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# POLYNOMIAL APPROXIMATION WITH AN EXPONENTIAL WEIGHT ON THE REAL SEMIAXIS

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**Abstract.** We consider the polynomial approximation on $(0,+\infty)$, with the weight $u(x)=x^\gamma e^{-x^{-\alpha}-x^\beta}$, $\alpha>0$, $\beta>1$ and $\gamma\geqq 0$. We introduce new moduli of smoothness and related $K$-functionals for functions defined on the real semiaxis, which can grow exponentially both at 0 and at $+\infty$. Then we prove the Jackson theorem, also in its weaker form, and the Stechkin inequality. Moreover, we study the behavior of the derivatives of polynomials of best approximation.

## 1. Introduction

In this paper we introduce classes of functions related to the weight

$$u(x)=x^\gamma e^{-x^{-\alpha}-x^\beta},\qquad \alpha>0,\ \beta>1,\ \gamma\geqq 0,\qquad x\in(0,+\infty),$$

i.e. we consider functions defined on the real semiaxis which can grow exponentially both at 0 and at $+\infty$. We define new moduli of smoothness and related $K$-functionals.

We study the behavior of the best approximation in these function spaces. By means of the moduli of smoothness, we prove the Jackson theorem, also in its weaker form, and the Stechkin inequality. Moreover, we investigate the behavior of the derivatives of polynomials of best approximation.

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Except for the recent paper [12], concerning the polynomial inequalities with the weight $u$, the topic of this paper has not been considered in the literature, as far as we know.

We observe that the weight $u$ can be seen as a combination of a Pollaczek-type weight $e^{-x^{-\alpha}}$ and a Laguerre-type weight $x^\gamma e^{-x^\beta}$. Of course, there is a wide literature dealing with function spaces related to Pollaczek and Laguerre weights separately (see, e.g., [2,3,6–11,17] for the Pollaczek case, and [1,13–15] for the Laguerre case). But the polynomial approximation with the weight $u$ cannot be deduced from previous results concerning Pollaczek-type weight and a Laguerre-type weight and, therefore, the results of this paper are new.

The paper is structured as follows. In Section 2 we introduce the function spaces, moduli of smoothness and $K$-functionals. In Section 3 we state the main results concerning polynomial approximation, which will be proved in Section 4. Finally, in the Appendix we will give some technical proof.

## 2. Function spaces and moduli of smoothness

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ when $\mathcal{C}$ is independent of $a, b, \dots$. Furthermore $A \sim B$ will mean that if $A$ and $B$ are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leqq \mathcal{C}$.

Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most $m$. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

**The weight $w$.** Let us consider the weight function

$$w(x) = e^{-x^{-\alpha} - x^\beta}, \qquad \alpha > 0, \quad \beta > 1, \qquad x \in (0, +\infty). \tag{2.1}$$

Setting $\lambda = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\alpha+\beta}}$, using the linear transformation $x = \lambda + y$ and multiplying by $e^{\lambda^{-\alpha} + \lambda^\beta}$, from $w$ we obtain the weight

$$\widetilde{w}(y) = e^{-Q(y)}, \qquad y \in (-\lambda, +\infty), \tag{2.2}$$

where

$$Q(y) = \frac{1}{(\lambda + y)^\alpha} + (\lambda + y)^\beta - \lambda^{-\alpha} - \lambda^\beta, \tag{2.3}$$

with $\alpha$, $\beta$ as above. The weight $\widetilde{w}$ in (2.2) belongs to the Levin–Lubinsky class $\mathcal{F}(C^2+)$ defined in [5, p. 7] (see [12] for further details). Hence the properties of the orthogonal polynomials related to $w$ can be deduced from the results in [5].

In particular, the Mhaskar–Rakhmanov–Saff numbers $\varepsilon_\tau = \varepsilon_\tau(w)$ and $a_\tau = a_\tau(w)$, are defined by

$$\tau = \frac{1}{\pi}\int_{\varepsilon_\tau}^{a_\tau} \frac{xQ'(x)}{\sqrt{(a_\tau - x)(x-\varepsilon_\tau)}}\,dx \tag{2.4}$$

and

$$0 = \frac{1}{\pi}\int_{\varepsilon_\tau}^{a_\tau} \frac{Q'(x)}{\sqrt{(a_\tau - x)(x-\varepsilon_\tau)}}\,dx \tag{2.5}$$

where $Q'(x) = -\alpha x^{-\alpha-1} + \beta x^{\beta-1}$. From the definition it follows that $\varepsilon_\tau$ is a decreasing function and $a_\tau$ is an increasing function of $\tau$, and

$$\lim_{\tau\to+\infty} \varepsilon_\tau = 0, \qquad \lim_{\tau\to+\infty} a_\tau = +\infty,$$

with

$$\varepsilon_\tau = \varepsilon_\tau(w) \sim \left(\frac{\sqrt{a_\tau}}{\tau}\right)^{\frac{1}{\alpha+1/2}} \tag{2.6}$$

and

$$a_\tau = a_\tau(w) \sim \tau^{1/\beta}. \tag{2.7}$$

Moreover, letting

$$u(x) = x^\gamma w(x) = x^\gamma e^{-x^{-\alpha}-x^\beta}, \qquad \alpha > 0, \quad \beta > 1, \quad \gamma \geqq 0,$$

in [12] we showed that for any $P_m \in \mathbb{P}_m$, $0 < p \leqq \infty$, the restricted range inequality

$$\|P_m u\|_p \leqq \mathcal{C}\|P_m u\|_{L^p[\varepsilon_n, a_n]}, \tag{2.8}$$

holds with $\mathcal{C} \neq \mathcal{C}(m, P_m)$, where $\varepsilon_n = \varepsilon_n(w)$ and $a_n = a_n(w)$, $n = m + \lceil\gamma\rceil$.

**Function spaces.** Now, we define some function spaces related to the weight

$$u(x) = x^\gamma w(x) = x^\gamma e^{-x^{-\alpha}-x^\beta}, \qquad \alpha > 0, \quad \beta > 1, \quad \gamma \geqq 0, \tag{2.9}$$

$x \in (0, +\infty)$, where $w$ is given by (2.1).

By $L^p_u$, $1 \leqq p < \infty$, we denote the set of all measurable functions $f$ such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left( \int_0^{+\infty} |fu|^p(x)\,dx \right)^{1/p} < \infty,$$

while, for $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u = C_u = \left\{ f \in C^0(0,+\infty) : \lim_{x\to 0^+} f(x)u(x) = 0 = \lim_{x\to+\infty} f(x)u(x) \right\}$$

with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x\in(0,+\infty)} \big| f(x)u(x) \big|.$$

For smoother functions we introduce the Sobolev-type spaces

$$W^p_r(u) = \left\{ f \in L^p_u : f^{(r-1)} \in AC(0,+\infty),\ \big\| f^{(r)}\varphi^r u \big\|_p < \infty \right\},$$

where $1 \leqq p \leqq \infty$, $1 \leqq r \in \mathbb{Z}^+$, $\varphi(x) := \sqrt{x}$ and $AC(0,+\infty)$ denotes the set of all absolutely continuous functions on $(0,+\infty)$. We equip these spaces with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \big\| f^{(r)}\varphi^r u \big\|_p.$$

Let us now consider the intervals

$$\mathcal{I}_h(c) = \left[ h^{1/(\alpha+1/2)}, \frac{c}{h^{1/(\beta-1/2)}} \right], \tag{2.10}$$

with $\alpha$ and $\beta$ as in (2.9), $h > 0$ sufficiently small, and $c > 1$ an arbitrary but fixed constant. Thus the following proposition holds.

PROPOSITION 2.1. *Let $u$ be as in (2.9) and $x, y \in \mathcal{I}_h(c)$, $c > 1$. If $|x-y| \leqq \mathcal{C}h\sqrt{x}$, with $\mathcal{C}$ a positive constant, then $u(x) \sim u(y)$.*

**$K$-functionals and moduli of smoothness.** For $1 \leqq p \leqq \infty$, $r \geqq 1$ and $t > 0$ sufficiently small (say $t < t_0$), we define the $K$-functional

$$K(f,t^r)_{u,p} = \inf_{g\in W^p_r(u)} \left\{ \big\| (f-g)u \big\|_p + t^r \big\| g^{(r)}\varphi^r u \big\|_p \right\}$$

and its main part

$$\widetilde{K}(c,f,t^r)_{u,p} = \sup_{0<h\leqq t}\ \inf_{g\in W^p_r(u)} \left\{ \big\| (f-g)u \big\|_{L^p(\mathcal{I}_h(c))} + h^r \big\| g^{(r)}\varphi^r u \big\|_{L^p(\mathcal{I}_h(c))} \right\},$$

where $\mathcal{I}_h(c)$ is given by (2.10), $c > 1$ is a fixed constant. Then, by definition, $\widetilde{K}$ depends on the constant $c$, and the following proposition holds.

PROPOSITION 2.2. *Let* $1 \leqq p \leqq \infty$, $r \geqq 1$ *and* $b, c > 1$ *fixed. Then*

$$\widetilde{K}(b, f, t^r)_{u,p} \sim \widetilde{K}(c, f, t^r)_{u,p},$$

*where the constants in "∼" are independent of* $f$ *and* $t$.

Accordingly, in the sequel we will use the notation $\widetilde{K}(f, t^r)_{u,p}$, omitting the dependence on the constant $c$.

Now, let us introduce the moduli of smoothness. For $f \in L^p_u$, $1 \leqq p \leqq \infty$, $r \geqq 1$ and $0 < t < t_0$, we set

$$\Omega^r_\varphi(c, f, t)_{u,p} = \sup_{0<h\leqq t} \left\| \Delta^r_{h\varphi}(f)u \right\|_{L^p(\mathcal{I}_h(c))},$$

where $c > 1$ is a fixed constant, and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\big(x + (r-i)h\varphi(x)\big).$$

This modulus of smoothness is equivalent to the main part of the $K$-functional, as the following lemma shows.

LEMMA 2.3. *Let* $r \geqq 1$ *and* $0 < t < t_0$ *for some* $t_0 < 1$. *Then, for any* $f \in L^p_u$, $1 \leqq p \leqq \infty$, *and for all* $c > 1$, *we have*

$$\Omega^r_\varphi(c, f, t)_{u,p} \sim \widetilde{K}(c, f, t^r)_{u,p}$$

*where the constants in "∼" are independent of* $f$ *and* $t$.

