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Effective field theories and integrable deformations in two space-time dimensions

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Preface

The research that I carried on during my Ph.D. program under the supervision of Prof. Roberto Tateo is essentially centered around the investigation of classical and quantum aspects of particle physics theories using the powerful tools provided by integrability. In a broad sense, integrable models are usually referred to as "*exactly solvable*", meaning that all the physical observables can be found non-perturbatively both in analytical form or by means of quadrature techniques. The *exact solvability* makes them perfectly suited laboratories

- to study simplified versions of complicated physical theories at full non-perturbative level;
- to get insight into interesting physical properties and mathematical structures which would be inaccessible from other directions.

The latter two points represents the guidelines of my research program which is mainly divided into two different projects.

The first project, which represented also my first contact with the universe of integrable systems, concerns the investigation of the conformal spectrum of single trace-operators in the $3-$dimensional $\mathcal{N} = 6$ *super Chern-Simons* theory at finite values of the coupling constant, using integrability techniques. The outcomes of this research resulted in the publication of [1] and another work still on-going.

The second project is, instead, related to the exploration of $2-$dimensional Quantum Field Theories through the study of some exactly solvable irrelevant deformations of QFTs. This dissertation will be entirely based on the results obtained in this context, which led to the publication of [2], [3] and [4].

[1] D. Bombardelli, A. Cavaglià, R. Conti, R. Tateo, "*Exploring the spectrum of planar AdS_4/CFT_3 at finite coupling*", JHEP **04** (2018) 117 [arXiv:1803.04748].

[2] R. Conti, L. Iannella, S. Negro, R. Tateo, "*Generalised Born-Infeld models, Lax operators and the* $\mathrm{T}\overline{\mathrm{T}}$ *perturbation*", JHEP **11** (2018) 007 [arXiv:1806.11515].

[3] R. Conti, S. Negro, R. Tateo, "*The* $\mathrm{T}\bar{\mathrm{T}}$ *perturbation and its geometric interpretation*", JHEP **02** (2019) 085 [arXiv:1809.09593].

[4] R. Conti, S. Negro, R. Tateo, "*Conserved currents and* $\mathrm{T}\bar{\mathrm{T}}_{\mathrm{s}}$ *irrelevant deformations of 2D integrable field theories*", JHEP **11** (2019) 120 [arXiv:1904.09141].

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Chapter 1

Introduction

1.1 The Space of Quantum Field Theories

Generally speaking, the long term goal of the research program summarised in this dissertation, is the exploration of the "Space of $2-$dimensional Quantum Field Theories (QFTs)", here denoted by Σ following the convention of [1], and of its associated geometric structure through the investigation of some peculiar irrelevant deformations of QFTs which turn out to be *exactly solvable*.

The idea of equipping Σ with a geometric structure is concretely realised in the context of the Renormalisation Group (RG) approach á la Wilson [2]. Following the *path integral formulation*, the generic element of Σ is a QFT represented by a quasi-local action $\mathcal{A}[\Phi]$, where Φ denotes a collection of fundamental fields, taking values on a $d-$dimensional manifold M, together with their derivatives. The whole information on the theory is encoded in the partition function

$$\mathcal{Z} = \int \mathcal{D}\Phi \, e^{-\mathcal{A}[\Phi]/\hbar} \ , \tag{1.1}$$

from which the correlation functions between all the operators of the theory can be extracted. The first step of the renormalisation procedure is to set a cut-off energy Λ_0 and define a regularised partition function

$$\mathcal{Z}^{(\text{reg.})} = \int_{C^\infty(M)_{<\Lambda_0}} \mathcal{D}\Phi \, e^{-\mathcal{A}[\Phi]/\hbar} \ , \tag{1.2}$$

in which the path integral is restricted to the space of smooth functions $C^\infty(M)_{<\Lambda_0}$ on M whose energy is at most Λ_0. Notice that ruling out the modes with energy higher than Λ_0, we have automatically prevented the emergence of UV divergences arising from any perturbative loop integral. Then, the second step consists in integrating out the modes with energy between Λ_0 and a generic scale $\Lambda < \Lambda_0$. This operation induces a flow in Σ, which can be abstractly written at the level of the action as

$$\boxed{\frac{\partial \mathcal{A}}{\partial l} = B(\mathcal{A}) \ , \quad l = \log \Lambda \ ,} \tag{1.3}$$

with $\mathcal{A}|_{\Lambda=\Lambda_0} = \mathcal{A}^{(0)}$ initial condition. Denoting by $T_{\mathcal{A}}\Sigma$ the tangent space to Σ at the point $\mathcal{A} \in \Sigma$, then $B(\mathcal{A}) \in T_{\mathcal{A}}\Sigma$ represents the tangent vector which generates the so-called RG *flow equation* and the integral curves solution to it are referred to as RG *trajectories*. All the QFTs lying on the same RG trajectory represent the same physical theory as observed at different scales.

To translate (1.3) into ordinary differential equations for some functions, one needs to parametrise the generic element $\mathcal{A} \in \Sigma$ in terms of a set of coordinates $\{\alpha_k\}_{k\geq 0}$. The usual way to do that, is to start from a so-called *fixed* or *critical point*[1] of the RG flow and add to it an infinite set of local operators[2] $\{\mathcal{O}_k(\mathbf{x})\}_{k\geq 0}$ with classical mass dimension d_k

$$\mathcal{A} = \mathcal{A}^\star + \sum_{k\geq 0} \alpha_k \int d\mathbf{x}\, \mathcal{O}_k(\mathbf{x}) \ . \tag{1.4}$$

where α_k plays the role of coupling constant associated to $\mathcal{O}_k(\mathbf{x})$.

Observation 1. Due to quantum corrections, the classical mass dimension of the operator $\mathcal{O}_k(\mathbf{x})$ gets modified and the overall scaling dimension Δ_k becomes

$$\Delta_k = d_k + \gamma_k \ , \tag{1.5}$$

where γ_k is the so-called *anomalous dimension*.

In the set of coordinates $\{\alpha_k\}_{k\geq 0}$, the RG flow equation (1.3) becomes a set of ordinary differential equations for the functions $\{\alpha_k(l)\}_{k\geq 0}$

$$\beta_k\left(\{\alpha_j\}\right) = \frac{\partial \alpha_k}{\partial l} = B_k\left(\{\alpha_j\}\right) \ , \quad (k \geq 0) \ , \tag{1.6}$$

where $B_k\left(\{\alpha_j\}\right)$ is the coordinate expression of $B(\mathcal{A})$ and $\beta_k = \frac{\partial \alpha_k}{\partial l}$ is the so-called $\beta-$function.

Observation 2. A theory represented by a set of coordinates $\{\alpha_k^\star\}_{k\geq 0}$ is a fixed point of the RG flow if the $\beta-$ functions are all vanishing, *i.e.*

$$\beta_k\left(\{\alpha_j^\star\}\right) = 0 \ . \tag{1.7}$$

Close to a fixed point, *i.e.* the difference $\delta\alpha_k = \alpha_k - \alpha_k^\star \ , \ (k \geq 0)$ is small, the RG flow equations (1.6) can be linearised as

$$\beta_i\left(\{\alpha_j\}\right) = b_{ij}\, \delta\alpha_j + \mathcal{O}\left(\delta\alpha^2\right) \ , \quad (i \geq 0) \ , \tag{1.8}$$

where b_{ij} is a constant, infinite dimensional matrix, which is diagonalisable on the basis of eigenvectors $\{\sigma_k\}_{k\geq 0}$ with eigenvalues $\{\Delta_k - d\}_{k\geq 0}$. Therefore, in the basis $\{\sigma_k\}_{k\geq 0}$, equation (1.8) becomes

$$\frac{\partial \sigma_k}{\partial l} = (\Delta_k - d)\, \sigma_k + \mathcal{O}(\sigma^2) \ , \quad (k \geq 0) \ , \tag{1.9}$$

[1] The fixed or critical point is an element $\mathcal{A}^\star \in \Sigma$ which is invariant under the action of the RG flow, namely

$$\frac{\partial \mathcal{A}^\star}{\partial l} = 0 \ .$$

In $d = 2$ there is a theorem which states that all fixed points are CFTs, while for $d > 2$ the problem is still under debate. The trivial fixed point is called *Gaussian fixed point* and it corresponds to the free theory, *i.e.* all the couplings are set to zero.

[2] The generic local operator $\mathcal{O}_k(\mathbf{x})$ is a monomial involving a number d_k, *i.e.* the mass dimension, of powers of the fundamental fields of the theory and their derivatives.

which gives, at the first perturbative order

$$\boxed{\sigma_k(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_k - d} \sigma_k(\Lambda_0)\ .} \tag{1.10}$$

Expression (1.10) allows to classify the set of local operators related to a given critical point according to the behavior of $\{\sigma_k(\Lambda)\}_{k\geq 0}$ as the energy scale is lowered from the cut-off Λ_0 to $\Lambda < \Lambda_0$, namely as we flow towards the IR:

- $\Delta_k > d$ (*irrelevant*): as the energy scale Λ is lowered from the cut-off scale Λ_0, the couplings $\sigma_k(\Lambda)$ become smaller and smaller. The corresponding local operator is called *irrelevant* since its inclusion in the action (1.4) induces an RG flow which drives the theory back to the critical point in the IR. Notice that, at fixed d, one can construct an infinite tower of irrelevant operators simply by adding powers of fields and their derivatives. Therefore, any critical point is defined on an infinite-dimensional subspace in the whole space of the theories, called *critical surface*, parametrised by the couplings of the irrelevant operators.

- $\Delta_k < d$ (*relevant*): contrary to the previous case, here the couplings $\sigma_k(\Lambda)$ grow as the energy scale Λ is lowered. The corresponding local operators are called *relevant* because their presence in the action (1.4) drives the theory away from the critical surface in the IR, along a so-called *renormalized trajectory*, which lead either to an other fixed point or to a limiting cycle. Since at fixed d, there is a finite amount of relevant operators, it follows that the critical surface has finite co-dimension.

- $\Delta_k = d$ (*marginal*): the couplings are unchanged under the RG flow transformation, and the corresponding operators are called *marginal*. However, close to a generic critical point, quantum corrections may produce logarithmic modifications of Δ_k, transforming the marginal operator into a *marginally relevant* or *marginally irrelevant* operator, depending on the sign of $\Delta_k - d$.

To remove the cut-off, sending it to infinity, one needs to apply the standard renormalisation procedure (see, for example [3]), through the introduction of *counterterms*. When this cannot be done sistematically (without changing the form of the action) the theory is said to be "*non-renormalisable*". Non-renormalisable theories still make sense provided the cut-off is kept finite, namely their validity is limited to the low energies regime. Generically speaking, they are called *Effective Field Theories* (EFTs), typical examples being lattice-regularised models.

So far, the action has been taken as characterising element to uniquely identify a QFT. A different, yet useful perspective is provided by the *bootstrap formulation*, in which the fundamental objects that define a QFT are the set of local fields $\{\mathcal{O}_k(\mathbf{x})\}_{k\geq 0}$, which span the vector space $\mathcal{F}$

$$\mathcal{F} = \text{span}\,\{\mathcal{O}_k(\mathbf{x})\}_{k\geq 0}\ , \tag{1.11}$$

along with the set of all correlation functions

$$\Big\{\langle \mathcal{O}_{k_1}(\mathbf{x}_1)\ldots\mathcal{O}_{k_n}(\mathbf{x}_n)\rangle \mid \{\mathcal{O}_k(\mathbf{x})\}_{k\geq 0} \in \mathcal{F}\Big\}\ , \tag{1.12}$$

equipped with an *Operator Product Expansion* (OPE) algebra

$$\mathcal{O}_i(\mathbf{x})\mathcal{O}_j(\mathbf{x}') \underset{\mathbf{x}\to\mathbf{x}'}{\sim} C_{ij}^k(\mathbf{x}-\mathbf{x}')\mathcal{O}_k(\mathbf{x}') \ , \tag{1.13}$$

which is specified by the set of real-valued functions $\left\{C_{ij}^k\right\}_{(i,j,k)\in\mathbb{N}}$, called *structure constants*. Since the generic element $\mathcal{A}\in\Sigma$ can be expressed in terms of local fields as in (1.3), it appears natural to identify $\mathcal{F}$ with the tangent space $T\Sigma$, modulo the subspace $\partial\mathcal{F}$ consisting of local fields which are derivatives of other local fields (since do not contribute to the dynamics)

$$\boxed{\hat{\mathcal{F}} = T\Sigma \ , \quad \hat{\mathcal{F}} = \mathcal{F}/\partial\mathcal{F} \ .} \tag{1.14}$$

This identification may be used to translate the properties of local fields into geometric notions at the level of Σ.

1.2 Irrelevant deformations, Effective Field Theories and holography

A good starting point for the exploration of Σ is provided by the subset $\Sigma^{\mathrm{Int}}\subset\Sigma$ made of Integrable Quantum Field Theories (IQFTs). Massive IQFTs initially emerged from the factorisable S-matrix context [4]. Starting from the set of exact two-body amplitudes, integrability also provides powerful tools for the study of the finite-size effects such as the *Thermodynamic Bethe Ansatz* (TBA) [5, 6] or the *Non-Linear Integral Equations* (NLIEs) [7, 8]. Through conformal perturbation theory and the TBA method, these scattering models were then interpreted as perturbations, by relevant operators, of CFTs [6]. The deformation of a CFT, considered as the UV fixed point, by a relevant operator can also lead to a model with a massless sub-sector [9], defining, therefore, a critical flow connecting the initial fixed point to a second non-trivial CFT, in the IR. From the point of view of the ultraviolet CFT, these relevant perturbations correspond to super-renormalizable interactions. An arbitrary Green function can have primitive UV divergencies only in finitely many orders of CPT (see, for example [10]).

In general, the situation is very different when it comes to deformations of CFTs or, more generally QFTs, by an irrelevant operator, which usually lead to EFTs with finite UV cut-off, for which the quantisation procedure is problematic. However, in the past three years, the study of irrelevant deformations is experiencing a period of renewed interest thanks to the groundbreaking discoveries of [1, 11] in which it has been found an exactly solvable perturbation generated by the irrelevant operator $\mathrm{T\bar{T}}$, rigorously defined by A. B. Zamolodchikov in [12] for a generic QFT on flat space-time. Besides laying the basis of the $\mathrm{T\bar{T}}$ deformation for a generic QFT, [1, 11] showed that, at least for massive IQFTs, the exact solvability of this deformation is encoded in the introduction of a specific *Castillejo-Dalitz-Dyson* (CDD) phase factor [13] in the exact $S-$matrix of the original theory. Although the proper definition of $\mathrm{T\bar{T}}$ deformation has been given in recent times [1, 11], the emergence of the $\mathrm{T\bar{T}}$ and other irrelevant deformations in relation with the CDD ambiguity dates back to the beginning of the nineties. The first occurrence was in the study of the massless flow from

tri-critical to critical Ising model pionereed by [9], in which the $\mathrm{T\bar{T}}$ operator – at the critical point[1] – appeared as leading attracting operator in the (EFT) action associated to the low energy IR regime. The exploration of the IR fixed point led to the proposal of a possible connection between CDD factors and irrelevant operators [14].
Several years later, A. B. Zamolodchikov [12] gave an off-critical definition of the $\mathrm{T\bar{T}}$ operator, up to total derivative terms

$$\lim_{\mathbf{z}'\to\mathbf{z}} \mathbf{T}(\mathbf{z})\bar{\mathbf{T}}(\mathbf{z}') - \mathbf{\Theta}(\mathbf{z})\mathbf{\Theta}(\mathbf{z}') \equiv \mathrm{T\bar{T}}(\mathbf{z}) + \text{derivatives} \,, \tag{1.15}$$

which is valid for any 2−dimensional QFT on a flat space-time and with *local translational* and *rotational* symmetry. In particular, he showed that the expectation value of the $\mathrm{T\bar{T}}$ operator on a generic eigenvalue of the energy and momentum factorizes as

$$\langle n|\, \mathrm{T\bar{T}}(\mathbf{z})\, |n\rangle = \langle n|\, \mathbf{T}(\mathbf{z})\, |n\rangle \, \langle n|\, \bar{\mathbf{T}}(\mathbf{z})\, |n\rangle - \langle n|\, \mathbf{\Theta}(\mathbf{z})\, |n\rangle \, \langle n|\, \mathbf{\Theta}(\mathbf{z})\, |n\rangle \ . \tag{1.16}$$

The latter property is commonly referred to as *factorisation property* and it will prove to be fundamental in the definition of the $\mathrm{T\bar{T}}$ deformation [1, 11].
Another important step toward the identification of the $\mathrm{T\bar{T}}$ deformation was made in [15, 16] where it was noticed that the energy levels of a $c = n$ CFT[2] modified by the inclusion of a non-trivial CDD factor (cf. equation (1.19) below) take the same analytic form of the energy levels of an infinitely long critical bosonic string, *i.e.* the Nambu-Goto string in 26 dimensions, which represents a simple theory of quantum gravity. For this reason [15] referred to the CDD factor modification of the $S-$matrix (1.19) as "*gravitational dressing*".
Subsequently [17] adapted the analysis of [15, 16] to encompass the open string case and extended the CDD deformation to a generic 2−dimensional CFT. Moreover, the authors of [17] interpreted the results of [15, 16] in terms of a $\mathrm{T\bar{T}}$ deformation and suggested a simple iterative scheme to generate the action deformed by the $\mathrm{T\bar{T}}$ operator.
Finally, in 2016, [1, 11] merged together all these observations accumulated over time and, using the factorisation property found in [12], formulated the $\mathrm{T\bar{T}}$ deformation of a generic QFT in terms of a partial differential equation (PDE) for the energy levels $\{E_n(R,\tau)\}_{n\geq 0}$, *i.e.* an *inviscid inhomogeneous Burgers* equation,

$$\partial_\tau E_n(R,\tau) = E_n(R,\tau)\partial_R E_n(R,\tau) + \frac{1}{R}P_n^2(R) \,, \tag{1.17}$$

where τ is the deformation parameter associated to the $\mathrm{T\bar{T}}$ flow. At the same time, they provided a classical geometric interpretation of the $\mathrm{T\bar{T}}$ deformation which is enclosed in a flow equation for the classical action

$$\partial_\tau \mathcal{L}(\mathbf{x},\tau) = \mathrm{T\bar{T}}(\mathbf{x},\tau) \,, \tag{1.18}$$

where the $\mathrm{T\bar{T}}$ operator plays the role of tangent vector to the curve $\mathcal{L}(\mathbf{x},\tau)$ as the parameter τ is varied. Notice that equation (1.18) can be considered as the mathematical formulation of the iterative scheme

[1] The name "$\mathrm{T\bar{T}}$" comes precisely from its definition at the critical point in which it is just the product of the holomorphic $\mathbf{T}$ and anti-holomorphic components $\bar{\mathbf{T}}$ of the stress-energy tensor. Conventionally, also its off-critical extension (1.15) is still named $\mathrm{T\bar{T}}$, even if an additional term appears.
[2] Here n denotes the number of independent bosonic species.

proposed in [17]. In [11], equation (1.18) has been explicitly solved for some simple cases. Interestingly, starting from a theory of N non-interacting bosonic fields, the $\mathrm{T\bar{T}}$ deformation leads to the Nambu-Goto Lagrangian in static gauge, giving further evidence in favour of the relation between $\mathrm{T\bar{T}}$ and string theory. Furthermore, [1, 11] connected this irrelevant deformation with a specific CDD factor which modifies the $S-$matrix by an amount

$$S(\theta) \to S(\theta)\, f(\theta) \;, \quad f(\theta) = \exp\left(\mathrm{i}\tau\, m^2 \sinh(\theta)\right) \;. \tag{1.19}$$

A different perspective of looking at the solvability of the $\mathrm{T\bar{T}}$ deformation was provided by [18]. In fact, interpreting the modification of the partition function under an infinitesimal deformation as a random fluctuation of the space-time geometry, [18] argued that the associated local action is a total derivative of quasi-local fields, affecting therefore only the finite-size properties of the system (see also the comment in [19] about *prime forms*). In addition, linear diffusion equations for the $\mathrm{T\bar{T}}-$deformed partition functions on some simple geometries were derived. Further investigations on the partition functions carried on in [20] showed that the partition function of a $\mathrm{T\bar{T}}-$deformed CFT on a torus preserves interesting modular properties. Even more interestingly, [21] proved, under certain (weak) assumptions, the uniqueness of the previous statement, namely that $\mathrm{T\bar{T}}$ is the only deformation of a CFT that preserves modular properties of the torus partition function.

A further step in the direction marked by [18] was made by [22, 23] in which it has been proposed a path integral formulation of the $\mathrm{T\bar{T}}-$deformed theory. The authors of [22] proved that the $\mathrm{T\bar{T}}$ deformation of a QFT can be equivalently interpreted as coupling it to a flat space-time Jackiw-Teitelbolm (JT) like[1] gravity. The ideas of [18, 22] are in spirit similar to the interpretation of the $\mathrm{T\bar{T}}$ deformation – at classical level – as a field-dependent space-time coordinate transformation [24], even though the precise link is still not completely transparent.

Shortly after [1, 11], [25] opened the way to a new perspective on the $\mathrm{T\bar{T}}$ deformation in the framework of the *AdS*/CFT correspondence. Restricting the original theory to be a CFT, the $\mathrm{T\bar{T}}-$deformed spectrum exhibits a square root whose argument depend on the energy levels of the original CFT. Depending on the sign of the deformation parameter, as the volume is varied there is either a finite number of complex energy levels ("*good sign*"), *i.e.* the ground state plus a finite number of excited states, or an infinite number of complex energy levels plus a finite amount of them which is real ("*bad sign*").

The authors [25] pointed out that in the holographic dual, the $\mathrm{T\bar{T}}$ deformation of a CFT for the "bad sign" behaves as a geometric cut-off which places the boundary of AdS at a finite radial distance in the bulk. Afterwards, [26, 27] proposed a bulk interpretation of the $\mathrm{T\bar{T}}$ deformation of a CFT for the "good sign" in terms of a marginal current-current deformation of the worldsheet on AdS_3. Motivated by these works, a lot of effort have been devoted so far to investigate the $\mathrm{T\bar{T}}$ deformation in the AdS/CFT context.

Another interesting direction of research consists in the exploration of other solvable irrelevant deformations. The first step was made in [1] in which it was pointed out the existence of a whole family of irrelevant deformations generated by scalar local fields $\{X_s\} \in \hat{\mathcal{F}}$ with mass dimension

[1] For the path integral on the plane, the gravity theory is almost JT gravity, but for other geometries it is not anymore.

$d_s = 2(s+1)$ and Lorentz spin[1] 0, which preserve the integrability of the original IQFT, namely $\{X_s\} \in T\Sigma^{\text{Int}}$. These operators are uniquely defined in terms of bi-linear combinations of the level$-s$ currents of the integrable hierarchy, up to total derivative terms, as

$$\lim_{\mathbf{z}' \to \mathbf{z}} \mathbf{T}_{s+1}(\mathbf{z})\bar{\mathbf{T}}_{s+1}(\mathbf{z}') - \mathbf{\Theta}_{s-1}(\mathbf{z})\bar{\mathbf{\Theta}}_{s-1}(\mathbf{z}') \equiv X_s(\mathbf{z}) + \text{derivatives} , \tag{1.20}$$

where X_1 coincides with the T$\bar{\text{T}}$ operator. The authors of [1] proved that the introduction of the local operators $\{X_s\}$ in the action causes a modification of the original exact $S-$matrix $S(\theta)$ by a CDD factor which generalises (1.19), *i.e.*

$$S(\theta) \to S(\theta) f(\theta) , \quad f(\theta) = \exp\left(\mathbb{i} \sum_s \tau^{(s)} \gamma_s^2 \sinh(s\theta) \right) , \tag{1.21}$$

where $\{\tau^{(s)}\}$ is a set of independent deformation parameters.
Afterwards, [28, 29] considered a Lorentz-breaking deformation of a CFT generated by the so-called J$\bar{\text{T}}$ operator, built out of a bi-linear combination of the anti-holomorphic component of the stress-energy tensor and a $U(1)$ current. Soon after, also the holographic dual of the J$\bar{\text{T}}$ deformation has been identified [30] and much of the work done for the T$\bar{\text{T}}$ deformation was extended to this case (see [21,31]). Along the same line, [32–36] investigated deformations generated by a combination of the T$\bar{\text{T}}$ and J$\bar{\text{T}}$ operators. More recently, it has started the exploration of new families of deformations involving other symmetries that the original theory may eventually possess, such as supersymmetry [37–42] or non-Noetherian symmetries, tipically present in the hierarchy of IQFTs [43–45].

1.3 Overview of the thesis

This thesis is a collection of the works [24, 43, 46] that I have done during my Ph.D. program at the University of Turin under the supervision of Prof. Roberto Tateo. The main focus of the dissertation is on the field theoretical aspects of irrelevant deformations of 2$-$dimensional QFTs, in particular in relation with the integrable structures.

The first part is devoted to the investigation of classical and quantum aspects related to the T$\bar{\text{T}}$ deformation of 2$-$dimensional QFTs. To make the contents as self contained as possible, in section 2.1 we introduce the T$\bar{\text{T}}$ operator for a generic 2$-$dimensional QFT on flat spacetime and, reviewing the result of [12], we show that, under broad assumptions, it is a well defined local operator, up to total derivative terms. Then, in section 2.2 we move on to axiomatically define the action of the T$\bar{\text{T}}$ operator on the quantised energy levels of the original QFT, which is taken as the definition of the T$\bar{\text{T}}$ deformation [1, 11]. Exploiting the nice properties of the T$\bar{\text{T}}$ operator found in [12], we show that the evolution of the energy levels can be equivalently recast into a simple first order PDE, *i.e.* an inviscid Burgers equation, and we derive explicitly the expressions of the T$\bar{\text{T}}-$deformed spectra for

[1]The Lorentz spin of a local operator is defined as the difference between the total number of holomorphic derivative ∂_z and the total number of anti-holomorphic derivative $\partial_{\bar{z}}$ appearing in it. Since these operators are spin 0, the corresponding deformations do not break the Lorentz invariance of the original theory.

some particular theories [1, 11]. In the framework of IQFTs, in section 3.4 we review the fundamental results of [1, 11] in which the $T\bar{T}$ deformation is interpreted as a modification of the exact and factorisable $S-$matrix by a non-trivial CDD factor which preserves the Lorentz invariance of the original theory. Starting from the Burgers equation as operational definition of the $T\bar{T}$ deformation, in section 2.3 we derive the evolution equations for the classical Hamiltonian and Lagrangian densities [11, 46, 47] and we explicitly solve them for some specific models. This analysis allows to extract useful information about the relation between the classical and quantum sides of the deformation. Interestingly, the classical outcomes of section 2.3.5 also led to conjecture [46] the partition function and heat-kernel of a generic $2-$dimensional $T\bar{T}-$deformed Yang-Mills theory (see appendix B.1).
To make a more concrete step forward in the understanding of the classical structure of the deformation, we restrict our attention to IFTs, and in particular to the *sine-Gordon* model, which it is often used throughout the thesis as study case. In section 4.2, we prove that the $T\bar{T}-$deformed sine-Gordon theory admits a Lax pair representation [46], from which it automatically descends that the $T\bar{T}$ deformation preserves the classical integrability of the sine-Gordon model. Exploiting the well known relation between the sine-Gordon equation and pseudo-spherical soliton surfaces embedded in a $3-$dimensional Euclidean ambient space (see also appendix C.1 for a brief introduction to the subject), in section 4.4 we construct the soliton surfaces associated to the $T\bar{T}-$deformed sine-Gordon model [24]. Since the intrinsic properties of the surfaces, *i.e.* the *Gaussian* and *mean curvatures*, are unchanged, we infer that the $T\bar{T}-$deformation acts as a reparametrisation of the solitonic surfaces and, at the level of the solutions, as a coordinate transformation. In section 4.5, we explicitly construct a space-time coordinate transformation which maps the Equations of Motion (EoM) of the sine-Gordon theory onto the $T\bar{T}-$deformed one and viceversa [24]. From simple considerations, we show that the coordinate transformation is easily generalisable to any bosonic theory and sigma models. From one hand, this map provides a powerful tool to generate solutions to the deformed EoMs starting from the original ones. In section 4.7, we explicitly compute the $T\bar{T}-$deformed version of some simple sine-Gordon soliton solutions and, in section 4.8, we comment on the emergence of singularities. From the other hand, in section 4.6 we show that the coordinate transformation gives access to the full integrable structure of the $T\bar{T}-$deformed IFT, by means of a geometrical construction which allows to derive the conserved charges of the deformed hierarchy [43]. With this technology, one can take the coordinate transformation as an equivalent definition of the $T\bar{T}$ deformation at the classical level.

Precisely this consideration opens the way to the second part of the thesis, concerning the investigation of generalisations of the $T\bar{T}$ deformation involving conserved currents of the hierarchy of a generic classical Integrable Field Theory (IFT) with Lorentz spin different than 1. Inspired by the classical geometric interpretation of the $T\bar{T}$ deformation in terms of a coordinate transformation, in sections 5.1 and 5.2 we construct consistent coordinate transformations in which the components of the stress-energy tensor are replaced by those of an arbitrary higher spin conserved current [43]. The setup is also extended to encompass $U(1)_L \times U(1)_R$ conserved currents (section 5.3). By means of a simple example, in section 5.4 we investigate the effect of these deformations on the Lorentz invariance of the original theory, which gets broken. Restricting for simplicity to deformations of the massless free boson theory, in section 5.2.1 we explicitly derive the deformed integrable hierarchies using the method previously developed in the $\mathrm{T\bar{T}}$ context. This classical analysis suggests that the quantisation of the models associated to these generalised coordinate transformations is again generated by a modification of the original $S-$matrix by a family of phase factors which, contrary to (1.21), are not CDD factors and break explicitly the Lorentz invariance of the original theory. In section 6.1 we identify this class of phase factors in the large volume approximation, then, in section 6.2 we perform a finite-size analysis in the framework of the NLIEs and derive the evolution equations for the conserved charges, which turn out to be much more complicated compared to the inviscid Burgers equation emerging in the $\mathrm{T\bar{T}}$ case. In section 6.3 we compute the expressions of the conserved charges by explicitly solving the evolution equations in the CFT limit and compare them with the classical outcomes finding agreement. Finally, in section 6.5 we extend the setup to encompass deformations involving the topological charge. We discuss various examples and, in particular, we provide the phase factor which generates the well-studied $\mathrm{J\bar{T}}$ deformation of a CFT.

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Part I

The $T\bar{T}$ deformation

Chapter 2

The $\mathrm{T\bar{T}}$ deformation: definition and properties

In this introductory chapter we start by reviewing the fundamental results of [1, 11, 12]. We first introduce the $\mathrm{T\bar{T}}$ operator for a generic QFT on a $2-$dimensional Euclidean space-time and we prove that it defines a local operator up to total derivative terms. Then, we show that its expectation value on a generic eigenstate of the energy and momentum operators enjoys the fundamental *factorisation property*, which leads to define the $\mathrm{T\bar{T}}$ deformation on the energy levels in terms of an inviscid inhomogenous Burgers equation. From the latter definition, we derive the corresponding flow equation for the classical Hamiltonian and Lagrangian densities. Finally, we explicitly compute the $\mathrm{T\bar{T}}-$deformed Hamiltonian and Lagrangian densities for some simple theories and we comment on the relation between the classical and quantum sides of the deformation.

2.1 The $\mathrm{T\bar{T}}$ operator

We start by considering the composite operator

$$\boxed{\mathcal{O}(\mathbf{z},\mathbf{z}') = \mathbf{T}(\mathbf{z})\bar{\mathbf{T}}(\mathbf{z}') - \mathbf{\Theta}(\mathbf{z})\mathbf{\Theta}(\mathbf{z}')\ ,} \tag{2.1}$$

defined using the *point splitting*, where $\{\mathbf{T}(\mathbf{z}),\bar{\mathbf{T}}(\mathbf{z}),\mathbf{\Theta}(\mathbf{z})\}$ are the chiral components of the stress-energy tensor (cf. Appendix A) of a generic QFT defined on a flat $2-$dimensional Euclidean space-time with *local translational* and *rotational symmetry*.

We will work under the following assumptions:

1. The basis of eigenstates[1] $\{|n\rangle\}_{n\in\mathbb{N}}$ associated to the energy and momentum operators is *non-degenerate*,

$$\hat{E}\,|n\rangle = E_n\,|n\rangle\ ,\quad \hat{P}\,|n\rangle = P_n\,|n\rangle\ ,\quad (n\in\mathbb{N})\ , \tag{2.2}$$

[1] We assume the normalisation $\langle n|n\rangle = 1$.

with

$$E_n = E_{n'} \text{ and } P_n = P_{n'} \iff n = n' \ . \tag{2.3}$$

2. The vector space $\mathcal{F}$ spanned by the set of local fields $\{\mathcal{O}_k\}_{k\in\mathbb{N}} \in \mathcal{F}$, is *complete* w.r.t. the *Operator Product Expansion* (OPE), *i.e.*

$$\mathcal{O}_i(\mathbf{z})\mathcal{O}_j(\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} C_{ij}^k(\mathbf{z}-\mathbf{z}')\mathcal{O}_k(\mathbf{z}') \ , \tag{2.4}$$

and the structure constants are assumed to depend on $\mathbf{z}$ and $\mathbf{z}'$ only through the difference $(\mathbf{z}-\mathbf{z}')$ due to the local translational and rotational symmetry.

3. Assuming also *global translational symmetry*, the expectation value of any local field $\mathcal{O}_k \in \mathcal{F}$ is a constant independent of the coordinates

$$\langle n| \, \mathcal{O}_k(\mathbf{z}) \, |n\rangle = c_k \in \mathbb{C} \ , \quad (k \in \mathbb{N}) \ . \tag{2.5}$$

Combining (2.5) with (2.4) one has

$$\langle n| \, \mathcal{O}_i(\mathbf{z})\mathcal{O}_j(\mathbf{z}') \, |n\rangle \underset{\mathbf{z}\to\mathbf{z}'}{\sim} G_{ij}(\mathbf{z}-\mathbf{z}') \ . \tag{2.6}$$

4. There exists at least one space-time direction $\mathbf{e} = (e, \bar{e})$ such that – for the vacuum state $(n = 0)$ – any 2-point function factorises when $\mathbf{z}$ and $\mathbf{z}'$ are infinitely separated from each other along $\mathbf{e}$, *i.e.*

$$\lim_{t\to\infty} \langle \mathcal{O}_i(\mathbf{z} + t\,\mathbf{e})\mathcal{O}_j(\mathbf{z})\rangle = \langle \mathcal{O}_i(\mathbf{z})\rangle\langle \mathcal{O}_j(\mathbf{z})\rangle \ . \tag{2.7}$$

where we use the shorthand notation $\langle 0| \star |0\rangle \equiv \langle \star \rangle$ to denote the expectation values on the vacuum state.

Observation 3. While 2. is a local statement, assumptions 3. and 4. are global requirements which restrict the QFT to be defined on a flat geometry, *i.e.* either an infinite plane or an infinitely long cylinder. See [48] for the definition of the T$\bar{\mathrm{T}}$ operator on a generic curved manifold.

Theorem 2.1.1. *Expression (2.1) defines a local operator up to total derivative terms in the collision limit* $\mathbf{z}' \to \mathbf{z}$.

