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ORIGINAL PAPER

Comparative analysis between three theoretical approaches through empirical experiences at university level

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Abstract
Research into mathematics education at university level includes a wide range of theoretical approaches. This poses considerable challenges to researchers in terms of understanding and harmonizing the compatibility and commensurability of those approaches. The research community has already problematised and studied these challenges using networking theories. The networking theories framework is taken as a starting point in this study to contrast different approaches and to broaden the comparison of different frameworks. In particular, three case studies framed in the Action, Process, Object, Schema Theory, in the Problem-Solving approach, and in the Anthropological Theory of the Didactic are analysed. The differences and possible similarities between the three with regard to the research questions addressed, their objects of study, their empirical bases, as well as their research ends are considered. The analysis offers an insight into the potential for collaboration and the networking of theories in the field of university mathematics education.

1 Introduction

Research into mathematics education at university level is incipient. However, a considerable number of scientific papers have been published in the field. Although it is true that Klein (2016) addressed aspects related to university education providing a new perspective, the International Commission on Mathematical Instruction (ICMI) Study of 1998 on the teaching and learning of mathematics in university institutions is considered to be one of the pioneering studies in the field (Holton, 2001). The creation of the Delta Conferences on the teaching and learning of Undergraduate Mathematics and Statistics in 1997; the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education in 1999, and a specific group for mathematics education at university level at the seventh Congress of the European Research in Mathematics Education (CERME 7) in Rzeszów (Poland) in 2011 were also important highlights. Relevant works in this research field include the special issues published in ZDM on the *Impact of University Teacher Education Programs on Teacher Change and Mathematics Teaching Practice* (2017) and in *Exploring and Strengthening University Mathematics Courses for Secondary Teacher Preparation* (2023) at the intersection of undergraduate mathematics and teacher education. Finally, the launch in 2015 of the International Journal of Research of Undergraduate Mathematics Education (IJRUME) dedicated to the research of teaching and learning processes at the post-secondary level shows that the field is well-established.

The growing complexity of mathematics education at university level, led to an increase in theories and approaches used to model different phenomena related to teaching and learning processes. This diversity of approaches, gave rise to the need, not only at university level, but in the entire research community in mathematics education, to find a systematic manner of addressing different theories in research (Bikner-Ahsbahs, 2016; Bikner-Ahsbahs et al., 2014) while sharing a common problem.

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One of the first researchers to deal with this diversity of theories was Bauersfeld. He proposed to analyse a teaching and learning situation using different theoretical approaches (Bauersfeld, 1992). The work of Cobb et al. (1996) is also relevant as it was the first to explicitly compare two frameworks: constructivism and activity theory.

Bikner-Ahsbahs and Prediger (2006) and Prediger et al. (2008) laid the foundations for a systematic approach to address the networking of theories based on the following four main principles: (1) regarding the diversity of theories as a form of scientific richness, (2) acknowledging the specificity of theories, (3) looking for the connectivity of theories and research results and (4) developing theory and theory use to inform practice.

The publication of special issues in journals devoted this question such as the ones published in ZDM in 2006 and 2008, (ZDM, vol. 38(1) and 40(2)) show its importance in the research community. Regarding the level of undergraduate mathematics, the introduction to the papers of TWG14 of CERME 12 (González-Martín et al., 2022) also highlights this diversity, and stresses the relevance of some of the approaches such as the Action, Process, Object, Schema (APOS) Theory, the Anthropological Theory of the Didactic (ATD), or the problem-solving approach.

Although several studies analyse these three theoretical frameworks through the lens of networking theories, they typically compare them in pairs without carrying out a comprehensive analysis of all three of them. This paper presents three case studies framed in different theoretical approaches that have widely been used in university mathematics education: APOS theory, the problem-solving approach, and the ATD. This study aims to characterise different aspects, such as their objects of study, empirical bases, research questions, and research ends in order to study potential collaborations in the research field.

The studies analysing the three approaches in pairs were taken as a starting point. First, the work of Bosch et al. (2017) exploring the possible networking of APOS and the ATD by regarding the theories as research praxeologies was considered. The second contribution taken into account was the one by Rodríguez et al. (2008) addressing the analysis of the ATD and problem-solving. In this work, Rodríguez et al. (2008) considered the notion of "metacognition" as the object of study. This notion, initially developed in problem-solving, allowed the authors to address and describe new research phenomena when analysing it through the ATD framework. Bosch et al. (2017) and Rodríguez et al. (2008) focus on the theoretical foundations of the approaches and their possible connections or disconnections at the conceptual level. In order to complement this initial line of research, the possible networking of the three approaches at the level of the research tasks they explore, and how they do so, is analysed. In other words, the level of the "research tasks" is referred to so as to analyse to what extent the different approaches have a common research programme with respect to research questions, research ends, empirical bases and objects of study.

2 Methodology and research questions

Summaries of case studies framed in APOS, problem-solving, and the ATD are presented in order to explore the possible networking at the task level, thus continuing and extending the dialogue between theories started by Bosch et al. (2017) and Rodríguez et al. (2008). These case studies are regarded as relevant examples and as empirical material to explore diverse aspects of each approach.

