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Lagrange interpolation with exponential weights on $(-1, 1)$

G. Mastroianni*, I. Notarangelo

Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy

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Abstract

In order to approximate functions defined on $(-1, 1)$ with exponential growth for $|x| \to 1$, we consider interpolation processes based on the zeros of orthonormal polynomials with respect to exponential weights. Convergence results and error estimates in weighted L^p metric and uniform metric are given. In particular, in some function spaces, the related interpolating polynomials behave essentially like the polynomial of best approximation.

Keywords: Orthogonal polynomials; Lagrange interpolation; Approximation by polynomials; Exponential weights

1. Introduction

There is an extensive literature concerning the mean convergence on $(-1, 1)$ of Lagrange interpolation based on the zeros of orthogonal polynomials w.r.t. "doubling" weights (for instance Jacobi or generalized Jacobi weights). To this regard, we recall [17,18,22,9,28], among others. These interpolation processes are useful in the (weighted) polynomial approximation of

* Corresponding author.
E-mail addresses: mastroianni@unibas.it, giuseppe.mastroianni@unibas.it (G. Mastroianni), incoronata.notarangelo@unibas.it (I. Notarangelo).

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locally continuous functions, having algebraic singularities at the endpoints ± 1 and at some inner points. Nevertheless, these processes are not suitable in order to approximate functions having exponential growth close to ± 1. This last topic has received few attention and, as far as we know, we recall [2,7,8,25,29].

In this paper we propose two interpolation processes, which behave like the best approximation in wide subspaces of weighted L^p-spaces and then turn out to be suitable for approximating functions with exponential growth at ± 1.

Taking into account the properties of the considered functions, it is natural to choose weighted spaces L^p_u with

$$u(x) = (1-x^2)^{\mu} \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}, \quad \alpha > 0,\ \mu \geq 0,\ x \in (-1,1),$$

and a weight of the form

$$\sigma(x) = (1-x^2)^{\lambda} \mathrm{e}^{-(1-x^2)^{-\alpha}}, \quad \alpha > 0,\ \lambda \geq 0,\ x \in (-1,1),$$

for the orthonormal systems $\{p_m(\sigma)\}_{m\in\mathbb{N}}$.

Let us denote by $S_m(\sigma, f)$ the mth Fourier sum of $f \in L^1_\sigma$ in the previous orthonormal system and by $L_m(\sigma, f)$ the mth Lagrange polynomial of $f \in C^0(-1,1)$ based on the zeros of $p_m(\sigma)$. Unfortunately, as in the case of exponential weights on unbounded intervals (see, e.g., [19,13]), the sequence $\{S_m(\sigma, f)\}_{m\in\mathbb{N}}$ converges to f in L^p_u for a restricted class of functions (see [15,16]). Therefore, we cannot expect good approximation properties for the polynomial $L_m(\sigma, f)$, which is the discrete version of $S_m(\sigma, f)$. In fact, the associated Lebesgue constants in L^p_u are "big" (see [2,7]).

On the other hand, bounded projectors, or projectors having the minimal order $\log m$, are required in several contexts. So, the aim of this paper is to overcome this gap.

With $a_m = a_m\left(\sqrt{\sigma}\right)$ the Mhaskar–Rakhmanov–Saff number related to $\sqrt{\sigma}$ and $\theta \in (0,1)$ fixed, we denote by χ_θ the characteristic function of the interval $[-a_{\theta m}, a_{\theta m}]$. Then we are going to consider the interpolating polynomial $\mathcal{L}_m(\sigma, f) = L_m(\sigma, \chi_\theta f)$. Analogously, letting $L^*_{m,2}(\sigma, f)$ denote the Lagrange polynomial, interpolating f at the zeros of $(a_m^2 - \cdot^2)p_m(\sigma)$, we are going to study $\mathcal{L}^*_{m,2}(\sigma, f) = L^*_{m,2}(\sigma, \chi_\theta f)$. The behavior of these interpolation processes is stated in Theorems 3.1, 3.2, 3.5, 3.7 and 3.8, where error estimates are given.

The operators related to these processes are not projectors of continuous functions into the space of all the polynomials of degree at most $m-1$ and $m+1$, $\mathbb{P}_{m-1}$ and $\mathbb{P}_{m+1}$, but they do are projectors into some subsets $\mathcal{P}_{m-1} \subset \mathbb{P}_{m-1}$ and $\mathcal{P}^*_{m+1} \subset \mathbb{P}_{m+1}$. We will show that these subspaces fulfill the same density properties of $\mathbb{P}_{m-1}$ or $\mathbb{P}_{m+1}$ and that the Marcinkiewicz-type inequalities hold for polynomials belonging to them.

The paper is structured as follows. In Section 2 we recall some basic facts concerning the best weighted polynomial approximation, give the definitions of the interpolation processes and state some preliminary results. In Section 3 we state the main results, proved in Section 4.

2. Basic facts and preliminary results

In the sequel $\mathcal{C}$ will stand for a positive constant that can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$.

2.1. Function spaces and best polynomial approximation

Let w be defined as

$$w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}, \tag{2.1}$$

we consider the weight function

$$u(x) = v^{\mu}(x)\sqrt{w(x)} = (1-x^2)^{\mu}\mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}, \tag{2.2}$$

where $\alpha > 0$, $\mu \geq 0$, $x \in (-1, 1)$.

The weight w belongs to a wide class of exponential weights defined in [4,5], and in [15], it was checked that u belongs to the same class of w. In particular, setting $Q(x) = -\log u(x)$, we can define the Mhaskar–Rakhmanov–Saff number $\bar{a}_\tau = \bar{a}_\tau(u)$, $1 \leq \tau \in \mathbb{R}$, as the positive root of

$$\tau = \frac{2}{\pi}\int_0^1 \bar{a}_\tau t\, Q'(\bar{a}_\tau t)\frac{\mathrm{d}t}{\sqrt{1-t^2}}.$$

The number $\bar{a}_\tau$ is an increasing function of τ, with $\lim_{\tau\to+\infty}\bar{a}_\tau = 1$ and

$$C_1\tau^{-\frac{1}{\alpha+1/2}} \leq 1-\bar{a}_\tau \leq C_2\tau^{-\frac{1}{\alpha+1/2}},$$

where C_1 and C_2 are positive constants independent of τ and α is fixed (see [5, pp. 13,31]).

We can associate to the weight u the following function spaces. For $1 \leq p < \infty$, by L^p_u we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_{-1}^1 |fu|^p(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

For $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u := C_u = \left\{f \in C^0(-1,1) : \lim_{x\to\pm1} f(x)u(x) = 0\right\},$$

and we equip this space with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x\in(-1,1)} |f(x)u(x)|.$$

Note that the Weierstrass theorem implies the limit conditions in the definition of C_u.

In the sequel, if $u = 1$, we will simply write L^p. Moreover, we will use the notation $L^p(I)$, meaning that the norm is extended to $I \subset (-1, 1)$.

The Sobolev-type subspaces of L^p_u are given by

$$W^p_r(u) = \left\{f \in L^p_u : f^{(r-1)} \in AC(-1,1),\ \|f^{(r)}\varphi^r u\|_p < \infty\right\}, \quad 1 \leq r \in \mathbb{Z},$$

where $1 \leq p \leq \infty$, $\varphi(x) := \sqrt{1-x^2}$ and $AC(-1, 1)$ denotes the set of all functions which are absolutely continuous on every closed subset of $(-1, 1)$. We equip these spaces with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)}\varphi^r u\|_p.$$

In order to introduce some further subspaces of L_u^p, for $1 \le p \le \infty, r \ge 1$ and for a sufficiently small $t > 0$ (say $t < t_0$), we define the main part of the rth modulus of smoothness as

$$\Omega_\varphi^r(f,t)_{u,p} = \sup_{0<h\le t} \left\| \Delta_{h\varphi}^r (f)\, u \right\|_{L^p[-h^*,h^*]},$$

where $h^* = 1 - \mathrm{B}\, h^{1/(\alpha+1/2)}$, $\mathrm{B} > 1$ is a fixed constant, and

$$\Delta_{h\varphi}^r f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + (r-2i)\frac{h\varphi(x)}{2}\right).$$

Then the complete rth modulus of smoothness is given by

$$\begin{aligned}\omega_\varphi^r(f,t)_{u,p} &= \Omega_\varphi^r(f,t)_{u,p} + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,u\|_{L^p[-1,-t^*]} \\ &\quad + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,u\|_{L^p[t^*,1]}\end{aligned}$$

with $t^* = 1 - \mathrm{B}\, t^{1/(\alpha+1/2)}$ and $\mathrm{B} > 1$ a fixed constant. We emphasize that the behavior of $\omega_\varphi^r(f,t)_{u,p}$ is independent of the constant B.

We also remark that for any $f \in W_r^p(u)$, with $r \ge 1$ and $1 \le p \le \infty$, we have (see [14])

$$\Omega_\varphi^r(f,t)_{u,p} \le \mathcal{C} \sup_{0<h\le t} h^r \|f^{(r)}\varphi^r u\|_{L^p[-h^*,h^*]}, \quad \mathcal{C} \ne \mathcal{C}(f,t). \tag{2.3}$$

By means of the rth modulus of smoothness, for $1 \le p \le \infty$, we can define the Zygmund spaces

$$Z_s^p(u) := Z_{s,r}^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty,\ r > s \right\}, \quad 0 < s \in \mathbb{R},$$

equipped with the norm

$$\|f\|_{Z_{s,r}^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s}.$$

In the sequel we will denote these subspaces briefly by $Z_s^p(u)$, without the second index r and with the assumption $r > s$. We note that $\Omega_\varphi^r(f,t)_{u,p} \sim \omega_\varphi^r(f,t)_{u,p}$ for any $f \in Z_s^p(u), r > s$, and so in the definition of the Zygmund-type spaces $\omega_\varphi^r(f,t)_{u,p}$ can be replaced by $\Omega_\varphi^r(f,t)_{u,p}$ (see [14]).

Let us denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by

$$E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|_p$$

the error of best polynomial approximation in L_u^p, $1 \le p \le \infty$. A polynomial realizing the infimum in the previous definition is called polynomial of best approximation for $f \in L_u^p$. Moreover, we say that $P \in \mathbb{P}_m$ is a polynomial of quasi best approximation for $f \in L_u^p$ if

$$\|(f-P)\,u\|_p \le \mathcal{C}\, E_m(f)_{u,p},$$

with $\mathcal{C}$ independent of m and f.

The next theorem can be deduced from the results in [14].

Theorem 2.1. *Let* $u(x) = (1-x^2)^\mu e^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, *with* $\alpha > 0$ *and* $\mu \geq 0$. *For any* $f \in L^p_u$, $1 \leq p \leq \infty$, *the inequalities*

$$E_m(f)_{u,p} \leq \mathcal{C}\, \omega^r_\varphi \left(f, \frac{1}{m}\right)_{u,p} \tag{2.4}$$

and

$$\omega^r_\varphi \left(f, \frac{1}{m}\right)_{u,p} \leq \frac{\mathcal{C}}{m^r} \sum_{i=0}^{m} (1+i)^{r-1} E_i(f)_{u,p},$$

hold with $\mathcal{C}$ *independent of* m *and* f.

From the previous theorem, we deduce the following estimates for the error of best approximation

$$E_m(f)_{u,p} \leq \frac{\mathcal{C}}{m^r} \|f\|_{W^p_r(u)}, \quad \forall f \in W^p_r(u),\ r \geq 1, \tag{2.5}$$

and

$$E_m(f)_{u,p} \leq \frac{\mathcal{C}}{m^s} \|f\|_{Z^p_s(u)}, \quad \forall f \in Z^p_s(u),\ s > 0, \tag{2.6}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$ and $1 \leq p \leq \infty$.

2.2. Interpolation operators and polynomial spaces

Now, with w as in (2.1) and $v^\lambda(x) = (1-x^2)^\lambda$, we consider the weight

$$\sigma(x) = v^\lambda(x) w(x) = (1-x^2)^\lambda e^{-(1-x^2)^{-\alpha}}, \quad \alpha > 0,\ \lambda \geq 0, \tag{2.7}$$

and the corresponding sequence of orthonormal polynomials with positive leading coefficients $\{p_m(\sigma)\}_{m\in\mathbb{N}}$. We denote by $x_k = x_{m,k}(\sigma)$, $1 \leq k \leq \lfloor m/2 \rfloor$, the positive zeros of $p_m(\sigma)$ and by $x_{-k} = -x_k$ the negative ones. If m is odd, then $x_0 = 0$ is a zero of $p_m(\sigma)$.