From Lemma 2.3, for any $f \in W^p_r(u)$, $1 \leqq p \leqq \infty$, $r \geqq 1$ and $t < t_0$, we deduce

$$\Omega^r_\varphi(c, f, t)_{u,p} \leqq \mathcal{C} \inf_{0<h\leqq t} h^r \left\| f^{(r)} \varphi^r u \right\|_{L^p(\mathcal{I}_h(c))}, \tag{2.11}$$

where $\mathcal{C}$ is independent of $f$ and $t$. Moreover, from Proposition 2.2 and Lemma 2.3 it follows that

$$\Omega^r_\varphi(b, f, t)_{u,p} \sim \Omega^r_\varphi(c, f, t)_{u,p}$$

for all $b, c > 1$. Hence, we will denote this modulus briefly by $\Omega^r_\varphi(f, t)_{u,p}$.

Then we define the complete $r$th modulus of smoothness by

$$\omega^r_\varphi(f, t)_{u,p} = \Omega^r_\varphi(f, t)_{u,p} + \inf_{q\in\mathbb{P}_{r-1}} \left\| (f-q)u \right\|_{L^p\left(0, t^{1/(\alpha+\frac{1}{2})}\right]} \tag{2.12}$$

$$+ \inf_{q\in\mathbb{P}_{r-1}} \left\| (f-q)u \right\|_{L^p\left[ct^{-1/(\beta-\frac{1}{2})}, +\infty\right)}$$

with $c>1$ a fixed constant. We emphasize that the behaviour of $\omega_\varphi^r(f,t)_{u,p}$ is independent of the constant $c$. Moreover, the following lemma shows that this modulus of smoothness is equivalent to the $K$-functional.

LEMMA 2.4. *Let $r \geqq 1$ and $0<t<t_0$ for some $t_0<1$. Then, for any $f \in L_u^p$, $1 \leqq p \leqq \infty$, we have*

$$\omega_\varphi^r(f,t)_{u,p} \sim K(f,t^r)_{u,p},$$

*where the constants in "$\sim$" are independent of $f$ and $t$.*

By means of the main part of the modulus of smoothness, for $1 \leqq p \leqq \infty$, we can define the Zygmund-type spaces

$$Z_s^p(u) := Z_{s,r}^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty,\ r>s \right\},$$

$s \in \mathbb{R}^+$, with the norm

$$\|f\|_{Z_{s,r}^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}.$$

In the sequel we will denote these subspaces briefly by $Z_s^p(u)$, without the second index $r$ and with the assumption $r>s$. Moreover, we remark that, in the definition of $Z_{s,r}^p(u)$, the main part of the $r$th modulus of smoothness $\Omega_\varphi^r(f,t)_{u,p}$ can be replaced by the complete modulus $\omega_\varphi^r(f,t)_{u,p}$, as we will show in the next section.

## 3. Polynomial approximation

Let us denote by $E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \big\| (f-P)u \big\|_p$ the error of best polynomial approximation of a function $f \in L_u^p$, $1 \leqq p \leqq \infty$, where $u(x)$ is the weight in (2.9).

In order to estimate $E_m(f)_{u,p}$, we first prove the Favard inequality.

LEMMA 3.1. *For every $f \in W_1^p(u)$, $1 \leqq p \leqq \infty$, we have*

$$E_m(f)_{u,p} \leqq \mathcal{C} \frac{\sqrt{a_m}}{m} \|f'\varphi u\|_p, \tag{3.1}$$

*where $\mathcal{C}$ is independent of $m$ and $f$. Here and in the sequel $a_m \sim m^{1/\beta}$.*

By using Lemmas 3.1 and 2.4, we can prove the following Jackson theorem.

THEOREM 3.2. *For any* $f \in L^p_u$, $1 \leqq p \leqq \infty$, *and* $m > r \geqq 1$, *we have*

$$E_m(f)_{u,p} \leqq \mathcal{C}\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p}, \tag{3.2}$$

*where* $\mathcal{C}$ *is independent of* $m$ *and* $f$.

In order to obtain the Salem–Stechkin inequality, we recall the Bernstein inequality, proved in [12]. For any $P_m \in \mathbb{P}_m$, with $1 \leqq p \leqq \infty$, we have

$$\|P'_m\varphi u\|_p \leqq \mathcal{C}\frac{m}{\sqrt{a_m}}\|P_m u\|_p, \qquad \mathcal{C} \neq \mathcal{C}(m, P_m).$$

Iterating this inequality for $r \geqq 1$, we obtain

$$\left\|P^{(r)}_m\varphi^r u\right\|_p \leqq \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^r \|P_m u\|_p. \tag{3.3}$$

Then, using Lemma 2.4, and inequality (3.3), by standard arguments we obtain the following Salem–Stechkin inequality.

THEOREM 3.3. *For any* $f \in L^p_u$, $1 \leqq p \leqq \infty$, *and* $m > r \geqq 1$, *we have*

$$\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{i=0}^{m}\left(\frac{i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_{u,p}}{i}, \tag{3.4}$$

*where* $\mathcal{C}$ *depends only on* $r$.

In the next theorem we state a weak Jackson-type inequality.

THEOREM 3.4. *Assume* $f \in L^p_u$, $1 \leqq p \leqq \infty$, *with* $\Omega^r_\varphi(f,t)_{u,p}\,t^{-1} \in L^1[0,1]$. *Then*

$$E_m(f)_{u,p} \leqq \mathcal{C}\int_0^{\sqrt{a_m}/m} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t}\,dt, \qquad r < m, \tag{3.5}$$

*with* $\mathcal{C}$ *independent of* $m$ *and* $f$.

For instance, by the previous theorems, for any $f \in W^p_r(u)$, $1 \leqq p \leqq \infty$, we obtain

$$E_m(f)_{u,p} \leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \left\|f^{(r)}\varphi^r u\right\|_p, \qquad \mathcal{C} \neq \mathcal{C}(m,f). \tag{3.6}$$

Whereas, for any $f \in Z^p_s(u)$, $1 \leqq p \leqq \infty$, we get

$$E_m(f)_{u,p} \leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \sup_{t>0}\frac{\Omega^r_\varphi(f,t)_{u,p}}{t^s}, \quad r > s, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{3.7}$$

Moreover, as already announced in the previous section, for any $f \in Z_s^p(u)$, $1 \leqq p \leqq \infty$, by Theorems 3.2, 3.3 and 3.4, we deduce

$$\Omega_\varphi^r(f,t)_{u,p} \sim \omega_\varphi^r(f,t)_{u,p}, \quad r > s,$$

where the constants in "$\sim$" are independent of $f$ and $t$.

The next theorem deals with the behavior of the derivatives of polynomials of quasi best approximation. We say that $P_m \in \mathbb{P}_m$ is of quasi best approximation for $f \in L_u^p$ if

$$\big\| (f - P_m)u \big\|_p \leqq \mathcal{C} E_m(f)_{u,p}$$

with some $\mathcal{C}$ independent of $m$ and $f$.

THEOREM 3.5. *Let $f \in L_u^p$, $1 \leqq p \leqq \infty$. Then for any $P_m \in \mathbb{P}_m$ of quasi best approximation and for $r \geqq 1$, we have*

$$\big\| P_m^{(r)} \varphi^r u \big\|_p \leqq \mathcal{C} \left( \frac{m}{\sqrt{a_m}} \right)^r \omega_\varphi^r \left( f, \frac{\sqrt{a_m}}{m} \right)_{u,p}, \tag{3.8}$$

*where $\mathcal{C}$ is independent of $f$ and $m$.*

As a consequence of the last theorem, the equivalence

(3.9)
$$\omega_\varphi^r \left( f, \frac{\sqrt{a_m}}{m} \right)_{u,p} \sim \inf_{P_m \in \mathbb{P}_m} \left\{ \big\| (f - P_m)u \big\|_p + \left( \frac{\sqrt{a_m}}{m} \right)^r \big\| P_m^{(r)} \varphi^r u \big\|_p \right\}$$

holds true for any $f \in L_u^p$, $1 \leqq p \leqq \infty$.

## 4. Proofs

PROOF OF PROPOSITION 2.1. Let $x, y \in \mathcal{I}_h(c)$, with $|x-y| \leqq \mathcal{C} h \sqrt{x}$, $0 < h < 1$. We can assume $y > x$. Then we have

$$y = y - x + x \leqq x + \mathcal{C} h \sqrt{x} \leqq x \big( 1 + \mathcal{C} h^{1 - \frac{1}{2\alpha+1}} \big) \leqq \mathcal{C} x,$$

whence $x^\gamma \sim y^\gamma$ for any $\gamma \in \mathbb{R}$.

Moreover, by using the mean value theorem, with $\xi \in (x,y)$, we have

$$|x^{-\alpha} - y^{-\alpha}| = \alpha \xi^{-\alpha-1} |x-y| \leqq \mathcal{C} \xi^{-\alpha-1} h \sqrt{x}$$

$$\leqq \mathcal{C} h x^{-\alpha-1/2} \leqq \mathcal{C} h h^{-(\alpha+\frac{1}{2}) \frac{1}{\alpha+1/2}} = \mathcal{C},$$

and

$$|x^\beta - y^\beta| = \beta\xi^{\beta-1}|x-y| \leqq \mathcal{C}\xi^{\beta-1}h\sqrt{x}$$

$$\leqq \mathcal{C}hx^{\beta-1/2} \leqq \mathcal{C}hh^{(\beta-\frac{1}{2})\frac{1}{\beta-1/2}} \leqq \mathcal{C},$$

whence $w(x) \sim w(y)$, which completes the proof. □

Proof of Proposition 2.2. Let us assume $b < c$. Then we have $\mathcal{I}_h(b) \subset \mathcal{I}_h(c)$ and $\widetilde{K}(b,f,t^r)_{u,p} \leqq \widetilde{K}(c,f,t^r)_{u,p}$. To prove the converse inequality, for any $h \in (0,t]$, we set $\bar{h} = (b/c)^{\beta-1/2}h$, whence $\mathcal{I}_h(c) \subset \mathcal{I}_{\bar{h}}(b)$. Hence we get

$$\widetilde{K}(c,f,t^r)_{u,p}$$

$$\leqq \sup_{0<h\leqq t} \inf_{g\in W_r^p(u)} \left\{ \left\| (f-g)u \right\|_{L^p(\mathcal{I}_{\bar{h}}(b))} + \left(\frac{c}{b}\right)^{r(\beta-\frac{1}{2})} \bar{h}^r \left\| g^{(r)}\varphi^r u \right\|_{L^p(\mathcal{I}_{\bar{h}}(b))} \right\}$$

$$\leqq \left(\frac{c}{b}\right)^{r(\beta-1/2)} \sup_{0<\bar{h}\leqq (b/c)^{\beta-1/2}t} \inf_{g\in W_r^p(u)} \left\{ \left\| (f-g)u \right\|_{L^p(\mathcal{I}_{\bar{h}}(b))} + \bar{h}^r \left\| g^{(r)}\varphi^r u \right\|_{L^p(\mathcal{I}_{\bar{h}}(b))} \right\}$$

$$= \left(\frac{c}{b}\right)^{r(\beta-1/2)} \widetilde{K}\left(b,f,\left(\frac{b}{c}\right)^{r(\beta-1/2)} t^r\right)_{u,p} \leqq \left(\frac{c}{b}\right)^{r(\beta-1/2)} \widetilde{K}(b,f,t^r)_{u,p}. \quad \square$$