This study is framed within the Networking Theories approach. It focuses primarily on comparing and contrasting strategies (see Fig. 1), and seeks to explore the similarities and differences observed in the approaches. While this is the general framework, the specific methodology employed in the study is adapted from Bosch et al. (2017), in which the analysis of these similarities and differences is made in terms of research praxeologies. In Bosch et al. (2017), the praxeological model is used to analyse the research activity: the different situations are examined with regard to the tasks and techniques used that belong to the *praxis* and the *logos* blocks made up of technology and theory. This study aims to

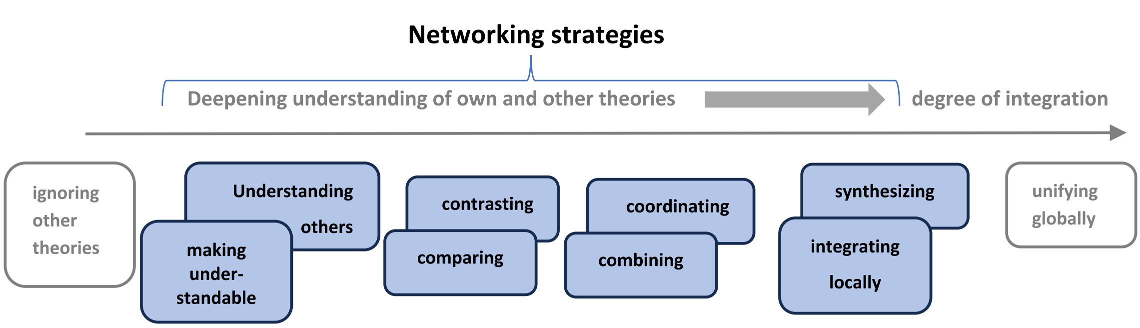


Fig. 1 Networking strategies (Prediger et al., 2008, p. 170)

expand this approach by explicitly identifying the research questions, research ends, empirical bases and objects of study considered in the sense of Gascón and Nicolás (2019). This may be seen as an extension of the method used in Bosch et al. (2017). In said study, theories are modelled as research praxeologies since the definition of research questions and the empirical basis associated can be considered as types of tasks and techniques (respectively), while the research ends, and the delimitation and definition of the objects of study are related to the technologies and theories.

First of all, three recent works framed in each of the three approaches were selected as the initial empirical material of this study. They were chosen because they were proposed as representative research studies in each theory or approach, and thus provide information about the research questions, objects of study, empirical bases, and research ends that characterise them. Secondly, the fundamental ideas of the study were discussed as well as the theoretical approaches concerning the problems posed. All these activities are undertaken with the aim of examining the commensurability of the approaches with the possibility of coordinating and combining theories (see Fig. 1) and approaches in future projects.

The research questions formulated are the following:

- RQ1: Which are the research questions addressed by empirical studies framed within APOS, problem-solving and the ATD? What are the issues raised? Which are the more or less explicit research ends behind these questions?
- RQ2: Which is the empirical material considered in the studies framed within APOS theory, problem-solving and the ATD? What is taken as the object of study?
- RQ3: Given the research ends, the objects of study and the empirical material considered in each approach, to what extent are research programmes commensurable? Is undertaking collaborative or coordinated work possible?

3 Case studies

3.1 Case study 1: APOS

3.1.1 Introduction

There is a growing body of research dealing with the teaching and learning of multivariable calculus, including its definition and geometric representation, and its differential and integral calculus. A survey can be found in Martínez-Planell and Trigueros (2021). However, research addressing the optimisation of two-variable functions is scarce. Given the importance of optimisation to applications in science, technology, engineering, mathematics, economics, and other fields, it represents a significant gap which this research aims to bridge. This study was motivated by repeated classroom observations of student difficulties when optimising functions on compact domains. While this study mainly focuses on the didactics of multivariable optimisation, it contributes to the understanding of the notion of Schema, as interpreted in APOS theory, and the recently defined notion of types of transformations between Schema components. Very little research in the literature explores these theoretical ideas.

3.1.2 Theoretical framework: schemas in APOS

Although APOS theory (Arnon et al., 2013) was employed in this study, only the Schemas part is described, since it is the part of the theory is used. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas that are interconnected in a way that allows an individual to recognise when a problem situation falls under the scope of the Schema. Another important idea in APOS is that of genetic decomposition (GD). This is a model of how students may construct a particular mathematical notion. When modelling the development of student understanding with Schemas, the researcher chooses what component structures to consider, and describes the relations that interconnect these components, and the types of relations students construct. A correspondence relation is constructed when the individual notices that in some circumstances one component structure is used regarding another but is not yet able to explain or justify the relation. Transformation relations are constructed when the individual groups different component structures and justifies the grouping in terms of some of these structures. When a relation between components is consistently used in different problem situations involving the components, in such a way that it seems that in certain situations one structure interchangeably be used for the other, then it is said that the relation between components is a conservation relation.

In this GD, the following components and relations are considered:

Components: Set topology (s) and one variable function (1f) as Schemas, two-variable function (2f), partial derivatives (∂f), critical points and extrema of two-variable functions (cp), and second partial derivatives (2∂f) as Processes. Lagrange multipliers were not included as a component because they deserve a separate study.

Relations between components: eight relations were underscored. The first two emphasise the relation between set topology and two-variable function (s-2f) and the same relation when used to compute a partial (s–2f–∂f). The next four are shown in the application of optimisation theorems (cp-∂f: critical point theorem; cp-2∂f: second derivative test; cp-2f: the use of ad-hoc methods; s-2f-cp: compact-continuous theorem); and the final two relations refer to separating problems into interior and boundary (s-2f-1f) and using one-variable function optimisation in the boundary (cp-1f-s). Due to space restrictions, only one of them is described.

1. Critical points, one-variable function, and set topology (cp-1f-s) evidenced by the conversion of $f|\partial D$ (f restricted to the boundary) to g:[a, b] → R with g continuous, identification and evaluation of critical points on the boundary of the closed set, and comparing extrema found on the boundary with extrema found in the interior of the domain. That is, this relation shows an understanding of the computations involved in changing an optimisation problem on a boundary of a set in R^2 to a one-variable optimisation problem, when possible.