Proof. First we differentiate (2.1) w.r.t. $\bar{z}$

$$\partial_{\bar{z}}\mathcal{O}(\mathbf{z},\mathbf{z}') = (\partial_{\bar{z}}\mathbf{T}(\mathbf{z}))\,\bar{\mathbf{T}}(\mathbf{z}') - (\partial_{\bar{z}}\boldsymbol{\Theta}(\mathbf{z}))\,\boldsymbol{\Theta}(\mathbf{z}') \ , \tag{2.8}$$

then we use the continuity equations (A.13) in the *rhs* of (2.8) to trade $\partial_{\bar{z}}\mathbf{T}(\mathbf{z})$ with $\partial_z\boldsymbol{\Theta}(\mathbf{z})$

$$\partial_{\bar{z}}\mathcal{O}(\mathbf{z},\mathbf{z}') = (\partial_z\boldsymbol{\Theta}(\mathbf{z}))\,\bar{\mathbf{T}}(\mathbf{z}') - (\partial_{\bar{z}}\boldsymbol{\Theta}(\mathbf{z}))\,\boldsymbol{\Theta}(\mathbf{z}') + \left[\boldsymbol{\Theta}(\mathbf{z})\left(\partial_{z'}\bar{\mathbf{T}}(\mathbf{z}') - \partial_{\bar{z}'}\boldsymbol{\Theta}(\mathbf{z}')\right)\right] \ , \tag{2.9}$$

where the additional term between square brackets in (2.9) is identically zero due to (A.13). Equation (2.9) can be more conveniently rewritten as

$$\partial_{\bar{z}}\mathcal{O}(\mathbf{z},\mathbf{z}') = (\partial_z + \partial_{z'})\,\boldsymbol{\Theta}(\mathbf{z})\bar{\mathbf{T}}(\mathbf{z}') - (\partial_{\bar{z}} + \partial_{\bar{z}'})\,\boldsymbol{\Theta}(\mathbf{z})\boldsymbol{\Theta}(\mathbf{z}') \ . \tag{2.10}$$

Analogously, the derivative of (2.1) w.r.t. z becomes

$$\partial_z \mathcal{O}(\mathbf{z},\mathbf{z}') = (\partial_z + \partial_{z'})\,\mathbf{T}(\mathbf{z})\bar{\mathbf{T}}(\mathbf{z}') - (\partial_{\bar z} + \partial_{\bar z'})\,\mathbf{T}(\mathbf{z})\mathbf{\Theta}(\mathbf{z}') \ . \tag{2.11}$$

Using the OPEs between the components of the stress-energy tensor

$$\mathrm{T}(\mathbf{z})\Theta(\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} A_k(\mathbf{z}-\mathbf{z}')\mathcal{O}_k(\mathbf{z}') \quad , \qquad \Theta(\mathbf{z})\bar{\mathrm{T}}(\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} B_k(\mathbf{z}-\mathbf{z}')\mathcal{O}_k(\mathbf{z}') \ , \tag{2.12}$$

$$\Theta(\mathbf{z})\Theta(\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} C_k(\mathbf{z}-\mathbf{z}')\mathcal{O}_k(\mathbf{z}') \quad , \qquad \mathrm{T}(\mathbf{z})\bar{\mathrm{T}}(\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} D_k(\mathbf{z}-\mathbf{z}')\mathcal{O}_k(\mathbf{z}') \ , \tag{2.13}$$

the r.h.s of expressions (2.10)-(2.11) become

$$\partial_{\bar z}\mathcal{O}(\mathbf{z},\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} B_k(\mathbf{z}-\mathbf{z}')\left(\partial_{z'}\mathcal{O}_k(\mathbf{z}')\right) - \sum_{k\geq 0} C_k(\mathbf{z}-\mathbf{z}')\left(\partial_{\bar z'}\mathcal{O}_k(\mathbf{z}')\right) \ , \tag{2.14}$$

$$\partial_{z}\mathcal{O}(\mathbf{z},\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} D_k(\mathbf{z}-\mathbf{z}')\left(\partial_{z'}\mathcal{O}_k(\mathbf{z}')\right) - \sum_{k\geq 0} A_k(\mathbf{z}-\mathbf{z}')\left(\partial_{\bar z'}\mathcal{O}_k(\mathbf{z}')\right) \ . \tag{2.15}$$

If we consider instead the OPE of (2.1)

$$\mathcal{O}(\mathbf{z},\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} F_k(\mathbf{z}-\mathbf{z}')\mathcal{O}_k(\mathbf{z}') = \sum_{k\geq 0} \left(D_k(\mathbf{z}-\mathbf{z}') - C_k(\mathbf{z}-\mathbf{z}')\right)\mathcal{O}_k(\mathbf{z}') \ , \tag{2.16}$$

and we differentiate it w.r.t. z and $\bar z$ we get

$$\partial_{\bar z}\mathcal{O}(\mathbf{z},\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} \left(\partial_{\bar z}F_k(\mathbf{z}-\mathbf{z}')\right)\mathcal{O}_k(\mathbf{z}') \ , \tag{2.17}$$

$$\partial_{z}\mathcal{O}(\mathbf{z},\mathbf{z}') \underset{\mathbf{z}\to\mathbf{z}'}{\sim} \sum_{k\geq 0} \left(\partial_{z}F_k(\mathbf{z}-\mathbf{z}')\right)\mathcal{O}_k(\mathbf{z}') \ . \tag{2.18}$$

Comparing expressions (2.14)-(2.15) with (2.17)-(2.18) one realises that they are compatible *iff* the *rhs* of (2.17)-(2.18) contain only coordinate derivatives of local operators. Therefore, the $k-$th coefficient $F_k(\mathbf{z}-\mathbf{z}')$ in (2.16) must be a constant unless $\mathcal{O}_k$ itself is a coordinate derivative of another local operator. We conclude that, up to total derivative terms, (2.1) defines a local operator which we refer to as T$\bar{\text{T}}$

$$\boxed{\lim_{\mathbf{z}'\to\mathbf{z}} \mathcal{O}(\mathbf{z},\mathbf{z}') \equiv \mathrm{T}\bar{\mathrm{T}}(\mathbf{z}) + \text{derivatives} \ .} \tag{2.19}$$

Theorem 2.1.2. *The expectation value of (2.19) on a generic eigenstate $|n\rangle$ of the energy and momentum operators fulfils*

$$\boxed{\langle n|\,T\bar{T}(\mathbf{z})\,|n\rangle = \langle n|\,\mathbf{T}(\mathbf{z})\,|n\rangle\,\langle n|\,\bar{\mathbf{T}}(\mathbf{z})\,|n\rangle - \langle n|\,\mathbf{\Theta}(\mathbf{z})\,|n\rangle\,\langle n|\,\mathbf{\Theta}(\mathbf{z})\,|n\rangle \ ,} \tag{2.20}$$

which will be referred to as factorization property.

Proof. Consider the expectation value of (2.1) on the $n-$th eigenstate $|n\rangle$

$$\mathcal{C}_n(\mathbf{z},\mathbf{z}') = \langle n|\,\mathcal{O}(\mathbf{z},\mathbf{z}')\,|n\rangle = \langle n|\,\mathbf{T}(\mathbf{z})\bar{\mathbf{T}}(\mathbf{z}')\,|n\rangle - \langle n|\,\mathbf{\Theta}(\mathbf{z})\mathbf{\Theta}(\mathbf{z}')\,|n\rangle \ , \tag{2.21}$$

and differentiate it w.r.t. $\bar{z}$

$$\begin{aligned}\partial_{\bar{z}}\mathcal{C}_n(\mathbf{z},\mathbf{z}') &= \langle n|\left(\partial_{\bar{z}}\mathbf{T}(\mathbf{z})\right)\bar{\mathbf{T}}(\mathbf{z}')|n\rangle - \langle n|\left(\partial_{\bar{z}}\mathbf{\Theta}(\mathbf{z})\right)\mathbf{\Theta}(\mathbf{z}')|n\rangle \\ &= \langle n|\left(\partial_{z}\mathbf{\Theta}(\mathbf{z})\right)\bar{\mathbf{T}}(\mathbf{z}')|n\rangle - \langle n|\left(\partial_{\bar{z}}\mathbf{\Theta}(\mathbf{z})\right)\mathbf{\Theta}(\mathbf{z}')|n\rangle \ . \end{aligned} \tag{2.22}$$

where in the last line we used the continuity equations (A.13) to trade $\partial_{\bar{z}}\mathbf{T}(\mathbf{z})$ with $\partial_z\mathbf{\Theta}(\mathbf{z})$. Using (2.6), we replace $(\partial_z,\partial_{\bar{z}})$ with $(-\partial_{z'},-\partial_{\bar{z}'})$ in (2.22) which becomes

$$\begin{aligned}\partial_{\bar{z}}\mathcal{C}_n(\mathbf{z},\mathbf{z}') &= \langle n|\,\mathbf{\Theta}(\mathbf{z})\left(\partial_{z'}\bar{\mathbf{T}}(\mathbf{z}')\right)|n\rangle - \langle n|\,\mathbf{\Theta}(\mathbf{z})\left(\partial_{\bar{z}'}\mathbf{\Theta}(\mathbf{z}')\right)|n\rangle \\ &= \langle n|\,\mathbf{\Theta}(\mathbf{z})\left(\partial_{z'}\bar{\mathbf{T}}(\mathbf{z}') - \partial_{\bar{z}'}\mathbf{\Theta}(\mathbf{z}')\right)|n\rangle = 0\ , \end{aligned} \tag{2.23}$$

where again we used (A.13). Following the same procedure one can show that

$$\partial_z\mathcal{C}_n(\mathbf{z},\mathbf{z}') = 0\ , \tag{2.24}$$

which proves that $\mathcal{C}_n(\mathbf{z},\mathbf{z}')$ is a constant. From this fact it immediately descends that (2.20) holds for the vacuum state ($n=0$). Indeed, from one hand we can take the limit

$$\lim_{\mathbf{z}'\to\mathbf{z}}\mathcal{C}_0(\mathbf{z},\mathbf{z}') = \langle \mathrm{T}\bar{\mathrm{T}}(\mathbf{z})\rangle\ , \tag{2.25}$$

where we used (2.19) and the fact that the expectation value of a total derivative is identically vanishing. From the other hand, applying (2.7) leads to

$$\lim_{t\to\infty}\mathcal{C}_0(\mathbf{z}+t\,\mathbf{e},\mathbf{z}) = \langle\mathbf{T}(\mathbf{z})\rangle\langle\bar{\mathbf{T}}(\mathbf{z})\rangle - \langle\mathbf{\Theta}(\mathbf{z})\rangle\langle\mathbf{\Theta}(\mathbf{z})\rangle\ . \tag{2.26}$$

Since $\mathcal{C}_0(\mathbf{z},\mathbf{z}')$ is constant, the expressions (2.25) and (2.26) coincide, therefore

$$\langle \mathrm{T}\bar{\mathrm{T}}(\mathbf{z})\rangle = \langle\mathbf{T}(\mathbf{z})\rangle\langle\bar{\mathbf{T}}(\mathbf{z})\rangle - \langle\mathbf{\Theta}(\mathbf{z})\rangle\langle\mathbf{\Theta}(\mathbf{z})\rangle\ . \tag{2.27}$$

For $n>0$, instead, there is an additional complication as the asymptotic factorization (2.7) no longer holds. If fact, the generic 2−point functions $\langle n|\,\mathcal{O}_i(\mathbf{z})\mathcal{O}_j(\mathbf{z}')\,|n\rangle$ receive contributions from non-diagonal matrix elements which exponentially grow as the points are taken infinitely apart. Concretely, the spectral decomposition of the *lhs* of (2.21) – in cartesian coordinates $\mathbf{x}$ and $\mathbf{x}'=(x'^1,x'^2)$ – becomes

$$\begin{aligned}&\langle n|\,\mathbf{T}(\mathbf{z})\bar{\mathbf{T}}(\mathbf{z}')\,|n\rangle = \sum_{n'\geq 0}\langle n|\,\mathbf{T}(\mathbf{z})\,|n'\rangle\,\langle n'|\,\bar{\mathbf{T}}(\mathbf{z}')\,|n\rangle\,e^{(E_n-E_{n'})|x^2-x'^2|}e^{i(P_n-P_{n'})|x^1-x'^1|} \\ &= \langle n|\,\mathbf{T}(\mathbf{z})\,|n\rangle\,\langle n|\,\bar{\mathbf{T}}(\mathbf{z}')\,|n\rangle + \sum_{n'\neq n}\langle n|\,\mathbf{T}(\mathbf{z})\,|n'\rangle\,\langle n'|\,\bar{\mathbf{T}}(\mathbf{z}')\,|n\rangle\,e^{(E_n-E_{n'})|x^2-x'^2|}e^{i(P_n-P_{n'})|x^1-x'^1|}\ ,\end{aligned} \tag{2.28}$$

and similarly for $\langle n|\,\mathbf{\Theta}(\mathbf{z})\mathbf{\Theta}(\mathbf{z}')\,|n\rangle$, where in the last line we used the fact that the basis of eigenstates $\{n\}_{n\in\mathbb{N}}$ is non-degenerate. However, the sum over $n'\neq n$ in the last line of (2.28) needs to cancel out with an analogous sum coming from the spectral decomposition of $\langle n|\,\mathbf{\Theta}(\mathbf{z})\mathbf{\Theta}(\mathbf{z}')\,|n\rangle$, to ensure that $\mathcal{C}_n(\mathbf{z},\mathbf{z}')$ does not depend on the coordinates. This leads to

$$\mathcal{C}_n(\mathbf{z},\mathbf{z}') = \langle n|\,\mathbf{T}(\mathbf{z})\,|n\rangle\,\langle n|\,\bar{\mathbf{T}}(\mathbf{z}')\,|n\rangle - \langle n|\,\mathbf{\Theta}(\mathbf{z})\,|n\rangle\,\langle n|\,\mathbf{\Theta}(\mathbf{z}')\,|n\rangle\ , \tag{2.29}$$

and equating the limit $\mathbf{z}'\to\mathbf{z}$ in (2.21) and (2.29) we find (2.20).

Observation 4. Applying the relations (A.7)-(A.8), the operator (2.1) in cartesian coordinates $\mathbf{x}$ (cf. (A.1)) becomes

$$\mathcal{O}(\mathbf{x},\mathbf{x}') = -\frac{\pi^2}{2}\epsilon_{\mu\lambda}\epsilon_{\nu\rho}\boldsymbol{T}^{\mu\nu}(\mathbf{x})\boldsymbol{T}^{\lambda\rho}(\mathbf{x}') \ , \tag{2.30}$$

where $\epsilon_{\mu\nu}$ is the Levi-Civita tensor. Taking the limit $\mathbf{x}' \to \mathbf{x}$ in (2.30), it descends that the T$\bar{\text{T}}$ operator in cartesian coordinates is proportional to the determinant of the stress-energy tensor

$$\text{T}\bar{\text{T}}(\mathbf{x}) = -\pi^2 \det\left[\boldsymbol{T}^{\mu\nu}(\mathbf{x})\right] \ , \tag{2.31}$$

and the factorization property (2.20) becomes

$$\langle n| \det\left[\boldsymbol{T}^{\mu\nu}(\mathbf{x})\right] |n\rangle = \det\left[\langle n| \boldsymbol{T}^{\mu\nu}(\mathbf{x}) |n\rangle\right] \ . \tag{2.32}$$

Observation 5. The results presented in this section can be straightforwardly generalised to the family of composite operators [1]

$$\mathcal{O}_s(\mathbf{z},\mathbf{z}') = \mathbf{T}_{s+1}(\mathbf{z})\bar{\mathbf{T}}_{s+1}(\mathbf{z}') - \mathbf{\Theta}_{s-1}(\mathbf{z})\bar{\mathbf{\Theta}}_{s-1}(\mathbf{z}') \ . \tag{2.33}$$

where $\left\{\mathbf{T}_{s+1}, \mathbf{\Theta}_{s-1}, \bar{\mathbf{T}}_{s+1}, \bar{\mathbf{\Theta}}_{s-1}\right\}$ are the chiral components of the level$-s$ conserved current of the hierarchy (see appendix A).

2.2 The T$\bar{\text{T}}$ deformation and the Burgers equation

In this section, we define the action of the T$\bar{\text{T}}$ operator, *i.e.* the T$\bar{\text{T}}$ deformation, on the energy levels of a generic 2$-$dimensional QFT confined on an infinitely long cylinder of circumference R, *i.e.*

$$(x^1, x^2) \sim (x^1 + R, x^2) \ , \tag{2.34}$$

where $\mathbf{x} = (x^1, x^2)$ is a set of cartesian coordinates. Then, using the factorization property (2.20), we write the evolution equation of the energy levels in terms of an inviscid Burgers equation.

The Hilbert space of the theory is defined on constant$-x^2$ slices, namely the energy and momentum operators $\hat{E}$ and $\hat{P}$ – and their corresponding eigenvalues $\{E_n\}_{n\in\mathbb{N}}$ and $\{P_n\}_{n\in\mathbb{N}}$ – depend only on the compactified size R

$$\hat{E}(R)\,|n\rangle = E_n(R)\,|n\rangle \ , \quad \hat{P}(R)\,|n\rangle = P_n(R)\,|n\rangle \ , \tag{2.35}$$

and the quantisation of the momentum implies

$$P_n(R) = \frac{2\pi\, k_n}{R} \ , \quad (k_n \in \mathbb{Z}) \ . \tag{2.36}$$

In this setup, the expectation values of the Euclidean components of the stress-energy tensor on the eigenstates of the energy and momentum operators are related to their eigenvalues as follows (cf. (A.15) for $\mathrm{s}=1$)

$$E_n(R) = -R\,\langle n| \boldsymbol{T}^{22} |n\rangle \ , \quad \partial_R E_n(R) = -\langle n| \boldsymbol{T}^{11} |n\rangle \ , \quad P_n(R) = -\mathbb{i}R\,\langle n| \boldsymbol{T}^{12} |n\rangle \ , \tag{2.37}$$

where we used the global translational invariance on the cylinder:

$$\langle n| \boldsymbol{T}^{\mu\nu} |n\rangle = \langle n| \boldsymbol{T}^{\mu\nu}(\mathbf{x}) |n\rangle = \frac{1}{R} \langle n| \int_0^R \boldsymbol{T}^{\mu\nu}(\mathbf{x})\, dx^1 |n\rangle \ . \tag{2.38}$$

Using (2.31)-(2.32) and (2.37)-(2.38) one finds

$$\langle n| \mathrm{T}\bar{\mathrm{T}} |n\rangle = -\pi^2 \det\left[\langle n| \boldsymbol{T}^{\mu\nu} |n\rangle\right] = -\frac{\pi^2}{R}\left(E_n(R)\partial_R E_n(R) + \frac{1}{R}P_n^2(R)\right) \ , \tag{2.39}$$

where again we denoted

$$\langle n| \mathrm{T}\bar{\mathrm{T}} |n\rangle = \frac{1}{R} \langle n| \int_0^R \mathrm{T}\bar{\mathrm{T}}(\mathbf{x})\, dx^1 |n\rangle \ . \tag{2.40}$$

Let us consider an infinitesimal transformation in the small parameter τ which modifies the $n-$th energy eigenvalue $E_n(R)$ by an amount

$$E_n(R,\tau) = E_n(R) - \tau\, \frac{R}{\pi^2} \langle n| \mathrm{T}\bar{\mathrm{T}} |n\rangle + \mathcal{O}(\tau^2) \ , \tag{2.41}$$

leaving the $n-$th momentum eigenvalue $P_n(R)$ unchanged. Iterating (2.41) leads to the definition of the T$\bar{\mathrm{T}}$ deformation – at finite values of τ – as

$$\boxed{\partial_\tau E_n(R,\tau) = -\frac{R}{\pi^2}\, {}_\tau\langle n| \mathrm{T}\bar{\mathrm{T}} |n\rangle_\tau \ ,} \tag{2.42}$$

where $\{|n\rangle_\tau\}_{n\in\mathbb{N}}$ are the eigenstates of the deformed energy operator $\hat{E}(R,\tau)$

$$\hat{E}(R,\tau)\, |n\rangle_\tau = E_n(R,\tau)\, |n\rangle_\tau \ , \quad \hat{P}(R)\, |n\rangle_\tau = P_n(R)\, |n\rangle_\tau \ , \tag{2.43}$$

and

$${}_\tau\langle n| \mathrm{T}\bar{\mathrm{T}} |n\rangle_\tau = \frac{1}{R}\, {}_\tau\langle n| \int_0^R \mathrm{T}\bar{\mathrm{T}}(\mathbf{x},\tau)\, dx^1 |n\rangle_\tau \ , \tag{2.44}$$

has the geometrical interpretation of tangent vector to the curve described by $E_n(R,\tau)$ as τ varies. Interestingly, using (2.39), the flow equation (2.42) can be recast into a PDE for $E_n(R,\tau)$ in the variables R and τ

$$\boxed{\partial_\tau E_n(R,\tau) = E_n(R,\tau)\partial_R E_n(R,\tau) + \frac{1}{R}P_n^2(R) \ ,} \tag{2.45}$$

which has the form of a $1-$dimensional inviscid Burgers equation. The momentum quantisation (2.36) implies

$$\partial_R P(R) = -\frac{P(R)}{R} \ , \tag{2.46}$$

from which it descends that (2.45) can be more conveniently rewritten as

$$\partial_\tau E(R,\tau) = \frac{1}{2}\partial_R\left(E^2(R,\tau) - P^2(R)\right) \ , \tag{2.47}$$

where we dropped the subscript n, since (2.45) has the same expression for all n. Using the method of characteristics, it is possible to construct a general solution to (2.47) in implicit form as

$$\boxed{E^2(R,\tau) - P^2(R) = E^2(\mathcal{R}_0) - P^2(\mathcal{R}_0) \ ,} \tag{2.48}$$

with the additional constraint

$$\boxed{\partial_\tau R|_{\mathcal{R}_0=\text{const.}} = -E(R,\tau)\ ,} \tag{2.49}$$

where $\mathcal{R}_0$ is a redefinition of the size R which depends on the energy and momentum of the deformed theory through

$$\boxed{\mathcal{R}_0^2 = (R+\tau E(R,\tau))^2 - (\tau P(R))^2\ .} \tag{2.50}$$

Observation 6. To verify that (2.48)-(2.50) is indeed solution to (2.47), we differentiate both sides of (2.48) w.r.t. τ keeping $\mathcal{R}_0$ fixed and, thus, letting R depend on τ according to (2.50). The result of this operation is

$$\frac{d}{d\tau}\left(E^2(R,\tau)-P^2(R)\right)=0 \quad\longrightarrow\quad \partial_\tau E^2(R,\tau)+(\partial_\tau R)\,\partial_R\left(E^2(R,\tau)-P^2(R)\right)=0\ , \tag{2.51}$$

which coincides with (2.47) provided (2.49) holds.

From (2.48)-(2.50) it descends that the spectrum of the T$\bar{\text{T}}$−deformed theory is uniquely derived from the spectrum of the original theory confined on a cylinder with a circumference modified according to (2.50). For this reason, the T$\bar{\text{T}}$ deformation is referred to as exactly solvable, even if the original theory is not integrable. To see concretely the effect of the T$\bar{\text{T}}$ deformation, now we explicitly compute the T$\bar{\text{T}}$−deformed energy levels of a CFT, reviewing the result previously obtained in [1,11].

2.2.1 T$\bar{\text{T}}$ deformation of a CFT

Let us consider a CFT with central charge c defined on a cylinder of circumference R. In terms of the holomorphic and anti-holomorphic weights

$$h^{(\pm)} = h_0^{(\pm)} + n^{(\pm)}\ , \quad \left(n^{(\pm)}\in\mathbb{N}\right)\ , \tag{2.52}$$

$h_0^{(\pm)}$ being the highest weights, the energies of the (anti)-holomorphic sectors are

$$I_1^{(\pm)}(R) = \frac{2\pi a_1^{(\pm)}}{R}\ , \quad a_1^{(\pm)} = h^{(\pm)} - \frac{c}{24}\ , \tag{2.53}$$

and the total energy and momentum are

$$E^{\text{CFT}}(R) = I_1^{(+)}(R)+I_1^{(-)}(R) = \frac{2\pi}{R}\left(h^{(+)}+h^{(-)}-\frac{c}{12}\right)\ , \tag{2.54}$$

$$P^{\text{CFT}}(R) = I_1^{(+)}(R)-I_1^{(-)}(R) = \frac{2\pi}{R}\left(h^{(+)}-h^{(-)}\right)\ . \tag{2.55}$$

Using expressions (2.54)-(2.55) as $\tau=0$ initial condition for the energy and momentum, (2.48)-(2.50) become a quadratic equation for $E^{\text{CFT}}(R,\tau)$ from which one easily extract the solution

$$\boxed{E^{\text{CFT}}(R,\tau) = \frac{R}{2\tau}\left(-1+\sqrt{1+\frac{4\tau}{R}E^{\text{CFT}}(R)+\frac{4\tau^2}{R^2}\left(P^{\text{CFT}}(R)\right)^2}\right)}\ . \tag{2.56}$$

Observation 7. Restricting to the zero momentum case, *i.e.* $h^{(+)} = h^{(-)} = h$, formula (2.56) becomes

$$E^{\text{CFT}}(R,\tau) = \frac{R}{2\tau}\left(-1+\sqrt{1+\frac{8\pi C\tau}{R^2}}\right) , \quad C = 2h - \frac{c}{12} . \tag{2.57}$$

Notice that, as soon as $\tau \neq 0$, the pole of (2.54) at $R = 0$ resolves into a pair of square root branch points at $R^{(\star,\pm)} = \pm\sqrt{-8\pi C\tau}$. For $C\tau < 0$, one of the branch points moves from $R = 0$ rightwards on the positive $R-$axis, producing a singularity at physical values of the radius such that the analytic continuation of (2.57) to the region $R^{(\star,-)} < R < R^{(\star,+)}$ is complex (see Figure 2.1a). Conversely, for $C\tau > 0$, the branch points move away from $R = 0$ along the imaginary $R-$axis and no singularities appear in the energy (see Figure 2.1b). Thus, there are either infinitely-many real energy levels and a finite number of complex ones or viceversa, depending on whether τ is positive or negative, respectively.

Observation 8. Setting $c = D - 2$, the square root term in (2.56) becomes

$$\begin{aligned} &\frac{R}{2\tau}\sqrt{1+\frac{4\tau}{R}E^{\text{CFT}}(R)+\frac{4\tau^2}{R^2}\left(P^{\text{CFT}}(R)\right)^2} \\ &\qquad = \sqrt{\frac{R^2}{4\tau^2}+\frac{2\pi}{\tau}\left(h^{(+)}+h^{(-)}-\frac{D-2}{12}\right)+\frac{4\pi^2}{R^2}\left(h^{(+)}-h^{(-)}\right)^2} , \end{aligned} \tag{2.58}$$

which has the same form of the quantised spectrum of the Nambu-Goto string in the critical dimension.

As noticed in [1], adding to (2.54) a bulk term proportional to R

$$E^{\text{TOT}}(R) = E^{\text{BULK}}(R) + E^{\text{CFT}}(R) , \quad E^{\text{BULK}}(R) = F_0 R , \quad (F_0 \in \mathbb{R}) , \tag{2.59}$$

and solving (2.48)-(2.50) using (2.55) and (2.59) as $\tau = 0$ initial condition, one gets

$$\begin{aligned} &\boxed{E^{\text{TOT}}(R,\tau) = E^{\text{BULK}}(R,\tau) + E^{\text{CFT}}(R,\tilde{\tau})} \\ &\qquad = \frac{F_0 R}{1-\tau F_0} + \frac{R}{2\tilde{\tau}}\left(-1+\sqrt{1+\frac{4\tilde{\tau}}{R}E^{\text{CFT}}(R)+\frac{4\tilde{\tau}^2}{R^2}\left(P^{\text{CFT}}(R)\right)^2}\right) , \end{aligned} \tag{2.60}$$

where the bulk term transforms as

$$\boxed{E^{\text{BULK}}(R,\tau) = \frac{R E^{\text{BULK}}(R)}{R - \tau E^{\text{BULK}}(R)}} , \tag{2.61}$$

and $\tilde{\tau}$ is a redefinition of the deformation parameter depending on F_0

$$\tilde{\tau} = \tau(1 - \tau F_0) . \tag{2.62}$$

Observation 9. The deformed bulk term (2.61) introduces a Landau-type pole singularity at $\tau_{\text{LP}} = \frac{1}{F_0}$ in the evolution of the energy levels.

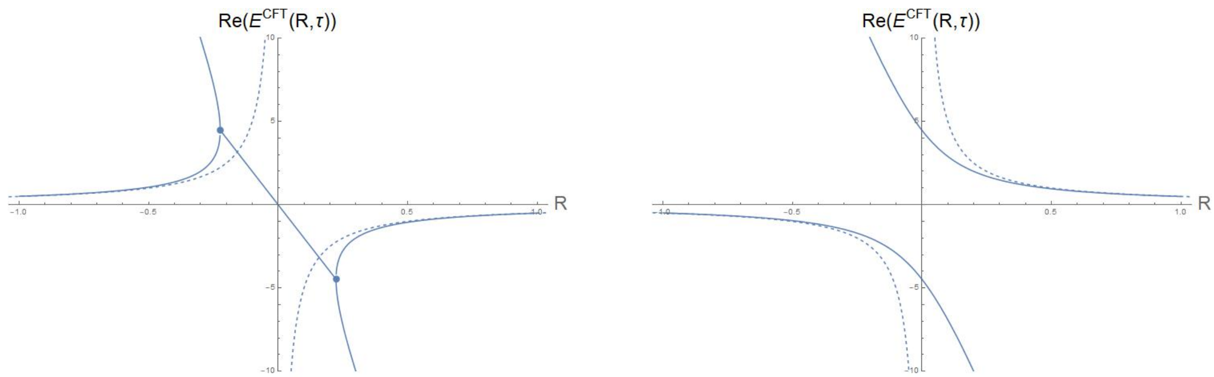


Figure 2.1: The real part of the energy of a T$\bar{\text{T}}-$deformed CFT for $\tau = 0.025$ with $C = -\frac{1}{2\pi}$ (a) or $C = \frac{1}{2\pi}$ (b). The blue dots in (a) represent the critical values $R^{(\star,\pm)}$.

Observation 10. The presence of (2.61) allows to tune the deformation parameter τ such that (2.60) becomes a pure square root without additional terms. There exists a unique value $\tau_0 = \frac{1}{2F_0}$ such that (2.60) evaluated at $\tau = \tau_0$ is

$$E^{\text{TOT}}(R,\tau_0) = \frac{R}{2\tilde{\tau}_0}\sqrt{1 + \frac{4\tilde{\tau}_0}{R}E^{\text{CFT}}(R) + \frac{4\tilde{\tau}_0^2}{R^2}\left(P^{\text{CFT}}(R)\right)^2}\ , \quad \tilde{\tau}_0 = \tau_0(1-\tau_0 F_0)\ . \tag{2.63}$$

In addition, the expectation value of the T$\bar{\text{T}}$ operator becomes state- and size-independent at $\tau = \tau_0$

$$_{\tau_0}\langle n|\,\text{T}\bar{\text{T}}\,|n\rangle_{\tau_0} = -\frac{\pi^2}{2R}\partial_R\left(\left(E^{\text{TOT}}(R,\tau_0)\right)^2 - \left(P^{\text{CFT}}(R)\right)^2\right) = -\left(\frac{\pi}{2\tilde{\tau}_0}\right)^2\ . \tag{2.64}$$

2.3 The classical T$\bar{\text{T}}$ flow equation

Starting from the definition (2.42) of the T$\bar{\text{T}}$ deformation, in this section we derive the corresponding flow equation for the classical densities. For this purpose, we consider the theory confined on a torus with characteristic lengths R and L using the identifications

$$(x^1,x^2) \sim (x^1+R, x^2+L)\ , \tag{2.65}$$

where the energy and momentum operator are defined on slices at fixed x^2 with eigenvalue equations (2.43), and

$$\hat{E}(R,\tau) = \int_0^R \hat{\mathcal{H}}(\mathbf{x},\tau)\,dx^1\ , \quad \hat{P}(R) = \int_0^R \hat{\mathcal{P}}(\mathbf{x})\,dx^1\ . \tag{2.66}$$

The partition function of the deformed system is then

$$Z(R,L,\tau) = \text{Tr}\left[e^{-L\hat{E}(R,\tau)}\right] = \sum_{n\geq 0} e^{-LE_n(R,\tau)}\ , \tag{2.67}$$

where L is the inverse temperature. Multiplying both sides of (2.42) by the factor $Le^{-LE_n(R,\tau)}$ and summing over $n \geq 0$ yields

$$\partial_\tau Z(R,L,\tau) = -\frac{Z(R,L,\tau)}{\pi^2}\left\langle \int_0^L dx^2 \int_0^R dx^1\, \mathrm{T\bar{T}}(\mathbf{x},\tau)\right\rangle_{\text{ens.}} , \tag{2.68}$$

where $\langle\star\rangle_{\text{ens.}}$ is the ensemble average defined as

$$\langle\hat{F}\rangle_{\text{ens.}} = \frac{1}{Z}\sum_{n\geq 0} F_n e^{-LE_n(R,\tau)} , \tag{2.69}$$

$\{F_n\}_{n\in\mathbb{N}}$ being the set of eigenvalues of the generic operator $\hat{F}$ on the eigenstates $\{|n\rangle\}_{n\in\mathbb{N}}$ of the energy and momentum operators. Rewriting equation (2.68) as

$$\partial_\tau \log Z(R,L,\tau) = -\frac{1}{\pi^2}\left\langle \int_0^L dx^2 \int_0^R dx^1\, \mathrm{T\bar{T}}(\mathbf{x},\tau)\right\rangle_{\text{ens.}} , \tag{2.70}$$

we realise that the *lhs* of (2.70) can be expressed as an ensemble average

$$\partial_\tau \log Z(R,L,\tau) = -\frac{L}{Z}\sum_{n\geq 0}\left(\partial_\tau E_n(R,\tau)\right) e^{-LE_n(R,\tau)} = -\left\langle \int_0^L dx^2 \int_0^R dx^1\, \partial_\tau \hat{\mathcal{H}}(\mathbf{x},\tau)\right\rangle_{\text{ens.}} . \tag{2.71}$$

In conclusion, plugging (2.71) back in (2.70) we find

$$\left\langle \int_0^L dx^2 \int_0^R dx^1\, \partial_\tau \hat{\mathcal{H}}(\mathbf{x},\tau)\right\rangle_{\text{ens.}} = \frac{1}{\pi^2}\left\langle \int_0^L dx^2 \int_0^R dx^1\, \mathrm{T\bar{T}}(\mathbf{x},\tau)\right\rangle_{\text{ens.}} , \tag{2.72}$$

which, in turn, implies that the corresponding classical densities fulfils, up to total derivative terms, the following differential equation

$$\boxed{\partial_\tau \mathcal{H}(\mathbf{x},\tau) = \frac{1}{\pi^2}\mathrm{T\bar{T}}(\mathbf{x},\tau)\ .} \tag{2.73}$$

Next, we want to prove that (2.73) implies a similar relation for the Lagrangian density $\mathcal{L}(\mathbf{x},\tau)$. For this purpose, we consider an Hamiltonian density $\mathcal{H}(\mathbf{x},\tau)$ which depends on $\mathbf{x}$ through the set of independent fields[1] $\{\phi(\mathbf{x}), \phi'(\mathbf{x}), \pi(\mathbf{x})\}$, where $\phi'(\mathbf{x}) = \partial_{x^1}\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ is the conjugated momentum. Since the fields do not depend on τ when considered off-shell, the dependence on τ in (2.73) is only explicit, *i.e.*

$$\frac{d}{d\tau}\mathcal{H}(\mathbf{x},\tau) = \partial_\tau \mathcal{H}(\mathbf{x},\tau)\ . \tag{2.74}$$

Instead, the Lagrangian density $\mathcal{L}(\mathbf{x},\tau)$ depend on $\mathbf{x}$ through the fields $\left\{\phi(\mathbf{x}), \phi'(\mathbf{x}), \dot{\phi}(\mathbf{x},\tau)\right\}$, where $\dot{\phi}(\mathbf{x},\tau) = \partial_{x^2}\phi(\mathbf{x},\tau)$, which are not independent. In fact, the relation

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} , \tag{2.75}$$

[1] For simplicity, we consider a theory involving a single scalar field $\phi(\mathbf{x})$ and only first derivatives of $\phi(\mathbf{x})$. However, it is easy to extend the argument to a generic field theory involving higher-derivatives.

fixes $\dot{\phi}(\mathbf{x},\tau)$ as a function of the independent fields $\{\phi(\mathbf{x}),\phi'(\mathbf{x}),\pi(\mathbf{x})\}$ and τ. Therefore, one has

$$\frac{d}{d\tau}\mathcal{L}(\mathbf{x},\tau)=\partial_\tau\mathcal{L}(\mathbf{x},\tau)+\left(\partial_\tau\dot{\phi}(\mathbf{x},\tau)\right)\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\ . \tag{2.76}$$

Differentiating both sides of the Legendre transform w.r.t. τ and using (2.74)-(2.76) one gets

$$\begin{aligned}\frac{d}{d\tau}\mathcal{H}(\mathbf{x},\tau)&=\frac{d}{d\tau}\Big(\pi(\mathbf{x})\,\dot{\phi}(\mathbf{x},\tau)-\mathcal{L}(\mathbf{x},\tau)\Big)\\ \longrightarrow\quad\partial_\tau\mathcal{H}(\mathbf{x},\tau)&=\pi(\mathbf{x})\big(\partial_\tau\dot{\phi}(\mathbf{x},\tau)\big)-\partial_\tau\mathcal{L}(\mathbf{x},\tau)-\big(\partial_\tau\dot{\phi}(\mathbf{x},\tau)\big)\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=-\partial_\tau\mathcal{L}(\mathbf{x},\tau)\ .\end{aligned} \tag{2.77}$$

The latter result implies that the Lagrangian density fulfils, again up to total derivative terms, the following differential equation

$$\boxed{\partial_\tau\mathcal{L}(\mathbf{x},\tau)=-\frac{1}{\pi^2}\mathrm{T}\bar{\mathrm{T}}(\mathbf{x},\tau)\ .} \tag{2.78}$$

The geometrical meaning of both (2.73) and (2.78) is transparent: the local operator $\mathrm{T}\bar{\mathrm{T}}(\mathbf{x},\tau)$ is the tangent vector to the curve $\mathcal{H}(\mathbf{x},\tau)$ and $\mathcal{L}(\mathbf{x},\tau)$ as τ varies. In the next section, we explicitly solve (2.78) for a non-interacting massless scalar field, reviewing the result first obtained in [11]. Subsequently, we generalise the result to sigma models and also to Yang-Mills theories [46]. For the extension to fermionic theories we refer the reader to [47, 49].

2.3.1 Massless free scalar field

Consider the action of a massless scalar field ϕ minimally coupled to gravity through a generic $2-$dimensional metric tensor $g_{\mu\nu}$. In cartesian coordinates $\mathbf{x}$ the action is

$$\mathcal{A}\,[\phi]=\int_{\mathbb{R}^2}\mathcal{L}_g(\mathbf{x})\,dx^1\wedge dx^2\ , \tag{2.79}$$

with Lagrangian density

$$\mathcal{L}_g(\mathbf{x})=\frac{1}{4}\sqrt{g}\,g^{\mu\nu}\partial_\mu\phi(\mathbf{x})\,\partial_\nu\phi(\mathbf{x})\ ,\quad \sqrt{g}=\det\left[g_{\mu\nu}\right]\ ,\quad ((\mu,\nu)\in\{1,2\})\ . \tag{2.80}$$

To start, we first analyse the case of a flat Euclidean space-time, *i.e.* $g_{\mu\nu}=\delta_{\mu\nu}$, and we denote $\mathcal{L}(\mathbf{x})=\mathcal{L}_\delta(\mathbf{x})$. At the end of the section, we extend the result to a generic curved background $g_{\mu\nu}$. In accordance with the notation of [11], we work in the set of complex coordinates $\mathbf{z}$ defined through (A.1). Correspondingly the Lagrangian density $\mathcal{L}(\mathbf{x})$ becomes

$$\mathcal{L}(\mathbf{z})=\partial_z\phi\,\partial_{\bar{z}}\phi\ . \tag{2.81}$$

From now on, we drop the explicit dependence of the field on $\mathbf{z}$.