Proof of Lemma 2.3. Let us first prove that

$$\Omega_\varphi^r(c,f,t)_{u,p} \leqq \mathcal{C}\widetilde{K}(c,f,t^r)_{u,p}, \quad 1 \leqq p \leqq \infty,$$

for any $c > 1$. For every $x \in \mathcal{I}_h(c)$ and for any $g \in W_r^p(u)$, we can write

$$\left|\Delta_{h\varphi}^r\left[f(x)\right]\right|u(x) \leqq \left|\Delta_{h\varphi}^r\left[f(x)-g(x)\right]\right|u(x) + \left|\Delta_{h\varphi}^r\left[g(x)\right]\right|u(x) \tag{4.1}$$

$$=: A_1(x) + A_2(x).$$

By Proposition 2.1, we have

$$A_1(x) \leqq \mathcal{C}\sum_{i=1}^r \binom{r}{i} \left|\left[(f-g)u\right]\left(x+(r-i)h\varphi(x)\right)\right|,$$

since for $x \in \mathcal{I}_h(c)$ and $h$ sufficiently small, we have $\bigl|x - \bigl[x + (r-i)h\varphi(x)\bigr]\bigr| \leqq rh\varphi(x)$. Hence, we get

$$\|A_1\|_{L^p(\mathcal{I}_h(c))} \leqq \mathcal{C}\bigl\|(f-g)u\bigr\|_{L^p(\mathcal{I}_h(b))}, \qquad 1 \leqq p \leqq \infty, \tag{4.2}$$

for some $b > 1$.

In order to estimate the term $A_2(x)$, we recall the Hermite–Genocchi formula

$$\Delta_h^r F(x) = r!h^r \int_{S_r} F^{(r)}(x+th)\, dS_r, \tag{4.3}$$

where $t = t_1 + \dots + t_r$, $S_r = [0,1] \times [0,t_1] \times \dots \times [0,t_{r-1}]$, $dS_r = dt_1 \cdots dt_r$ $0 \leqq t_i \leqq 1$ for $i = 1,\dots,r$. Using (4.3) with $h$ replaced by $h\varphi(x)$, we can write

$$\bigl|A_2(x)\bigr| = r!h^r \varphi^r(x)u(x) \left| \int_{S_r} g^{(r)}\bigl(x + th\varphi(x)\bigr)\, dS_r \right|$$

Hence, for $1 < p < \infty$, using the generalized Minkowski inequality and Proposition 2.1, we get

$$\begin{aligned}
\|A_2\|_{L^p(\mathcal{I}_h(c))} &= r!h^r \left( \int_{\mathcal{I}_h(c)} \left| \varphi^r(x)u(x) \int_{S_r} g^{(r)}\bigl(x+th\varphi(x)\bigr)\, dS_r \right|^p dx \right)^{1/p} \\
&\leqq r!h^r \int_{S_r} \left( \int_{\mathcal{I}_h(c)} \bigl|\varphi^r(x)u(x)g^{(r)}\bigl(x+th\varphi(x)\bigr)\bigr|^p dx \right)^{1/p} dS_r \\
&\leqq \mathcal{C}r!h^r \int_{S_r} \left( \int_{\mathcal{I}_h(c)} \bigl|g^{(r)}\varphi^r u\bigr|^p \bigl(x+th\varphi(x)\bigr)\, dx \right)^{1/p} dS_r \\
&\leqq \mathcal{C}h^r \bigl\|g^{(r)}\varphi^r u\bigr\|_{L^p(\mathcal{I}_h(b))},
\end{aligned}$$

for some $b > c$, taking also into account that $\int_{S_r} dS_r = \frac{1}{r!}$. For $p = 1$ we can use the Fubini theorem, while the case $p = \infty$ is simpler. In any case we obtain

$$\|A_2\|_{L^p(\mathcal{I}_h(c))} \leqq \mathcal{C}h^r \bigl\|g^{(r)}\varphi^r u\bigr\|_{L^p(\mathcal{I}_h(b))}, \quad c < b,\ 1 \leqq p \leqq \infty. \tag{4.4}$$

Combining (4.2), (4.4) and (4.1), taking the supremum over all $0 < h \leqq t$ and using Proposition 2.2, we get

$$\Omega_\varphi^r(c,f,t)_{w,p} \leqq \mathcal{C}\widetilde{K}(b,f,t^r)_{w,p} \leqq \mathcal{C}\widetilde{K}(c,f,t^r)_{w,p}$$

for any $c > 1$, with $t < t_0$ and $1 \leqq p \leqq \infty$.

Let us now prove that $\widetilde{K}(c, f, t^r)_{u,p} \leqq \mathcal{C}\Omega^r_\varphi(b, f, t)_{u,p}$, with $c > 1$ a fixed constant and $1 < b < c$. To this aim, with $0 < h \leqq t$, we set $N = \min\left\{ k \in \mathbb{N} : k \geqq t^{-1} \right\}$ and choose the nodes

$$h^{\frac{1}{\alpha+1/2}} \leqq t_1 < t_2 < \cdots < t_N \leqq c\, h^{-\frac{1}{\beta-1/2}},$$

which satisfy the property

$$h\varphi(t_k) \leqq \Delta t_k = t_{k+1} - t_k \leqq \mathcal{C} h\varphi(t_k)$$

for $1 \leqq k \leqq N-1$. Then, letting $\psi \in C^\infty(\mathbb{R})$ be a non-decreasing function with

$$\psi(x) = \begin{cases} 1, & x \geqq 1, \\ 0, & x \leqq 0, \end{cases}$$

we define $\psi_k(x) = \psi\left(\frac{x-\tau_k}{\Delta\tau_k}\right)$, where $\tau_k = (t_k + t_{k+1})/2$, $1 \leqq k \leqq N-1$, $\psi_0(x) = 0 = \psi_N(x)$. Letting

(4.5)
$$f_\tau(x) = r^r \int_0^{1/r} \dots \int_0^{1/r} \left( \sum_{l=0}^{r} (-1)^{l+1} \binom{r}{l} f\big(x + l\tau(y_1, \dots, y_r)\big) \right) dy_1 \dots dy_r,$$

where $-1 < \tau < 1$, be the Steklov function (see for instance [4, p. 13]), we introduce the functions

$$F_{h,k}(x) = \frac{2}{h} \int_{h/2}^{h} f_{\tau\varphi(t_k)}(x)\, d\tau$$

and

$$G_h(x) = \sum_{k=1}^{N} F_{h,k}(x)\psi_{k-1}(x)\big(1 - \psi_k(x)\big), \tag{4.6}$$

with $\psi_0(x) = 1$ and $\psi_N(x) = 0$.

With this function $G_h$, proceeding as in [4, pp. 14–16] (see also [1]), we can prove that the inequalities

$$\big\| (f - G_h)u \big\|_{L^p(\mathcal{I}_h(c))} \leqq \mathcal{C}\Omega^r_\varphi(b, f, h)_{u,p}, \tag{4.7}$$

$$\big\| G_h^{(r)} \varphi^r u \big\|_{L^p(\mathcal{I}_h(c))} \leqq \mathcal{C} h^{-r}\, \Omega^r_\varphi(b, f, h)_{u,p}, \tag{4.8}$$

hold for $1 \leqq p \leqq \infty$ and $b < c$. Taking supremum over all $0 < h \leqq t$ and using Proposition 2.2, we get our claim. $\square$

Now, let $G \in L^p_u$, $1 \leqq p \leqq \infty$. Setting $t^* = t^{1/(\alpha+1/2)}$ and $t^{**} = ct^{-1/(\beta-1/2)}$, $c > 1$, consider the functions

$$\Gamma_r(x) = \frac{1}{(r-1)!} \int_x^{t^*} G(y)(y-x)^{r-1}\, dy$$

and

$$\widetilde{\Gamma}_r(x) = \frac{1}{(r-1)!} \int_{t^{**}}^{x} G(y)(x-y)^{r-1}\, dy,$$

with $r \geqq 1$ an integer. In order to prove Lemma 2.4, we will need the following proposition.

Proposition 4.1. *Let $G \in L^p_u$, $1 \leqq p \leqq \infty$. Then the inequalities*

$$\|\Gamma_r u\|_{L^p(0,t^*)} \leqq \mathcal{C} t^r \|G\varphi^r u\|_{L^p(0,t^*)} \tag{4.9}$$

*and*

$$\|\widetilde{\Gamma}_r u\|_{L^p(t^{**},+\infty)} \leqq \mathcal{C} t^r \|G\varphi^r u\|_{L^p(t^{**},+\infty)}, \tag{4.10}$$

*hold with $\mathcal{C} \neq \mathcal{C}(G,t)$ and $\varphi(x) = \sqrt{x}$.*

Proof. We first prove inequality (4.9). Since, by definition,

$$\Gamma_1(x) = \int_x^{t^*} G(y)\, dy$$

and

$$\Gamma_r(x) = \int_x^{t^*} \Gamma_{r-1}(y)\, dy, \qquad r \geqq 2,$$

for our aim it suffices to show that

$$\|\Gamma_1 u\|_{L^p(0,t^*)} \leqq \mathcal{C} t \|G\varphi u\|_{L^p(0,t^*)}.$$

In fact from the last inequality we get

$$\|\Gamma_2 u\|_{L^p(0,t^*)} \leqq \mathcal{C} t \|\Gamma_1 \varphi u\|_{L^p(0,t^*)} \leqq \mathcal{C} t^2 \big\| G\varphi^2 u \big\|_{L^p(0,t^*)}$$

and the rest of the proof follows by induction.

