Normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities

Jack Borthwick[*1], Xiaojun Chang [†2], Louis Jeanjean[‡3], and Nicola Soave[§4]

[1]*Université de Franche-Comté, Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, 16 route de Gray, 25000 Besançon, France*

[2]*School of Mathematics and Statistics & Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, Jilin, PR China*

[3]*Université de Franche-Comté, Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, 16 route de Gray, 25000 Besançon, France*

[4]*Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, 10123 Torino, Italy*

Abstract

In this paper we are concerned with the existence of normalized solutions for nonlinear Schrödinger equations on noncompact metric graphs with localized nonlinearities. In a L^2-supercritical regime, we obtain the existence of solutions for any prescribed mass. This result is obtained through an approach which could prove successful to treat more general equations on noncompact graphs.

1 Introduction and main results

This paper is devoted to the existence of bound states of prescribed mass for the L^2-*supercritical* nonlinear Schrödinger equation (NLS) on $\mathcal{G}$

$$-u'' + \lambda u = \kappa(x)|u|^{p-2}u, \tag{1.1}$$

coupled with Kirchhoff conditions at the vertices, see (1.4) below. Here λ appears as a Lagrange multiplier, $p > 6$, $\mathcal{G}$ is a noncompact connected metric graph, and κ is the characteristic function of the compact core $\mathcal{K}$ of $\mathcal{G}$. We recall, see [4], that if $\mathcal{G}$ is a metric graph with a finite number of vertices its compact core $\mathcal{K}$ is defined as the metric subgraph of $\mathcal{G}$ consisting of all the bounded edges of $\mathcal{G}$.

---

[*]jack.borthwick@math.cnrs.fr

[†]changxj100@nenu.edu.cn

[‡]louis.jeanjean@univ-fcomte.fr

[§]nicola.soave@unito.it

A solution with prescribed mass is often referred to as a *normalized solution*. For Equation (1.1) they correspond to critical points of the NLS energy functional $E(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E(u, \mathcal{G}) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 \, dx - \frac{1}{p} \int_{\mathcal{K}} |u|^p \, dx \tag{1.2}$$

under a mass constraint

$$\int_{\mathcal{G}} |u|^2 \, dx = \mu > 0. \tag{1.3}$$

The study of NLS on metric graphs has attracted much attention over the last few decades. Among the physical motivations to consider Schrödinger equations on a metric graph, the most prominent one comes from the study of quantum graphs, see for example [6, 11, 29, 32] and the references therein. In particular, the investigation of NLS with localized nonlinearities appeared in optical fibers and Bose-Einstein condensates in non-regular traps (see [24, 32]).

Recently, much effort has been devoted to the existence of normalized solutions of NLS on metric graphs, in the L^2-*subcritical* (i.e., $p \in (2,6)$) or L^2-*critical regimes* (i.e., $p = 6$). In these two regimes, the energy functional $E(\cdot, \mathcal{G})$ is bounded from below on the mass constraint and one may develop minimization methods. A relevant notion is indeed the one of ground states, namely of solutions which minimize the energy functional on the constraint. For the existence of ground states solutions, see [1, 2, 3, 4, 5, 33, 35] for noncompact $\mathcal{G}$, and [14, 16] for the compact case; some studies are also conducted on the existence of local minimizers, see [36]. Regarding problems with a localized nonlinearity as in Equation (1.1), existence or non-existence of ground states solutions was discussed in [39, 42] for the L^2-subcritical case, and in [18, 19] for the L^2-critical case. Moreover, in the L^2-subcritical regime, genus theory was applied in [38] to obtain the existence of multiple bound states solutions at negative energy levels.

In the L^2-supercritical regime on general metric graphs, i.e., when $p > 6$, the energy functional $E(\cdot, \mathcal{G})$ is always unbounded from below. Moreover, due to the fact that graphs are not scale invariant, the techniques based on scalings, usually employed in the Euclidean setting and related to the validity of a Pohozaev identity (see [26] or [8, 25, 40, 41]), do not work. These two features make the search for normalized solutions in the L^2-supercritical regime delicate. Recently, in [15], this issue was considered on compact metric graphs. The existence of solutions was proved for small values of $\mu > 0$, see [15, Theorem 1.1] for a precise statement. In the present paper we pursue this study by allowing the graph to be noncompact. Although in our proofs we make use of the fact that the nonlinearity in Equation (1.1) only acts on a compact part of the graph, we believe that our approach could prove fruitful to study more general situations.

Basic notations and main result

Throughout this paper, we consider a connected noncompact metric graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is the set of edges and $\mathcal{V}$ is the set of vertices. We assume that $\mathcal{G}$ has a finite number of edges and the compact part $\mathcal{K}$ is non-empty. Without loss of generality, we may assume that $\{e_1, \cdots, e_{m_1}\}$ are the bounded edges, $\{l_1, \cdots, l_{m_2}\}$ are the half-lines, and $\{v_1, \cdots, v_{m_3}\}$ are the finite vertices of $\mathcal{G}$. Here m_1, m_2, m_3 are positive integers. Moreover, any bounded edge e is identified with a closed bounded interval I_e, typically $[0, \ell_e]$ (where ℓ_e is the length of e), while each unbounded edge is identified with a closed half-line $I_e = [0, +\infty)$. For details, one can refer to [3, 4, 11]. A function on metric graph $u : \mathcal{G} \to \mathbb{R}$ is identified with a vector of functions $\{u_e\}$, where each u_e is defined on the corresponding interval I_e such that $u|_e = u_e$. Endowing each edge with Lebesgue measure, one can define the space $L^p(\mathcal{G})$ in a natural way, with norm

$$\|u\|_{L^p(\mathcal{G})}^p = \sum_e \|u_e\|_{L^p(e)}^p.$$

The Sobolev space $H^1(\mathcal{G})$ consists of the set of continuous functions $u : \mathcal{G} \to \mathbb{R}$ such that $u_e \in H^1([0, \ell_e])$ for every edge e (note that ℓ_e can be $+\infty$); the norm in $H^1(\mathcal{G})$ is defined as

$$\|u\|_{H^1(\mathcal{G})}^2 = \sum_e \|u_e'\|_{L^2(e)}^2 + \|u_e\|_{L^2(e)}^2.$$

2

We shall study the existence of critical points of the functional $E(u, \mathcal{G})$ constrained on the L^2-sphere

$$H^1_\mu(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) : \int_\mathcal{G} |u|^2 \, dx = \mu \right\}.$$

If $u \in H^1_\mu(\mathcal{G})$ is such a critical point, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies the following problem:

$$\begin{cases} -u'' + \lambda u = \kappa(x)|u|^{p-2}u & \text{on every edge e of } \mathcal{G}, \\ \sum\limits_{e \succ v} u'_e(v) = 0 & \text{at every vertex } v \in \mathcal{K}, \end{cases} \tag{1.4}$$

where $e \succ v$ means that the edge e is incident at v, and the notation $u'_e(v)$ stands for $u'_e(0)$ or $-u'_e(\ell_e)$, according to whether the vertex v is identified with 0 or ℓ_e (namely, the sum involves the outer derivatives at v). The second equation is the so-called *Kirchhoff condition*.

Our main result can be stated as follows.

Theorem 1.1. *Let $\mathcal{G}$ be any noncompact metric graph having a non-empty compact core $\mathcal{K}$, and let $p > 6$. Then, for any $\mu > 0$, problem (1.4) with the mass constraint (1.3) has a positive solution u associated to a Lagrange multiplier $\lambda > 0$. This solution corresponds to a critical point of $E(\cdot, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$ at a strictly positive energy level.*

Let us present the main ideas to prove Theorem 1.1.

It is simple to check that the functional $E(\cdot, \mathcal{G})$ admits a mountain pass geometry in $H^1_\mu(\mathcal{G})$ for any $\mu > 0$. However, the existence of a bounded Palais-Smale sequence at the mountain pass level, a key preliminary step to obtain a critical point, is not directly available. To overcome this issue, following the strategy presented in [15], we first consider the family of functionals $E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ given by

$$E_\rho(u, \mathcal{G}) = \frac{1}{2} \int_\mathcal{G} |u'|^2 dx - \frac{\rho}{p} \int_\mathcal{K} |u|^p dx, \qquad \forall u \in H^1(\mathcal{G}), \forall \rho \in \left[\frac{1}{2}, 1\right]. \tag{1.5}$$

As in [15] we shall rely on a parametrized version of a mountain-pass theorem on a constraint established in [12]. Roughly, this theorem, recalled here as Theorem 2.4, guarantees, for almost every value of a parameter, the existence of a bounded Palais-Smale sequence with second-order information. Applying Theorem 2.4 on the family $E_\rho(\cdot, \mathcal{G})$ we deduce that, for almost every $\rho \in \left[\frac{1}{2}, 1\right]$, these functionals admit a bounded Palais-Smale sequence $\{u_{\rho,n}\}$ of non negative functions with "approximate Morse index" information.

The next step is to show that, for such $\rho \in \left[\frac{1}{2}, 1\right]$, the sequence $\{u_{\rho,n}\}$ converges to some u_ρ. If this is the case, as a consequence of the Morse type information on $\{u_{\rho,n}\}$ one can directly deduce that u_ρ has a Morse index $m(u_\rho)$ at most 2, see Definition 3.4. In [15] the convergence of $\{u_{\rho,n}\}$ was immediate due to the compactness of $\mathcal{G}$. In our setting, despite the fact that the nonlinearity acts only on the compact core, the non-compactness of the graph prevents us from directly concluding that bounded Palais-Smale sequences converge. This kind of difficulty is regularly encountered in the search of normalized solutions in L^2-supercritical problems when the underlying domain is noncompact, see for example [28, 31]. To overcome it one needs to benefit from information on the sign of an associated sequence of *almost* Lagrange multipliers. In our problem we derive this information by using the "approximate Morse index" information carried by the sequence $\{u_{\rho,n}\}$. This argument seems new and we suspect that it could be used in other problems. In order to emphasize its generality we present it, in Section 2, in a general framework.