The goal of this section is to solve the flow equation (2.78), namely to find the deformed Lagrangian $\mathcal{L}(\mathbf{z},\tau)$ using (2.81) as initial condition at $\tau=0$. Using the definition of the T$\bar{\text{T}}$ operator (2.19), the *rhs* of (2.78) becomes

$$\partial_\tau\mathcal{L}(\mathbf{z},\tau)=-4\,\left(\mathrm{T}(\mathbf{z},\tau)\bar{\mathrm{T}}(\mathbf{z},\tau)-\Theta^2(\mathbf{z},\tau)\right)\ , \tag{2.82}$$

where $\{\mathrm{T}(\mathbf{z},\tau),\bar{\mathrm{T}}(\mathbf{z},\tau),\Theta(\mathbf{z},\tau)\}$ are the rescaled chiral components of the stress-energy tensor of the deformed theory, defined according to (A.9). For a generic Lagrangian theory involving a single scalar field, they are obtained from the Lagrangian density through

$$\mathrm{T} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_{\bar{z}}\phi)}\,\partial_z\phi\ , \quad \bar{\mathrm{T}} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_z\phi)}\,\partial_{\bar{z}}\phi\ ,$$
$$\Theta = \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial(\partial_z\phi)}\,\partial_z\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\bar{z}}\phi)}\,\partial_{\bar{z}}\phi - 2\mathcal{L}\right)\ . \tag{2.83}$$

Using the expressions (2.83), equation (2.82) becomes a PDE for the function $\mathcal{L}(\mathbf{z},\tau)$ in the variables $(\tau,\partial_z\phi,\partial_{\bar{z}}\phi)$

$$\partial_\tau\mathcal{L}(\mathbf{z},\tau) = -\left(\frac{\partial\mathcal{L}(\mathbf{z},\tau)}{\partial(\partial_{\bar{z}}\phi)}\,\partial_z\phi\right)\left(\frac{\partial\mathcal{L}(\mathbf{z},\tau)}{\partial(\partial_z\phi)}\,\partial_{\bar{z}}\phi\right)$$
$$+\frac{1}{4}\left(\frac{\partial\mathcal{L}(\mathbf{z},\tau)}{\partial(\partial_z\phi)}\,\partial_z\phi + \frac{\partial\mathcal{L}(\mathbf{z},\tau)}{\partial(\partial_{\bar{z}}\phi)}\,\partial_{\bar{z}}\phi - 2\mathcal{L}(\mathbf{z},\tau)\right)^2\ . \tag{2.84}$$

To solve (2.84), we expand the solution $\mathcal{L}(\mathbf{z},\tau)$ as a Taylor series around $\tau=0$ as

$$\mathcal{L}(\mathbf{z},\tau) = \sum_{n\geq 0}\mathcal{L}^{(n)}(\mathbf{z})\,\tau^n\ , \tag{2.85}$$

where the coefficients $\{\mathcal{L}^{(n)}(\mathbf{z})\}_{n\in\mathbb{N}}$ are functions of $(\partial_z\phi,\partial_{\bar{z}}\phi)$ to be determined and $\mathcal{L}^{(0)}(\mathbf{z}) = \mathcal{L}(\mathbf{z})$. Plugging the ansatz (2.85) into (2.84) one finds the recurrence relation

$$\mathcal{L}^{(n+1)}(\mathbf{z}) = -\frac{4}{n+1}\sum_{m=0}^{n}\mathrm{T}^{(m)}(\mathbf{z})\,\bar{\mathrm{T}}^{(n-m)}(\mathbf{z}) - \Theta^{(m)}(\mathbf{z})\,\Theta^{(n-m)}(\mathbf{z})\ , \tag{2.86}$$

with

$$\mathrm{T}^{(n)}(\mathbf{z}) = -\frac{1}{2}\frac{\partial\mathcal{L}^{(n)}(\mathbf{z})}{\partial(\partial_{\bar{z}}\phi)}\,\partial_z\phi\ , \quad \bar{\mathrm{T}}^{(n)}(\mathbf{z}) = -\frac{1}{2}\frac{\partial\mathcal{L}^{(n)}(\mathbf{z})}{\partial(\partial_z\phi)}\,\partial_{\bar{z}}\phi\ , \tag{2.87}$$

$$\Theta^{(n)}(\mathbf{z}) = \frac{1}{4}\left(\frac{\partial\mathcal{L}^{(n)}(\mathbf{z})}{\partial(\partial_z\phi)}\,\partial_z\phi + \frac{\partial\mathcal{L}^{(n)}(\mathbf{z})}{\partial(\partial_{\bar{z}}\phi)}\,\partial_{\bar{z}}\phi - 2\mathcal{L}^{(n)}(\mathbf{z})\right)\ , \tag{2.88}$$

which furnishes $\mathcal{L}^{(n+1)}(\mathbf{z})$ in terms of $\mathcal{L}^{(n)}(\mathbf{z})$. Solving iteratively (2.86) with initial condition $\mathcal{L}^{(0)}(\mathbf{z}) = \mathcal{L}(\mathbf{z})$, one can fix the $n-$th coefficient $\mathcal{L}^{(n)}(\mathbf{z})$ as

$$\mathcal{L}^{(n)}(\mathbf{z}) = \binom{1/2}{n+1}\,2^{2n+1}\big(\mathcal{L}(\mathbf{z})\big)^{n+1}\ . \tag{2.89}$$

From the latter expression, one finds that the series (2.85) admits the resummation

$$\boxed{\mathcal{L}(\mathbf{z},\tau) = \frac{1}{2\tau}\left(-1+\sqrt{1+4\tau\,\mathcal{L}(\mathbf{z})}\right)\ .} \tag{2.90}$$

Observation 11. Notice that the square root term in (2.90), *i.e.*

$$\frac{1}{2\tau}\sqrt{1+4\tau\,\partial_z\phi\,\partial_{\bar{z}}\phi} = \frac{1}{2\tau}\sqrt{1+\tau\,\partial_\mu\phi\,\partial^\mu\phi}\ ,$$

is equivalent to the 3−dimensional Nambu-Goto Lagrangian density[1]

$$\mathcal{L}^{\text{NG}} = \sqrt{\det\left[\mathfrak{h}_{\mu\nu}\right]} \,, \quad \mathfrak{h}_{\mu\nu} = \eta_{\alpha\beta}\,\partial_\mu X^\alpha\,\partial_\nu X^\beta \,, \quad ((\mu,\nu) = \{1,2\} \,,\, (\alpha,\beta) = \{1,2,3\}) \,, \tag{2.91}$$

with target space metric $\eta_{\alpha\beta} = \delta_{\alpha\beta}$, upon a gauge choice called *static-gauge*, which constrains the directions $\{X^\mu\}_{\mu=1,2}$ to coincide with the worldsheet coordinates $\mathbf{x}$ and the transverse direction X^3 – w.r.t. to the worldsheet – to be proportional to the scalar field ϕ, in formulae

$$X^\mu = x^\mu \,, \quad X^3 = \sqrt{\tau}\phi \,, \quad (\mu = 1,2) \,. \tag{2.92}$$

Interestingly, also at the level of the classical action it emerges a connection between the T$\bar{\text{T}}$ deformation and the Nambu-Goto theory.

From the latter observation it follows that (2.90) can be more elegantly rewritten in cartesian coordinates $\mathbf{x}$ as

$$\mathcal{L}(\mathbf{x},\tau) = \frac{1}{2\tau}\left(-1 + \sqrt{\det\left[\delta_{\mu\nu} + \tau\,\partial_\mu\phi\,\partial_\nu\phi\right]}\right) \,, \tag{2.93}$$

which suggests a natural generalisation to the case of a theory coupled to a generic metric $g_{\mu\nu}$

$$\boxed{\mathcal{L}_g(\mathbf{x},\tau) = \frac{1}{2\tau}\left(-\sqrt{g} + \sqrt{\det\left[g_{\mu\nu} + \tau\,\partial_\mu\phi\,\partial_\nu\phi\right]}\right) \,.} \tag{2.94}$$

Indeed, it can be checked that (2.94) fulfils (2.78), provided the stress-energy tensor is computed via the Hilbert construction as

$$\boldsymbol{T}^{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\partial\mathcal{L}_g}{\partial g_{\mu\nu}} \,. \tag{2.95}$$

2.3.2 Interacting scalar field

In this section we will extend the result of 2.3.1 to the case of a scalar field ϕ interacting with a generic potential term V depending on ϕ only[2]. The starting point is the action

$$\mathcal{A}^V[\phi] = \int_{\mathbb{R}^2} \mathcal{L}^V_g(\mathbf{x})\,dx^1 \wedge dx^2 \,, \tag{2.96}$$

with Lagrangian density

$$\mathcal{L}^V_g(\mathbf{x}) = \sqrt{g}\left(\frac{1}{4}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V\right) \,. \tag{2.97}$$

Following the strategy of section 2.3.1, we first consider the case of a flat Euclidean space-time, *i.e.* $g_{\mu\nu} = \delta_{\mu\nu}$ and we denote $\mathcal{L}^V(\mathbf{x}) = \mathcal{L}^V_\delta(\mathbf{x})$. Afterwards, we generalise the results to an arbitrary curved background $g_{\mu\nu}$.

[1]The D−dimensional Nambu Goto theory describes the dynamics of a 2−dimensional surface, called worldsheet, embedded in a D−dimensional target space. $\{X^\alpha(\mathbf{x})\}_{\alpha=1}^D$ denotes the embedding of the worldsheet in the target space, *i.e.* the parametrisation of the worldsheet, while $\mathfrak{h}_{\mu\nu}$ represents the metric induced on the worldsheet.

[2]We only require the potential not to depend on derivatives of ϕ.

The PDE describing the evolution of the deformed Lagrangian is again (2.84)

$$\partial_\tau \mathcal{L}^V(\mathbf{z},\tau) = -\left(\frac{\partial \mathcal{L}^V(\mathbf{z},\tau)}{\partial(\partial_{\bar z}\phi)}\,\partial_z\phi\right)\left(\frac{\partial \mathcal{L}^V(\mathbf{z},\tau)}{\partial(\partial_z\phi)}\,\partial_{\bar z}\phi\right) + \frac{1}{4}\left(\frac{\partial \mathcal{L}^V(\mathbf{z},\tau)}{\partial(\partial_z\phi)}\,\partial_z\phi + \frac{\partial \mathcal{L}^V(\mathbf{z},\tau)}{\partial(\partial_{\bar z}\phi)}\,\partial_{\bar z}\phi - 2\mathcal{L}^V(\mathbf{z},\tau)\right)^2 , \tag{2.98}$$

with $\tau = 0$ initial condition

$$\mathcal{L}^V(\mathbf{z}) = \partial_z\phi\,\partial_{\bar z}\phi + V \; . \tag{2.99}$$

To solve (2.98), we again expand the solution around $\tau = 0$ as a Taylor series

$$\mathcal{L}^V(\mathbf{z},\tau) = \sum_{n\geq 0} \mathcal{L}^{(n)}(\mathbf{z})\,\tau^n \; , \tag{2.100}$$

with $\mathcal{L}^{(0)}(\mathbf{z}) = \mathcal{L}^V(\mathbf{z})$. Plugging (2.100) into (2.98), we arrive at the recurrence relation (2.86), from which the coefficients $\{\mathcal{L}^{(n)}(\mathbf{z})\}_{n\in\mathbb{N}}$ can be extracted in terms of $\mathcal{L}(\mathbf{z}) = \partial_z\phi\,\partial_{\bar z}\phi$ and V as

$$\begin{aligned}
\mathcal{L}^{(1)}(\mathbf{z}) &= -\big(\mathcal{L}(\mathbf{z})\big)^2 + V^2 \; , \\
\mathcal{L}^{(2)}(\mathbf{z}) &= 2\big(\mathcal{L}(\mathbf{z})\big)^3 + \big(\mathcal{L}(\mathbf{z})\big)^2 V + V^3 \; , \\
\mathcal{L}^{(3)}(\mathbf{z}) &= -5\big(\mathcal{L}(\mathbf{z})\big)^4 - 4\big(\mathcal{L}(\mathbf{z})\big)^3 V + V^4 \; , \\
\mathcal{L}^{(4)}(\mathbf{z}) &= 14\big(\mathcal{L}(\mathbf{z})\big)^5 + 15\big(\mathcal{L}(\mathbf{z})\big)^4 V + 2\big(\mathcal{L}(\mathbf{z})\big)^3 V^2 + V^5 \; , \\
\mathcal{L}^{(5)}(\mathbf{z}) &= -42\big(\mathcal{L}(\mathbf{z})\big)^6 - 56\big(\mathcal{L}(\mathbf{z})\big)^5 V - 15\big(\mathcal{L}(\mathbf{z})\big)^4 V^2 + V^6 \; , \\
&\;\;\vdots \; .
\end{aligned} \tag{2.101}$$

After some algebraic manipulations, one finds that $\{\mathcal{L}^{(n)}(\mathbf{z})\}_{n\in\mathbb{N}}$ can be written in closed form as

$$\mathcal{L}^{(n)}(\mathbf{z}) = V^{n+1} + \frac{(-4)^n}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+2)}\big(\mathcal{L}(\mathbf{z})\big)^{n+1}\,{}_3F_2\left(1-n, \frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}-n, -n; -\frac{V}{\mathcal{L}(\mathbf{z})}\right) , \tag{2.102}$$

and the series (2.100) becomes

$$\mathcal{L}^V(\mathbf{z},\tau) = \frac{V}{1-\tau V} + \sum_{n\geq 0} L^{(n)}(\mathbf{z},\tau) \; , \tag{2.103}$$

with

$$L^{(n)}(\mathbf{z},\tau) = \frac{(-4)^n}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+2)}\,\tau^n\big(\mathcal{L}(\mathbf{z})\big)^{n+1}\,{}_3F_2\left(1-n, \frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}-n, -n; -\frac{V}{\mathcal{L}(\mathbf{z})}\right) . \tag{2.104}$$

The expressions of $\{L^{(n)}(\mathbf{z},\tau)\}_{n\in\mathbb{N}}$ are quite involved and it seems not to be possible to resum the series $\sum_{n\geq 0} L^{(n)}(\mathbf{z},\tau)$ appearing in (2.103). However, computing the partial sums

$S_m(\mathbf{z},\tau) = \sum_{n=0}^{m} L^{(n)}(\mathbf{z},\tau)$ for some values of $m \geq 0$

$$\begin{aligned}
S_0(\mathbf{z},\tau) &= \mathcal{L}(\mathbf{z}) \,, \\
S_1(\mathbf{z},\tau) &= \mathcal{L}(\mathbf{z}) - \tau\big(\mathcal{L}(\mathbf{z})\big)^2 \,, \\
S_2(\mathbf{z},\tau) &= \mathcal{L}(\mathbf{z}) - \tau(1-\tau V)\big(\mathcal{L}(\mathbf{z})\big)^2 + 2\tau^2\big(\mathcal{L}(\mathbf{z})\big)^3 \,, \\
S_3(\mathbf{z},\tau) &= \mathcal{L}(\mathbf{z}) - \tau(1-\tau V)\big(\mathcal{L}(\mathbf{z})\big)^2 + 2\tau^2(1-2\tau V)\big(\mathcal{L}(\mathbf{z})\big)^3 - 5\tau^3\big(\mathcal{L}(\mathbf{z})\big)^4 \,, \\
S_4(\mathbf{z},\tau) &= \mathcal{L}(\mathbf{z}) - \tau(1-\tau V)\big(\mathcal{L}(\mathbf{z})\big)^2 + \tau^2(1-\tau V)^2\big(\mathcal{L}(\mathbf{z})\big)^3 - 5\tau^3(1-3\tau V)\big(\mathcal{L}(\mathbf{z})\big)^4 \\
&+ 14\tau^4\big(\mathcal{L}(\mathbf{z})\big)^5 \,, \\
&\vdots \,,
\end{aligned} \tag{2.105}$$

one realises that $S_m(\mathbf{z},\tau)$ can be expressed in closed form as

$$S_m(\mathbf{z},\tau) = \mathcal{R}_m(\mathbf{z},\tau) + \sum_{n=0}^{m} \frac{(-4)^{n-1}}{\sqrt{\pi}} \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n+1)} \tilde{\tau}^{n-1}\big(\mathcal{L}(\mathbf{z})\big)^n \,, \tag{2.106}$$

where

$$\tilde{\tau} = \tau(1-\tau V) \,, \tag{2.107}$$

is a $V-$dependent redefinition of the deformation parameter and $\{\mathcal{R}_m(\mathbf{z},\tau)\}_{m\in\mathbb{N}}$ are such that $\lim_{m\to\infty}\mathcal{R}_m(\mathbf{z},\tau) = 0$. Using (2.106), one has

$$\begin{aligned}
\sum_{n\geq 0} L^{(n)}(\mathbf{z},\tau) &= \lim_{m\to\infty} S_m(\mathbf{z},\tau) = \sum_{n\geq 0} \frac{(-4)^{n-1}}{\sqrt{\pi}} \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n+1)} \tilde{\tau}^{n-1}\big(\mathcal{L}(\mathbf{z})\big)^n \\
&= \frac{1}{2\tilde{\tau}}\left(-1+\sqrt{1+4\tilde{\tau}\,\mathcal{L}(\mathbf{z})}\right) \,,
\end{aligned} \tag{2.108}$$

consequently (2.103) becomes

$$\boxed{\mathcal{L}^V(\mathbf{z},\tau) = \frac{V}{1-\tau V} + \frac{1}{2\tilde{\tau}}\left(-1+\sqrt{1+4\tilde{\tau}\mathcal{L}(\mathbf{z})}\right) \,.} \tag{2.109}$$

Comparing (2.90) and (2.109) we notice that the presence of the potential causes a redefinition of the deformation parameter ($\tau \to \tilde{\tau}$) and the introduction of an additional term $\frac{V}{1-\tau V}$.

It is then straightforward to generalise (2.109) to the case of an arbitrary metric $g_{\mu\nu}$. In $\mathbf{x}$ coordinates one has

$$\boxed{\mathcal{L}^V_g(\mathbf{x},\tau) = \sqrt{g}\,\frac{V}{1-\tau V} + \frac{1}{2\tilde{\tau}}\left(-\sqrt{g}+\sqrt{\det\left[g_{\mu\nu}+\tilde{\tau}\,\partial_\mu\phi\,\partial_\nu\phi\right]}\right) \,,} \tag{2.110}$$

which indeed fulfils (2.78).

2.3.3 $\sigma-$models

In this section we extend the results of sections 2.3.1 and 2.3.2 to the general case of $\sigma-$models. The starting point is the action

$$\mathcal{A}^\sigma[\underline{\phi}] = \int_{\mathbb{R}^2} \mathcal{L}^\sigma_{g,G}(\mathbf{x})\, dx^1 \wedge dx^2 \,, \tag{2.111}$$

with Lagrangian density

$$\mathcal{L}^{\sigma}_{g,G}(\mathbf{x}) = \sqrt{g}\left(\frac{1}{4}g^{\mu\nu}G^{ij}\partial_\mu\phi_i\,\partial_\nu\phi_j + V\right) \;, \quad (\mu,\nu = 1,2\;,\; i,j = 1,\dots N)\;, \tag{2.112}$$

where $\underline{\phi} = (\phi_1,\dots,\phi_N)$ is a set of independent scalar fields, G_{ij} is an $N\times N$ matrix which couples the scalar fields, and V is a potential term which again depends on the fields $\underline{\phi}$ only.

Putting together the results obtained in sections 2.3.1 and 2.3.2, it is natural to conjecture that the deformed version of (2.112) is obtained from (2.110) replacing the matrix $\partial_\mu\phi\,\partial_\nu\phi$ with

$$h_{\mu\nu} = G^{ij}\partial_\mu\phi_i\,\partial_\nu\phi_j\;, \tag{2.113}$$

thus obtaining

$$\boxed{\mathcal{L}^{\sigma}_{g,G}(\mathbf{x},\tau) = \sqrt{g}\,\frac{V}{1-\tau V} + \frac{1}{2\tilde{\tau}}\left(-\sqrt{g} + \sqrt{\det\left[g_{\mu\nu} + \tilde{\tau}h_{\mu\nu}\right]}\right)\;.} \tag{2.114}$$

Indeed, it is possible to check that (2.114) fulfils (2.78).

2.3.4 Hamiltonian description

Knowing the expression of the $T\bar{T}$−deformed Lagrangian density for the most general bosonic field theory, it is instructive to translate it in the Hamiltonian formalism. For simplicity, we restrict to the flat case $g_{\mu\nu} = \delta_{\mu\nu}$ and we set $G_{ij} = \delta_{ij}$. The Lagrangian densities (2.112) and (2.114) become, respectively

$$\mathcal{L}^{\sigma}(\mathbf{x}) \equiv \mathcal{L}^{\sigma}_{\delta,\delta}(\mathbf{x},\tau) = \frac{1}{4}\partial_\mu\underline{\phi}\cdot\partial^\mu\underline{\phi} + V\;, \tag{2.115}$$

and

$$\mathcal{L}^{\sigma}(\mathbf{x},\tau) \equiv \mathcal{L}^{\sigma}_{\delta,\delta}(\mathbf{x},\tau) = \frac{V}{1-\tau V} + \frac{1}{2\tilde{\tau}}\left(-1 + \mathcal{S}^{\sigma}(\mathbf{x},\tau)\right)\;, \tag{2.116}$$

where we defined

$$\mathcal{S}^{\sigma}(\mathbf{x},\tau) = \sqrt{1 + \tilde{\tau}\,\partial_\mu\underline{\phi}\cdot\partial^\mu\underline{\phi} + \frac{1}{2}\tilde{\tau}^2\,\epsilon_{\mu\rho}\,\epsilon_{\nu\lambda}\left(\partial^\mu\underline{\phi}\cdot\partial^\nu\underline{\phi}\right)\left(\partial^\rho\underline{\phi}\cdot\partial^\lambda\underline{\phi}\right)}\;, \tag{2.117}$$

and "$\cdot$" denotes the standard scalar product of $\mathbb{R}^N$, *i.e.* $\underline{v}\cdot\underline{w} = \sum_{i=1}^N v_i\,w_i\,,\,(\forall\,\underline{v},\underline{w}\in\mathbb{R}^N)$. The conjugated momenta $\underline{\pi} = (\pi^1,\dots,\pi^N)$ associated to the fields $\underline{\phi}$ are obtained through

$$\underline{\pi} = \mathbb{i}\,\frac{\partial\mathcal{L}^{\sigma}(\mathbf{x},\tau)}{\partial\underline{\dot{\phi}}}\;, \tag{2.118}$$

where again we adopt the shorthand notation $\underline{\dot{\phi}} = \partial_{x^2}\underline{\phi} = (\partial_{x^2}\phi_1,\dots,\partial_{x^2}\phi_N)$ and $\underline{\phi}' = \partial_{x^1}\underline{\phi} = (\partial_{x^1}\phi_1,\dots,\partial_{x^1}\phi_N)$. To compute the Hamiltonian density, we shall first express $\underline{\dot{\phi}}$ in terms of $\underline{\phi}'$ and $\underline{\pi}$ using (2.118). Inverting (2.118) one finds

$$\underline{\dot{\phi}} = -2\,\frac{\left(1+\tilde{\tau}\left(\underline{\phi}'\cdot\underline{\pi}\right)\right)\left(\underline{\pi}+\tilde{\tau}\,\underline{\phi}'\left(\underline{\phi}'\cdot\underline{\pi}\right)\right)}{\sqrt{1+4\tilde{\tau}\left(\frac{1}{4}|\underline{\phi}'|^2+|\underline{\pi}|^2\right)+4\tilde{\tau}^2\left(\underline{\phi}'\cdot\underline{\pi}\right)^2}}\;, \tag{2.119}$$

and the deformed Hamiltonian density is

$$\boxed{\mathcal{H}^{\sigma}(\mathbf{x},\tau) = \mathbb{i}\,\underline{\pi}\cdot\dot{\underline{\phi}} + \mathcal{L}^{\sigma}(\mathbf{x},\tau) = \frac{V}{1-\tau V} + \frac{1}{2\tilde{\tau}}\left(-1+\sqrt{1+4\tilde{\tau}\,\mathcal{H}^{\sigma}(\mathbf{x})+4\tilde{\tau}^{2}\left(\mathcal{P}^{\sigma}(\mathbf{x})\right)^{2}}\right)\ ,} \tag{2.120}$$

where

$$\mathcal{H}^{\sigma}(\mathbf{x}) = \frac{1}{4}|\underline{\phi}'|^{2} + |\underline{\pi}|^{2}\ , \quad \mathcal{P}^{\sigma}(\mathbf{x}) = -\underline{\phi}'\cdot\underline{\pi}\ , \tag{2.121}$$

are the Hamiltonian and momentum density of the original theory, defined according to (A.15).

Observation 12. The momentum density of the deformed theory coincides with the original one

$$\boxed{\mathcal{P}^{\sigma}(\mathbf{x},\tau) = -\frac{\mathbb{i}}{2\mathcal{S}^{\sigma}(\mathbf{x},\tau)}\,\underline{\phi}'\cdot\dot{\underline{\phi}} = -\underline{\phi}'\cdot\underline{\pi} = \mathcal{P}^{\sigma}(\mathbf{x})\ ,} \tag{2.122}$$

where in the second equality we used (2.119). It is interesting to notice that, the equivalence between the original and the deformed momentum density becomes manifest only when we perform the Legendre transformation.

From (2.120) we realise that the classical Hamiltonian density of a T$\bar{\text{T}}$−deformed σ−model has the same formal structure of the energy spectrum of a T$\bar{\text{T}}$−deformed CFT with bulk term (2.60). In particular, an expression analogous to (2.60) is formally obtained from (2.120) substituting the potential V with the constant term F_0 and replacing the Hamiltonian and momentum density with the corresponding integrated quantities averaged over the volume R

$$\big(\mathcal{H}^{\sigma}(\mathbf{x},\tau),\mathcal{P}^{\sigma}(\mathbf{x},\tau)\big) \longrightarrow \frac{1}{R}\big(E^{\sigma}(R,\tau),P^{\sigma}(R,\tau)\big)\ , \tag{2.123}$$

and analogously for the $\tau=0$ quantities, where

$$E^{\sigma}(R,\tau) = \int_{0}^{R}\mathcal{H}^{\sigma}(\mathbf{x},\tau)\,dx^{1}\ , \quad P^{\sigma}(R,\tau) = \int_{0}^{R}\mathcal{P}^{\sigma}(\mathbf{x},\tau)\,dx^{1}\ . \tag{2.124}$$

2.3.5 Yang-Mills theories

As a further example, we apply the T$\bar{\text{T}}$ deformation to a 2−dimensional Yang-Mills theory with generic gauge group G. The Yang-Mills action is

$$\mathcal{A}^{\text{YM}}[\mathcal{F}] = \int_{\mathbb{R}^{2}}\mathcal{L}^{\text{YM}}_{g}(\mathbf{x})\,dx^{1}\wedge dx^{2}\ , \tag{2.125}$$

with Lagrangian density

$$\mathcal{L}^{\text{YM}}_{g}(\mathbf{x}) = \frac{1}{4}\mathcal{F}^{a}_{\mu\nu}\,\mathcal{F}^{\mu\nu}_{a}\ , \quad \mathcal{F}^{\mu\nu}_{a} = g^{\mu\rho}\mathcal{F}_{\rho\sigma\,a}\,g^{\sigma\nu}\ , \tag{2.126}$$

where $\mathcal{F}^{a}_{\mu\nu} = \partial_{\mu}\mathcal{A}^{a}_{\nu} - \partial_{\nu}\mathcal{A}^{a}_{\mu} + f^{a}{}_{bc}\,\mathcal{A}^{b}_{\mu}\,\mathcal{A}^{b}_{\nu}$ are the components of the field-strength 2−form

$$\mathbf{F} = d\mathbf{A} = \mathcal{F}^{a}_{\mu\nu}\,\mathbf{t}_{a}\,dx^{\mu}\wedge dx^{\nu}\ , \quad \mathbf{A} = \mathcal{A}^{a}_{\mu}\,\mathbf{t}_{a}\,dx^{\mu}\ , \tag{2.127}$$

and $\{\mathbf{t}_a\}_{a=1}^{\dim(\mathfrak{g})}$ are the generators of the algebra $\mathfrak{g}$ of G which fulfils

$$[\mathbf{t}_a, \mathbf{t}_b] = \mathbb{i}\, f_{ab}{}^{c}\, \mathbf{t}_c \ , \tag{2.128}$$

where $\{f_{ab}{}^{c}\}$ are the structure constants.

The goal is to solve the flow equation (2.78) using $\mathcal{L}_g^{\text{YM}}(\mathbf{x})$ as $\tau = 0$ initial condition. For simplicity, let us start our analysis from the flat case, *i.e.* $g_{\mu\nu} = \delta_{\mu\nu}$, and denote $\mathcal{L}^{\text{YM}}(\mathbf{x}) = \mathcal{L}_\delta^{\text{YM}}(\mathbf{x})$.

First of all notice that, contrary to the cases discussed in the previous sections, here the stress-energy tensor is not uniquely defined due to the presence of an internal symmetry. In fact, from one hand the Noether procedure leads to the canonical stress-energy tensor

$$\boldsymbol{T}_{\text{N}}^{\mu\nu} = \frac{\partial \mathcal{L}^{\text{YM}}}{\partial(\partial_\mu \mathcal{A}_\rho^a)}\, \mathcal{A}_a^{\nu\rho} - \delta^{\mu\nu}\mathcal{L}^{\text{YM}} \ , \tag{2.129}$$

which is neither symmetric nor gauge invariant. From the other hand, using the Belinfante-Rosenfeld procedure, one can add to (2.129) a total derivative term to construct a new symmetric and gauge invariant object which can be shown to be equivalent to the Hilbert stress-energy tensor. In formulae

$$\boldsymbol{T}^{\mu\nu} \equiv \boldsymbol{T}_{\text{H}}^{\mu\nu} = \frac{\partial \mathcal{L}^{\text{YM}}}{\partial(\partial_\mu \mathcal{A}_\rho^a)}\, \mathcal{F}_a^{\nu\rho} - \delta^{\mu\nu}\mathcal{L}^{\text{YM}} = 2\frac{\partial \mathcal{L}^{\text{YM}}}{\partial \mathcal{F}_{\mu\rho}^a}\, \mathcal{F}_a^{\nu\rho} - \delta^{\mu\nu}\mathcal{L}^{\text{YM}} \ , \tag{2.130}$$

where in the last equality we used

$$\frac{\partial \mathcal{L}^{\text{YM}}}{\partial(\partial_\mu \mathcal{A}_\nu^a)} = \frac{\partial \mathcal{L}^{\text{YM}}}{\partial \mathcal{F}_{\rho\sigma}^b}\frac{\partial \mathcal{F}_{\rho\sigma}^b}{\partial(\partial_\mu \mathcal{A}_\nu^a)} = \frac{\partial \mathcal{L}^{\text{YM}}}{\partial \mathcal{F}_{\rho\sigma}^b}\left(\delta_\rho^\mu\, \delta_\sigma^\nu - \delta_\sigma^\mu\, \delta_\rho^\nu\right)\delta_a^b = 2\frac{\partial \mathcal{L}^{\text{YM}}}{\partial \mathcal{F}_{\mu\nu}^a} \ . \tag{2.131}$$

Observation 13. In principle, one may expect that (2.129) and (2.130) give rise to different kind of deformations at classical level and, a priori, there is no reason to choose one or the other. However, the factorisation property (2.32) implies

$$\boxed{\langle n|\det\left[\boldsymbol{T}_{\text{H}}^{\mu\nu}(\mathbf{x})\right]|n\rangle = \det\left[\langle n|\,\boldsymbol{T}_{\text{N}}^{\mu\nu}(\mathbf{x}) + \text{derivatives}\,|n\rangle\right] = \langle n|\det\left[\boldsymbol{T}_{\text{N}}^{\mu\nu}(\mathbf{x})\right]|n\rangle \ ,} \tag{2.132}$$

since the expectation value of total derivative terms vanishes. Therefore, (2.129) and (2.130) generate the same deformation at the quantum level. Equivalently, it should be possible to prove that the deformed classical Lagrangians corresponding to (2.129) and (2.130) differ only by total derivative terms, thus they are equivalent.

For simplicity, we will perform the computation using (2.130). Equation (2.78) becomes a PDE for the function $\mathcal{L}^{\text{YM}}(\mathbf{x},\tau)$ in the variables $(\tau, \{\mathcal{F}_{\mu\nu}^a\})$

$$\partial_\tau \mathcal{L}^{\text{YM}}(\mathbf{x},\tau) = \frac{1}{2}\epsilon_{\mu\rho}\,\epsilon_{\nu\sigma}\left(\frac{\partial \mathcal{L}^{\text{YM}}(\mathbf{x},\tau)}{\partial \mathcal{F}_{\mu\lambda}^a}\,\mathcal{F}_a^{\nu\lambda} - \delta^{\mu\nu}\mathcal{L}^{\text{YM}}(\mathbf{x},\tau)\right)\left(\frac{\partial \mathcal{L}^{\text{YM}}(\mathbf{x},\tau)}{\partial \mathcal{F}_{\rho\lambda}^a}\,\mathcal{F}_a^{\sigma\lambda} - \delta^{\rho\sigma}\mathcal{L}^{\text{YM}}(\mathbf{x},\tau)\right) . \tag{2.133}$$

Setting up a perturbative computation, one can write the solution to (2.133) in closed form as

$$\boxed{\mathcal{L}^{\mathrm{YM}}(\mathbf{x},\tau)=\frac{3}{4\tau}\left[-1+{}_3F_2\left(-\frac{1}{2},-\frac{1}{4},\frac{1}{4};\frac{1}{3},\frac{2}{3};\frac{256}{27}\tau\,\mathcal{L}^{\mathrm{YM}}(\mathbf{x})\right)\right]\,.} \tag{2.134}$$

The Lagrangian (2.134) seems quite involved, however, as we shall see, the Hamiltonian associated to it takes a very nice form. To perform the Legendre transform, we must first compute the conjugated momenta

$$\pi_a^{\mu}=\mathbb{i}\,\frac{\partial\mathcal{L}^{\mathrm{YM}}(\mathbf{x},\tau)}{\partial\dot{\mathcal{A}}_{\mu}^{a}}\,,\qquad \dot{\mathcal{A}}_{\mu}^{a}=\partial_2\mathcal{A}_{\mu}^{a}\,, \tag{2.135}$$

and express $\{\mathcal{F}_{\mu\nu}^{a}\}$ in terms of $\{\pi_a^{\mu}\}$. Actually, due to the antisymmetry of $\{\mathcal{F}_{\mu\nu}^{a}\}$ in $\mu\leftrightarrow\nu$, for each $a=1,\dots,\dim(\mathfrak{g})$ there is just one non-vanishing component $\mathcal{F}_{12}^{a}=-\mathcal{F}_{21}^{a}$, thus the only non-trivial conjugated momenta are

$$\pi_a^{1}=\mathbb{i}\,\mathcal{F}_a^{12}\,{}_3F_2\left(\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{4}{3},\frac{5}{3};\frac{128}{27}\tau\,\mathcal{F}_{12}^{b}\,\mathcal{F}_b^{12}\right)\,, \tag{2.136}$$

since $\pi_a^2=0$. Inverting the relation (2.136) perturbatively around $\tau=0$, one finds

$$\mathcal{F}_a^{12}=\frac{-\mathbb{i}\,\pi_a^{1}}{\left(1-\frac{1}{2}\tau\,\delta^{bc}\,\pi_b^1\pi_c^1\right)^2}\,, \tag{2.137}$$

and the deformed Hamiltonian density is then

$$\boxed{\mathcal{H}^{\mathrm{YM}}(\mathbf{x},\tau)=\mathbb{i}\,\pi_a^1\,\mathcal{F}_{12}^{a}+\mathcal{L}^{\mathrm{YM}}(\mathbf{x},\tau)=\frac{\mathcal{H}^{\mathrm{YM}}(\mathbf{x})}{1-\tau\,\mathcal{H}^{\mathrm{YM}}(\mathbf{x})}\,,} \tag{2.138}$$

where

$$\mathcal{H}^{\mathrm{YM}}(\mathbf{x})=\frac{1}{2}\,\delta^{bc}\,\pi_b^1\pi_c^1\,, \tag{2.139}$$

is the Yang-Mills Hamiltonian density. As for the momentum density, both the original and the deformed ones are identically zero, since $\boldsymbol{T}^{12}(\mathbf{x},\tau)=\boldsymbol{T}^{21}(\mathbf{x},\tau)=0$

$$\boxed{\mathcal{P}^{\mathrm{YM}}(\mathbf{x},\tau)=\mathcal{P}^{\mathrm{YM}}(\mathbf{x})=0\,.} \tag{2.140}$$

Observation 14. The Hamiltonian transforms under the TT̄ flow in the same way as a potential term depending on the field only (cf. sections 2.3.2 and 2.3.3). We interpret this fact as a consequence of the quasi-topological nature of Yang-Mills theory in 2−dimensions. The latter feature also explains the emergence in the Lagrangian density (2.134) of the same functional expression found in [50] in the context of the deformation of a 1−dimensional free particle mechanical system generated by the square of the stress-energy tensor, *i.e.* the square of the Hamiltonian. In fact, since the only non-vanishing components of $\{\mathcal{F}_{\mu\nu}^{a}\}$ are $\{\mathcal{F}_{12}^{a}\}$, the stress-energy tensor is diagonal with coincident eigenvalues, *i.e.* $\boldsymbol{T}^{11}(\mathbf{x},\tau)=\boldsymbol{T}^{22}(\mathbf{x},\tau)$, thus the TT̄ operator is equivalent to the Hamiltonian squared.