The object of study is the students' construction of knowledge of two-variable function optimisation. The following research questions were developed: (Q1) How does the schema development inferred from the students compare to that modelled by the GD? (Q2) How do the constructions described in the GD become evident in the students' work?

3.1.3 Methodology for the APOS case study

A GD was designed for two-variable function optimisation. A total of 22 engineering students from two introductory multivariable calculus courses were chosen to participate. The students in each course were chosen according to their performance in such a manner that three of them were over-average, five were average, and three were under-average (as determined by their teacher). This variety of students was chosen to enable deducing a wide range of different mental constructions from observations. Two experienced teachers were in charge of the course. The students participated in semi-structured interviews that took place approximately 1 week after completing the course. The interviews lasted for about 1 h. During the interviews, the students solved problems, explaining their thoughts as they went along. The interviewer asked questions for clarification whenever necessary. The interviews were recorded, transcribed, translated into English, analysed individually, and a group discussions was held until a consensus was reached.

An interview instrument was designed to provide information on the constructions in the GD. It consisted of 17 questions distributed among five problems. Only one question is discussed.

1. Let $f(x,y) = x^2 - 2x + y^2 + 2y$. Let $S_1 = \{(x,y) : x^2 + y^2 \leq 9\}$, and $S_2 = \{(x,y) : x^2 + y^2 < 9\}$. Discuss the strategy (but do not carry it out!) you would use to find the maximum and minimum values of f on each of these sets.

3.1.4 Results and discussion of the APOS case study

Of the 22 participating students, 11 constructed mostly correspondence relations or no relations between components (this is called the intra-stage of Schema development; Arnon et al., 2013). Four students started to construct transformation relations and interrelated most components (inter-stage of Schema development). Seven students interrelated all components mainly with conservation relations (trans-stage of Schema development).

R2 is an example of a student constructing only correspondence relations. In question 1:

R2: … if I solve the equations $f_x = 0$ and $f_y = 0$, then they will be the critical points, now we should check which of them are inside S_1 I mean $x^2 + y^2 \leq 9$, hmm and which of them are on the boundary of S_1 I mean to satisfy the equation $x^2 + y^2 = 9$ …

It is observed that R2 relates set topology and function (s-2f): "I first need to find the critical points of f inside S_1 [interior of a set] … then I need to find the critical points $f(x,y)$ on the boundary of f." He also relates these components to partial derivatives (s-2f-∂f) and critical points (cp-∂f): "for the inside of the circle, hmm if I solve equations $f_x = 0$ and $f_y = 0$, then they will be the critical points". It is noted that he suggests that after setting both partials equal to zero "now we should check which of them are inside S_1… and which of them are on the boundary of S_1." He does not establish a difference for finding extrema in the interior and in the boundary of disk S_1. But there is a difference, which means he will not be able to justify this. Hence, in this problem s-2f-∂f and cp-∂f are correspondence rather than transformation relations. The student seems to compute and set partial derivatives equal to zero as a memorised procedure.

In short, when considering all the interview questions, and analysing all the responses of R2's regarding the components and relations in the GD, it is observed that the student did not establish any relations or, at the most, constructed correspondence relations.

At the other extreme, some students, like A3, constructed conservation relations between all Schema components.

A3: [Question 1] … I first find the critical points of f in the interior of S_1, then I need to evaluate f in the boundary of S_1. Finally, I need to compare the values of function f at all the points I have already found to see which of them is the biggest value for f … and… which of them is min.

It is worthy of note that A3 has recognised the need to separate a problem into interior and boundary ("I first find the critical points of f in the interior of S_1, then I need to evaluate f in the boundary of S_1"; s-2f-1f). Next, consistent with a transformation relation in this question, he explains the need to separate the problem by applying different techniques in each case:

A3: … on the circle $x^2 + y^2 = 9$, I can use Lagrange multipliers, hmm or I can change function $f(x, y)$ by substituting $y^2 = 9 - x^2$. Then I will have a one-variable function based on only x on the interval $[-3,3]$. Hm both ways will give me the same answer… The next one $[S_2]$ does not contain the boundary, so I only need the values of f at the critical points which are inside the circle. Then, I should compare the values for finding max and min, hmm in S_2, if I have a critical point then I can use the second derivative test to see if it is a local min or max or a saddle point.

He was consistent throughout the interview in his application and justification of all relations. Hence, the relations were conservation relations. Seven students showed the same type of constructions as A3.

To summarise, 11 of 22 students showed missing or only correspondence relations. This underscores that constructing the optimisation Schema is challenging for students. It is commonly taken for granted that students understand the main ideas and key role of topology in optimisation, but evidence shows otherwise. This study stresses the importance of explicitly discussing the topology of the domain set during instruction. Further, the study contributes examples of correspondence, transformation, and conservation relations, helping clarify these ideas.

3.2 Case study 2: problem solving approach

3.2.1 Introduction

Several authors have explored problem-solving by formulating a research approach around the following essential questions: *How to solve a mathematical problem?* and *How can teachers guide students in constructing strategies to solve problems?* (Lester, 1994; Liljedahl & Cai, 2021; Rodríguez et al., 2008; Schoenfeld, 2007). To address and answer these questions, which are still considered today, numerous researchers have focused on metacognitive aspects. Others have analysed cognitive processes, identifying different kinds of reasoning involved. Still others have integrated socio-cultural perspectives to analyse the teaching and learning activities around problem-solving (Holton, 2001; Schoenfeld, 2007).