These zeros are located as follows (see [5, pp. 22–23])

$$-a_m(1 - c\delta_m) \leq x_{-\lfloor m/2 \rfloor} < \cdots < x_1 < x_2 < \cdots < x_{\lfloor m/2 \rfloor} \leq a_m(1 - c\delta_m),$$

where $0 < c \neq c(m)$, a_m is the Mhaskar–Rakhmanov–Saff number related to the weight $\sqrt{\sigma}$, satisfying (see [4, p. 4] and also [15])

$$1 - a_m \sim m^{-\frac{1}{\alpha+1/2}}, \tag{2.8}$$

and

$$\delta_m := \left(\frac{1-a_m}{m}\right)^{2/3}. \tag{2.9}$$

Let us denote by $L_m(\sigma, f)$ the mth Lagrange polynomial interpolating a function $f \in C^0(-1,1)$ at the zeros of $p_m(\sigma)$. It is well-known that

$$L_m(\sigma, f, x) = \sum_{|k| \leq \lfloor m/2 \rfloor} \ell_k(\sigma, x) f(x_k),$$

with

$$\ell_k(\sigma, x) = \frac{p_m(\sigma, x)}{p'_m(\sigma, x_k)(x - x_k)}, \quad |k| \le \lfloor m/2 \rfloor,$$

and $L_m(\sigma) : C^0(-1, 1) \to \mathbb{P}_{m-1}$ is a projector, i.e. $L_m(\sigma, P) = P$ for any $P \in \mathbb{P}_{m-1}$.

Following an idea used for gaussian rules on $(0, +\infty)$ in [10,11] and for Lagrange interpolation on unbounded intervals in [19], we are going to define a "truncated" interpolation process.

With a_m the Mhaskar–Rakhmanov–Saff number related to $\sqrt{\sigma}$, for a fixed $\theta \in (0, 1)$, we define an index $j = j(m, \theta)$ by means of the equation

$$x_j = \min_{1 \le k \le \lfloor m/2 \rfloor} \{x_k : x_k \ge a_{\theta m}\}, \tag{2.10}$$

with m sufficiently large (say $m > m_0$), and we introduce the interpolation operator $\mathcal{L}_m(\sigma)$, defined by

$$\mathcal{L}_m(\sigma, f, x) = \sum_{|k| \le j} \ell_k(\sigma, x) f(x_k) \tag{2.11}$$

for any $f \in C_u$. By definition, $\mathcal{L}_m(\sigma, f) \in \mathbb{P}_{m-1}$ and $\mathcal{L}_m(\sigma, f, x_i) = f(x_i)$ for $|i| \le j$, while $\mathcal{L}_m(\sigma, f, x_i) = 0$ for $|i| > j$. Therefore $\mathcal{L}_m(\sigma, P) \ne P$ in general for $P \in \mathbb{P}_{m-1}$.

Now, consider the following collection of polynomials

$$\mathcal{P}_{m-1} = \{Q \in \mathbb{P}_{m-1} : Q(x_i) = 0, \ |i| > j\}$$

with $x_i = x_{m,i}(\sigma)$ and $j = j(m)$ defined as in (2.10). Naturally, $\mathcal{P}_{m-1}$ depends on the weight σ and on the parameter $\theta \in (0, 1)$. Moreover, for any $f \in C_u, \mathcal{L}_m(\sigma, f) \in \mathcal{P}_{m-1}$ and $\mathcal{L}_m(\sigma, L_m(\sigma, f)) = \mathcal{L}_m(\sigma, f)$. It is also easily seen that

$$\mathcal{P}_{m-1} = \mathcal{L}(\sigma)(\mathbb{P}_{m-1}) = \left\{ \sum_{|k| \le j} \ell_k(\sigma, x) P(x_k) \right\}_{P \in \mathbb{P}_{m-1}}$$

and the operator $\mathcal{L}_m(\sigma) : C^0(-1, 1) \to \mathcal{P}_{m-1}$ is a projector, i.e. $\mathcal{L}_m(\sigma, Q) = Q$ for any $Q \in \mathcal{P}_{m-1}$.

The next theorem shows that $\bigcup_m \mathcal{P}_{m-1}$ is dense in L^p_u, $1 \le p \le \infty$, and the corresponding error of best approximation

$$\widetilde{E}_{m-1}(f)_{u,p} = \inf_{Q \in \mathcal{P}_{m-1}} \|(f - Q) u\|_p$$

is strictly connected with $E_M(f)_{u,p}$, where

$$M = \left\lfloor \left(\frac{\theta}{\theta + 1} \right) \frac{m}{s} \right\rfloor \sim m, \tag{2.12}$$

with $s \ge 1$ fixed and $\theta \in (0, 1)$ as in (2.10).

Theorem 2.2. *Let $\sigma = v^\lambda w$ and $u = v^\mu \sqrt{w}$ with arbitrarily fixed parameters $\alpha > 0, \lambda, \mu \ge 0$. Then, for every function $f \in L^p_u$, $1 \le p \le \infty$, there exists a polynomial sequence $\{q^*_{m-1}\}_m$, with $q^*_{m-1} \in \mathcal{P}_{m-1}$, such that*

$$\lim_{m \to \infty} \left\| \left(f - q^*_{m-1} \right) u \right\|_p = 0. \tag{2.13}$$

Furthermore, letting χ_θ *denote the characteristic function of* $[-a_{\theta m}, a_{\theta m}]$*, with* $a_{\theta m} = a_{\theta m}\left(\sqrt{\sigma}\right), \theta \in (0,1)$*, we have*

$$\lim_{m\to\infty} \left\| \left(f - \chi_\theta q^*_{m-1}\right) u \right\|_p = 0. \tag{2.14}$$

Finally, for any $f \in L^p_u$*,* $1 \le p \le \infty$*, we get*

$$\widetilde{E}_{m-1}(f)_{u,p} \le \mathcal{C}\left\{E_M(f)_{u,p} + \mathrm{e}^{-\mathrm{A}M^\gamma}\|fu\|_p\right\}, \tag{2.15}$$

where $\gamma = 2\alpha/(2\alpha+1)$*,* M *is defined by (2.12), and* $\mathcal{C}$ *and* A *are positive constants independent of* m *and* f*.*

Let us now introduce a further interpolation operator. Following an idea due to J. Szabados in [27], we denote by $L^*_{m,2}(\sigma, f)$ the Lagrange polynomial, interpolating f at the zeros of $p_m(\sigma)$ and at the two extra points $x_{\pm(\lfloor m/2\rfloor+1)} := \pm a_m$, $a_m = a_m\left(\sqrt{\sigma}\right)$. Then, for any $f \in C^0(-1,1)$, we have

$$L^*_{m,2}(\sigma, f, x) = \sum_{|k|\le\lfloor m/2\rfloor+1} \ell^*_k(\sigma, x) f(x_k)$$

where

$$\ell^*_k(\sigma, x) = \frac{p_m(\sigma, x)}{p'_m(\sigma, x_k)(x-x_k)} \frac{(a_m^2 - x^2)}{(a_m^2 - x_k^2)}, \qquad 1 \le |k| \le \lfloor m/2\rfloor,$$

and

$$\ell^*_{\pm(\lfloor m/2\rfloor+1)}(\sigma, x) = \frac{a_m \pm x}{2a_m} \frac{p_m(\sigma, x)}{p_m(\sigma, \pm a_m)}.$$

In analogy with (2.11), the "truncated" version of $L^*_{m,2}(\sigma)$ is given by

$$\mathcal{L}^*_{m,2}(\sigma, f, x) = \sum_{|k|\le j} \ell^*_k(\sigma, x) f(x_k), \tag{2.16}$$

where the index j is defined by (2.10). We note that this operator has the same form of the one defined in [12] for Lagrange interpolation on the real line.

By arguments similar to those used for the operator $\mathcal{L}_m(\sigma)$, it is easily seen that the operator $\mathcal{L}^*_{m,2}(\sigma)$ is a projector from the space of all continuous on $(-1,1)$ functions into the set of polynomials

$$\mathcal{P}^*_{m+1} = \{Q \in \mathbb{P}_{m+1} : Q(\pm a_m) = Q(x_i) = 0,\ |i| > j\} = \mathcal{L}^*_{m,2}(\sigma)(\mathbb{P}_{m+1}),$$

where $x_i = x_{m,i}(\sigma)$, $a_m = a_m\left(\sqrt{\sigma}\right)$ and $j = j(m)$ is given by (2.10). Moreover, $\mathcal{L}^*_{m,2}(\sigma, Q) = Q$ for any $Q \in \mathcal{P}^*_{m+1}$ and, in analogy with Theorem 2.2, $\bigcup_m \mathcal{P}^*_{m+1}$ is dense in L^p_u, $1 \le p \le \infty$. In particular, setting

$$\widetilde{E}^*_{m+1}(f)_{u,p} = \inf_{Q\in\mathcal{P}^*_{m+1}} \|(f-Q)u\|_p$$

for any $f \in L^p_u$, $1 \le p \le \infty$, we have

$$\widetilde{E}^*_{m+1}(f)_{u,p} \le \mathcal{C}\left\{E_M(f)_{u,p} + \mathrm{e}^{-\mathrm{A}M^\gamma}\|fu\|_p\right\} \tag{2.17}$$

where $\gamma = 2\alpha/(2\alpha+1)$, M is defined by (2.12), and $\mathcal{C}$ and A are positive constants independent of m and f.

In the next section, we are going to study the behavior of the previous defined polynomial operators in some suitable function spaces.

3. Main results

Let us first consider the Lagrange polynomial $L_m(w, f)$, based on the zeros of $p_m(w)$, in the space $C_{\sqrt{w}}$, where $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}, \alpha > 0$. Since the sequence $\{p_m(w)\sqrt{w}\}_m$ is not uniformly bounded (see formula (4.10) in Section 4) and, moreover, the zeros of $p_m(w)$ are not arc sine distributed, then we cannot expect that the mth Lebesgue constant

$$\|L_m(w)\|_\infty = \max_{x\in(-1,1)} \sqrt{w(x)} \sum_{|k|\le\lfloor m/2\rfloor} \frac{|\ell_k(w,x)|}{\sqrt{w(x_k)}}$$

has the desired order $\log m$ (for a more complete discussion on this topic see [9, p. 251] and also [18]). In fact, from a result due to S.B. Damelin in [2], it follows that

$$\|L_m(w)\|_\infty \sim \delta_m^{-1/4} \sim m^{\frac{1}{6}\left(\frac{2\alpha+3}{2\alpha+1}\right)}.$$

Now, concerning the interpolation process $\{\mathcal{L}_m(\sigma)\}_{m\in\mathbb{N}}$ in C_u, denoting by χ_j the characteristic function of $[-x_j, x_j]$, with j as in (2.10), we have $\mathcal{L}_m(\sigma, f) = L_m(\sigma, \chi_j f)$ for any $f \in C_u$ and the following theorem holds.

Theorem 3.1. *Let* $\sigma(x) = (1-x^2)^\lambda \mathrm{e}^{-(1-x^2)^{-\alpha}}$ *and* $u(x) = (1-x^2)^\mu \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}, \alpha > 0, \lambda, \mu \ge 0$. *Then, for every* $f \in C_u$, *we have*

$$\left\|\chi_j \mathcal{L}_m(\sigma, f)\, u\right\|_\infty \le \mathcal{C}_\theta (\log m)\|\chi_j f u\|_\infty, \quad \mathcal{C}_\theta \ne \mathcal{C}_\theta(m, f), \tag{3.1}$$

if and only if

$$\frac{\lambda}{2} + \frac{1}{4} \le \mu \le \frac{\lambda}{2} + \frac{5}{4}. \tag{3.2}$$

Moreover, from (3.1), *it follows that*

$$\left\|\left[f - \chi_j \mathcal{L}_m(\sigma, f)\right] u\right\|_\infty \le \mathcal{C}_\theta \left\{(\log m)\, E_M(f)_{u,\infty} + \mathrm{e}^{-\mathrm{A}M^\gamma} \|fu\|_\infty\right\}, \tag{3.3}$$

where $\gamma = 2\alpha/(2\alpha+1)$, $M = \mathrm{c}m$, $0 < \mathrm{c} < 1$, *is defined by* (2.12), $\mathrm{A} \ne \mathrm{A}(m, f)$ *and here and in the following we denote by* $\mathcal{C}_\theta$ *a constant, independent of* m *and* f, *having the form* $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-\nu}(1/\theta)\right)$ *with some fixed* ν.

We also remark that the previous statement seems to be the best possible in the sense that it can be shown that, even if conditions (3.2) are satisfied, the quantities

$$\left\|\chi_j L_m(\sigma, f)\, u\right\|_\infty \quad \text{and} \quad \left\|L_m\left(\sigma, \chi_j f\right) u\right\|_\infty$$

diverge with an order greater than $\log m$ for $m \to \infty$, but we will omit the details (the proofs are based on inequalities (4.9)–(4.12) in Section 4).

The behavior of $\mathcal{L}_m(\sigma, f)$ in the L^p_u-norm is expressed by the following theorem.