Having been able to show, for almost every $\rho \in \left[\frac{1}{2}, 1\right]$, the convergence of $\{u_{\rho,n}\}$ to some u_ρ with Morse index at most 2 (see Proposition 3.5 for a precise statement), our proof follows the lines of [15]. Namely, taking a sequence $\rho_n \to 1^-$, we adapt the blow-up analysis, developed in [15], to show that $\{u_n\} := \{u_{\rho_n}\}$ is bounded. Then, we use some arguments previously developed in the proof of Proposition 3.5 to deduce that $\{u_n\}$ converges and to end the proof of Theorem 1.1.

The rest of the paper is mainly devoted to the proof of Theorem 1.1. Before proceeding, we would like to discuss some open problems and possible further directions of research.

3

Remark 1.2. Since they are obtained through a mountain pass procedure on a codimension 1 manifold, it is natural to expect that the solutions we found are orbitally unstable. This is what happens in mass-superlinear problems when the underlying equation is autonomous and set on all the space. See, for example, [39] in that direction. However, here things are more complex since we have little information on the variational characterizations of the solutions that we obtain. In comparison to [39], we do not know if our solutions can be characterized as a *ground state* or as a global minimizer on some additional constraint. More globally, the issue of orbital stability (and the related issue of the monotonicity of the mass with respect to the Lagrange multiplier) appears complex *on a general non-compact graph*. So far, orbital stability/instability of bound states on metric graphs has only been characterized for some specific examples, see [2, 14, 34] and references therein.

Remark 1.3. Another challenging issue would consist in characterizing precisely the properties of the solutions in terms of the number and the location of local and global maximum points, at least in the limit of large/small mass parameter. This is also related to multiplicity issues. We refer to [7, 13] for some recent results in the subcritical case, however it seems unlikely that they are directly adaptable to the supercritical case (the former exploits a doubly constrained minimization argument; the latter uses some techniques specifically suited to deal with the cubic NLS).

On the other hand, it is not difficult to prove the following basic result, which actually holds for *any* one-signed critical point of $E(\cdot, \mathcal{G})$ on S_μ.

Proposition 1.4. *For $p > 6$ and $\mu > 0$, let $u \in H^1_\mu(\mathcal{G})$ be a critical point of $E(\cdot, \mathcal{G})$ defined in* (1.2) *under the mass constraint* (1.3), *and suppose that $u > 0$ and $\lambda > 0$. If $\bar{x} \in \mathcal{G}$ is a local maximum point of u, then $\bar{x} \in \mathcal{K}$, the compact core. Furthermore, on any half-line $\ell_i \simeq [0, +\infty)$ one has $u(x) = c_i e^{-\sqrt{\lambda} x}$ for some $c_i > 0$.*

The proof follows directly by integrating the equation on each half-line, and using the fact that $\lambda > 0$.

Remark 1.5. Finally, we make a short remark on extending our methods to study the existence of bound states in cases where the nonlinearity acts on the full graph, and not only its compact core. In this direction, our min-max geometric argument still works, but some issues arise in studying the compactness of Palais-Smale sequences. Considering what happens in the subcritical and critical cases (see e.g. [3, 4, 5]), it is expected that topological and metric properties of the underlying graph play a role.

The paper is organized as follows. In Section 2 we recall in Theorem 2.4 the main result of [15] and explore some of its consequences. In particular, we show that the second order information on the Palais-Smale sequences can be used to get uniform bounds from below on the associated sequences of almost Lagrange parameter, see Lemma 2.5. In Section 3 we show that $E_\rho(\cdot, \mathcal{G})$ has a mountain pass critical point for almost every $\rho \in \left[\frac{1}{2}, 1\right]$ and we prove Proposition 3.5. Section 4 presents the proof of Theorem 1.1.

2 Some preliminary definitions and results

In this section, we recall [12, Theorem 1] and explore some of its consequences.

Let $(E, \langle \cdot, \cdot \rangle)$ and $(H, (\cdot, \cdot))$ be two *infinite-dimensional* Hilbert spaces and assume that $E \hookrightarrow H \hookrightarrow E'$, with continuous injections. For simplicity, we assume that the continuous injection $E \hookrightarrow H$ has norm at most 1 and identify E with its image in H. Set

$$\begin{cases} \|u\|^2 = \langle u, u \rangle, \\ |u|^2 = (u, u), \end{cases} \quad u \in E,$$

and we define for $\mu > 0$:

$$S_\mu = \{u \in E \,|\, |u|^2 = \mu\}.$$

In the context of this paper, we shall have $E = H^1(\mathcal{G})$ and $H = L^2(\mathcal{G})$. Clearly, S_μ is a smooth submanifold of E of codimension 1. Its tangent space at a given point $u \in S_\mu$ can be considered as the closed codimension 1 subspace of E given by:

$$T_u S_\mu = \{v \in E \mid (u,v) = 0\}.$$

In the following definition, we denote by $\|\cdot\|_*$ and $\|\cdot\|_{**}$, respectively, the operator norm of $\mathcal{L}(E,\mathbb{R})$ and of $\mathcal{L}(E,\mathcal{L}(E,\mathbb{R}))$.

Definition 2.1. Let $\phi : E \to \mathbb{R}$ be a C^2-functional on E and $\alpha \in (0,1]$. We say that ϕ' and ϕ'' are α-Hölder continuous on bounded sets if for any $R > 0$ one can find $M = M(R) > 0$ such that, for any $u_1, u_2 \in B(0,R)$:

$$||\phi'(u_1) - \phi'(u_2)||_* \le M||u_1 - u_2||^\alpha, \quad ||\phi''(u_1) - \phi''(u_2)||_{**} \le M||u_1 - u_2||^\alpha. \tag{2.1}$$

Definition 2.2. Let ϕ be a C^2-functional on E. For any $u \in E$, we define the continuous bilinear map:

$$D^2\phi(u) = \phi''(u) - \frac{\phi'(u) \cdot u}{|u|^2}(\cdot, \cdot).$$

Definition 2.3. Let ϕ be a C^2-functional on E. For any $u \in S_\mu$ and $\theta > 0$, we define the *approximate Morse index* by

$$\tilde{m}_\theta(u) = \sup\left\{\dim L \mid L \text{ is a subspace of } T_u S_\mu \text{ such that: } D^2\phi(u)[\varphi, \varphi] < -\theta\|\varphi\|^2, \quad \forall \varphi \in L \setminus \{0\} \right\}.$$

If u is a critical point for the constrained functional $\phi|_{S_\mu}$ and $\theta = 0$, we say that this is the *Morse index of u as constrained critical point*.

The following abstract theorem was established in [12]. Its derivation is based on a combination of ideas from [22, 23, 27] implemented in a convenient geometric setting. Note that related theorems, but dealing with unconstrained functionals, were recently developed in [10, 30].

Theorem 2.4 (Theorem 1 in [12]). *Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \qquad \rho \in I,$$

where $B(u) \ge 0$ for every $u \in E$, and

$$\text{either } A(u) \to +\infty \text{ or } B(u) \to +\infty \text{ as } u \in E \text{ and } \|u\| \to +\infty. \tag{2.2}$$

Suppose moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0,1]$. Finally, suppose that there exist $w_1, w_2 \in S_\mu$ (independent of ρ) such that, setting

$$\Gamma = \left\{\gamma \in C([0,1], S_\mu) \mid \gamma(0) = w_1, \quad \gamma(1) = w_2\right\},$$

we have

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\rho(\gamma(t)) > \max\{\Phi_\rho(w_1), \Phi_\rho(w_2)\}, \quad \rho \in I. \tag{2.3}$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$,

(i) $\Phi_\rho(u_n) \to c_\rho$;

(ii) $||\Phi'_\rho|_{S_\mu}(u_n)||_* \to 0$;

(iii) $\{u_n\}$ is bounded in E;

(iv) $\tilde{m}_{\zeta_n}(u_n) \le 1$.

It is easily observed, see [12, Remarks 1.3], from Theorem 2.4 (ii)-(iii) that

$$\Phi'_\rho(u_n) + \lambda_n(u_n, \cdot) \to 0 \text{ in } E' \text{ as } n \to +\infty \tag{2.4}$$

where we have set

$$\lambda_n := -\frac{1}{\mu}(\Phi'_\rho(u_n) \cdot u_n). \tag{2.5}$$

Also, Theorem 2.4 (iv) directly implies that if there exists a subspace $W_n \subset T_{u_n} S_\mu$ such that

$$D^2\Phi_\rho(u_n)[w,w] = \Phi''_\rho(u_n)[w,w] + \lambda_n(w,w) < -\zeta_n ||w||^2, \quad \text{for all } w \in W_n \setminus \{0\}, \tag{2.6}$$

then necessarily $\dim W_n \le 1$. We called the sequence $\{\lambda_n\} \subset \mathbb{R}$ defined in (2.5) the sequence of *almost Lagrange multipliers*.

We shall see in Section 3 that if $\{u_n\} \subset S_\mu$ converges to some $u \in S_\mu$ then information on the Morse index of u as constrained critical point can be obtained. First, let us show how information on the sequence $\{\lambda_n\} \subset \mathbb{R}$ can be derived.