Looking at problem solving as a cognitive activity, several studies have focused on what a solver does during resolution and explored heuristics (Liljedahl & Cai, 2021). With the aim of further deepening the understanding of resolution processes, both at a strategic and cognitive level, this study presents the analysis of a problem-solving situation. It was conducted at university level, focusing on backward reasoning (BR) heuristic, which adopts a central role in advanced mathematical thinking where abstract processes prevail (Tall, 2002).

3.2.2 Theoretical framework: heuristics and reasoning involved in problem-solving

From a cognitive perspective, it is observed that to effectively tackle problem-solving tasks, beyond forward reasoning (FR), typical of deduction, other types of reasoning like BR, can be used. BR is one of the methods of reasoning that takes on a significant role in discovery phases. Starting from the conclusion of the problem (that is, what needs to be found, shown, constructed), using BR means proceeding through logical correspondences to the initial premises until reaching a known result. Working on BR holds significant potential for enhancing mathematical argumentation, inquiry, and proof processes (Hintikka & Remes, 1974; Polya, 1945; Schoenfeld, 1992).

Four epistemic dimensions of BR were identified. They shape the BR epistemic dimension model and play a crucial role in identifying, understanding, and interpreting BR in resolution processes, distinguishing it from the other ways of reasoning involved (Barbero et al., 2020; Beaney, 2018). These dimensions are the following:

- *Search for cause-effect relationships*. BR involves a process of working in the “reverse direction”, aiming to uncover the underlying principles of a problem by exploring cause-and-effect relationships between ideas. Through it, connections can be identified between basic notions and the problem itself.
- *Breakdown*. BR concerns actions that allow the problem to be reduced to its fundamental components, identifying the properties involved and showing the relationships between the more complex and simpler objects. Breaking down a given concept into its primary elements allows analysing it in detail, and making its logical structure clear. For instance, analysing a geometric configuration by analysing and associating its constituent elements.
- *Transformative*. BR is involved in the interpretation of concepts and in conversions between semiotic registers, such as in transformations of geometric entities into algebraic language; the transformative and interpretative dimension emerges during the analysis of utterances and their translation into logical form.
- *Introduction of new elements*. BR has a strong component of creativity and discovery characterized by the inclusion of new elements in the solution. The inclusion and development of these elements depends on both the problem characteristics and the solver’s needs.

The four dimensions turn out to be different sides of the same construct. Reasoning backwards in solving a problem consists, in fact, of breaking down an entity, translating it into mathematical language, identifying its relevant elements and finding its principles, and inserting auxiliary elements into the process where necessary. These processes lead to something known from which the solver can then proceed progressively.

BR underlies different problem-solving strategies. By looking at the heuristic techniques, it is possible to identify those involving the "thing sought": working backward, assuming the problem solved strategy, beginning at the end of the problem, applying the Diaeresis method, and Reductio ad Absurdum (Polya, 1945).

To effectively integrate the reflection on BR in problem solving, BR needs to be fully understood from a strategic and cognitive perspective, and investigating how it relates to different resolution strategies is required. The following two research questions were formulated: (Q1) How does BR appear in the heuristic techniques related to it? (Q2) Are there any other strategies in which it emerges?

3.2.3 Methodology for the problem-solving case study

The data reported in this paper was collected during an experiment involving 66 undergraduate students attending didactic courses of bachelor's and master's degree programmes in Mathematics. Part of the courses focused on problem solving theories. In a 2-h class session, the students developed resolution protocols stressing their thinking processes and strategies solving four mathematical problems. The problems encompassed various mathematical aspects (graphical representation of functions, geometrical constructions, algebraic representation of geometrical elements, and combinatorial calculus). They were "tied to a proof problem" for which it was necessary to bring formal and intuitive knowledge and non-routine procedures into play (Schoenfeld, 1992, p. 350). The selection was guided by the requirement of the use of BR in their resolution and the inclusion of auxiliary constructions or novel elements. To fulfil these criteria, the chosen problems prominently incorporate a visual component and entail geometric development. Figure 2 shows one of the chosen problems.

As with all four problems of the design experiment, several approaches related to BR can be developed to solve it (Barbero, 2020). To identify the students' reasoning and procedures followed in the resolution protocol, *epistemic actions*, namely, the different mental processes in which knowledge is used or constructed, were identified: recognising previously acquired knowledge as relevant to the task; combining different knowledge elements to implement a strategy, justify a conjecture, or discover a solution to the problem; assembling and integrating prior knowledge to generate a new construct or idea (Hershkowitz et al., 2001).

Each epistemic action in which the students' resolution protocols were broken down was classified using the BR dimensions: whether the student breaks down the construction with the intention of analysing it, looks for a cause-effect relationship, uses algebraic language to refer to geometric entities, or introduces auxiliary elements (Barbero et al., 2020).

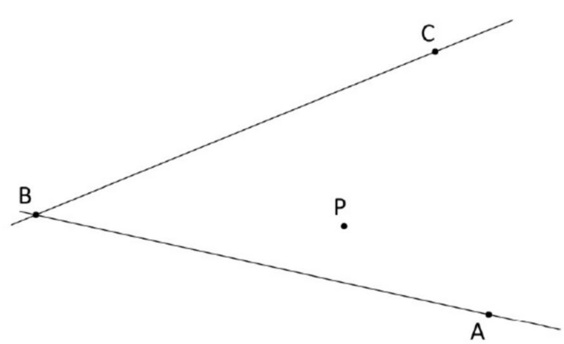
3.2.4 Results and discussion of the problem-solving case study

Upon examining the whole group resolving the four problems, six strategies are identified in which BR develops: four are strictly related to BR (assuming the problem solved, applying the Diaeresis method, starting at the end of the

Problem: Geometrical Construction

Given an ABC angle and a P point inside the angle, construct a QT segment, using only a ruler and compass, so that it passes through P and QP is twice PT.