Let us first prove (4.9) for $r = 1$ and $p = \infty$. Since $u$ is an increasing function on $(0, t^*)$, we have

$$\big|\Gamma_1(x)\big|\, u(x) = u(x)\bigg|\int_x^{t^*} G(y)\, dy\bigg|$$

$$\leqq \sqrt{u(x)} \int_x^{t^*} \big|G(y)\big|\sqrt{y}u(y)\frac{u^{-1/2}(y)}{\sqrt{y}}\, dy$$

$$\leqq \mathcal{C}\|G\varphi u\|_{L^\infty(0,t^*)}\sqrt{u(x)}\int_x^{t^*} \frac{u^{-1/2}(y)}{\sqrt{y}}\, dy,$$

whence, taking supremum over all $x \in (0, t^*)$, we get our claim, since

$$\sqrt{u(x)}\int_x^{t^*} \frac{u^{-1/2}(y)}{\sqrt{y}}\, dy \leqq \mathcal{C}e^{-\frac{1}{2x^\alpha}} \int_x^{t^*} \frac{e^{\frac{1}{2y^\alpha}}}{\sqrt{y}}\, dy \tag{4.11}$$

$$\leqq \mathcal{C}\frac{2}{\alpha}e^{-\frac{1}{2x^\alpha}} \int_x^{t^*} y^{\alpha+1/2}\left(\frac{\alpha}{2y^{\alpha+1}}e^{\frac{1}{2y^\alpha}}\right) dy$$

$$\leqq \mathcal{C}te^{-\frac{1}{2x^\alpha}}\left[-\int_x^{t^*} -\frac{\alpha}{2y^{\alpha+1}}e^{\frac{1}{2y^\alpha}}\, dy\right] \leqq \mathcal{C}t.$$

Now, consider the case $r = 1$ and $1 < p < \infty$. Using Hölder inequality with $q = \frac{p}{p-1}$, by (4.11) we obtain

$$\|\Gamma_1 u\|^p_{L^p(0,t^*)} = \int_0^{t^*} \bigg|u(x)\int_x^{t^*} (G\varphi u)(y)\, (\varphi u)^{-\frac{1}{p}-\frac{1}{q}}(y)\, dy\bigg|^p dx$$

$$\leqq \int_0^{t^*} u(x)\int_x^{t^*} |G\varphi u|^p(y)\varphi^{-1}(y)u^{-1}(y)\, dy\bigg(u(x)\int_x^{t^*} \varphi^{-1}(y)u^{-1}(y)\, dy\bigg)^{p-1} dx$$

and then, using the Fubini theorem,

$$\|\Gamma_1 u\|^p_{L^p(0,t^*)} \leqq \mathcal{C}t^{p-1}\int_0^{t^*} u(x)\int_x^{t^*} |G\varphi u|^p(y)\, \varphi^{-1}(y)\, u^{-1}(y)\, dy\, dx$$

$$\leqq \mathcal{C}t^{p-1}\int_0^{t^*} |G\varphi u|^p(y)\bigg[\varphi^{-1}(y)\, u^{-1}(y)\int_0^y u(x)\, dx\bigg]\, dy.$$

Taking into account that, for $y \in (0, t^*)$,

$$\varphi^{-1}(y)\, u^{-1}(y)\int_0^y u(x)\, dx \leqq \mathcal{C}y^{-1/2-\gamma}e^{y^{-\alpha}}\int_0^y x^\gamma e^{-x^{-\alpha}}\, dx \tag{4.12}$$

$$\leqq \mathcal{C} y^{\alpha+1/2} e^{y^{-\alpha}} \int_0^y d\big(e^{-x^{-\alpha}}\big) \leqq \mathcal{C} t,$$

we get

$$\|\Gamma_1 u\|^p_{L^p(0,t^*)} \leqq \mathcal{C} t^p \|G\varphi u\|^p_{L^p(0,t^*)}.$$

In a simpler way we can show that

$$\|\Gamma_1 u\|_{L^1(0,t^*)} \leqq \mathcal{C} t \|G\varphi u\|_{L^1(0,t^*)},$$

and then (4.9) holds for $r = 1$ and $1 \leqq p \leqq \infty$.

Concerning inequality (4.10), as in the first part of this proof, we note that, by definition,

$$\widetilde{\Gamma}_1(x) = \int_{t^{**}}^x G(y)\, dy$$

and

$$\widetilde{\Gamma}_r(x) = \int_{t^{**}}^x \widetilde{\Gamma}_{r-1}(y)\, dy, \qquad r \geqq 2.$$

So, for our aim it suffices to show that

$$\|\widetilde{\Gamma}_1 u\|_{L^p(t^{**},\infty)} \leqq \mathcal{C} t \|G\varphi u\|_{L^p(t^{**},+\infty)}.$$

But this last inequality with minor changes is proved in [4, Lemma 11.4.1, pp. 186–187]. □

Proof of Lemma 2.4. We first prove that $\omega^r_\varphi(f,t)_{u,p} \leqq \mathcal{C} K(f,t^r)_{u,p}$. Taking into account Lemma 2.3, it suffices to show that, for any $g \in W^p_r(u)$, the second and the third term in (2.12) are dominated by $\mathcal{C} t^r \big\|g^{(r)}\varphi^r u\big\|_p$. We estimate only the second term, because the other one can be handled in an analogous way.

Let $T$ be the Taylor polynomial of $g \in W^p_r(u)$ of degree $r-1$ about $t^{1/(\alpha+1/2)}$. We have

$$\inf_{q\in\mathbb{P}_{r-1}} \big\|(f-q)u\big\|_{L^p\left(0,t^{1/(\alpha+1/2)}\right)} \leqq \big\|(g-T)u\big\|_{L^p\left(0,t^{1/(\alpha+1/2)}\right)} + \big\|(f-g)u\big\|_p$$

and

$$(g-T)(x)u(x) = \frac{u(x)}{(r-1)!} \int_x^{t^{1/(\alpha+1/2)}} (x-y)^{r-1} g^{(r)}(y)\, dy.$$

Then, using Proposition 4.1, we get

$$\inf_{q\in\mathbb{P}_{r-1}} \big\|(f-q)u\big\|_{L^p\left(0,t^{1/(\alpha+1/2)}\right)} \leqq \mathcal{C} t^r \big\|g^{(r)}\varphi^r u\big\|_p + \big\|(f-g)u\big\|_p$$

and the first part of our claim follows taking the infimum over all $g \in W_r^p(u)$.

Now, let us prove the converse inequality, i.e. $K(f,t^r)_{u,p} \leqq \mathcal{C}\omega_\varphi^r(f,t)_{u,p}$. To this aim we are going to construct a function $\Gamma_t \in W_r^p(u)$, combining the function $G_t$ defined in the proof of Lemma 2.3, and the following two polynomials. By definition, there exist $P_1, P_2 \in \mathbb{P}_{r-1}$, such that

$$\left\| (f-P_1)u \right\|_{L^p\left(0,t^{1/(\alpha+1/2)}\right)} + t^r \left\| P_1^{(r)} \varphi^r u \right\|_{L^p\left(0,t^{1/(\alpha+1/2)}\right)} \leqq \mathcal{C}\omega_\varphi^r(f,t)_{u,p}$$

and

$$\left\| (f-P_2)u \right\|_{L^p\left(ct^{-1/(\beta-1/2)},+\infty\right)} + t^r \left\| P_2^{(r)} \varphi^r u \right\|_{L^p\left(ct^{-1/(\beta-1/2)},+\infty\right)} \leqq \mathcal{C}\omega_\varphi^r(f,t)_{u,p}.$$

Now set

$$x_1 = x_2/2, \quad x_2 = t^{1/(\alpha+1/2)}, \quad x_3 = ct^{-1/(\beta-1/2)}, \quad x_4 = 2x_3.$$

Given a non-decreasing function $\psi \in C^\infty$, with $\psi(x) = 1$ for $x \geqq 1$, $\psi(x) = 0$ for $x \leqq 0$, we define $\psi_i(x) = \psi\left(\frac{x-x_i}{x_{i+1}-x_i}\right)$, $i = 1,2,3$ and the function

$$\Gamma_t(x) = \left(1-\psi_1(x)\right)P_1(x) + \psi_1(x)\left(1-\psi_3(x)\right)G_t(x) + \psi_3(x)P_2(x),$$

where $G_t$ is given by (4.6), with $h$ replaced by $t$. Hence

$$\Gamma_t(x) = \begin{cases} P_1(x) & \text{if } x \leqq x_1 \\ \left(1-\psi_1(x)\right)P_1(x) + \psi_1(x)G_t(x) & \text{if } x_1 \leqq x \leqq x_2 \\ G_t(x) & \text{if } x_2 \leqq x \leqq x_3 \\ \left(1-\psi_3(x)\right)G_t(x) + \psi_3(x)P_2(x) & \text{if } x_3 \leqq x \leqq x_4 \\ P_2(x) & \text{if } x \geqq x_4 \end{cases}$$

and $\Gamma_t \in W_r^p(u)$.

Then it is not difficult to show that

$$K(f,t^r)_{u,p} \leqq \left\| (f-\Gamma_t)u \right\|_p + t^r \left\| \Gamma_t^{(r)} \varphi^r u \right\|_p \leqq \mathcal{C}\omega_\varphi^r(f,t)_{u,p},$$

so we omit the details. □

In order to prove the Favard inequality in Lemma 3.1 we use arguments analogous to those in [13,14,16]. So, in this section we will describe the main steps of the procedure and, in the Appendix, we will give some technical proofs.

First of all, we recall some results about orthogonal polynomials associated with the weight $w(x)$ and their zeros. Letting $A \in \mathbb{Z}^+$ to be fixed in the sequel, consider the sequence $\left\{p_m\left(w^{1/A}\right)\right\}_m$ of orthonormal polynomials with positive leading coefficient. Let

$$\tilde{\varepsilon}_m < x_1 < x_2 < \cdots < x_m < \tilde{a}_m, \tag{4.13}$$

be the zeros of $p_m\left(w^{1/A}\right)$ (see [5, pp. 380–381]), where the M–R–S numbers

$$\tilde{\varepsilon}_m = \varepsilon_m\left(w^{1/(2A)}\right) = \varepsilon_{2Am}(w)$$

and

$$\tilde{a}_m = a_m\left(w^{1/(2A)}\right) = a_{2Am}(w)$$

satisfy (2.6) and (2.7), i.e.

$$\tilde{\varepsilon}_m \sim \left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{\alpha+1/2}}, \qquad \tilde{a}_m \sim m^{1/\beta}. \tag{4.14}$$

The distance between two consecutive zeros of $p_m\left(w^{1/A}\right)$ is given by (see [5, p. 315])

$$\Delta x_k := x_{k+1} - x_k \sim \frac{(x_k - \tilde{\varepsilon}_{2m})(\tilde{a}_{2m} - x_k)}{m\sqrt{(x_k - \tilde{\varepsilon}_m + \delta_m)(\tilde{a}_m - x_k + \tilde{a}_m m^{-2/3})}}, \tag{4.15}$$

$k = 1, \ldots, m$, with

$$\Delta x_1 \sim x_1 - \tilde{\varepsilon}_m \sim \delta_m \sim \left(\tilde{\varepsilon}_m \frac{\sqrt{\tilde{a}_m}}{m}\right)^{2/3} \tag{4.16}$$

and

$$\Delta x_{m-1} \sim \tilde{a}_m - x_m \sim \tilde{a}_m m^{-2/3}.$$

Now, let $\theta \in (0, 1)$ be fixed and consider the interval $[\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$. Then, from (4.15), we obtain

$$\Delta x_k \sim \frac{\sqrt{\tilde{a}_m}}{m}\varphi(x_k), \qquad x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}], \tag{4.17}$$

where the constants in "$\sim$" depend only on $\theta$.

