^0 + g_N^{\infty} := \widetilde{\omega} g_N + (1-\widetilde{\omega}) g_N$. Then the support of $\mathbf{e}^+ g_N^{\infty} * \mathbf{e}^+ u'$ is contained in that of $(1-\widetilde{\omega})$ (note that $\mathbf{e}^+ u'$ is supported in $\mathbb{R}_+$), hence $\omega(r)(\mathbf{e}^+ g_N^\infty * \mathbf{e}^+ u')(r) \equiv 0$ for every $u \in C_0^\infty(\mathbb{R}_+)$. If we define G_N by

$$(G_N u)(r) = \omega(r)(\mathbf{e}^+ g_n^0 * \mathbf{e}^+ u')(r)$$

then G_N has the required properties. In fact, by a simple application of Hölder inequality, $u \mapsto \mathbf{e}^+ u'$ extends to a continuous map $\mathcal{K}^{0,\gamma}(\mathbb{R}_+) \to L^1(\mathbb{R})$, hence $G_N : \mathcal{K}^{0,\gamma}(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ continuously. Moreover, $\mathbf{e}^+ g_n^0 * \mathbf{e}^+ u'$ belongs to $C^{2m}(\mathbb{R})$ and is supported in $\overline{\mathbb{R}}_+$, hence its restriction to $\mathbb{R}_+$ belongs to $r^m C^m(\overline{\mathbb{R}}_+)$ by Taylor expansion. The continuity of (4.15) follows from the closed graph theorem. The integral kernel of G_N is

$$k_N(r,s) = \omega(r)\mathbf{e}^+ g_N^0(r-s)\omega'(s), \qquad r, s > 0,$$

which is the restriction of a $C^{2m}(\mathbb{R} \times \mathbb{R})$-function to $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$.

Now let us focus on the terms of the finite sum in (4.16). Consider first the case $\gamma = 0$.

$$\begin{aligned}\omega(r)(\mathbf{e}^+ r^{-\mu-1+k} * \mathbf{e}^+ u')(r) &= \omega(r)\int_0^r (r-s)^{-\mu-1+k} u'(s)\,ds \\ &= \omega(r)\int_1^{+\infty} (t-1)^{-\mu-1+k} t^{-(k-\mu)} u'\Big(\frac{r}{t}\Big)\frac{dt}{t} \\ &= \omega(r)(b_{k-\mu} \star u')(r),\end{aligned} \tag{4.17}$$

where $\star$ denotes the Mellin-convolution and $b_\nu(r) := \theta_{[1,+\infty)}(r)(r-1)^{\nu-1} r^{-\nu}$ with $\theta_{[1,+\infty)}$ denoting the characteristic function of the interval $[1,+\infty)$. Note that $b_\nu \in L^2(\mathbb{R}_+)$ whenever $\operatorname{Re}\nu > 1/2$. Since the Mellin transform induces an isomorphism $\mathcal{M}_0 : L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2})$, we can write

$$b_{k-\mu} \star u' = (\mathcal{M}_0^{-1} h_{k-\mu}) \star u' = \mathrm{op}_M^0(h_{k-\mu})u'$$

with

$$\begin{aligned}h_\nu(z) = (\mathcal{M}_0 b_\nu)(z) &= \int_1^{+\infty} r^{z-\nu}(r-1)^{\nu-1}\,\frac{dr}{r} \\ &= \int_0^1 t^{-z}(1-t)^{\nu-1}\,dt = B(1-z,\nu) = \frac{\Gamma(1-z)\Gamma(\nu)}{\Gamma(1-z+\nu)} = \Gamma(\nu) f_\nu(z),\end{aligned}$$

where $B(\zeta, w)$ is the beta-function. Altogether we have obtained the claimed result for all μ with $\operatorname{Re}\mu < -1/2$. For the other μ we will argue by holomorphic extension.

Fixing $u \in C_0^\infty(\mathbb{R}_+)$ and $r > 0$, the left-hand side of (4.14) is an entire function of μ. As we shall see below, the same is true for any term in the finite sum on the right-hand side. Given $M \in \mathbb{N}$ and choosing $N > 2m + M + 1$, we have shown above that $(G_N u)(r)$ depends holomorphically on μ with $\operatorname{Re}\mu < M$. Since (4.14) holds true for $\operatorname{Re}\mu < -1/2$, it thus holds true for $\operatorname{Re}\mu < M$ by the identity theorem for holomorphic functions. Let us show now that

$$[\mathrm{op}_M^0(f_m)u'](r) = \int_{\Gamma_{1/2}} t^{-z} f_m(z)(\mathcal{M}_0 u')(z) đz$$

is an entire function of m. To this end it suffices to show that $\partial_m f_m(z)(\mathcal{M}_0 u')(z)$ belongs to $L^1(\Gamma_{1/2})$ locally uniformly in $m \in \mathbb{C}$. Since $\mathcal{M}_0 u' \in \mathscr{S}(\Gamma_{1/2})$ and

$$f_m(z) = f_{m+L}(z) \prod_{\ell=0}^{L-1} (1 - z + m + \ell)$$

for every $L \in \mathbb{N}$, it is enough to consider m with $\operatorname{Re} m \geq 1$. For these m,

$$f_m(1/2 + i\tau) = \frac{1}{\Gamma(m)} \int_0^{+\infty} e^{it\tau} e^{-t/2} (1 - e^{-t})^{m-1} \, dt,$$

cf. (7.7) and (7.5). Using that $\sup_{t \geq 0} |(1 - e^{-t})^{m-1}| \leq 1$ uniformly for $\operatorname{Re} m \geq 1$ we conclude that

$$\sup_{\tau \in \mathbb{R}} |\partial_\tau^k f_m(1/2 + i\tau)| \leq \frac{c_k}{|\Gamma(m)|}$$

for suitable constants c_k. Hence $\partial_z^k f_m \in L^\infty(\Gamma_{1/2})$ locally uniformly in m with $\operatorname{Re} m \geq 1$. Finally, an elementary calculation shows that

$$\partial_m f_m(z) = \partial_z f_m(z) + \psi(1 - z) f_m(z),$$

where $\psi(w) = \Gamma'(w)/\Gamma(w)$ is the digamma function. Noting that $|\psi(1/2 - i\tau)| \sim 1 + \log\langle\tau\rangle$ we obtain our claim.

To complete the proof it remains to observe that

$$\mathrm{op}_M^0(f_{k-\mu})u = \mathrm{op}_M^\gamma(f_{k-\mu})u, \qquad \gamma > -\frac{1}{2},$$

for every $u \in C_0^\infty(\mathbb{R}_+)$, since $f_{k-\mu}$ is holomorphic in the half-plane $\operatorname{Re} z > 0$ and using standard properties of Mellin pseudodifferential operators. □

Corollary 4.7 *For every integer $L \geq 1$,*

$$\omega \mathrm{op}^+(l_+^\mu)\omega' = \omega r^{-\mu} \sum_{k=0}^{L+\lfloor \operatorname{Re}\mu \rfloor - 1} \binom{\mu}{k} (r\delta)^k \mathrm{op}_M^0(f_{k-\mu})\omega' + R_L \tag{4.18}$$

(in case $L+\lfloor \operatorname{Re}\mu \rfloor \leq 0$, the finite sum is defined to be zero), where $R_L \in \mathscr{L}(L^2(\mathbb{R}_+))$ has the mapping properties

$$R_L : L^2(\mathbb{R}_+) \longrightarrow \mathcal{K}^{L,0;\infty}_{O_L}(\mathbb{R}_+), \qquad R_L^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{K}^{L,0;\infty}_{T_L}(\mathbb{R}_+),$$

where $O_L \in \operatorname{As}(0, (-L, 0])$ is the empty asymptotic type while

$$T_L = \{(-\ell, 0) \mid \ell = 0, 1, \ldots, L-1\} \in \operatorname{As}(0, (-L, 0]) \tag{4.19}$$

denotes Taylor asymptotics of depth L.

Proof Starting out from the identity (4.14) we obtain (4.18) with

$$R_L = \omega r^{-\mu} \sum_{k=\max(0, L+\lfloor \operatorname{Re}\mu \rfloor)}^{N} (r\delta)^k \operatorname{op}_M^0(f_{k-\mu})\omega' + G_N,$$

where we can choose N as big as we like. Taking N large enough, G_N has the indicated mapping properties. The same is true for the other terms, using standard mapping properties of Mellin pseudodifferential operators with meromorphic symbols. □

Inserting the representation (4.18) for every $\mu = -j$ in (4.12) and arguing similarly as in the proof of the previous corollary, we find

$$\omega \operatorname{op}^+(a)\omega' = \sum_{j=0}^{L-1} \sum_{k=0}^{L-j-1} \binom{-j}{k} \omega \operatorname{op}_M^0(h_j)\, \omega_0\, r^j (\delta r)^k \operatorname{op}_M^0(f_{k+j})\, \omega' + S_L \tag{4.20}$$

for every positive integer L, where $S_L \in \mathscr{L}(L^2(\mathbb{R}_+))$ is a remainder satisfying

$$S_L, S_L^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{K}^{L,0;\infty}_{T_L}(\mathbb{R}_+)$$

with T_L being as in (4.19).

Proposition 4.8 *Let $L > 0$ be an arbitrary integer number and*

$$b_j(r, z) := \{a_j^+(r)g^+(z) + a_j^-(r)g^-(z)\} f_j(z), \qquad j \in \mathbb{N}, \tag{4.21}$$

with $f_j(z)$ as in Example 3.7. Then, for any two cut-off functions ω, ω',

$$\omega \operatorname{op}^+(a)\omega' = \omega \sum_{j=0}^{L-1} r^j \operatorname{op}_M(b_j)\omega' + C_L,$$

where $C_L \in \mathscr{L}(L^2(\mathbb{R}_+))$ is a remainder satisfying

$$C_L : L^2(\mathbb{R}_+) \longrightarrow \mathcal{K}^{L,0;\infty}_{T_L}(\mathbb{R}_+), \qquad C_L^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{K}^{L,0;\infty}_{P_L}(\mathbb{R}_+), \tag{4.22}$$

where T_L is as in (4.19) and $P_L \in \mathrm{As}(0,(-L,0])$ is the asymptotic type defined as

$$P_L = \{(-\ell, 1) \mid \ell = 0, 1, \ldots, L-1\}. \tag{4.23}$$

Before proving this result let us make a remark. Evidently, the map $a \mapsto b_j = b_j(a)$ is a continuous map from $S^0_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R})$ to $S^0_{\mathrm{cl}}(\mathbb{R}\times\Gamma_{1/2})$. Consequently, the mapping $a \mapsto C_L = C_L(a)$ is continuous with values in $\mathscr{L}(L^2(\mathbb{R}_+))$. If we denote by X the subspace of $\mathscr{L}(L^2(\mathbb{R}_+))$ consisting of all operators satisfying (4.22) then X is a Fréchet space in an obvious way. By the closed graph theorem the map $a \mapsto C_L(a) : S^0_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R}) \to X$ is then continuous, too.