Note: the Q point belongs to BA and the T point belongs to BC.



Solve the construction problem. Detail your entire thinking process using the resolution protocol technique.

Fig. 2 Geometrical construction problem

problem, and working backward), and two are not directly related to BR (identifying analogies and using an algebraic language). They appear intertwined throughout the resolution. In the construction problem shown in Fig. 2 three different behaviours are observed: 33 students start at the end of the problem analysing the solution (the desired geometric configuration), and then apply the Diaeresis method analysing its details in depth. At the same time, they introduce auxiliary elements, with the objective of obtaining a known configuration. 12 students start at the end of the problem and identify an analogy recognizing the final configuration as a particular component of a known theorem (segment trisection configuration and Thales' theorem) or configuration (QT as median of the triangle which has barycentre P), they then work backward reversing it in order to obtain the desired configuration; 16 students proceed mixing the two exemplified resolution pathways by first applying the Diaeresis method and then identifying an analogy from the final configuration exploration.

A student showing the first behaviour draws the segment QT, then adds further segments, and draws circumferences and lines parallel to the sides of the angle. After analysing the configuration, they identify the elements that constitute the configuration of the segment trisection.

Recurring patterns in the evolution of BR in the six resolution strategies were identified using the epistemic dimensions model:

- When the students start at the end of the problem, BR emerges in the breakdown dimension. Generally, this strategy is immediately followed by another strategy that may involve either BR or FR.
- When the students assume the problem solved, BR emerges in the breakdown dimension; when the configuration of the problem does not reveal "the thing sought", auxiliary elements are initially introduced.
- When the students apply the Diaeresis method, BR emerges in the breakdown dimension together with the introduction of auxiliary elements. Those elements are introduced as the student proceed with the in-depth analysis of the problem.
- When the students work backward, BR emerges in search of the cause-effect relationships dimension, and auxiliary elements are introduced throughout the resolution process.
- When the students identify analogies, BR emerges first with the introduction of an auxiliary element, and then develops into combining the breakdown and the search for cause-effect relationships dimensions.
- When the students use an algebraic language, BR emerges with a combination of the breakdown and the transformative dimension. During the evolution of this strategy BR and FR are alternated.

The breakdown dimension of BR allows students to observe relationships between elements, which are then encapsulated within a more complex structure. The search for rules and relations is guided by the formulation of hypotheses that enables algebraically expressing the problem, or reconducting it to a known structure. When components are missing to establish useful relationships, some auxiliary elements are introduced to identify familiar patterns or known properties that can lead students towards solving the problem.

The cause-effect relationship dimension of BR leads the students to work backward or to identify analogies. On the one hand, it allows looking for previously required elements to reach a particular structure, and on the other hand, it allows acting in combination with the breakdown dimension, to recognize familiar elements, problems, or configurations between the outcome of the problem and known theorems. In this case, introducing auxiliary elements becomes crucial, as it provides access to different patterns that, when analysed, allow for the formulation of hypotheses.

Finally, the transformative dimension of BR, associated with the breakdown dimension, guides students to identify an appropriate mathematical language to explore and manipulate the elements of the problem. The formulation of the solution in algebraic terms and its resolution require forward and backward movements in reasoning and a higher degree of abstraction.

The BR epistemic dimension model enables describing what occurs at an epistemic level when students apply resolution strategies related or not to BR. This finding has been complemented by a strategic, cognitive, and linguistic analysis to characterize BR and identify indicators that allow, on the one hand, monitoring and observing its evolution during problem-solving, and, on the other hand, having elements to be able to intervene at the educational level in the future for its propoer implementation (Barbero, 2020, 2023).

3.3 Case study 3: the ATD

3.3.1 Introduction

The summary of a case study framed within the ATD is provided in this study. More specifically, the design, implementation and analysis of a Study and Research Path (SRP) following the Didactic Engineering (DE) methodology (Barquero & Bosch, 2015) is presented. This case study, described in depth in Florensa et al. (2018), was implemented in a third-year course on General Elasticity in a Mechanical Engineering Degree at an engineering school in Barcelona (Spain) during the year 2015–16. However, the SRP implementation has been replicated up to now.

3.3.2 Theoretical framework and methodology: DE and SRPs in the ATD

SRPs are inquiry-based teaching formats with an associated methodology regarding its design and analysis. An SRP is initiated by an open generating question (Q_0) asked to a community of study (a group of students and teacher(s)). This leads to moments of study of available information in different sources, along with moments of research and creation of new solutions. This includes the adaptation of the data and available answers obtained to generate and answer to the specific question. This dialectic between study and research is one of the crucial characteristics of SRPs. A second important specificity of SRPs, compared to other inquiry-based proposals is the relevance of the generating question: the main didactic goal is to generate an answer to the question and not to encounter specific pieces of knowledge during the inquiry process.