Theorem 3.2. *Let $u = v^{\mu}\sqrt{w}$ and $\sigma = v^{\lambda}w$. For any $f \in C_u$ and for $1 \le p < \infty$, there exists a constant $\mathcal{C}_\theta$, depending on $\theta \in (0, 1)$, such that*

$$\left\| \chi_j \mathcal{L}_m(\sigma, f)\, u \right\|_p \le \mathcal{C}_\theta \|\chi_j f u\|_\infty, \quad \mathcal{C}_\theta \ne \mathcal{C}_\theta(m, f), \tag{3.4}$$

if and only if

$$\frac{v^{\mu}}{\sqrt{v^{\lambda}\varphi}} \in L^p, \qquad \frac{\sqrt{v^{\lambda}\varphi}}{v^{\mu}} \in L^1, \tag{3.5}$$

where $\varphi(x) = \sqrt{1 - x^2}$.

Moreover, from (3.4), for any $f \in C_u$, we get

$$\left\| \left[f - \chi_j \mathcal{L}_m(\sigma, f) \right] u \right\|_p \le \mathcal{C}_\theta \left\{ E_M(f)_{u,\infty} + \mathrm{e}^{-\mathrm{A}M^{\gamma}} \|f u\|_\infty \right\} \tag{3.6}$$

with $\mathcal{C}_\theta$, A independent of m and f, M as in (2.12) and $\gamma = 2\alpha/(2\alpha + 1)$.

Finally, setting $\Delta x_k = x_{k+1} - x_k$, for any $f \in C^0(-1, 1)$ and for $1 < p < \infty$, we have

$$\left\| \chi_j \mathcal{L}_m(\sigma, f)\, u \right\|_p \le \mathcal{C}_\theta \left(\sum_{|k| \le j} \Delta x_k \, |f(x_k) u(x_k)|^p \right)^{1/p}, \quad \mathcal{C}_\theta \ne \mathcal{C}_\theta(m, f), \tag{3.7}$$

if and only if

$$\frac{v^{\mu}}{\sqrt{v^{\lambda}\varphi}} \in L^p, \qquad \frac{\sqrt{v^{\lambda}\varphi}}{v^{\mu}} \in L^q, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.8}$$

The next proposition will be useful in order to estimate the error of our interpolation processes in L^p_u metric.

Proposition 3.3. *Let $f \in L^p_u$, such that $\Omega_\varphi(f, t)_{u,p}\, t^{-1-1/p} \in L^1(0, 1)$ for $1 < p < \infty$. Then we have*

$$\left(\sum_{|k| \le j} \Delta x_k \, |f(x_k) u(x_k)|^p \right)^{1/p} \le \mathcal{C} \left\{ \|\chi_j f u\|_p + \frac{1}{m^{1/p}} \int_0^{1/m} \frac{\Omega_\varphi(f, t)_{u,p}}{t^{1+1/p}} \, \mathrm{d}t \right\} \tag{3.9}$$

with $\mathcal{C}$ independent of m and f.

Then, as a consequence of (3.7) and Proposition 3.3, we obtain the following.

Corollary 3.4. *Let $1 < p < \infty$ and $f \in L^p_u$ such that $\Omega^r_\varphi(f, t)_{u,p}\, t^{-1-1/p} \in L^1(0, 1)$. Then, under the assumptions (3.8), we have*

$$\left\| \left[f - \chi_j \mathcal{L}_m(\sigma, f) \right] u \right\|_p \le \mathcal{C}_\theta \left\{ \frac{1}{m^{1/p}} \int_0^{1/m} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+1/p}} \, \mathrm{d}t + \mathrm{e}^{-\mathrm{A}m^{\gamma}} \|f u\|_p \right\}, \tag{3.10}$$

where $\gamma = 2\alpha/(2\alpha + 1)$, $\mathcal{C}_\theta$, A are independent of m and f, but depend on θ.

We remark that, under the assumptions of Corollary 3.4, the Lagrange interpolation is well defined, since in [24] it was proved that if $f \in L^p_u$, $1 < p < \infty$, is such that

$$\int_0^1 \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+1/p}} \, \mathrm{d}t < \infty, \quad r \ge 1,$$

then f is continuous on $(-1, 1)$.

In particular, if

$$\sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty,$$

with $1/p < s \in \mathbb{R}$ and $r > s$, i.e. if $f \in Z_s^p(u)$, $1 < p < \infty$, then, for $m > m_0$, (3.10) becomes

$$\left\| \left[f - \chi_j \mathcal{L}_m(\sigma, f) \right] u \right\|_p \le \frac{\mathcal{C}_\theta}{m^s} \|f\|_{Z_s^p(u)}, \tag{3.11}$$

and, if $1 \le s \in \mathbb{N}$, then the norm $\|f\|_{Z_s^p(u)}$ can be replaced by the Sobolev-type norm $\|f\|_{W_s^p(u)}$, taking (2.3) into account. Therefore, in the considered classes of functions, the sequence $\left\{\chi_{j(m)} \mathcal{L}_m(\sigma, f)\right\}_m$ converges to f in the L_u^p-norm with the order of the best polynomial approximation, given by (2.6).

Let us now consider the polynomial sequence $\{\mathcal{L}_m(\sigma, f)\}_m$, in which we drop one of the "truncations". In this case a theorem, analogous to Theorem 3.2, holds with the restriction $1 < p < 4$.

Theorem 3.5. *Let $u = v^\mu \sqrt{w}$ and $\sigma = v^\lambda w$. Then, for any $f \in C_u$, there exists a constant $\mathcal{C}_\theta$, depending on $\theta \in (0,1)$ and independent of m and f, such that*

$$\|\mathcal{L}_m(\sigma, f) u\|_p \le \mathcal{C}_\theta \|\chi_j f u\|_\infty, \quad \mathcal{C}_\theta \ne \mathcal{C}_\theta(m, f), \tag{3.12}$$

if and only if

$$\frac{v^\mu}{\sqrt{v^\lambda}} \in L^p, \qquad \frac{\sqrt{v^\lambda}}{v^\mu} \in L^1, \quad 1 \le p < 4. \tag{3.13}$$

Moreover, we get

$$\|\mathcal{L}_m(\sigma, f) u\|_p \le \mathcal{C}_\theta \left(\sum_{|k| \le j} \Delta x_k \, |f(x_k) u(x_k)|^p \right)^{1/p}, \tag{3.14}$$

if and only if

$$\frac{v^\mu}{\sqrt{v^\lambda}} \in L^p, \qquad \frac{\sqrt{v^\lambda}}{v^\mu} \in L^q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < 4. \tag{3.15}$$

Note that the special case $p = 2$ and $\mu = \lambda = 0$ has been treated in [1] using a truncated gaussian rule. Nevertheless, this procedure cannot be applied in the general case $p \ne 2$. From Theorem 3.5, we can deduce consequences similar to those related to Theorem 3.2; we omit the details about this topic and state the following.

Corollary 3.6. *Let $1 < p < 4$. For any polynomial $P_{m-1} \in \mathcal{P}_{m-1}$, the equivalence*

$$\|P_{m-1} u\|_p \sim \left(\sum_{|k| \le j} \Delta x_k |P_{m-1} u|^p (x_k) \right)^{1/p}$$

holds if and only if assumptions (3.15) are satisfied. Here the constants in "$\sim$" depend on θ and are independent of m and P_{m-1}.

Let us now consider the second interpolation polynomial $\mathcal{L}^*_{m,2}(\sigma, f)$. Its behavior is stated by the following theorems.

Theorem 3.7. *Let* $\sigma(x) = (1-x^2)^\lambda e^{-(1-x^2)^{-\alpha}}$ *and* $u(x) = (1-x^2)^\mu e^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, $\alpha > 0, \lambda, \mu \geq 0$. *Then, for every* $f \in C_u$, *we have*

$$\left\| \mathcal{L}^*_{m,2}(\sigma, f)\, u \right\|_\infty \leq \mathcal{C}_\theta (\log m) \|\chi_j f u\|_\infty, \quad \mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f), \tag{3.16}$$

if and only if

$$0 \leq \mu - \frac{\lambda}{2} + \frac{3}{4} \leq 1. \tag{3.17}$$

Moreover, from (3.16), *it follows that*

$$\left\| \left[f - \mathcal{L}^*_{m,2}(\sigma, f) \right] u \right\|_\infty \leq \mathcal{C}_\theta \left\{ (\log m)\, E_M(f)_{u,\infty} + e^{-\mathrm{A}M^\gamma} \|f u\|_\infty \right\}, \tag{3.18}$$

where $\gamma = 2\alpha/(2\alpha + 1)$, $M = \mathrm{c}m$, $0 < \mathrm{c} < 1$, *is defined by* (2.12), $\mathcal{C} \neq \mathcal{C}(m, f)$ *and* $\mathrm{A} \neq \mathrm{A}(m, f)$.

Theorem 3.8. *Let* $u = v^\mu \sqrt{w}$ *and* $\sigma = v^\lambda w$. *Then, for any* $f \in C_u$ *and for* $1 \leq p < \infty$, *there exists a constant* $\mathcal{C}_\theta$, *depending on* $\theta \in (0, 1)$, *such that*

$$\left\| \mathcal{L}^*_{m,2}(\sigma, f)\, u \right\|_p \leq \mathcal{C}_\theta \|\chi_j f u\|_\infty, \quad \mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f), \tag{3.19}$$

if and only if

$$\frac{v^{\mu+1}}{\sqrt{v^\lambda \varphi}} \in L^p, \qquad \frac{\sqrt{v^\lambda \varphi}}{v^{\mu+1}} \in L^1. \tag{3.20}$$

Moreover, for any $f \in C_u$ *and for* $1 < p < \infty$, *we have*

$$\left\| \mathcal{L}^*_{m,2}(\sigma, f)\, u \right\|_p \leq \mathcal{C}_\theta \left(\sum_{|k| \leq j} \Delta x_k\, |f(x_k) u(x_k)|^p \right)^{1/p}, \quad \mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f),$$

if and only if

$$\frac{v^{\mu+1}}{\sqrt{v^\lambda \varphi}} \in L^p, \qquad \frac{\sqrt{v^\lambda \varphi}}{v^{\mu+1}} \in L^q, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.21}$$

From the previous theorem we can deduce an analogous of Corollary 3.4. Therefore, as in (3.11), under the assumptions (3.21), for any $f \in Z^p_s(u)$, $1 < p < \infty$ and $1/p < s \in \mathbb{R}$, and for $m > m_0$, we have

$$\left\| \left[f - \mathcal{L}^*_{m,2}(\sigma, f) \right] u \right\|_p \leq \frac{\mathcal{C}_\theta}{m^s} \|f\|_{Z^p_s(u)},$$

and, for any $f \in W^p_r(u)$, $1 < p < \infty$ and $1 \leq r \in \mathbb{Z}$, we get

$$\left\| \left[f - \mathcal{L}^*_{m,2}(\sigma, f) \right] u \right\|_p \leq \frac{\mathcal{C}_\theta}{m^r} \|f\|_{W^p_r(u)}, \tag{3.22}$$

where $\mathcal{C}_\theta$ is independent of m and f in both cases. So, the polynomial sequence $\left\{\mathcal{L}^*_{m,2}(\sigma, f)\right\}_m$ converges to the function f in the L^p_u-norm with the order of the best approximation in the considered classes of functions (see (2.6) and (2.5)).

Furthermore, the operator $\mathcal{L}^*_{m,2}(\sigma)$ is uniformly bounded in Sobolev-type spaces, as the next theorem shows.

Theorem 3.9. *Let σ, u be the above defined weights and let conditions (3.21) be fulfilled. Then, for every $f \in W^p_r(u)$, with $r \geq 1$ and $1 < p < \infty$, we have*

$$\left\|\mathcal{L}^*_{m,2}(\sigma, f)\right\|_{W^p_r(u)} \leq \mathcal{C}_\theta \, \|f\|_{W^p_r(u)}. \tag{3.23}$$

Moreover, for every $f \in W^p_s(u)$, $s > r \geq 1$, we get

$$\left\|f - \mathcal{L}^*_{m,2}(\sigma, f)\right\|_{W^p_r(u)} \leq \frac{\mathcal{C}_\theta}{m^{s-r}} \|f\|_{W^p_s(u)}, \tag{3.24}$$

where in both cases $\mathcal{C}_\theta$ is independent of m and f.

Note that in the previous theorem $W^p_r(u)$ can be replaced by $Z^p_s(u)$, with the assumption $s > 1/p$ (which is necessary for the continuity of f, see [24]).

As a further consequence of Theorem 3.8, in analogy with Corollary 3.6, the following Marcinkiewicz equivalence holds in the subspace $\mathcal{P}^*_{m+1}$.