Lemma 2.5. *Let $\{u_n\} \subset S_\mu$, $\{\lambda_n\} \subset \mathbb{R}$ and $\{\zeta_n\} \subset \mathbb{R}^+$ with $\zeta_n \to 0^+$. Assume that the following conditions hold:*

(i) For large enough $n \in \mathbb{N}$, all subspaces $W_n \subset E$ with the property

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 < -\zeta_n \|\varphi\|^2, \quad \text{for all } \varphi \in W_n \setminus \{0\}, \tag{2.7}$$

satisfy: $\dim W_n \le 2$.

(ii) There exist $\lambda \in \mathbb{R}$, a subspace Y of E with $\dim Y \ge 3$ and $a > 0$ such that, for large enough $n \in \mathbb{N}$,

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda |\varphi|^2 \le -a||\varphi||^2, \quad \text{for all } \varphi \in Y. \tag{2.8}$$

Then $\lambda_n > \lambda$ for all large enough $n \in \mathbb{N}$. In particular, if (2.8) holds for any $\lambda < 0$, then $\liminf_{n\to\infty} \lambda_n \ge 0$.

Proof. Suppose by contradiction that $\lambda_n \le \lambda$ along a subsequence. From (2.8) we have, for all $\varphi \in Y \setminus \{0\}$,

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 = \Phi''_\rho(u_n)[\varphi, \varphi] + \lambda |\varphi|^2 + (\lambda_n - \lambda)|\varphi|^2$$
$$\le -a||\varphi||^2 + (\lambda_n - \lambda)|\varphi|^2.$$

Now, since $\zeta_n \to 0^+$, there exists $n_0 \in \mathbb{N}$ such that: $\forall n \ge n_0, \zeta_n < a$. Thus, for an arbitrary $n \ge n_0$, we obtain

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 \le -\zeta_n \|\varphi\|^2, \quad \text{for all } \varphi \in Y \setminus \{0\},$$

in contradiction with (2.7) since $\dim Y > 2$. $\qquad \square$

3   Mountain pass solutions for approximating problems

Theorem 2.4 will be a key ingredient in the proof of Theorem 1.1. To show that the family $E_\rho(\cdot \, \mathcal{G})$ enters into the framework of Theorem 2.4 we first need to show that it has a mountain pass geometry on $H^1_\mu(\mathcal{G})$ uniformly with respect to $\rho \in \left[\frac{1}{2}, 1\right]$.

Lemma 3.1. *For every $\mu > 0$, there exist $w_1, w_2 \in S_\mu$ independent of $\rho \in \left[\frac{1}{2}, 1\right]$ such that*

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_\rho(\gamma(t), \mathcal{G}) > \max\{E_\rho(w_1, \mathcal{G}), E_\rho(w_2, \mathcal{G})\}, \quad \forall \rho \in \left[\frac{1}{2}, 1\right],$$

where $\Gamma := \left\{\gamma \in C([0,1], H^1_\mu(\mathcal{G})) \mid \gamma \text{ is continuous, } \gamma(0) = w_1, \gamma(1) = w_2\right\}$.

Proof. For any $\mu, k > 0$, denote

$$A_{\mu,k} := \{u \in H^1_\mu(G) \,|\, \int_G |u'|^2 dx < k\},$$

$$\partial A_{\mu,k} := \{u \in H^1_\mu(G) \,|\, \int_G |u'|^2 dx = k\}.$$

First note that, since the graph is noncompact, for any $\mu, k > 0$, we have $A_{\mu,k} \neq \emptyset$ (and, similarly, $A_{\mu,k} \neq \emptyset$). Indeed, take any function $v \in C_c^\infty(\mathbb{R})$ with $\|v\|^2_{L^2(\mathbb{R})} = \mu$, and consider $v_t(x) = t^{\frac{1}{2}} v(tx)$, for $t > 0$. We have that $\|v_t\|^2_{L^2(\mathbb{R})} = \mu$ and $\|v'_t\|^2_{L^2(\mathbb{R})} = t^2 \|v'\|^2_{L^2(\mathbb{R})}$ for every $t > 0$. Since G contains at least a half-line, it contains arbitrarily large intervals. Therefore, any function v_t can be regarded as a function in $H^1_\mu(G)$, still denoted by v_t, with the support entirely located on the half-line l_1; and, in particular, $v_t \in A_{\mu,k}$ for t small enough.

Now, by the Gagliardo-Nirenberg inequality on metric graphs (see Proposition 2.1 in [4]), we obtain

$$E_\rho(u, G) \geq \frac{1}{2} \|u'\|^2_{L^2(G)} - \rho \frac{C_p}{p} \mu^{\frac{p}{4}+\frac{1}{2}} \|u'\|^{\frac{p}{2}-1}_{L^2(G)}, \qquad \forall u \in H^1_\mu(G). \tag{3.1}$$

Then, for any $\mu > 0$ and $u \in \partial A_{\mu,k_0}$ with $k_0 = \frac{1}{2}(\frac{p}{2C_p})^{\frac{4}{p-6}} \mu^{-\frac{p+2}{p-6}}$, we have

$$\inf_{u \in \partial A_{\mu,k_0}} E_\rho(u, G) \geq k_0 \left(\frac{1}{2} - \frac{C_p}{p} \mu^{\frac{p+2}{4}} k_0^{\frac{p-6}{4}} \right) =: \alpha > 0, \tag{3.2}$$

for every $\rho \in [1/2, 1]$.

Next, observe that for any $u \in A_{\mu,k}$

$$E_\rho(u, G) \leq \frac{1}{2} \int_G |u'|^2 dx \leq \frac{1}{2} k.$$

Thus, recalling that $A_{k,\mu} \neq \emptyset$ for all $\mu, k > 0$, it is possible to choose a $w_1 \in H^1_\mu(G)$ such that

$$\|w'_1\|^2_{L^2(G)} < k_0 \quad \text{and} \quad E_\rho(w_1, G) \leq \frac{\alpha}{2} \quad \forall \rho \in \left[\frac{1}{2}, 1 \right]. \tag{3.3}$$

Moreover, we also observe that we can identify any bounded edge, say e_1, with the interval $[-\ell_1/2, \ell_1/2]$. It follows that any compactly supported H^1 function w on such interval, with mass μ, can be seen as a function in $H^1_\mu(G)$. Defining $w_t(x) := t^{1/2} w(tx)$, with $t > 1$, it is not difficult to check that $w_t \in H^1_\mu(G)$ (notice in particular that the support of w_t is shrinking on e_1 as t becomes larger), and that

$$E_\rho(w_t, G) = \frac{t^2}{2} \int_{e_1} |w'|^2 dx - \frac{\rho t^{\frac{p-2}{2}}}{p} \int_{e_1} |w|^p dx \leq \frac{t^2}{2} \left(\int_{e_1} |w'|^2 dx - \frac{t^{\frac{p-6}{2}}}{p} \int_{e_1} |w|^p dx \right),$$

for every $\rho \in [\frac{1}{2}, 1]$. Since $p > 6$, then the right-hand side tends to $-\infty$ as $t \to +\infty$, and in particular there exists $t_2 > 0$ large enough such that $\|w'_t\|^2_{L^2(G)} = t^2 \|w'\|^2_{L^2(G)} > 2k_0$ and $E_\rho(w_t, G) < 0$ for all $t > t_2$ and $\rho \in [\frac{1}{2}, 1]$. We set $w_2 := w_{2t_2}$, and we point out that

$$\|w'_2\|^2_{L^2(G)} > 2k_0 \quad \text{and} \quad E_\rho(w_2, G) < 0 \quad \forall \rho \in \left[\frac{1}{2}, 1 \right]. \tag{3.4}$$

At this point the thesis follows easily. Let Γ and c_ρ be defined as in the statement of the lemma for our choice of w_1 and w_2; the fact that $\Gamma \neq 0$ is straightforward, since

$$\gamma_0(t) := \frac{\mu^{1/2}}{\|(1-t)w_1 + tw_2\|_{L^2(G)}} [(1-t)w_1 + tw_2] \qquad t \in [0, 1]$$

7

belongs to Γ. By (3.3) and (3.4), we note that for every $\gamma \in \Gamma$ there exists $t_\gamma \in [0,1]$ such that $\gamma(t_\gamma) \in \partial A_{\mu,k_0}$, by continuity. Therefore, for any $\rho \in [1/2,1]$, we have for every $\gamma \in \Gamma$

$$\max_{t \in [0,1]} E_\rho(\gamma(t), \mathcal{G}) \geq E_\rho(\gamma(t_\gamma), \mathcal{G}) \geq \inf_{u \in \partial A_{\mu,k_0}} E_\rho(u, \mathcal{G}) \geq \alpha$$

(see (3.2)) and thus $c_\rho \geq \alpha$ while

$$\max\{E_\rho(w_1, \mathcal{G}), E_\rho(w_2, \mathcal{G})\} = E_\rho(w_1, \mathcal{G}) < \frac{\alpha}{2}. \qquad \square$$

The following lemma, together with Lemma 2.5, will be central to establish the convergence of the Palais-Smale sequences provided by the application of Theorem 2.4.