In the sequel we will need the following propositions, proved in the Appendix.

Proposition 4.2. *Let* $x_i$, $1 \leqq i \leqq m-1$, *be an arbitrary zero of* $p_m\big(w^{1/A}\big)$, $x_0 = \tilde{\varepsilon}_m$, $x_{m+1} = \tilde{a}_m$, *and* $x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$, $\theta \in (0,1)$. *Then we have*

$$x_i^\gamma \leqq \mathcal{C}\big(1 + |i-k|\big)^{2\gamma} x_k^\gamma \tag{4.18}$$

*and*

$$\int_{x_i}^{x_{i+1}} x^\gamma\, dx \leqq \mathcal{C}\big(1 + |i-k|\big)^{2\gamma+1} \int_{x_k}^{x_{k+1}} x^\gamma\, dx \tag{4.19}$$

*where* $\mathcal{C}$ *is independent of* $m$ *and* $k$ *in both cases.*

Let us denote by $\ell_k\big(w^{1/A}\big)$ the $k$th fundamental Lagrange polynomial based on the zeros of $p_m\big(w^{1/A}\big)$ and the two extra points $\tilde{\varepsilon}_m$ and $\tilde{a}_m$. For $1 \leqq k \leqq m$ we have

$$\ell_k\big(w^{1/A}, x\big) = \frac{p_m\big(w^{1/A}, x\big)}{p_m'\big(w^{1/A}, x_k\big)(x - x_k)} \frac{(\tilde{\varepsilon}_m - x)(\tilde{a}_m - x)}{(\tilde{\varepsilon}_m - x_k)(\tilde{a}_m - x_k)}. \tag{4.20}$$

Proposition 4.3. *Let* $x \in [\tilde{\varepsilon}_{2m}, \tilde{a}_{2m}]$ *and* $k$ *be an index such that* $x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$, $\theta \in (0,1)$. *Then we have*

$$\big|\ell_k\big(w^{1/A}, x\big)\big|^{2A} w(x) \leqq \mathcal{C} \frac{w(x_k)}{\big(1 + |k-d|\big)^{A/2}}, \tag{4.21}$$

*where* $x_d$, $1 \leqq d \leqq m$, *is a zero closest to* $x$ *and* $\mathcal{C}$ *is independent of* $m$ *and* $k$.

We are now able to prove Lemma 3.1. We divide the proof into four steps.

*First step.* For any $f \in W_1^p(u)$, $1 \leqq p \leqq \infty$, introduce the function $f_\theta$, $\theta \in (0,1)$, defined as

$$f_\theta(x) = \begin{cases} f(\tilde{\varepsilon}_{\theta m}), & 0 < x < \tilde{\varepsilon}_{\theta m}, \\ f(x), & \tilde{\varepsilon}_{\theta m} \leqq x \leqq \tilde{a}_{\theta m}, \\ f(\tilde{a}_{\theta m}), & x > \tilde{a}_{\theta m}. \end{cases} \tag{4.22}$$

Obviously $f_\theta \in W_1^p(u)$ and

$$E_{2A(m+1)}(f)_{u,p} \leqq \big\|(f - f_\theta)u\big\|_p + E_{2A(m+1)}(f_\theta)_{u,p}, \tag{4.23}$$

where $2A(m+1)$ is the degree of a suitable polynomial which we will use in the proof of Lemma 3.1.

Lemma 4.4. *For any $f \in W_1^p(u)$, $1 \leqq p \leqq \infty$, we have*

$$\left\| (f - f_\theta)u \right\|_p \leqq \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_p \tag{4.24}$$

*with $\mathcal{C}$ independent of $m$ and $f$, $a_m \sim m^{1/\beta}$.*

*Second step.* Now, we approximate the function $f_\theta$ by means of step functions. For $x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$, $\theta \in (0,1)$, we set

$$M_k = \max_{x \in [x_{k-1}, x_k]} f_\theta(x), \qquad m_k = \min_{x \in [x_{k-1}, x_k]} f_\theta(x), \tag{4.25}$$

and

$$x_+^0 = \begin{cases} 1, & x \geqq 0, \\ 0, & x < 0. \end{cases}$$

Hence we introduce the functions

$$(S^+ f_\theta)(x) = f(\tilde{\varepsilon}_{\theta m}) + \sum_{x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]} (x - x_k)_+^0 [M_{k+1} - M_k] \tag{4.26}$$

and

$$(S^- f_\theta)(x) = f(\tilde{\varepsilon}_{\theta m}) + \sum_{x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]} (x - x_k)_+^0 [m_{k+1} - m_k]. \tag{4.27}$$

By definition, we have

$$(S^- f_\theta)(x) = f_\theta(x) = (S^+ f_\theta)(x), \qquad x \in (0, +\infty) \setminus [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}], \tag{4.28}$$

and

$$(S^- f_\theta)(x) \leqq f_\theta(x) \leqq (S^+ f_\theta)(x), \qquad x \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]. \tag{4.29}$$

Moreover, if $x \in [x_{k-1}, x_k]$, with $x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$, we get

$$(S^+ f_\theta)(x) - (S^- f_\theta)(x) = M_k - m_k. \tag{4.30}$$

Lemma 4.5. *For any $f \in W_1^p(u)$, $1 \leqq p \leqq \infty$, we have*

$$\left\| (S^+ f_\theta - S^- f_\theta)u \right\|_p \leqq \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_{L^p[\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]} \tag{4.31}$$

*with $\mathcal{C}$ independent of $m$ and $f$, $a_m \sim m^{1/\beta}$.*

*Third step.* Now we introduce some polynomials of one-sided approximation for $S^+f_\theta$ and $S^-f_\theta$. To this aim recall that $\ell_k\left(w^{1/A}\right)$ denotes the $k$th fundamental Lagrange polynomial based on the zeros of $p_m\left(w^{1/A}\right)$ and the two extra points $\tilde{\varepsilon}_m$ and $\tilde{a}_m$.

Let $k$ be an index satisfying $x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$, $\theta \in (0,1)$. Proceeding in analogy with [13,18], we are going to construct the polynomials $p_k^\pm \in \mathbb{P}_{2A(m+1)}$ such that

$$p_k^-(x) \leqq (x-x_k)_+^0 \leqq p_k^+(x),$$

and

$$p_k^+(x) - p_k^-(x) = \ell_k^{2A}\left(w^{1/A}, x\right), \tag{4.32}$$

for $x \in [\tilde{\varepsilon}_{2m}, \tilde{a}_{2m}]$.

So, with $x_i$, $i = 1, \dots m$, the zeros of $p_m\left(w^{1/A}\right)$, $x_0 = \tilde{\varepsilon}_m$ and $x_{m+1} = \tilde{a}_m$, define the polynomial $p_k^+$ by

$$p_k^+(x_i) = \begin{cases} 0, & 0 \leqq i \leqq k-1 \\ 1, & k \leqq i \leqq m+1, \end{cases}$$

$$\frac{d^\nu}{dx^\nu} p_k^+(x_i) = 0, \qquad i \neq k,\ \nu = 1, \dots, 2A-1,$$

whereas the polynomial $p_k^-$ is given by

$$p_k^-(x_i) = \begin{cases} 0, & 0 \leqq i \leqq k \\ 1, & k+1 \leqq i \leqq m+1, \end{cases}$$

$$\frac{d^\nu}{dx^\nu} p_k^-(x_i) = 0, \qquad i \neq k,\ \nu = 1, \dots, 2A-1.$$

By means of $p_k^\pm$ and choosing $A = \lceil 4\gamma + 8 \rceil$, introduce the polynomials

$$Q^\pm(x) = f(\tilde{\varepsilon}_{\theta m}) + \sum_{\Delta M_k > 0} p_k^\pm(x) \Delta M_k + \sum_{\Delta M_k < 0} p_k^\mp(x) \Delta M_k \in \mathbb{P}_{2A(m+1)} \tag{4.33}$$

and

$$q^\pm(x) = f(\tilde{\varepsilon}_{\theta m}) + \sum_{\Delta m_k > 0} p_k^\pm(x) \Delta m_k + \sum_{\Delta m_k < 0} p_k^\mp(x) \Delta m_k \in \mathbb{P}_{2A(m+1)}, \tag{4.34}$$

where $\Delta M_k = M_{k+1} - M_k$ and $\Delta m_k = m_{k+1} - m_k$, $k$ is such that $x_k \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$, $\theta \in (0,1)$.

By definition, in $[\tilde{\varepsilon}_{2m}, \tilde{a}_{2m}]$, we have

$$q^- \leqq S^- f_\theta \leqq q^+, \qquad Q^- \leqq S^+ f_\theta \leqq Q^+,$$

and then

$$q^- \leqq S^- f_\theta \leqq f_\theta \leqq S^+ f_\theta \leqq Q^+. \tag{4.35}$$

Then the following lemma, proved in the Appendix, holds.

LEMMA 4.6. *For any $f \in W_1^p(u)$, $1 \leqq p \leqq \infty$, we have*

$$\left\| (Q^+ - Q^-)u \right\|_p \leqq \mathcal{C} \frac{\sqrt{a_m}}{m} \|f'\varphi u\|_{L^p[\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]} \tag{4.36}$$

*and*

$$\left\| (q^+ - q^-)u \right\|_p \leqq \mathcal{C} \frac{\sqrt{a_m}}{m} \|f'\varphi u\|_{L^p[\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]}, \tag{4.37}$$

*where in both cases $\mathcal{C}$ is independent of $m$ and $f$.*

*Fourth step.* Now we are able to prove the Favard inequality.

PROOF OF LEMMA 3.1. By Lemma 4.4 we have

$$E_{2A(m+1)}(f)_{u,p} \leqq E_{2A(m+1)}(f_\theta)_{u,p} + \left\| (f - f_\theta)u \right\|_p$$

$$\leqq E_{2A(m+1)}(f_\theta)_{u,p} + \mathcal{C} \frac{\sqrt{a_m}}{m} \|f'\varphi u\|_p.$$

For the first summand on the right-hand side, by (4.35) and using Lemmas 4.5 and 4.6, we obtain

$$E_{2A(m+1)}(f_\theta)_{u,p} \leqq \left\| (Q^+ - f_\theta)u \right\|_p$$

$$\leqq \left\| (Q^+ - Q^-)u \right\|_p + \left\| (S^+ f_\theta - S^- f_\theta)u \right\|_p + \left\| (q^+ - q^-)u \right\|_p$$

$$\leqq \mathcal{C} \frac{\sqrt{a_m}}{m} \|f'\varphi u\|_p,$$

and the Favard inequality (3.1) follows. □

PROOF OF THEOREM 3.4. First of all we observe that

$$[\varepsilon_m, a_m] \subset \mathcal{I}_h(c) = \left[h^{1/(\alpha+1/2)}, c h^{-1/(\beta-1/2)}\right]$$

for some $c>1$ and with $h=\varepsilon_m^{\alpha+1/2}\sim\sqrt{a_m}/m$. Let us prove that, for $1\leqq p\leqq\infty$ and $m>r$, the inequality

$$(4.38)\qquad \widetilde{E}_m(f)_{u,p}:=\inf_{P_m\in\mathbb{P}_m}\big\|(f-P_m)u\big\|_{L^p[\varepsilon_m,a_m]}\leqq\mathcal{C}\Omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p}$$

holds with $\mathcal{C}\neq\mathcal{C}(f,m)$.