Proof For convenience of notation let us set $f_{kj} := \binom{-j}{k} f_{k+j}$ (where $f_{00}=1$). We shall show below that (4.20) implies that

$$\begin{aligned}\omega \mathrm{op}^+(a)\omega' &\equiv \omega \sum_{j=0}^{L-1}\sum_{k=0}^{L-j-1} r^j(\delta r)^k \mathrm{op}^0_M(h_j)\mathrm{op}^0_M(f_{kj})\,\omega' \\ &= \omega\,\mathrm{op}^0_M(h_0)\,\omega' + \sum_{\ell=1}^{L-1} r^\ell \sum_{\substack{j+k=\ell,\\ j\ge 1}} \delta^k \mathrm{op}^0_M(h_j f_{kj})\end{aligned} \tag{4.24}$$

modulo a remainder of the described type (note that $f_{k0}=0$ for all $k\ge 1$). Since

$$f_{\ell-j,j}(z) = (-1)^{\ell-j}\binom{\ell-1}{\ell-j}\prod_{p=1}^{\ell}(p-z)^{-1} = (-1)^{\ell-j}\binom{\ell-1}{\ell-j} f_\ell(z)$$

for all $1\le j\le \ell$, we find that

$$\sum_{\substack{j+k=\ell,\\ j\ge 1}} \delta^k h_j f_{kj} = \sum_{j=1}^{\ell}\delta^{\ell-j} h_j f_{\ell-j,j} = \sum_{j=1}^{\ell}\binom{\ell-1}{\ell-j}(-\delta)^{\ell-j}(c_j^+ g^+ + c_j^- g^-) f_\ell.$$

Thus the claim holds true due to Remark 4.4.

It remains to show (4.24). Since multiplication with a function from $C_0^\infty(\overline{\mathbb{R}}_+)$ preserves $\mathcal{K}^{L,0;\infty}_{\mathcal{T}_L}(\mathbb{R}_+)$, it is enough to analyze the operators

$$A_{kj} := \omega\,\mathrm{op}^0_M(g^\pm)\,\omega_0\, r^{j+k}\mathrm{op}^0_M(f_{kj})\,\omega'.$$

Since $g^\pm(z+N)\equiv g^\pm(z)$ for every integer N, we can write

$$A_{kj} := \omega\, r^{j+k}\mathrm{op}^0_M(g^\pm)\,\omega_0\,\mathrm{op}^0_M(f_{kj})\,\omega' + C_{kj}$$

with

$$C_{kj} := \underbrace{\omega \left(\mathrm{op}_M^0(g^\pm) - \mathrm{op}_M^{j+k}(g^\pm)\right) r^{j+k}\, \omega_0}_{=:C_{kj}^\pm} \omega_0' \, \mathrm{op}_M^0(f_{kj})\, \omega',$$

where ω_0' is chosen such that $\omega_0\omega_0' = \omega_0$. By [20, Remark 2.3.70] and the pole structure of $g^\pm$, both $C_{kj}^\pm$ and $(C_{kj}^\pm)^*$ map $L^2(\mathbb{R}_+)$ into $\mathscr{S}_T^0(\mathbb{R}_+)$ with the Taylor asymptotic type $T = \{(-j, 0) \mid j \in \mathbb{N}_0\} \in \mathrm{As}(0, (-\infty, 0])$. Hence C_{kj} has the required mapping property. Moreover, $C_{kj}^* = \omega' \,\mathrm{op}_M^0(f_{kj}^{(*)})\, \omega_0'(C_{kj}^\pm)^*$ with

$$f_{kj}^{(*)}(z) = \overline{f_{kj}(1-\overline{z})} \in M^{-\infty}_{\{(-\ell,0)\mid 0\le \ell\le j+k-1\}};$$

thus standard mapping properties of Mellin operators yield that also C_{kj}^* has the required mapping property. Next, we write

$$A_{kj} := \omega\, r^{j+k} \mathrm{op}_M^0(g^\pm)\mathrm{op}_M^0(f_{kj})\, \omega' + C_{kj} + r^{j+k}\widetilde{C}_{kj}$$

with

$$\widetilde{C}_{kj} := \omega\, \mathrm{op}_M^0(g^\pm)\, (1-\omega_0)\, \mathrm{op}_M^0(f_{kj})\, \omega'.$$

Using [20, Lemma 2.3.73], both $\widetilde{C}_{kj}$ and its adjoint map $L^2(\mathbb{R}_+)$ into $\mathscr{S}_T^0(\mathbb{R}_+)$. The same is then true for $r^{j+k}\widetilde{C}_{kj}$. This completes the proof. □

Given an arbitrary asymptotic type $\mathcal{Q}$ for Mellin symbols, we have the decompostion

$$M_{\mathcal{Q}}^\mu = M_{\mathcal{O}}^\mu + M_{\mathcal{Q}}^{-\infty}$$

for every $\mu \in \mathbb{R}$, cf. (3.4). Applying such a decompostion to the symbols $g^\pm(z)$ and $f_j(z)$, we obtain a decomposition

$$b_j(r,z) = b_j^0(r,z) + b_j^{-\infty}(r,z),$$

with $b_j^0(r,z) \in M_{\mathcal{O}}^{-j}(\overline{\mathbb{R}}_+)$ and $b_j^{-\infty}(r,z) \in M_{\mathcal{Q}_j}^{-\infty}(\overline{\mathbb{R}}_+)$, where

$$\mathcal{Q}_j = \{(\ell,0) \mid \ell \in \mathbb{Z}\setminus[1,j]\} \cup \{(\ell,1) \mid \ell = 1,\ldots,j\}. \tag{4.25}$$

For the following observation we shall employ the so-called *kernel cut-off* operator (see [19, Section 1.3.1], for instance). It is a $C^\infty(\overline{\mathbb{R}}_+)$-linear map

$$V : \bigcup_{\mu\in\mathbb{R}} S_{\mathrm{cl}}^\mu(\overline{\mathbb{R}}_+ \times \Gamma_{1/2}) \longrightarrow \bigcup_{\mu\in\mathbb{R}} M_{\mathcal{O}}^\mu(\overline{\mathbb{R}}_+) \tag{4.26}$$

that restricts to continuous maps $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2}) \to M^{\mu}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$ for every μ and has the property that $(I - V)(a) \in S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2})$ and $(I - V)(a) \in M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$ if $a \in M^{\mu}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$. The symbol classes $S^{\mu}(\overline{\mathbb{R}}_+ \times \Gamma_\delta)$ are defined using the identification of Γ_δ with $\mathbb{R}$ given by $\delta + i\tau \mapsto \tau$; similarly other symbol classes involving a complex line Γ_δ are obtained.

Lemma 4.9 *Let $a_k \in S^{-k}_{\mathrm{cl}}(\mathbb{R}^n, E)$. Then the series $\sum_{k=0}^{\infty} a_k$ converges absolutely in $S^0_{\mathrm{cl}}(\mathbb{R}^n, E)$ if and only if $\sum_{k=\ell}^{\infty} a_k$ converges absolutely in $S^{-\ell}(\mathbb{R}^n, E)$ for every ℓ.*

Proof Let $a_k^{(j)}$ denote the homogeneous components of a_k. Then absolute convergence of the series in $S^0_{\mathrm{cl}}(\mathbb{R}^n, E)$ means:

(i) For every continuous seminorm $\|\cdot\|$ of $C^{\infty}(S^{n-1}, E)$ and every j the series $\sum_{k=0}^{\infty} \|a_k^{(j)}\|$ is finite.
(ii) Let χ be a zero excision function. For every ℓ and every continuous seminorm $\|\cdot\|$ of $S^{-\ell}(\mathbb{R}^n, E)$ the series $\sum_{k=0}^{\infty} \left\| a_k - \chi \sum_{j=0}^{\ell-1} a_k^{(j)} \right\|$ is finite.

However, the series in (i) is always finite since it is equal to the sum $\sum_{k=0}^{j} \|a_k^{(j)}\|$. The series in (ii) is equal to $\sum_{k=0}^{\ell-1} \left\| a_k - \chi \sum_{j=0}^{\ell-1} a_k^{(j)} \right\| + \sum_{k=\ell}^{\infty} \|a_k\|$. The first term in this expression is always finite. This yields the claim. □

Proposition 4.10 *There exists a Mellin symbol $h(r, z) \in M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+)$ and symbols $\widetilde{h}_j(r, z) \in M^{-j}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$ and $\widetilde{h}_j^{-\infty} \in M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$ such that, for every integer L,*

$$h(r, z) - \sum_{j=0}^{L-1} r^j b_j^0(r, z) = r^L \widetilde{h}_L(r, z) + \sum_{j=0}^{L-1} r^j \widetilde{h}_j^{-\infty}(r, z).$$

Proof Let us consider, by restriction, b_j^0 as an element of $S^{-j}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2})$. Let χ be an excision function and $\chi_j(1/2 + i\tau) = \chi(\tau/c_j)$ with $c_j > 0$. By a diagonal sequence argument we can choose an increasing sequence of c_j with $c_j \to \infty$ as $j \to \infty$ such that the series $f_k := \sum_{j=k}^{\infty} r^{j-k} \chi_j b_j^0$ converge in $S^{-k}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{1/2})$ for every k.

Now define $\widetilde{h}_k = V(f_k) \in M^{-k}_{\mathcal{O}}(\overline{\mathbb{R}}_+)$. Then

$$\widetilde{h}_0 - \sum_{j=0}^{L-1} r^j b_j^0 = \underbrace{V\Big(f_0 - \sum_{j=0}^{L-1} r^j \chi_j b_j^0\Big)}_{=V(r^L f_L) = r^L \widetilde{h}_L} + \sum_{j=0}^{L-1} r^j \big\{V((1 - \chi_j) b_j^0) - (I - V)(b_j^0)\big\}.$$

The claim follows by choosing $h = \widetilde{h}_0$ and $\widetilde{h}_j^{-\infty} = V((1 - \chi_j) b_j^0) - (I - V)(b_j^0)$. □

Define $\widetilde{d}_j(r,z) := b_j^{-\infty}(r,z) - \widetilde{h}_j^{-\infty}(r,z) \in M_{\mathcal{Q}_j}^{-\infty}(\overline{\mathbb{R}}_+)$. Then, by construction of $h(r,z)$ and Proposition 4.8,

$$\begin{aligned}\widetilde{G}_k &:= \omega \mathrm{op}^+(a)\omega' - \omega \mathrm{op}_M^0(h)\omega' - \omega \sum_{j=0}^{k-1} r^j \mathrm{op}_M^0(\widetilde{d}_j)\omega' \\ &= -r^L\, \omega \mathrm{op}_M^0(\widetilde{h}_L)\omega' + \omega \sum_{j=k}^{L-1} r^j \mathrm{op}_M^0(\widetilde{d}_j)\omega' + C_L\end{aligned}$$

for arbitrary $L \geq k$. Since the $\widetilde{d}_j$ are smoothing symbols and L can be chosen arbitrarily large, we conclude that

$$\widetilde{G}_k : L^2(\mathbb{R}_+) \to \mathcal{K}_{\mathcal{T}_k}^{\infty,0;\infty}(\mathbb{R}_+), \qquad \widetilde{G}_k^* : L^2(\mathbb{R}_+) \to \mathcal{K}_{\mathcal{P}_k}^{\infty,0;\infty}(\mathbb{R}_+), \tag{4.27}$$

i.e., $\widetilde{G}_k \in C_G(\mathbb{R}_+, \boldsymbol{g}_k)$ is a Green operator. Now let $\widetilde{d}_{jk}(z) := \partial_r^k \widetilde{d}_j(0,z)/k!$ be the Taylor-coefficients of $\widetilde{d}_j(r,z)$ in $r=0$ and $d_\ell(z) := \sum_{j+k=\ell} \widetilde{d}_{jk}(z)$. Then we observe that

$$\omega \sum_{j=0}^{k-1} r^j \mathrm{op}_M^0(\widetilde{d}_j)\omega' = \omega \sum_{j=0}^{k-1} r^j \mathrm{op}_M^0(d_j)\omega' + \widetilde{G}_k'$$

with a remainder $\widetilde{G}_k'$ satisfying (4.27). Summing up, we obtain the representation

$$\mathrm{op}^+(a) = \omega \mathrm{op}_M^0(h)\omega' + (1-\omega)\mathrm{op}(a)(1-\omega'') + \omega \sum_{j=0}^{k-1} r^j \mathrm{op}_M^0(d_j)\omega' + G_k, \tag{4.28}$$

where G_k is a Green operator satisfying (4.27). Hence $\mathrm{op}^+(a)$ belongs to the cone algebra as claimed.