Chapter 3

The CDD factor analysis

In this chapter we review the main result of [1, 11], namely we show that, in the framework of massive IFTs, the $\mathrm{T\bar{T}}$ deformation arises as a modification of the exact $S-$matrix by a non-trivial CDD factor. Following [11], we carry on the explicit computation for the sine-Gordon model using the language of the NLIE, although the same conclusions could be obtained using the TBA. Before going into the details of the CDD factor modification, we first briefly review the sine-Gordon model and its associated NLIE. For completeness, we also derive the massless limit which will be useful in the second part of the thesis (see section 6.3).

3.1 The sine-Gordon model

The sine-Gordon model is a 2$-$dimensional relativistic field theory. In Minkowsky signature and using the conventions of [46], the Lagrangian density in cartesian coordinates $\mathbf{x}_{\mathrm{M}} = (x, t_x) \equiv (x, t)$ is

$$\mathcal{L}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}}) = \frac{1}{4}\left((\partial_x\phi)^2 - (\partial_t\phi)^2\right) + V^{\mathrm{sG}} , \quad V^{\mathrm{sG}} = \frac{4m^2}{\beta^2}\sin\left(\frac{\beta\phi}{2}\right)^2 = \frac{2m^2}{\beta^2}\left(1-\cos\left(\beta\phi\right)\right) , \tag{3.1}$$

Switching to light-cone coordinates $\tilde{\mathbf{x}}_{\mathrm{M}} = (x^+, x^-)$ according to (A.6), (3.1) becomes

$$\mathcal{L}^{\mathrm{sG}}(\tilde{\mathbf{x}}_{\mathrm{M}}) = \partial_+\phi\,\partial_-\phi + V^{\mathrm{sG}} . \tag{3.2}$$

where we denoted $\partial_\pm = \partial_{x^\pm}$. The Euler-Lagrange equation associated to (3.1) or, equivalently (3.2), is the famous sine-Gordon equation, *i.e.*

$$\partial_x^2\phi - \partial_t^2\phi = \frac{4m^2}{\beta}\sin\left(\beta\phi\right) , \tag{3.3}$$

in cartesian and

$$\partial_+\partial_-\phi = \frac{m^2}{\beta}\sin\left(\beta\phi\right) , \tag{3.4}$$

in complex coordinates, respectively. A remarkable feature of the sine-Gordon equation is the existence of soliton and multi-soliton solutions which, ultimately, reflects the integrability of the theory. A powerful tool to generate soliton solutions is provided by the so-called *auto-Bäcklund transform* which relates two solutions ϕ and φ to equation (3.4) through

$$\begin{cases} \partial_+\varphi = \partial_+\phi + \frac{2ma}{\beta}\sin\left(\frac{\beta}{2}(\varphi+\phi)\right) \\ \partial_-\varphi = -\partial_-\phi + \frac{2m}{\beta a}\sin\left(\frac{\beta}{2}(\varphi-\phi)\right) \end{cases} \quad , \quad (a \in \mathbb{R}) \; . \tag{3.5}$$

Expression (3.5) can be used to create a 1−kink solution moving with velocity v starting from the vacuum $\phi = 0$ as follows

$$\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}) = 4\arctan\left(e^{\frac{2m}{\beta}\frac{x-vt}{\sqrt{1-v^2}}}\right) \quad , \quad v = \frac{1-a^2}{1+a^2} \; , \tag{3.6}$$

and, in turn, the 2−kink solution moving with velocities v_1 and v_2 can be obtained from the 1−kink setting $\phi = \phi_{\text{1-kink}}$,

$$\phi_{\text{2-kink}}(\mathbf{x}_{\text{M}}) = 4\arctan\left(\frac{a_1+a_2}{a_2-a_1}\,\frac{e^{\frac{2m}{\beta}\frac{x-v_1t}{\sqrt{1-v_1^2}}} - e^{\frac{2m}{\beta}\frac{x-v_2t}{\sqrt{1-v_2^2}}}}{1+e^{\frac{2m}{\beta}\frac{x-v_1t}{\sqrt{1-v_1^2}}}\,e^{\frac{2m}{\beta}\frac{x-v_2t}{\sqrt{1-v_2^2}}}}\right) \quad , \quad v_i = \frac{1-a_i^2}{1+a_i^2} \; , \tag{3.7}$$

and so on. The auto-Bäcklund transform allows to compute all the multi-kink solutions and can be seen as a fingerprint of the integrability of the model.

Another important feature is that the sine-Gordon equation is equivalent to a *Zero Curvature Representation*

$$\partial_+ L_-^{\text{sG}} - \partial_- L_+^{\text{sG}} = \left[L_+^{\text{sG}}, L_-^{\text{sG}}\right] \; , \tag{3.8}$$

for the $\mathfrak{su}(2)$ connection $\Omega = L_+^{\text{sG}}\,dx^+ + L_-^{\text{sG}}\,dx^-$ where the components

$$\begin{aligned} L_+^{\text{sG}}(\tilde{\mathbf{x}}_{\text{M}}) &= \frac{\beta}{2}\partial_+\phi\,\mathbf{u}_3 + \mathbb{i}\,m\lambda\left[\cos\left(\frac{\beta\phi}{2}\right)\mathbf{u}_1 - \sin\left(\frac{\beta\phi}{2}\right)\mathbf{u}_2\right] \; , \\ L_-^{\text{sG}}(\tilde{\mathbf{x}}_{\text{M}}) &= -\frac{\beta}{2}\partial_-\phi\,\mathbf{u}_3 + \frac{\mathbb{i}\,m}{\lambda}\left[\cos\left(\frac{\beta\phi}{2}\right)\mathbf{u}_1 + \sin\left(\frac{\beta\phi}{2}\right)\mathbf{u}_2\right] \; , \end{aligned} \tag{3.9}$$

form the so-called *Lax pair* and $\lambda \in \mathbb{C}$ is the spectral parameter. In (3.9), $\{\mathbf{u}_i\}_{i=1}^3$ are the generators of $\mathfrak{su}(2)$

$$\mathbf{u}_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix} \; , \quad \mathbf{u}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; , \quad \mathbf{u}_3 = \frac{1}{2}\begin{pmatrix} \mathbb{i} & 0 \\ 0 & -\mathbb{i} \end{pmatrix} \; , \tag{3.10}$$

which fulfil the commutation relations

$$[\mathbf{u}_i, \mathbf{u}_j] = \epsilon_{ij}{}^k\mathbf{u}_k \; . \tag{3.11}$$

The existence of a Zero Curvature Representation is another evidence of the integrability of the sine-Gordon model and allows to derive the infinite tower of conserved charges of the integrable hierarchy.

3.2 The sine-Gordon NLIE

Integrability provides powerful tools to compute the finite-size spectrum of the sine-Gordon model such as the TBA and the NLIE. Since in this thesis we will use only the second approach, we start this section by introducing the NLIE of sine-Gordon. Following the notation of [51], we consider the Lagrangian density

$$\mathcal{L}^{\text{sG}}(\mathbf{x}_{\text{M}}) = \frac{1}{8\pi}\left((\partial_t\varphi)^2 - (\partial_x\varphi)^2\right) + 2\mu\cos(\sqrt{2}\beta\varphi) \ , \quad \left(\mu \propto (m)^{2-2\beta^2}\right) \ , \tag{3.12}$$

where m is the sine-Gordon *soliton mass* and β is the coupling constant. Confining the theory on an infinite cylinder of circumference R, and imposing periodic boundary conditions on the field, *i.e.* $\varphi(x+R,t) = \varphi(x,t)$, it emerges the so-called *quasi-momentum* or *vacuum parameter* $\alpha_0 \in [-1/2, 1/2]$ [52, 53]. Due to the periodicity of the potential in (3.12), the Hilbert space splits into orthogonal subspaces $\mathcal{H}_{\alpha_0}$ labelled by α_0, namely a shift of the field

$$\varphi \to \varphi + \frac{2\pi}{\sqrt{2}\beta} \ , \tag{3.13}$$

which leaves (3.12) invariant, corresponds to a rotation

$$|\Psi_{\alpha_0}\rangle \to e^{2\pi\mathbb{i}\,\alpha_0}\,|\Psi_{\alpha_0}\rangle \ , \quad (\,|\Psi_{\alpha_0}\rangle \in \mathcal{H}_{\alpha_0}) \ , \tag{3.14}$$

in the states of the Hilbert space.

The NLIE of the sine-Gordon model confined on an infinite cylinder of circumference R is [8, 51, 54–56]

$$f_\nu(\theta) = \nu(R, \alpha_0\,|\,\theta) \\ - \int_{\mathcal{C}_1} d\theta'\,\mathcal{K}(\theta-\theta')\,\log\left(1+e^{-f_\nu(\theta')}\right) + \int_{\mathcal{C}_2} d\theta'\,\mathcal{K}(\theta-\theta')\,\log\left(1+e^{f_\nu(\theta')}\right) \ , \tag{3.15}$$

where

1. $f_\nu(\theta)$ is the *counting function*, which depend on the *rapidity variable* θ. It represents the unknown function of the NLIE.
2. $\nu(R,\alpha_0\,|\,\theta)$ is the *driving term*, defined as

   $$\nu(R,\alpha_0\,|\,\theta) = 2\pi\mathbb{i}\,\alpha_0 - \mathbb{i}mR\sinh(\theta) \ . \tag{3.16}$$

   For future convenience, we explicitly report the parametric dependence of ν on R and α_0.
3. $\mathcal{K}(\theta)$ is the *kernel*, defined as the logarithmic derivative of the sine-Gordon $S-$matrix $S^{\text{sG}}(\theta)$

   $$\mathcal{K}(\theta) = \frac{1}{2\pi\mathbb{i}}\partial_\theta\log S^{\text{sG}}(\theta) \ , \tag{3.17}$$

   with

   $$\log S^{\text{sG}}(\theta) = -\mathbb{i}\int_{\mathbb{R}+}\frac{dp}{p}\sin(p\theta)\,\frac{\sinh\left(\pi p(\zeta-1)/2\right)}{\cosh\left(\pi p/2\right)\,\sinh\left(\pi p\,\zeta/2\right)} \ , \quad \zeta = \frac{\beta^2}{1-\beta^2} \ . \tag{3.18}$$

4. $\mathcal{C}_1$ and $\mathcal{C}_2$ are state-dependent integration contours in the θ complex plane. For the ground-state in an arbitrary subspace $\mathcal{H}_{\alpha_0}$, one may choose them to be straight lines slightly displaced from the real axis
$$\mathcal{C}_1 = \mathbb{R} + \mathbb{i}0^+ \;, \quad \mathcal{C}_2 = \mathbb{R} - \mathbb{i}0^+ \;. \tag{3.19}$$
Equations describing excited states have the same form [51, 53, 56, 57] but the integration contours $\mathcal{C}_1$ and $\mathcal{C}_2$ encircle a certain number of singularities $\{\theta_i\}$ such that $\left(1+e^{f(\theta_i)}\right) = 0$. In the following we will ignore the subtleties related to the choice of the integration contours, since the arguments we are going to present are not sensitive to them.

Observation 15. One may also consider twisted boundary conditions of the form
$$\varphi(x+R,t) = \varphi(x,t) + \frac{2\pi}{\sqrt{2}\beta} n \;, \quad (n \in \mathbb{Z}) \;, \tag{3.20}$$
which correspond, in the infinite volume limit, to field configurations with non-trivial topological charge
$$Q_x = \frac{\sqrt{2}\beta}{2\pi} \int_0^R \partial_x \varphi \, dx \;. \tag{3.21}$$
Energy levels in the twisted sectors are described by the same NLIE at specific values of α_0. Furthermore, α_0 can also be related to a background charge (cf. equation (3.33)). Therefore, (3.15) also describes minimal models of the Virasoro algebra, $\mathcal{M}_{p,q}$ perturbed by the operator Φ_{13} [58].

The eigenvalues $\{I_k^{(\pm)}\}_{k\in 2\mathbb{N}+1}$ of the quantum charges $\{\hat{I}_k^{(\pm)}\}_{k\in 2\mathbb{N}+1}$ of the sine-Gordon hierarchy, can be obtained from the counting function as follows
$$I_k^{(\pm)}(R,m) = \left(\frac{2\pi}{R}\right)^k \frac{b_k^{(\pm)}(r)}{C_k} \;, \quad r = mR \;, \quad (k \in 2\mathbb{N}+1) \;, \tag{3.22}$$
with
$$b_k^{(\pm)}(r) = -\int_{\mathcal{C}_1} \frac{d\theta}{2\pi\mathbb{i}} \left(\pm\frac{r}{2}e^{\pm\theta}\right)^k \log\left(1+e^{-f_\nu(\theta)}\right) + \int_{\mathcal{C}_2} \frac{d\theta}{2\pi\mathbb{i}} \left(\pm\frac{r}{2}e^{\pm\theta}\right)^k \log\left(1+e^{f_\nu(\theta)}\right) \;, \tag{3.23}$$
and
$$C_k = \frac{1}{2k}\left(\frac{4\pi}{\beta^2}\right)^{\frac{k+1}{2}} \frac{\Gamma\left(\frac{k}{2}(\zeta+1)\right)}{\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{k}{2}\zeta\right)} \left(\frac{\Gamma\left(1+\frac{\zeta}{2}\right)}{\Gamma\left(\frac{3+\zeta}{2}\right)}\right)^k \;. \tag{3.24}$$
From the previous definition, it follows that $\{I_k^{(\pm)}(R)\}_{k\in 2\mathbb{N}+1}$ fulfil the following reflection property
$$I_{-k}^{(\pm)}(R) = I_k^{(\mp)}(R) \;, \quad (k \in 2\mathbb{N}+1) \;, \tag{3.25}$$
which allows to extend the range of values of k from $2\mathbb{N}+1$ to $2\mathbb{Z}+1$. To encompass also the negative branch, we define a bold index $\mathbf{k}$ which takes values in $\mathbf{k} \in 2\mathbb{Z}+1$, and we denote $k = |\mathbf{k}|$, $k' = -|\mathbf{k}|$.

Finally, let us introduce the alternative set of conserved charges[1]

$$\mathcal{E}_{\mathbf{k}}(R) = I_{\mathbf{k}}^{(+)}(R) + I_{\mathbf{k}}^{(-)}(R)\,, \quad \mathcal{P}_{\mathbf{k}}(R) = I_{\mathbf{k}}^{(+)}(R) - I_{\mathbf{k}}^{(-)}(R)\,, \tag{3.26}$$

where $E(R) = \mathcal{E}_1(R)$ and $P(R) = \mathcal{P}_1(R)$ are the total energy and momentum of the state, respectively.

3.3 The CFT limit of the NLIE

The CFT limit of (3.15) results in a pair of decoupled NLIEs corresponding to the right- $(+)$ and the left- $(-)$ mover sectors. They are obtained from (3.15) by sending simultaneously $m \to 0$ and $\theta \to \pm\infty$ as

$$m = \hat{m}\epsilon\,, \quad \theta = \hat{\theta} \pm \log(\epsilon)\,, \quad \epsilon \to 0^+\,, \tag{3.27}$$

such that $\hat{m}e^{\pm\hat{\theta}}$ remains finite. The resulting equations are identical to (3.15)

$$\begin{aligned} f_{\nu^{(\pm)}}(\theta) &= \nu^{(\pm)}(R, \alpha_0^{(\pm)}\,|\,\theta) \\ &- \int_{\mathcal{C}_1} d\theta'\, \mathcal{K}(\theta-\theta')\, \log\left(1+e^{-f_{\nu^{(\pm)}}(\theta')}\right) + \int_{\mathcal{C}_2} d\theta'\, \mathcal{K}(\theta-\theta')\, \log\left(1+e^{f_{\nu^{(\pm)}}(\theta')}\right)\,, \end{aligned} \tag{3.28}$$

but for the driving term, in which $m\sinh(\theta)$ is replaced by $\frac{\hat{m}}{2}e^{\theta}$ and $-\frac{\hat{m}}{2}e^{-\theta}$, respectively

$$\nu^{(\pm)}(R, \alpha_0^{(\pm)}\,|\,\theta) = 2\pi\mathbb{i}\,\alpha_0^{(\pm)} \mp \mathbb{i}\frac{\hat{m}R}{2}\,e^{\pm\theta}\,, \tag{3.29}$$

and $\hat{m}$ sets the energy scale. In (3.29), we also take into account for two independent vacuum parameters $\alpha_0^{(\pm)}$ in the two chiral sectors.

Observation 16. The NLIEs (3.28) are, in principle, suitable for the description of twisted boundary conditions and more general states in the $c = 1$ CFT, compared to the set strictly emerging from the massless limit of the sine-Gordon model. For example, they can accommodate states with odd fermionic numbers of the massless Thirring model, which require anti-periodic boundary conditions.

Observation 17. The integration contours $\mathcal{C}_1$, $\mathcal{C}_2$ are still state-dependent. In particular, they should be deformed away from the initial ground-state configuration $\mathcal{C}_1 = \mathbb{R} + \mathbb{i}0^+$, $\mathcal{C}_2 = \mathbb{R} - \mathbb{i}0^+$ when the parameters $\pm\alpha_0^{(\pm)}$ are analytically extended to large negative values.

Similarly to the massive case, we define the eigenvalues of the quantum operators associated to the conserved currents of the hierarchy as

$$I_k^{(\pm)}(R) = \left(\frac{2\pi}{R}\right)^k \frac{\hat{b}_k^{(\pm)}}{C_k} = \left(\frac{1}{R}\right)^k 2\pi a_k^{(\pm)}\,, \quad \hat{r} = \hat{m}R\,, \quad (k \in 2\mathbb{N}+1)\,, \tag{3.30}$$

[1] The explicit dependence on some parameters, *e.g.* the mass m, will be sometime omitted not to weigh down the notation.

with

$$\hat{b}_k^{(\pm)} = -\int_{\mathcal{C}_1} \frac{d\theta}{2\pi \mathbb{i}} \left(\pm\frac{\hat{r}}{2}e^{\pm\theta}\right)^k \log\left(1+e^{-f_{\nu(\pm)}(\theta)}\right) + \int_{\mathcal{C}_2} \frac{d\theta}{2\pi \mathbb{i}} \left(\pm\frac{\hat{r}}{2}e^{\pm\theta}\right)^k \log\left(1+e^{f_{\nu(\pm)}(\theta)}\right) \;, \quad (3.31)$$

where the $a_k^{(\pm)}$ are state-dependent constants, *i.e.* they do not depend on R and $\hat{m}$. Using again the reflection property (3.25), we can extend the discussion to $\mathbf{k} \in 2\mathbb{Z}+1$. Some of the state-dependent coefficients $I_k^{(\pm)}(2\pi)$ can be found in [52]. In particular, the energy and momentum of a generic state are

$$\begin{aligned} E(R) &= I_1^{(+)}(R) + I_1^{(-)}(R) = \frac{2\pi}{R}\left(n^{(+)} - \frac{c_0^{(+)}}{24}\right) + \frac{2\pi}{R}\left(n^{(-)} - \frac{c_0^{(-)}}{24}\right) \;, \\ P(R) &= I_1^{(+)}(R) - I_1^{(-)}(R) = \frac{2\pi}{R}\left(h^{(+)} - h^{(-)}\right) \;, \end{aligned} \quad (3.32)$$

with effective central charges

$$c_0^{(\pm)} = 1 - 24\beta^2\left(\alpha_0^{(\pm)}\right)^2 = 1 - 24 h_0^{(\pm)} \;, \quad (3.33)$$

where $h_0^{(\pm)}$ are the (anti)-holomorphic highest weights and $h^{(\pm)} = h_0^{(\pm)} + n^{(\pm)}$, $(n^{(\pm)} \in \mathbb{N})$.

For the current purposes, it is convenient to think about the massless limit of the sine-Gordon model as a relevant perturbation of the compactified free boson with Lagrangian given by (3.12) with $\mu = 0$ (see [51] for more details) and twisted boundary conditions (3.20), where $\mathbf{r} = (\sqrt{2}\beta)^{-1}$ is the compactification radius. Then, the highest weights are now labelled by a pair of integers $(n, \tilde{n})$, where $\frac{\tilde{n}\beta^2}{\sqrt{4\pi}}$ is the quantised charge associated to the total conjugated momentum

$$Q_t = \frac{\sqrt{2}}{2\pi\beta}\int_0^R \partial_t\varphi\, dx = \frac{2\sqrt{2}}{\beta}\int_0^R \Pi\, dx \;, \quad (3.34)$$

and n is the winding number corresponding to the topological charge

$$Q_x = \frac{\sqrt{2}\beta}{2\pi}\int_0^R \partial_x\varphi\, dx \;. \quad (3.35)$$

The combinations

$$Q_0^{(\pm)} = \pi\left(Q_x \pm Q_t\right) = \pi\left(n \pm \tilde{n}\beta^2\right) \;, \quad (3.36)$$

are the two different charges, associated to the $U(1)_L \times U(1)_R$ symmetry of the $c = 1$ compactified boson. Notice that $Q_0^{(\pm)}$ differ from the standard Kac-Moody $U(1)_L \times U(1)_R$ charges by a multiplicative factor β which breaks explicitly the $\beta \leftrightarrow 1/\beta$ symmetry. We adopted this unconventional definition for the topological charges since, as a reminiscence of the sine-Gordon model, it emerges more naturally from the current setup. The anagolous of the Bloch wave states in (3.14) are now created by the action on the CFT vacuum state of the vertex operators

$$\mathcal{V}_{(n,\tilde{n})}(\mathbf{z}) = \exp\left(\frac{\sqrt{2}}{2\pi\beta}Q_0^{(+)}\,\phi(z) + \frac{\sqrt{2}}{2\pi\beta}Q_0^{(-)}\,\bar{\phi}(\bar{z})\right) \;, \quad \varphi(\mathbf{z}) = \phi(z) + \bar{\phi}(\bar{z}) \;, \quad (3.37)$$

with left and right conformal dimensions given by

$$h_0^{(\pm)} = \frac{1}{4\pi^2\beta^2}\left(Q_0^{(\pm)}\right)^2 = \frac{1}{4}\left(\frac{n}{\beta} \pm \tilde{n}\beta\right)^2 . \tag{3.38}$$

Considering (3.33), in section 6.4 we will make the following identification:

$$Q_0^{(\pm)} = 2\pi\alpha_0^{(\pm)}\beta^2 . \tag{3.39}$$

However, relation (3.39) is valid only at formal level since $\alpha_0^{(\pm)}$ are continuous parameters which can also account, for example, for twisted boundary conditions while, at fixed β, the charges $Q_0^{(\pm)}$ can only assume the discrete set of values given in (3.36). Using (3.39) in (3.37) we find

$$\mathcal{V}_{(n,\tilde{n})}(\mathbf{z}) = \exp\left(\sqrt{2}\beta\alpha_0^{(+)}\,\phi(z) + \sqrt{2}\beta\alpha_0^{(-)}\,\bar{\phi}(\bar{z})\right) , \tag{3.40}$$

which, for $\tilde{n} = 0$ and under the field-shift (3.20), display the same quasi-periodicity properties of the finite volume sine-Gordon Bloch states.

3.4 The CDD factor and the T$\bar{\text{T}}$ deformation

Following [1, 11], in this section we prove that, in massive IFTs, the T$\bar{\text{T}}$ deformation acts as a twist of the exact $S-$matrix by a non-trivial CDD factor. Even though the argument is expected to apply in general, we carry out the explicit computation in the sine-Gordon model using the language of the NLIE [11].

Let us modify the sine-Gordon $S-$matrix by

$$\boxed{S^{\text{sG}}(\theta-\theta') \to S^{\text{sG}}(\theta-\theta')\,f(\theta-\theta') \;, \quad f(\theta) = e^{\mathbb{i}\delta(\theta)} \;,} \tag{3.41}$$

with

$$\boxed{\delta(\theta) = \tau\, m^2 \sinh(\theta) \;.} \tag{3.42}$$

The function $f(\theta)$ fulfils

$$f(\theta)f(-\theta) = 1 \;, \quad f(\mathbb{i}\pi+\theta)f(\mathbb{i}\pi-\theta) = 1 \;, \tag{3.43}$$

thus it represents a CDD factor. It is easy to show that the transformation (3.41) affects the kernel $\mathcal{K}(\theta-\theta')$ as

$$\mathcal{K}(\theta-\theta') \to \mathcal{K}(\theta-\theta') + \frac{1}{2\pi}\partial_\theta\delta(\theta-\theta') = \mathcal{K}(\theta-\theta') + \tau\frac{m^2}{2\pi}\cosh(\theta-\theta') \;. \tag{3.44}$$

which, in turn, causes a modification of the driving term in (3.15)

$$\nu = \nu(\mathcal{R}_0, \alpha_0\,|\,\theta-\theta_0) \;, \tag{3.45}$$

where $\mathcal{R}_0$ and θ_0 are defined through

$$\mathcal{R}_0 \cosh(\theta_0) = R + \tau E(R,\tau) \, , \quad \mathcal{R}_0 \sinh(\theta_0) = \tau P(R) \, . \tag{3.46}$$

and $E(R,\tau) = I_1^{(+)}(R,\tau) - I_1^{(-)}(R,\tau)$ denotes the energy of the deformed theory.[1] Formula (3.45) tells that the solutions of the deformed NLIE are obtained from the original ones simply by a redefinition of the length R and by a rapidity shift.

Using the deformed counting function, we move on to derive the evolution equations for the conserved charges. From the definitions (3.22)-(3.24), it follows immediately that the deformed level-$\mathbf{k}$ charges are related to the original ones through

$$I_{\mathbf{k}}^{(\pm)}(R,\tau) = e^{\pm \mathbf{k}\theta_0} I_{\mathbf{k}}^{(\pm)}(\mathcal{R}_0) \, , \tag{3.47}$$

which, in terms of the combinations (3.26), can be recast into a Lorentz boost with rapidity $\mathbf{k}\theta_0$

$$\begin{pmatrix} \mathcal{E}_{\mathbf{k}}(R,\tau) \\ \mathcal{P}_{\mathbf{k}}(R) \end{pmatrix} = \begin{pmatrix} \cosh(\mathbf{k}\theta_0) & \sinh(\mathbf{k}\theta_0) \\ \sinh(\mathbf{k}\theta_0) & \cosh(\mathbf{k}\theta_0) \end{pmatrix} \begin{pmatrix} \mathcal{E}_{\mathbf{k}}(\mathcal{R}_0) \\ \mathcal{P}_{\mathbf{k}}(\mathcal{R}_0) \end{pmatrix} . \tag{3.48}$$

The conservation of the norm of the vector $(\mathcal{E}_{\mathbf{k}}, \mathcal{P}_{\mathbf{k}})$ leads to

$$\boxed{\mathcal{E}_{\mathbf{k}}^2(R,\tau) - \mathcal{P}_{\mathbf{k}}^2(R,\tau) = \mathcal{E}_{\mathbf{k}}^2(\mathcal{R}_0) - \mathcal{P}_{\mathbf{k}}^2(\mathcal{R}_0) \, .} \tag{3.49}$$

Observation 18. Using (3.48) with $\mathbf{k} = 1$ in (3.46) one can derive the following expressions for $\cosh(\theta_0)$ and $\sinh(\theta_0)$

$$\cosh(\theta_0) = \frac{R + \tau E(R,\tau)}{\mathcal{R}_0} = \frac{\mathcal{R}_0 - \tau E(\mathcal{R}_0)}{R} \, , \quad \sinh(\theta_0) = \frac{\tau P(R)}{\mathcal{R}_0} = \frac{\tau P(\mathcal{R}_0)}{R} \, , \tag{3.50}$$

which imply

$$\boxed{(\mathcal{R}_0)^2 = (R + \tau E(R,\tau))^2 - (\tau P(R))^2 \, , \quad R^2 = (\mathcal{R}_0 - \tau E(\mathcal{R}_0))^2 - (\tau P(\mathcal{R}_0))^2 \, .} \tag{3.51}$$

To find the evolution equations of the level$-\mathbf{k}$ energy and momentum $\mathcal{E}_{\mathbf{k}}(R,\tau)$ and $\mathcal{P}_{\mathbf{k}}(R,\tau)$, we differentiate both sides of (3.47) w.r.t. τ keeping $\mathcal{R}_0$ fixed

$$\partial_\tau I_{\mathbf{k}}^{(\pm)}(R,\tau) + R' \partial_R I_{\mathbf{k}}^{(\pm)}(R,\tau) = \pm \mathbf{k} \theta_0' \, I_{\mathbf{k}}^{(\pm)}(R,\tau) \, , \tag{3.52}$$

with $R' \equiv \partial_\tau R$ and $\theta_0' \equiv \partial_\tau \theta_0$. To determine R' and θ_0' we rewrite (3.46) as

$$\begin{cases} \mathcal{R}_0 = R\, e^{-\theta_0} + 2\tau I_1^{(+)}(\mathcal{R}_0) \\ \mathcal{R}_0 = R\, e^{\theta_0} + 2\tau I_1^{(-)}(\mathcal{R}_0) \end{cases} , \tag{3.53}$$

and, differentiating both equations in (3.53) w.r.t. τ we obtain the following set of equations for R' and θ_0'

$$\begin{cases} 0 = R'\, e^{-\theta_0} - R\theta_0'\, e^{-\theta_0} + 2 I_1^{(+)}(\mathcal{R}_0) \\ 0 = R'\, e^{\theta_0} + R\theta_0'\, e^{\theta_0} + 2 I_1^{(-)}(\mathcal{R}_0) \end{cases} , \tag{3.54}$$

[1] The deformed charges $\{I_{\mathbf{k}}^{(\pm)}(R,\tau)\}_{\mathbf{k}\in 2\mathbb{Z}+1}$ are computed from (3.22)-(3.24) using the driving term (3.45).

whose solution is

$$R'|_{\mathcal{R}_0=\text{const.}} = -E(R,\tau)\ , \quad \theta_0'|_{\mathcal{R}_0=\text{const.}} = \frac{P(R)}{R}\ . \tag{3.55}$$

Plugging (3.55) into (3.52) we find the evolution equations for the level$-\mathbf{k}$ energy and momentum

$$\boxed{\begin{cases} \partial_\tau \mathcal{E}_{\mathbf{k}}(R,\tau) - E(R,\tau)\,\partial_R \mathcal{E}_{\mathbf{k}}(R,\tau) = \frac{\mathbf{k}}{R}\,P(R)\,\mathcal{P}_{\mathbf{k}}(R,\tau) \\ \partial_\tau \mathcal{P}_{\mathbf{k}}(R,\tau) - E(R,\tau)\,\partial_R \mathcal{P}_{\mathbf{k}}(R,\tau) = \frac{\mathbf{k}}{R}\,P(R)\,\mathcal{E}_{\mathbf{k}}(R,\tau) \end{cases} .} \tag{3.56}$$

In particular, setting $\mathbf{k} = 1$ in (3.56) and using (2.46) – which is equivalent to the momentum quantisation – one has

$$\boxed{\partial_\tau E(R,\tau) = \frac{1}{2}\partial_R\left(E^2(R,\tau) - P^2(R)\right)\ ,} \tag{3.57}$$

which coincide with the Burgers equation (2.47). Therefore, we proved that the modification (3.41) of the $S-$matrix leads to the definition of the T$\bar{\text{T}}$ deformation of the energy levels.

Chapter 4

The $\mathrm{T\bar{T}}$ deformation and integrability

As already discussed in the previous chapter, the $\mathrm{T\bar{T}}$ deformation preserves the integrability of the original theory, both at classical and quantum level. The aim of this chapter is to investigate the integrable structure of a $\mathrm{T\bar{T}}-$deformed IFT, using the sine-Gordon (sG) model as an example. We start to look for the existence of soliton solutions in the $\mathrm{T\bar{T}}-$deformed model by explicitly solving the deformed EoM [46]. Then, we construct the *Zero Curvature Representation* for the $\mathrm{T\bar{T}}-$deformed theory [46], which definitively proves the integrability of the model. Finally, exploiting the unique relation between integrable equations and surfaces embedded in $N-$dimensional ambient space (with $N \geq 3$), we infer that the $\mathrm{T\bar{T}}$ deformation acts, at the classical level, as a space-time coordinate transformation depending non-trivially on the field configuration itself [24]. This geometric interpretation gives a powerful tool to generate solutions to the $\mathrm{T\bar{T}}-$deformed EoMs [24] and to construct the integrable structure of the deformed theory [43].

4.1 Simple soliton solution in $\mathrm{T\bar{T}}-$deformed sine-Gordon

From section 2.3.2, we know that the $\mathrm{T\bar{T}}$ deformation of (3.1) is (2.110), which we rewrite as

$$\mathcal{L}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}},\tau) = \frac{V^{\mathrm{sG}}}{1-\tau V^{\mathrm{sG}}} + \frac{1}{2\tilde{\tau}}\left(-1+\mathcal{S}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}},\tau)\right) \ , \tag{4.1}$$

with

$$\mathcal{S}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}},\tau) = \sqrt{1+\tau(1-\tau V^{\mathrm{sG}})\left(\phi_x^2-\phi_t^2\right)} \ , \tag{4.2}$$

where hereafter we adopt the shorthand notation $\phi_x = \partial_x\phi$ and $\phi_t = \partial_t\phi$ to denote space and time derivatives, respectively. The Euler-Lagrange equation associated to (4.1) is

$$\partial_x\left(\frac{\partial\mathcal{L}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}},\tau)}{\partial\phi_x}\right) - \partial_t\left(\frac{\partial\mathcal{L}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}},\tau)}{\partial\phi_t}\right) = \frac{\partial\mathcal{L}^{\mathrm{sG}}(\mathbf{x}_{\mathrm{M}},\tau)}{\partial\phi} \ , \tag{4.3}$$

which explicitly gives the following EoM

$$\begin{aligned}\left(1-\tau V^{\mathrm{sG}}\right)^2(\phi_{xx}-\phi_{tt}) - \tau\left(1-\tau V^{\mathrm{sG}}\right)^3\left(\phi_{xx}\,\phi_t^2 - 2\phi_{xt}\,\phi_x\,\phi_t + \phi_{tt}\,\phi_x^2\right)&\\ = \frac{1}{2}\tau(V^{\mathrm{sG}})'\left(1-\tau V^{\mathrm{sG}}\right)\left(3+2\mathcal{S}^{\mathrm{sG}}\right)\left(\phi_x^2-\phi_t^2\right) + \left(1+\mathcal{S}^{\mathrm{sG}}\right)(V^{\mathrm{sG}})' \ ,&\end{aligned} \tag{4.4}$$

where $(V^{\text{sG}})' = \frac{dV^{\text{sG}}}{d\phi}$.