Lemma 3.2. *For any $\lambda < 0$, there exists a subspace Y of $H^1(\mathcal{G})$ with $\dim Y = 3$ such that*

$$\int_{\mathcal{G}} |w'|^2 \, dx + \lambda \int_{\mathcal{G}} |w|^2 \, dx \leq \frac{\lambda}{2} \|w\|_{H^1(\mathcal{G})}^2, \qquad \forall w \in Y. \tag{3.5}$$

Proof. Take $\phi \in C_0^\infty(\mathbb{R}^+)$ with $\operatorname{supp} \phi \subset [\frac{3}{4}, \frac{5}{4}]$ such that $\int_0^{+\infty} |\phi|^2 dx = 1$. Recalling that l_1 is an arbitrary half-line of $\mathcal{G}$, we define the following functions for $\sigma > 0$:

$$\psi_{0,\sigma}(x) = \begin{cases} 0 & \text{for } x \in \mathcal{G} \setminus l_1 \\ \sigma^{\frac{1}{2}} \phi(\sigma x) & \text{for } x \in l_1, \end{cases}$$

$$\psi_{1,\sigma}(x) = \begin{cases} 0 & \text{for } x \in \mathcal{G} \setminus l_1 \\ \sigma^{\frac{1}{2}} \phi(\sigma x - 1) & \text{for } x \in l_1, \end{cases}$$

and

$$\psi_{2,\sigma}(x) = \begin{cases} 0 & \text{for } x \in \mathcal{G} \setminus l_1 \\ \sigma^{\frac{1}{2}} \phi(\sigma x - 2) & \text{for } x \in l_1. \end{cases}$$

Clearly, for any $\sigma > 0$, $\int_{\mathcal{G}} |\psi_{j,\sigma}|^2 \, dx = 1$ for all $j = 0, 1, 2$, and the three functions are mutually orthogonal in $H^1(\mathcal{G})$, having disjoints supports. Assume that $w = \sum_{j=0}^{2} \theta_j \psi_{j,\sigma}$. Then

$$\int_{\mathcal{G}} |w'|^2 \, dx + \lambda \int_{\mathcal{G}} |w|^2 \, dx = \sigma^2 \Big(\sum_{j=0}^{2} \theta_j^2 \int_0^{+\infty} |\phi'|^2 \, dx \Big) + \lambda \Big(\sum_{j=0}^{2} \theta_j^2 \int_0^{+\infty} |\phi|^2 \, dx \Big)$$

$$= (\sigma^2 \alpha + \lambda) \sum_{j=0}^{2} \theta_j^2,$$

where $\alpha := \int_0^{+\infty} |\phi'|^2 \, dx > 0$. Similarly, $\|w\|_{H^1(\mathcal{G})}^2 = (\sigma^2 \alpha + 1) \sum_{j=0}^{2} \theta_j^2$. Therefore, provided that $w \neq 0$,

$$\frac{\int_{\mathcal{G}} |w'|^2 \, dx + \lambda \int_{\mathcal{G}} |w|^2 \, dx}{\|w\|_{H^1(\mathcal{G})}^2} = \frac{\sigma^2 \alpha + \lambda}{\sigma^2 \alpha + 1} \leq \frac{\lambda}{2}$$

for every $\sigma > 0$ sufficiently small, and for every $\theta_0, \theta_1, \theta_2 \in \mathbb{R}$. $\qquad \square$

Remark 3.3. The validity of the previous lemmata relies on the existence of at least one half-line, thus on the non-compact character of the graph. Results like Lemma 3.2 are expected to hold broadly in problems set on a non-compact domain.

Finally, we recall a definition. In what follows, χ_A denotes the characteristic function of A.

Definition 3.4. For any graph $\mathcal{F}$, any subgraph $\mathcal{F}' \subset \mathcal{F}$, and any solution $U \in C(\mathcal{F}) \cap H^1_{\mathrm{loc}}(\mathcal{F})$, not necessarily in $H^1(\mathcal{F})$, of

$$\begin{cases} -U'' + \lambda U = \rho|U|^{p-2}U\chi_{\mathcal{F}'} & \text{in } \mathcal{F}, \\ \sum_{e \succ v} U'(\mathrm{v}) = 0 & \text{for any vertex v of } \mathcal{F}, \end{cases} \tag{3.6}$$

with $\lambda, \rho \in \mathbb{R}$, we consider

$$Q(\varphi; U, \mathcal{F}) := \int_{\mathcal{F}} \left(|\varphi'|^2 + (\lambda - (p-1)\rho|U|^{p-2}\chi_{\mathcal{F}'})\varphi^2 \right) dx, \quad \forall \varphi \in H^1(\mathcal{F}) \cap C_c(\mathcal{F}). \tag{3.7}$$

The *Morse index of U*, denoted by $m(U)$, is the maximal dimension of a subspace $W \subset H^1(\mathcal{F}) \cap C_c(\mathcal{F})$ such that $Q(\varphi; U, \mathcal{F}) < 0$ for all $\varphi \in W \setminus \{0\}$.

The main aim of this section is to prove the following result.

Proposition 3.5. *For any fixed $\mu > 0$ and almost every $\rho \in I_\rho$, there exists $(u_\rho, \lambda_\rho) \in H^1_\mu(\mathcal{G}) \times \mathbb{R}^+$ which solves*

$$\begin{cases} -u''_\rho + \lambda_\rho u_\rho = \rho\kappa(x)u_\rho^{p-1}, & u_\rho > 0 \quad \text{in } \mathcal{G}, \\ \sum_{e \succ v} u'_\rho(\mathrm{v}) = 0 & \text{for any vertex v}. \end{cases} \tag{3.8}$$

Moreover, $E_\rho(u_\rho, \mathcal{G}) = c_\rho$ and $m(u_\rho) \leq 2$.

Proof. For simplicity, we omit the dependence of the functionals $E_\rho(\cdot \mathcal{G})$ on $\mathcal{G}$. We apply Theorem 2.4 to the family of functionals E_ρ, with $E = H^1(\mathcal{G})$, $H = L^2(\mathcal{G})$, $S_\mu = H^1_\mu(\mathcal{G})$, and Γ defined in Lemma 3.1. Setting

$$A(u) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 \, dx \quad \text{and} \quad B(u) = \frac{\rho}{p} \int_{\mathcal{G}} |u|^p \, dx,$$

assumption (2.2) holds, since we have that

$$u \in H^1_\mu(\mathcal{G}), \ \|u\| \to +\infty \quad \Longrightarrow \quad A(u) \to +\infty.$$

Let E'_ρ and E''_ρ denote respectively the free first and second derivatives of E_ρ. Clearly, E'_ρ and E''_ρ are both of class C^1, and hence locally Hölder continuous, on $H^1_\mu(\mathcal{G})$, which implies that assumption (2.1) holds.

Thus, taking into account Lemma 3.1, by Theorem 2.4 and the considerations just after it, for almost every $\rho \in [1/2, 1]$, there exist a bounded sequence $\{u_{n,\rho}\} \subset H^1_\mu(\mathcal{G})$, that we shall denote simply by $\{u_n\}$ from now on, and a sequence $\{\zeta_n\} \subset \mathbb{R}^+$ with $\zeta_n \to 0^+$, such that

$$E'_\rho(u_n) + \lambda_n u_n \to 0 \quad \text{in the dual of} \quad H^1_\mu(\mathcal{G}), \tag{3.9}$$

where

$$\lambda_n := -\frac{1}{\mu}E'_\rho(u_n)u_n. \tag{3.10}$$

Moreover, if the inequality

$$\int_{\mathcal{G}} \left[|\varphi'|^2 + \left(\lambda_n - (p-1)\rho\kappa(x)|u_n|^{p-2}\right)\varphi^2 \right] dx = E''_\rho(u_n)[\varphi, \varphi] + \lambda_n\|\varphi\|^2_{L^2(\mathcal{G})} < -\zeta_n\|\varphi\|^2_{H^1(\mathcal{G})} \tag{3.11}$$

holds for any $\varphi \in W_n \setminus \{0\}$ in a subspace W_n of $T_{u_n}S_\mu$, then the dimension of W_n is at most 1. In addition, as explained in [12, Remark 1.4], since $u \in H^1_\mu(\mathcal{G}) \implies |u| \in H^1_\mu(\mathcal{G})$, $w_1, w_2 \geq 0$, the map $u \mapsto |u|$ is continuous, and $E_\rho(u) = E_\rho(|u|)$, it is possible to choose $\{u_n\}$ with the property that $u_n \geq 0$ on $\mathcal{G}$.

9

The sequence $\{u_n\}$ being bounded, it follows by (3.10) that $\{\lambda_n\}$ is bounded. Then, passing to a subsequence, there exists $\lambda_\rho \in \mathbb{R}$ such that $\lim_{n\to+\infty} \lambda_n = \lambda_\rho$. Furthermore, there exists $u_\rho \in H^1(\mathcal{G})$ such that

$$u_n \rightharpoonup u_\rho \ \text{ in } H^1(\mathcal{G}), \tag{3.12}$$

$$u_n \to u_\rho \ \text{ in } L^r_{\mathrm{loc}}(\mathcal{G}), r > 2, \tag{3.13}$$

$$u_n(x) \to u_\rho(x) \ \text{ for a.e. } x \in \mathcal{G}, \tag{3.14}$$

which implies that $u_\rho \geq 0$. Also, recalling (3.9) and that $\lim_{n\to+\infty} \lambda_n = \lambda_\rho$, we see that u_ρ satisfies

$$-u_\rho'' + \lambda_\rho u_\rho = \rho\kappa(x) u_\rho^{p-1} \tag{3.15}$$

with the Kirchhoff condition at the vertices. In order to show that $u_\rho \not\equiv 0$ we shall first prove that $\lambda_\rho \geq 0$. Here the Morse type information (3.11) proves decisive. Firstly, since the codimension of $T_{u_n} S_\mu$ is 1, we infer that if the inequality (3.11) holds for every $\varphi \in V_n \setminus \{0\}$ for a subspace V_n of $H^1(\mathcal{G})$, then the dimension of V_n is at most 2. Secondly, by Lemma 3.2, we have that for every $\lambda < 0$, there exist a subspace Y of E with $\dim Y \geq 3$ and $a > 0$ such that, for $n \in \mathbb{N}$ large,

$$E_\rho''(u_n)[\varphi,\varphi] + \lambda|\varphi|^2 \leq \int_\mathcal{G} |\varphi'|^2\,dx + \lambda\int_\mathcal{G} |\varphi|^2 \leq -a\|\varphi\|^2, \qquad \text{for all } \varphi \in Y\setminus\{0\}.$$

Therefore, Lemma 2.5 implies that $\lambda_\rho \geq 0$.