The formula (4.6) for the conormal symbols follows directly from construction. In fact, for $0 \leq \ell \leq k-1$,

$$\begin{aligned}\sigma_M^{-\ell}(\mathrm{op}^+(a))(z) &= \frac{1}{\ell!}(\partial_r^\ell h)(0,z) + d_\ell(z) \\ &= \frac{1}{\ell!}\partial_r^\ell \Big(h(r,z) + \sum_{j=0}^{k-1} r^j \widetilde{d}_j(r,z) - r^k \widetilde{h}_k(r,z)\Big)\Big|_{r=0} \\ &= \frac{1}{\ell!}\partial_r^\ell\big|_{r=0} \sum_{j=0}^{k-1} r^j b_j(r,z) = \sum_{i+j=\ell} \frac{\partial_r^i b_j(0,z)}{i!};\end{aligned} \tag{4.29}$$

substitution of the explicit expression of $b_j(r,z)$, cf. (4.21), yields (4.6).

4.3 A parameter-dependent extension

Let us extend the above construction to the case of symbols $a(y,\eta,r,\rho)$ that depend on an additional parameter $(y,\eta)\in U\times\mathbb{R}^q$. Specifically, we require that

$$a(y,\eta,r,\rho)\in S^0_{\mathrm{cl}}(U_y\times\mathbb{R}^q_\eta,S^0_{\mathrm{cl}}(\mathbb{R}_r\times\mathbb{R}_\rho))$$

in the sense of Sect. 3.1. The homogeneous components of $a(y,\eta,r,\rho)$ as a classical symbol in the variables (r,ρ), are then of the form

$$a_{(-j)}(y,\eta,r,\rho)=\big\{a_j^+(y,\eta,r)\theta^+(\rho)+a_j^-(y,\eta,r)\theta^-(\rho)\big\}(i\rho)^{-j} \tag{4.30}$$

with $a_j^\pm(y,\eta,r)\in S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,C^\infty(\mathbb{R}))$; for every $N\in\mathbb{N}$,

$$a(y,\eta,r,\rho)-\chi(\rho)\sum_{j=0}^{N-1}a_{(-j)}(y,\eta,r,\rho)\in S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,S^{-N}(\mathbb{R}\times\mathbb{R})).$$

Following the steps in Sect. 4.2, we obtain the following variant of Proposition 4.8.

Proposition 4.11 *Let $a(y,\eta,r,\rho)\in S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,S^0_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R}))$. Then there exist symbols $h(y,\eta,r,z)\in S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+))$ and $\widetilde{d}_j(y,\eta,r,z)\in S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,M^{-\infty}_{\mathcal{Q}_j}(\overline{\mathbb{R}}_+))$ such that, for every positive integer k,*

$$\omega\,\mathrm{op}^+(a)(y,\eta)\,\omega'=\omega\,\mathrm{op}^0_M(h)(y,\eta)\,\omega'+\omega\sum_{j=0}^{k-1}r^j\mathrm{op}^0_M(\widetilde{d}_j)(y,\eta)\,\omega'+\widetilde{g}_k(y,\eta),$$

where $\widetilde{g}_k(y,\eta)\in S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,\mathscr{L}(L^2(\mathbb{R}_+)))$ with

$$\widetilde{g}_k(y,\eta)\in\bigcap_{s,\sigma\in\mathbb{R}}S^0_{\mathrm{cl}}(U\times\mathbb{R}^q;\mathscr{L}(L^2(\mathbb{R}_+),\mathcal{K}^{s,0;\sigma}_{T_k}(\mathbb{R}_+))),$$
$$\widetilde{g}_k(y,\eta)^*\in\bigcap_{s,\sigma\in\mathbb{R}}S^0_{\mathrm{cl}}(U\times\mathbb{R}^q;\mathscr{L}(L^2(\mathbb{R}_+),\mathcal{K}^{s,0;\sigma}_{P_k}(\mathbb{R}_+))).$$

Moreover, the symbols $\widetilde{d}_j(y,\eta,r,z)$ have the form

$$\widetilde{d}_j(y,\eta,r,z)=a_j^+(y,\eta,r)\widetilde{d}_j^+(z)+a_j^-(y,\eta,r)\widetilde{d}_j^-(z),\qquad \widetilde{d}_j^\pm(z)\in M^{-\infty}_{\mathcal{Q}_j}, \tag{4.31}$$

while

$$h(y,\eta,r,z)=\sum_{j=0}^{\infty}r^j\Big(a_j^+(y,\eta,r)V\big(\chi(\tau/c_j)h_j^+\big)(z)+a_j^-(y,\eta,r)V\big(\chi(i\tau/c_j)h_j^-\big)(z)\Big) \tag{4.32}$$

with absolute convergence in $S^0_{\mathrm{cl}}(U\times\mathbb{R}^q,M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+))$, where $h_j^\pm(1/2+i\tau)\in S^{-j}_{\mathrm{cl}}(\Gamma_{1/2})$, V is the kernel cut-off operator, and $(c_j)\subset(0,+\infty)$ is a numerical sequence tending to $+\infty$ sufficiently fast.

5 The edge algebra on the half-space

Let $U \subset \mathbb{R}^q$ be an open set. Given weight-data $\mathbf{g} = (\gamma, \gamma - \mu, (-N, 0])$ with $\gamma, \mu \in \mathbb{R}$ and $N \in \mathbb{N}$, the space of edge symbols

$$R^\mu(U \times \mathbb{R}^q, \mathbf{g}), \qquad \mathbf{g} = (\gamma, \gamma - \mu, (-N, 0]),$$

is a particular subspace of the space $S^\mu(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+))$ of operator-valued symbols; see Sect. 7.2 of the appendix for an introduction to the underlying concept of operator-valued symbols.

Edge symbols are at the base of the so-called edge algebra, which is a calculus of pseudo-differential operators on manifolds with edges (here, in the special case of manifolds with boundary). For further details we refer the reader to the literature, for example [19, 20].

5.1 Green symbols

Green symbols are operator-valued symbols taking values in the class of Green operators on $\mathbb{R}_+$, cf. Section 7.1.5 of the appendix.

Definition 5.1 Given $\mathbf{g} = (\gamma_0, \gamma_1, (-\theta, 0])$ and asymptotic types $P_0 \in \mathrm{As}(-\gamma_0, (-\theta, 0])$ and $P_1 \in \mathrm{As}(\gamma_1, (-\theta, 0])$, we define

$$R^\mu_G(U \times \mathbb{R}^q; \mathbf{g})_{P_0,P_1} \subset S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{K}^{0,\gamma_0}(\mathbb{R}_+), \mathcal{K}^{0,\gamma_1}(\mathbb{R}_+))$$

as the space of all symbols $g(y, \eta)$ that satisfy

$$g(y, \eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{K}^{0,\gamma_0}(\mathbb{R}_+), \mathcal{K}^{s,\gamma_1}_{P_1}(\mathbb{R}_+)^\rho),$$
$$g(y, \eta)^* \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{K}^{0,-\gamma_1}(\mathbb{R}_+), \mathcal{K}^{s,-\gamma_0}_{P_0}(\mathbb{R}_+)^\rho),$$

for every choice of $s, \rho \in \mathbb{R}$; here $*$ indicates the pointwise adjoint. The space $R^\mu_G(U \times \mathbb{R}^q; \mathbf{g})$ is obtained by passing to the union over all asymptotic types.

Let us consider a function $k(y, \eta; r, s) \in S^{\mu+1}_{\mathrm{cl}}(U \times \mathbb{R}^q, E)$ with

$$E = \mathscr{S}^{\gamma_1}_{P_1}(\mathbb{R}_+) \,\widehat{\otimes}_\pi\, \mathcal{K}^{0,-\gamma_0}(\mathbb{R}_+) \;\cap\; \mathcal{K}^{0,\gamma_1}(\mathbb{R}_+) \,\widehat{\otimes}_\pi\, \mathscr{S}^{-\gamma_0}_{\overline{P}_0}(\mathbb{R}_+),$$

where the notation is as in Proposition 7.2 below. Define $g(y, \eta)$ by

$$(g(y, \eta)u)(r) = \int_0^\infty k(y, \eta; [\eta]r, [\eta]s)u(s)\, ds.$$

Then $g(y, \eta) \in R^\mu_G(U \times \mathbb{R}^q; \mathbf{g})_{P_0,P_1}$. Moreover, its principal symbol is the integral operator with kernel $k_{(\mu+1)}(y, \eta; |\eta|r, |\eta|s)$, where $k_{(\mu+1)}$ is the principal symbol of k. Similarly to Proposition 7.2 it can be shown that any Green symbol $g(y, \eta) \in R^\mu_G(U \times \mathbb{R}^q; \mathbf{g})_{P_0,P_1}$ is of this form.

5.2 Edge degenerate symbols and Mellin quantization

A symbol $p(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times U \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ is called edge-degenerate if there exists a symbol $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+q}_{(\tilde{\rho},\tilde{\eta})})$ such that

$$p(r, y, \rho, \eta) = \tilde{p}(r, y, r\rho, r\eta). \tag{5.1}$$

To any symbol $a(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ there exists an edge degenerate symbol $p(r, y, \rho, \eta)$ of the same order μ such that

$$a(r, y, \rho, \eta) - r^{-\mu} p(r, y, \rho, \eta) \in S^{-\infty}(\mathbb{R}_+ \times U \times \mathbb{R}^{1+q}_{(\rho,\eta)}); \tag{5.2}$$

in fact, the symbol $\tilde{p}$ associated with p is uniquely determined modulo $S^{-\infty}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+q}_{(\tilde{\rho},\tilde{\eta})})$ and has the asymptotic expansion

$$\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \sim \sum_{j=0}^{\infty} \chi(\tilde{\rho}, \tilde{\eta}) r^j a_{(\mu-j)}(r, y, \tilde{\rho}, \tilde{\eta}).$$

Throughout the paper it will be useful to use the following definition: If ω is a cut-off function then define

$$\omega_\eta(r) = \omega(r[\eta]), \qquad r \geq 0, \;\; \eta \in \mathbb{R}^q. \tag{5.3}$$

The following result essentially follows from the fact that the difference in (5.2) is smoothing on $\mathbb{R}_+$ together with the presence of functions of the form $(1 - \omega_\eta)$ which localize the operator away from $r = 0$; we skip presenting all technical details.

Proposition 5.2 *Let $a(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ and let $p(r, y, \rho, \eta)$ be the edge-degenerate symbol associated with a. Moreover, let σ, σ' be cut-off functions. Then*

$$g(y, \eta) := \sigma(1 - \omega_\eta)\{\mathrm{op}^+(a)(y, \eta) - r^{-\mu}\mathrm{op}(p)(y, \eta)\}(1 - \omega'_\eta)\sigma'$$

belongs to $R^{\mu}_{G}(U \times \mathbb{R}^q, (\gamma, \gamma, (-\infty, 0]))_{\mathcal{O},\mathcal{O}}$ for every $\gamma \in \mathbb{R}$.

We will say that $h(r, y, z, \eta) = \tilde{h}(r, y, z, r\eta)$ with $\tilde{h}(r, y, z, \tilde{\eta}) \in M^{\mu}_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^q_{\tilde{\eta}})$ is a *Mellin quantization* of an edge-degenerate symbol $p(r, y, \rho, \eta) = \tilde{p}(r, y, r\rho, r\eta)$ if for some (and then for all) $\gamma \in \mathbb{R}$

$$\mathrm{op}^{\gamma}_{M}(h)(y, \eta) - \mathrm{op}(p)(y, \eta) \in S^{-\infty}(U \times \mathbb{R}^q, L^{-\infty}(\mathbb{R}_+))$$

where $L^{-\infty}(\mathbb{R}_+)$ is the space of smoothing operators on $\mathbb{R}_+$, i.e, those operators with an integral kernel belonging to $C^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+)$.