The goal of this section is to find a class of soliton solutions to (4.4). We will proceed by parametrising the field ϕ using three generic functions F, X and T as follows

$$F(\phi) = X(x) + T(t) \ . \tag{4.5}$$

Then, all the derivatives of ϕ can be expressed in terms of the derivatives of F, X, T as follows

$$\begin{gathered}\phi_x = \frac{X_x}{F'} \ , \quad \phi_t = \frac{T_t}{F'} \ , \\ \phi_{xx} = \frac{X_{xx}}{F'} - X_x^2 \frac{F''}{F'^3} \ , \quad \phi_{tt} = \frac{T_{tt}}{F'} - T_t^2 \frac{F''}{F'^3} \ , \quad \phi_{xt} = -X_x T_t \frac{F''}{F'^3} \ ,\end{gathered} \tag{4.6}$$

where $F^{(k)} = \frac{dF}{d\phi}$ and $F' = F^{(1)}$, $F'' = F^{(2)}$. Using the relations (4.6), the EoM (4.4) becomes

$$\begin{aligned}\left(1-\tau V^{\text{sG}}\right)^2 F'^2 \left(X_{xx} - T_{tt}\right) - \tau \left(1-\tau V^{\text{sG}}\right)^3 \left(X_{xx} T_t^2 + T_{tt} X_x^2\right) = \left(1-\tau V^{\text{sG}}\right)^2 F'' \left(X_x^2 - T_t^2\right) \\ + \frac{1}{2}\tau (V^{\text{sG}})' \left(1-\tau V^{\text{sG}}\right) \left(3 + 2\mathcal{S}^{\text{sG}}\right) F' \left(X_x^2 - T_t^2\right) + \left(1+\mathcal{S}^{\text{sG}}\right) (V^{\text{sG}})' F'^3 \ ,\end{aligned} \tag{4.7}$$

and (4.2) reads

$$\mathcal{S}^{\text{sG}} = \sqrt{1 + \tau \frac{1-\tau V^{\text{sG}}}{F'^2} \left(X_x^2 - T_t^2\right)} \ . \tag{4.8}$$

From (4.8), we extract the combination $X_x^2 - T_t^2$

$$X_x^2 - T_t^2 = \frac{(\mathcal{S}^{\text{sG}})^2 - 1}{\tau\left(1-\tau V^{\text{sG}}\right)} F'^2 \ , \tag{4.9}$$

then, we compute the second order derivatives X_{xx} and T_{tt} using the chain rule,[1]

$$\begin{aligned}X_{xx} &= -T_{tt} \\ &= \frac{F'\left[2\mathcal{S}^{\text{sG}}(\mathcal{S}^{\text{sG}})'\left(1-\tau V^{\text{sG}}\right) + \tau\left((\mathcal{S}^{\text{sG}})^2 - 1\right)(V^{\text{sG}})'\right] + 2F''\left((\mathcal{S}^{\text{sG}})^2 - 1\right)\left(1-\tau V^{\text{sG}}\right)}{2\tau\left(1-\tau V^{\text{sG}}\right)^2} F' \ ,\end{aligned} \tag{4.10}$$

where $(\mathcal{S}^{\text{sG}})' = \frac{\partial \mathcal{S}^{\text{sG}}}{\partial \phi}$. Equation (4.10) implies $X_{xx} = -T_{tt} = c_0$, with $c_0 \in \mathbb{R}$ arbitrary constant which we set to zero, *i.e.* $c_0 = 0$. In this way one has

$$\begin{aligned}X_x &= 2\alpha_1 \quad , \quad & T_t &= 2\alpha_2 \\ X &= 2\alpha_1 x - 2k_x \quad , \quad & T_t &= 2\alpha_2 t - 2k_t \ ,\end{aligned} \tag{4.11}$$

with $\alpha_1, \alpha_2, k_x, k_t \in \mathbb{R}$ integration constants. Using (4.9), equations (4.7) and (4.10) become, respectively

$$0 = 2\left((\mathcal{S}^{\text{sG}})^2 - 1\right)\left(1-\tau V^{\text{sG}}\right) F'' + \tau (V^{\text{sG}})' F' \left(\mathcal{S}^{\text{sG}} + 1\right)^2 \left(2\mathcal{S}^{\text{sG}} - 1\right) \ , \tag{4.12}$$

$$0 = 2\left((\mathcal{S}^{\text{sG}})^2 - 1\right)\left(1-\tau V^{\text{sG}}\right) F'' + \left[2\mathcal{S}^{\text{sG}}(\mathcal{S}^{\text{sG}})'\left(1-\tau V^{\text{sG}}\right) + \tau\left((\mathcal{S}^{\text{sG}})^2 - 1\right)(V^{\text{sG}})'\right] F' \ , \tag{4.13}$$

[1] This part relies fundamentally on the fact that the variables are separate.

which can be combined to give

$$(\mathcal{S}^{\text{sG}})'\left(1-\tau V^{\text{sG}}\right)=\tau\mathcal{S}^{\text{sG}}\left(\mathcal{S}^{\text{sG}}+1\right)(V^{\text{sG}})' \quad\longrightarrow\quad \mathcal{S}^{\text{sG}}=\frac{1-c}{c-\tau V^{\text{sG}}}\,, \tag{4.14}$$

where $c\in\mathbb{R}$ is an arbitrary integration constant. Plugging expression (4.14) for $\mathcal{S}^{\text{sG}}$ into (4.12), or equivalently (4.13), we obtain the following equation

$$2\left(c-\tau V^{\text{sG}}\right)\left(2c-1-\tau V^{\text{sG}}\right)F''+\tau\left(3c-2-\tau V^{\text{sG}}\right)(V^{\text{sG}})'F'=0\,, \tag{4.15}$$

whose solution is

$$F'(\phi)=\beta\tilde{k}\,\frac{c-\tau V^{\text{sG}}}{\sqrt{1-2c+\tau V^{\text{sG}}}}\,,\qquad(\tilde{k}\in\mathbb{R})\,. \tag{4.16}$$

Integrating further (4.16) one gets

$$F(\phi)=2k_\phi+\beta\tilde{k}\,\frac{(1+4\tau\kappa)\,\mathbf{F}\left(\frac{\beta\phi}{2}\middle|-\frac{m^2}{\beta^2\kappa}\right)-8\tau\kappa\,\mathbf{E}\left(\frac{\beta\phi}{2}\middle|-\frac{m^2}{\beta^2\kappa}\right)}{2\sqrt{\tau\kappa}}\,, \tag{4.17}$$

where $k_\phi\in\mathbb{R}$ is an integration constant, κ is related to c through $c=\frac{1}{2}-2\tau\kappa$, and $\mathbf{F}(z|m)$, $\mathbf{E}(z|m)$ are the incomplete elliptic integrals of the first and second kind, respectively.

Plugging (4.11) and (4.16) into (4.8) using (4.14), one gets the following equation

$$\left(\frac{1-c}{c-\tau V^{\text{sG}}}\right)^2=1+4\tau\left(1-\tau V^{\text{sG}}\right)\left(\alpha_1^2-\alpha_2^2\right)\frac{1-2c+\tau V^{\text{sG}}}{\tilde{k}^2\left(c-\tau V^{\text{sG}}\right)^2}\,, \tag{4.18}$$

which allows to fix $\tilde{k}$ as

$$\tilde{k}=\pm\frac{2\sqrt{\tau}}{\beta}\sqrt{\alpha_1^2-\alpha_2^2}\,. \tag{4.19}$$

In conclusion, we found a class of moving soliton solutions $\phi(\mathbf{x}_{\text{M}},\tau)$ which fulfil the implicit relation

$$\boxed{\frac{1}{\sqrt{\kappa}}\left[(1+4\tau\kappa)\,\mathbf{F}\left(\frac{\beta\phi(\mathbf{x}_{\text{M}},\tau)}{2}\middle|-\frac{m^2}{\beta^2\kappa}\right)-8\tau\kappa\,\mathbf{E}\left(\frac{\beta\phi(\mathbf{x}_{\text{M}},\tau)}{2}\middle|-\frac{m^2}{\beta^2\kappa}\right)\right]=\pm2\,\frac{\alpha_1x+\alpha_2t-k}{\sqrt{\alpha_1^2-\alpha_2^2}}\,.} \tag{4.20}$$

where k is a constant defined as $k=k_x+k_t+k_\phi$. They correspond to the $T\bar{T}$ deformation of a particular family of elliptic solutions to the sine-Gordon equation [59, 60]

$$\phi(\mathbf{x}_{\text{M}})=\pm\frac{2}{\beta}\,\mathbf{am}\left(2\sqrt{\kappa}\,\frac{\alpha_1x+\alpha_2t-k}{\sqrt{\alpha_1^2-\alpha_2^2}}\middle|-\frac{m^2}{\beta^2\kappa}\right)\,, \tag{4.21}$$

where $\mathbf{am}\,(x|k)$ is the amplitude of Jacobi elliptic function. Expressions (4.21) correspond to "*staircase*" type solutions (see Figure 4.1a), in which the parameter κ is related to the amplitude of each step.

Probably the most physically interesting solution belonging to (4.20) is the $T\bar{T}$ deformation of the sine-Gordon 1−kink (3.6), which is recovered with an appropriate scaling of the parameters and it reads

$$\boxed{8\tau\frac{m^2}{\beta^2}\cos\left(\frac{\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}},\tau)}{2}\right)+\log\left(\tan\left(\frac{\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}},\tau)}{4}\right)\right)=\frac{2m}{\beta}\,\frac{\alpha_1x+\alpha_2t-k}{\sqrt{\alpha_1^2-\alpha_2^2}}\,.} \tag{4.22}$$

In Figure 4.3 we represented (4.22) for different values of τ, setting $m = \beta = 1$, $\alpha_1 = 1$, $\alpha_2 = \frac{3}{5}$ and $k = 0$. Notice that for negative values of τ (Figure 4.3a) the solution stretches w.r.t the original one (Figure 4.3b), while for positive values of τ (Figures 4.3c and 4.3d) it bends and becomes a multi-valued function. The transition between the single- and the multi-valued regimes (Figure 4.3c) corresponds to a shock wave singularity for the solution and it occurs at the critical value $\tau = \tau^\star$. To estimate $\tau^\star$, we consider the norm of the gradient of $\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau)$ and we look for the values of τ which make it divergent. Setting $m = \beta = 1$, one has

$$|\nabla_{\mathbf{x}}\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau)| = \sqrt{(\partial_x\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau))^2 - (\partial_t\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau))^2} = \pm\frac{4\sin\left(\frac{\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}},\tau)}{2}\right)}{1 - 4\tau\left[1 - \cos\left(\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau)\right)\right]} \; , \tag{4.23}$$

which is divergent if and only if the field configuration $\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau)$ fulfils

$$1 - 4\tau\left[1 - \cos\left(\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau)\right)\right] = 0 \quad \longrightarrow \quad \phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau) = \arccos\left(\frac{4\tau - 1}{4\tau}\right) + 2\pi n \; , \quad (n \in \mathbb{Z}) \; . \tag{4.24}$$

The fact that the solution $\phi_{\text{1-kink}}(\mathbf{x}_{\text{M}}, \tau)$ lies in the strip $[2\pi n, 2\pi(1+n)]$, $(n \in \mathbb{Z})$, imposes the following constraint

$$0 \leq \arccos\left(\frac{4\tau - 1}{4\tau}\right) \leq 2\pi \quad \longrightarrow \quad \tau \geq \frac{1}{8} \; , \tag{4.25}$$

from which it descends that the critical value is $\tau^\star = \frac{1}{8}$.

As for the general solution (4.20), from figure 4.1b we see that once the $\mathrm{T\bar{T}}$ deformation is turned on, it displays a deformed shape similar to that observed for the $1-$kink solution, with a shock-wave singularities at $\tau^\star \simeq 1/8$. In section 4.7, we will explicitly compute the critical value $\tau^\star$ from a completely different perspective.

Observation 19. Switching to light-cone coordinates $\tilde{\mathbf{x}}_{\text{M}} = (x^+, x^-)$ according to (A.6) and setting $m = \beta = 1$, we notice that (4.22) fulfils

$$\begin{cases} \partial_+\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau) = \dfrac{2a\sin\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}},\tau)}{2}\right)}{1 - 4\tau + 4\tau\cos\left(\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau)\right)} \\ \partial_-\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau) = \dfrac{\frac{2}{a}\sin\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}},\tau)}{2}\right)}{1 - 4\tau + 4\tau\cos\left(\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau)\right)} \end{cases} \; , \quad a = \sqrt{\frac{1 - v}{1 + v}} \; , \tag{4.26}$$

which have the form of a first-step Bäcklund transformation from the vacuum solution. Unfortunately, equations (4.26) do not contain much information and the complete form of the Bäcklund transformation is expected to be very complicated.

4.2 $\mathrm{T\bar{T}}-$deformed sine-Gordon Lax pair

Let us now switch to Euclidean signature and use complex coordinates $\mathbf{z} = (z, \bar{z})$ defined through (A.6). The Lagrangian density of $\mathrm{T\bar{T}}$ deformed sine-Gordon is (2.109), which we rewrite as

$$\mathcal{L}^{\text{sG}}(\mathbf{z}, \tau) = \frac{V}{1 - \tau V} + \frac{1}{2\tilde{\tau}}\left(-1 + \mathcal{S}^{\text{sG}}(\mathbf{z}, \tau)\right) \; , \tag{4.27}$$

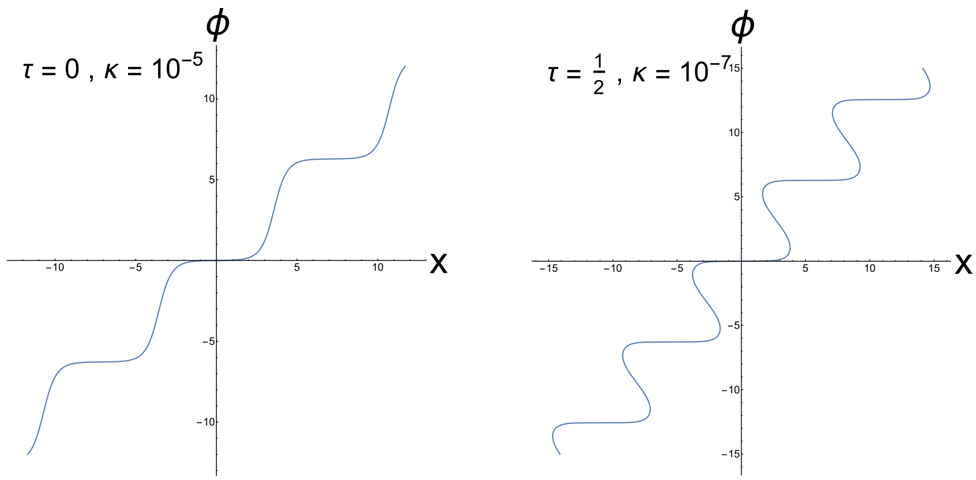


Figure 4.1: The general solution (4.20) in the stationary limit ($\alpha_1 = 1\,,\, \alpha_2 = 0\,,\, k = 0$) for the undeformed (a) and the deformed (b) theory, for small values of κ.

with

$$\mathcal{S}^{\rm sG}(\mathbf{z},\tau) = \sqrt{1 + 4\tilde{\tau}\,\partial_z\phi\,\partial_{\bar{z}}\phi}\ , \quad \tilde{\tau} = \tau(1 - \tau\, V^{\rm sG})\ . \tag{4.28}$$

The Euler-Lagrange equation associated to (4.27) is

$$\partial_z\left(\frac{\partial \mathcal{L}^{\rm sG}(\mathbf{z},\tau)}{\partial(\partial_z\phi)}\right) + \partial_{\bar{z}}\left(\frac{\partial \mathcal{L}^{\rm sG}(\mathbf{z},\tau)}{\partial(\partial_{\bar{z}}\phi)}\right) = \frac{\partial \mathcal{L}^{\rm sG}(\mathbf{z},\tau)}{\partial\phi}\ , \tag{4.29}$$

and, using

$$\frac{\partial \mathcal{S}^{\rm sG}(\mathbf{z},\tau)}{\partial(\partial_z\phi)} = \frac{4\tilde{\tau}\,\partial_{\bar{z}}\phi}{2\mathcal{S}^{\rm sG}(\mathbf{z},\tau)}\ , \quad \frac{\partial \mathcal{S}^{\rm sG}(\mathbf{z},\tau)}{\partial(\partial_{\bar{z}}\phi)} = \frac{4\tilde{\tau}\,\partial_z\phi}{2\mathcal{S}^{\rm sG}(\mathbf{z},\tau)}\ , \quad \frac{\partial \mathcal{S}^{\rm sG}(\mathbf{z},\tau)}{\partial\phi} = \frac{\tau^2\,(V^{\rm sG})'}{\tilde{\tau}}\frac{1 - (\mathcal{S}^{\rm sG}(\mathbf{z},\tau))^2}{2\mathcal{S}^{\rm sG}(\mathbf{z},\tau)}\ , \tag{4.30}$$

with $(V^{\rm sG})' = \frac{dV^{\rm sG}}{d\phi}$, it can be recast into the following compact form

$$\partial_z\left(\frac{\partial_{\bar{z}}\phi}{\mathcal{S}^{\rm sG}(\mathbf{z},\tau)}\right) + \partial_{\bar{z}}\left(\frac{\partial_z\phi}{\mathcal{S}^{\rm sG}(\mathbf{z},\tau)}\right) = \frac{(V^{\rm sG})'}{4\mathcal{S}^{\rm sG}(\mathbf{z},\tau)}\left(\frac{1 + \mathcal{S}^{\rm sG}(\mathbf{z},\tau)}{1 - \tau V^{\rm sG}}\right)^2\ . \tag{4.31}$$

Starting from (4.31), we look for a pair of $\mathfrak{su}(2)$ matrices

$$L^{\rm sG}(\mathbf{z},\tau) = \begin{pmatrix} -a(\mathbf{z},\tau) & b(\mathbf{z},\tau) \\ c(\mathbf{z},\tau) & a(\mathbf{z},\tau) \end{pmatrix}\ , \quad \bar{L}^{\rm sG}(\mathbf{z},\tau) = \begin{pmatrix} \bar{a}(\mathbf{z},\tau) & \bar{b}(\mathbf{z},\tau) \\ \bar{c}(\mathbf{z},\tau) & -\bar{a}(\mathbf{z},\tau) \end{pmatrix}\ , \tag{4.32}$$

such that the zero-curvature condition

$$\partial_z\bar{L}^{\rm sG}(\mathbf{z},\tau) - \partial_{\bar{z}}L^{\rm sG}(\mathbf{z},\tau) = \left[L^{\rm sG}(\mathbf{z},\tau), \bar{L}^{\rm sG}(\mathbf{z},\tau)\right]\ , \tag{4.33}$$

is satisfied if and only if ϕ is a solution to (4.31). In terms of the Lax pair's components, (4.33) is equivalent to the following three equations

$$\partial_z \bar{a} + \partial_{\bar{z}} a = b\bar{c} - c\bar{b} \,, \tag{4.34a}$$

$$\partial_{\bar{z}} b - \partial_z \bar{b} = 2a\bar{b} + 2\bar{a}b \,, \tag{4.34b}$$

$$\partial_z \bar{c} - \partial_{\bar{z}} c = 2a\bar{c} + 2\bar{a}c \,. \tag{4.34c}$$

where we dropped the explicit dependence of the functions a, b, $\bar{a}$ and $\bar{b}$. We choose (rather arbitrarily) the first equation (4.34a) to match manifestly with the EoMs (4.31). A reasonable parametrisation is

$$a = \gamma \, \frac{\partial_z \phi}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \,, \quad \bar{a} = \gamma \, \frac{\partial_{\bar{z}} \phi}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \,, \quad b\bar{c} - c\bar{b} = \gamma \, \frac{(V^{\text{sG}})'}{8\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \left(\frac{1+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}{1-\tau V^{\text{sG}}} \right)^2 \,, \tag{4.35}$$

with $\gamma \in \mathbb{C}$ an arbitrary constant to be fixed, from which the equations (4.34) become

$$b\bar{c} - c\bar{b} = \gamma \, \frac{(V^{\text{sG}})'}{8\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \left(\frac{1+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}{1-\tau V^{\text{sG}}} \right)^2 \,, \tag{4.36a}$$

$$\partial_{\bar{z}} b - \partial_z \bar{b} = \gamma \, \frac{\partial_z \phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \, \bar{b} + \gamma \frac{\partial_{\bar{z}} \phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \, b \,, \tag{4.36b}$$

$$\partial_z \bar{c} - \partial_{\bar{z}} c = \gamma \, \frac{\partial_z \phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \, \bar{c} + \gamma \frac{\partial_{\bar{z}} \phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \, c \,. \tag{4.36c}$$

In order to determine the functions b, c, $\bar{b}$ and $\bar{c}$ which solve (4.36), one can perform a perturbative expansion around $\tau = 0$ trying to recognize some pattern in the terms. After some computations one is lead to the following ansatz:

$$b = -m \left[\mu e^{-\mathbb{i}\frac{\beta\phi}{2}} B_+ + \tilde{\mu} \, e^{\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_z \phi)^2 \, B_- \right] \,, \quad c = -m \left[\frac{1}{\tilde{\mu}} \, e^{\mathbb{i}\frac{\beta\phi}{2}} B_+ + \frac{1}{\mu} \, e^{-\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_z \phi)^2 \, B_- \right] \,, \tag{4.37a}$$

$$\bar{b} = -m \left[\tilde{\mu} \, e^{\mathbb{i}\frac{\beta\phi}{2}} B_+ + \mu \, e^{-\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_{\bar{z}} \phi)^2 \, B_- \right] \,, \quad \bar{c} = -m \left[\frac{1}{\mu} \, e^{-\mathbb{i}\frac{\beta\phi}{2}} B_+ + \frac{1}{\tilde{\mu}} \, e^{\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_{\bar{z}} \phi)^2 \, B_- \right] \,, \tag{4.37b}$$

where $\gamma = -\frac{\mathbb{i}\beta}{2}$, $B_\pm$ are two unknown functions to be fixed and $\mu, \tilde{\mu} \in \mathbb{C}$ are arbitrary complex numbers, which can be regarded as two independent spectral parameters. Plugging the ansatz (4.37) into (4.36), the functions $B_\pm$ are fixed as

$$B_+ = \frac{\tau \left(1+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)\right)^2}{8\tilde{\tau}\, \mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \,, \quad B_- = \frac{\tau}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \,, \tag{4.38}$$

and the Lax pair are

$$L^{\text{sG}}(\mathbf{z},\tau) = \begin{pmatrix} \mathbb{i}\frac{\partial_z \phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} & -m\mu \, e^{-\mathbb{i}\frac{\beta\phi}{2}} B_+ - m\tilde{\mu} \, e^{\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_z \phi)^2 \, B_- \\ -\frac{m}{\tilde{\mu}} \, e^{\mathbb{i}\frac{\beta\phi}{2}} B_+ - \frac{m}{\mu} \, e^{-\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_z \phi)^2 \, B_- & -\mathbb{i}\frac{\partial_z \phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \end{pmatrix} \,,$$

$$\bar{L}^{\text{sG}}(\mathbf{z},\tau) = \begin{pmatrix} -\mathbb{i}\frac{\partial_{\bar{z}} \phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} & -m\tilde{\mu} \, e^{\mathbb{i}\frac{\beta\phi}{2}} B_+ - m\mu \, e^{-\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_{\bar{z}} \phi)^2 \, B_- \\ -\frac{m}{\mu} \, e^{-\mathbb{i}\frac{\beta\phi}{2}} B_+ - \frac{m}{\tilde{\mu}} \, e^{\mathbb{i}\frac{\beta\phi}{2}} \, (\partial_{\bar{z}} \phi)^2 \, B_- & \mathbb{i}\frac{\partial_{\bar{z}} \phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \end{pmatrix} \,. \tag{4.39}$$

Notice that there exist a global $SL(2,\mathbb{C})$ transformation acting on $(L^{\text{sG}}, \bar{L}^{\text{sG}})$ as

$$\hat{L}^{\text{sG}}(\mathbf{z},\tau) = M^{-1} L^{\text{sG}}(\mathbf{z},\tau) M \ , \quad \hat{\bar{L}}^{\text{sG}}(\mathbf{z},\tau) = M^{-1} \bar{L}^{\text{sG}}(\mathbf{z},\tau) M \ , \tag{4.40}$$

with

$$M = \begin{pmatrix} \sqrt{\tilde{\mu}\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\tilde{\mu}\lambda}} \end{pmatrix} \equiv \begin{pmatrix} (\tilde{\mu}\mu)^{\frac{1}{4}} & 0 \\ 0 & (\tilde{\mu}\mu)^{-\frac{1}{4}} \end{pmatrix} \in SL(2,\mathbb{C}) \ , \quad \lambda = \sqrt{\frac{\mu}{\tilde{\mu}}} \ , \tag{4.41}$$

such that $(\hat{L}^{\text{sG}}, \hat{\bar{L}}^{\text{sG}})$ is a Lax pair representation equivalent to $(L^{\text{sG}}, \bar{L}^{\text{sG}})$ and depending on a single parameter λ. This implies that the two spectral parameters μ and $\tilde{\mu}$ are not independent.

In conclusion, we found that a one-parameter Lax pair representation for the T$\bar{\text{T}}-$deformed sine-Gordon model is given by

$$\begin{aligned}
L^{\text{sG}}(\mathbf{z},\tau) &= \begin{pmatrix} \mathbb{i}\frac{\partial_z\phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} & -m\lambda\, e^{-\mathbb{i}\frac{\beta\phi}{2}} B_+ - \frac{m}{\lambda}\, e^{\mathbb{i}\frac{\beta\phi}{2}} \left(\partial_z\phi\right)^2 B_- \\ -m\lambda\, e^{\mathbb{i}\frac{\beta\phi}{2}} B_+ - \frac{m}{\lambda}\, e^{-\mathbb{i}\frac{\beta\phi}{2}} \left(\partial_z\phi\right)^2 B_- & -\mathbb{i}\frac{\partial_z\phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \end{pmatrix} \\
&= \frac{\beta}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \partial_z\phi\, \mathbf{u}_3 + 2\mathbb{i}\, m \left[F_+ \cos\frac{\beta\phi}{2}\, \mathbf{u}_1 - F_- \sin\frac{\beta\phi}{2}\, \mathbf{u}_2 \right] \ , \\
\bar{L}^{\text{sG}}(\mathbf{z},\tau) &= \begin{pmatrix} -\mathbb{i}\frac{\partial_{\bar{z}}\phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} & -\frac{m}{\lambda}\, e^{\mathbb{i}\frac{\beta\phi}{2}} B_+ - m\lambda\, e^{-\mathbb{i}\frac{\beta\phi}{2}} \left(\partial_{\bar{z}}\phi\right)^2 B_- \\ -\frac{m}{\lambda}\, e^{-\mathbb{i}\frac{\beta\phi}{2}} B_+ - m\lambda\, e^{\mathbb{i}\frac{\beta\phi}{2}} \left(\partial_{\bar{z}}\phi\right)^2 B_- & \mathbb{i}\frac{\partial_{\bar{z}}\phi}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \end{pmatrix} \\
&= -\frac{\beta}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)} \partial_{\bar{z}}\phi\, \mathbf{u}_3 + 2\mathbb{i}\, m \left[\bar{F}_+ \cos\frac{\beta\phi}{2}\, \mathbf{u}_1 + \bar{F}_- \sin\frac{\beta\phi}{2}\, \mathbf{u}_2 \right] \ ,
\end{aligned} \tag{4.42}$$

where $\{\mathbf{u}_i\}_{i=1}^3$ are the generators of the $\mathfrak{su}(2)$ algebra defined in (3.10) and

$$\begin{aligned}
F_+ &= \lambda\, B_+ + \frac{1}{\lambda}\, (\partial_z\phi)^2 B_- \ , \quad F_- = \lambda\, B_+ - \frac{1}{\lambda}\, (\partial_z\phi)^2 B_- \ , \\
\bar{F}_+ &= \frac{1}{\lambda}\, B_+ + \lambda\, (\partial_{\bar{z}}\phi)^2 B_- \ , \quad \bar{F}_- = \frac{1}{\lambda}\, B_+ - \lambda\, (\partial_{\bar{z}}\phi)^2 B_- \ .
\end{aligned} \tag{4.43}$$

As a consistency check one can easily verify that, in the $\tau \to 0$ limit, expression (4.42) correctly reproduces the Lax pair of the sine-Gordon model (3.9).
The existence of a Lax pair representation for the T$\bar{\text{T}}-$deformed sine-Gordon theory explicitly proves that the sine-Gordon integrable structure is preserved along the T$\bar{\text{T}}$ flow.

Observation 20. From (4.42), one can obtain the Lax pair of the T$\bar{\text{T}}$-deformed sinh-Gordon (shG) model, by a simple field redefinition $\phi^{\text{shG}} = -\mathbb{i}\phi^{\text{sG}}$.

Observation 21. Taking the massless limit, *i.e.* $m \to 0$, of the sine-Gordon Lagrangian one gets the Lax pair of the T$\bar{\text{T}}-$deformed massless scalar field

$$L(\mathbf{z},\tau) \ = \ \begin{pmatrix} \mathbb{i}\frac{\partial_z\phi}{4\mathcal{S}(\mathbf{z},\tau)} & 0 \\ 0 & -\mathbb{i}\frac{\partial_z\phi}{4\mathcal{S}(\mathbf{z},\tau)} \end{pmatrix} = \frac{\partial_z\phi}{2\mathcal{S}(\mathbf{z},\tau)}\, \mathbf{u}_3 \ , \tag{4.44}$$

$$\bar{L}(\mathbf{z},\tau) \ = \ \begin{pmatrix} -\mathbb{i}\frac{\partial_{\bar{z}}\phi}{4\mathcal{S}(\mathbf{z},\tau)} & 0 \\ 0 & \mathbb{i}\frac{\partial_{\bar{z}}\phi}{4\mathcal{S}(\mathbf{z},\tau)} \end{pmatrix} = -\frac{\partial_{\bar{z}}\phi}{2\mathcal{S}(\mathbf{z},\tau)}\, \mathbf{u}_3 \ , \tag{4.45}$$

with

$$\mathcal{S}(\mathbf{z},\tau) = \sqrt{1 + 4\tau\, \partial_z\phi\, \partial_{\bar{z}}\phi} \; . \tag{4.46}$$

4.3 Classical integrable equations and embedded surfaces

It is a well known fact that integrable equations in 2 dimensions are related to surfaces embedded in an $N-$dimensional ambient space. In this respect, the two oldest examples are the sine-Gordon and Liouville equations, whose discovery dates back to the 19th century [61, 62]. They appear as the Gauss-Mainardi-Codazzi (GMC) system of equations (C.13) for pseudo-spherical and minimal surfaces embedded in $\mathbb{R}^3$, respectively. As proved by Bonnet [63], any surface embedded in $\mathbb{R}^3$ is uniquely determined (up to its position in the ambient space) by two rank-2 symmetric tensors: the metric tensor $g_{\mu\nu}$ (C.4) and the second fundamental tensor $d_{\mu\nu}$ (C.5). Their intuitive role is to measure, respectively, the length of an infinitesimal curve and the displacement of its endpoint from the tangent plane at the starting point. These two objects can be used to study the motion of a frame anchored to the surface, which results in a system of linear differential equations, known as *Gauss-Weingarten* equations (C.8, C.9). The GMC system appears then as the consistency condition for this linear system, effectively constraining the "moduli space" consisting of the tensors $g_{\mu\nu}$ and $d_{\mu\nu}$.

The search for a general correspondence originated in the works of Lund, Regge, Pohlmeyer and Getmanov [64–66] and was subsequently formalised by Sym [67–71] who showed that any integrable system whose associated linear problem is based on a semi-simple Lie algebra $\mathfrak{g}$ can be put in the form of a GMC system for a surface embedded in a N-dimensional ambient space,[1] with $N = \dim(\mathfrak{g})$. In this section, we will shortly review Sym's results for the general setup and then we will focus on the case of the sine-Gordon model. We will use the following conventions

$$\mathbf{z} = (z^1, z^2) = (z, \bar{z}) \; , \quad \partial_\mu f(\mathbf{z}) \equiv \frac{\partial}{\partial z^\mu} f(\mathbf{z}) \; , \quad (\forall f \,:\, \mathbb{C} \to \mathbb{R} \; , \quad \mu = 1, 2) \; .$$

Let us consider a generic $2-$dimensional system of non-linear partial differential equations for a set of real fields $\{\phi_i(\mathbf{z})\}$. We assume that this system admits a *Zero Curvature Representation* (ZCR) for a pair of functions (L_1, L_2) taking values in a d-dimensional representation of a semi-simple Lie algebra[2] $\mathfrak{g}$

$$\partial_2 L_1 - \partial_1 L_2 + [L_1, L_2] = 0 \; , \tag{4.47}$$

where the functions (L_1, L_2) depend on $\mathbf{z}$ through the fields $\{\phi_i(\mathbf{z})\}$ and their derivatives w.r.t. $\mathbf{z}$

$$L_\mu \equiv L_\mu(\mathbf{z}|\lambda) \equiv L_\mu \left(\{\phi_i(\mathbf{z})\}, \{\partial_\nu \phi_i(\mathbf{z})\}, \ldots |\lambda\right) \; , \quad (\mu = 1, 2) \; , \tag{4.48}$$

and they additionally depend on a real spectral parameter λ. The ZCR can be interpreted as the compatibility condition for a system of first-order linear partial differential equations involving an

[1] An interesting additional result of Sym concerns the existence of the same kind of connection for spin systems and σ-models.

[2] Here we abuse notations by denoting with $\mathfrak{g}$ both the algebra and its d-dimensional representation. The same applies for the associated Lie Group G.

auxiliary $d \times d$ matrix-valued function $\Phi \equiv \Phi(\mathbf{z}|\lambda)$

$$\partial_\mu \Phi = L_\mu \Phi \ , \quad (\mu = 1, 2) \ , \tag{4.49}$$

commonly known as *associated linear problem*. Assuming $\Phi(\mathbf{z}_0|\lambda) \in G$ as initial condition, G being the Lie group associated to $\mathfrak{g}$, equation (4.49) allows, in principle, to recover a single-valued function $\Phi(\mathbf{z}|\lambda) \in G$, $\forall \mathbf{z} \in \mathbb{R}^2$. This function can then be used to construct the following object

$$r(\mathbf{z}|\lambda) = \Phi^{-1}(\mathbf{z}|\lambda) \frac{\partial}{\partial \lambda} \Phi(\mathbf{z}|\lambda) \ , \tag{4.50}$$

which is interpreted as the coordinate description of a $\lambda-$family of surfaces embedded into the N-dimensional affine space $\mathfrak{g}$. Moreover, equipping $\mathfrak{g}$ with a non-degenerate scalar product, *i.e.* the Killing form of the semi-simple Lie algebra, we can transform $\mathfrak{g}$ into an $N-$dimensional flat space. In other words, we can find a basis $\{\mathbf{e}^i\}_{i=1}^N$ of $\mathfrak{g}$ orthonormal w.r.t. the Killing form and then extract the coefficients $\{r_i\}_{i=1}^N$ from the identity

$$r(\mathbf{z}|\lambda) = \sum_{i=1}^N r_i \, \mathbf{e}^i = \Phi^{-1}(\mathbf{z}|\lambda) \frac{\partial}{\partial \lambda} \Phi(\mathbf{z}|\lambda) \ . \tag{4.51}$$

The row vector $\mathbf{r} = (r_1, r_2, \dots, r_N)^{\mathrm{T}}$ represents the position vector of a family of surfaces embedded in an N-dimensional flat space,[1] parametrised by λ. These are called *solitonic surfaces* and satisfy the following properties:

1. their GMC system reduces to the ZCR (4.47). This means that any integrable system whose EoMs can be represented as a ZCR depending on a spectral parameter λ, can be associated to a particular class of surfaces;

2. they are invariant w.r.t. $\lambda-$independent gauge transformation of the pair (L_1, L_2). This fact provides a way to prove the equivalence of distinct soliton systems up to gauge transformations and independent coordinate redefinitions, see [70];

3. their metric tensor (induced by the flat space $\mathfrak{g}$) is explicitly computed from the pair (L_1, L_2) as

$$g_{\mu\nu} = \mathrm{Tr}\left[\mathbf{Ad}\left(\frac{\partial L_\mu}{\partial \lambda} \right) \mathbf{Ad}\left(\frac{\partial L_\nu}{\partial \lambda} \right) \right] \ , \tag{4.52}$$

   where $\mathbf{Ad}$ denotes the adjoint representation of the algebra $\mathfrak{g}$. Consequently, any intrinsic property of the soliton surface is determined uniquely by the ZCR.

4.4 Embedded surfaces and the T$\bar{\text{T}}$ deformation

Let us now consider the specific case of the sine-Gordon equation (3.4). The functions (L_1, L_2) for this model takes values in a 2$-$dimensional representation of the Lie algebra $\mathfrak{su}(2)$ and corresponds to the

[1] The signature of this space depends on the real form chosen for the algebra; for example $\mathfrak{sl}\,(2) \simeq \mathfrak{so}\,(2,1)$ give rise to surfaces in Minkowski space $\mathbb{R}^{2,1}$.

Lax pair (3.9)

$$L_1^{\text{sG}}(\mathbf{z}) = L^{\text{sG}}(\mathbf{z}) = \frac{\beta}{2}\,\partial_z\phi\;\mathbf{u}^3 + \mathbb{i}\,m\lambda\left[\cos\left(\frac{\beta\phi}{2}\right)\mathbf{u}^1 - \sin\left(\frac{\beta\phi}{2}\right)\mathbf{u}^2\right] , \tag{4.53}$$

$$L_2^{\text{sG}}(\mathbf{z}) = \bar{L}^{\text{sG}}(\mathbf{z}) = -\frac{\beta}{2}\,\partial_{\bar{z}}\phi\;\mathbf{u}^3 + \frac{\mathbb{i}\,m}{\lambda}\left[\cos\left(\frac{\beta\phi}{2}\right)\mathbf{u}^1 + \sin\left(\frac{\beta\phi}{2}\right)\mathbf{u}^2\right] . \tag{4.54}$$

Since $\dim\left(\mathfrak{su}\left(2\right)\right) = 3$, the solitonic surfaces corresponding to the sine-Gordon model are embedded in the Euclidean plane $\mathbb{R}^3$ ($\mathfrak{su}(2)$ is compact). As mentioned in section 4.3, Bonnet theorem [63] tells us that any surface in $\mathbb{R}^3$ is completely specified (modulo its position) by its first and second fundamental quadratic forms which are (see appendix C.1)

$$\text{I}^{\text{sG}}(\mathbf{z}) = g_{\mu\nu}^{\text{sG}}(\mathbf{z})\,dz^\mu dz^\nu = 2m^2\left[dz^2 - \frac{2}{\lambda^2}\cos\left(\beta\phi\right)dz\,d\bar{z} + \frac{1}{\lambda^4}d\bar{z}^2\right] , \tag{4.55}$$

$$\text{II}^{\text{sG}}(\mathbf{z}) = d_{\mu\nu}^{\text{sG}}(\mathbf{z})\,dz^\mu dz^\nu = 2m^2\frac{\sqrt{2}}{\lambda}\sin\left(\beta\phi\right)dz\,d\bar{z}\;. \tag{4.56}$$

From (4.55) and (4.56) one can then extract the Gaussian and the mean curvatures using (C.7):

$$K^{\text{sG}} = \text{Det}\left[d_{\mu\rho}^{\text{sG}}\,\left(g^{\text{sG}}\right)^{\rho\nu}\right] = -\frac{\lambda^2}{2}\;,\qquad H^{\text{sG}} = d_{\mu\nu}^{\text{sG}}\,\left(g^{\text{sG}}\right)^{\nu\mu} = \frac{\lambda}{\sqrt{2}}\cot\left(\beta\phi\right)\;, \tag{4.57}$$

with $g_{\mu\nu}^{\text{sG}}\left(g^{\text{sG}}\right)^{\nu\rho} = \delta_\mu^\rho$. Since K^{sG} is constant negative, the solitonic surfaces we are dealing with are pseudo-spherical, with the spectral parameter λ playing the role of Gaussian curvature. In figure 4.2, we reported examples of pseudo-spherical surfaces corresponding to 1−kink, 2−kink and stationary breather solutions of the sine-Gordon model.