Now from (3.9) and the fact that $\lambda_n \to \lambda_\rho$, we deduce that

$$\int_\mathcal{G} \left(u_n'\varphi' + \lambda_\rho u_n\varphi\right) dx - \rho\int_\mathcal{K} u_n^{p-1}\varphi\,dx = o(1)\|\varphi\|_{H^1(\mathcal{G})}.$$

Moreover, by (3.15),

$$\int_\mathcal{G} \left(u_\rho'\varphi' + \lambda_\rho u_\rho\varphi\right) dx - \rho\int_\mathcal{K} u_\rho^{p-1}\varphi\,dx = 0$$

Therefore, taking the difference and exploiting (3.12)-(3.14) we infer that

$$\int_\mathcal{G} |(u_n - u_\rho)'|^2\,dx + \lambda_\rho\int_\mathcal{G} |u_n - u_\rho|^2\,dx \to 0 \tag{3.16}$$

as $n \to \infty$. If we assume that $u_\rho \equiv 0$ then we deduce that

$$\int_\mathcal{G} |u_n'|^2\,dx + \lambda_\rho\int_\mathcal{G} |u_n|^2\,dx \to 0. \tag{3.17}$$

If $\lambda_\rho > 0$, this is not possible since $\|u_n\|_{L^2(\mathcal{G})}^2 = \mu > 0$. If $\lambda_\rho = 0$, then (3.17) contradicts the fact that the mountain-pass level satisfies $c_\rho > 0$. Hence, $u_\rho \not\equiv 0$. Appealing to the Kirchhoff condition and the uniqueness theorem for ODEs, we have in fact that $u_\rho > 0$ in $\mathcal{G}$: indeed, assume by contradiction that there exists $x_0 \in \mathcal{G}$ such that $u_\rho(x_0) = 0$. If x_0 stays in the interior of some edge, then by $u_\rho \geq 0$ on $\mathcal{G}$ it follows that $u_\rho'(x_0) = 0$. If instead x_0 is a vertex, then by $u_\rho(x) \geq 0$ and by the Kirchhoff condition we also get $u_\rho'(x_0) = 0$. Hence, by uniqueness, $u_\rho \equiv 0$ on any edge containing x_0; but then, by repeating this argument a finite number of times (since $\mathcal{G}$ has finitely many vertices and edges), we deduce that $u_\rho \equiv 0$ on $\mathcal{G}$, which is the desired contradiction.

Now we claim that $\lambda_\rho > 0$. If not, then $\lambda_\rho = 0$. Identifying l_1 with $[0,+\infty)$, we have that u_ρ is a C^2 function on $[0,+\infty)$ such that $u_\rho''(x) = 0$ for all $x \in (0,+\infty)$. It follows that $u_\rho > 0$ on $[0,+\infty)$,

$$u_\rho(x) = u_\rho(0) + u_\rho'(0)x, \quad \text{and} \quad u_\rho \in L^2(\mathcal{G}),$$

which are incompatible. Hence the claim holds.

Having proved that $\lambda_\rho > 0$, we get from (3.16) that $u_n \to u_\rho$ strongly in $H^1(\mathcal{G})$.

It remains to show that the Morse index $m(u_\rho)$ is at most 2. Since the tangent space $T_u H^1_\mu(\mathcal{G})$ has codimension 1, it suffices to show that $u_\rho \in S_\mu$ has Morse index at most 1 as a constrained critical point. If not, in view of Definition 2.3 we may assume by contradiction that there exists a $W_0 \subset T_u S_\mu$ with $\dim W_0 = 2$ such that

$$D^2 E_\rho(u_\rho)[w,w] < 0 \quad \text{for all } w \in W_0 \backslash \{0\}. \tag{3.18}$$

Then, since W_0 is of finite dimension, there exists $\beta > 0$ such that

$$D^2 E_\rho(u_\rho)[w,w] < -\beta \quad \text{for all } w \in W_0 \backslash \{0\} \text{ with } \|w\|_{H^1(\mathcal{G})} = 1,$$

using the homogeneity of $D^2 E_\rho(u_\rho)$, we deduce that

$$D^2 E_\rho(u_\rho)[w,w] < -\beta \|w\|^2_{H^1(\mathcal{G})} \quad \text{for all } w \in W_0 \backslash \{0\}.$$

Now, from [12, Corollary 1] or using directly that E'_ρ and E''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0,1]$, it follows that there exists $\delta_1 > 0$ small enough such that, for any $v \in S_\mu$ satisfying $\|v - u\| \leq \delta_1$,

$$D^2 E_\rho(v)[w,w] < -\frac{\beta}{2} \|w\|^2_{H^1(\mathcal{G})} \quad \text{for all } w \in W_0 \backslash \{0\}. \tag{3.19}$$

Hence, noting that $\|u_n - u_\rho\|_{H^1(\mathcal{G})} \leq \delta_1$ for $n \in \mathbb{N}$ large enough, by (3.18)-(3.19) and $\zeta_n \to 0^+$ we get

$$D^2 E_\rho(u_n)[w,w] < -\frac{\beta}{2} \|w\|^2_{H^1(\mathcal{G})} < -\zeta_n \|w\|^2_{H^1(\mathcal{G})} \quad \text{for all } w \in W_0 \backslash \{0\} \tag{3.20}$$

for any such large n. Therefore, since $\dim W_0 > 1$ and recalling that

$$E''_\rho(u_n)[w,w] + \lambda_\rho \|w\|^2_{L^2(\mathcal{G})} = D^2 E_\rho(u_n)[w,w],$$

Equation (3.20) provides a contradiction with Equation (3.11). Thus we infer that $\tilde{m}_0(u_\rho) \leq 1$. $\qquad \square$

4   Proof of Theorem 1.1

From Proposition 3.5 we know that there exists a sequence $\rho_n \to 1^-$, and a corresponding sequence of critical points $u_{\rho_n} \in H^1_\mu(\mathcal{G})$ of $E_{\rho_n}(\cdot, \mathcal{G})$ constrained to $H^1_\mu(\mathcal{G})$, lying at the mountain-pass level c_{ρ_n}, with a Morse index $m(u_{\rho_n})$ lesser or equal to 2. Additionally, the associated Lagrange multipliers satisfy $\lambda_n > 0$. To obtain the existence of solutions for the original problem, at a strictly positive energy level, it clearly suffices to show that $\{u_{\rho_n}\}$ converges. In this direction the key point is to show that $\{u_{\rho_n}\}$ is bounded in $H^1(\mathcal{G})$.

By Lemma 3.1 and the monotonicity of c_ρ,

$$E_1(w_1, \mathcal{G}) \leq E_\rho(w_1, \mathcal{G}) \leq c_\rho \leq c_{1/2}, \qquad \forall \rho \in \left[\frac{1}{2}, 1\right],$$

which implies that c_{ρ_n} is bounded. In addition, thanks to the Kirchhoff condition

$$\int_\mathcal{G} \left(|u'_n|^2 + \lambda_n u_n^2\right) dx = \int_\mathcal{K} |u_n|^p dx,$$

whence it follows that

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_\mathcal{G} |u'_n|^2 dx = c_n + \frac{\lambda_n \mu}{p}.$$

Therefore, if $\{\lambda_n\}$ is bounded, then $\{u_{\rho_n}\}$ is bounded as well. We shall thus assume that $\lambda_n \to +\infty$, up to extraction of a subsequence and, in order to reach a contradiction, develop a blow-up analysis for $\{u_{\rho_n}\}$ as in [15] (see also [20, 21, 37]).

Precisely, writing for simplicity $u_n := u_{\rho_n}$ for $\rho_n \to 1^-$, we consider the behavior of a sequence of solutions $\{u_n\} \in H^1(\mathcal{G})$ of the following problem:

$$\begin{cases} -u_n'' + \lambda_n u_n = \rho_n \kappa(x) u_n^{p-1} & \text{on } \mathcal{G}, \\ u_n > 0 & \text{on } \mathcal{G}, \\ \sum_{e \succ v} u_{e,n}'(v) = 0 & \forall v \in \mathcal{V}, \end{cases} \tag{4.1}$$

where $m(u_n) \leq 2$ for all $n \in \mathbb{N}$, and it is assumed that $\lambda_n \to +\infty$.

Firstly, we note that the local maximum points of u_n stay on the compact part $\mathcal{K}$ of $\mathcal{G}$.

Lemma 4.1. *Let $x_n \in \mathcal{G}$ be a local maximum point of u_n. Then $x_n \in \mathcal{K}$, and*

$$u_n(x_n) \geq \lambda_n^{\frac{1}{p-2}}.$$

Proof. Let x_n be in the interior of some half line $l_i \in \mathcal{E}$. In coordinates, $l_i \simeq [0, +\infty)$ and $x_n \in (0, +\infty)$, and the restriction of u_n on l_i is a C^2 function on $[0, +\infty)$, by regularity. Since x_n is a local maximum point, we have $u_n''(x_n) \leq 0$, which implies that $\lambda_n u_n(x_n) \leq 0$. A contradiction. Thus $x_n \in \mathcal{K}$, and at this point the estimate can be proved as in [15, Lemma 4.1]. $\square$

In the following, we denote by $B_r(x_0) = \{x \in \mathcal{G} : \text{dist}(x, x_0) < r\}$. Moreover, we denote by $\mathcal{G}_m$ the star-graph with $m \geq 1$ half-lines glued together at their common origin 0. In particular, $\mathcal{G}_1 = \mathbb{R}^+$, and $\mathcal{G}_2$ is isometric to $\mathbb{R}$.