5.3 Edge symbols

We define $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \mathbf{g})$, $\mathbf{g} = (\gamma, \gamma - \mu, (-N, 0]))$, as the space of all symbols of the form $m(y, \eta) + g(y, \eta)$ with arbitrary $g \in R^{\mu}_{G}(U \times \mathbb{R}^q, \mathbf{g})$ and

$$m(y, \eta) = \omega_\eta \sum_{j=0}^{N-1} \sum_{|\alpha|=0}^{j} r^{-\mu+j} \mathrm{op}_M^{\gamma_{j\alpha}}(h_{j\alpha})(y)\eta^\alpha \omega'_\eta, \tag{5.4}$$

where ω, ω' denote arbitrary cut-off functions and $\omega_\eta, \omega'_\eta$ are as in (5.3). Moreover, $h_{j\alpha}(y) \in C^\infty(U, M^{-\infty}_{\mathcal{R}_{j\alpha}})$ with arbitrary asymptotic types $\mathcal{R}_{j\alpha}$ and weights $\gamma_{j\alpha} \in \mathbb{R}$ such that $\gamma - j \le \gamma_{j\alpha} \le \gamma$ and $\mathrm{Re}\, p \ne \frac{1}{2} - \gamma_{j\alpha}$ for every $(p, n) \in R_{j\alpha}$.

The space of edge symbols $R^{\mu}(U \times \mathbb{R}^q, \mathbf{g})$, $\mathbf{g} = (\gamma, \gamma - \mu, (-N, 0])$, consists of all symbols of the form

$$\begin{aligned} a(y, \eta) = {} & \sigma\left\{\omega_\eta r^{-\mu} \mathrm{op}_M^{\gamma}(h)(y, \eta)\omega'_\eta + (1 - \omega_\eta) r^{-\mu} \mathrm{op}(p)(y, \eta)(1 - \omega''_\eta)\right\}\sigma' \\ & + (1 - \sigma)\mathrm{op}(q)(y, \eta)(1 - \sigma'') + m(y, \eta) + g(y, \eta) \end{aligned}$$

where $m + g \in R^{\mu}_{M+G}(U \times \mathbb{R}^q, \mathbf{g})$, $p(r, y, \tau, \eta) = \tilde{p}(r, y, r\tau, r\eta)$ is edge-degenerate of order μ, h is a Mellin quantization of p, and $q(r, y, \tau, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R} \times U \times \mathbb{R}^{1+q}_{\tau,\eta})$. Moreover, $\sigma, \sigma', \sigma''$, and $\omega, \omega', \omega''$ are arbitrary cut–off functions satisfying $\omega'' \prec \omega \prec \omega'$ and $\sigma'' \prec \sigma \prec \sigma'$.

Using the representation of $m(y, \eta)$ as in (5.4), we associate with an edge symbol $a(y, \eta)$ the conormal symbols

$$\sigma_M^{\mu-j}(a)(y, z, \eta) = \frac{1}{j!} \partial_r^j h(r, y, z, r\eta)\Big|_{r=0} + \sum_{|\alpha|=0}^{j} h_{j\alpha}(y, z)\eta^\alpha, \tag{5.5}$$

where $j = 0, \ldots, N - 1$.

As already mentioned,

$$R^{\mu}(U \times \mathbb{R}^q, \mathbf{g}) \subset S^{\mu}(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(\mathbb{R}_+), \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)).$$

Moreover, given $a(y, \eta) \in R^{\mu}(U \times \mathbb{R}^q, \mathbf{g})$ and an asymptotic type $P \in \mathrm{As}(\gamma, (-N, 0])$ there exists an asymptotic type $Q \in \mathrm{As}(\gamma - \mu, (-N, 0])$ such that

$$a(y, \eta) \in S^{\mu}(U \times \mathbb{R}^q; \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+), \mathcal{K}_Q^{s-\mu,\gamma-\mu}(\mathbb{R}_+));$$

the asymptotic type Q is determined by the Green part of a and the conormal symbols of a.

6 Truncated operators on the half-space

The main scope of this section is to show that if $a(y, r, \rho, \eta) \in S^0_{\mathrm{cl}}(U \times \mathbb{R}_+ \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ is independent of r for large r then $\mathrm{op}^+(a)(y, \eta)$ is an edge symbol, i.e.

$$\mathrm{op}^+(a)(y, \eta) \in R^0(U \times \mathbb{R}^q_\eta, (0, 0, (-\infty, 0]));$$

cf. Theorem 6.6, below. In order to keep the exposition more lean, we shall focus on the case of y-independent symbols. Having verified this special situation, the general case easily follows by the tensor-product argument outlined in Remark 3.3, using the fact that

$$S^0_{\mathrm{cl}}(U \times \mathbb{R}_+ \times \mathbb{R}^{1+q}_{(\rho,\eta)}) = C^\infty(U)\,\widehat{\otimes}_\pi\, S^0_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{(\rho,\eta)}).$$

6.1 First observations for the analysis of truncated operators

Let $a(r, \rho, \eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ be independent of r for large r. If $\sigma, \sigma', \sigma''$ are cut-off functions on the half-axis with $\sigma''\sigma = \sigma''$ and $\sigma\sigma' = \sigma$, then we have

$$\mathrm{op}^+(a)(\eta) = \sigma\,\mathrm{op}^+(a)(\eta)\sigma' + (1 - \sigma)\mathrm{op}(a)(\eta)(1 - \sigma'') + g(\eta), \tag{6.1}$$

where

$$g(\eta) := \sigma\,\mathrm{op}^+(a)(\eta)(1 - \sigma') + (1 - \sigma)\mathrm{op}^+(a)(\eta)\sigma''$$

has an integral kernel $k(\eta, r, r') \in \mathscr{S}(\mathbb{R}^q_\eta, \mathscr{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+))$ due to the disjoint supports of $\sigma, 1 - \sigma'$ and $1 - \sigma, \sigma''$, respectively.

Similarly, for any choice of cut-off functions $\omega'' \prec \omega \prec \omega'$, we find

$$\sigma\,\mathrm{op}^+(a)(\eta)\sigma' = \sigma\Big\{\omega_\eta \mathrm{op}^+(a)(\eta)\omega'_\eta + \\ +(1 - \omega_\eta)\mathrm{op}(a)(\eta)(1 - \omega''_\eta)\Big\}\sigma' + g(\eta) \tag{6.2}$$

with a Green symbol

$$g(\eta) \in R^\mu_G(\mathbb{R}^q_\eta, (0, 0, (-\infty, 0]))_{T,T}$$

with $T = \{(-j, 0) \mid j \in \mathbb{N}_0\}$ being the Taylor asymptotic type; here, ω_η is defined by $\omega_\eta(r) = \omega(r[\eta])$ and analogously for the other cut-off functions.

Observing Proposition 5.2, the crucial term in the analysis of $\mathrm{op}^+(a)(\eta)$ is thus the operator family

$$\omega_\eta \mathrm{op}^+(a)(\eta)\omega'_\eta. \tag{6.3}$$

6.2 Decoupling of parameter-dependent symbols

Given a symbol $a(r,\rho,\eta)\in S^{\mu}(\mathbb{R}\times\mathbb{R}^{1+q}_{(\rho,\eta)})$, $\mu\in\mathbb{R}$, define

$$\mathfrak{a}(\eta,r,\rho):=a([\eta]^{-1}r,[\eta]\rho,\eta). \tag{6.4}$$

We shall prove the following result:

Proposition 6.1 *The mapping $a\mapsto\mathfrak{a}$ induces a continuous operator*

$$S^{\mu}_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R}^{1+q}_{(\rho,\eta)})\longrightarrow S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta,S^{\mu}_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R})). \tag{6.5}$$

If $\chi(\rho)$ is an arbitrary zero excision function we have, for every $N\in\mathbb{N}$,

$$\mathfrak{a}(\eta,r,\rho)\equiv\chi(\rho)\sum_{k=0}^{N-1}\Big\{\mathfrak{a}^+_k(r,y,\eta)\theta^+(\rho)+\mathfrak{a}^-_k(r,y,\eta)\theta^-(\rho)\Big\}(i\rho)^{\mu-k}$$
$$\bmod\ S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta,S^{\mu-N}(\mathbb{R}\times\mathbb{R})) \tag{6.6}$$

with coefficients

$$\mathfrak{a}^{\pm}_k(\eta,r)=[\eta]^{\mu-k}\sum_{\ell+|\alpha|=k}a^{\pm}_{\ell\alpha}([\eta]^{-1}r)\eta^{\alpha},$$
$$a^{\pm}_{\ell\alpha}(r)=\frac{1}{\alpha!}e^{\pm i(\ell-\mu+|\alpha|)\pi/2}(\partial^{\alpha}_{\eta}a_{(\mu-\ell)})(r,\pm1,0), \tag{6.7}$$

where $a_{(\mu-\ell)}$ is the homogeneous component of degree $\mu-\ell$ of a and $\theta^{\pm}(\rho)$ denotes the characteristic function of $\mathbb{R}_{\pm}$.

For the proof we need the following two lemmas.

Lemma 6.2 *The mapping $a\mapsto\mathfrak{a}$ defines a continuous operator*

$$S^{\mu}(\mathbb{R}\times\mathbb{R}^{1+q}_{\rho,\eta})\longrightarrow S^{\mu}(\mathbb{R}^q_\eta,S^{\mu}(\mathbb{R}\times\mathbb{R})). \tag{6.8}$$

Proof Clearly there exist constants $C_1,C_2>0$ such that

$$C_1\langle\rho\langle\eta\rangle,\eta\rangle\le\langle\rho[\eta],\eta\rangle\le C_2\langle\rho\langle\eta\rangle,\eta\rangle\qquad\forall\ (\rho,\eta).$$

Then the result follows easily from chain-rule and the identity $\langle\rho\langle\eta\rangle,\eta\rangle=\langle\rho\rangle\langle\eta\rangle$. □

Lemma 6.3 *Let $p(\eta,\xi)\in S^{\mu}(\mathbb{R}^q_\eta,S^{\mu}(\mathbb{R}^m_\xi))$ and assume that there exists a sequence of symbols $p_j(\eta,\xi)\in S^{\mu-j}_{\mathrm{cl}}(\mathbb{R}^q_\eta,S^{\mu-j}_{\mathrm{cl}}(\mathbb{R}^m_\xi))$ such that, for every $N\in\mathbb{N}$,*

$$p(\eta,\xi)-\sum_{j=0}^{N-1}p_j(\eta,\xi)\ \in\ S^{\mu-N}(\mathbb{R}^q_\eta,S^{\mu-N}(\mathbb{R}^m_\xi)).$$

Then $p(\eta,\xi)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m_\xi))$.

Proof To show that $p(\eta,\xi)$ belongs to $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q, S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m))$ it suffices to show the existence of sequences

$$q_\ell \in S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^q, S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m)), \qquad \ell\in\mathbb{N},$$

and

$$r_{N,\ell}(\eta,\xi)\in S^{\mu-N}(\mathbb{R}^q, S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^m)), \qquad \ell, N\in\mathbb{N},$$

such that, for every choice of $M, N\in\mathbb{N}$,

$$\Big(p(\eta,\xi)-\sum_{\ell=0}^{N-1} q_\ell(\eta,\xi)\Big)-\sum_{j=0}^{M-1} r_{N,j}(\eta,\xi) \;\in\; S^{\mu-N}(\mathbb{R}^q, S^{\mu-M}(\mathbb{R}^m)). \tag{6.9}$$

This is satisfied by defining $q_\ell=p_\ell$ and $r_{N,\ell}=0$ for $0\le\ell<N$ and $r_{N,\ell}=p_\ell$ for $\ell\ge N$. For example, if $M>N$, the left-hand side in (6.9) equals

$$\Big(p(\eta,\xi)-\sum_{\ell=0}^{N-1} q_\ell(\eta,\xi)\Big)-\sum_{j=N}^{M-1} p_j(\eta,\xi)=p(\eta,\xi)-\sum_{\ell=0}^{M-1} q_\ell(\eta,\xi),$$

hence belongs to $S^{\mu-M}(\mathbb{R}^q, S^{\mu-M}(\mathbb{R}^m))$. □

Proof of Proposition 6.1 We only need to show that $\mathfrak{a}(\eta,r,\rho)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^{\mu}_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R}))$; the continuity is then a consequence of Lemma 6.2 and the closed graph theorem.