Corollary 3.10. *Let $1 < p < \infty$. For any polynomial $P_{m+1} \in \mathcal{P}^*_{m+1}$, the equivalence*

$$\|P_{m+1}u\|_p \sim \left(\sum_{|k|\leq j} \Delta x_k |P_{m+1}u|^p(x_k)\right)^{1/p}$$

holds if and only if assumptions (3.21) are satisfied. Here the constants in "$\sim$" depend on θ and are independent of m and P_{m+1}.

To conclude this section, we want to emphasize that, in the previous theorems, the constants $\mathcal{C}_\theta$, depending on $\log^{-1}(1/\theta)$, $\theta \in (0, 1)$, appear. The parameter θ is crucial in our results, since it cannot assume the value 1; in other words, the "truncation" of the sums in (2.11) and (2.16) seems to be essential (for further details see Section 4).

4. Proofs

First of all we recall some restricted range inequalities, which will be used in the proofs. Letting σ be as in (2.7), we consider $\sqrt{\sigma} = v^{\lambda/2}\sqrt{w}$, $\lambda \geq 0$, and its related Mhaskar–Rakhmanov–Saff number $a_m = a_m\left(\sqrt{\sigma}\right)$. Then, for any polynomial $P_m \in \mathbb{P}_m$, $1 \leq p \leq \infty$, the inequalities

$$\left\|P_m\sqrt{\sigma}\right\|_p \leq \mathcal{C}\left\|P_m\sqrt{\sigma}\right\|_{L^p[-a_m,a_m]} \tag{4.1}$$

and

$$\left\|P_m\sqrt{\sigma}\right\|_{L^p\{|x|\geq a_{\eta m}\}} \leq \mathcal{C}\mathrm{e}^{-\mathrm{A}m^\gamma}\left\|P_m\sqrt{\sigma}\right\|_p, \quad \eta > 1,\ \gamma = 2\alpha/(2\alpha+1), \tag{4.2}$$

hold with $\mathcal{C}$ and A positive constants independent of P_m (see [5, pp. 15–16], [4, Theorem 1.7, p. 12] and [6, Lemma 2.3]).

In order to avoid considering Mhaskar–Rakhmanov–Saff numbers related to different weights in the proofs, the following remark will be useful. Let $u = v^{\mu}\sqrt{w} = v^{\mu-\lambda/2}\sqrt{\sigma}$, $\mu \geq 0$, be the weight in (2.2), and $a_m = a_m\left(\sqrt{\sigma}\right)$. For any $P_m \in \mathbb{P}_m$, from (4.1) we can deduce (see [15])

$$\|P_m u\|_p \leq \mathcal{C}\begin{cases}\|P_m u\|_{L^p[-a_{sm},a_{sm}]}, & s>1, \text{ if } \mu-\lambda/2<0,\\ \|P_m u\|_{L^p[-a_m,a_m]}, & \text{otherwise}\end{cases} \tag{4.3}$$

where $\mathcal{C}$ is independent of P_m. Using the same argument, we can deduce an inequality of the form (4.2), where $\sqrt{\sigma}$ and $a_{\eta m}$ are replaced by u and $a_{\eta s m}$, respectively.

Now, using the previous inequalities, we are able to prove the following.

Proposition 4.1. *Let u be the weight in (2.2), $\bar{a}_M$ its MRS number and $\eta > 1$. For any function $f \in L^p_u$, $1 \leq p \leq \infty$, we have*

$$\|fu\|_{L^p\{|x|\geq \bar{a}_{\eta M}\}} \leq \mathcal{C}\left\{E_M(f)_{u,p} + \mathrm{e}^{-\mathrm{A}M^{\gamma}}\|fu\|_p\right\}, \tag{4.4}$$

where $\mathcal{C}$, A are positive constants independent of f and M, and $\gamma = 2\alpha/(2\alpha+1)$.

Proof. Letting $P_M \in \mathbb{P}_M$ be the polynomial of best approximation of f in L^p_u-metric, by (4.2), we get

$$\begin{aligned}\|fu\|_{L^p\{|x|\geq \bar{a}_{\eta M}\}} &\leq \|(f-P_M)u\|_{L^p\{|x|\geq \bar{a}_{\eta M}\}} + \|P_M u\|_{L^p\{|x|\geq \bar{a}_{\eta M}\}}\\ &\leq E_M(f)_{u,p} + \mathcal{C}\mathrm{e}^{-\mathrm{A}M^{\gamma}}\|P_M u\|_p,\end{aligned}$$

whence our claim follows. □

Now, given $f \in L^p_u$, $1 \leq p \leq \infty$, let $f_j = \chi_j f$, where χ_j is the characteristic function of $[-x_j, x_j]$ and $x_j = \min_{1\leq k\lfloor m/2\rfloor}\{x_k : x_k \geq a_{\theta m}\}$ as in (2.10), for a fixed $\theta \in (0,1)$ and for $m > m_0$, As a consequence of Proposition 4.1, we can estimate the L^p_u-distance between f and f_j by (4.4) as

$$M = \begin{cases}\left\lfloor \dfrac{\theta m}{s(\theta+1)}\right\rfloor, & s>1, \text{ if } \mu-\lambda/2<0,\\ \left\lfloor \dfrac{\theta m}{\theta+1}\right\rfloor, & \text{otherwise,}\end{cases}$$

taking (4.3) into account.

Let us now recall some properties of the orthonormal system $\{p_m(\sigma)\}_{m\in\mathbb{N}}$, related to the weight $\sigma(x) = (1-x^2)^{\lambda}\mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0, \lambda \geq 0$, which will be used in the sequel. The distance between two consecutive zeros of $p_m(\sigma)$ satisfies (see [5, p. 23])

$$\Delta x_k = x_{k+1} - x_k \sim \frac{a_{2m}^2 - x_k^2}{m\sqrt{a_m^2 - x_k^2 + a_m^2\delta_m}}, \quad -\lfloor m/2\rfloor \leq k \leq \lfloor m/2\rfloor - 1, \tag{4.5}$$

and

$$1 - \frac{x_{\lfloor m/2\rfloor}}{a_m} \sim \delta_m, \tag{4.6}$$

where δ_m is given by (2.9).

Taking into account the definition of the operators $\chi_{j(m)}\mathcal{L}_m(\sigma)$ and $\mathcal{L}^*_{m,2}(\sigma)$, we are interested in the behavior of the orthonormal polynomials and their zeros in some subintervals of the related Mhaskar–Rakhmanov–Saff interval $\left[-a_m\left(\sqrt{\sigma}\right), a_m\left(\sqrt{\sigma}\right)\right]$. The next lemma is useful to this aim.

Lemma 4.2. *Let* θ_1 *and* θ_2 *be fixed such that* $0 < \theta_1 < \theta_2 \le 1$ *and let* $x \in [-a_{\theta_1 m}, a_{\theta_1 m}]$. *Then we have*

$$\varphi_m(x) := \frac{a_{2m}^2 - x^2}{m\sqrt{|a_m^2 - x^2| + a_m^2\delta_m}} \sim \frac{\sqrt{a_{\theta_2 m}^2 - x^2}}{m}$$

and

$$a_{\theta_2 m}^2 - x^2 \le 1 - x^2 \le \left(1 + \frac{\mathcal{C}}{\log(\theta_2/\theta_1)}\right)(a_{\theta_2 m}^2 - x^2),$$

where the constants in "$\sim$" depend on θ_1 *and* θ_2, *while* $\mathcal{C}$ *does not.*

In particular, for $x \in [-a_{\theta m}, a_{\theta m}]$, $\theta \in (0, 1)$, *we get*

$$a_m^2 - x^2 \sim 1 - x^2, \tag{4.7}$$

where the constants in "$\sim$" depend on $\log^{-1}(1/\theta)$.

We omit the proof of the lemma, which follows from standard computation and is essentially based on the following inequality. For every fixed $r, s > 0$, with $r < s$, one has (see [5, formula (3.5.3), p. 81] and [4, p. 27])

$$c_1(1 - a_{sm})\log\left(\frac{s}{r}\right) \le \left(1 - \frac{a_{rm}}{a_{sm}}\right) \le c_2(1 - a_{rm})\log\left(\frac{s}{r}\right),$$

where $c_1, c_2 > 0$ are independent of m, r, s and $(1 - a_{rm}) \sim (1 - a_{sm})$.

From Lemma 4.2, we deduce that the zeros of $p_m(\sigma)$ are arc sine distributed in every subinterval of the Mhaskar–Saff interval of the form $[-a_{\bar\theta m}, a_{\bar\theta m}]$ with $\bar\theta \in (0, 1)$ fixed. Namely, from (4.5), for $|x_k| \le x_j$, j as in (2.10), with $a_{\theta m} \le x_j < a_{\bar\theta m}$ and $0 < \theta < \bar\theta < 1$, by Lemma 4.2 and (4.6), we deduce (see also [14] and [5, p. 32])

$$\Delta x_k \sim \frac{\sqrt{a_m^2 - x_k^2}}{m} \sim \frac{\varphi(x_k)}{m}, \quad |k| \le j, \tag{4.8}$$

where $\varphi(x_k) = \sqrt{1 - x_k^2}$ and the constants in "$\sim$" depend on $\log^{-1}(1/\theta)$. We note that (4.8) is not true in general for $|k| \le \lfloor m/2 \rfloor$ (we refer to [29] for precise estimates of the distance between consecutive zeros related to a wider class of weights).

Concerning the polynomials $p_m(\sigma)$, $m \in \mathbb{N}$, the equivalences

$$\sup_{x\in(-1,1)}\left|p_m(\sigma, x)\sqrt{\sigma(x)}\sqrt[4]{|a_m^2 - x^2|}\right| \sim 1 \tag{4.9}$$

and

$$\sup_{x\in(-1,1)}\left|p_m(\sigma, x)\sqrt{\sigma(x)}\right| \sim \delta_m^{-1/4} \sim \left(\frac{m}{1 - a_m}\right)^{1/6}, \tag{4.10}$$

were proved in [4, formulas (1.38) and (1.39), p. 10], where $a_m = a_m\left(\sqrt{\sigma}\right)$ and $\delta_m = \delta_m\left(\sqrt{\sigma}\right)$ satisfy (2.8) and (2.9). Moreover, in [5, pp. 22–23] the relation

$$\frac{1}{\left|p_m'(\sigma, x_k)\right|\sqrt{\sigma(x_k)}} \sim \Delta x_k \sqrt[4]{a_m^2 - x_k^2}, \quad |k| \le \lfloor m/2 \rfloor, \tag{4.11}$$

where x_k are the zeros of $p_m(\sigma)$ and $\Delta x_k = x_{k+1} - x_k$, was shown.

Now, let $\theta \in (0, 1)$ fixed. By Lemma 4.2, from (4.9) and (4.7), we deduce the inequality

$$|p_m(\sigma, x)| \sqrt{\sigma(x)\varphi(x)} \le \mathcal{C}_\theta, \quad |x| \le a_{\theta m}, \tag{4.12}$$

where $\varphi(x) = \sqrt{1 - x^2}$ and

$$\mathcal{C}_\theta = \mathcal{C}\left(1 + \frac{1}{\log(1/\theta)}\right)^{1/4}$$

with $\mathcal{C}$ independent of m and θ.

In the next lemma we show a rough inequality for the Lebesgue constant in the space C_u associated with the Lagrange interpolation at the zeros of $p_m(\sigma)$, for arbitrary parameters of the weights, that can be easily deduced from (4.9)–(4.12). This will be useful to prove Theorem 2.2.

Lemma 4.3. *Let $\sigma = v^\lambda w$ and $u = v^\mu \sqrt{w}$ be the weights in (2.7) and (2.2). Then, for every $\lambda, \mu \ge 0$ and for any $f \in C_u$, we have*

$$\|L_m(\sigma, f)\, u\|_\infty \le \mathcal{C}\, m^\tau\, \|f u\|_\infty,$$

for some $\tau > 0$, where $\mathcal{C}$ is independent of m and f.