Since $m(u_n) \leq 2$, we can describe the asymptotic behavior of the sequence $\{u_n\}$ near a local maximum point as $\lambda_n \to +\infty$. From now on, we denote by χ_A the characteristic function of a set A.

Proposition 4.2. *Let $x_n \in \mathcal{G}$ be such that, for some $R_n \to \infty$,*

$$u_n(x_n) = \max_{B_{R_n \tilde{\varepsilon}_n}(x_n)} u_n \quad \text{where } \tilde{\varepsilon}_n = (u_n(x_n))^{-\frac{p-2}{2}} \to 0.$$

Suppose moreover that

$$\limsup_{n \to \infty} \frac{\text{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n} = +\infty. \tag{4.2}$$

Then, up to extraction of a subsequence, points (i)-(iv) in [15, Theorem 4.2] holds. If, instead of (4.2), we suppose that

$$\limsup_{n \to \infty} \frac{\text{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n} < +\infty; \tag{4.3}$$

then, up to extraction of a subsequence :

(i') *$x_n \to v \in \mathcal{V}$, and all the x_n lie on the same edge $e_1 \simeq [0, \rho_1]$, where $e_1 \subset \mathcal{K}$ and the vertex v is identified by the coordinate 0 on e_1.*

(ii') *Let $e_2 \simeq [0, r_2], \ldots, e_j \simeq [0, r_j]$ be the other bounded edges of $\mathcal{G}$ having v as a vertex (if any), and let $l_{j+1} \simeq [0, +\infty), \ldots, l_m \simeq [0, +\infty)$ be the half-lines of $\mathcal{G}$ having v as a vertex (if any), where v is identified by the coordinate 0 on each e_i and l_i. Setting $\varepsilon_n = \lambda_n^{-\frac{1}{2}}$, we have that*

$$\frac{\tilde{\varepsilon}_n}{\varepsilon_n} \to (0, 1],$$

$$\limsup_{n \to \infty} \frac{\text{dist}(x_n, \mathcal{V})}{\varepsilon_n} < +\infty, \tag{4.4}$$

12

and the scaled sequence defined by

$$v_n(y) := \varepsilon_n^{\frac{2}{p-2}} u_n(\varepsilon_n y) \quad \text{for } y \in \frac{e_i}{\varepsilon_n}, \text{ with } i = 1,\ldots,j, \quad \text{or for } y \in \frac{l_i}{\varepsilon_n}, \text{ with } i = j+1,\ldots,m$$

converges to a limit V in $C^0_{\text{loc}}(\mathcal{G}_m)$ as $n \to \infty$. Denoting by V_i the restriction of V to the i-th half-line ℓ_i of $\mathcal{G}_m$, and by $v_{i,n}$ the restriction of v_n to e_i/ε_n, we have moreover that $v_{i,n} \to V_i$ in $C^2_{\text{loc}}([0,+\infty))$. Finally, $V \in H^1(\mathcal{G}_m)$ is a positive solution to the NLS-type equation on the star-graph

$$\begin{cases} -V'' + V = V^{p-1}\chi_{\ell_1 \cup \cdots \cup \ell_j}, \quad V > 0 \quad \text{in } \mathcal{G}_m, \\ \sum_{i=1}^m V_i'(0^+) = 0, \\ V(x) \to 0 \qquad\qquad\qquad\qquad \text{as } \text{dist}(x,0) \to \infty \end{cases}$$

with a global maximum point $\bar{x}$ located on ℓ_1, whose coordinate is

$$\bar{x} = \lim_{n\to\infty} \bar{x}_n \in [0,+\infty), \quad \text{where} \quad \bar{x}_n := \frac{\text{dist}(x_n, \mathcal{V})}{\varepsilon_n}.$$

(iii') There exists $\phi_n \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$, with $\text{supp}\,\phi_n \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some $\bar{R} > 0$, such that

$$Q(\phi_n; u_n, \mathcal{G}) < 0.$$

(iv') For all $R > 0$ and $q \geq 1$, we have that

$$\lim_{n\to\infty} \lambda_n^{\frac{1}{2}-\frac{q}{p-2}} \int_{B_{R\varepsilon_n}(x_n)} u_n^q \, dx = \lim_{n\to\infty} \int_{B_R(\bar{x}_n)} v_n^q \, dy = \int_{[0,\bar{x}+R]} V_1^q \, dy + \sum_{i=2}^m \int_{[0,R-\bar{x}]} V_i^q \, dy = \int_{B_R(\bar{x})} V^q \, dy$$

(where $B_R(\bar{x}_n)$ and $B_R(\bar{x})$ denote the balls in the scaled and in the limit graphs, respectively).

The proof of the proposition can be obtained by making minor modifications to the one of Theorem 4.2 in [15]. Roughly speaking, with respect to the setting in [15], we have to take into account the fact that the points x_n (which belong to the compact core of $\mathcal{G}$, by Proposition 4.1) may accumulate on a vertex v of $\mathcal{G}$ which belongs to some half-line. We will need the following two statements, which correspond to Lemmata 4.3 and 4.5 in [15].

Lemma 4.3. *Let $U \in H^1_{\text{loc}}(\mathcal{G}_m)$ be a solution to*

$$\begin{cases} -U'' + \lambda U = \rho U^{p-1}\chi_{\ell_1 \cup \cdots \cup \ell_j} & \text{in } \mathcal{G}_m, \\ U > 0 & \text{in } \mathcal{G}_m, \\ \sum_{i=1}^m U_i'(0) = 0, \end{cases} \tag{4.5}$$

for some $p > 2$, $\rho, \lambda > 0$, where U_i denotes the restriction of U on the i-th half-line ℓ_i of $\mathcal{G}_m$, and $1 < j \leq m$. Suppose that U is bounded, and that U is stable outside a compact set K, in the sense that $Q(\varphi; U, \mathcal{G}_m) \geq 0$ for all $\varphi \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m \setminus K)$. Then $U(x) \to 0$ as $\text{dist}(x,0) \to +\infty$, and $U \in H^1(\mathcal{G})$.

Proof. On $\ell_1,\ldots,\ell_j$, one can argue exactly as in [15] (which in turn follows [21, Theorem 2.3]). On $\ell_{j+1},\ldots,\ell_m$, we note that U satisfies $U'' = \lambda U$ for some $\lambda > 0$, and, since U is bounded, we deduce that the restriction of U on each of these half-lines has the form $U(x) = ce^{-\sqrt{\lambda}x}$, for some $c > 0$. $\qquad\square$

Lemma 4.4. *Let $U \in H^1(\mathcal{G}_m)$ be any non-trivial solution of (4.5). Then its Morse index $m(U)$ is strictly positive.*

Proof. Thanks to the Kirchhoff condition

$$\int_{\mathcal{G}_m} \left(|U'|^2 + \lambda U^2 \right) dx = \sum_{i=1}^{j} \int_{\ell_i} \rho |U|^p \, dx.$$

Therefore

$$Q(U;U,\mathcal{G}_m) = (2-p) \sum_{i=1}^{j} \int_{\ell_i} |U|^p \, dx < 0$$

(recall that $j > 1$), and the thesis follows by density of $H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$ in $H^1(\mathcal{G}_m)$. $\qquad\square$

Proof of Proposition 4.2. We only emphasize the main differences with respect to [15]. Since $\{x_n\} \subset \mathcal{K}$ by Lemma 4.1, and $\mathcal{G}$ has a finite number of vertices and edges, up to extraction of a subsequence, the maximum points x_n belong to the same bounded edge $e_1 \simeq [0, r_1]$, and converge. If (4.2) holds, then the proof proceeds exactly as the one of [15, Theorem 4.2 under assumption (4.3)]. If instead (4.3) holds, then $x_n \to v \in \mathcal{V}$, since $\tilde{\varepsilon}_n \to 0$. Contrary to [15], we have to take into account the possibility that v is on some half-line of $\mathcal{G}$; thus, the proof does not directly follows from [15], but can be easily adjusted. By the previous discussion, (i') holds, and we can suppose that

$$\frac{d_n}{\tilde{\varepsilon}_n} \to \eta \in [0, +\infty), \quad d_n := \text{dist}(x_n, \mathcal{V}) = x_n.$$

Let

$$\tilde{u}_{i,n}(y) := \tilde{\varepsilon}_n^{\frac{2}{p-2}} u_n(\tilde{\varepsilon}_n y) \quad \text{for } y \in \tilde{e}_{i,n} := \frac{e_i}{\tilde{\varepsilon}_n}, \text{ with } i = 1, \ldots, j, \text{ and for } y \in \tilde{l}_{i,n} := \frac{l_i}{\tilde{\varepsilon}_n}, \text{ with } i = j+1, \ldots, m,$$

and $\tilde{u}_n = (\tilde{u}_{1,n}, \ldots, \tilde{u}_{m,n})$. Note that $\tilde{u}_n$ is defined on a graph $\mathcal{G}_{m,n}$ consisting in j expanding edges and $m - j$ half-lines, glued together at their common origin, which is identified with the coordinate 0 on each edge or half-line. In the limit $n \to \infty$, this graph converges to the star-graph $\mathcal{G}_m$. Clearly, for every $a > \eta + 1$ and large enough n

$$\tilde{u}_{1,n}\left(\frac{d_n}{\tilde{\varepsilon}_n}\right) = 1 = \max_{B_a(0)} \tilde{u}_n \tag{4.6}$$