Let $c(r)\in C^\infty(\mathbb{R})$. Then $\mathfrak{c}(\eta,r):=c(r[\eta]^{-1})$ belongs to $S^0_{\mathrm{cl}}(\mathbb{R}^q, C^\infty(\mathbb{R}))$. In fact, a Taylor expansion of c centered in $r=0$ together with Lemma 6.2 shows that the homogeneous components of $\mathfrak{c}(\eta,r)$ are given by $\mathfrak{c}_{(-j)}(\eta,r)=\frac{(\partial_r^j c)(0)}{j!}(r|\eta|^{-1})^j$. By a tensor-product argument, cf. Remark 3.3, it remains to study the case when a does not depend on the variable r.

Let $a_{(\mu-\ell)}(\rho,\eta)$ be the homogeneous components of $a(\rho,\eta)$; then

$$r_N(\rho,\eta):=a(\rho,\eta)-\chi_0(\rho,\eta)\sum_{\ell=0}^{N-1} a_{(\mu-\ell)}(\rho,\eta)\in S^{\mu-N}(\mathbb{R}^{1+q}_{(\rho,\eta)}), \tag{6.10}$$

where $\chi_0(\rho,\eta)$ is a zero excision function. It will be convenient to introduce the symbols

$$a_\ell(\rho,\eta):=\chi_0(\rho,\eta)a_{(\mu-\ell)}(\rho,\eta)\in S^{\mu-\ell}(\mathbb{R}^{1+q}_{(\rho,\eta)}).$$

We may choose χ_0 such that $\chi_0(\rho,\eta)\chi(\rho/[\eta])=\chi(\rho/[\eta])$ or, equivalently,

$$\chi_0([\eta]\rho,\eta)\chi(\rho)=\chi(\rho), \qquad (\rho,\eta)\in\mathbb{R}^{1+q}. \tag{6.11}$$

Substituting ρ by $[\eta]\rho$ in (6.10), we find

$$\mathfrak{a}(\eta,\rho)=\sum_{\ell=0}^{N-1}\mathfrak{a}_\ell(\eta,\rho)+\mathfrak{r}_N(\eta,\rho).$$

Due to Lemma 6.2, $\mathfrak{r}_N(\eta,\rho)\in S^{\mu-N}(\mathbb{R}^q,S^{\mu-N}(\mathbb{R}))$. Moreover, $(1-\chi)(\rho)\mathfrak{a}_\ell(\eta,\rho)$ is compactly supported in ρ and is homogeneous of degree $\mu-\ell$ in η for large $|\eta|$, since $\mathfrak{a}_\ell(\eta,\rho)=a_{(\mu-\ell)}(|\eta|\rho,\eta)$ provided $|\eta|$ is large. In particular, $(1-\chi)(\rho)\mathfrak{a}_\ell(\eta,\rho)\in S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^q,S^{-\infty}(\mathbb{R}))$ for every ℓ.

We shall now show that

$$\chi(\rho)\mathfrak{a}_\ell(\rho,\eta)=\chi(\rho)a_{(\mu-\ell)}([\eta]\rho,\eta)\in S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R},S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^q));$$

the identity holds true due to (6.11). By Taylor-expansion in η of $a_{(\mu-\ell)}(\rho,\eta)$ for $\rho\neq 0$ we find

$$\begin{aligned}a_{(\mu-\ell)}([\eta]\rho,\eta)=&\sum_{|\alpha|\le L-1}\frac{1}{\alpha!}[\eta]^{\mu-|\alpha|-\ell}(\partial_\eta^\alpha a_{(\mu-\ell)})(\rho,0)\eta^\alpha\\&+\sum_{|\gamma|=L}\frac{\eta^\gamma}{L!}\int_0^1(1-t)^L[\eta]^{\mu-\ell-L}(\partial_\eta^\gamma a_{(\mu-\ell)})\Big(\rho,t\frac{\eta}{[\eta]}\Big)\,dt\end{aligned}$$

Evidently,

$$\sum_{|\alpha|=j}\frac{1}{\alpha!}[\eta]^{\mu-\ell-j}(\partial_\eta^\alpha a_{(\mu-\ell)})(\rho,0)\eta^\alpha\in S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^q,S^{(\mu-\ell-j)}(\mathbb{R})),\tag{6.12}$$

hence multiplication with $\chi(\rho)$ yields a element in $S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^q,S^{\mu-\ell-j}_{\mathrm{cl}}(\mathbb{R}))$ which is homogeneous of degree $\mu-\ell-j$ for large ρ. The second term, after multiplication with $\chi(\rho)$, yields a symbol in $S^{\mu-\ell}_{\mathrm{cl}}(\mathbb{R}^q,S^{\mu-\ell-L}(\mathbb{R}))$; note the homogeneity of degree $\mu-\ell$ in η for large $|\eta|$.

Now Lemma 6.3 yields $\mathfrak{a}(\eta,\rho)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q,S^{\mu}_{\mathrm{cl}}(\mathbb{R}))$.

The homogeneous component with respect to the variable ρ of degree $\mu-k$ of $\mathfrak{a}(\eta,\rho)$ is obtained by summing up all terms from (6.12) with $j+l=k$; this yields the expression

$$[\eta]^{\mu-k}\sum_{\ell+|\alpha|=k}\frac{1}{\alpha!}(\partial_\eta^\alpha a_{(\mu-\ell)})(\rho,0)\eta^\alpha.$$

To find the expansion (6.6) it remains to observe that

$$(\partial_\eta^\alpha a_{(\mu-\ell)})(\rho,0)=(\partial_\eta^\alpha a_{(\mu-\ell)})(\rho/|\rho|,0)|\rho|^{\mu-\ell-|\alpha|},\qquad \rho\neq 0,$$

and

$$|\rho|^z = e^{-\mathrm{sign}(\rho) i z\pi/2}(i\rho)^z, \qquad z \in \mathbb{C},\ 0 \neq \rho \in \mathbb{R}. \tag{6.13}$$

This completes the proof. □

6.3 Operators of zero order

Let $a(r,\rho,\eta) \in S^0_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ be given. Denote by $a_{(-\ell)}(r,\rho,\eta)$ the homogeneous component of degree $-\ell$. The symbol $a(r,\rho,\eta)$ is, for every fixed η, a classical symbol in (r,ρ) and we have

$$a(r,\rho,\eta) \equiv \chi(\rho) \sum_{k=0}^{N-1} \Big\{ a_k^+(r,\eta)\theta^+(\rho) + a_k^-(r,\eta)\theta^-(\rho) \Big\} (i\rho)^{-k} \\ \bmod\ C^\infty(\mathbb{R}^q_\eta, S^{-N}(\mathbb{R} \times \mathbb{R})) \tag{6.14}$$

with coefficients

$$\begin{aligned} a_k^\pm(r,\eta) &= \sum_{n+|\alpha|=k} a^\pm_{n\alpha}(r)\eta^\alpha, \\ a^\pm_{n\alpha}(r) &= \frac{1}{\alpha!}(\pm i)^{n+|\alpha|}(\partial_\eta^\alpha a_{(-n)})(r,\pm 1, 0). \end{aligned} \tag{6.15}$$

We associate with a the decoupled symbol

$$\mathfrak{a}(\eta, r, \rho) \in S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta, S^0_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R})),$$

cf. (6.4) and Proposition 6.1. We now apply Proposition 4.11 to the symbol $\mathfrak{a}(\eta, r, \rho)$ and find, for every positive integer k, a representation

$$\omega\, \mathrm{op}^+(\mathfrak{a})(\eta)\, \omega' = \omega\, \mathrm{op}^0_M(\mathfrak{h})(\eta)\, \omega' + \omega \sum_{j=0}^{k-1} r^j \mathrm{op}^0_M(\widetilde{\mathfrak{d}}_j)(\eta)\, \omega' + \widetilde{\mathfrak{g}}_k(\eta), \tag{6.16}$$

with Mellin symbols $\mathfrak{h}(\eta, r, z)$, $\widetilde{\mathfrak{d}}_j(\eta, z)$ and remainder $\widetilde{\mathfrak{g}}_k(\eta)$ as described in Proposition 4.11.

Theorem 6.4 *Let $a(r,\rho,\eta) \in S^0_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ and ω, ω' be arbitrary cut-off functions. Moreover, let $h(r,z,\eta) = \widetilde{h}(r,z,r\eta)$ for $\widetilde{h}(r,z,\tilde{\eta}) \in M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_{\tilde{\eta}})$ be a Mellin quantisation of a. Then, for every positive integer k,*

$$\omega_\eta\, \mathrm{op}^+(a)(\eta)\, \omega'_\eta = \omega_\eta \mathrm{op}^0_M(h)(\eta)\, \omega'_\eta + \omega_\eta \sum_{j=0}^{k-1} r^j \left(\sum_{|\alpha| \le j} \mathrm{op}^0_M(d_{\alpha j})\eta^\alpha \right) \omega'_\eta + g_k(\eta)$$

for Mellin symbols $d_{\alpha j}(z) \in M^{-\infty}_{\mathcal{Q}_j}$ *and Green symbols* $g_k(\eta) \in R^0_G(\mathbb{R}^q, \boldsymbol{g}_k)_{P_k,T_k}$, $\boldsymbol{g}_k = (0, 0, (-k, 0])$*; here* T_k, $P_k \in \mathrm{As}(0, (-k, 0])$ *are as in* (4.19) *and* (4.23)*, respectively, while* $\mathcal{Q}_j$ *is as in* (4.25).

Proof Starting point is the above formula (6.16). By conjugation with $\kappa_{[\eta]}$, where κ denotes the group action from (7.1), we obtain

$$\begin{aligned}\omega_\eta\, \mathrm{op}^+(a)(\eta)\, \omega'_\eta &= \kappa_{[\eta]}\big(\omega\, \mathrm{op}^+(\mathfrak{a})(\eta)\, \omega'\big)\kappa^{-1}_{[\eta]}\\ &= \omega_\eta\, \mathrm{op}^0_M(H)(\eta)\, \omega'_\eta + \omega_\eta \sum_{j=0}^{k-1} r^j \mathrm{op}^0_M(\widetilde{d}_j)(\eta)\, \omega'_\eta + \kappa_{[\eta]}\widetilde{\mathfrak{g}}_k(\eta)\kappa^{-1}_{[\eta]},\end{aligned} \tag{6.17}$$

where $H(r, z, \eta) = \mathfrak{h}(\eta, r[\eta], z)$ and $\widetilde{d}_j(r, z, \eta) = [\eta]^j \widetilde{\mathfrak{d}}_j(\eta, r[\eta], z)$.