Proceeding as in the proof of Lemma 2.3, for any function $f\in L_u^p$, we can construct a function $g_m$ such that

$$(4.39)\qquad \big\|(f-g_m)u\big\|_{L^p[\varepsilon_m,a_m]}\leqq\mathcal{C}\Omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p}$$

and

$$(4.40)\qquad \big\|g_m^{(r)}\varphi^r u\big\|_{L^p[\varepsilon_m,a_m]}\leqq\mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^r\Omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p},$$

where $\mathcal{C}\neq\mathcal{C}(f,m)$. Namely, $g_m$ is the function $G_h$ in (4.6), with $h=\varepsilon_m^{\alpha+1/2}$ $\sim\sqrt{a_m}/m$.

Then we define the function $\widetilde{g}_m$ as

$$\widetilde{g}_m(x)=\begin{cases}T_{r-1}(g_m,x), & 0<x\leqq\varepsilon_m,\\ g_m(x), & \varepsilon_m\leqq x\leqq a_m,\\ \widetilde{T}_{r-1}(g_m,x), & x\geqq a_m.\end{cases}$$

where $T_{r-1}(g_m),\widetilde{T}_{r-1}(g_m)\in\mathbb{P}_{r-1}$ are the Taylor polynomials of $g_m$ about $\varepsilon_m$ and $a_m$, respectively. Hence, by (4.39), we get

$$\begin{aligned}(4.41)\quad \widetilde{E}_m(f)_{u,p}&\leqq\big\|(f-\widetilde{g}_m)u\big\|_{L^p[\varepsilon_m,a_m]}+\inf_{P_m\in\mathbb{P}_m}\big\|(\widetilde{g}_m-P_m)u\big\|_{L^p[\varepsilon_m,a_m]}\\ &\leqq\big\|(f-g_m)u\big\|_{L^p[\varepsilon_m,a_m]}+\inf_{P_m\in\mathbb{P}_m}\big\|(\widetilde{g}_m-P_m)u\big\|_p\\ &\leqq\mathcal{C}\Omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p}+\inf_{P_m\in\mathbb{P}_m}\big\|(\widetilde{g}_m-P_m)u\big\|_p.\end{aligned}$$

For the second summand on the right-hand side, since $\widetilde{g}_m\in W_r^p(u)$, by (3.6) and (4.40), we obtain

$$(4.42)\qquad \inf_{P_m\in\mathbb{P}_m}\big\|(\widetilde{g}_m-P_m)u\big\|_p\leqq\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r\big\|\widetilde{g}_m^{(r)}\varphi^r u\big\|_p$$

$$= \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \left\|g_m^{(r)}\varphi^r u\right\|_{L^p[\varepsilon_m,a_m]} \leqq \mathcal{C}\Omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p}.$$

Combining (4.41) and (4.42), inequality (4.38) follows.

Therefore, for any $f \in L_u^p$, $1 \leqq p \leqq \infty$, there exist polynomials $P^*_{2^k m} \in \mathbb{P}_{2^k m}$, $k = 1, 2, \dots$, such that

$$\left\|\left(P^*_{2^{k+1}m} - P^*_{2^k m}\right)u\right\|_p \leqq \left\|\left(P^*_{2^{k+1}m} - P^*_{2^k m}\right)u\right\|_{L^p[\varepsilon_n,a_n]}$$

$$\leqq \mathcal{C}\Omega_\varphi^r\left(f,\frac{\sqrt{a_{2^k m}}}{2^k m}\right)_{u,p},$$

using the restricted range inequality (2.8) with $n = 2^{k+1}m + \lceil\gamma\rceil \sim 2^{k+1}m$. Then the series

$$\sum_{k=0}^{\infty}\left\|\left(P^*_{2^{k+1}m} - P^*_{2^k m}\right)u\right\|_p$$

converges, since it is dominated by

$$\sum_{k=0}^{\infty}\Omega_\varphi^r\left(f,\frac{\sqrt{a_{2^k m}}}{2^k m}\right)_{u,p} \sim \int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,dt < \infty.$$

So the equality

$$(f - P_m^*)u = \sum_{k=0}^{\infty}\left(P^*_{2^{k+1}m} - P^*_{2^k m}\right)u$$

holds a.e. in $(0,+\infty)$. It follows that

$$\left\|\left(f - P_m^*\right)u\right\|_p \leqq \mathcal{C}\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,dt < \infty,$$

and then we get (3.5). □

Proof of Theorem 3.5. The proof is based on the same argument as in [4, pp. 84–86], so we will give only the main steps.

Let $h = \sqrt{a_m}/m$. Using the restricted range inequality (2.8), with $n = m - r + \lceil\gamma + r/2\rceil$, we get

$$\left\|P_m^{(r)}(h\varphi)^r u\right\|_p \leqq \mathcal{C}\left\|P_m^{(r)}(h\varphi)^r u\right\|_{L^p[\varepsilon_n,a_n]} \tag{4.43}$$

$$\leqq \mathcal{C}\left\|\left[P_m^{(r)}(h\varphi)^r - \Delta_{h\varphi}^r(P_m)\right]u\right\|_{L^p[\varepsilon_n,a_n]} + \mathcal{C}\left\|\Delta_{h\varphi}^r(P_m)u\right\|_{L^p[\varepsilon_n,a_n]}$$

$$=: A_1 + A_2.$$

Observe that $x \in [\varepsilon_n, a_n]$ implies $x + rh\varphi(x) \in [\varepsilon_{2m}, a_{2m}]$ for $m$ sufficiently large.

Let us consider the term $A_2$. By Theorem 3.2 we get

$$A_2 \leqq \mathcal{C}\big\| \Delta^r_{h\varphi}(P_m - f)u\big\|_{L^p[\varepsilon_{2m},a_{2m}]} + \mathcal{C}\big\| \Delta^r_{h\varphi}(f)u\big\|_{L^p[\varepsilon_{2m},a_{2m}]} \tag{4.44}$$

$$\leqq \mathcal{C}E_m(f)_{u,p} + \mathcal{C}\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \leqq \mathcal{C}\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p}.$$

While, concerning the term $A_1$, by using the Hermite–Genocchi formula (4.3), with $h$ replaced by $h\varphi(x)$, we have

$$F(x) := \left[\big(h\varphi(x)\big)^r P^{(r)}_m(x) - \Delta^r_{h\varphi}\big(P_m(x)\big)\right]u(x)$$

$$= r!\big(h\varphi(x)\big)^r u(x) \int_{S_r} \left[P^{(r)}_m(x) - P_m\big(x + th\varphi(x)\big)\right] dS_r$$

$$= -r!\big(h\varphi(x)\big)^r u(x) \int_{S_r} \int_x^{x+th\varphi(x)} P^{(r+1)}_m(z)\, dz\, dS_r,$$

whence, by Proposition 2.1, we get

$$\big|F(x)\big| \leqq \mathcal{C}r!h^r \int_{S_r} \int_x^{x+th\varphi(x)} \big|P^{(r+1)}_m(z)\big|\varphi^r(z)u(z)\, dz\, dS_r$$

$$\leqq \mathcal{C}h^r \int_x^{x+rh\varphi(x)} \big|P^{(r+1)}_m(z)\big|\varphi^r(z)u(z)\, dz$$

$$= \mathcal{C}h^{r+1} \frac{1}{rh\varphi(x)} \int_x^{x+rh\varphi(x)} \big|P^{(r+1)}_m(z)\big|\varphi^{r+1}(z)u(z)\, dz,$$

since $\int_{S_r} dt_1 \cdots dt_r = \frac{1}{r!}$. Then, using the boundedness of the Hardy–Littlewood maximal function for $1 < p \leqq \infty$ and Fubini's theorem for $p = 1$, we obtain

$$A_1 = \|F\|_{L^p[\varepsilon_{2m},a_{2m}]} \leqq \mathcal{C}h^{r+1}\big\|P^{(r+1)}_m(h\varphi)^{r+1}u\big\|_{L^p[\varepsilon_{2m},a_{2m}]},$$

with $h = \sqrt{a_m}/m$. By (4.43) and (4.44), it follows that

$$\big\|P^{(r)}_m(h\varphi)^r u\big\|_p \leqq \mathcal{C}\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} + \mathcal{C}h^{r+1}\big\|P^{(r+1)}_m(h\varphi)^{r+1}u\big\|_{L^p[\varepsilon_{2m},a_{2m}]}.$$

Proceeding as in [4, p. 84], one can show that the second term on the right-hand side of the last inequality is bounded by $\mathcal{C}\omega_\varphi^r\big(f, \frac{\sqrt{a_m}}{m}\big)_{u,p}$, whence we get (3.8). □

## Appendix

Proof of Proposition 4.2. For $\theta \in (0,1)$ and $m$ sufficiently large, let us define the indices $j_1 = j_1(\theta, m)$ and $j_2 = j_2(\theta, m)$ as follows

$$x_{j_1} = \max_k \{x_k \leqq \tilde{\varepsilon}_{\theta m}\} \quad \text{and} \quad x_{j_2} = \min_k \{x_k \geqq \tilde{a}_{\theta m}\}.$$

We first prove inequality (4.18). Let us consider the case $x_i, x_k \in [x_{j_1}, x_{j_2}]$. If $x_i \leqq x_k$ we have $x_i^\gamma \leqq x_k^\gamma$, while, if $x_i > x_k$, by (4.17), we get

$$1 < \frac{x_i}{x_k} = \frac{x_i - x_k}{x_k} + 1$$

$$\leqq \mathcal{C}\big(1 + |i-k|\big)\frac{\Delta x_i}{\Delta x_k} + 1 \leqq \mathcal{C}\big(1 + |i-k|\big)\sqrt{\frac{x_i}{x_k}}$$

and then

$$\frac{x_i}{x_k} \leqq \mathcal{C}\big(1 + |i-k|\big)^2.$$

Let us now consider the case $x_i > x_{j_2} \geqq x_k$. Recalling the previous case we have

$$1 < \frac{x_i}{x_k} = \frac{x_i}{x_{j_2}}\frac{x_{j_2}}{x_k} \leqq \mathcal{C}\frac{\tilde{a}_m}{\tilde{a}_{\theta m}}\big(1 + |j_2 - k|\big)^2 \leqq \mathcal{C}\big(1 + |i-k|\big)^2.$$