Proof of Theorem 2.2. We first prove (2.13). By Theorem 2.1, there exists a sequence $\{q_{m-1}\}_m, q_{m-1} \in \mathbb{P}_{m-1}$, converging to $f \in L_u^p, 1 \le p \le \infty$, for $m \to \infty$. Now, with $M = M_m = \lfloor(\theta/(\theta+1))m/s\rfloor, s > 1$ fixed, the subsequence $\{q_M\}_M$ converges to $f \in L_u^p$ and we construct the sequence $\{q_{m-1}^*\}$ as

$$q_{m-1}^*(x) = \mathcal{L}_m(\sigma, q_{M_m}, x) \in \mathcal{P}_{m-1}.$$

Hence we get

$$\begin{aligned}
\left\|\left(f - q_{m-1}^*\right)u\right\|_p &\le \|(f - q_M)\,u\|_p + \left\|\left(q_M - q_{m-1}^*\right)u\right\|_p \\
&= \|(f - q_M)\,u\|_p + \|[L_m(\sigma, q_M) - \mathcal{L}_m(\sigma, q_M)]\,u\|_p \\
&= \|(f - q_M)\,u\|_p + \left\|\sum_{|k|>j} \ell_k(\sigma) q_M(x_k) u\right\|_p .
\end{aligned} \tag{4.13}$$

Concerning the second summand at the right-hand side, by using Lemma 4.3 and inequality (4.2), we have

$$\begin{aligned}
\left\|\sum_{|k|>j} \ell_k(\sigma) q_M(x_k) u\right\|_p &\le \mathcal{C} m^\tau \|q_M u\|_{L^\infty\{|x| \ge a_{\theta m}\}} \\
&\le \mathcal{C} \mathrm{e}^{-\mathrm{A} M^\gamma} \|q_M u\|_\infty,
\end{aligned} \tag{4.14}$$

where $\gamma = 2\alpha/(2\alpha+1), \tau > 0, \mathrm{A} > 0$. Then, by (4.13) and (4.14), the limit relation (2.13) follows.

Moreover, by Proposition 4.1, (4.13) and (4.14), we get

$$\begin{aligned}\left\|\left(f-\chi_j q_{m-1}^*\right)u\right\|_p &\le \left\|\left(f-\chi_j f\right)u\right\|_p + \left\|\chi_j\left(f-q_{m-1}^*\right)u\right\|_p\\ &\le \mathcal{C}\left\{E_M(f)_{u,p} + \mathrm{e}^{-\mathrm{A}M^\gamma}\|fu\|_p\right\} + \|(f-q_M)u\|_p\\ &\quad + \mathcal{C}\mathrm{e}^{-\mathrm{A}M^\gamma}\|q_M u\|_\infty,\end{aligned}$$

and then (2.14).

In particular, from the previous inequality we obtain (2.15), letting $\{q_{m-1}\}_m$ be a sequence of polynomials of quasi best approximation for f and using the Nikolskii inequality (see [5, pp. 293–312] and also [24])

$$\|q_M u\|_\infty \le \mathcal{C}M^{\frac{1}{p}\left(\frac{2\alpha+2}{2\alpha+1}\right)}\|q_M u\|_p. \quad \Box \tag{4.15}$$

Proof of Theorem 3.1. Let us first prove that conditions (3.2) imply inequality (3.1). We can write

$$\begin{aligned}\left\|\chi_j \mathcal{L}_m(\sigma)\right\|_\infty &= \sup_{f\in C_u}\frac{\left\|\chi_j L_m\left(\sigma,\chi_j f\right)u\right\|_\infty}{\|\chi_j f u\|_\infty}\\ &= \max_{x\in[-x_j,x_j]}\sum_{|k|\le j}\frac{|\ell_k(\sigma,x)|\,u(x)}{u(x_k)},\end{aligned} \tag{4.16}$$

where χ_j is the characteristic function of $[-x_j, x_j]$.

Taking into account that, for $x, x_k \in [-x_j, x_j]$, the relation (4.7) holds, using (4.11) and (4.12), we have

$$\frac{|\ell_k(\sigma,x)|\,u(x)}{u(x_k)} \le \mathcal{C}_\theta \frac{\Delta x_k v^{\mu-\lambda/2-1/4}(x)}{|x-x_k|v^{\mu-\lambda/2-1/4}(x_k)}$$

with $\mathcal{C}_\theta$ depending on θ and $k \neq d$, x_d being a zero closest to x. Since (see [5, pp. 320–321])

$$\frac{|\ell_d(\sigma,x)|\,u(x)}{u(x_d)} \sim 1,$$

it follows that

$$\begin{aligned}\left\|\chi_j \mathcal{L}_m(\sigma)\right\|_\infty &\le \mathcal{C}_\theta\left[1 + v^{\mu-\lambda/2-1/4}(x)\sum_{|k|\le j,\ k\ne d}\frac{\Delta x_k}{|x-x_k|v^{\mu-\lambda/2-1/4}(x_k)}\right]\\ &\le \mathcal{C}_\theta \log m\end{aligned}$$

since the zeros x_k, $|k| \le j$, are arc sine distributed by (4.8) and $0 \le \mu - \lambda/2 - 1/4 \le 1$ by assumption (see, e.g., [9, p. 243, formula (4.1.13)]). Then inequality (3.1) follows.

Now, let us prove by contradiction that inequality (3.1) implies conditions (3.2), i.e. $0 \le \mu - \lambda/2 - 1/4 \le 1$. Let us first suppose $\mu - \lambda/2 - 1/4 < 0$. Setting $\bar{x} = (x_{j-1} + x_j)/2$ and recalling a result in [18] (see also [9, pp. 250–251] and [26]), we have

$$\left\|\chi_j \mathcal{L}_m(\sigma)\right\|_\infty \ge \mathcal{C}\left\|\chi_j p_m(\sigma)u\right\|_\infty \ge \mathcal{C}\,|p_m(\sigma,\bar{x})|\,u(\bar{x}).$$

Then by using the formula (see [4, formula (12.7), p. 134])

$$|p_m(\sigma,x)|\sqrt{\sigma(x)\sqrt{a_m^2-x^2}} \sim \frac{|x-x_k|}{\Delta x_k}, \quad x \in (x_k, x_{k+1}), \tag{4.17}$$

we get

$$|p_m(\sigma,\bar{x})|\,u(\bar{x}) \sim |p_m(\sigma,\bar{x})|\sqrt{\sigma(\bar{x})\sqrt{a_m^2-\bar{x}^2}}\,v^{\mu-\lambda/2-1/4}(\bar{x}) \sim (1-a_{\theta m})^{\mu-\lambda/2-1/4},$$

since $a_m^2-\bar{x}^2 \sim 1-\bar{x}^2 \sim 1-a_{\theta m}$, and then

$$\left\|\chi_j \mathcal{L}_m(\sigma)\right\|_\infty \geq \mathcal{C}(1-a_{\theta m})^{\mu-\lambda/2-1/4} \sim m^{\frac{-(\mu-\lambda/2-1/4)}{\alpha+1/2}}$$

which is a contradiction.

Let us now suppose $\mu-\lambda/2-1/4 > 1$. We choose $\bar{x} \in (-1/2,1/2)$ and $\theta_0 \ll \theta$. By (4.16) and (4.11), we have

$$\begin{aligned}\left\|\chi_j \mathcal{L}_m(\sigma)\right\|_\infty &\geq \mathcal{C}\,|p_m(\sigma,\bar{x})|\,u(\bar{x}) \sum_{|k|\leq j} \frac{\Delta x_k}{(1-x_k^2)^{\mu-\lambda/2-1/4}|\bar{x}-x_k|}\\ &\geq \mathcal{C} \sum_{a_{\theta_0 m}\leq |x_k|\leq a_{\theta m}} \frac{\Delta x_k}{(1-x_k^2)^{\mu-\lambda/2-1/4}|\bar{x}-x_k|}\\ &\geq \mathcal{C}(1-a_{\theta_0 m})^{-(\mu-\lambda/2-1/4)} \sum_{a_{\theta_0 m}\leq |x_k|\leq a_{\theta m}} \frac{\Delta x_k}{|\bar{x}-x_k|},\end{aligned}$$

which yields again a contradiction.

Finally, let us prove that inequality (3.1) implies the error bound (3.3). Letting $Q \in \mathcal{P}_{m-1}$ be a polynomial realizing the infimum in the definition of $\widetilde{E}_{m-1}(f)_{u,\infty}$, for any $f \in C_u$, we have

$$\begin{aligned}\left\|\left[f-\chi_j\mathcal{L}_m(\sigma,f)\right]u\right\|_\infty &\leq \left\|\left(f-\chi_j f\right)u\right\|_\infty + \left\|\chi_j\,(f-Q)\,u\right\|_\infty\\ &\quad+\left\|\chi_j\mathcal{L}_m(\sigma,f-Q)\,u\right\|_\infty\\ &\leq \left\|\left(f-\chi_j f\right)u\right\|_\infty + \widetilde{E}_{m-1}(f)_{u,\infty}\\ &\quad+\left\|\chi_j\mathcal{L}_m(\sigma,f-Q)\,u\right\|_\infty.\end{aligned}$$

Hence, by Proposition 4.1, Theorem 2.2 and inequality (3.1), we obtain (3.3). □

The following lemma states a Marcinkiewicz-type inequality.

Lemma 4.4. *Let σ be the weight in (2.7), $\theta \in (0,1)$, and $x_k = x_{m,k}(\sigma)$, with $|k| \leq j$ and j given by (2.10). Moreover, let $v^\beta(x) = (1-x^2)^\beta$, with $\beta > -1$. Then there exists $\bar{\theta} \in (\theta,1)$ such that, for any polynomial $P_{lm} \in \mathbb{P}_{lm}$, with l a fixed integer, and for $1 \leq p < \infty$, we have*

$$\sum_{|k|\leq j} \Delta x_k \left|P_{lm}(x_k)v^\beta(x_k)\right|^p \leq \mathcal{C}_{\bar{\theta}} \int_{-a_{\bar{\theta}m}}^{a_{\bar{\theta}m}} \left|P_{lm}(x)v^\beta(x)\right|^p \mathrm{d}x, \tag{4.18}$$

where $\mathcal{C}_{\bar{\theta}}$ depends on $\bar{\theta}$, but is independent of m and P_{lm}.

Proof. Let $a_{\theta m} \leq x_j < x_{j+1} < a_{\bar{\theta}m}$. By (4.8) and Lemma 4.2, we have

$$\sum_{|k|\leq j} \Delta x_k\,|P_{lm}(x_k)|^p \leq \mathcal{C}_{\bar{\theta}}\left\{\int_{-x_j}^{x_{j+1}} |P_{lm}(x)|^p\,\mathrm{d}x + \frac{1}{m^p}\int_{-x_j}^{x_{j+1}} \left|\sqrt{a_{\bar{\theta}m}^2-x^2}\,P'_{lm}(x)\right|^p \mathrm{d}x\right\}.$$

Hence, using the unweighted Bernstein inequality in $[-a_{\bar\theta m}, a_{\bar\theta m}]$, we get

$$\sum_{|k|\le j} \Delta x_k \, |P_{lm}(x_k)|^p \le \mathcal{C}_{\bar\theta} \int_{-a_{\bar\theta m}}^{a_{\bar\theta m}} |P_{lm}(x)|^p \, \mathrm{d}x. \tag{4.19}$$

Since there exists a polynomial $Q_m \in \mathbb{P}_m$, such that (see [17])

$$v^\beta(x) \sim Q_m(x), \quad x \in [-a_{\bar\theta m}, a_{\bar\theta m}] \subset [-1 + m^{-2}, 1 - m^{-2}],$$

using (4.19) with the polynomial $P_{lm} Q_m$, we obtain

$$\sum_{|k|\le j} \Delta x_k \left| P_{lm}(x_k) v^\beta(x_k) \right|^p \le \mathcal{C}_{\bar\theta} \int_{-a_{\bar\theta m}}^{a_{\bar\theta m}} \left| P_{lm}(x) v^\beta(x) \right|^p \mathrm{d}x,$$

which was our claim. □

In order to prove Theorem 3.2, we recall some properties of the Hilbert transform $\mathcal{H}$ extended to the interval $(-1, 1)$, defined by

$$\mathcal{H}(f, y) = \int_{-1}^{1} \frac{f(x)}{x - y} \, \mathrm{d}x, \quad y \in (-1, 1),$$

where the integral is understood in the Cauchy principal value sense. The inversion formula

$$\int_{-1}^{1} \mathcal{H}(f) g = -\int_{-1}^{1} \mathcal{H}(g) f \tag{4.20}$$

holds for any $f \in L^p$ and $g \in L^q$, $1 < p < \infty$, $1/p + 1/q = 1$. Moreover, if $f \in L^\infty$ and $g \in L \log^+ L$, i.e. $\int_{-1}^{1} |g(x)| \log^+ |g(x)| \, \mathrm{d}x < \infty$, the inversion (4.20) is still true (see [23]) and

$$\| f H(g) \|_1 \le \| f \|_\infty \| g (1 + \log^+ |g|) \|_1. \tag{4.21}$$

If v^β is a Jacobi weight of the form $v^\beta(x) = (1 - x^2)^\beta$, then, for any measurable function f such that $f v^\beta \in L^p$, $1 < p < \infty$, the inequality

$$\left\| \mathcal{H}(f) \, v^\beta \right\|_p \le \mathcal{C} \| f v^\beta \|_p, \quad \mathcal{C} \ne \mathcal{C}(f), \tag{4.22}$$

holds if and only if (see [20,3,21])

$$-\frac{1}{p} < \beta < 1 - \frac{1}{p}.$$

Moreover, denoting by $L^p(\log^+ L^p)$, $1 \le p < \infty$, the collection of all the functions f such that

$$\| f \log^+ |f| \|_p = \left(\int_{-1}^{1} \left[|f(x)| \log^+ |f(x)| \right]^p \mathrm{d}x \right)^{1/p} < \infty,$$

the following lemma holds.