The conception, implementation and analysis of an SRP cannot be detached from the DE methodology (Barquero & Bosch, 2015) even if its initial conception as a research methodology has evolved considerably over the past decade. DE is a methodology that consists of four-stages: preliminary analysis, a priori analysis, in vivo analysis and a posteriori analysis. During the preliminary analysis the goal is to characterise a didactic phenomenon (a regular didactic fact that can be observed in different institutions and situations) and to determine how the phenomenon is related to how knowledge is conceived and organised. In this same stage, the elaboration of an alternative explicit model of knowledge at stake (Chevallard, 2006) that is supposed to modify the didactic phenomenon is a crucial point of the approach. In the second stage, the a priori analysis, the researchers conceive an SRP: a generating question that will start the study process, the possible derived questions and answers and the sources of information in order to modify through amongst others. The third stage, the *in-vivo* analysis takes place once the SRP is implemented and consists of the data collection and management of the study process by the teachers involved. Finally, in the a posteriori analysis the researchers determine to what extent the new conception of knowledge materialised by the SRP implementation has modified the didactic phenomenon studied.

3.3.3 Results and discussion of the SRP design and implementations

The case study described below is presented in depth in Florensa et al. (2018). The starting point of the study consists of two didactic phenomena that were characterised in a third-year Elasticity course (6 ECTS) of a 4-year bachelor's degree in mechanical engineering. The traditional organisation of a course in Elasticity includes the notions of stress and strain, the introduction of stress and strain tensors and the generalised Hooke's law establishing the linear relationship between stress and strain. This course is taught in diverse engineering degrees in which mathematics plays an important role. In particular, the eigenvalues and eigenvectors of the tensors are crucial values to analyse the mechanical behaviour of the parts studied. A major change took place in higher education degrees in 2008 in Europe when competence-based programmes were introduced throughout the European Higher Education Area (EHEA). However, the organisation and the contents of the course had remained unchanged despite the EHEA changes proposed.

The first didactic phenomenon characterised during the study is *monumentalism* (Chevallard, 2015). This phenomenon consists of the fact that the contents to be taught are important because of their intrinsic relevance and their study appears as the visit of relevant knowledge monuments. The main *raison d'être* of the course in Elasticity was to introduce the General Elasticity Model to the students: stress and strain tensors, the principal stresses and strains, their relationship and failure criteria were the main objects of study. Related to this first phenomenon, the second phenomenon is the *algorithmisation* of the activity. This phenomenon was identified in the initial organisation of the course. Although the real problems tackled by the General Elasticity model can only be solved by using Finite Element Methods (FEM) and numerical analysis, the course (consistent with the textbooks used in the field, such as Reddy, 2013) was organised around traditional problems solved using paper-and-pencil settings in which the main task was to determine the eigenvalues and eigenvectors of the stress and strain tensors. However, no interest was paid to the mechanical engineering activity. Because of this *algorithmisation,* the main activity was to determine the characteristic polynomial of a tensor, and the eigenvalues and eigenvectors in a contextualised setting, a type of tasks far removed from the ones used in engineering.

In order to modify both didactic phenomena, the *raison d'être* of the course was changed: from presenting the model to "designing mechanical parts working under the elastic regime". To implement this new conception, the course was re-structured: the traditional structure was kept during the first 7 weeks to present the model and to promote analytical-solvable cases, while an SRP was designed and implemented in the remaining 8 weeks. The use of an SRP aimed to change the activity of the course putting the design of mechanical parts at the centre. To do so, the generating question chosen of the first implementation was "How to redesign the different parts of a mountain-bike to make it lighter?". The teachers of the course asked their students this questions, but it was presented as a commission of a (fictitious) bike company to whom we the students were supposed to address their answers to. The students worked in different groups to redesign different parts of the original bike introducing FEM simulation software, different materials available and testing equipment. The final answer to the generating question was a redesigned part of the bike including a technical report addressed to the fictious bike company.

Managing the SRP was challenging as important changes were introduced to the didactic contract: in the SRP, the validation of the answers represented a shared responsibility between teachers and students, the new questions were raised by the students and the dynamic nature of knowledge mobilised in the SRP, compared to the static conception of the previous organisation, represented three of these changes. One of the tools implemented to manage the evolution of knowledge during the SRP implementation were the Question–Answer maps (Q–A maps) (Florensa et al., 2018). These maps initially used as an SRP modelling tool by researchers in previous experiments, were adapted and used by teachers and students in this SRP to describe the questions derived from the initial question, the partial answers generated.

The analysis of the SRP implementation was twofold. On the one hand, it was performed in the context of the ecological conditions and constraints facilitating or hindering its implementation. On the other hand, the changes in the activity of the students and the mobilised knowledge were also considered.

One of the findings of the experiment was that the change of the *raison d'être* of the course, embodied by the implementation of the SRP, drastically modified the role played by the General Elasticity Model. The *algorithmisation* of the traditional organisation of the course in which mathematical procedures played a key role (paper-and-pencil eigenvalues and eigenvectors calculations) shifted to a more functional use of the General Elasticity Model backed using FEM simulations. The model changed its role: from an object to be studied to a construct that underpins the simulations used to make decisions regarding mechanical design. This is a major shift in the role of the mathematical activity in the course in Elasticity. From being a prevailing activity, it moved to taking on the role of ensuring the functionality and consistency of the model. This change is also related to the phenomenon of *monumentalisation* of the model itself. The General Elasticity model became the piece of knowledge mobilised to the main activity which was the redesign of a bike part.

4 Discussion

The analysis of three case studies in this paper enable exploring the distinctive features of different theoretical approaches by investigating their research questions, objects of study, and empirical bases.

The first case study presented in this paper, the APOS case study, explores a didactic phenomenon, namely, students' recurrent difficulties with optimisation on closed and bounded domains in a multivariable calculus course. It encompasses two primary focal points. First, it involves a didactic exploration of the didactic phenomena identified. Second, it strives to enrich the theoretical framework through the development of the Schema concept. Research starts by proposing a model of mental structures (actions, processes, objects, schemas) a student may construct in order to understand the mathematical notions considered. The research questions enquire into what structures students tend to construct with regard to the GD (the model) of the particular mathematical notion.