(since $u_n(x_n) = \max_{B_{R_n \tilde{\varepsilon}_n}(x_n)} u_n$ for some $R_n \to +\infty$),

$$-\tilde{u}_n'' + \tilde{\varepsilon}_n^2 \lambda_n \tilde{u}_n = \rho_n \tilde{u}_n^{p-1} \chi_{\tilde{e}_{1,n} \cup \cdots \cup \tilde{e}_{j,n}}, \quad \tilde{u}_n > 0$$

on any edge of $\mathcal{G}_{m,n}$, and the Kirchhoff condition at the origin holds. Also, by Lemma 4.1,

$$\tilde{\varepsilon}_n^2 \lambda_n \in (0,1] \quad \forall n.$$

Thus, by elliptic estimates, for $i = 1, \ldots, j$ we have that $\tilde{u}_{i,n} \to \tilde{u}_i$ in $C^2_{\text{loc}}([0,+\infty))$ for every $i = 1, \ldots, j$, and the limit $\tilde{u}_i$ solves

$$-\tilde{u}_i'' + \tilde{\lambda} \tilde{u}_i = \tilde{u}_i^{p-1}, \quad \tilde{u}_i \geq 0 \quad \text{in } (0,+\infty) \tag{4.7}$$

for some $\tilde{\lambda} \in [0,1]$. Analogously, for $i = j+1, \ldots, m$ we have that $\tilde{u}_{i,n} \to \tilde{u}_i$ in $C^2_{\text{loc}}([0,+\infty))$, and the limit $\tilde{u}_i$ solves

$$-\tilde{u}_i'' + \tilde{\lambda} \tilde{u}_i = 0, \quad \tilde{u}_i \geq 0 \quad \text{in } (0,+\infty). \tag{4.8}$$

Moreover, since $\tilde{u}_n$ is continuous on $\mathcal{G}_{m,n}$ and by uniform convergence, $\tilde{u}_{i_1}(0) = \tilde{u}_{i_2}(0)$ for every $i_1 \neq i_2$, so that $\tilde{u} \simeq (\tilde{u}_1, \ldots, \tilde{u}_m)$ can be regarded as a function defined on $\mathcal{G}_m$. Since the convergence $\tilde{u}_{i,n} \to \tilde{u}_i$ takes

14

place in C^2 up to the origin, the Kirchhoff condition also passes to the limit. Now we exclude the case that $\tilde{u} \equiv 0$ on some half-line of $\mathcal{G}_m$. By local uniform convergence, we have that

$$\tilde{u}_1(\eta) = \lim_{n \to \infty} \tilde{u}_{1,n}\left(\frac{d_n}{\tilde{\varepsilon}_n}\right) = 1,$$

and η is a global maximum point of $\tilde{u}$, in view of (4.6). This implies that $\tilde{u}_1 > 0$ in $(0, +\infty)$, by the uniqueness theorem for ODEs. In turn, the Kirchhoff condition, and the uniqueness theorem for ODEs again, ensure that $\tilde{u}_i > 0$ on $(0, +\infty)$ for every i. Finally, we claim that

$$\text{the Morse index of } \tilde{u} \text{ is at most 2.} \tag{4.9}$$

The proof of this claim exploits the fact that $m(u_n) \leq 2$ for every n, is analogue to the one of [15, Eq. (4.10)], and hence is omitted.

The case $\tilde{\lambda} = 0$ can be ruled out by phase-plane analysis (the equation $\tilde{u}_1'' + \tilde{u}_1^{p-1} = 0$ on $\ell_1 = [0, +\infty)$ has only periodic sign-changing solutions, save the trivial one). Therefore,

$$0 < \liminf_{n \to \infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq \limsup_{n \to \infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq 1, \tag{4.10}$$

which proves the first estimate in (4.4). Furthermore, $\tilde{u}$ is bounded and hence, by Lemma 4.3, $\tilde{u} \to 0$ as $|x| \to +\infty$, and $\tilde{u} \in H^1(\mathcal{G}_m)$.

At this point it is equivalent, but more convenient, to work with v_n defined in point (ii') of the proposition, rather than with $\tilde{u}_n$. By (4.3) and (4.10),

$$\limsup_{n \to \infty} \frac{\text{dist}(x_n, \mathcal{V})}{\varepsilon_n} < +\infty.$$

Thus, similarly to what was done before, one can show that v_n converges, in $C^0_{\text{loc}}(\mathcal{G}_m)$ and in $C^2_{\text{loc}}([0, +\infty))$ on every half-line, to a limit function $V \simeq (V_1, \ldots, V_m)$, which solves

$$\begin{cases} -V'' + V = V^{p-1}\chi_{\ell_1 \cup \cdots \cup \ell_j}, & V \geq 0 \quad \text{in } \mathcal{G}_m, \\ \sum_{i=1}^m V_i'(0^+) = 0; \end{cases} \tag{4.11}$$

furthermore, V has a positive global maximum on the half-line ℓ_1, $V_1(\bar{x}) \geq 1$ (thus $V > 0$ in $\mathcal{G}_m$), and has finite Morse index $m(V) \leq 2$. Moreover, by Lemma 4.3, $V \to 0$ as $|x| \to \infty$. Thus, (ii') is proved. Point (iv') follows directly by local uniform convergence. Finally, point (iii') is a consequence of Lemma 4.4. This implies that there exists $\phi \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$ such that $Q(\phi; V, \mathcal{G}_m) < 0$; thus, defining

$$\phi_{i,n}(x) = \varepsilon_n^{\frac{1}{2}}\phi_i\left(\frac{x - x_n}{\varepsilon_n}\right),$$

it is not difficult to deduce that for sufficiently large n we have $Q(\phi_{i,n}; u_n, \mathcal{G}) < 0$, and $\text{supp}\,\phi_{i,n} \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some positive $\bar{R}$. $\qquad\square$

In what follows, we focus on the global behavior of $\{u_n\}$ and show the way that $\{u_n\}$ decays exponentially away from local maximum points. In fact, by similar arguments to those used to establish Theorem 4.6 in [15] (see also [17, 20, 21, 37]), we have the following result.

Proposition 4.5. *There exist $k \in \{1, 2\}$, and sequences of points $\{P_n^i\}$, $i \in \{1, k\}$, such that*

$$\lambda_n \text{dist}(P_n^1, P_n^2) \to +\infty \quad (\text{if } k = 2), \tag{4.12}$$

$$u_n(P_n^i) = \max_{B_{R_n\lambda_n^{-1/2}}(P_n^i)} u_n \quad \text{for some } R_n \to +\infty, \text{ for every } i, \tag{4.13}$$

and constants $C_1, C_2 > 0$ such that

$$u_n(x) \leq C_1 \lambda_n^{\frac{1}{p-2}} \sum_{i=1}^{k} e^{-C_2 \lambda_n^{\frac{1}{2}} \mathrm{dist}(x,P_n^i)} + C_1 \lambda_n^{\frac{1}{p-2}} \sum_{j=1}^{m_3} e^{-C_2 \lambda_n^{\frac{1}{2}} \mathrm{dist}(x,v_j)}, \quad \forall x \in \mathcal{G} \setminus \bigcup_{i=1}^{k} B_{R\lambda_n^{-1/2}}(P_n^i),$$

where $v_1, \ldots, v_{m_3}$ are all the vertices of $\mathcal{G}$.

Proof. Having proved Proposition 4.2, we can adjust the proof of [15, Theorem 4.6] with minor changes.

Step 1) There exist $k \in \{1, 2\}$, and sequences of points $\{P_n^i\}$, $i \in \{1, k\}$, such that (4.12) *and* (4.13) *hold, and moreover*

$$\lim_{R \to +\infty} \left(\limsup_{n \to \infty} \lambda_n^{-\frac{1}{p-2}} \max_{d_n(x) \geq R\lambda_n^{-1/2}} u_n(x) \right) = 0. \tag{4.14}$$

where $d_n(x) = \min\{\mathrm{dist}(x, P_n^i) : i = 1, k\}$ is the distance function from $\{P_1^n, P_k^n\}$.

The proof of this point, as in [15], can be directly adapted from [21, Theorem 3.2].

Step 2) Conclusion of the proof. Here we argue as in the second step of the proof of [15, Theorem 4.6]. Therein, a comparison argument is used based on the following estimates: on one side, for all small $\varepsilon \in (0, 1)$, there exist $R > 0$ and $n_R \in \mathbb{N}$ such that

$$\max_{d_n(x) > R\lambda_n^{-1/2}} u_n(x) \leq \lambda_n^{\frac{1}{p-2}} \varepsilon, \quad \forall n \geq n_R. \tag{4.15}$$

Moreover, in $A_n := \{d_n(x) > R\lambda_n^{-1/2}\}$ it results that

$$-u_n'' + \frac{\lambda_n}{2} u_n \leq 0. \tag{4.16}$$

Once we have checked that these estimates are satisfied also in the present setting, we can follow [15] verbatim.

Inequality (4.15) is given by the first step of this proof. Regarding (4.16), its validity on $\mathcal{K} \cap A_n$ follows from (4.15), since on this set

$$u_n'' = (\lambda_n - u_n^{p-2}) u_n$$

and $\lambda_n - u_n^{p-2} \geq \lambda_n/2$ provided that $\varepsilon > 0$ is small enough. On $\mathcal{G} \setminus \mathcal{K}$, instead, we can directly observe that

$$u_n'' = \lambda_n u_n \geq \frac{\lambda_n}{2} u_n$$

since $\lambda_n > 0$ and $u_n \geq 0$. Therefore, (4.15) and (4.16) holds, and the thesis follows. $\square$

Now, we can give

Proof of Theorem 1.1. We have already observed that if $\{\lambda_n\}$ is bounded then also $\{u_{\rho_n}\}$ is bounded. We shall thus assume by contradiction that $\lambda_n \to +\infty$, up to extraction of a subsequence. Then, Propositions 4.2 and 4.5 hold for $u_n := u_{\rho_n}$ and by similar arguments as in the proof of Proposition 5.1 in [15], we can get a contradiction.