Lemma 4.5 (*P. Nevai, [22]*)**.** *Let* $1 < p < \infty$, $v^\beta(x) = (1 - x^2)^\beta$ *be a Jacobi weight,* G *a function such that* $|G(x)| \le 1$ *almost everywhere in* $[-1, 1]$ *and* $G(x) = 0$ *for* $x \notin [-1, 1]$. *If a* *function* $g \in L^p(\log^+ L)^p$ *satisfies* $gv^\beta \in L^p$ *then we have*

$$\sup_G \left\| gv^\beta \mathcal{H}(Gv^{-\beta}) \right\|_p \le \mathcal{C} \left\{ 1 + \left\| g \left(1 + v^\beta + \log^+ |g| \right) \right\|_p \right\}$$

with $\mathcal{C} \ne \mathcal{C}(g)$.

In the next proofs, we will need also the following lemma, which can be proved applying arguments similar to those used in [14] for the weight $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$.

Lemma 4.6. *Let* $\sqrt{\sigma(x)} = v^{\lambda/2}(x)\sqrt{w(x)} = (1-x^2)^{\lambda/2}\mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, *with* $\alpha > 0, \lambda \ge 0$, *and* $a_m = a_m\left(\sqrt{\sigma}\right)$. *Then, for any sufficiently large* m, *there exists a polynomial* $R_{lm} \in \mathbb{P}_{lm}$, *where* l *is a fixed integer, satisfying*

$$R_{lm}(x) \sim \sqrt{\sigma(x)}$$

and

$$\left| R'_{lm}(x) \right| \sqrt{1-x^2} \le \mathcal{C}m\sqrt{\sigma(x)}$$

for $|x| \le a_{sm}$, $s \ge 1$ *a fixed integer, with* $\mathcal{C}$ *independent of* m, $\sqrt{\sigma}$ *and* R_{lm}.

Finally, we recall a well-known fact. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and set $\|\mathbf{a}\|_p = \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}$ if $1 \le p < \infty$ and $\|\mathbf{a}\|_\infty = \max_{1\le i \le n} |a_i|$. Moreover, let $\Gamma_n : \mathbb{R}^n \to \mathbb{R}$ be defined by $\Gamma_n = \sum_{i=1}^n a_i b_i$, with $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and n fixed. Then the norm of the functional Γ_n is

$$\|\Gamma_n\|_p = \sup_{\|\mathbf{a}\|_p \le 1} |\Gamma_n(\mathbf{a})| = \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}, \tag{4.23}$$

where $1/p + 1/q = 1$ if $1 < p < \infty$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

Proof of Theorem 3.2. First of all we observe that (3.4) can be written more precisely as

$$\left\| \chi_j \mathcal{L}_m(\sigma, f)\, u \right\|_p \le \mathcal{C}_\theta \max_{|k| \le j} |f(x_k)u(x_k)| \le \mathcal{C}_\theta \|\chi_j f u\|_\infty, \quad \mathcal{C}_\theta \ne \mathcal{C}_\theta(m, f).$$

Let us first prove that inequality (3.4) implies condition (3.5) and that (3.7) implies (3.8). To this aim, with $\eta \in (0, 1/4)$ fixed, we consider the interval $[-\eta, \eta]$ and, letting $f \in C_u$, introduce a piecewise linear function F_m such that

$$F_m(x_k) = \begin{cases} 0 & \text{if } x_k \in [-\eta, \eta] \\ |f(x_k)| \mathrm{sgn}\left\{ p'_m(\sigma, x_k) \right\} \mathrm{sgn}\{-x_k\} & \text{otherwise.} \end{cases}$$

Obviously $F_m \in C_u$. Then, for $x \in [-\eta, \eta]$, we can write

$$\mathcal{L}_m(\sigma, F_m, x)\, u(x) = \sum_{\eta < |x_k| \le x_j} \frac{p_m(\sigma, x)u(x)|fu|(x_k)}{\left| p'_m(\sigma, x_k) \right| u(x_k)\, |x - x_k|},$$

since $\mathrm{sgn}\{-x_k\} = \mathrm{sgn}\{x - x_k\}$, $|x| \le \eta < |x_k|$. By (4.11) and (4.7), it follows that

$$|\mathcal{L}_m(\sigma, F_m, x)|\, u(x) \ge \mathcal{C} \frac{|p_m(\sigma, x)u(x)|}{2} \sum_{\eta < |x_k| \le x_j} \Delta x_k \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k) |fu|(x_k).$$

Hence, using (4.17), for $1 \le p < \infty$, we get

$$\|\mathcal{L}_m(\sigma, F_m)u\|_{L^p[-\eta,\eta]} \ge \mathcal{C} \sum_{\eta<|x_k|\le x_j} \Delta x_k \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k)|fu|(x_k)$$
$$= \mathcal{C} \sum_{|k|\le j} \Delta x_k \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k)|F_m u|(x_k). \tag{4.24}$$

Now, if (3.4) holds for any $f \in C_u$ and for $1 \le p < \infty$, with $\mathbf{a} = (a_{-j}, \dots, a_j)$ and $a_k = F_m(x_k)u(x_k)$ for $|k| \le j$, we have

$$\left\| \chi_j \mathcal{L}_m(\sigma, F_m)u \right\|_p \le \mathcal{C} \max_{|k|\le j} |a_k| = \mathcal{C}\|\mathbf{a}\|_\infty.$$

Then by (4.24), taking into account (4.23), it follows that

$$\sup_{\|\mathbf{a}\|_\infty \le 1} \sum_{|k|\le j} \Delta x_k \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k) a_k \le \mathcal{C}$$

i.e.

$$\sum_{|k|\le j} \Delta x_k \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k) \le \mathcal{C}. \tag{4.25}$$

Since

$$\sum_{|k|\le j} \Delta x_k \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k) \ge \int_{-a_{\theta m}}^{a_{\theta m}} \frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x)\,\mathrm{d}x,$$

taking the supremum on all $m \in \mathbb{N}$, we deduce $v^{\lambda/2-1/4-\mu} \in L^1$, which is the second condition in (3.5).

Now, proceeding in an analogous way, inequality (3.7) implies

$$\left\| \chi_j \mathcal{L}_m(\sigma, F_m)u \right\|_p \le \mathcal{C}\|\mathbf{a}\|_p$$

where $1 < p < \infty$ and $a_k = \Delta x_k |F_m u|(x_k)$, $|k| \le j$, and then

$$\left(\sum_{|k|\le j} \Delta x_k \left[\frac{\sqrt{v^\lambda \varphi}}{v^\mu}(x_k) \right]^q \right)^{1/q} \le \mathcal{C}, \quad 1/q + 1/p = 1. \tag{4.26}$$

Hence we deduce $v^{\lambda/2-1/4-\mu} \in L^q$, which is the second condition in (3.8).

Moreover, using a result in [18], inequality (3.4) implies

$$\left\| \chi_j \mathcal{L}_m(\sigma) \right\|_{u,p} = \sup_{\|\chi_j f u\|_\infty = 1} \left\| \chi_j \mathcal{L}_m(\sigma, f)u \right\|_p \ge \mathcal{C}\|\chi_j p_m(\sigma)u\|_p.$$

Then, by (4.17), we get

$$\sup_m \|\chi_j p_m(\sigma)u\|_p \ge \mathcal{C} \left(\int_{-1}^1 \left[\frac{v^\mu}{\sqrt{v^\lambda \varphi}}(x) \right]^p \mathrm{d}x \right)^{1/p},$$

i.e. the first condition in (3.5) has to be fulfilled. Analogously we can show that inequality (3.7) implies the first condition in (3.8).

Let us now prove that (3.5) and (3.8) imply (3.4) and (3.7), respectively. To this aim, setting

$$g(x) = \chi_j(x)\,|\mathcal{L}_m(\sigma, f, x)u(x)|^{p-1}\,\mathrm{sgn}\,\{\mathcal{L}_m(\sigma, f, x)\},$$
$$c_k = \Delta x_k\, p_m'(\sigma, x_k)u(x_k)\frac{\sqrt{v^\lambda\varphi}}{v^\mu}(x_k)$$

and

$$G(y) = \frac{\sqrt{v^\lambda\varphi}}{v^\mu}(y)\int_{-1}^{1}\chi_j(x)p_m(\sigma, x)g(x)u(x)\frac{\mathrm{d}x}{x-y},$$

we can write

$$\left\|\chi_j\mathcal{L}_m(\sigma, f)\,u\right\|_p^p = \sum_{|k|\le j}\Delta x_k\frac{f(x_k)u(x_k)}{c_k}G(x_k) \tag{4.27}$$

with $c_k \sim 1$, by virtue of (4.11) and $1 - x_k^2 \sim a_m^2 - x_k^2$, $|k| \le j$.

Now, considering the quantities at the right-hand side of (4.27) and taking into account (4.23), in order to prove (3.4), it suffices to show that

$$\sum_{|k|\le j}\Delta x_k\,|G(x_k)| \le \mathcal{C}\left\|\chi_j\mathcal{L}_m(\sigma, f)\,u\right\|_p^{p-1}, \quad \mathcal{C} \ne \mathcal{C}(m). \tag{4.28}$$

While, in order to prove (3.7), it suffices to show that

$$\left(\sum_{|k|\le j}\Delta x_k\,|G(x_k)|^q\right)^{1/q} \le \mathcal{C}\left\|\chi_j\mathcal{L}_m(\sigma, f)\,u\right\|_p^{p-1}, \quad q = \frac{p}{p-1},\ \mathcal{C} \ne \mathcal{C}(m). \tag{4.29}$$

In any case, the following lemma completes the proof.

Lemma 4.7. *The conditions* (3.5) *and* (3.8) *imply inequalities* (4.28) *and* (4.29)*, respectively.*

Proof. For any $R_{lm} \in \mathbb{P}_{lm}$, l a fixed integer, we set

$$\begin{aligned}\Pi(y) &= \int_{-1}^{1}\chi_j(x)\frac{p_m(\sigma, x)R_{lm}(x) - p_m(\sigma, y)R_{lm}(y)}{x-y}\frac{g(x)u(x)}{R_{lm}(x)}\,\mathrm{d}x\\ &= \mathcal{H}(\chi_j p_m(\sigma)gu, y) - p_m(\sigma, y)R_{lm}(y)\mathcal{H}\left(\chi_j\frac{gu}{R_{lm}}, y\right).\end{aligned}$$

Note that Π is a polynomial of degree at most $m + lm - 1$ and we can use inequality (4.18), since we assumed $v^{\lambda/2-1/4-\mu} \in L^q$, $1 \le q < \infty$. Hence we get

$$\begin{aligned}A_m &:= \left(\sum_{|k|\le j}\Delta x_k\,|G(x_k)|^q\right)^{1/q}\\ &= \left(\sum_{|k|\le j}\Delta x_k\left|\frac{\sqrt{v^\lambda\varphi}}{v^\mu}(x_k)\Pi(x_k)\right|^q\right)^{1/q}\\ &\le \mathcal{C}_\theta\left(\int_{-1}^{1}\chi_{\bar\theta}(y)\left|\frac{\sqrt{v^\lambda\varphi}}{v^\mu}(y)\Pi(y)\right|^q\mathrm{d}y\right)^{1/q},\end{aligned}$$

letting $\chi_{\bar\theta}$ be the characteristic function of $[-a_{\bar\theta m}, a_{\bar\theta m}]$, with $\bar\theta \in (\theta, 1)$. From the definition of Π, it follows that

$$\begin{aligned} A_m \;\le\; &\mathcal{C}_\theta \left\{ \left\| \chi_{\bar\theta} \frac{\sqrt{v^\lambda \varphi}}{v^\mu} \mathcal{H}(\chi_j p_m(\sigma) g u) \right\|_q + \left\| \chi_{\bar\theta} \frac{\sqrt{v^\lambda \varphi}}{v^\mu} p_m(\sigma) R_{lm} \mathcal{H}\left(\chi_j \frac{gu}{R_{lm}}\right) \right\|_q \right\} \\ &=: \{B_1 + B_2\}. \end{aligned} \tag{4.30}$$