A crucial task in the APOS case study is the explicit modelling of the knowledge considered in terms of a GD. This model enables researchers to describe the process of students constructing mathematical notions. Thus, the APOS framework investigates the students' construction of knowledge, focusing on a set of Schema components (e.g., function of one or two variables) and the relationships established between them. As a result, the object of study in APOS is the process of knowledge construction.

Regarding the empirical basis of the APOS case study, it comprises activities designed to help students build some constructions, and a series of semi-structured interviews in which mathematical questions are asked. By analysing the interview transcriptions, the researchers identify the types of relationships the students establish between the different components of the optimisation Schema, as described in the GD, enabling them to track the students' progress. The researchers' main findings revolve around the development and establishment of these relationships.

The conclusions of the APOS case study address the need to explicitly consider the topology of domain sets as suggested by the difficulties encountered during the construction of relationships in the context of understanding optimisation in two variables. Regarding the research ends, the study aims to provide recommendations to teachers. The results are formulated as guidance to highlight specific elements of GD during the teaching process, which may facilitate the construction of a particular domain of knowledge. The results also contribute to develop the notion of Schema in the theory.

The second case study explores a teaching and learning process framed within the problem-solving approach. The research questions for this study examine BR in connection with different heuristic strategies and epistemological dimensions used by students while solving non-routine problems. The case study considers the analysis of the solving process of a model in terms of BR heuristics.

The objects of study are the different heuristic strategies, how they appear, how they are used during problem-solving, and how they can be encouraged. In this case in particular, BR heuristics is considered as a strategy to be promoted at the undergraduate level because of its potential to promote mathematical argumentation, inquiry, and proof. There is not an explicit formulation of a didactic phenomenon to be characterised. In contrast to APOS and the ATD, the problem-solving framework does not propose or consider a methodology to explicitly model the knowledge in question, nor the institutional conditions and constraints facilitating or hindering the implementation of said activities.

As far as the empirical basis is concerned, the case study analyses the students' work using a geometric construction problem as an example to associate their problem-solving strategies with the epistemic dimensions of BR. The students' reports served as the primary object of analysis. Both the written content and the actions the students described by the students were carefully examined to identify the various mental processes involved in knowledge construction. This was complemented by direct observations conducted during the session. The results analysis focused on examining the students' resolution strategies by first separating their work into epistemic actions (Hershkowitz, 2001) and then analysing each epistemic action corresponding to the epistemic dimensions of cause-effect, breakdown, transformation, and the introduction of new elements. This led to recognising how BR emerges in different problem-solving strategies, even in those that are not directly related to it.

The conclusions of the problem-solving approach are formulated in terms of the heuristics and epistemological dimensions used during the process and the relationships between them. Regarding the research ends, this study sheds light on the processes through which BR can effectively be integrated into problem-solving. This may result in pedagogical recommendations to help implement BR in classroom activities.

Finally, the third case study, framed within the context of the ATD, explores two didactic phenomena identified, *algorithmisation* and *monumentalism*, in a course in Elasticity taught in a Mechanical Engineering Degree. The epistemological approach of the ATD stresses the interdependence between the conception of the knowledge to be taught and the didactic phenomena studied. Thus, the object of study is the design and implementation of an SRP using the DE methodology as a way to modify the conception of knowledge at the institution and, hence, to partially overcome the previously identified didactic phenomena. To investigate the feasibility of modifying the didactic phenomena identified, an SRP is designed to alter the activity and the underlying rationale in the field of Elasticity. The SRP itself becomes both the teaching and research instrument through which the modification of the knowledge at stake is intended.

The ATD adopts an institutional perspective, where the cognitive aspect is no longer the central focus, in contrast with the previous two cases studies. Instead, the analysis and research questions shift towards examining the role of the initial conception of modelled knowledge, the activity proposed, and its relation to the didactic phenomena identified during the preliminary analysis. Another crucial aspect of the study is to consider the ecological dimension in the core of the analysis. In the APOS and the problem-solving case studies, the conditions and constraints affecting the implementation of the study process were not considered or taken for granted.

The empirical material analysed encompasses the students' work, including follow-up reports and a final report on the SRP, which includes the intended answer for the client. Additionally, semi-structured interviews are conducted with both the students and the teachers involved in the experiment to understand the rationale they attribute to the course and the activity carried out. Further analysed material consists of the curricula and textbooks as a way to characterise the prevailing epistemology.

The analysis of the empirical material unfolds in two main directions. First, it identifies how the implementation of the SRP modifies the studied phenomena. Second, it examines the implementation from an ecological and management perspective, considering the extent to which existing conditions and constraints facilitate or hinder the implementation of the SRP. Moreover, it explores the new tools and actions necessary to bring about the changes in responsibilities and activities resulting from the implementation.

The work framed within the ATD aims to describe and better understand certain didactic phenomena by altering the activity within educational institutions, placing the study of modification, evolution, and institutional relativity of knowledge at the centre of the analysis.

After exploring the different case studies and their distinctive features a summary of the findings is shown in Tables 1 and 2.

5 Conclusions: distinctive features of networking theories at university level

First of all, this study highlights the extent to which significant differences are observed between the three approaches regarding their objects of study, research questions, empirical bases, and research ends considered. At the first level of analysis, the only point where there is some consensus regarding the empirical basis used is that student reports are key in all three cases. However, in the case of the ATD, considering the institutional conditions and constraints broadens significantly broadens the empirical basis. This is a consequence of its ecological perspective, which includes the different epistemological and didactic conceptions of knowledge as critical questions to be studied.