Finally, in case $x_i < x_{j_1} \leqq x_k$, we get

$$1 < \frac{x_k}{x_i} = \frac{x_k}{x_{j_1}}\frac{x_{j_1}}{x_i} \leqq \mathcal{C}\frac{\tilde{\varepsilon}_{\theta m}}{\tilde{\varepsilon}_m}\big(1 + |j_1 - k|\big)^2 \leqq \mathcal{C}\big(1 + |i-k|\big)^2.$$

In any case we obtain

$$\left(\frac{x_i}{x_k}\right)^{\pm 1} \leqq \mathcal{C}\big(1 + |i-k|\big)^2,$$

whence

$$x_i^\gamma \leqq \mathcal{C}\big(1 + |i-k|\big)^{2\gamma} x_k^\gamma.$$

In order to prove inequality (4.19), we first show that

$$\frac{\Delta x_i}{\Delta x_k} \leqq \mathcal{C}\big(1+|i-k|\big). \tag{4.45}$$

In case $x_i, x_k \in [x_{j_1}, x_{j_2}]$, by (4.17) and (4.18), we easily get

$$\frac{\Delta x_i}{\Delta x_k} \sim \sqrt{\frac{x_i}{x_k}} \leqq \mathcal{C}\big(1+|i-k|\big).$$

In case $x_i > x_{j_2} \geqq x_k$, recalling the previous case, by (4.15), we have

$$\frac{\Delta x_i}{\Delta x_k} = \frac{\Delta x_i}{\Delta x_{j_2}}\frac{\Delta x_{j_2}}{\Delta x_k} \leqq \mathcal{C}\big(1+|j_2-k|\big)\frac{\Delta x_i}{\Delta x_{j_2}} \leqq \mathcal{C}\big(1+|i-k|\big)$$

since, by (4.15),

$$\begin{aligned}
\frac{\Delta x_i}{\Delta x_{j_2}} &\sim \frac{(x_i-\tilde\varepsilon_{2m})(\tilde a_{2m}-x_i)}{(x_{j_2}-\tilde\varepsilon_{2m})(\tilde a_{2m}-x_{j_2})}\sqrt{\frac{(x_{j_2}-\tilde\varepsilon_m+\delta_m)(\tilde a_m-x_{j_2}+\tilde a_m m^{-2/3})}{(x_i-\tilde\varepsilon_m+\delta_m)(\tilde a_m-x_i+\tilde a_m m^{-2/3})}}\\
&\leqq \left(\frac{x_i-\tilde\varepsilon_{2m}}{x_{j_2}-\tilde\varepsilon_{2m}}\right)\sqrt{\frac{\tilde a_m-x_{j_2}+\tilde a_m m^{-2/3}}{\tilde a_m-x_i+\tilde a_m m^{-2/3}}}\\
&\leqq \mathcal{C}\left(\frac{\tilde a_m}{\tilde a_{\theta m}-\tilde\varepsilon_{2m}}\right)\sqrt{1+\frac{x_i-x_{j_2}}{\tilde a_m-x_i+\tilde a_m m^{-2/3}}}\\
&\leqq \mathcal{C}\sqrt{1+\frac{\big(1+|j_2-i|\big)\Delta x_i}{\big(1+|m-i|\big)\Delta x_i}} \leqq \mathcal{C}.
\end{aligned}$$

Finally, in case $x_i < x_{j_1} \leqq x_k$, proceeding in analogy with the previous case, we get

$$1 < \frac{\Delta x_i}{\Delta x_k} = \frac{\Delta x_i}{\Delta x_{j_1}}\frac{\Delta x_{j_1}}{\Delta x_k} \leqq \mathcal{C}\frac{\Delta x_i}{\Delta x_{j_1}}\big(1+|j_1-k|\big) \leqq \mathcal{C}\big(1+|i-k|\big),$$

since

$$\begin{aligned}
\frac{\Delta x_i}{\Delta x_{j_1}} &\sim \frac{(x_i-\tilde\varepsilon_{2m})(\tilde a_{2m}-x_i)}{(x_{j_1}-\tilde\varepsilon_{2m})(\tilde a_{2m}-x_{j_1})}\sqrt{\frac{(x_{j_1}-\tilde\varepsilon_m+\delta_m)(\tilde a_m-x_{j_1}+\tilde a_m m^{-2/3})}{(x_i-\tilde\varepsilon_m+\delta_m)(\tilde a_m-x_i+\tilde a_m m^{-2/3})}}\\
&\leqq \left(\frac{\tilde a_{2m}-x_i}{\tilde a_{2m}-x_{j_1}}\right)\sqrt{\frac{x_{j_1}-\tilde\varepsilon_m+\delta_m}{x_i-\tilde\varepsilon_m+\delta_m}} \leqq \mathcal{C}\sqrt{1+\frac{\big(1+|j_1-i|\big)\Delta x_i}{(1+i)\Delta x_i}} \leqq \mathcal{C}
\end{aligned}$$

whence (4.45) follows.

Finally, from (4.18) and (4.45), we deduce

$$\frac{\int_{x_i}^{x_{i+1}} x^\gamma\,dx}{\int_{x_k}^{x_{k+1}} x^\gamma\,dx} \sim \left(\frac{x_i}{x_k}\right)^\gamma \frac{\Delta x_i}{\Delta x_k} \leqq \mathcal{C}\big(1+|i-k|\big)^{2\gamma+1},$$

which completes the proof. □

Proof of Lemma 4.4. Since

$$(f-f_\theta)(x)u(x) = \begin{cases} u(x)\displaystyle\int_0^{\tilde{\varepsilon}_{\theta m}} f'(y)\,dy, & x\in(0,\tilde{\varepsilon}_{\theta m}),\\ 0, & x\in[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}],\\ u(x)\displaystyle\int_{\tilde{a}_{\theta m}}^{\infty} f'(y)\,dy, & x\in(\tilde{a}_{\theta m},+\infty),\end{cases}$$

inequality (4.24) follows from Proposition 4.1. □

Proof of Lemma 4.5. Set $y_k=(x_{k-1}+x_k)/2$, with $k$ such that $x_k\in[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]$. Then, for $x\in[y_{k-1},y_k]$, by (4.30), Proposition 2.1 and (4.17), we have

$$\big|(S^+f_\theta - S^-f_\theta)(x)\big|\,u(x) \leqq u(x)\int_{x_{k-1}}^{x_{k+1}} \big|f_\theta'(y)\big|\,dy$$

$$\leqq \mathcal{C}\int_{x-c\frac{\sqrt{a_m}}{m}\sqrt{x}}^{x+c\frac{\sqrt{a_m}}{m}\sqrt{x}} \big|f'(y)u(y)\big|\,dy,$$

for some $c>0$. Hence, for $p=\infty$, we get

$$\big\|(S^+f_\theta - S^-f_\theta)u\big\|_\infty = \max_{x\in[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]}\big|(S^+f_\theta - S^-f_\theta)(x)u(x)\big|$$

$$\leqq \mathcal{C}\max_k \max_{x\in[y_{k-1},y_{k+1}]}\int_{x-c\frac{\sqrt{a_m}}{m}\sqrt{x}}^{x+c\frac{\sqrt{a_m}}{m}\sqrt{x}} \big|f'(y)u(y)\big|\,dy$$

$$\leqq \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_{L^\infty[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]}$$

For $1<p<\infty$, using the boundedness of the Hardy–Littlewood maximal function, we obtain

$$\big\|(S^+f_\theta - S^-f_\theta)u\big\|_p^p \leqq \mathcal{C}\int_{\tilde{\varepsilon}_{\theta m}}^{\tilde{a}_{\theta m}}\left|\int_{x-c\frac{\sqrt{a_m}}{m}\sqrt{x}}^{x+c\frac{\sqrt{a_m}}{m}\sqrt{x}} \big|f'(y)u(y)\big|\,dy\right|dx$$

$$\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \int_{\tilde{\varepsilon}_{\theta m}}^{\tilde{a}_{\theta m}} \left| \frac{m}{\sqrt{a_m}\sqrt{x}} \int_{x-c\frac{\sqrt{a_m}}{m}\sqrt{x}}^{x+c\frac{\sqrt{a_m}}{m}\sqrt{x}} \left| f'(y)\varphi(y)u(y)\right| dy \right| dx$$

$$\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \|f'\varphi u\|^p_{L^p[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]}.$$

We omit the proof for $p=1$, which follows by Fubini theorem. □

Proof of Proposition 4.3. First of all we observe that, letting $x_d \sim x$ be a zero closest to $x$, by an extension of an inequality of Erdős and Turán (see [5, p. 361]), we have

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right| w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \sim 1, \qquad k\in\{d-1,d,d+1\}.$$

For $k \neq d-1,d,d+1$ such that $x_k \in [\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]$, using the relations (see [5, p. 325])

$$\sup_{x\in(0,+\infty)} \left|p_m\left(w^{1/A},x\right)\right| w^{1/(2A)}(x)\sqrt[4]{\left|(\tilde{a}_m-x)(x-\tilde{\varepsilon}_m)\right|} \sim 1 \tag{4.46}$$

and

$$\frac{1}{\left|p'_m\left(w^{1/A},x_k\right)\right| w^{1/(2A)}(x_k)} \sim \Delta x_k \sqrt[4]{(\tilde{a}_m-x)(x-\tilde{\varepsilon}_m)} \tag{4.47}$$

we get

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right| w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \leqq \mathcal{C}\frac{\Delta x_k}{|x-x_k|}\left(\frac{\left|(\tilde{a}_m-x)(x-\tilde{\varepsilon}_m)\right|}{(\tilde{a}_m-x_k)(x_k-\tilde{\varepsilon}_m)}\right)^{3/4}$$

$$\leqq \mathcal{C}\frac{\Delta x_k}{|x-x_k|}\left(\frac{|x-\tilde{\varepsilon}_m|}{x_k-\tilde{\varepsilon}_m}\right)^{3/4},$$

since for $|\tilde{a}_m - x| \leqq \mathcal{C}\tilde{a}_m$ and $(\tilde{a}_m - x_k) \geqq \tilde{a}_m - \tilde{a}_{\theta m} \geqq \mathcal{C}\tilde{a}_m$ for $x\in[\tilde{\varepsilon}_{2m},\tilde{a}_{2m}]$ and $x_k \in [\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]$.