Let us first estimate the term B_1 for $q = 1$. Using the inversion formula (4.20) and subsequently (4.12), we obtain

$$B_1 \le \mathcal{C}_\theta \int_{-1}^{1} \chi_{\bar\theta}(x) \left| g(x) \frac{v^\mu}{\sqrt{v^\lambda \varphi}}(x) \mathcal{H}\left(G_1 \frac{\sqrt{v^\lambda \varphi}}{v^\mu}, x\right) \right| \mathrm{d}x,$$

for some G_1 such that $|G_1(y)| \le 1$. Recalling the definition of g,

$$g(x) = \chi_j(x) \, |\mathcal{L}_m(\sigma, f, x) u(x)|^{p-1} \operatorname{sgn}\{\mathcal{L}_m(\sigma, f, x)\},$$

if $p = 1$, i.e. $|g(x)| \le 1$, we use the estimate (4.21), since one of the functions $\chi_{\bar\theta} v^{\mu-\lambda/2-1/4}$ or $G_1 v^{\lambda/2+1/4-\mu}$ is bounded and the other one belongs to $L \log^+ L$. Therefore $B_1 \le \mathcal{C}_\theta$ for $p = 1$. Otherwise, if $1 < p < \infty$, we use the Hölder inequality and Lemma 4.5, under the assumption $v^{\mu-\lambda/2-1/4} \in L^p$, obtaining

$$\begin{aligned} B_1 &\le \mathcal{C}_\theta \left(\int_{-1}^{1} |g(x)|^{\frac{p}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \left\| \frac{v^\mu}{\sqrt{v^\lambda \varphi}} \mathcal{H}\left(G_1 \frac{\sqrt{v^\lambda \varphi}}{v^\mu}\right) \right\|_p \\ &\le \mathcal{C}_\theta \left\| \chi_j \mathcal{L}_m(\sigma, f) u \right\|_p^{p-1}. \end{aligned}$$

Let us now estimate the term B_1 for $q > 1$. In this case, since by assumption (3.8), we have $-1/q < \lambda/2 + 1/4 - \nu < 1 - 1/q$, we can use the boundedness of the Hilbert transform (4.22). Hence, with $x_j < a_{\bar\theta m}$, by (4.12), we get

$$\begin{aligned} B_1 &= \left\| \chi_{\bar\theta} \frac{\sqrt{v^\lambda \varphi}}{v^\mu} \mathcal{H}(\chi_j p_m(\sigma) g u) \right\|_q \\ &\le \mathcal{C}_\theta \left\| \chi_j \frac{\sqrt{v^\lambda \varphi}}{v^\mu} p_m(\sigma) g u \right\|_q \\ &\le \mathcal{C}_\theta \|g\|_q = \mathcal{C}_\theta \left\| \chi_j \mathcal{L}_m(\sigma, f) u \right\|_p^{p-1}. \end{aligned}$$

In order to estimate the term B_2 in (4.30), taking into account Lemma 4.6, we choose $R_{lm} \in \mathbb{P}_{lm}$ such that

$$|R_{lm}(y)| \sim \sqrt{\sigma(y) \varphi(y)}, \qquad y \in [-a_{\bar\theta m}, a_{\bar\theta m}].$$

Using also (4.12), we deduce

$$B_2 \le \mathcal{C}_\theta \left\| \chi_{\bar\theta} \frac{\sqrt{v^\lambda \varphi}}{v^\mu} \mathcal{H}\left(\chi_j \frac{gu}{R_{lm}}\right) \right\|_q,$$

whence, proceeding in analogy with the estimate of B_1, our claim follows. □

By virtue of the previous lemma, recalling (4.28) and (4.29), the proof for the inequalities (3.4) and (3.7) is complete.

Finally, in order to show that the error estimate (3.6) follows from (3.4), we can proceed as was done in the proof of (3.3). We omit the details. □

Proof of Proposition 3.3. Let us set $I_k = [x_k, x_{k+1}]$ and $|I_k| = x_{k+1} - x_k = \Delta x_k$, $|k| \le j$. Using a well-known embedding theorem (see [17, p. 283]), we can write

$$\Delta x_k \, |f(x_k)|^p \le \mathcal{C} \left\{ \|f\|^p_{L^p(I_k)} + \left[(\Delta x_k)^{1/p} \int_0^{|I_k|} \frac{\omega(f,t)_{L^p(I_k)}}{t^{1+1/p}} \, \mathrm{d}t \right]^p \right\} \tag{4.31}$$

for $1 < p < \infty$, where

$$\omega(f,t)_{L^p(I_k)} = \sup_{0<h\le t} \left(\int_{I_k} \left| f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \right|^p \mathrm{d}x \right)^{1/p}.$$

Since $|I_k| \sim \varphi(x_k)/m$, by making a change of variables in the integral in (4.31) and recalling the properties of the ordinary modulus of smoothness, we get

$$(\Delta x_k)^{1/p} \int_0^{|I_k|} \frac{\omega(f,t)_{L^p(I_k)}}{t^{1+1/p}} \, \mathrm{d}t \sim \frac{1}{m^{1/p}} \int_0^{1/m} \frac{\omega\left(f, t\varphi(x_k)\right)_{L^p(I_k)}}{t^{1+1/p}} \, \mathrm{d}t.$$

Moreover, taking into account that $u(x) \sim u(x_k)$ for $x \in I_k$, we have

$$u(x_k)\omega(f, t\varphi(x_k))_{L^p(I_k)} \sim \sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p(I_k)}.$$

Then inequality (4.31) becomes

$$\begin{aligned} &\Delta x_k \, |f(x_k)u(x_k)|^p \\ &\le \mathcal{C} \left\{ \|fu\|^p_{L^p(I_k)} + \left[\frac{1}{m^{1/p}} \int_0^{1/m} \frac{\sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p(I_k)}}{t^{1+1/p}} \, \mathrm{d}t \right]^p \right\}. \end{aligned}$$

Using the Minkowski inequality, it follows that

$$\begin{aligned} &\left(\sum_{|k|\le j} \Delta x_k \, |f(x_k)u(x_k)|^p \right)^{1/p} \\ &\le \mathcal{C} \left\{ \|fu\|_{L^p[-x_j,x_j]} + \frac{1}{m^{1/p}} \left(\sum_{|k|\le j} \left[\int_0^{1/m} \frac{\sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p(I_k)}}{t^{1+1/p}} \, \mathrm{d}t \right]^p \right)^{1/p} \right\} \\ &\le \mathcal{C} \left\{ \|fu\|_{L^p[-x_j,x_j]} + \frac{1}{m^{1/p}} \int_0^{1/m} \frac{\left(\sum_{|k|\le j} \left[\sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p(I_k)} \right]^p \right)^{1/p}}{t^{1+1/p}} \, \mathrm{d}t \right\} \\ &\le \mathcal{C} \left\{ \|fu\|_{L^p[-x_j,x_j]} + \frac{1}{m^{1/p}} \int_0^{1/m} \frac{\sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p[-x_j,x_j]}}{t^{1+1/p}} \, \mathrm{d}t \right\}, \end{aligned}$$

whence we deduce inequality (3.9), since $t \le 1/m$ and for some constant B we have $[-x_j, x_j] \subset \left[-1 + \mathrm{B}h^{\frac{1}{\alpha+1/2}}, 1 - \mathrm{B}h^{\frac{1}{\alpha+1/2}}\right] =: \mathcal{I}_h$, by (2.8), and then

$$\sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p[-x_j,x_j]} \le \sup_{0<h\le t} \left\| \Delta_{h\varphi}(f)\, u \right\|_{L^p(\mathcal{I}_h)} = \Omega_\varphi(f,t)_{u,p}. \quad \square$$

Proof of Corollary 3.4. Let P_M, with M as in (2.12), be a polynomial of quasi best approximation for $f \in L^p_u$. Then we can write

$$\begin{aligned} &\left\| \left[f - \chi_j \mathcal{L}_m(\sigma, f) \right] u \right\|_p \\ &\quad \le \|(f - P_M)\, u\|_p + \left\| \left[P_M - \chi_j \mathcal{L}_m(\sigma, P_M) \right] u \right\|_p + \left\| \chi_j \mathcal{L}_m(\sigma, f - P_M)\, u \right\|_p \\ &\quad \le \mathcal{C} E_M(f)_{u,p} + \left\| \left[P_M - \chi_j \mathcal{L}_m(\sigma, P_M) \right] u \right\|_p + \left\| \chi_j \mathcal{L}_m(\sigma, f - P_M)\, u \right\|_p \\ &\quad =: \mathcal{C} E_M(f)_{u,p} + I_1 + I_2. \end{aligned}$$

For the term I_1, using the error estimate (3.6) and the Nikolskii inequality (4.15), we get

$$I_1 \le \mathcal{C}_\theta \mathrm{e}^{-\mathrm{A}M^\gamma} \|P_M u\|_\infty \le \mathcal{C}_\theta \mathrm{e}^{-\mathrm{A}M^\gamma} \|P_M u\|_p \le \mathcal{C}_\theta \mathrm{e}^{-\mathrm{A}M^\gamma} \|f u\|_p.$$

Since, for the term I_2 we can use inequality (3.7), Proposition 3.3 and the inequality (see [17, p. 280])

$$\int_0^{1/M} \frac{\Omega^r_\varphi(f - P_M, t)_{u,p}}{t^{1+1/p}}\, \mathrm{d}t \le \mathcal{C} \int_0^{1/M} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+1/p}}\, \mathrm{d}t,$$

our claim follows. $\square$

Proof of Theorem 3.5. We can use the same arguments of the proof of Theorem 3.2, noting that, since the equivalence $a_m^2 - x^2 \sim 1 - x^2$ does not hold for any $x \in [-a_m, a_m]$, we will use (4.9) and (4.11). In this case we will also use inequality (4.3), reducing the norm to the interval $[-a_{sm}, a_{sm}]$, $s \ge 1$. Therefore in this case (4.27) is replaced by

$$\|\mathcal{L}_m(\sigma, f)\, u\|^p_{L^p[-a_{sm},a_{sm}]} = \sum_{|k|\le j} \Delta x_k \frac{f(x_k)u(x_k)}{\bar{c}_k} \bar{G}(x_k)$$

$$\bar{c}_k = \Delta x_k p_m'(\sigma, x_k) u(x_k) v^{\lambda/2-\mu}(x_k) \left| a_m^2 - x_k^2 \right|^{1/4} \sim 1$$

and

$$\bar{G}(y) = v^{\lambda/2-\mu}(y) \left| a_m^2 - y^2 \right|^{1/4} \int_{-a_{sm}}^{a_{sm}} p_m(\sigma, x) g(x) u(x) \frac{\mathrm{d}x}{x - y}, \qquad s \ge 1,$$

where

$$g(x) = |\mathcal{L}_m(\sigma, f, x) u(x)|^{p-1} \operatorname{sgn}\{\mathcal{L}_m(\sigma, f, x)\}.$$

Proceeding as in the previous proofs of Theorem 3.2 and Lemma 4.7, we can show that (3.12) and (3.14) follow from (3.13) and (3.15), respectively. The restriction $p < 4$ is due to the fact that the factor $\left| a_m^2 - \cdot^2 \right|^{1/4}$ has to belong to L^q, with $1/p + 1/q = 1$, and its reciprocal to L^p. We omit these details and prove that (3.12) and (3.14) imply (3.13) and (3.15), respectively.

So, in analogy with (4.25), inequality (3.12) implies

$$\sum_{|k|\le j} \Delta x_k v^{\lambda/2-\mu}(x_k) \left| a_m^2 - x_k^2 \right|^{1/4} \le \mathcal{C}$$

and then

$$\int_{-a_{\theta m}}^{a_{\theta m}} v^{\lambda/2-\mu}(x) \left|a_m^2 - x^2\right|^{1/4} \mathrm{d}x \le \mathcal{C},$$

whence we deduce (3.13), taking the supremum on $m \in \mathbb{N}$.