According to (4.31) there exists $\widetilde{d}^{\pm}_j(z) \in M^{-\infty}_{\mathcal{Q}_j}$ such that

$$\widetilde{\mathfrak{d}}_j(\eta, r, z) = \mathfrak{a}^+_j(\eta, r)\widetilde{d}^+_j(z) + \mathfrak{a}^-_j(\eta, r)\widetilde{d}^-_j(z)$$

with the coefficients $\mathfrak{a}^{\pm}_j(\eta, r)$ from (6.7). From (6.15) it follows that

$$[\eta]^j \mathfrak{a}^{\pm}_j(\eta, r[\eta]) = a_j(r, \eta), \tag{6.18}$$

hence

$$\widetilde{d}_j(\eta, r, z) = a^+_j(r, \eta)\widetilde{d}^+_j(z) + a^-_j(r, \eta)\widetilde{d}^-_j(z).$$

By substituting $a^{\pm}_j(r, \eta)$ by its Taylor-polynomial in r of degree $k - 1 - j$ and rearranging summands one shows that

$$\omega_\eta \sum_{j=0}^{k-1} r^j \mathrm{op}^0_M(\widetilde{d}_j)(\eta)\, \omega'_\eta = \omega_\eta \sum_{j=0}^{k-1} r^j \left(\sum_{|\alpha|\le j} \mathrm{op}^0_M(d_{\alpha j})\eta^\alpha\right) \omega'_\eta + g'_k(\eta)$$

with resulting symbols

$$d_{\alpha j}(z) = \sum_{i=|\alpha|}^{j} \frac{1}{(j-i)!}\left\{\partial_r^{j-i} a^+_{i-|\alpha|,\alpha}(0)\widetilde{d}^+_i(z) + \partial_r^{j-i} a^-_{i-|\alpha|,\alpha}(0)\widetilde{d}^-_i(z)\right\}$$

and a Green symbol $g'_k(\eta)$ of the required form.

Now consider the term $g_k(\eta) := g'_k(\eta) + \kappa_{[\eta]}\widetilde{\mathfrak{g}}_k(\eta)\kappa^{-1}_{[\eta]}$. Since the above representation is valid for every choice of k, we find that

$$g_k(\eta) = \omega_\eta \sum_{j=k}^{2k-1} r^j \left(\sum_{|\alpha|\le j} \mathrm{op}^0_M(d_{\alpha j})\eta^\alpha\right) \omega'_\eta + g_{2k}(\eta).$$

The first term on the right-hand side is easily seen to be a Green symbol as required. Now let X_k denote the Fréchet space of all operators A in $\mathscr{L}(L^2(\mathbb{R}_+))$ such that $A : L^2(\mathbb{R}_+) \to \mathscr{S}^0_{T_k}(\mathbb{R}_+)$ and $A^* : L^2(\mathbb{R}_+) \to \mathscr{S}^0_{P_k}(\mathbb{R}_+)$. By Proposition 7.2 any such A is an integral operator with integral kernel

$$k_A(r,s) \in \mathscr{S}^0_{T_k}(\mathbb{R}_+) \,\widehat{\otimes}_\pi\, L^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \,\widehat{\otimes}_\pi\, \mathscr{S}^0_{P_k}(\mathbb{R}_+);$$

the map is continuous. Correspondingly, $\widetilde{\mathfrak{g}}_{2k}(\eta)$ is represented by an η-dependent kernel

$$\mathfrak{k}(\eta,r,s) \in S^0_{\mathrm{cl}}\big(\mathbb{R}^q_\eta; \mathscr{S}^0_{T_k}(\mathbb{R}_+) \,\widehat{\otimes}_\pi\, L^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \,\widehat{\otimes}_\pi\, \mathscr{S}^0_{P_k}(\mathbb{R}_+)\big).$$

But then $\kappa_{[\eta]}\widetilde{\mathfrak{g}}_k(\eta)\kappa^{-1}_{[\eta]}$ is represented by the integral kernel

$$k(\eta,r,s) = [\eta]\mathfrak{k}(\eta,[\eta]r,[\eta]s).$$

hence, as explained after Definition 5.1, $g_k(\eta)$ is a Green symbol as required.

It remains to consider the term $\omega_\eta\, \mathrm{op}^0_M(H)(\eta)\, \omega'_\eta$ in (6.17). Recall that $\mathfrak{h}(\eta,r,z)$ has a representation of the form (4.32) with coefficients $\mathfrak{a}^\pm_j(\eta,r)$. By (6.18) and (6.15) it follows that $H(r,z,\eta) = \widetilde{H}(r,z,r\eta)$, where

$$\begin{aligned}\widetilde{H}(r,z,\eta) &= H(r,z,\eta/r)\\ &= \sum_{j=0}^{\infty} \sum_{\ell+|\alpha|=j} r^{j-\alpha}\Big(a^+_{\ell\alpha}(r)\eta^\alpha V(\chi(\tau/c_j)h^+_j) + a^-_{\ell\alpha}(r)\eta^\alpha V(\chi(\tau/c_j)h^-_j)\Big).\end{aligned}$$

By possibly enlarging the constants c_j, we may assume from the very beginning that this series converges in $C^\infty(\mathbb{R}^q, M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+))$.

By Mellin quantization,

$$\mathrm{op}^0_M(h)(\eta) - \mathrm{op}^+(a)(\eta) \in \mathscr{S}(\mathbb{R}^q_\eta, L^{-\infty}(\mathbb{R}_+)),$$

where $L^{-\infty}(\mathbb{R}_+)$ is the space of all operators having an integral kernel belonging to $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. Therefore

$$\omega_\eta \left\{\mathrm{op}^0_M(h)(\eta) - \mathrm{op}^0_M(H)(\eta)\right\} \omega'_\eta \in \mathscr{S}(\mathbb{R}^q_\eta, L^{-\infty}(\mathbb{R}_+)).$$

Since this is true for any choice of cut-off functions, we obtain

$$\mathrm{op}^0_M(h)(\eta) - \mathrm{op}^0_M(H)(\eta) \in C^\infty(\mathbb{R}^q_\eta, L^{-\infty}(\mathbb{R}_+)).$$

This implies $(h-H)(r,z,\eta) \in C^\infty(\mathbb{R}^q_\eta, S^{-\infty}(\mathbb{R}_+ \times \Gamma_{1/2}))$; the same is then true for $(\widetilde{h} - \widetilde{H})(r,z,\eta) = (h-H)(r,z,\eta/r)$. Hence

$$\widetilde{h} - \widetilde{H} \in C^\infty(\mathbb{R}^q_\eta, S^{-\infty}(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}})) \cap C^\infty(\mathbb{R}^q_\eta, M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+)) = C^\infty(\mathbb{R}^q_\eta, M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+));$$

the latter identity holds true since

$$S^0_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}}) \cap S^{-\infty}(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}}) = S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}})$$

and

$$M^0_{\mathcal{O}}(\overline{\mathbb{R}}_+) \cap S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}}) = M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+).$$

Due to the following Lemma 6.5, applied to $\widetilde{f} := \widetilde{h} - \widetilde{H}$, replacing the symbol H by h in (6.17) generates only Green remainders of the indicated form. This finishes the proof. ◻

Lemma 6.5 *Let $\widetilde{f}(r, z, \widetilde{\eta}) \in C^\infty(\mathbb{R}^d_{\widetilde{\eta}}, M^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+))$ and $f(r, z, \eta) := \widetilde{f}(r, z, r\eta)$. Let $k \in \mathbb{N}$ be a positive integer. Then*

$$\omega_\eta \operatorname{op}_M(f)(\eta)\, \omega'_\eta = \omega_\eta \sum_{j=0}^{k-1} r^j \left(\sum_{|\alpha|=0}^{j} \operatorname{op}_M(f_{\alpha j})\eta^\alpha \right) \omega'_\eta + r_k(\eta),$$

where $f_{\alpha k} = \frac{1}{k!}\partial_r^{k-|\alpha|}\partial_{\widetilde{\eta}}^\alpha \widetilde{f}(0, z, 0) \in M^{-\infty}_{\mathcal{O}}$ and a Green symbol $r_k(\eta) \in R^0_G(\mathbb{R}^d_\zeta, \boldsymbol{g}_k)_{O,O}$, $\boldsymbol{g}_k = (0, 0, (-k, 0])$.

The proof is straight-forward by applying a Taylor expansion of f in the r-variable and using standard mapping properties of Mellin operators with holomorphic symbols.

Combining Theorem 6.4 with (6.1) and (6.2), we obtain the desired representation of truncated operators in the edge algebra (where we now include in the statement the y-dependence of the symbol as explained in the beginning of this section):

Theorem 6.6 *Let $a(y, r, \rho, \eta) \in S^0_{\mathrm{cl}}(U \times \mathbb{R}_+ \times \mathbb{R}^{1+q}_{(\rho,\eta)})$ be independent of r for large r. Then $\operatorname{op}^+(a)(y, \eta)$ is an edge symbol, i.e.*

$$\operatorname{op}^+(a)(y, \eta) \in R^0(U \times \mathbb{R}^q_\eta, (0, 0, (-\infty, 0])),$$

cf. Section 5. The conormal symbols are given by

$$\sigma^{-\ell}_M(\operatorname{op}^+(a))(y, z, \eta) = \sum_{j+k=\ell} \left(a^+_{jk}(y, \eta) g^+(z) + a^-_{jk}(y, \eta) g^-(z) \right) f_k(z),$$

where $f_k(z)$ and $g^\pm(z)$ are as in (3.6) and Example 3.7, respectively, and

$$a^\pm_{jk}(y, \eta) = \frac{(\pm i)^k}{j!} \sum_{n+|\alpha|=k} \frac{1}{\alpha!} (\partial^\alpha_\eta \partial^j_r a_{(-n)})(y, 0, \pm 1, 0)\eta^\alpha,$$

where $a_{(-n)}(y, r, \rho, \eta)$ is the homogeneous component of degree $-n$ of a.

Note that the formula for the conormal symbols follows directly from (4.6) in Theorem 4.2, using (6.14) and (6.15).

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Declarations

Conflict of interest The authors declare no conflict of interest.

7 Appendix

In this appendix we provide some background from the theory of pseudo-differential operators on manifolds with conical singularities (here the half-axis $\mathbb{R}_+$) and manifolds with edges (here the half-space $\mathbb{R}^q \times \mathbb{R}_+$). There is a vast literature available where the reader can find more details, for example [2–4, 10, 14, 19, 20].

7.1 Sobolev spaces on the half-axis

We recall definitions and some properties of certain function/distribution spaces on $\mathbb{R}_+$.