Since $\{\lambda_n\}$ is bounded, passing to a subsequence, there exists $\lambda^* \in \mathbb{R}$ such that $\lim_{n \to +\infty} \lambda_n = \lambda^*$. Also, since $\{u_n\}$ is bounded and $u_n > 0$ on $\mathcal{G}$, there exists $u^* \in H^1(\mathcal{G})$ such that

$$u_n \rightharpoonup u^* \ \text{ in } \ H^1(\mathcal{G}),$$
$$u_n \to u^* \ \text{ in } \ L^r_{\mathrm{loc}}(\mathcal{G}), r > 2,$$
$$u_n(x) \to u^*(x) \ \text{ for a.e. } x \in \mathcal{G},$$

16

which implies that $u^* \geq 0$ and u^* satisfies

$$-(u^*)'' + \lambda^* u^* = \kappa(x)(u^*)^{p-1}$$

together with the Kirchhoff condition

$$\sum_{e_i \succ v_j} (u^*)'(v_j) + \sum_{l_i \succ v_j} (u^*)'(v_j) = 0, \qquad \forall j = 1, \cdots, m_3.$$

By similar arguments to those in the proof of Proposition 3.5, we can deduce that $\lambda^* > 0$, $u_n \to u^*$ strongly in $H^1(\mathcal{G})$, and that $u^* > 0$. This completes the proof. $\qquad \square$

References

[1] R. Adami, F. Boni and S. Dovetta. Competing nonlinearities in NLS equations as source of threshold phenomena on star graphs. *J. Funct. Anal.* 283(1): Paper No. 109483, 34 pp. 2022.

[2] R. Adami, C. Cacciapuoti, D. Finco and D. Noja. Constrained energy minimization and orbital stability for the NLS equation on a star graph. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 31(6): 1289-1310, 2014.

[3] R. Adami, E. Serra and P. Tilli. NLS ground states on graphs. *Calc. Var. Partial Differential Equations* 54 (1): 743-761, 2015.

[4] R. Adami, E. Serra and P. Tilli. Threshold phenomena and existence results for NLS ground states on metric graphs. *J. Funct. Anal.* 271 (1): 201-223, 2016.

[5] R. Adami, E. Serra and P. Tilli. Negative energy ground states for the L^2-critical NLSE on metric graphs. *Comm. Math. Phys.* 352 (1): 387-406, 2017.

[6] R. Adami, E. Serra and P. Tilli. Nonlinear dynamics on branched structures and networks. *Riv. Math. Univ. Parma (N.S.)* 8(1):109-159, 2017.

[7] R. Adami, E. Serra and P. Tilli. Multiple positive bound states for the subcritical NLS equation on metric graphs. *Calc. Var. Partial Differential Equations* 58(1), Paper No. 5, 16pp., 2019.

[8] T. Bartsch and N. Soave. A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. *J. Funct. Anal.* 272 (12): 4998-5037, 2017.

[9] J. Bellazzini, L. Jeanjean and T. Luo. Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations. *Proc. Lond. Math. Soc.*, 107 (2): 303-339, 2013.

[10] J. Bellazzini and D. Ruiz. Finite energy traveling waves for the Gross-Pitaevskii equation in the subsonic regime. *Amer. J. Math.*, 145 (1): 109-149, 2023.

[11] G. Berkolaiko and P. Kuchment. *Introduction to quantum graphs*, Vol. 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[12] J. Borthwick, X. J. Chang, L. Jeanjean and N. Soave. Bounded Palais-Smale sequences with Morse type information for some constrained functionals. arXiv:2210.12626, 2022.

[13] G. Berkolaiko, J. Marzuola, D. Pelinovsky. Edge-localized states on quantum graphs in the limit of large mass. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 38(5): 1295-1335, 2021.

[14] C. Cacciapuoti, S. Dovetta and E. Serra. Variational and stability properties of constant solutions to the NLS equation on compact metric graphs. *Milan J. Math.* 86(2): 305-327, 2018.

[15] X. J. Chang, L. Jeanjean and N. Soave. Normalized solutions of L^2-supercritical NLS equations on compact metric graphs. arXiv:2204.01043, 2022. To appear in *Ann. Inst. H. Poincaré C Anal. Non Linéaire*.

[16] S. Dovetta. Existence of infinitely many stationary solutions of the L^2- subcritical and critical NLSE on compact metric graphs. *J. Differential Equations* 264 (7): 4806-4821, 2018.

[17] S. Dovetta, M. Ghimenti, A. M. Micheletti and A. Pistoia. Peaked and low action solutions of NLS equations on graphs with terminal edges. *SIAM J. Math. Anal.* 52 (3): 2874-2894, 2020.

[18] S. Dovetta and L. Tentarelli. L^2-critical NLS on noncompact metric graphs with localized nonlinearity: topological and metric features. *Calc. Var. Partial Differential Equations* 58(3), Paper No. 108, 26 pp., 2019.

[19] S. Dovetta and L. Tentarelli. Ground states of the L^2-critical NLS equation with localized non-linearity on a tadpole graph. *Discrete and continuous models in the theory of networks (Oper. Theory Adv. Appl.)* 281: 113-125, 2020.

[20] P. Esposito, G. Mancini, S. Santra and P. N. Srikanth. Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index. *J. Differential Equations* 239 (1): 1-15, 2007.

[21] P. Esposito and M. Petralla. Pointwise blow-up phenomena for a Dirichlet problem. *Commun. Partial Differ. Equations* 36 (7-9): 1654-1682, 2011.

[22] G. Fang and N. Ghoussoub. Second-order information on Palais-Smale sequences in the mountain pass theorem. *Manuscripta Math.* 75 (1): 81-95, 1992.

[23] G. Fang and N. Ghoussoub. Morse-type information on Palais-Smale sequences obtained by min-max principles. *Comm. Pure Appl. Math.* 47: 1595-1653, 1994.

[24] S. Gnutzmann, U. Smilansky and S. Derevyanko. Stationary scattering from a nonlinear network. *Phys. Rev. A* 83 (3): 033831, 2011.

[25] N. Ikoma and K. Tanaka. A note on deformation argument for L^2 normalized solutions of nonlinear Schrödinger equations and systems. *Adv. Differ. Equ.* 24 (11-12): 609-646, 2019.

[26] L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.* 28 (10): 1633-1659, 1997.

[27] L. Jeanjean. On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A* 129: 787-809, 1999.

[28] L. Jeanjean and S-S. Lu. A mass supercritical problem revisited. *Calc. Var. Partial Differential Equations* 59 (5), Paper No. 174, 43 pp., 2020.

[29] A. Kairzhan, D. Noja and D. E. Pelinovsky. Standing waves on quantum graph. J. Phys. A: Math. Theor. 55 243001, 2022.

[30] R. Lopez-Soriano, A. Malchiodi and D. Ruiz. Conformal metrics with prescribed Gaussian and geodesic curvatures. *Ann. Sci. Ec. Norm. Supér.* 55 (5): 1289-1328, 2022.

[31] J. Mederski and J. Schino. Least energy solutions to a cooperative system of Schrödinger equations with prescribed L2-bounds: at least L2-critical growth. *Calc. Var. Partial Differential Equations* 61 (1), Paper No. 10, 31 pp., 2022.

[32] D. Noja. Nonlinear Schrödinger equation on graphs: recent results and open problems. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2007):20130002, 20, 2014.

[33] D. Noja and D. E. Pelinovsky. Standing waves of the quintic NLS equation on the tadpole graph. *Calc. Var. Partial Differential Equations* 59 (5): Paper No. 173, 30 p., 2020.

[34] D. Noja, D. Pelinovsky and G. Shaikhova. Bifurcations and stability of standing waves in the nonlinear Schrödinger equation on the tadpole graph. *Nonlinearity* 28, 2343-2378, 2015.

[35] D. Pierotti and N. Soave. Ground states for the NLS equation with combined nonlinearities on noncompact metric graphs. *SIAM J. Math. Anal.* 54 (1): 768-790, 2022.

[36] D. Pierotti, N. Soave and G. Verzini. Local minimizers in absence of ground states for the critical NLS energy on metric graphs. *Proc. Royal Soc. Edinburgh, Sect. A: Math.* 151 (2): 705-733, 2021.

[37] D. Pierotti and G. Verzini. Normalized bound states for the nonlinear Schrödinger equation in bounded domains. *Calc. Var. Partial Differential Equations* 56 (5): Paper No. 133, 27 p., 2017.

[38] E. Serra and L. Tentarelli. Bound states of the NLS equation on metric graphs with localized nonlinearities. *J. Differential Equations* 260 (7): 5627-5644, 2016.

[39] E. Serra and L. Tentarelli. On the lack of bound states for certain NLS equations on metric graphs. *Nonlinear Anal.* 145: 68-82, 2016.

[40] N. Soave. Normalized ground states for the NLS equation with combined nonlinearties. *J. Differential Equations* 269(9): 6941-6987, 2020.

[41] N. Soave. Normalized ground states for the NLS equation with combined nonlinearties: the Sobolev critical case. *J. Funct. Anal.* 279(6): 108610, 43 p., 2020.

[42] L. Tentarelli. NLS ground states on metric graphs with localized nonlinearities. *J. Math. Anal. Appl.* 433 (1): 291-304, 2016.