Table 1 Research questions and research ends of the different approaches

Approach	Research questions	Research ends
APOS	What structures (A, P, O, S) do students tend to construct with regard to a GD of a particular mathematical notion or topic?	Give pedagogical recommendations to instructors for the design of activities and the teaching of the notion or topic considered
Problem-Solving	How do students use different heuristic strategies and epistemological dimensions when solving non-routine problems? How to teach to promote BR heuristics?	Give pedagogical suggestions to instructors to help them teach problem solving and to promote specific heuristics
ATD	To what extent does the implementation of an SRP bring about changes in the prevailing epistemology that is the basis of didactic phenomena? Which are the institutional conditions and constraints enabling or hindering the implementation of the study process?	Study the conditions and constraints facilitating and hindering the modification of the prevailing epistemology and consequently analyse the modification of didactic phenomena

Secondly, the two cognitive approaches, APOS and problem-solving share the same object of study: both place the students' construction of knowledge at the heart of their analysis. However, APOS explicitly models the knowledge considered to analyse the students' reports with regard to the elements of the GD. This contrasts with the proposal of problem-solving, where the analysis is made in terms of epistemic actions, heuristic strategies, and epistemic dimensions. Even if these elements are well defined prior to the analysis, they are not explicitly related to the knowledge in question, and to how it is considered. This explicit modelling of knowledge is also taken into account in the ATD where students' reports are analysed in terms of praxeologies. By analysing theoretical approaches as research praxeologies (Bosch et al., 2017), it is observed that the problem-solving approach seems to lack an explicit type of task related to knowledge modelling. In order to extend the networking strategy proposed by Bosch et al. (2017), it is considered that it would be interesting to start from the task component and explore the way APOS and the ATD would model mathematical contents mobilised in the activities proposed in the problem-solving approach. The interpretation of the epistemic actions dealing with the construction of APOS structures and the interpretation of BR relevant to the APOS mechanism of process reversion seem promising.

A second aspect that should be emphasised is that both cognitive approaches (APOS and problem-solving) share the research ends they formulate with regard to suggestions and recommendations to practitioners, while the ATD remains behind in this aspect. It focuses on the characterization of didactic phenomena and the options to modify them. Promising collaboration may hence be expected between the two cognitive approaches, especially with respect to suggesting indications and guidelines to teachers.

In the case of APOS, attention is focused on the mental structures (actions, processes, objects, schemas) the students construct with respect to a GD of a given mathematical notion. In the case of problem-solving and, in particular, the study of the BR heuristic, the focus is on the students' use of the heuristic in connection with an epistemic dimension (cause-effect, breakdown, transformation, and the introduction of new elements). In problem-solving, no notion is analogous to the GD in APOS or the reference epistemological model in the ATD. However, the use of epistemic actions (Hershkowitz et al., 2001) to parse an interview transcript into episodes that are later analysed in terms of epistemic dimensions and heuristics suggests possible networking where the different epistemic actions are reinterpreted as the different APOS conceptualisations and can be used to do the parsing. In this case, if a mathematical notion or problem is given, a GD may also be developed for the notion or problem. It would be interesting to explore if activities based on this GD help students with other situations involving BR.

Table 2 Empirical bases and objects of study of the different approaches

Approach	Empirical basis	Object of study
APOS	Student work	Structures (A, P, O, S) constructed when learning a mathematical notion or topic
Problem- Solving	Student work	The relation between different heuristic strategies and epistemological dimensions when students solve non-routine problems
ATD	Institutional and student work (all stages and institutions involved in the process of didactic transposition)	Institutional praxeologies mobilised during the implementation of a study process, their emergence, organization, and ecology

An example of possible networking between APOS and the ATD, starting from the task component, would be to consider the domain of the optimisation of two-variable functions, particularly when the domain region is a compact set. The ATD epistemological analysis of the knowledge at stake through the study of institutional documents like textbooks and syllabi, might be used to show that the observed student behaviour characterised by the APOS is a consequence of how the domain of topology of domain sets is taught and conceived in the teaching institution considered. It would entail showing that some optimisation techniques are mainly taught as action-techniques (Bosch et al., 2017) that only promote a rigid application of optimisation techniques, lacking justification and being isolated from other mathematical knowledge.

As Rodríguez et al. (2008) point out, "[…] it always seems possible to find an initial problematic question—a 'practical problem' which can be formulated in the words and culture of the considered educational institutions—that can be meaningful in the different approaches since they are appearing as part of the reality modelled by them. This problematic question can thus be taken as a 'common base' for the comparison of approaches by looking at how each frame transforms it into a research problem and what kind of admissible answer can be provided." This suggests possible networking at the task level between problem-solving and the ATD: the design and analysis of SRPs could include the study of techniques that might be analysed under the prism of the epistemic dimensions. Likewise, a generating question designed for an SRP could be studied with regard to the heuristic strategies it foments and the epistemological dimensions in which these strategies appear.

This study is considered to be an extension of the dialogue between theories initiated by Bosch et al. (2017) in their networking study of APOS and the ATD. They incorporated the perspective of problem-solving by extending the networking study of Rodríguez et al. (2008). The analysis highlights important differences, mainly in the need to model the knowledge at stake and how this knowledge is considered. The role assigned to the institutional influence is also stressed. However, shared aspects were identified in the research ends and in the empirical bases. Those aspects should be considered in further research.

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