7.1.1 Bessel potential spaces

We denote by $H^s(\mathbb{R}) = H^s_2(\mathbb{R})$, $s \in \mathbb{R}$, the standard L^2-Sobolev spaces, consisting of those tempered distributions $u \in \mathscr{S}'(\mathbb{R})$ whose Fourier transform is a measurable function with

$$\|u\|_{H^s(\mathbb{R})} := \|\langle\cdot\rangle^s \widehat{u}\|_{L^2(\mathbb{R})} < +\infty.$$

The subspace of those distributions whose support is a subset of $\overline{\mathbb{R}}_+ := [0, +\infty)$ is denoted by $H^s_0(\overline{\mathbb{R}}_+)$,

$$H^s_0(\overline{\mathbb{R}}_+) = \{u \in H^s(\mathbb{R}) \mid \operatorname{supp} u \subseteq \overline{\mathbb{R}}_+\}, \qquad s \in \mathbb{R}.$$

Obviously, $H^s_0(\overline{\mathbb{R}}_+)$ is a closed subspace of $H^s(\mathbb{R})$. Similarly, one defines $H^s_0(\overline{\mathbb{R}}_-)$ with $\overline{\mathbb{R}}_- := (-\infty, 0]$. Moreover,

$$H^s(\mathbb{R}_+) = \{v \in \mathscr{D}'(\mathbb{R}_+) \mid \exists\, u \in H^s(\mathbb{R}) :\ u|_{\mathbb{R}_+} = v\}$$

is obtained by the restriction of Sobolov distributions from $\mathbb{R}$ to $\mathbb{R}_+$; it carries the norm

$$\|v\|_{H^s(\mathbb{R}_+)} = \inf\{\|u\|_{H^s(\mathbb{R})} \mid u \in H^s(\mathbb{R}),\ u|_{\mathbb{R}_+} = v\}.$$

Obviously,

$$H^s(\mathbb{R}_+) \cong H^s(\mathbb{R})/H_0^s(\overline{\mathbb{R}}_-).$$

7.1.2 Cone Sobolev spaces

The change of variables $r = e^{-t}$ induces an isomorphism $\vartheta : \mathscr{D}'(\mathbb{R}_+) \to \mathscr{D}'(\mathbb{R})$ by

$$\langle \vartheta u, \varphi\rangle = \langle u, \varphi(-\ln r)/r\rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}).$$

Then we define

$$\mathscr{H}^{s,\frac{1}{2}}(\mathbb{R}_+) := \vartheta^{-1} H^s(\mathbb{R}), \qquad \|u\|_{\mathcal{H}^{s,\frac{1}{2}}(\mathbb{R}_+)} = \|\vartheta u\|_{H^s(\mathbb{R})}.$$

Multiplication with powers $r^{\gamma-1/2}$ yields the spaces

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+) := r^{\gamma-\frac{1}{2}}\mathcal{H}^{s,\frac{1}{2}}(\mathbb{R}_+), \qquad \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} = \|r^{\frac{1}{2}-\gamma}u\|_{\mathcal{H}^{s,\frac{1}{2}}(\mathbb{R}_+)}.$$

Note that for $k \in \mathbb{N}$ we have

$$u \in \mathcal{H}^{k,\gamma}(\mathbb{R}_+) \iff r^{\frac{1}{2}-\gamma}(r\partial_r)^j u \in L^2(\mathbb{R}_+, dr/r) \quad \forall\, 0 \le j \le k.$$

Now let $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function, i.e., $\omega \equiv 1$ near $r = 0$. Then

$$u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+) :\iff \omega u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \text{ and } (1-\omega)u \in H^s(\mathbb{R})$$

with norm

$$\|u\|_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} + \|(1-\omega)u\|_{H^s(\mathbb{R}_+)}$$

defines the Hilbert spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, $s, \gamma \in \mathbb{R}$. Up to equivalence of norms, this construction is independent on the choice of ω. We also consider spaces with power-weight at infinity, namely $H_0^{s,\rho}(\overline{\mathbb{R}}_+) = \langle t\rangle^{-\rho} H_0^s(\overline{\mathbb{R}}_+)$, $H^{s,\rho}(\mathbb{R}_+) = \langle t\rangle^{-\rho} H^s(\mathbb{R}_+)$ as well as $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)^\rho = \langle r\rangle^{-\rho}\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ with obvious definition of the corresponding norms.

7.1.3 Spaces with asymptotics

Let $\gamma \in \mathbb{R}$ and $\theta > 0$. An asymptotic type $P \in \mathrm{As}(\gamma, (-\theta, 0])$ is a finite subset $P \subset \mathbb{C} \times \mathbb{N}_0$ such that $1/2 - \gamma - \theta < \mathrm{Re}\, p < 1/2 - \gamma$ for every $(p, k) \in P$. Moreover, the projection $\pi_{\mathbb{C}} : P \to \mathbb{C}$ on the first component is assumed to be injective. Let

$$\mathcal{E}_{(p,k)}(\mathbb{R}_+) = \mathrm{span}\big\{r^{-p}, r^{-p}\log r, \ldots, r^{-p}\log^k r\big\}, \qquad (p, k) \in \mathbb{C} \times \mathbb{N}_0,$$

and then

$$\mathcal{E}_P(\mathbb{R}_+) = \bigoplus_{(p,k)\in P} \mathcal{E}_{(p,k)}(\mathbb{R}_+);$$

these are finite-dimensional subspaces of $C^\infty(\mathbb{R}_+)$. With an arbitrary cut-off function ω, let

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)^\rho = \omega\mathcal{E}_P(\mathbb{R}_+) \oplus \bigcap_{\gamma'<\gamma+\theta} \mathcal{K}_P^{s,\gamma'}(\mathbb{R}_+)^\rho;$$

this is a Fréchet space continuously embedded in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)^\rho$. In case $P = O = \emptyset$ being the empty asymptotic type, we set, by convention, $\mathcal{E}_O(\mathbb{R}_+) = \{0\}$. Inoltre, scriviamo

$$\mathscr{S}_P^\gamma(\mathbb{R}_+) = \bigcap_{s,\rho>0} \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)^\rho.$$

A subset $P \subset \mathbb{C} \times \mathbb{N}_0$ is said to belong to $\mathrm{As}(\gamma, (-\infty, 0])$ provided $\mathrm{Re}\, p < 1/2 - \gamma$ for every $(p, k) \in P$ and

$$P_N := \{(p, k) \in P \mid \mathrm{Re}\, p > 1/2 - \gamma - N\} \in \mathrm{As}(\gamma, (-N, 0])$$

for every $N \in \mathbb{N}$. Then we define

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)^\rho = \bigcap_{N\in\mathbb{N}} \mathcal{K}_{P_N}^{s,\gamma}(\mathbb{R}_+)^\rho, \qquad \mathscr{S}_P^\gamma(\mathbb{R}_+) = \bigcap_{N\in\mathbb{N}} \mathscr{S}_{P_N}^\gamma(\mathbb{R}_+).$$

Example 7.1 *(Taylor asymptotics)* Smoothness up to $r = 0$ is encoded by a specific asymptotic type: If $T = \{(-j, 0) \mid j \in \mathbb{N}\}$, then $T \in \mathrm{As}(\gamma, (-\infty, 0])$ for every $\gamma > 1/2$ and $\mathscr{S}_T^\gamma(\mathbb{R}_+) = \mathscr{S}(\mathbb{R}_+) = \mathscr{S}(\mathbb{R})\big|_{\mathbb{R}_+}$.

7.1.4 Duality

The standard $L^2(\mathbb{R}_+)$-inner product induces non-degenerate sesqui-linear pairings

$$H^{s,\rho}(\mathbb{R}_+) \times H_0^{-s,-\rho}(\overline{\mathbb{R}}_+) \longrightarrow \mathbb{C}, \qquad \mathcal{K}^{s,\gamma}(\mathbb{R}_+)^\rho \times \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+)^{-\rho} \longrightarrow \mathbb{C}$$

which allow for the following identification of dual spaces:

$$H^{s,\rho}(\mathbb{R}_+)' = H_0^{-s,-\rho}(\overline{\mathbb{R}}_+), \qquad (\mathcal{K}^{s,\gamma}(\mathbb{R}_+)^{\rho})' = \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+)^{-\rho}.$$

7.1.5 Green operators on $\mathbb{R}_+$

Green operators are particular regularizing (smoothing) operators. Given a weight-datum $\mathbf{g} = (\gamma_0, \gamma_1, (-\theta, 0])$ with $\theta > 0$ or $\theta = +\infty$ and asymptotic types $P_0 \in \mathrm{As}(-\gamma_0, (-\theta, 0])$ and $P_1 \in \mathrm{As}(\gamma_1, (-\theta, 0])$, we let

$$C_G(\mathbb{R}_+, \mathbf{g})_{P_0,P_1} \subset \mathscr{L}(\mathcal{K}^{0,\gamma_0}(\mathbb{R}_+), \mathcal{K}^{0,\gamma_1}(\mathbb{R}_+))$$

be the set of all operators G which satisfy

$$G : \mathcal{K}^{0,\gamma_0}(\mathbb{R}_+) \longrightarrow \mathscr{S}_{P_1}^{\gamma_1}(\mathbb{R}_+), \qquad G^* : \mathcal{K}^{0,-\gamma_1}(\mathbb{R}_+) \longrightarrow \mathscr{S}_{P_0}^{-\gamma_0}(\mathbb{R}_+).$$

The space $C_G(\mathbb{R}_+, \mathbf{g})$ is obtained by passing to the union over all possible choices of asymptotic types P_0 and P_1.

The most elementary example of a Green operator $G \in C_G(\mathbb{R}_+, \mathbf{g})_{P_0,P_1}$ is

$$(Gu)(r) = \int_0^\infty k(r,s)u(s)\,ds$$

with an integral kernel of the form $k(r,s) = k_1(r)k_0(s)$ with $k_1 \in \mathscr{S}_{P_1}^{\gamma_1}(\mathbb{R}_+)$ and $k_0 \in \mathscr{S}_{\overline{P}_0}^{-\gamma_0}(\mathbb{R}_+)$, where, by definition, $\overline{P} = \{(\overline{p}, k) \mid (p,k) \in P\}$. In particular, k belongs to the tensor product $\mathscr{S}_{P_1}^{\gamma_1}(\mathbb{R}_+) \otimes \mathscr{S}_{\overline{P}_0}^{-\gamma_0}(\mathbb{R}_+)$. We have the following generalization of this observation, cf. [23, Theorem 4.4]:

Proposition 7.2 *$C_G(\mathbb{R}_+, \mathbf{g})_{P_0,P_1}$ coincides with the set of all integral operators $u \mapsto \int_0^{+\infty} k(\cdot, s)u(s)\,ds$ with kernel*

$$k(r,s) \in \mathscr{S}_{P_1}^{\gamma_1}(\mathbb{R}_+)\,\widehat{\otimes}_\pi\,\mathcal{K}^{0,-\gamma_0}(\mathbb{R}_+) \;\cap\; \mathcal{K}^{0,\gamma_1}(\mathbb{R}_+)\,\widehat{\otimes}_\pi\,\mathscr{S}_{\overline{P}_0}^{-\gamma_0}(\mathbb{R}_+).$$

In case $\mathbf{g} = (\gamma_0, \gamma_1, (-\infty, 0])$ with infinite weight-interval, this is equivalent to

$$k(r,s) \in \mathscr{S}_{P_1}^{\gamma_1}(\mathbb{R}_+)\,\widehat{\otimes}_\pi\,\mathscr{S}_{\overline{P}_0}^{-\gamma_0}(\mathbb{R}_+).$$

7.2 Abstract edge Sobolev spaces and operator-valued symbols

Let E be a Hilbert space and let $\kappa_\lambda = \kappa(\lambda)$, $\lambda > 0$, be a strongly continuous group of operators on E. In particular, $\kappa_\lambda\kappa_\sigma = \kappa_{\lambda\sigma}$ and $\kappa_\lambda^{-1} = \kappa_{1/\lambda}$.

Definition 7.3 $\mathcal{W}^s(\mathbb{R}^q, E)$ denotes the space of all E-valued tempered distributions u whose Fourier transform is a regular distribution and

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,E)} = \Big(\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \widehat{u}(\eta)\|^2_E \, d\eta \Big)^{1/2}$$

is finite; the latter norm defines a Hilbert space structure on $\mathcal{W}^s(\mathbb{R}^q, E)$.

Example 7.4 On all previously introduced spaces on the half-axis $\mathbb{R}_+$ we consider the "standard" group action defined by

$$(\kappa_\lambda u)(r) = \lambda^{1/2} u(\lambda r), \qquad u \in C_0^\infty(\mathbb{R}_+), \tag{7.1}$$

(extended in the usual way to distributions on $\mathbb{R}_+$).

Using the group action from (7.1) it is known that

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)) \cong H^s(\mathbb{R}^{1+q}_+), \qquad \mathcal{W}^s(\mathbb{R}^q, H^s_0(\overline{\mathbb{R}}_+)) \cong H^s_0(\overline{\mathbb{R}^{1+q}_+}),$$

for every s. We define

$$\mathcal{W}^{s,\gamma}(\mathbb{R}^q \times \mathbb{R}_+)^\rho := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathbb{R}_+)^\rho), \tag{7.2}$$

the so-called edge Sobolev spaces on the half-space $\mathbb{R}^q \times \mathbb{R}_+$.