Analogously, as in (4.26), inequality (3.14) implies

$$\left(\sum_{|k|\le j} \Delta x_k \left[v^{\lambda/2-\mu}(x_k) \left|a_m^2 - x_k^2\right|^{1/4} \right]^q \right)^{1/q} \le \mathcal{C}, \quad 1/p + 1/q = 1,$$

whence condition $\lambda/2 - \mu > -1/q$ in (3.15) follows. Finally, using a result in [18], we have

$$\|\mathcal{L}_m(\sigma)\|_{u,p} = \sup_{\|\chi_j f u\|_\infty} \|\mathcal{L}_m(\sigma, f)\, u\|_p \ge \mathcal{C}\|p_m(\sigma)u\|_p \ge \mathcal{C}\|v^{\mu-\lambda/2}|a_m^2 - \cdot^2|^{-1/4}\|_p$$

whence we get the remaining conditions in (3.15), i.e. $\mu - \lambda/2 > -1/p$ and $p < 4$. □

Proof of Corollary 3.6. For any $P_{m-1} \in \mathcal{P}_{m-1}$, by definition, we have $\mathcal{L}_m(\sigma, P_{m-1}) = P_{m-1}$. Hence, under the assumptions (3.15), for $1 < p < 4$, from (3.14), it follows that

$$\|P_{m-1}u\|_p \le \mathcal{C}_\theta \left(\sum_{|k|\le j} \Delta x_k \,|P_{m-1}(x_k)u(x_k)|^p \right)^{1/p}.$$

On the other hand, the converse inequality can be proved by the same arguments used in the proof of Lemma 4.4, taking into account that, by Lemma 4.6, there exists a polynomial $R_{lm} \in \mathbb{P}_{lm}$, for some fixed l, such that

$$R_{lm}(x) \sim u(x), \quad x \in [-a_{\bar\theta m}, a_{\bar\theta m}],\ a_{\theta m} \le x_j < a_{\bar\theta m}.$$

Then inequality (4.18) holds also with v^β replaced by u. □

Proof of Theorem 3.7. We can proceed in analogy with the proof of Theorem 3.1. Taking into account inequality (4.3), we set

$$\left\|\mathcal{L}_{m,2}^*(\sigma)\right\|_{u,\infty} = \sup_{\|\chi_j f u\|_\infty = 1} \left\|\chi_s \mathcal{L}_{m,2}^*(\sigma, f)\, u\right\|_\infty$$

where χ_s is the characteristic function of $[-a_{sm}, a_{sm}]$, $a_{sm} = a_{sm}\left(\sqrt{\sigma}\right)$, $s \ge 1$.

Using (4.11), we have

$$\begin{aligned}
\left\|\mathcal{L}_{m,2}^*(\sigma)\right\|_{u,\infty} &= \max_{x\in[-a_{sm},a_{sm}]} \sum_{|k|\le j} \left| \frac{a_m^2 - x^2}{a_m^2 - x_k^2} \right| \frac{|\ell_k(\sigma, x)|\, u(x)}{u(x_k)} \\
&\sim \max_{x\in[-a_{sm},a_{sm}]} \left| p_m(\sigma, x) u(x)(a_m^2 - x^2)\right| \sum_{|k|\le j} \frac{\Delta x_k}{(a_m^2 - x_k^2)^{3/4} v^{\mu-\lambda/2}(x_k)|x - x_k|} \\
&=: \max_{x\in[-a_{sm},a_{sm}]} S(x).
\end{aligned}$$

We note that $|a_m^2 - x^2| \le \mathcal{C}(1 - x^2)$, since if $s > 1$ we have $a_{sm}^2 - a_m^2 \sim 1 - a_{sm}^2$. While, concerning the factor $(a_m^2 - x_k^2)$, by Lemma 4.2, we have $a_m^2 - x_k^2 \sim \mathcal{C}_\theta(1 - x_k^2)$ for $|k| \le j$. Thus, using (4.9), we deduce

$$S(x) \le \mathcal{C}_\theta v^{\mu-\lambda/2+3/4}(x) \sum_{|k|\le j} \frac{\Delta x_k}{v^{\mu-\lambda/2+3/4}(x_k)|x - x_k|}, \quad x \in [-a_{sm}, a_{sm}].$$

Now, if $x_j \le |x| \le a_{sm}$, we have $v^{\mu-\lambda/2+3/4}(x) \le v^{\mu-\lambda/2+3/4}(x_k)$, since $\mu - \lambda/2 + 3/4 \ge 0$, by (3.17), and then

$$S(x) \le \mathcal{C}_\theta \sum_{|k|\le j} \frac{\Delta x_k}{|x-x_k|} \le \mathcal{C}_\theta \log m.$$

Otherwise, if $|x| \le x_j$, using well known arguments (see, e.g., [9, p. 243, formula (4.1.13)]) we have

$$S(x) \le \mathcal{C}_\theta v^{\mu-\lambda/2+3/4}(x) \sum_{|k|\le j} \frac{\Delta x_k}{v^{\mu-\lambda/2+3/4}(x_k)|x-x_k|} \le \mathcal{C}_\theta \log m$$

since the zeros x_k, $|k| \le j$, are arc sine distributed by (4.8) and $0 \le \mu - \lambda/2 - 1/4 \le 1$ by assumption (3.17).

In order to prove that inequality (3.16) implies (3.17), it suffices to compute $S(x)$ in the point $(x_{j-1} + x_j)/2$ and in some arbitrary $\bar{x} \in [-1/2, 1/2]$, considering the sum for $a_{\theta_0 m} \le |x_k| \le a_{\theta m}$, with $0 < \theta_0 \ll \theta < 1$, in analogy with the proof of Theorem 3.1.

We omit the proof of (3.18), which is similar to that of (3.3), taking (2.17) into account. □

Proof of Theorem 3.8. We can use arguments analogous to those in the proof of Theorem 3.2. Therefore we will only show that conditions (3.20) imply inequality (3.19) and vice-versa, omitting the rest of the proof.

Using the restricted range inequality (4.3), with $s > 1$, for any $f \in C_u$, in analogy with (4.27), we can write

$$\|\mathcal{L}_m(\sigma, f)u\|_p^p \le \|\mathcal{L}_m(\sigma, f)u\|_{L^p[-a_{sm}, a_{sm}]}^p = \sum_{|k|\le j} \Delta x_k \frac{f(x_k)u(x_k)}{c_k^*} G^*(x_k)$$

$$c_k^* = \Delta x_k p_m'(\sigma, x_k)u(x_k)v^{\lambda/2-\mu}(x_k)\left|a_m^2 - x_k^2\right|^{1/4} \sim 1$$

and

$$G^*(y) = v^{\lambda/2-\mu}(y)\left|a_m^2 - y^2\right|^{-3/4} \int_{-a_{sm}}^{a_{sm}} p_m(\sigma, x)\left|a_m^2 - x^2\right| g^*(x)u(x)\frac{\mathrm{d}x}{x-y},$$

where $s \ge 1$ and

$$g^*(x) = \left|\mathcal{L}_{m,2}(\sigma, f, x)u(x)\right|^{p-1} \operatorname{sgn}\left\{\mathcal{L}_{m,2}(\sigma, f, x)\right\}.$$

Then, in order to prove (3.19), it suffices to show that

$$\sum_{|k|\le j} \Delta x_k \left|G^*(x_k)\right| \le \mathcal{C}\|g^*\|_p, \quad \mathcal{C} \ne \mathcal{C}(m),\ 1 \le p < \infty.$$

This can be done proceeding in analogy with the proof of Lemma 4.7, under the assumptions (3.20) and recalling that $a_m^2 - x_k^2 \sim 1 - x_k^2$ for $|k| \le j$, $|a_m^2 - x^2| \le \mathcal{C}(1-x^2)$ for $x \in [-a_{sm}, a_{sm}]$.

Moreover, inequality (3.19) implies conditions (3.20). In fact, in this case (4.25) becomes

$$\mathcal{C} \ge \sum_{|k|\le j} \Delta x_k v^{\lambda/2-\mu-3/4}(x_k) \ge \int_{-a_{\theta m}}^{a_{\theta m}} v^{\lambda/2-\mu-3/4}(x)\mathrm{d}x$$

whence we get $v^{\lambda/2-\mu-3/4} \in L^1$, taking the supremum on $m \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned}\mathcal{C} &\geq \left\|\mathcal{L}^*_{m,2}(\sigma)\right\|_{u,p} = \sup_{\|\chi_j f u\|_\infty=1} \left\|\mathcal{L}^*_{m,2}(\sigma, f)\, u\right\|_p \\ &\geq \mathcal{C}\|p_m(\sigma)(a_m^2 - \cdot^2)u\|_p \\ &\geq \mathcal{C}\|v^{\mu-\lambda/2}|a_m^2 - \cdot^2|^{3/4}\|_{L^p[-a_m,a_m]}\end{aligned}$$

and taking the supremum on $m \in \mathbb{N}$, we get $v^{\mu-\lambda/2+3/4} \in L^p$, $1 \leq p < \infty$.

We omit the proof of the second part of the theorem. □

Proof of Theorem 3.9. By definition, the norm in (3.23) is

$$\left\|\mathcal{L}^*_{m,2}(\sigma, f)\right\|_{W^p_r(u)} = \left\|\mathcal{L}^*_{m,2}(\sigma, f)\, u\right\|_p + \left\|\left[\mathcal{L}^*_{m,2}(\sigma, f)\right]^{(r)} \varphi^r u\right\|_p.$$

Concerning the first term on the right-hand side, by Theorem 3.8 and Proposition 3.3, for m sufficiently large, we have

$$\left\|\mathcal{L}^*_{m,2}(\sigma, f)\, u\right\|_p \leq \mathcal{C}_\theta \left\{\|fu\|_p + \frac{\|f^{(r)}\varphi^r u\|_p}{m^r}\right\}$$

for $f \in W^p_r(u)$, $r \geq 1$ and $1 < p < \infty$. Moreover, since

$$\left\|\mathcal{L}^*_{m,2}(\sigma, f)^{(r)}\, \varphi^r u\right\|_p \leq \|f^{(r)}\varphi^r u\|_p + \left\|\left[f - \mathcal{L}^*_{m,2}(\sigma, f)\right]^{(r)} \varphi^r u\right\|_p,$$

to complete the proofs of (3.23) and (3.24) it suffices to estimate the norm

$$\left\|\left[f - \mathcal{L}^*_{m,2}(\sigma, f)\right]^{(r)} \varphi^r u\right\|_p$$

for $f \in W^p_r(u)$ and $f \in W^p_s(u)$, $s > r$, respectively.

For the first case, we recall that, letting $P_m \in \mathbb{P}_m$ be a polynomial of quasi best approximation, we have (see [14, Theorem 3.7])

$$\left\|P_m^{(r)}\varphi^r u\right\|_p \leq \mathcal{C}\, m^r\, \omega^r_\varphi\left(f, \frac{1}{m}\right)_{u,p} \leq \mathcal{C}\|f^{(r)}\varphi^r u\|_p,$$

and, moreover, for any $Q_m \in \mathbb{P}_m$, the following Bernstein inequality

$$\left\|q_m^{(r)}\varphi^r u\right\|_p \leq \mathcal{C}\, m^r\ \|q_m u\|_p \tag{4.32}$$

holds (see [14,24]). Hence, by the Jackson inequality (2.4) and since

$$\left\|\left[f - \mathcal{L}^*_{m,2}(\sigma, f)\right] u\right\|_p \leq \frac{\mathcal{C}_\theta}{m^r}\left\{\|fu\|_p + \|f^{(r)}\varphi^r u\|_p\right\}, \tag{4.33}$$

we get

$$\begin{aligned}\left\|\left[f - \mathcal{L}^*_{m,2}(\sigma, f)\right]^{(r)} \varphi^r u\right\|_p &\leq \left\|[f - P_m]^{(r)}\varphi^r u\right\|_p + \left\|\left[P_m - \mathcal{L}^*_{m,2}(\sigma, f)\right]^{(r)}\varphi^r u\right\|_p \\ &\leq \mathcal{C}_\theta\|f^{(r)}\varphi^r u\|_p + \mathcal{C}_\theta m^r \left\{\|[P_m - f]\, u\|_p \right. \\ &\qquad \left. + \left\|\left[f - \mathcal{L}^*_{m,2}(\sigma, f)\right] u\right\|_p\right\} \\ &\leq \mathcal{C}_\theta\|f\|_{W^p_r(u)}.\end{aligned}$$

Concerning inequality (3.24), we assume $f \in W_s^p(u)$, $s > r$, and set $Q_{m+1} = L_{m,2}^*(\sigma, f)$. By (3.22) we have

$$f - Q_{m+1} = \sum_{k=0}^{\infty} \left(Q_{2^{k+1}(m+1)} - Q_{2^k(m+1)} \right) \quad \text{in } (-1, 1).$$

Then, by the Bernstein inequality (4.32), we get

$$\begin{aligned} \left\| (f - Q_{m+1})^{(r)} \varphi^r u \right\|_p &\le \sum_{k=0}^{\infty} \left\| \left(Q_{2^{k+1}(m+1)} - Q_{2^k(m+1)} \right)^{(r)} \varphi^r u \right\|_p \\ &\le \mathcal{C}_\theta \sum_{k=0}^{\infty} \left(2^{k+1} m \right)^r \left\| \left[Q_{2^{k+1}(m+1)} - Q_{2^k(m+1)} \right] u \right\|_p , \end{aligned}$$

whence, by (4.33), it follows that

$$\left\| (f - Q_{m+1})^{(r)} \varphi^r u \right\|_p \le \mathcal{C}_\theta \sum_{k=0}^{\infty} \left(2^{k+1} m \right)^{r-s} \| f \|_{W_s^p(u)} \le \frac{\mathcal{C}_\theta}{m^{s-r}} \| f \|_{W_s^p(u)},$$

which completes the proof. □

We omit the proof of Corollary 3.10, which follows from Theorem 3.8 using arguments analogous to those in the proof of Corollary 3.6.

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