Moreover observe that, from (4.15) we can deduce

$$\Delta x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k-\tilde{\varepsilon}_m}, \qquad x_k\in[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}] \tag{4.48}$$

and

$$\Delta x_i \geqq \mathcal{C}\frac{\sqrt{a_m}}{m}\sqrt{x_i-\tilde{\varepsilon}_m}, \qquad x_i \leqq \tilde{a}_{\theta m}. \tag{4.49}$$

Now, we distinguish two cases: $x > x_k$ and $x < x_k$. In the first case we have

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \leqq \mathcal{C}\frac{\Delta x_k}{x-x_k}\left(\frac{x-x_k}{x_k-\tilde{\varepsilon}_m}+1\right)^{3/4}$$

$$=\mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^{1/4}\left(\frac{\Delta x_k}{x_k-\tilde{\varepsilon}_m}+\frac{\Delta x_k}{x-x_k}\right)^{3/4} \leqq \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^{1/4},$$

since $\Delta x_k \leqq x - x_k$, $k \neq d, d\pm 1$, and $\Delta x_k \leqq \mathcal{C}(x_k-\tilde{\varepsilon}_m)$, using (4.48) and (4.16). Moreover, using

$$x-x_k \geqq \sum_{i=k}^{d-1}\Delta x_i \geqq (d-k)\min_{k\leqq i\leqq d-1}\Delta x_i \geqq \mathcal{C}(d-k)\Delta x_k,$$

we get

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \leqq \frac{\mathcal{C}}{\left(1+|d-k|\right)^{1/4}}, \tag{4.50}$$

for $x > x_k$.

Now, consider the case $x < x_k$. If $\tilde{\varepsilon}_m < x < x_k$, we can proceed analogously to the previous case, taking into account that

$$|x-x_k| \geqq \sum_{i=d+1}^{k-1}\Delta x_i \geqq \mathcal{C}\frac{\sqrt{a_m}}{m}\sum_{i=d+1}^{k-1}\sqrt{x_i-\tilde{\varepsilon}_m}$$

$$\geqq \mathcal{C}\left(|d-k|\right)\frac{\sqrt{a_m}}{m}\sqrt{x-\tilde{\varepsilon}_m},$$

by (4.49) and since $x_d-\tilde{\varepsilon}_m \sim x-\tilde{\varepsilon}_m$. Hence, by (4.48), we get

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \leqq \mathcal{C}\frac{\Delta x_k}{|x-x_k|}\left(\frac{x-\tilde{\varepsilon}_m}{x_k-\tilde{\varepsilon}_m}\right)^{3/4} \tag{4.51}$$

$$\leqq \frac{\mathcal{C}}{\left(1+|d-k|\right)}\left(\frac{x-\tilde{\varepsilon}_m}{x_k-\tilde{\varepsilon}_m}\right)^{1/4} \leqq \frac{\mathcal{C}}{\left(1+|d-k|\right)},$$

since $x-\tilde{\varepsilon}_m < x_k-\tilde{\varepsilon}_m$.

Finally, if $\tilde{\varepsilon}_{2m} \leqq x \leqq \tilde{\varepsilon}_m$, with $x < x_k$, we can write

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \leqq \mathcal{C}\frac{\Delta x_k}{x_k-x}\left(\frac{\tilde{\varepsilon}_m-x}{x_k-\tilde{\varepsilon}_m}\right)^{3/4}$$

$$\leqq \mathcal{C}\frac{\Delta x_k}{(x_k-x)^{1/4}(x_k-\tilde\varepsilon_m)^{3/4}} \leqq \mathcal{C}\frac{\Delta x_k}{(x_k-\tilde\varepsilon_m)},$$

since $\tilde\varepsilon_m - x < x_k - x$ and $x_k - x \geqq x_k - \tilde\varepsilon_m$. Since $x_k - \tilde\varepsilon_m \geqq x_k - x_1 \geqq \mathcal{C}|k-d|\Delta x_1$, by (4.48) and (4.16), we get

$$\frac{\left|\ell_k\left(w^{1/A},x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \leqq \mathcal{C}\frac{\Delta x_k}{(x_k-\tilde\varepsilon_m)} \leqq \mathcal{C}\frac{\sqrt{a_m}}{m\sqrt{x_k-\tilde\varepsilon_m}} \tag{4.52}$$

$$\leqq \frac{\mathcal{C}}{\left(1+|d-k|\right)^{1/2}}\frac{\sqrt{a_m}}{m\sqrt{\Delta x_1}} \leqq \frac{\mathcal{C}}{\left(1+|d-k|\right)^{1/2}}.$$

Combining (4.50), (4.51) and (4.52), we obtain (4.21). □

Proof of Lemma 4.6. Since inequalities (4.36) and (4.37) can be proved using similar arguments, we are going to show the proof only for (4.36).

We first observe that, by (2.8), with $Q^\pm \in \mathbb{P}_{2A(m+1)}$, we have

$$\left\|(Q^+-Q^-)u\right\|_p \leqq \mathcal{C}\left\|(Q^+-Q^-)u\right\|_{L^p[\tilde\varepsilon_{sm},\tilde a_{sm}]}$$

for some $s>1$ and for $1\leqq p\leqq\infty$. Then we assume $x\in[\tilde\varepsilon_{sm},\tilde a_{sm}]$.

Letting $x_i$, $i=1,\dots,m$, be the zeros of $p_m\left(w^{1/A}\right)$, we set $x_0=\tilde\varepsilon_{sm}$, $x_{m+1}=\tilde a_{sm}$ and

$$y_i=\frac{x_{i-1}+x_i}{2},\ i=1,\dots,m, \qquad y_0=\tilde\varepsilon_{sm}, \qquad y_{m+1}=\tilde a_{sm}.$$

Let us first prove (4.36) for $p=\infty$. By (4.33) and (4.32), we have

$$\left\|(Q^+-Q^-)u\right\|_\infty \leqq \mathcal{C}\max_{i=1,\dots,m+1}\ \max_{x\in[y_{i-1},y_i]}\left|(Q^+-Q^-)(x)u(x)\right|$$

$$\leqq \mathcal{C}\max_{i=1,\dots,m+1}\ \max_{x\in[y_{i-1},y_i]}\ \sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\ell_k^{2A}\left(w^{1/A},x\right)u(x)|\Delta M_k|.$$

Hence, by using Proposition 4.3 and Propositions 4.2 and 2.1, we get

$$\left\|(Q^+-Q^-)u\right\|_\infty$$

$$\leqq \mathcal{C}\max_{i=1,\dots,m+1}\ \sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{u(x_k)}{\left(1+|i-k|\right)^{A/2-2\gamma}}\int_{x_{k-1}}^{x_{k+1}}\left|f'(y)\right|dy$$

$$\leqq \mathcal{C}\max_{i=1,\dots,m+1}\ \sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{1}{\left(1+|i-k|\right)^{A/2-2\gamma}}\int_{x_{k-1}}^{x_{k+1}}\left|f'(y)u(y)\right|dy$$

$$\leqq \mathcal{C}\frac{\sqrt{a_m}}{m}\max_{i=1,\dots,m+1}\sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{1}{\big(1+|i-k|\big)^{A/2-2\gamma}}$$

$$\times\left(\frac{1}{2\Delta x_k}\int_{x_{k-1}}^{x_{k+1}}|f'\varphi u|(y)\,dy\right)\leqq \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_\infty$$

since $x_{k+1}-x_{k-1}\sim(\sqrt{a_m}/m)\sqrt{y}$, by (4.17), and $A>4\gamma+2$.

Now, let us consider the case $1<p<\infty$. In analogy with the previous case we have

$$\big\|(Q^+-Q^-)u\big\|_p^p\leqq \mathcal{C}\sum_{i=1}^{m+1}\big\|(Q^+-Q^-)u\big\|_{L^p[y_{i-1},y_i]}^p$$

$$\leqq \mathcal{C}\sum_{i=1}^{m+1}\int_{y_{i-1}}^{y_i}x^{\gamma p}\bigg|\sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{w(x_k)}{\big(1+|i-k|\big)^{A/2}}\int_{x_{k-1}}^{x_{k+1}}\big|f'(y)\big|\,dy\bigg|^p dx$$

$$\leqq \mathcal{C}\sum_{i=1}^{m+1}\int_{y_{i-1}}^{y_i}x^{\gamma p}\bigg|\sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{1}{\big(1+|i-k|\big)^{A/2}}\int_{x_{k-1}}^{x_{k+1}}\big|f'(y)w(y)\big|\,dy\bigg|^p dx.$$

By Hölder inequality, Proposition 4.2 and (4.17), we get

$$\big\|(Q^+-Q^-)u\big\|_p^p\leqq \mathcal{C}\sum_{i=1}^{m+1}\int_{y_{i-1}}^{y_i}x^{\gamma p}$$

$$\times\sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{1}{\big(1+|i-k|\big)^{(A/2-1)p}}\left[\int_{x_{k-1}}^{x_{k+1}}\big|f'(y)w(y)\big|\,dy\right]^p dx$$

$$\leqq \mathcal{C}\sum_{i=1}^{m+1}\sum_{k:\,x_k\in[\tilde\varepsilon_{\theta m},\tilde a_{\theta m}]}\frac{1}{\big(1+|i-k|\big)^{(A/2-1-(2\gamma+1))p}}A_k$$

where

$$A_k:=\int_{x_{k-1}}^{x_{k+1}}x^{\gamma p}\left[\int_{x_{k-1}}^{x_{k+1}}\big|f'(y)w(y)\big|\,dy\right]^p dx.$$

By Proposition 2.1 and (4.17), we have

$$A_k\leqq \mathcal{C}\int_{x_{k-1}}^{x_{k+1}}\left[\int_{x_{k-1}}^{x_{k+1}}\big|f'(y)u(y)\big|\,dy\right]^p dx$$

$$\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \int_{x_{k-1}}^{x_{k+1}} \left[\frac{m}{\sqrt{x}\sqrt{a_m}} \int_{x-c\frac{\sqrt{a_m}}{m}\sqrt{x}}^{x+c\frac{\sqrt{a_m}}{m}\sqrt{x}} |f'\varphi u|(y)\,dy\right]^p dx$$

$$=: \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \int_{x_{k-1}}^{x_{k+1}} \big|\mathcal{M}(F,x)\big|^p\,dx$$

where $c>0$ and $\mathcal{M}(F)$ is the Hardy–Littlewood maximal function of $F := f'\varphi u$. So, reversing the sums and using the boundedness of the maximal function, for $1<p\leqq\infty$, we obtain

$$\big\|(Q^+ - Q^-)u\big\|_p^p$$

$$\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \sum_{k:\,x_k\in[\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]} \int_{x_{k-1}}^{x_{k+1}} \big|\mathcal{M}(F,x)\big|^p\,dx$$

$$\times \sum_{i=1}^{m+1} \frac{1}{\big(1+|i-k|\big)^{(A/2-1-(2\gamma+1))p}}$$

$$\leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \int_{\tilde{\varepsilon}_{\theta m}}^{\tilde{a}_{\theta m}} \big|\mathcal{M}(F,x)\big|^p\,dx \leqq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^p \|f'\varphi u\|_p^p$$

since $A > 4\gamma + 4 + 4/p$.

For $p=1$ we can use the Fubini theorem. We omit the details. □

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