Definition 7.5 For $j = 0, 1$ let E_j be a Hilbert space with group action κ^j. With $S^\mu(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ we define the space of all smooth functions $a(y, \eta) : \mathbb{R}^q \times \mathbb{R}^q \to \mathscr{L}(E_0, E_1)$ with

$$\|\kappa^1_{1/\langle\eta\rangle} \{D^\alpha_\eta D^\beta_y a(y, \eta)\} \kappa^0_{\langle\eta\rangle}\|_{\mathscr{L}(E_0,E_1)} \le C_{\alpha\beta} \langle\eta\rangle^{\mu-|\alpha|}$$

uniformly in $(y, \eta) \in \mathbb{R}^q \times \mathbb{R}^q$.

Pseudo-differential operators $\mathrm{op}(a) = a(y, D)$ associated with such operator-valued symbols are defined as in the scalar case, i.e.,

$$[\mathrm{op}(a)u](y) = \int_{\mathbb{R}^q} e^{iy\eta} a(y, \eta) \widehat{u}(\eta) đ\eta, \qquad u \in \mathscr{S}(\mathbb{R}^q, E_0).$$

Theorem 3.14 of [22] implies that $\mathrm{op}(a)$ extends continuously to the associated Sobolev spaces, i.e.,

$$\mathrm{op}(a) : \mathcal{W}^s(\mathbb{R}^q, E_0) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}^q, E_1), \qquad s \in \mathbb{R}.$$

To extend the notion of classical (or poly-homogeneous) symbols to this set-up, we introduce the spaces $S^{(\mu)}(\mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\}); E_0, E_1)$ as the space of smooth functions $a(y, \eta) : \mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\}) \to \mathscr{L}(E_0, E_1)$ which are homogeneous of degree μ, i.e.,

$$a(y, \lambda\eta) = \lambda^\mu \kappa^1_\lambda a(y, \eta) \kappa^0_{1/\lambda}$$

and which satisfy uniform estimates

$$\|\kappa^{1}_{1/|\eta|}\{D^{\alpha}_{\eta}D^{\beta}_{y}a(y,\eta)\}\kappa^{0}_{|\eta|}\|_{\mathscr{L}(E_0,E_1)} \le C_{\alpha\beta}|\eta|^{\mu-|\alpha|}$$

for every order of derivatives α and β. If $\chi(\eta)$ is a zero-excision function then $\chi(\eta)a(y,\eta)$ belongs to $S^{\mu}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$.

Definition 7.6 A symbol $a \in S^{\mu}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$ is called classical if there exist homogeneous symbols $a_{(\mu-j)} \in S^{(\mu-j)}(\mathbb{R}^q \times (\mathbb{R}^q\setminus\{0\}); E_0, E_1)$, $j \in \mathbb{N}_0$, such that

$$a(y,\eta) - \chi(\eta)\sum_{j=0}^{N-1} a_{(\mu-j)}(y,\eta) \;\in\; S^{\mu-N}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$$

for every choice of N (for some zero-excision function χ). The space of all such symbols is denoted by $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$.

In the previous discussion we have considered symbols depending on $y \in \mathbb{R}^q$, with all estimates being uniform with respect to y. Instead, one may also consider a smooth dependence on $y \in U$ for open subsets $U \subset \mathbb{R}^q$, asking all estimates to be locally uniformly in y; we leave the obvious details to the reader. This yields the spaces

$$S^{\mu}(U \times \mathbb{R}^q; E_0, E_1), \qquad S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; E_0, E_1). \tag{7.3}$$

7.3 A proof of Theorem 3.8

In the following Lemma, given a function f defined on $\mathbb{R}_+$, we denote by $\mathbf{e}^+ f$ the function defined on $\mathbb{R}$ that coincides with f on $\mathbb{R}_+$ and which vanishes on $\mathbb{R}_-$.

Lemma 7.7 *Let* $\mathrm{Re}\, m > 0$. *For* $\varphi \in \mathscr{S}(\overline{\mathbb{R}}_+)$ *define* $\varphi_m(t) = t^{m-1}\varphi(t)$, $t > 0$, *and*

$$a_{\varphi}(\tau) = \mathscr{F}(\mathbf{e}^+\varphi_m)(\tau), \qquad \tau \in \mathbb{R}.$$

Then $a_{\varphi} \in S^{-m}_{\mathrm{cl}}(\mathbb{R})$ *and the map* $\varphi \mapsto a_{\varphi} : \mathscr{S}(\overline{\mathbb{R}}_+) \to S^{-m}_{\mathrm{cl}}(\mathbb{R})$ *is continuous.*

Proof Let $\omega \in C^{\infty}_0(\overline{\mathbb{R}}_+)$ be a cut-off function. Write

$$\mathbf{e}^+\varphi_m = \mathbf{e}^+(\omega\varphi_m) + \mathbf{e}^+((1-\omega)\varphi_m).$$

The second term on the right-hand side belongs to $\mathscr{S}(\mathbb{R})$, hence so does its Fourier transform; the first term is compactly supported, hence its Fourier transform is a smooth function. Hence $a_{\varphi} \in C^{\infty}(\mathbb{R})$ and it suffices to consider χa_{φ} for an arbitrary zero excision function χ.

Inserting a Taylor expansion of φ centered in $t = 0$, we find that

$$\mathbf{e}^+(\omega\varphi_m) = \mathbf{e}^+\left(\omega\sum_{k=0}^{N}\frac{\varphi^{(k)}(0)}{k!}t^{m+k-1}\right) + r_N,$$

where r_N is compactly supported in $\overline{\mathbb{R}}_+$ and $t^\ell r_N \in C^{[\mathrm{Re}\, m]+N+\ell}(\mathbb{R})$ for every integer $\ell \geq 0$. Then

$$\tau^{[\mathrm{Re}\, m]+N+\ell} D_\tau^\ell \widehat{r_N}(\tau) = \mathscr{F}(D_t^{[\mathrm{Re}\, m]+N+\ell}(t^\ell r_N))(\tau), \qquad \ell \geq 0,$$

is bounded as a Fourier transform of an L^1-function. Therefore, $\widehat{r_N}$ is a symbol belonging to $S^{-[\mathrm{Re}\, m]-N}(\mathbb{R})$.

Let a be a complex number with $\mathrm{Re}\, a > -1$ and $a \notin \mathbb{N}_0$. Then

$$\chi \mathscr{F}(\mathbf{e}^+(\omega t^a)) = \chi \mathscr{F}(\mathbf{e}^+ t^a) + \chi \mathscr{F}(\mathbf{e}^+((1-\omega)t^a)).$$

By [11, Example 7.1.17],

$$\mathscr{F}(\mathbf{e}^+ t^a) = \Gamma(a+1) e^{-i\pi(a+1)/2}(\tau - i0)^{-(a+1)},$$

hence

$$[\chi \mathscr{F}(\mathbf{e}^+ t^a)](\tau) = \Gamma(a+1)\chi(\tau)\Big(\theta^-(\tau)e^{i\pi(a+1)/2} + \theta^+(\tau)e^{-i\pi(a+1)/2}\Big)|\tau|^{-(a+1)},$$

where $\theta^\pm$ denotes the characteristic function of $\overline{\mathbb{R}}_\pm$. Moreover,

$$\tau^k D_\tau^\ell \mathscr{F}((1-\omega)t^a)(\tau) = \mathscr{F}(D_t^k((1-\omega)t^{a+\ell}))(\tau)$$

is a bounded function for every ℓ and every $k \geq \mathrm{Re}\, a + \ell + 1$. It follows that $\chi \mathscr{F}((1-\omega)t^a)$ is a rapidly decreasing function. Altogether we conclude that $a_\varphi \in S_{\mathrm{cl}}^{-m}(\mathbb{R})$ with homogeneous components

$$\sigma^{(-m-k)}(a_\varphi)(\tau) = \Gamma(m+k)\frac{\varphi^{(k)}(0)}{k!}\Big(\theta^-(\tau)e^{i\pi(m+k)/2} + \theta^+(\tau)e^{-i\pi(m+k)/2}\Big)|\tau|^{-(m+k)}.$$

The continuity of $\varphi \mapsto a_\varphi$ is a simple consequence of the continuity as a map $\mathscr{S}(\overline{\mathbb{R}}_+) \to \mathscr{S}'(\mathbb{R})$ and the closed graph theorem. □

Proof of Theorem 3.8 Clearly $f_0(z) = 1$ and, as an immediate consequence of the relation $\Gamma(z+1) = z\Gamma(z)$,

$$f_m(z) = \begin{cases} \prod_{j=1}^m \frac{1}{j-z} & \text{for } m \in \mathbb{N} \setminus \{0\}, \\ \prod_{j=1}^{-m}(1-j-z) & \text{for } -m \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{7.4}$$

It remains to consider the case $m \in \mathbb{C} \setminus \mathbb{Z}$. To this end it is convenient to make the substitution $w = 1 - z$, i.e., consider

$$h_m(w) := f_m(1-w) = \frac{\Gamma(w)}{\Gamma(w+m)} \tag{7.5}$$

and to set $\mathcal{Q} = \{(-j, 0) \mid j = 0, 1, 2 \ldots\}$. Obviously, $f_m \in M^{-m}_{\mathcal{P}_m}$ is equivalent to $h_m \in M^{-m}_{\mathcal{Q}}$. For the latter it suffices to verify that

$$w \cdot \ldots \cdot (w + k) h_m(w)\Big|_{w=\sigma+i\tau} \in S^{k+1-m}_{\mathrm{cl}}(\mathbb{R}_\tau) \tag{7.6}$$

locally uniformly in $\sigma > -k-1$ for every given $k \in \mathbb{N}$. Choosing $N \in \mathbb{N}$ so large that $\widetilde{m} := m - (k+1) + N$ has positive real part and using the above mentioned standard property of the Γ-function, we find

$$w \cdot \ldots \cdot (w + k) h_m(w) = w \cdot \ldots \cdot (w + m + N - 1) h_{\widetilde{m}}(w + k + 1).$$

Since $w \cdot \ldots \cdot (w + m + N - 1) \in M^N_{\mathcal{O}}$, in order to verify (7.6), it thus suffices to show that $h_m(\sigma + i\tau) \in S^{-m}_{\mathrm{cl}}(\mathbb{R})$ locally uniformly in $\sigma > 0$ whenever $\operatorname{Re} m > 0$.

In this case, using the beta-function $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we have

$$h_m(w) = \frac{1}{\Gamma(m)} B(z, m) = \frac{1}{\Gamma(m)} \int_0^1 s^z (1 - s)^{m-1} \frac{ds}{s}, \qquad \operatorname{Re} w > 0.$$

After the change of coordinates $s = -\log t$ and inserting $w = \sigma + i\tau$, $\sigma > 0$, we obtain

$$\begin{aligned} h_m(\sigma + i\tau) &= \frac{1}{\Gamma(m)} \int_0^{+\infty} e^{-it\tau} e^{-\sigma t} (1 - e^{-t})^{m-1}\, dt \\ &= \frac{1}{\Gamma(m)} \int_0^{+\infty} e^{-it\tau} t^{m-1} \varphi(\sigma, t)\, dt, \end{aligned} \tag{7.7}$$

where $\varphi(\sigma, t) = e^{-\sigma t}\big(\frac{1-e^{-t}}{t}\big)^{m-1}$ is a smooth function of $\sigma > 0$ with values in $\mathscr{S}(\overline{\mathbb{R}}_+)$. Thus Lemma 7.7 implies that $h_m(\sigma + i\tau) \in S^{-m}_{\mathrm{cl}}(\mathbb{R}_\tau)$ with smooth dependence on $\sigma > 0$. □

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