Let us now apply the Sym formalism sketched above to the T$\bar{\text{T}}$-deformed sine-Gordon model [46] and derive the geometric properties of the associated surfaces. The starting point is the Lax pair of the T$\bar{\text{T}}$-deformed theory (4.42)

$$\begin{aligned} L_1^{\text{sG}}(\mathbf{z},\tau) &= L^{\text{sG}}(\mathbf{z},\tau) = \frac{\beta}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\,\partial_z\phi\,\mathbf{u}^3 + 2\mathbb{i}\,m\left[F_+\cos\left(\frac{\beta\phi}{2}\right)\mathbf{u}^1 - F_-\sin\left(\frac{\beta\phi}{2}\phi\right)\mathbf{u}^2\right] ,\\ L_2^{\text{sG}}(\mathbf{z},\tau) &= \bar{L}^{\text{sG}}(\mathbf{z},\tau) = -\frac{\beta}{2\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\partial_{\bar{z}}\phi\,\mathbf{u}^3 + 2\mathbb{i}\,m\left[\bar{F}_+\cos\left(\frac{\beta\phi}{2}\right)\mathbf{u}^1 + \bar{F}_-\sin\left(\frac{\beta\phi}{2}\right)\mathbf{u}^2\right] , \end{aligned} \tag{4.58}$$

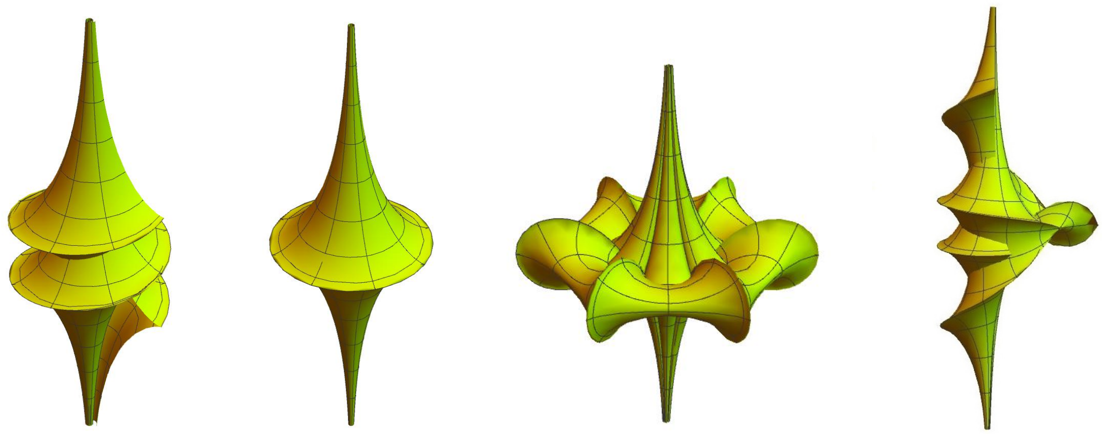
which again belongs to a 2−dimensional representation of the algebra $\mathfrak{su}(2)$ and thus the resulting solitonic surfaces are embedded in $\mathbb{R}^3$. Going through the computation of the first and second fundamental forms, one finds

$$\text{I}^{\text{sG}}(\mathbf{z},\tau) = g_{\mu\nu}^{\text{sG}}(\mathbf{z},\tau)\,dz^\mu dz^\nu = \frac{m^2}{2\left(\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)\right)^2}\left(\frac{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)+1}{1-\tau V^{\text{sG}}}\right)^2\hat{g}_{\mu\nu}\,dz^\mu dz^\nu\;, \tag{4.59}$$

$$\text{II}^{\text{sG}}(\mathbf{z},\tau) = d_{\mu\nu}^{\text{sG}}(\mathbf{z},\tau)\,dz^\mu dz^\nu = \frac{m^2\sin\left(\beta\phi\right)}{\sqrt{2}\lambda\left(1-\tau V^{\text{sG}}\right)}\left(\frac{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)+1}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\right)^2\hat{d}_{\mu\nu}\,dz^\mu dz^\nu\;, \tag{4.60}$$

where the matrices $\hat{g}_{\mu\nu}$ and $\hat{d}_{\mu\nu}$ are

$$\hat{g}_{\mu\nu} = \begin{pmatrix} \left(\frac{\mathcal{S}^{\text{sG}}+1}{2} - \frac{\mathcal{S}^{\text{sG}}-1}{2\lambda^2}\frac{\partial_z\phi}{\partial_{\bar{z}}\phi}\right)^2 + \frac{(\mathcal{S}^{\text{sG}})^2-1}{4\lambda^2}\frac{\beta^2V^{\text{sG}}}{m^2}\frac{\partial_z\phi}{\partial_{\bar{z}}\phi} & \frac{(\mathcal{S}^{\text{sG}})^2-1}{4}\left(\frac{\partial_{\bar{z}}\phi}{\partial_z\phi} + \frac{1}{\lambda^4}\frac{\partial_z\phi}{\partial_{\bar{z}}\phi}\right) - \frac{(\mathcal{S}^{\text{sG}})^2+1}{2\lambda^2}\cos\left(\beta\phi\right) \\ \frac{(\mathcal{S}^{\text{sG}})^2-1}{4}\left(\frac{\partial_{\bar{z}}\phi}{\partial_z\phi} + \frac{1}{\lambda^4}\frac{\partial_z\phi}{\partial_{\bar{z}}\phi}\right) - \frac{(\mathcal{S}^{\text{sG}})^2+1}{2\lambda^2}\cos\left(\beta\phi\right) & \left(\frac{\mathcal{S}^{\text{sG}}+1}{2\lambda^2} - \frac{\mathcal{S}^{\text{sG}}-1}{2}\frac{\partial_{\bar{z}}\phi}{\partial_z\phi}\right)^2 + \frac{(\mathcal{S}^{\text{sG}})^2-1}{4\lambda^2}\frac{\beta^2V^{\text{sG}}}{m^2}\frac{\partial_{\bar{z}}\phi}{\partial_z\phi} \end{pmatrix}_{\mu\nu} ,$$



(a) (b) (c) (d)

Figure 4.2: Pseudo-spherical solitonic surfaces associated to kink and breather solutions. Figure 4.2a represents the Dini surface, corresponding to a moving kink, while in Figure 4.2b the famous Beltrami pseudo-sphere is represented. The latter surface is obtained from Dini's surface by taking the stationary limit of the kink solution. Figures 4.2c and 4.2d correspond to the pseudo-spherical surfaces associated to a stationary breather and to a two-kink solution, respectively.

$$\hat{d}_{\mu\nu} = \begin{pmatrix} \tau\left(\partial_z\phi\right)^2 & \frac{(\mathcal{S}^{\text{sG}})^2+1}{4\left(1-\tau V^{\text{sG}}\right)} \\ \frac{(\mathcal{S}^{\text{sG}})^2+1}{4\left(1-\tau V^{\text{sG}}\right)} & \tau\left(\partial_{\bar{z}}\phi\right)^2 \end{pmatrix}_{\mu\nu} . \tag{4.61}$$

One easily verifies that, in the $\tau \to 0$ limit, $\text{I}^{\text{sG}}(\mathbf{z},\tau)$ and $\text{II}^{\text{sG}}(\mathbf{z},\tau)$ correctly reduces to the fundamental forms of sine-Gordon

$$\text{I}^{\text{sG}}(\mathbf{z},\tau) \underset{\tau\to 0}{\to} 2m^2 \begin{pmatrix} 1 & -\frac{1}{\lambda^2}\cos\left(\beta\phi\right) \\ -\frac{1}{\lambda^2}\cos\left(\beta\phi\right) & \frac{1}{\lambda^4} \end{pmatrix}_{\mu\nu} dz^\mu dz^\nu = \text{I}^{\text{sG}}(\mathbf{z}) \ , \tag{4.62}$$

$$\text{II}^{\text{sG}}(\mathbf{z},\tau) \underset{\tau\to 0}{\to} m^2\frac{\sqrt{2}}{\lambda}\sin\left(\beta\phi\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mu\nu} dz^\mu dz^\nu = \text{II}^{\text{sG}}(\mathbf{z}) \ . \tag{4.63}$$

Quite remarkably, although the dependence of the matrices (4.61) on τ is complicated, they recombine in such a way that the Gaussian and mean curvature do not depend explicitly on it. In fact, these two geometric invariants in the T$\bar{\text{T}}$−deformed theory are exactly the same as in the original sine-Gordon model

$$\boxed{K^{\text{sG}}(\mathbf{z},\tau) = -\frac{\lambda^2}{2} = K^{\text{sG}}(\mathbf{z}) \ , \quad H^{\text{sG}}(\mathbf{z},\tau) = \frac{\lambda}{\sqrt{2}}\cot\left(\beta\phi\right) = H^{\text{sG}}(\mathbf{z}) \ .} \tag{4.64}$$

Therefore, the solitonic surface corresponding to a particular solution of the T$\bar{\text{T}}$-deformed sine-Gordon EoM is the same as the one associated to the solution of the original EoM. We conclude that the net effect of the $\mathrm{T\bar{T}}$ deformation is to change the parametrisation of the surface, without affecting its intrinsic properties. At the level of the solutions, instead, the deformation acts as a non-trivial space-time coordinate transformation. As we have already seen in section 4.1 and shall see in section 4.7, the $\mathrm{T\bar{T}}-$deformed solutions generally possess critical values in τ corresponding to shock-wave phenomena, *i.e.* branching of the solutions. Examples of shock-wave phenomena and square root-type transitions in the classical energy – similar to the Hagedorn transition at quantum level – will be discussed in sections 4.7 and 4.8 for specific solutions of the deformed sine-Gordon model.

4.5 Coordinate transformation

We have inferred that there must exist a reference frame $\mathbf{w} = (w^1, w^2) = (w, \bar{w})$ in which the components of the first and second fundamental forms of the $\mathrm{T\bar{T}}-$deformed sine-Gordon model becomes trivially the ones of the original theory. Therefore, we look for a space-time coordinate transformation of the form

$$\Psi_\tau^{\mathrm{sG}} : \mathbb{C} \to \mathbb{C} : \mathbf{z} \to \mathbf{w} = \Psi_\tau^{\mathrm{sG}}(\mathbf{z}) \,, \tag{4.65}$$

such that

$$g_{\mu\nu}^{\mathrm{sG}}(\mathbf{w})\, dw^\mu dw^\nu = g_{\mu\nu}^{\mathrm{sG}}(\mathbf{z},\tau)\, dz^\mu dz^\nu \quad \Longrightarrow \quad g_{\rho\sigma}^{\mathrm{sG}}\left(\Psi_\tau^{\mathrm{sG}}(\mathbf{z})\right) \frac{\partial w^\rho}{\partial z^\mu}\frac{\partial w^\sigma}{\partial z^\nu} = g_{\mu\nu}^{\mathrm{sG}}(\mathbf{z},\tau) \,, \tag{4.66}$$

$$d_{\mu\nu}^{\mathrm{sG}}(\mathbf{w})\, dw^\mu dw^\nu = d_{\mu\nu}^{\mathrm{sG}}(\mathbf{z},\tau)\, dz^\mu dz^\nu \quad \Longrightarrow \quad d_{\rho\sigma}^{\mathrm{sG}}\left(\Psi_\tau^{\mathrm{sG}}(\mathbf{z})\right) \frac{\partial w^\rho}{\partial z^\mu}\frac{\partial w^\sigma}{\partial z^\nu} = d_{\mu\nu}^{\mathrm{sG}}(\mathbf{z},\tau) \,. \tag{4.67}$$

where $\mathcal{J}^\mu_\nu = \frac{\partial w^\mu}{\partial z^\nu}$ are the components of the Jacobian of the transformation (4.65). Using the expression (4.55)-(4.56) and (4.59)-(4.61), it is now a matter of simple algebraic manipulations to obtain the following expressions for $\mathcal{J}^\mu_\nu$ in the coordinates $\mathbf{z}$

$$\partial_z w \equiv \frac{\partial w^1}{\partial z^1} = \frac{\left(\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)+1\right)^2}{4\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)\,(1-\tau V^{\mathrm{sG}})} \quad , \quad \partial_{\bar{z}} \bar{w} \equiv \frac{\partial w^2}{\partial z^2} = \frac{\left(\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)+1\right)^2}{4\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)\,(1-\tau V^{\mathrm{sG}})} \,, \tag{4.68}$$

$$\partial_{\bar{z}} w \equiv \frac{\partial w^1}{\partial z^2} = \frac{\tau}{\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)}\, (\partial_{\bar{z}}\phi)^2 \quad , \quad \partial_z \bar{w} \equiv \frac{\partial w^2}{\partial z^1} = \frac{\tau}{\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)}\, (\partial_z\phi)^2 \,. \tag{4.69}$$

The latter expressions allows one to write the partial derivatives of the field ϕ in the coordinates $\mathbf{w}$ in terms of those in the coordinates $\mathbf{z}$ and viceversa. In fact, starting from

$$\begin{pmatrix} \partial_z\phi \\ \partial_{\bar{z}}\phi \end{pmatrix} = \mathcal{J} \begin{pmatrix} \partial_w\phi \\ \partial_{\bar{w}}\phi \end{pmatrix} \,, \tag{4.70}$$

one arrives at the following set of algebraic equations

$$\begin{cases} 4\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)(1-\tau V^{\mathrm{sG}})(\partial_z\phi) = \left(\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)+1\right)^2(\partial_w\phi) + 4\tau(1-\tau V)(\partial_{\bar{w}}\phi)(\partial_z\phi)^2 \\ 4\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)(1-\tau V^{\mathrm{sG}})(\partial_{\bar{z}}\phi) = \left(\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)+1\right)^2(\partial_{\bar{w}}\phi) + 4\tau(1-\tau V)(\partial_w\phi)(\partial_{\bar{z}}\phi)^2 \,, \end{cases} \tag{4.71}$$

which can be easily solved for $(\partial_w\phi, \partial_{\bar{w}}\phi)$ as

$$\begin{cases} \partial_w\phi = \frac{2(1-\tau V^{\mathrm{sG}})}{\mathcal{S}^{\mathrm{sG}}+1}\,\partial_z\phi \\ \partial_{\bar{w}}\phi = \frac{2(1-\tau V^{\mathrm{sG}})}{\mathcal{S}^{\mathrm{sG}}+1}\,\partial_{\bar{z}}\phi \end{cases} . \tag{4.72}$$

Finally the latter expression can be inverted for $(\partial_z\phi, \partial_{\bar{z}}\phi)$ as

$$\begin{cases} (\partial_w\phi)^2\,(\mathcal{S}^{\text{sG}}(\mathbf{z},\tau))^2 = (-\partial_w\phi + 2(1-\tau V^{\text{sG}})\,\partial_z\phi)^2 \\ (\partial_{\bar{w}}\phi)^2\,(\mathcal{S}^{\text{sG}}(\mathbf{z},\tau))^2 = (-\partial_{\bar{w}}\phi + 2(1-\tau V^{\text{sG}})\,\partial_{\bar{z}}\phi)^2 \end{cases} \longrightarrow \begin{cases} \partial_z\phi = \frac{\partial_w\phi}{1-\tau\mathcal{L}^{\text{sG}}(\mathbf{w})} \\ \partial_{\bar{z}}\phi = \frac{\partial_{\bar{w}}\phi}{1-\tau\mathcal{L}^{\text{sG}}(\mathbf{w})} \end{cases}, \tag{4.73}$$

where we used (4.28) and $\mathcal{L}^{\text{sG}}(\mathbf{w}) = \partial_w\phi\,\partial_{\bar{w}}\phi + V^{\text{sG}}$. Using (4.73), we can now derive the expression of $\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)$ in the coordinates $\mathbf{w}$

$$\mathcal{S}^{\text{sG}}\left(\left(\Psi^{\text{sG}}_\tau(\mathbf{w})\right)^{-1},\tau\right) = \sqrt{1+4\tilde{\tau}\,\frac{\mathcal{L}(\mathbf{w})}{(1-\tau\mathcal{L}^{\text{sG}}(\mathbf{w}))^2}} = \frac{1+\tau\,(\mathcal{L}(\mathbf{w})-V^{\text{sG}})}{1-\tau\,\mathcal{L}^{\text{sG}}(\mathbf{w})}\,, \quad \mathcal{L}(\mathbf{w}) = \partial_w\phi\,\partial_{\bar{w}}\phi\,, \tag{4.74}$$

from which we can write the Jacobian $\mathcal{J}$ and its inverse $\mathcal{J}^{-1}$ in the coordinates $\mathbf{w}$ as

$$\mathcal{J} = \frac{1}{(1-\tau V^{\text{sG}})^2 - (\tau\mathcal{L}(\mathbf{w}))^2}\begin{pmatrix} 1-\tau V^{\text{sG}} & \tau\,(\partial_w\phi)^2 \\ \tau\,(\partial_{\bar{w}}\phi)^2 & 1-\tau V^{\text{sG}} \end{pmatrix}, \tag{4.75}$$

$$\mathcal{J}^{-1} = \begin{pmatrix} 1-\tau V^{\text{sG}} & -\tau\,(\partial_w\phi)^2 \\ -\tau\,(\partial_{\bar{w}}\phi)^2 & 1-\tau V^{\text{sG}} \end{pmatrix}, \tag{4.76}$$

where the components of $\mathcal{J}^{-1}$ are $\left(\mathcal{J}^{-1}\right)^\mu_{\ \nu} = \frac{\partial z^\mu}{\partial w^\nu}$. Using (4.75) we express the partial derivatives w.r.t. $\mathbf{z}$ in terms of partial derivatives w.r.t. $\mathbf{w}$ as

$$\begin{pmatrix} \partial_z f \\ \partial_{\bar{z}} f \end{pmatrix} = \mathcal{J}\begin{pmatrix} \partial_w f \\ \partial_{\bar{w}} f \end{pmatrix} \longrightarrow \begin{cases} \partial_z f = \frac{1}{1-\tau\mathcal{L}^{\text{sG}}(\mathbf{w})}\,\partial_w f \\ \partial_{\bar{z}} f = \frac{1}{1-\tau\mathcal{L}^{\text{sG}}(\mathbf{w})}\,\partial_{\bar{w}} f \end{cases}, \quad \forall f : \mathbb{C}\to\mathbb{R}\,. \tag{4.77}$$

Applying the latter result to the EoM of the $\mathrm{T\bar{T}}$−deformed sine-Gordon theory (4.31) we get, separately for the *lhs* and the *rhs* , respectively

$$\partial_z\left(\frac{\partial_{\bar{z}}\phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\right) + \partial_{\bar{z}}\left(\frac{\partial_z\phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\right) = \frac{2\,\partial_w\partial_{\bar{w}}\phi}{(1+\tau\,(\mathcal{L}(\mathbf{w})-V^{\text{sG}}))^2} - 2\tau\,\frac{(V^{\text{sG}})'}{(1-\tau V^{\text{sG}})^2-(\tau\,\mathcal{L}(\mathbf{w}))^2}\,\frac{\mathcal{L}(\mathbf{w})}{(1+\tau\,(\mathcal{L}(\mathbf{w})-V^{\text{sG}}))}\,, \tag{4.78}$$

$$\frac{(V^{\text{sG}})'}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\left(\frac{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)+1}{1-\tau V^{\text{sG}}}\right)^2 = \frac{(V^{\text{sG}})'}{(1-\tau V^{\text{sG}})^2-(\tau\mathcal{L}(\mathbf{w}))^2}\,. \tag{4.79}$$

and equating (4.78) to (4.79) yields

$$\boxed{\frac{2\,\partial_w\partial_{\bar{w}}\phi - (V^{\text{sG}})'}{(1+\tau\,(\mathcal{L}(\mathbf{w})-V^{\text{sG}}))^2} = 0\,.} \tag{4.80}$$

Since the determinant of the inverse Jacobian in coordinates $\mathbf{w}$ is

$$\det\left[\mathcal{J}^{-1}\right] = \left(1-\tau V^{\text{sG}}\right)^2 - (\tau\mathcal{L}(\mathbf{w}))^2 = \left(1-\tau\mathcal{L}^{\text{sG}}(\mathbf{w})\right)\left(1+\tau\left(\mathcal{L}(\mathbf{w})-V^{\text{sG}}\right)\right)\,, \tag{4.81}$$

it follows that, for any field configuration $\phi(\mathbf{w})$ such that $\det\left[\mathcal{J}^{-1}\right] \neq 0$, the denominator appearing in (4.80) is automatically non-vanishing and (4.80) gives exactly the EoM of the sine-Gordon theory.

Therefore, the field-dependent coordinate transformation (4.65) maps solutions of the T$\bar{\text{T}}$−deformed sine-Gordon theory into solutions of the original theory. Naturally, it is also possible to show that the viceversa holds. To do that, we must prove that the sine-Gordon EoM in the coordinates $\mathbf{w}$ is mapped into (4.31). Therefore, we need to express the partial derivatives w.r.t. $\mathbf{w}$ in terms of partial derivatives w.r.t. $\mathbf{z}$. Using (4.72) we can write $\mathcal{J}$ and $\mathcal{J}^{-1}$ in the coordinates $\mathbf{z}$ as

$$\mathcal{J} = \begin{pmatrix} 1-\tau\left[\frac{\tilde{\tau}(\partial_z\phi\,\partial_{\bar{z}}\phi)^2}{\mathcal{S}^{\text{sG}}}\left(\frac{2}{1+\mathcal{S}^{\text{sG}}}\right)^2+\frac{V^{\text{sG}}}{1-\tau V^{\text{sG}}}\right] & \tau\,\frac{(\partial_z\phi)^2}{\mathcal{S}^{\text{sG}}} \\ \tau\,\frac{(\partial_{\bar{z}}\phi)^2}{\mathcal{S}^{\text{sG}}} & 1-\tau\left[\frac{\tilde{\tau}(\partial_z\phi\,\partial_{\bar{z}}\phi)^2}{\mathcal{S}^{\text{sG}}}\left(\frac{2}{1+\mathcal{S}^{\text{sG}}}\right)^2+\frac{V^{\text{sG}}}{1-\tau V^{\text{sG}}}\right] \end{pmatrix}, \tag{4.82}$$

$$\mathcal{J}^{-1} = \begin{pmatrix} 1-\tau V^{\text{sG}} & -\frac{1}{4\tau}\left(\frac{-1+\mathcal{S}^{\text{sG}}}{\partial_{\bar{z}}\phi}\right)^2 \\ -\frac{1}{4\tau}\left(\frac{-1+\mathcal{S}^{\text{sG}}}{\partial_z\phi}\right)^2 & 1-\tau V^{\text{sG}} \end{pmatrix}. \tag{4.83}$$

Expression (4.83) allows us to express the partial derivatives w.r.t. $\mathbf{w}$ in terms of partial derivatives w.r.t. $\mathbf{z}$ as follows

$$\begin{pmatrix} \partial_w f \\ \partial_{\bar{w}} f \end{pmatrix} = \mathcal{J}^{-1}\begin{pmatrix} \partial_z f \\ \partial_{\bar{z}} f \end{pmatrix} \quad\longrightarrow\quad \begin{cases} \partial_w f = (1-\tau V^{\text{sG}})\,\partial_z f - \frac{1}{4\tau}\left(\frac{-1+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}{\partial_{\bar{z}}\phi}\right)^2\partial_{\bar{z}} f \\ \partial_{\bar{w}} f = (1-\tau V^{\text{sG}})\,\partial_{\bar{z}} f - \frac{1}{4\tau}\left(\frac{-1+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}{\partial_z\phi}\right)^2\partial_z f \end{cases}, \tag{4.84}$$

$\forall\, f\,:\,\mathbb{C}\to\mathbb{R}$. Plugging the latter result separately into the *lhs* and the *rhs* of the sine-Gordon EoM (3.4) in the coordinates $\mathbf{w}$ one gets, respectively

$$\partial_w\partial_{\bar{w}}\phi = \left[(1-\tau V^{\text{sG}})\,\partial_z - \frac{1}{4\tau}\left(\frac{-1+\mathcal{S}^{\text{sG}}}{\partial_{\bar{z}}\phi}\right)^2\partial_{\bar{z}}\right]\left[(1-\tau V^{\text{sG}})\,\partial_{\bar{z}}\phi - \frac{1}{4\tau}\left(\frac{-1+\mathcal{S}^{\text{sG}}}{\partial_z\phi}\right)^2\partial_z\phi\right], \tag{4.85}$$

$$\frac{1}{2}(V^{\text{sG}})' = \frac{1}{2}(V^{\text{sG}})', \tag{4.86}$$

and equating (4.85) to (4.86), after some computations one finds

$$\boxed{\frac{1}{2}\,\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)\,\det\left[\mathcal{J}^{-1}\right]\left[\partial_z\left(\frac{\partial_{\bar{z}}\phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\right)+\partial_{\bar{z}}\left(\frac{\partial_z\phi}{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\right)-\frac{(V^{\text{sG}})'}{4\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}\left(\frac{1+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)}{1-\tau V^{\text{sG}}}\right)^2\right]=0\,.} \tag{4.87}$$

Exactly as before, since the determinant of the inverse Jacobian in the coordinates $\mathbf{z}$ is

$$\det\left[\mathcal{J}^{-1}\right] = \frac{\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)\,(\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)-1)\left((\mathcal{S}^{\text{sG}}(\mathbf{z},\tau))^2+\mathcal{S}^{\text{sG}}(\mathbf{z},\tau)-2\right)}{4\tau\,\partial_z\phi\,\partial_{\bar{z}}\phi}, \tag{4.88}$$

for any field configuration $\phi(\mathbf{z},\tau)$ such that $\det\left[\mathcal{J}^{-1}\right]\neq 0$, the factor in front of the square brackets in (4.87) is non-vanishing, therefore (4.87) is equivalent to (4.31).

Observation 22. Altough the coordinate transformation has been obtained exploiting the unique relation between integrable equations and embedded surfaces, we never used the explicit form of the potential in transforming the sine-Gordon EoM into the $\mathrm{T}\bar{\mathrm{T}}-$deformed one and viceversa. In other words, the coordinate transformation (4.65) with Jacobian (4.75)-(4.76) is valid for any scalar field theory consisting of a single scalar field interacting with a potential V depending on the field only. For this reason, from now on we will drop the superscript "sG", unless we explicitly refer to the sine-Gordon model.

Observation 23. The components of the inverse Jacobian in the coordinates $\mathbf{w}$ (4.76) and the Jacobian in the coordinates $\mathbf{z}$ (4.82), depend on the chiral components of the stress-energy tensor of the original and the $\mathrm{T}\bar{\mathrm{T}}-$deformed theory, respectively

$$\mathcal{J}^{-1} = \begin{pmatrix} 1 + 2\tau\,\Theta(\mathbf{w}) & 2\tau\,\mathrm{T}(\mathbf{w}) \\ 2\tau\,\bar{\mathrm{T}}(\mathbf{w}) & 1 + 2\tau\,\Theta(\mathbf{w}) \end{pmatrix} , \tag{4.89}$$

$$\mathcal{J} = \begin{pmatrix} 1 - 2\tau\,\Theta(\mathbf{z},\tau) & -2\tau\,\mathrm{T}(\mathbf{z},\tau) \\ -2\tau\,\bar{\mathrm{T}}(\mathbf{z},\tau) & 1 - 2\tau\,\Theta(\mathbf{z},\tau) \end{pmatrix} , \tag{4.90}$$

with

$$\mathrm{T}(\mathbf{w}) = -\frac{1}{2}\left(\partial_w \phi\right)^2 \ , \quad \bar{\mathrm{T}}(\mathbf{w}) = -\frac{1}{2}\left(\partial_{\bar{w}} \phi\right)^2 \ , \quad \Theta(\mathbf{w}) = -\frac{1}{2} V \ , \tag{4.91}$$

and

$$\mathrm{T}(\mathbf{z},\tau) = -\frac{\left(\partial_z \phi\right)^2}{2\mathcal{S}^V(\mathbf{z},\tau)} \ , \quad \bar{\mathrm{T}}(\mathbf{z},\tau) = -\frac{\left(\partial_{\bar{z}} \phi\right)^2}{2\mathcal{S}^V(\mathbf{z},\tau)} \ ,$$
$$\Theta(\mathbf{z},\tau) = -\frac{1}{2}\left[\frac{\tilde{\tau}\left(\partial_z\phi\,\partial_{\bar{z}}\phi\right)^2}{\mathcal{S}^V(\mathbf{z},\tau)}\left(\frac{2}{1+\mathcal{S}^V(\mathbf{z},\tau)}\right)^2 + \frac{V}{1-\tau V}\right] , \tag{4.92}$$

where $\mathcal{S}^V(\mathbf{z},\tau)$ is obtained from $\mathcal{S}^{\mathrm{sG}}(\mathbf{z},\tau)$ replacing $V^{\mathrm{sG}} \to V$. From (4.89)-(4.90), it follows that the Hessian matrix associated to the coordinate transformation (4.65) is symmetric if and only if the field configuration ϕ is evaluated on-shell. In fact, the second mixed partial derivatives coincide if and only if the continuity equations are fulfilled

$$\partial_{\bar{w}}\left(\partial_w z\right) = 2\tau\,\partial_{\bar{w}}\Theta(\mathbf{w}) \underset{\text{EoMs }(\mathbf{w})}{\equiv} 2\tau\,\partial_w\bar{\mathrm{T}}(\mathbf{w}) = \partial_w\left(\partial_{\bar{w}} z\right) \ , \tag{4.93}$$

$$\partial_{\bar{w}}\left(\partial_w \bar{z}\right) = 2\tau\,\partial_{\bar{w}}\mathrm{T}(\mathbf{w}) \underset{\text{EoMs }(\mathbf{w})}{\equiv} 2\tau\,\partial_w\Theta(\mathbf{w}) = \partial_w\left(\partial_{\bar{w}} \bar{z}\right) \ , \tag{4.94}$$

and, analogously

$$\partial_{\bar{z}}\left(\partial_z w\right) = -2\tau\,\partial_{\bar{z}}\Theta(\mathbf{z},\tau) \underset{\text{EoMs }(\mathbf{z})}{\equiv} -2\tau\,\partial_z\bar{\mathrm{T}}(\mathbf{z},\tau) = \partial_z\left(\partial_{\bar{z}} w\right) \ , \tag{4.95}$$

$$\partial_{\bar{z}}\left(\partial_z \bar{w}\right) = -2\tau\,\partial_{\bar{z}}\mathrm{T}(\mathbf{z},\tau) \underset{\text{EoMs }(\mathbf{z})}{\equiv} -2\tau\,\partial_z\Theta(\mathbf{z},\tau) = \partial_z\left(\partial_{\bar{z}} \bar{w}\right) \ . \tag{4.96}$$

Observation 24. Expressions (4.89)-(4.90) suggest a natural generalisation of the coordinate transformation to the case of N bosonic fields and sigma-models. In fact, considering the Lagrangian densities (2.115) and (2.116) in the coordinates $\mathbf{w}$ and $\mathbf{z}$, respectively, it is possible to prove that the coordinate transformation (4.65) with Jacobian and inverse Jacobian given by (4.90) with

$$\mathrm{T}(\mathbf{w}) = -\frac{1}{2}\sum_{i=1}^{N}(\partial_w\phi_i)^2 \ , \quad \bar{\mathrm{T}}(\mathbf{w}) = -\frac{1}{2}\sum_{i=1}^{N}(\partial_{\bar{w}}\phi_i)^2 \ , \quad \Theta(\mathbf{w}) = -\frac{1}{2}V \ , \tag{4.97}$$

and

$$\mathrm{T}(\mathbf{z},\tau) = -\frac{1}{2\mathcal{S}^\sigma}\sum_{i=1}^{N}(\partial_z\phi_i)^2 \ , \quad \bar{\mathrm{T}}(\mathbf{z},\tau) = -\frac{1}{2\mathcal{S}^\sigma}\sum_{i=1}^{N}(\partial_{\bar{z}}\phi_i)^2 \ , \tag{4.98}$$

$$\Theta(\mathbf{z},\tau) = \frac{-1+\mathcal{S}^\sigma - 2\tilde{\tau}\sum_{i=1}^{N}\partial_z\phi_i\,\partial_{\bar{z}}\phi_i}{4\tilde{\tau}\mathcal{S}^\sigma} - \frac{1}{2}\frac{V}{1-\tau V} \ , \tag{4.99}$$

maps the EoMs associated to (2.115) and (2.116) one into the other.

Observation 25. Moving from complex to cartesian coordinates according to (A.1) and using (A.7) and (A.9), we can translate expressions (4.89)-(4.90) in cartesian coordinates as

$$\mathcal{J}^{-1} = \begin{pmatrix} 1+2\tau\boldsymbol{T}^{22}(\mathbf{y}) & -2\tau\boldsymbol{T}^{12}(\mathbf{y}) \\ -2\tau\boldsymbol{T}^{12}(\mathbf{y}) & 1+2\tau\boldsymbol{T}^{11}(\mathbf{y}) \end{pmatrix} \ , \tag{4.100}$$

$$\mathcal{J} = \begin{pmatrix} 1-2\tau\boldsymbol{T}^{22}(\mathbf{x},\tau) & 2\tau\boldsymbol{T}^{12}(\mathbf{x},\tau) \\ 2\tau\boldsymbol{T}^{12}(\mathbf{x},\tau) & 1-2\tau\boldsymbol{T}^{11}(\mathbf{x},\tau) \end{pmatrix} \ , \tag{4.101}$$

which can be more compactly written in components as

$$\boxed{(\mathcal{J}^{-1})^\mu_{\ \nu} = \frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_\nu + \tau\widetilde{\boldsymbol{T}}^\mu_{\ \nu}(\mathbf{y}) \ , \quad \mathcal{J}^\mu_{\ \nu} = \frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \tau\widetilde{\boldsymbol{T}}^\mu_{\ \nu}(\mathbf{x},\tau) \ ,} \tag{4.102}$$

with

$$\widetilde{\boldsymbol{T}}^\mu_{\ \nu}(\mathbf{y}) = -g^{\mu\delta}\epsilon_{\delta\rho}\,\epsilon_{\sigma\nu}\,\boldsymbol{T}^{\rho\sigma}(\mathbf{y}) \ , \quad \widetilde{\boldsymbol{T}}^\mu_{\ \nu}(\mathbf{x},\tau) = -g^{\mu\delta}\epsilon_{\delta\rho}\,\epsilon_{\sigma\nu}\,\boldsymbol{T}^{\rho\sigma}(\mathbf{x},\tau) \ . \tag{4.103}$$

Observation 26. The Lagrangian density (2.116) transforms under the coordinate transformation (4.65) as

$$\begin{aligned} \mathcal{L}^\sigma\left((\Psi^\sigma_\tau)^{-1}(\mathbf{w}),\tau\right) &= \frac{V}{1-\tau V} + \frac{-1+\mathcal{S}^\sigma\left((\Psi^\sigma_\tau)^{-1}(\mathbf{w}),\tau\right)}{2\tilde{\tau}} \\ &= \frac{\mathcal{L}^\sigma(\mathbf{w}) + 4\tau\left(\mathrm{T}(\mathbf{w})\bar{\mathrm{T}}(\mathbf{w}) - \Theta^2(\mathbf{w})\right)}{\det\left[\mathcal{J}^{-1}\right]} \ , \end{aligned} \tag{4.104}$$

where

$$\det\left[\mathcal{J}^{-1}\right] = 1 - 2\tau V - 4\tau^2\left(\mathrm{T}(\mathbf{w})\bar{\mathrm{T}}(\mathbf{w}) - \Theta^2(\mathbf{w})\right) \ , \tag{4.105}$$

and $\{\mathrm{T}(\mathbf{w}), \bar{\mathrm{T}}(\mathbf{w}), \Theta(\mathbf{w})\}$ are the components of the original stress-energy tensor defined in (4.97). Therefore, the action transforms as

$$\begin{aligned} \int_{\mathbb{C}}\mathcal{L}^\sigma(\mathbf{z},\tau)\,dz\wedge d\bar{z} &= \int_{\mathbb{C}}\mathcal{L}^\sigma\left((\Psi^\sigma_\tau)^{-1}(\mathbf{w}),\tau\right)\det\left[\mathcal{J}^{-1}\right]dw\wedge d\bar{w} \\ &= \int_{\mathbb{C}}\left[\mathcal{L}^\sigma(\mathbf{w}) + 4\tau\left(\mathrm{T}(\mathbf{w})\bar{\mathrm{T}}(\mathbf{w}) - \Theta^2(\mathbf{w})\right)\right]dw\wedge d\bar{w} \\ &= \int_{\mathbb{C}}\left[\mathcal{L}^\sigma(\mathbf{w}) + \frac{\tau}{\pi^2}\,\mathrm{T}\bar{\mathrm{T}}(\mathbf{w})\right]dw\wedge d\bar{w} \ . \end{aligned}$$

which in conclusion leads to

$$\boxed{\int_{\mathbb{C}} \mathcal{L}^{\sigma}(\mathbf{z},\tau)\, dz \wedge d\bar{z} = \int_{\mathbb{C}} \left[\mathcal{L}^{\sigma}(\mathbf{w}) + \frac{\tau}{\pi^2}\mathrm{T}\bar{\mathrm{T}}(\mathbf{w})\right] dw \wedge d\bar{w}\ .} \tag{4.106}$$

Analogously, one can prove that also the viceversa holds, namely

$$\boxed{\int_{\mathbb{C}} \mathcal{L}^{\sigma}(\mathbf{w})\, dw \wedge d\bar{w} = \int_{\mathbb{C}} \left[\mathcal{L}^{\sigma}(\mathbf{z},\tau) - \frac{\tau}{\pi^2}\mathrm{T}\bar{\mathrm{T}}(\mathbf{z},\tau)\right] dz \wedge d\bar{z}\ .} \tag{4.107}$$

The validity of the latter statements can be extended to any theory [40].

4.6 T$\bar{\text{T}}$-deformed integrable hierarchy

The aim of this section is to introduce an efficient method, based on the field-dependent coordinate transformation derived in section 4.5, to reconstruct the local Integrals of Motion (IMs) associated to the T$\bar{\text{T}}$ deformation of a generic IFT. Let us consider the following pair of conjugated 1-forms

$$\mathfrak{I}_{\mathbf{k}} = \mathrm{T}_{\mathbf{k}+1}(\mathbf{z})\, dz + \Theta_{\mathbf{k}-1}(\mathbf{z})\, d\bar{z}\ , \quad \bar{\mathfrak{I}}_{\mathbf{k}} = \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{z})\, d\bar{z} + \bar{\Theta}_{\mathbf{k}-1}(\mathbf{z})\, dz\ , \quad (\mathbf{k} \in \mathbb{N})\ , \tag{4.108}$$

where $\{\mathrm{T}_{\mathbf{k}+1}(\mathbf{z}), \Theta_{\mathbf{k}-1}(\mathbf{z}), \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{z}), \bar{\Theta}_{\mathbf{k}-1}(\mathbf{z})\}$ are the rescaled chiral components of the level$-\mathbf{k}$ higher conserved currents of the integrable hierarchy (see appendix A). Using (A.13), it is easy to check that (4.108) are closed forms[1],

$$d\mathfrak{I}_{\mathbf{k}} = \left(\partial\Theta_{\mathbf{k}-1} - \bar{\partial}\mathrm{T}_{\mathbf{k}+1}\right) dz \wedge d\bar{z} = 0\ , \quad d\bar{\mathfrak{I}}_{\mathbf{k}} = \left(\partial\bar{\mathrm{T}}_{\mathbf{k}+1} - \bar{\partial}\bar{\Theta}_{\mathbf{k}-1}\right) dz \wedge d\bar{z} = 0\ . \tag{4.109}$$

Therefore, for any given integration contour C in the complex plane of z, the following integrals

$$\int_C \mathfrak{I}_{\mathbf{k}}\ , \quad \int_C \bar{\mathfrak{I}}_{\mathbf{k}}\ , \tag{4.110}$$

define local IMs, since they do not depend on smooth deformations of C which keep the end-points fixed. The set of conserved charges are then obtained from (4.110) by fixing a given Cauchy surface, *i.e.* $dx^2 = 0$ and $dz = d\bar{z} = dx^1$, and integrating over a finite strip $C = [0, R]$ along the direction x^1

$$\begin{aligned} I_{\mathbf{k}}(R) &= -\int_0^R \left[\mathrm{T}_{\mathbf{k}+1}(\mathbf{x}) + \Theta_{\mathbf{k}-1}(\mathbf{x})\right] dx^1 = \int_0^R \mathcal{I}_{\mathbf{k}}(\mathbf{x})\, dx^1\ , \\ \bar{I}_{\mathbf{k}}(R) &= -\int_0^R \left[\bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{x}) + \bar{\Theta}_{\mathbf{k}-1}(\mathbf{x})\right] dx^1 = \int_0^R \bar{\mathcal{I}}_{\mathbf{k}}(\mathbf{x})\, dx^1\ , \end{aligned} \tag{4.111}$$

according to the definitions (A.17) and (A.19).

From their very definition, differential forms are intrinsic objects which do not depend on the set of coordinates chosen on the manifold. Since they remain closed under any coordinate transformations, the idea is to construct the local IMs of the T$\bar{\text{T}}$-deformed theory starting from those of the original theory, using the change of coordinates introduced in [22, 24]. The strategy is the following:

[1] Thus they are locally exact by the Poincaré lemma.

1. Start from the 1-forms (4.108) expressed in $\mathbf{w}$ coordinates

$$\mathfrak{J}_{\mathbf{k}} = \mathrm{T}_{\mathbf{k}+1}(\mathbf{w})\, dw + \Theta_{\mathbf{k}-1}(\mathbf{w})\, d\bar{w} \ , \quad \bar{\mathfrak{J}}_{\mathbf{k}} = \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{w})\, d\bar{w} + \bar{\Theta}_{\mathbf{k}-1}(\mathbf{w})\, dw \ ; \tag{4.112}$$

   where $\{\mathrm{T}_{\mathbf{k}+1}(\mathbf{w}), \Theta_{\mathbf{k}-1}(\mathbf{w}), \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{w}), \bar{\Theta}_{\mathbf{k}-1}(\mathbf{w})\}$ are the rescaled chiral components of the level$-\mathbf{k}$ conserved currents of the original theory which fulfil the continuity equations

$$\partial_{\bar{w}} \mathrm{T}_{\mathbf{k}+1}(\mathbf{w}) = \partial_w \Theta_{\mathbf{k}-1}(\mathbf{w}) \ , \quad \partial_w \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{w}) = \partial_{\bar{w}} \bar{\Theta}_{\mathbf{k}-1}(\mathbf{w}) \ . \tag{4.113}$$

2. Use the explicit expressions of $\{\mathrm{T}(\mathbf{w}), \bar{\mathrm{T}}(\mathbf{w}), \Theta(\mathbf{w})\}$ in terms of the fundamental fields of the theory to explicitly invert the map $\mathbf{z} = \left(\Psi_\tau^{-1}\right)(\mathbf{w})$ at differential level. Then, using the fact that the basis $d\mathbf{w} = (dw, d\bar{w})$ transforms into $d\mathbf{z} = (dz, d\bar{z})$ as

$$\begin{pmatrix} dw \\ d\bar{w} \end{pmatrix} = \mathcal{J}^{\mathrm{T}} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} , \tag{4.114}$$

   with $\mathcal{J}$ given by (4.90), expressions (4.112) become

$$\begin{aligned} \mathfrak{J}_{\mathbf{k}} = & \frac{\mathrm{T}_{\mathbf{k}+1}\big(\Psi_\tau(\mathbf{z})\big) + 2\tau\big[\mathrm{T}_{\mathbf{k}+1}\big(\Psi_\tau(\mathbf{z})\big)\,\Theta\big(\Psi_\tau(\mathbf{z})\big) - \Theta_{\mathbf{k}-1}\big(\Psi_\tau(\mathbf{z})\big)\,\mathrm{T}\big(\Psi_\tau(\mathbf{z})\big)\big]}{\Delta\big(\Psi_\tau(\mathbf{z})\big)}\, dz \\ & + \frac{\Theta_{\mathbf{k}-1}\big(\Psi_\tau(\mathbf{z})\big) + 2\tau\big[\Theta_{\mathbf{k}-1}\big(\Psi_\tau(\mathbf{z})\big)\,\bar{\Theta}\big(\Psi_\tau(\mathbf{z})\big) - \mathrm{T}_{\mathbf{k}+1}\big(\Psi_\tau(\mathbf{z})\big)\,\bar{\mathrm{T}}\big(\Psi_\tau(\mathbf{z})\big)\big)}{\Delta\big(\Psi_\tau(\mathbf{z})\big)}\, d\bar{z} \ , \end{aligned} \tag{4.115}$$

   where $F\big(\Psi_\tau(\mathbf{z})\big)$ indicates that the fundamental fields in $\mathbf{w}$ coordinates involved in an arbitrary function F, have been replaced with fundamental fields in $\mathbf{z}$ coordinates according to the map $\mathbf{w} = \Psi_\tau(\mathbf{z})$.

3. Read the T$\bar{\text{T}}$-deformed higher conserved currents as components of (4.115) in $\mathbf{z}$ coordinates:

$$\begin{aligned} \mathrm{T}_{\mathbf{k}+1}(\mathbf{z},\tau) &= \frac{\mathrm{T}_{\mathbf{k}+1}\big(\Psi_\tau(\mathbf{z})\big) + 2\tau\big(\mathrm{T}_{\mathbf{k}+1}\big(\Psi_\tau(\mathbf{z})\big)\,\Theta\big(\Psi_\tau(\mathbf{z})\big) - \Theta_{\mathbf{k}-1}\big(\Psi_\tau(\mathbf{z})\big)\,\mathrm{T}\big(\Psi_\tau(\mathbf{z})\big)\big)}{\Delta\big(\Psi_\tau(\mathbf{z})\big)} \ , \\ \Theta_{\mathbf{k}-1}(\mathbf{z},\tau) &= \frac{\Theta_{\mathbf{k}-1}\big(\Psi_\tau(\mathbf{z})\big) + 2\tau\big(\Theta_{\mathbf{k}-1}\big(\Psi_\tau(\mathbf{z})\big)\,\bar{\Theta}\big(\Psi_\tau(\mathbf{z})\big) - \mathrm{T}_{\mathbf{k}+1}\big(\Psi_\tau(\mathbf{z})\big)\,\bar{\mathrm{T}}\big(\Psi_\tau(\mathbf{z})\big)\big)}{\Delta\big(\Psi_\tau(\mathbf{z})\big)} \ . \end{aligned} \tag{4.116}$$

By definition, the integrals (4.110) are invariant under coordinate transformations, provided the integration contour C is mapped into C' accordingly

$$\begin{aligned} \int_C \mathrm{T}_{\mathbf{k}+1}(\mathbf{z})\, dz + \Theta_{\mathbf{k}-1}(\mathbf{z})\, d\bar{z} &= \int_{C'} \mathrm{T}_{\mathbf{k}+1}(\mathbf{z},\tau)\, dz + \Theta_{\mathbf{k}-1}(\mathbf{z},\tau)\, d\bar{z} \ , \\ \int_C \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{z})\, dz + \bar{\Theta}_{\mathbf{k}-1}(\mathbf{z})\, d\bar{z} &= \int_{C'} \bar{\mathrm{T}}_{\mathbf{k}+1}(\mathbf{z},\tau)\, dz + \bar{\Theta}_{\mathbf{k}-1}(\mathbf{z},\tau)\, d\bar{z} \ . \end{aligned} \tag{4.117}$$

Since the contours C and C' are not, in general, homotopically equivalent, at finite volume R one has

$$\begin{aligned} I_{\mathbf{k}}(R) &= \int_0^R \mathcal{I}_{\mathbf{k}}(\mathbf{x})\, dx^1 \neq \int_0^R \mathcal{I}_{\mathbf{k}}(\mathbf{x},\tau)\, dx^1 = I_{\mathbf{k}}(R,\tau) \ , \\ \bar{I}_{\mathbf{k}}(R) &= \int_0^R \bar{\mathcal{I}}_{\mathbf{k}}(\mathbf{x})\, dx^1 \neq \int_0^R \bar{\mathcal{I}}_{\mathbf{k}}(\mathbf{x},\tau)\, dx^1 = \bar{I}_{\mathbf{k}}(R,\tau) \ , \end{aligned} \tag{4.118}$$

where we defined

$$\mathcal{I}_{\mathbf{k}}(\mathbf{z},\tau) = -\left[\mathrm{T}_{\mathbf{k}+1}(\mathbf{x},\tau) + \Theta_{\mathbf{k}-1}(\mathbf{x},\tau)\right] \ , \quad \bar{\mathcal{I}}(\mathbf{z},\tau) = -\left[\mathrm{T}_{\mathbf{k}+1}(\mathbf{x},\tau) + \Theta_{\mathbf{k}-1}(\mathbf{x},\tau)\right] \ . \tag{4.119}$$

according to the definition (A.17). Therefore, the deformation of the conserved charges arises from the fact that we are integrating the deformed densities (4.119) over the same contour of integration of the original theory.

Observation 27. The Lax pair representation $(L, \bar{L})$ can be written covariantly in terms of a Lie algebra-valued differential form as

$$\Omega = L(\mathbf{z})\, dz + \bar{L}(\mathbf{z})\, d\bar{z} \ , \tag{4.120}$$

and the ZCR (4.33) translates into the Maurer-Cartan equation for Ω

$$d\Omega = \Omega \wedge \Omega \ . \tag{4.121}$$

Being components of Ω, we can apply the same strategy discussed above for the conserved currents to derive the Lax pair of the T$\bar{\text{T}}$−deformed theory from the original ones.

In order to make the construction of the deformed hierarchy more concrete, in the following section we perform the explicit computation for the massless free boson theory and comment on more general cases.

4.6.1 The massless free boson

Consider the Lagrangian of a single massless boson field ϕ in complex coordinates $\mathbf{w}$

$$\mathcal{L}(\mathbf{w}) = \partial_w \phi\, \partial_{\bar{w}} \phi \ . \tag{4.122}$$

The EoMs are

$$\partial_w \partial_{\bar{w}} \phi = 0 \ , \tag{4.123}$$

therefore, without further external constraints, there exists an infinite number of options for the choice of the basis of conserved currents. For example, both

$$\mathrm{T}^{(\mathrm{POW})}_{\mathbf{k}+1}(\mathbf{w}) = -\frac{1}{2}\left(\partial_w \phi\right)^{\mathbf{k}+1} \quad , \qquad \Theta^{(\mathrm{POW})}_{\mathbf{k}-1}(\mathbf{w}) = 0 \ , \quad (\mathbf{k} \in \mathbb{N}) \ , \tag{4.124}$$

and[1]

$$\mathrm{T}^{(\mathrm{KG})}_{\mathbf{k}+1}(\mathbf{w}) = -\frac{1}{2}\left(\partial_w^{\frac{1+\mathbf{k}}{2}} \phi\right)^2 \quad , \qquad \Theta^{(\mathrm{KG})}_{\mathbf{k}-1}(\mathbf{w}) = 0 \ , \quad (\mathbf{k} \in 2\mathbb{N}+1) \ , \tag{4.125}$$

[1] The set of currents (4.125) can be obtained as the massless limit of the Klein-Gordon hierarchy:

$$\mathrm{T}^{(\mathrm{KG})}_{\mathbf{k}+1}(\mathbf{w}) = -\frac{1}{2}(\partial_w^{\frac{1+\mathbf{k}}{2}} \phi)^2 \ , \quad \Theta^{(\mathrm{KG})}_{\mathbf{k}-1}(\mathbf{w}) = -\frac{m^2}{2}(\partial_w^{\frac{\mathbf{k}-1}{2}} \phi)^2 \ , \quad (\mathbf{k} \in 2\mathbb{N}+1) \ ,$$

with Lagrangian $\mathcal{L}^{(\mathrm{KG})}(\mathbf{w}) = \partial_w \phi\, \partial_{\bar{w}} \phi + m^2 \phi^2$.

are possible sets of higher conserved currents since they fulfil (4.113) on-shell. In general, any linear combination of the form

$$\mathrm{T}^{(\mathrm{GEN})}_{\mathbf{k}+1}(\mathbf{w}) = \sum_{j=0}^{\mathbf{k}+1} c_j^{(\mathbf{k})}\,(\partial_w\phi)^{\mathbf{k}+1-j}\,\partial_w^j\phi\,, \quad \Theta^{(\mathrm{GEN})}_{\mathbf{k}-1}(\mathbf{w}) = \sum_{j=0}^{\mathbf{k}-1} \bar{c}_j^{(\mathbf{k})}\,(\partial_{\bar{w}}\phi)^{\mathbf{k}-1-j}\,\partial_{\bar{w}}^j\phi\,, \tag{4.126}$$

automatically defines a conserved current with spin $\mathbf{k}$.

For simplicity, we will consider the sets (4.124)-(4.125) separately instead of the generic combinations (4.126). Considering the set of currents (4.124), namely setting $\mathrm{T}_{\mathbf{k}+1} = \mathrm{T}^{(\mathrm{POW})}_{\mathbf{k}+1}$, $\Theta_{\mathbf{k}-1} = \Theta^{(\mathrm{POW})}_{\mathbf{k}-1}$ and the same for their complex conjugates in (4.116), and replacing the derivatives of ϕ in $\mathbf{w}$ coordinates with those in $\mathbf{z}$ coordinates according to (4.84)

$$\partial_w f = \partial_z f - \frac{1}{4\tau}\left(\frac{-1+\mathcal{S}}{\partial_{\bar{z}}\phi}\right)^2 \partial_{\bar{z}} f\,, \quad \partial_{\bar{w}} f = \partial_{\bar{z}} f - \frac{1}{4\tau}\left(\frac{-1+\mathcal{S}}{\partial_z\phi}\right)^2 \partial_z f\,, \quad \forall\, f : \mathbb{C}\to\mathbb{R}\,, \tag{4.127}$$

with $\mathcal{S}$ defined in (4.46), one finds

$$\mathrm{T}^{(\mathrm{POW})}_{\mathbf{k}+1}(\mathbf{z},\tau) = -\frac{(\partial_z\phi)^{\mathbf{k}+1}}{2\mathcal{S}}\left(\frac{2}{1+\mathcal{S}}\right)^{\mathbf{k}-1}\,, \quad \Theta^{(\mathrm{POW})}_{\mathbf{k}-1}(\mathbf{z},\tau) = -\tau\,\frac{(\partial_z\phi)^{\mathbf{k}+1}(\partial_{\bar{z}}\phi)^2}{2\mathcal{S}}\left(\frac{2}{1+\mathcal{S}}\right)^{\mathbf{k}+1}\,, \tag{4.128}$$

which coincides with the result first obtained in [11] through perturbative computations.

For the set of currents (4.125), we can again derive exactly the associated T$\bar{\mathrm{T}}$-deformed currents, however their analytic expressions are more and more involved as $\mathbf{k}$ increases and we were unable to find a compact formula valid for arbitrary spin $\mathbf{k} \in \mathbb{N}$. We report here, as an example, the level $\mathbf{k} = 3$ deformed current of the hierarchy (4.125):

$$\begin{aligned} \mathrm{T}_4(\mathbf{z},\tau) &= -\frac{(\partial_z\phi)^2}{2\mathcal{S}}\left(\frac{(\mathcal{S}-1)^4\partial_{\bar{z}}^2\phi - 16\tau^2(\partial_z\phi)^4\,\partial_z^2\phi}{4\tau(\mathcal{S}-1)\,(\mathcal{S}^2+1)\,(\partial_{\bar{z}}\phi)^3}\right)^2\,, \\ \Theta_2(\mathbf{z},\tau) &= -\frac{(\mathcal{S}-1)^2}{8\tau\,\mathcal{S}}\left(\frac{(\mathcal{S}-1)^4\partial_{\bar{z}}^2\phi - 16\tau^2(\partial_z\phi)^4\,\partial_z^2\phi}{4\tau(\mathcal{S}-1)\,(\mathcal{S}^2+1)\,(\partial_{\bar{z}}\phi)^3}\right)^2\,. \end{aligned} \tag{4.129}$$

Finally, it is important to stress that the method presented in this section is completely general and can be applied to a generic integrable model, provided the stress-energy tensor and the conserved currents are known in terms of fundamental fields. We have explicitly computed the T$\bar{\mathrm{T}}$-deformed conserved currents with $\mathbf{k} = 3, 5$ for the sine-Gordon model. Again the resulting expressions are extremely complicated and we will not present them here.

4.7 T$\bar{\mathrm{T}}$-deformed soliton solutions in the sine-Gordon model

The aim of this section is to develop a procedure to construct either analytically or numerically (in cases where the analytic solution is out of reach) solutions to the T$\bar{\mathrm{T}}$−deformed EoMs for a bosonic theory by means of the coordinate transformation Ψ_τ. Notice that we can directly write a solution to the $\mathrm{T}\bar{\mathrm{T}}-$deformed EoM, *i.e.* $\phi_0(\mathbf{z},\tau)$,

$$\partial_z\left(\frac{\partial_{\bar{z}}\phi}{\mathcal{S}^V(\mathbf{z},\tau)}\right)+\partial_{\bar{z}}\left(\frac{\partial_z\phi}{\mathcal{S}^V(\mathbf{z},\tau)}\right)=\frac{V'}{4\mathcal{S}^V(\mathbf{z},\tau)}\left(\frac{1+\mathcal{S}^V(\mathbf{z},\tau)}{1-\tau V}\right)^2 , \tag{4.130}$$

in terms of the corresponding solution to the original EoM, *i.e.* $\phi_0(\mathbf{z})$, as

$$\boxed{\phi_0(\mathbf{z},\tau)=\phi_0\big(\Psi_\tau(\mathbf{z})\big)\ .} \tag{4.131}$$

Now, setting $\mathbf{z}$ as a function of τ such that $\frac{d}{d\tau}\Psi_\tau(\mathbf{z})=0$ and differentiating both sides of (4.131) w.r.t. τ one gets

$$\frac{d}{d\tau}\phi_0(\mathbf{z},\tau)=0\quad\longrightarrow\quad\partial_\tau\phi_0(\mathbf{z},\tau)+(\partial_\tau z)\,\partial_z\phi_0(\mathbf{z},\tau)+(\partial_\tau\bar{z})\,\partial_{\bar{z}}\phi_0(\mathbf{z},\tau)=0\ . \tag{4.132}$$

To fix the derivatives $\partial_\tau z$ and $\partial_\tau\bar{z}$, we first integrate the inverse Jacobian as expressed in (4.89), getting

$$z=w(\mathbf{z})+2\tau\int^{\mathbf{w}=\Psi_\tau(\mathbf{z})}\bar{\mathrm{T}}(\mathbf{w})\,d\bar{w}+\bar{\Theta}(\mathbf{w})\,dw\ ,\quad \bar{z}=\bar{w}(\mathbf{z})+2\tau\int^{\mathbf{w}=\Psi_\tau(\mathbf{z})}\mathrm{T}(\mathbf{w})\,dw+\Theta(\mathbf{w})\,d\bar{w}\ . \tag{4.133}$$

Then, using the definition (4.112) and the coordinate independence property of differential forms, we arrive to

$$\partial_\tau z=2\int^{\mathbf{z}}\bar{\mathfrak{I}}_1\ ,\quad \partial_\tau\bar{z}=2\int^{\mathbf{z}}\mathfrak{I}_1\ . \tag{4.134}$$

In conclusion, a generic solution to the $\mathrm{T}\bar{\mathrm{T}}-$deformed EoM (4.130) fulfils the following differential equation

$$\boxed{\partial_\tau\phi_0(\mathbf{z},\tau)+2\left(\int^{\mathbf{z}}\bar{\mathfrak{I}}_1\right)\partial_z\phi_0(\mathbf{z},\tau)+2\left(\int^{\mathbf{z}}\mathfrak{I}_1\right)\partial_{\bar{z}}\phi_0(\mathbf{z},\tau)=0\ .} \tag{4.135}$$

Again we consider the sine-Gordon model as a study case, and we try to reconstruct the deformed versions of some typical soliton configurations. The idea is to start from a specific solution of the sine-Gordon EoM in the coordinates $\mathbf{w}$, *i.e.* $\phi_0(\mathbf{w})$, and implement the coordinate transformation to get the corresponding $\mathrm{T}\bar{\mathrm{T}}-$deformed version in the coordinates $\mathbf{z}$ according to (4.131).

For this strategy to work, we need to derive the map $\mathbf{w}=\Psi_\tau^{\mathrm{sG}}(\mathbf{z})$ evaluated on the solution ϕ_0. To do this, we first solve the sets of differential equations

$$\begin{cases}\dfrac{\partial z(\mathbf{w})}{\partial w}=1-\tau V_0^{\mathrm{sG}}\\[4pt] \dfrac{\partial z(\mathbf{w})}{\partial\bar{w}}=-\tau\left(\partial_{\bar{w}}\phi_0\right)^2\end{cases},\quad\begin{cases}\dfrac{\partial\bar{z}(\mathbf{w})}{\partial w}=-\tau\left(\partial_w\phi_0\right)^2\\[4pt] \dfrac{\partial\bar{z}(\mathbf{w})}{\partial\bar{w}}=1-\tau V_0^{\mathrm{sG}}\end{cases},\quad V_0^{\mathrm{sG}}=V^{\mathrm{sG}}\left(\phi_0\right)\ , \tag{4.136}$$

for the functions $\mathbf{z}$ in the variables $\mathbf{w}$, which gives the inverse map $\mathbf{z}=\left(\Psi_\tau^{\mathrm{sG}}\right)^{-1}(\mathbf{w})$. Then by inverting the latter relation we find the desired map $\mathbf{w}=\Psi_\tau^{\mathrm{sG}}(\mathbf{z})$.

Although this strategy has general validity, the integration of (4.136) and the inversion of the map cannot be done analytically for all the solutions. In the following sections, we shall apply this method to some of the simplest soliton solutions, namely the 1−kink, the two-kink and the stationary breather. As we will see, while for the 1−kink solution both the integration and the inversion processes can be easily done in analytic way, for the others the inversion of the map must be performed numerically. For configurations involving more than two solitons, the situation is even more complicated and not even the integration of (4.136) is anymore feasible analytically.

Observation 28. Throughout the following sections, all the computations will be carried on in Minkowsky signature using light-cone coordinates $\tilde{\mathbf{x}}_{\text{M}} = (x^+, x^-)$ and $\tilde{\mathbf{y}}_{\text{M}} = (y^+, y^-)$. Instead, the plots are displayed in cartesian coordinates $\mathbf{x}_{\text{M}} = (x, t_x) = (x, t)$. Moving from Euclidean to Minkowsky signature, equations (4.136) become

$$\begin{cases} \dfrac{\partial x^+(\tilde{\mathbf{y}}_{\text{M}})}{\partial y^+} = 1 - \tau V_0^{\text{sG}} \\ \dfrac{\partial x^+(\tilde{\mathbf{y}}_{\text{M}})}{\partial y^-} = -\tau \left(\dfrac{\partial \phi_0}{\partial y^-}\right)^2 \end{cases} , \quad \begin{cases} \dfrac{\partial x^-(\tilde{\mathbf{y}}_{\text{M}})}{\partial y^+} = -\tau \left(\dfrac{\partial \phi_0}{\partial y^+}\right)^2 \\ \dfrac{\partial x^-(\tilde{\mathbf{y}}_{\text{M}})}{\partial y^-} = 1 - \tau V_0^{\text{sG}} \end{cases} . \tag{4.137}$$

4.7.1 The 1−kink solution

Let us start with the 1−kink solution moving with velocity v. In light-cone coordinates $\tilde{\mathbf{y}}_{\text{M}}$ it is

$$\phi_{\text{1-kink}}(\tilde{\mathbf{y}}_{\text{M}}) = 4 \arctan\left(e^{\frac{m}{\beta}(a_+ y^+ + a_- y^-)}\right) , \quad a_\pm = a^{\pm 1} , \quad a = \sqrt{\frac{1-v}{1+v}} . \tag{4.138}$$

With the identification $\phi_0(\tilde{\mathbf{y}}_{\text{M}}) = \phi_{\text{1-kink}}(\tilde{\mathbf{y}}_{\text{M}})$, equations (4.137) can be easily integrated yielding

$$x^\pm(\tilde{\mathbf{y}}_{\text{M}}) \quad = \quad y^\pm - 4\tau a_\pm \frac{m}{\beta} \tanh\left[\frac{m}{\beta}\left(a_+ y^+ + a_- y^-\right)\right] , \tag{4.139}$$

where the constants of integration are fixed in a way such that the map reduces to the identity at $\tau = 0$, *i.e.* $\tilde{\mathbf{x}}_{\text{M}} = (\Psi_0^{\text{sG}})^{-1}(\tilde{\mathbf{y}}_{\text{M}}) = \tilde{\mathbf{y}}_{\text{M}}$. Using the fact that

$$\frac{m}{\beta}\left(a_+ y^+ + a_- y^-\right) = \log\left(\tan\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{y}}_{\text{M}})}{4}\right)\right) , \tag{4.140}$$

which descends immediately from (4.138), expressions (4.139) become

$$x^\pm(\tilde{\mathbf{y}}_{\text{M}}) = y^\pm + 4\tau a_\mp \frac{m}{\beta} \cos\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{y}}_{\text{M}})}{2}\right) , \tag{4.141}$$

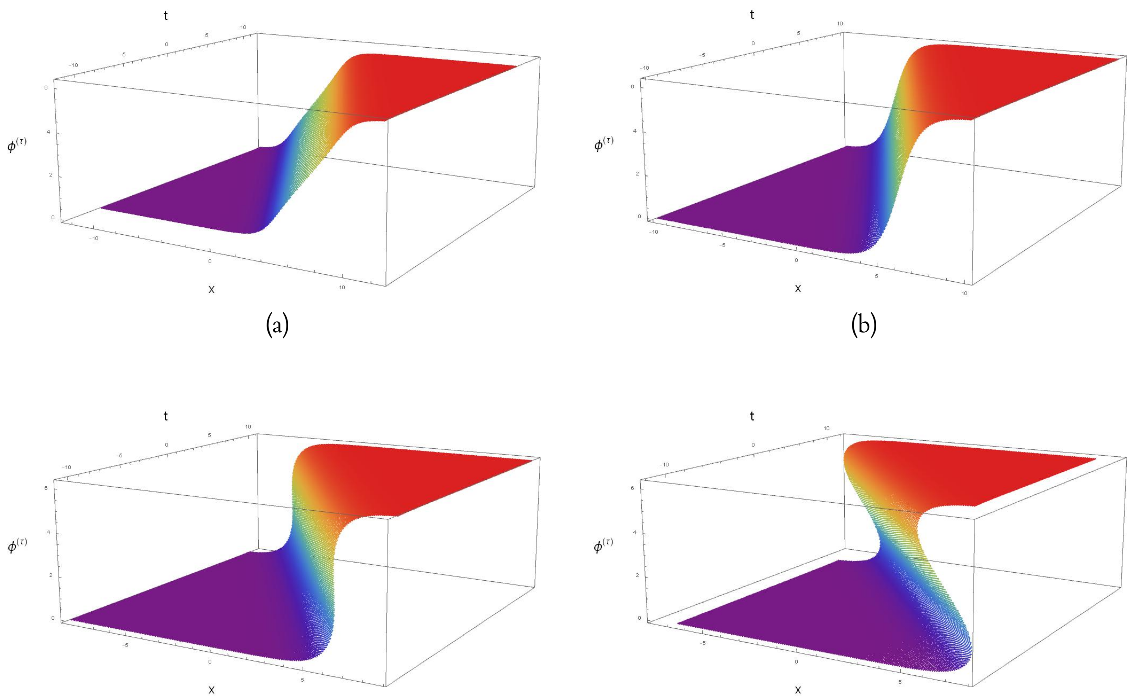
which are easily inverted as

$$y^\pm(\tilde{\mathbf{x}}_{\text{M}}) = x^\pm - 4\tau a_\mp \frac{m}{\beta} \cos\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau)}{2}\right) . \tag{4.142}$$

Finally, plugging (4.142) into (4.138) we find

$$\boxed{\frac{m}{\beta}\left(a_+ x^+ + a_- x^-\right) = 8\tau \frac{m^2}{\beta^2} \cos\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau)}{2}\right) + \log\left(\tan\left(\frac{\phi_{\text{1-kink}}(\tilde{\mathbf{x}}_{\text{M}}, \tau)}{4}\right)\right) ,} \tag{4.143}$$

which is exactly the same implicit relation defining the deformed 1−kink solution found in section 4.1 by explicitly solving the EoM of the T$\bar{\text{T}}$−deformed sine-Gordon model.



(a) (b) (c) (d)

Figure 4.3: The $T\bar{T}$-deformed moving 1−kink solution ($m = \beta = 1$, $a = 2$), for different values of the perturbation parameter τ. Figure 4.3b represents the undeformed solution. Figure 4.3a corresponds to $\tau = -1/4$, while Figures 4.3c and 4.3d correspond to $\tau = 1/8$ and $\tau = 1/3$, respectively. Notice that at $\tau = 1/8$ a shock-wave singularity occurs.

Observation 29. The coordinate transformation provide an interesting way to look at the emergence of the critical value $\tau^\star = \frac{1}{8}$, derived in section 4.1. In fact, it can be seen to arise as a singularity of the determinant of the Jacobian. Setting $m = \beta = 1$, one has

$$\begin{aligned} \det\left[\mathcal{J}\right] &= \left(1 - \tau V^{\text{sG}}(\phi_{\text{1-kink}}(\tilde{\mathbf{y}}_{\text{M}}))\right)^2 - \left(\tau \frac{\partial \phi_{\text{1-kink}}}{\partial y^+}(\tilde{\mathbf{y}}_{\text{M}}) \frac{\partial \phi_{\text{1-kink}}}{\partial y^-}(\tilde{\mathbf{y}}_{\text{M}})\right)^2 \\ &= 1 - 8\tau \,\text{sech}^2\left(a_+ y^+ + a_- y^-\right) , \end{aligned} \tag{4.144}$$

which implies

$$\det\left[\mathcal{J}\right] = 0 \quad \Longleftrightarrow \quad a_+ y^+ + a_- y^- = \pm \,\text{arccosh}\left(\sqrt{8\tau}\right) \quad \Longrightarrow \quad \tau \geq \frac{1}{8} , \tag{4.145}$$

imposing the reality of the argument function $a_+ y^+ + a_- y^-$. In conclusion we find again $\tau^\star = \frac{1}{8}$.

4.7.2 The two-kink solution

Let us consider now the solution which describes the scattering between two kinks moving with velocities v_1 and v_2

$$\phi_{\text{2-kink}}(\tilde{\mathbf{y}}_{\text{M}}) = 4\arctan\left(\frac{a_1+a_2}{a_2-a_1}\frac{e^{\frac{m}{\beta}(a_{1,+}y^++a_{1,-}y^-+k_1)}-e^{\frac{m}{\beta}(a_{2,+}y^++a_{2,-}y^-+k_2)}}{1+e^{\frac{m}{\beta}(a_{1,+}y^++a_{1,-}y^-+k_1)}e^{\frac{m}{\beta}(a_{2,+}y^++a_{2,-}y^-+k_2)}}\right) , \tag{4.146}$$

where again $a_{i,\pm} = a_i^{\pm 1}$, $a_i = \sqrt{\frac{1-v_i}{1+v_i}}$, $(i = 1,2)$, and $k_i \in \mathbb{R}$, $(i = 1,2)$, are constant phases. Plugging $\phi_0(\tilde{\mathbf{y}}_{\text{M}}) = \phi_{\text{2-kink}}(\tilde{\mathbf{y}}_{\text{M}})$ in (4.137), one immediately realises that the resulting sets of differential equations are more complicated to integrate compared to the 1−kink case.

To simplify the computation, it is useful to parametrise the solutions $\tilde{\mathbf{x}}_{\text{M}}(\tilde{\mathbf{y}}_{\text{M}})$ of (4.137) in terms of the combinations

$$u^i(\tilde{\mathbf{y}}_{\text{M}}) = \frac{m}{\beta}\left(a_{i,+}y^+ + a_{i,-}y^- + k_i\right) , \quad (i=1,2) . \tag{4.147}$$

Performing the change of variables $\mathbf{u}(\tilde{\mathbf{y}}_{\text{M}})$, with $\mathbf{u} = (u^1, u^2)$, one has

$$\begin{cases} \frac{\partial x^\pm}{\partial u^1} = \frac{\beta}{m}\frac{a_1}{a_1^2-a_2^2}\left(\frac{\partial x^\pm}{\partial y^+} - a_2^2\frac{\partial x^\pm}{\partial y^-}\right) \\ \frac{\partial x^\pm}{\partial u^2} = -\frac{\beta}{m}\frac{a_2}{a_1^2-a_2^2}\left(\frac{\partial x^\pm}{\partial y^+} - a_1^2\frac{\partial x^\pm}{\partial y^-}\right) \end{cases} , \tag{4.148}$$

and plugging (4.137) into (4.148) with the identification $\phi_0(\tilde{\mathbf{y}}_{\text{M}}) \equiv \phi_{\text{2-kink}}(\tilde{\mathbf{y}}_{\text{M}})$, we obtain two sets of differential equations which can be solved for $\tilde{\mathbf{x}}_{\text{M}}(\mathbf{u})$, giving

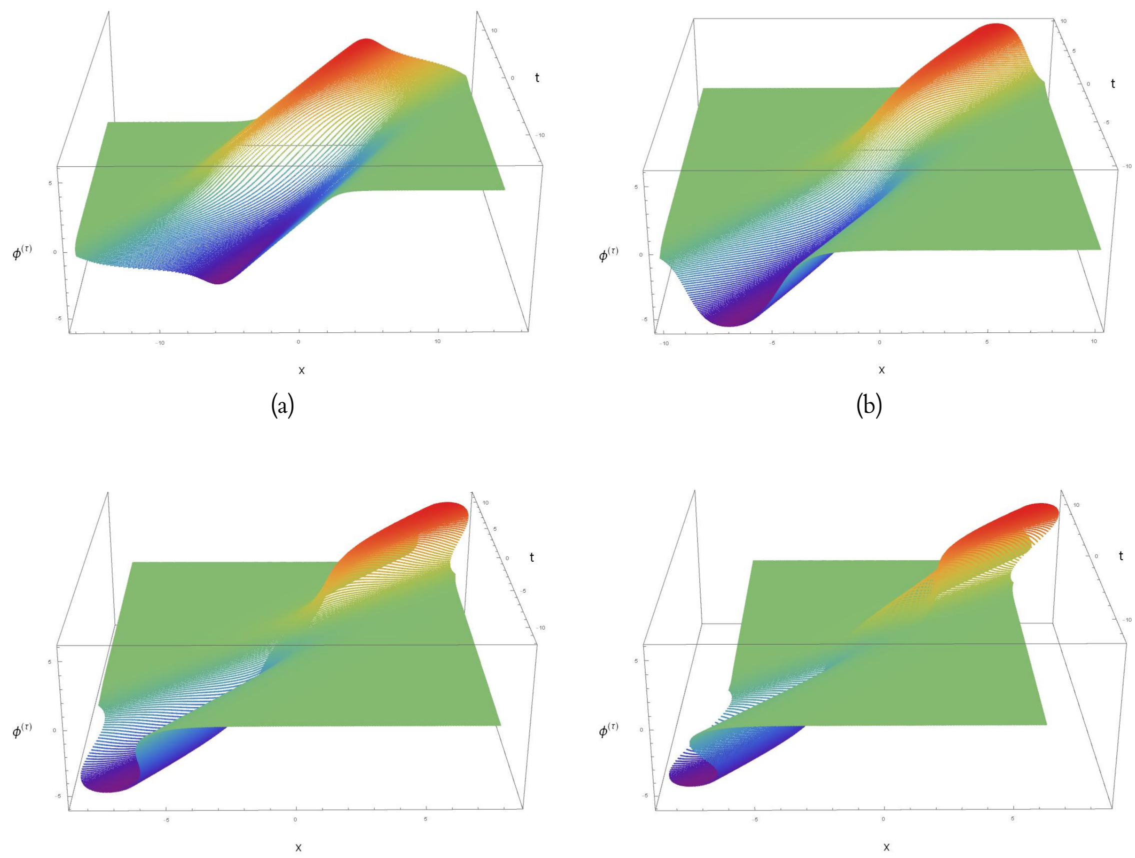
$$x^\pm(\mathbf{u}) = \pm\frac{\beta}{m}\frac{a_1a_2\left(a_{2,\mp}u^1 - a_{1,\mp}u^2\right)}{a_1^2-a_2^2} \pm 4\tau\frac{m}{\beta}\frac{\left(a_1^2-a_2^2\right)\left(a_{1,\mp}\tanh u^1 - a_{2,\mp}\tanh u^2\right)}{a_1^2+a_2^2-2a_1a_2\left(\operatorname{sech} u^1 \operatorname{sech} u^2 + \tanh u^1 \tanh u^2\right)} . \tag{4.149}$$

As in the previous section, the constants of integration in (4.149) are fixed by imposing that the map reduces to the identity at $\tau = 0$, *i.e.* $\tilde{\mathbf{x}}_{\text{M}} = \left(\Psi_0^{\text{sG}}\right)^{-1}(\tilde{\mathbf{y}}_{\text{M}}) = \tilde{\mathbf{x}}_{\text{M}}$. The last step to complete is the inversion of the relations (4.149) for $\mathbf{u}(\mathbf{z})$. Since this is analytically very complicated, we resort to numerical inversion. In Figure 4.4 the deformed solution $\phi_{\text{2-kink}}(\mathbf{z},\tau)$ is reported for different values of τ. The picture is quite similar to the 1−kink case. In fact, for negative values of τ (Figure 4.4a) the solution stretches w.r.t. the undeformed one (Figure 4.4b), while for positive values of τ (Figures 4.4c and 4.4d) it bends and again it becomes multi-valued. Unlike the 1−kink case, here it is not possible to find analytically the critical value of τ corresponding to the shock singularity.

4.7.3 The stationary breather

Another interesting solution is the stationary breather, *i.e.* the envelope speed is $v = 0$,

$$\phi_{\text{breather}}(\tilde{\mathbf{y}}_{\text{M}}) = 4\arctan\left(\tan\psi\,\frac{\sin\left(-\frac{m}{\beta}(y^+-y^-)\cos\psi + k^-\right)}{\cosh\left(\frac{m}{\beta}(y^++y^-)\sin\psi + k^+\right)}\right) , \tag{4.150}$$



(a) (b)

(c) (d)

Figure 4.4: The $T\bar{T}$-deformed two-kink solution ($m = \beta = 1\,,\ a_1 = 2\,,\ a_2 = 3$), for different values of the perturbation parameter τ. Figure 4.4b represents the undeformed solution. Figure 4.4a corresponds to $\tau = -1/4$, while Figures 4.4c and 4.4d correspond to τ, *i.e.* $\tau = 1/10$ and $\tau = 1/6$, respectively.

where ψ is a parameter related to the period T of one full oscillation via $T = \frac{2\pi}{\cos\psi}$ and $k^\pm$ are constant phases. Following the same strategy used in the two-kink case, we parametrise the solutions $\tilde{\mathbf{x}}_{\mathrm{M}}(\tilde{\mathbf{y}}_{\mathrm{M}})$ of (4.137) in terms of the combinations

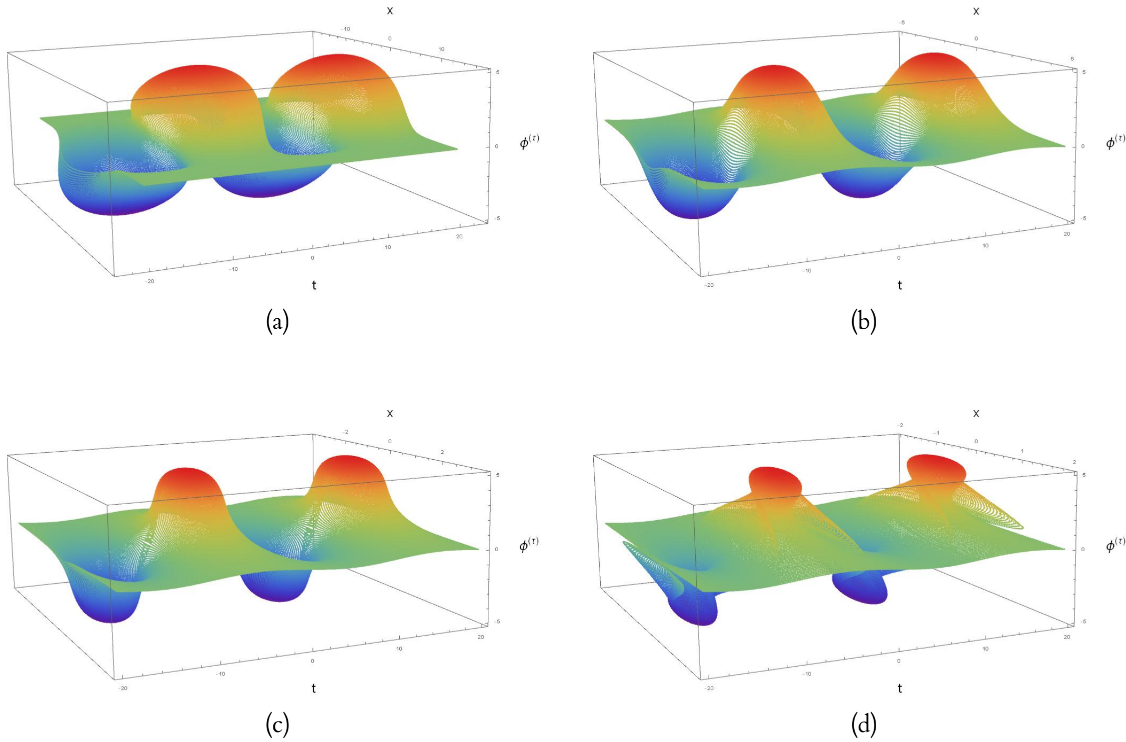
$$v^\pm(\tilde{\mathbf{y}}_{\mathrm{M}}) = \pm\frac{m}{\beta}(y^+ \pm y^-)\sin\psi + k^\pm\ . \tag{4.151}$$

Performing the change of variables $\mathbf{v}(\tilde{\mathbf{y}}_{\mathrm{M}})$, with $\mathbf{v} = (v^+, v^-)$ one finds

$$\begin{cases} \frac{\partial x^\pm}{\partial v^+} = \frac{\beta}{m}\frac{1}{2\sin\psi}\left(\frac{\partial x^\pm}{\partial y^+} + \frac{\partial x^\pm}{\partial y^-}\right) \\ \frac{\partial x^\pm}{\partial v^-} = \frac{\beta}{m}\frac{1}{2\cos\psi}\left(-\frac{\partial x^\pm}{\partial y^+} + \frac{\partial x^\pm}{\partial y^-}\right) \end{cases}\ , \tag{4.152}$$

and again plugging (4.137) into (4.152) with the identification $\phi_0(\tilde{\mathbf{y}}_{\mathrm{M}}) = \phi_{\text{breather}}(\tilde{\mathbf{y}}_{\mathrm{M}})$, one gets two sets of differential equations which can be solved for $\tilde{\mathbf{x}}_{\mathrm{M}}(\mathbf{v})$ giving

$$x^\pm(\mathbf{v}) = \frac{\beta}{m}\left(\frac{v^+}{2\sin\psi} \mp \frac{v^-}{2\cos\psi}\right) - 8\tau\frac{m}{\beta}\sin\psi\frac{\cos v^-}{\cosh v^+}\frac{\sec v^-\sinh v^+ \pm \operatorname{sech} v^+ \sin v^- \tan\psi}{1 + \left(\tan\psi\sin v^-\operatorname{sech} v^+\right)^2}\ , \tag{4.153}$$



(a) (b) (c) (d)

Figure 4.5: The $T\bar{T}$-deformed stationary breather solution with envelope speed $v = 0$ ($m = \beta = 1\,,\ \psi = \frac{2}{5}\pi$), for different values of the perturbation parameter τ. Figure 4.5b represents the undeformed solution, Figure 4.5a corresponds to $\tau = -1/2$, while Figures 4.5c and 4.5d correspond to $\tau = 1/10$ and $\tau = 1/5$, respectively.

where the constants of integration in (4.153) are again fixed according to the $\tau = 0$ initial condition. As in the two-kink case, also here the inversion of the relations (4.153) to get $\mathbf{v}(\tilde{\mathbf{x}}_{\mathrm{M}})$ is performed numerically. The deformed solution $\phi_{\text{breather}}(\tilde{\mathbf{x}}_{\mathrm{M}}, \tau)$ is displayed in Figure 4.5 for different values of τ. The result is similar to the previous cases: the solution stretches for negative values of τ (Figure 4.5a) and it bends for positive values of τ (Figure 4.5c and 4.5d) w.r.t. the undeformed one (Figure 4.5b). However, notice that in this case the wave-breaking occurs in both positive and negative directions of τ, and consequently the solution becomes multi-valued (Figures 4.5a and 4.5d) for $|\tau|$ sufficiently large.

4.8 Critical phenomena in the classical solutions

In this section we will discuss the emergence of critical phenomena in the classical solutions, *i.e.* the shock-wave singularity and the Hagedorn transition, and comment on the relations among them, using as a guide example the $T\bar{T}$-deformed elliptic solution (4.20) of the sine-Gordon model derived in section 4.1. For simplicity, we consider the stationary case ($\alpha = 1\ ,\ \beta = 0$) and we set $\rho = 1/\kappa > 0$ and

$m = \beta = 1$. The solution (4.20) is then

$$x = \frac{1}{2\sqrt{\rho}} \left[(\rho + 4\tau)\, \mathbf{F}\left(\frac{\phi(x)}{2} \middle| -\rho \right) - 8\tau\, \mathbf{E}\left(\frac{\phi(x)}{2} \middle| -\rho \right) \right] . \tag{4.154}$$

Due to the following properties of the elliptic functions

$$\begin{aligned} \mathbf{F}(z + n\pi\,|\gamma) &= \mathbf{F}(z\,|\gamma) + 2n\,\mathbf{K}(\gamma) , \\ \mathbf{E}(z + n\pi\,|\gamma) &= \mathbf{E}(z\,|\gamma) + 2n\,\mathbf{E}(\gamma) , \quad (z, \gamma \in \mathbb{C} ,\ n \in \mathbb{Z}) , \end{aligned} \tag{4.155}$$

where $\mathbf{E}(z)$ and $\mathbf{K}(z)$ are complete elliptic integrals of the first kind, the solution $\phi(x)$ can be interpreted as a stationary 1-kink with twisted boundary conditions

$$\phi(x + R) = \phi(x) + 2\pi , \tag{4.156}$$

living on a cylinder of radius R, which depend on ρ and τ through

$$R = \frac{1}{\sqrt{\rho}} \big((\rho + 4\tau)\ \mathbf{K}(-\rho) - 8\tau\, \mathbf{E}(-\rho) \big) . \tag{4.157}$$

The strategy is to keep fixed the cylinder on which the solution is defined, and to consider $\rho = \rho(\tau, R)$ as a function of τ and R, defined implicitly through (4.157). Differentiating both sides of (4.157) w.r.t. τ and R, and solving for $\partial_\tau \rho$ and $\partial_R \rho$ one finds

$$\partial_\tau \rho = -\frac{8\rho\,(1+\rho)\big(2\mathbf{E}(-\rho) - \mathbf{K}(-\rho)\big)}{(\rho + 4\tau)\,\mathbf{E}(-\rho)} , \quad \partial_R \rho = \frac{2\rho^{3/2}\,(1+\rho)}{(\rho + 4\tau)\,\mathbf{E}(-\rho)} . \tag{4.158}$$

Let us now compute the total energy of the solution (4.20) on the cylinder. First of all, the components of the Hilbert stress-energy tensor $T^{\mu\nu}(\mathbf{x}, \tau)$ are

$$T^{22}(\mathbf{x},\tau) \equiv \mathcal{H}(\mathbf{x},\tau) = \frac{V^{\text{sG}}}{1 - \tau V^{\text{sG}}} + \frac{1 + \tau(1 - \tau V^{\text{sG}})\,\phi_x^2 - \mathcal{S}^{\text{sG}}}{2\mathcal{S}^{\text{sG}}\,\tau(1 - \tau V^{\text{sG}})} = \frac{2\,(2 + \rho V^{\text{sG}})}{\rho\,(1 - 2\tau V^{\text{sG}}) - 4\tau} , \tag{4.159}$$

$$T^{12}(\mathbf{x},\tau) = T_{21}(\mathbf{x},\tau) \equiv \mathcal{P}(\mathbf{x},\tau) = -\frac{\phi_t\,\phi_x}{2\mathcal{S}^{\text{sG}}} = 0 , \tag{4.160}$$

$$T^{11}(\mathbf{x},\tau) = -\frac{V^{\text{sG}}}{1 - \tau V^{\text{sG}}} - \frac{1 - \tau(1 - \tau V^{\text{sG}})\,\phi_t^2 - \mathcal{S}^{\text{sG}}}{2\mathcal{S}^{\text{sG}}\,\tau(1 - \tau V^{\text{sG}})} = \frac{4}{\rho + 4\tau} , \tag{4.161}$$

where we used the following expressions for ϕ_t and ϕ_x derived from (4.154)

$$\phi_t = 0 , \quad \phi_x = \frac{2\sqrt{\rho}\,\sqrt{4 + \rho V^{\text{sG}}}}{\rho\,(1 - 2\tau V^{\text{sG}}) - 4\tau} , \tag{4.162}$$

and

$$\mathcal{S}^{\text{sG}} = \sqrt{1 + \tau(1 - \tau V^{\text{sG}})\,(\phi_x^2 - \phi_t^2)} = \frac{\rho + 4\tau}{\rho\,(1 - 2\tau V^{\text{sG}}) - 4\tau} . \tag{4.163}$$

Notice that the apparent pole singularity at $\tau = \frac{1}{V^{\rm sG}}$ in $T^{11}(\mathbf{x},\tau)$ and $T^{22}(\mathbf{x},\tau)$ disappears once (4.162) is used in (4.159) and (4.161). Finally the energy and momentum at finite volume R are

$$E(R,\tau) = \int_{x_0}^{x_0+R} \mathcal{H}(\mathbf{x},\tau)\,dx = \int_{\phi(x_0)=0}^{\phi(x_0+R)=2\pi} \frac{\mathcal{H}(\mathbf{x},\tau)}{\phi_x}\,d\phi = \frac{4}{\sqrt{\rho}}\left(2\,\mathbf{E}(-\rho) - \mathbf{K}(-\rho)\right) , \tag{4.164}$$

$$P(R,\tau) = \int_{x_0}^{x_0+R} \mathcal{P}(\mathbf{x},\tau)\,dx = \int_{\phi(x_0)=0}^{\phi(x_0+R)=2\pi} \frac{\mathcal{P}(\mathbf{x},\tau)}{\phi_x}\,d\phi = 0 , \tag{4.165}$$

$$K(R,\tau) = \int_{x_0}^{x_0+R} T^{11}(\mathbf{x},\tau)\,dx = \int_{\phi(x_0)=0}^{\phi(x_0+R)=2\pi} \frac{T^{11}(\mathbf{x},\tau)}{\phi_x}\,d\phi = \frac{4R}{\rho+4\tau} , \tag{4.166}$$

where $x_0 = 0 \,(\mathrm{mod}\, R)$. From (4.158), (4.164) and (4.166) one can prove that the energy fulfils the Burgers equation (2.45) with $P_n = 0$

$$\begin{aligned}\partial_\tau E(R,\tau) &= \frac{1}{2}E(R,\tau)\,\partial_R E(R,\tau) \\ &= -\frac{1}{R}\,\det\left[\int_{x_0}^{x_0+R} T^{\mu\nu}(\mathbf{x},\tau)\,dx\right] = -\int_{x_0}^{x_0+R} \det\left[T^{\mu\nu}(\mathbf{x},\tau)\right]dx ,\end{aligned} \tag{4.167}$$

where the last equality in (4.167) is the classical analogous of the factorization property (2.32) of the $\mathrm{T\bar{T}}$ operator.

Since the energy $E(R,\tau)$ fulfils a Burgers equation, it is expected to have a square root-type singularity.[1] The critical radius $R_c(\tau)$ corresponds to a value of R such that the first derivative of $E(R,\tau)$ w.r.t. R diverges. One easily checks that

$$\partial_R E(R,\tau) = -\frac{4}{\rho+4\tau} , \tag{4.169}$$

which is divergent at the radius $R_c(\tau)$ defined through the equation

$$\rho\big(\tau, R_c(\tau)\big) = -4\tau . \tag{4.170}$$

According to (4.157) and (4.164), the critical radius and the corresponding energy turn out to be

$$R_c(\tau) = 4\sqrt{-\tau}\,\mathbf{E}\,(4\tau) \ , \quad E_c(\tau) \equiv E(R_c,\tau) = \frac{2}{\sqrt{-\tau}}\big(\mathbf{K}(4\tau) - 2\,\mathbf{E}(4\tau)\big) . \tag{4.171}$$

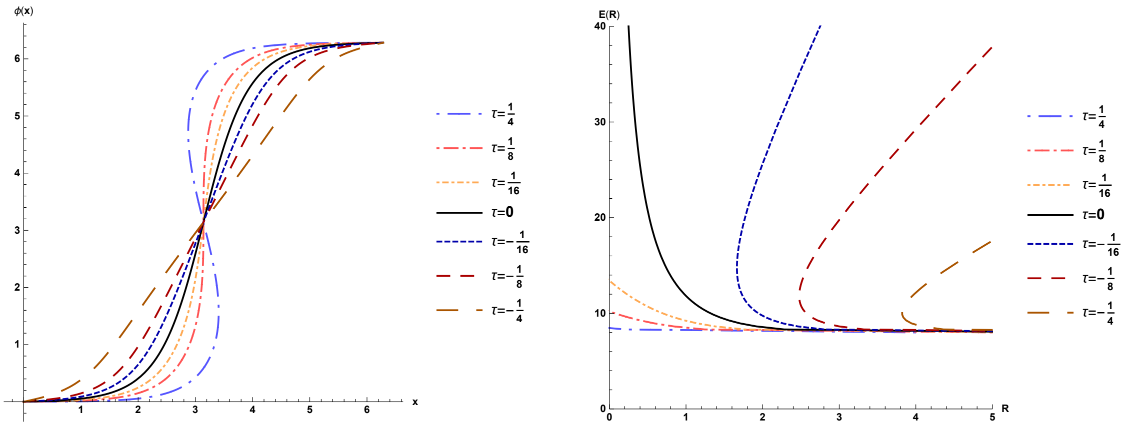
To find the behavior of $E(R,\tau)$ as a function of R close to the branch singularity R_c, we first expand R and $E(R,\tau)$ in powers of the small quantity $\varepsilon = \rho + 4\tau$

$$\begin{aligned} R - R_c &= \frac{R_c}{128\tau^2\,(1-4\tau)}\,\varepsilon^2 + \mathcal{O}(\varepsilon^3) , \\ E(R,\tau) - E_c(\tau) &= \frac{R_c}{16\tau^2\,(1-4\tau)}\,\varepsilon + \mathcal{O}(\varepsilon^2) , \end{aligned} \tag{4.172}$$

[1] It is worth to notice that the unperturbed energy $E(R)$ displays the following divergent behavior for small R

$$E(R) = \frac{\pi^2}{R} + 2R - \frac{R^3}{2\pi^2} + \mathcal{O}\left(R^7\right) , \tag{4.168}$$

which resembles that of a CFT.



(a) (b)

Figure 4.6: The kink solution to the T$\bar{\text{T}}$-deformed sG model on a cylinder of radius R (a) and the corresponding energies as functions of R (b).

then, removing ε, one finds

$$E(R,\tau) - E_c(\tau) = \pm \frac{\sqrt{R_c}}{\tau\sqrt{2-8\tau}}\sqrt{R-R_c} + \mathcal{O}\left(R-R_c\right) \ , \tag{4.173}$$

which gives a square root branch point at R_c for the energy.

Now we move on to discuss briefly the effect of the shock-wave singularities of the deformed solution on the Hamiltonian density. First of all, we must determine the critical values of τ at which the shock-wave singularities appear. In analogy with the computation performed for the 1−kink solution in section 4.7.1, we identify the zeros of Det $[\mathcal{J}^{-1}]$:

$$\text{Det}\left[\mathcal{J}^{-1}\right] = 0 \quad \Longleftrightarrow \quad x = \frac{\sqrt{\rho}}{2}\,\text{dn}^{-1}\left(\pm\sqrt{\frac{\rho+4\tau}{8\tau}}\,\middle|-\rho\right) \ , \tag{4.174}$$

where $\text{dn}^{-1}(z\,|\gamma)$ is the inverse of the Jacobi elliptic function $\text{dn}(z\,|\gamma)$. From the reality properties of $\text{dn}^{-1}(z\,|\gamma)$ it follows that x is real for

$$\boxed{\rho > 0 \quad \text{and} \quad \tau_1^* = \frac{\rho}{4+8\rho} < \tau < \frac{\rho}{4} = \tau_2^* \ ,} \tag{4.175}$$

where the critical values[1] τ_1^* and τ_2^* corresponds to shock-wave singularities of the solution at $\phi = \pi$ and $\phi = 0, 2\pi$, respectively. The Hamiltonian density (4.159) is indeed singular when

$$\tau = \frac{\rho}{4+2\rho V} = \frac{\rho}{4+8\rho\,\sin^2\left(\phi/2\right)} \ , \tag{4.176}$$

[1]Notice that, in the $\rho \to \infty$ limit, one recovers the 1-kink solution, and the critical range reduce to $\tau > \tau_1^* = \frac{1}{8}$, since $\tau_2^* \to \infty$.

which corresponds to the range of singular values of τ (4.175) as ϕ interpolates from 0 to 2π. However, it is important to stress that these branching singularities do not affect the total energy (4.164), which remains smooth in τ, since the singularities cancel out when dividing by ϕ_x in (4.164). In Figure 4.6 we displayed the behaviour of $\phi(x)$ (Figure 4.6a) and $E(R,\tau)$ (Figure 4.6b) for various values of τ. We see that the shock-wave phenomenon in the classical solution and the square root-type singularity in the total energy occur at positive and negative values of τ, respectively, therefore they are not related.

Part II

Generalised irrelevant deformations

Chapter 5

Generalised coordinate transformations

In this chapter, we introduce a family of irrelevant deformations which extend the $\mathrm{T}\bar{\mathrm{T}}$ deformation in the framework of IFTs. They are built from the higher conserved charges belonging to the hierarchy of a generic IFT and, to construct them, we adopt a different perspective compared to the $\mathrm{T}\bar{\mathrm{T}}$ example. In fact, in section 5.1, we define the action of these deformations at classical level, through a change of coordinates which generalises that associated to $\mathrm{T}\bar{\mathrm{T}}$. The main advantage of this geometric construction is the possibility to reconstruct the whole integrable structure of the deformed theories using the method described in section 4.6. Assuming that the theory is invariant under a parity transformation, in section 5.1, we consider a second complementary family of deformations which, due to the simpler structure, turns out to be extremely useful to identify the phase factors that generate both these family of deformations at the quantum level. Finally, in section 5.3, extending this general setup also to conserved charges associated to $U(1)_L \times U(1)_R$ Kac-Moody symmetry, we manage to include the $\mathrm{J}\bar{\mathrm{T}}$ deformation.

5.1 Generalised coordinate transformations: the $\mathrm{s} > 0$ case

From section 4.5, we know that the $\mathrm{T}\bar{\mathrm{T}}$ deformation can be interpreted, at the classical level, as a coordinate transformation which involve the components of the stress-energy tensor (see (4.89)-(4.90)). In addition we showed that the Hessian matrix associated to the coordinate transformation is symmetric on-shell, thus implying the commutativity of the second mixed partial derivatives. A somehow natural way to generalise the coordinate transformation Ψ_τ, is the following

$$\boxed{\Psi_\tau^{(s)} \,:\, \mathbb{C} \to \mathbb{C} \,:\, \mathbf{z} \to \mathbf{w} = \Psi_\tau^{(s)}(\mathbf{z}) \;, \quad s = |\mathbf{s}| > 0 \;,} \tag{5.1}$$

with associated Jacobian and inverse Jacobian given by

$$\mathcal{J}^{(s)} = \begin{pmatrix} \partial w & \partial \bar{w} \\ \bar{\partial} w & \bar{\partial} \bar{w} \end{pmatrix} = \frac{1}{\Delta^{(s)}(\mathbf{w})} \begin{pmatrix} 1 + 2\tau\, \Theta_{s-1}(\mathbf{w}) & -2\tau\, \mathrm{T}_{s+1}(\mathbf{w}) \\ -2\tau\, \bar{\mathrm{T}}_{s+1}(\mathbf{w}) & 1 + 2\tau\, \bar{\Theta}_{s-1}(\mathbf{w}) \end{pmatrix} , \tag{5.2}$$

$$\left(\mathcal{J}^{(s)}\right)^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1 + 2\tau\, \bar{\Theta}_{s-1}(\mathbf{w}) & 2\tau\, \mathrm{T}_{s+1}(\mathbf{w}) \\ 2\tau\, \bar{\mathrm{T}}_{s+1}(\mathbf{w}) & 1 + 2\tau\, \Theta_{s-1}(\mathbf{w}) \end{pmatrix} , \tag{5.3}$$

where we defined

$$\Delta^{(s)}(\mathbf{w}) = \det\left[\left(\mathcal{J}^{(s)}\right)^{-1}\right] = \left(1+2\tau\,\Theta_{s-1}(\mathbf{w})\right)\left(1+2\tau\,\bar{\Theta}_{s-1}(\mathbf{w})\right) - 4\tau^2\,\mathrm{T}_{s+1}(\mathbf{w})\,\bar{\mathrm{T}}_{s+1}(\mathbf{w}) \;, \tag{5.4}$$

and $s=1$ corresponds to the $\mathrm{T}\bar{\mathrm{T}}$ deformation, *i.e.* $\mathcal{J}^{(1)} = \mathcal{J}$.

Observation 30. The coordinate transformation (5.1) descends directly from (4.65) with the substitution $\boldsymbol{T} \to \boldsymbol{T}_s$ (see appendix A), which, in terms of the rescaled chiral components translates into

$$\{\mathrm{T},\Theta,\bar{\mathrm{T}},\Theta\} \equiv \{\mathrm{T}_2,\Theta_0,\bar{\mathrm{T}}_2,\Theta_0\} \to \{\mathrm{T}_{s+1},\Theta_{s-1},\bar{\mathrm{T}}_{s+1},\bar{\Theta}_{s-1}\} \;. \tag{5.5}$$

Notice that, in the definition of (5.2)-(5.3), we performed the substitution (5.5) only at the level of the original theory. Later on, we will prove in specific cases that (5.2) can be written in terms of the deformed chiral components exactly as (4.90), with the replacements (5.5).

Using the continuity equations (4.113) in (5.3), one finds that the second mixed partial derivatives again coincide when evaluated on the EoMs

$$\begin{aligned}
\partial_{\bar{w}}\left(\partial_w z\right) = 2\tau\,\partial_{\bar{w}}\bar{\Theta}_{s-1}(\mathbf{w}) \underset{\text{EoMs}\,(\mathbf{w})}{\equiv} 2\tau\,\partial_w\bar{\mathrm{T}}_{s+1}(\mathbf{w}) = \partial_w\left(\partial_{\bar{w}} z\right) \;, \\
\partial_{\bar{w}}\left(\partial_w \bar{z}\right) = 2\tau\,\partial_{\bar{w}}\mathrm{T}_{s+1}(\mathbf{w}) \underset{\text{EoMs}\,(\mathbf{w})}{\equiv} 2\tau\,\partial_w\Theta_{s-1}(\mathbf{w}) = \partial_w\left(\partial_{\bar{w}} \bar{z}\right) \;,
\end{aligned} \tag{5.6}$$

therefore, the coordinate transformation (5.1) is well defined on-shell.

Now we want to derive the integrable structure associated to the deformation induced by $\Psi_\tau^{(s)}$. To do that, we use the same strategy described in section 4.6, namely, we perform the coordinate transformation in the 1−forms (4.112). Using

$$\begin{pmatrix} dw \\ d\bar{w} \end{pmatrix} = \left(\mathcal{J}^{(s)}\right)^{\mathrm{T}} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} \;, \tag{5.7}$$

in (4.112) we obtain

$$\begin{aligned}
\mathfrak{I}_k &= \mathrm{T}^{(s)}_{k+1}(\mathbf{z},\tau)\,dz + \Theta^{(s)}_{k-1}(\mathbf{z},\tau)\,d\bar{z} \\
&= \frac{\mathrm{T}_{k+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) + 2\tau\left[\mathrm{T}_{k+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\Theta_{s-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) - \Theta_{k-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\mathrm{T}_{s+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\right]}{\Delta^{(s)}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)}\,dz \\
&+ \frac{\Theta_{k-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) + 2\tau\left[\Theta_{k-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\bar{\Theta}_{s-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) - \mathrm{T}_{k+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\bar{\mathrm{T}}_{s+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\right]}{\Delta^{(s)}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)}\,d\bar{z} \;,
\end{aligned} \tag{5.8}$$

from which we read off directly the components of the deformed currents

$$\begin{aligned}
\mathrm{T}^{(s)}_{k+1}(\mathbf{z},\tau) &= \frac{\mathrm{T}_{k+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) + 2\tau\left[\mathrm{T}_{k+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\Theta_{s-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) - \Theta_{k-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\mathrm{T}_{s+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\right]}{\Delta^{(s)}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)} \;, \\
\Theta^{(s)}_{k-1}(\mathbf{z},\tau) &= \frac{\Theta_{k-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) + 2\tau\left[\Theta_{k-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\bar{\Theta}_{s-1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right) - \mathrm{T}_{k+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\bar{\mathrm{T}}_{s+1}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\right]}{\Delta^{(s)}\left(\Psi_\tau^{(s)}(\mathbf{z})\right)} \;,
\end{aligned} \tag{5.9}$$

which fulfil the continuity equations

$$\partial_{\bar{z}} \mathrm{T}^{(s)}_{k+1}(\mathbf{z},\tau) = \partial_z \Theta^{(s)}_{k-1}(\mathbf{z},\tau) \ , \quad \partial_z \bar{\mathrm{T}}^{(s)}_{k+1}(\mathbf{z},\tau) = \partial_{\bar{z}} \bar{\Theta}^{(s)}_{k-1}(\mathbf{z},\tau) \ . \tag{5.10}$$

Then, following the convention (A.16), the level-k Hamiltonian and momentum density are[1]

$$\begin{aligned} \mathcal{H}^{(s)}_k(\mathbf{z},\tau) &= \mathcal{I}^{(s)}_k(\mathbf{z},\tau) + \bar{\mathcal{I}}^{(s)}_k(\mathbf{z},\tau) \\ &= \frac{1}{\Delta^{(s)}\big(\Psi^{(s)}_\tau(\mathbf{z})\big)} \Big[\mathcal{H}_k\big(\Psi^{(s)}_\tau(\mathbf{z})\big) + 2\tau\big(\mathcal{T}_{+,k}\big(\Psi^{(s)}_\tau(\mathbf{z})\big)\,\bar{\mathcal{T}}_{-,s}\big(\Psi^{(s)}_\tau(\mathbf{z})\big) + \text{c.c.}\big)\Big] \ , \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathcal{P}^{(s)}_k(\mathbf{z},\tau) &= \mathcal{I}^{(s)}_k(\mathbf{z},\tau) - \bar{\mathcal{I}}^{(s)}_k(\mathbf{z},\tau) \\ &= \frac{1}{\Delta^{(s)}\big(\Psi^{(s)}_\tau(\mathbf{z})\big)} \Big[\mathcal{P}_k\big(\Psi^{(s)}_\tau(\mathbf{z})\big) + 2\tau\big(\mathcal{T}_{-,k}\big(\Psi^{(s)}_\tau(\mathbf{z})\big)\,\bar{\mathcal{T}}_{-,s}\big(\Psi^{(s)}_\tau(\mathbf{z})\big) - \text{c.c.}\big)\Big] \ , \end{aligned} \tag{5.12}$$

where we defined the combinations

$$\mathcal{T}_{\pm,n} = \mathrm{T}_{n+1} \pm \bar{\Theta}_{n-1} \ , \quad \bar{\mathcal{T}}_{\pm,n} = \bar{\mathrm{T}}_{n+1} \pm \Theta_{n-1} \ , \tag{5.13}$$

Integrating (5.11) and (5.12) over the whole volume we find

$$\begin{aligned} \int \mathcal{H}^{(s)}_k(\mathbf{z},\tau)\, dz\wedge d\bar{z} &= \int \Big[\mathcal{H}_k(\mathbf{w}) + 2\tau\big(\mathcal{T}_{+,k}(\mathbf{w})\,\bar{\mathcal{T}}_{-,s}(\mathbf{w}) + \text{c.c.}\big)\Big]\, dw\wedge d\bar{w} \\ &= \int \mathcal{H}_k(\mathbf{w})\, dw\wedge d\bar{w} - 2\tau \int (\mathfrak{I}_k + \bar{\mathfrak{I}}_k)\wedge(\mathfrak{I}_s - \bar{\mathfrak{I}}_s) \ , \end{aligned} \tag{5.14}$$

$$\begin{aligned} \int \mathcal{P}^{(s)}_k(\mathbf{z},\tau)\, dz\wedge d\bar{z} &= \int \Big[\mathcal{P}_k(\mathbf{w}) + 2\tau\big(\mathcal{T}_{-,k}(\mathbf{w})\,\bar{\mathcal{T}}_{-,s}(\mathbf{w}) - \text{c.c.}\big)\Big]\, dw\wedge d\bar{w} \\ &= \int \mathcal{P}_k(\mathbf{w})\, dw\wedge d\bar{w} - 2\tau \int (\mathfrak{I}_k - \bar{\mathfrak{I}}_k)\wedge(\mathfrak{I}_s - \bar{\mathfrak{I}}_s) \ . \end{aligned} \tag{5.15}$$

We now interpret the result (5.14) as follows: $\mathcal{H}^{(s)}_k(\mathbf{z},\tau)\, dz\wedge d\bar{z}$ coincides with the corresponding bare quantity $\mathcal{H}_k(\mathbf{w})\, dw\wedge d\bar{w}$ deformed by the operator

$$\mathbf{\Phi}_{k,s}(\mathbf{w})\, dw\wedge d\bar{w} = -2\,(\mathfrak{I}_k + \bar{\mathfrak{I}}_k)\wedge(\mathfrak{I}_s - \bar{\mathfrak{I}}_s) \ , \tag{5.16}$$

$$\mathbf{\Phi}_{k,s}(\mathbf{w}) = 2\left(\mathcal{T}_{+,k}(\mathbf{w})\,\bar{\mathcal{T}}_{-,s}(\mathbf{w}) + \bar{\mathcal{T}}_{+,k}(\mathbf{w})\,\mathcal{T}_{-,s}(\mathbf{w})\right) \ , \tag{5.17}$$

together with a non-trivial dressing given by the change of variables (5.1). In the $s = 1$ case, *i.e.* the $\mathrm{T}\bar{\mathrm{T}}$ example, the operator (5.17) associated to the $k = 1$ Hamiltonian becomes

$$\mathbf{\Phi}_{1,1}(\mathbf{w}) = \mathrm{T}(\mathbf{w})\,\bar{\mathrm{T}}(\mathbf{w}) - \Theta^2(\mathbf{w}) \ , \tag{5.18}$$

which, but for the change of coordinates, coincides with the bare $\mathrm{T}\bar{\mathrm{T}}$ operator. In analogy with the $\mathrm{T}\bar{\mathrm{T}}$ result [22, 24], one may be tempted to interpret (5.17) as the perturbing operator of the level-k Hamiltonian. However, the coordinate transformation (5.1) also introduces $O(\tau)$ corrections which

[1] In the following, "c.c." denotes the replacement $\{\mathrm{T}_{k+1},\Theta_{k-1}\} \to \{\bar{\mathrm{T}}_{k+1},\bar{\Theta}_{k-1}\}$ and viceversa.

can, in principle, completely spoil this naive picture. In addition, even when the initial theory is a CFT and the bare operator (5.17) is completely symmetric in s and k

$$\mathbf{\Phi}_{k,s}(\mathbf{w}) = \mathrm{T}_{k+1}(w)\,\bar{\mathrm{T}}_{s+1}(\bar{w}) + \bar{\mathrm{T}}_{k+1}(\bar{w})\,\mathrm{T}_{s+1}(w)\ , \tag{5.19}$$

the change of variables spoils the $s \leftrightarrow k$ symmetry, since it involves only the level-s currents. Examples of this phenomena will neatly emerge from the study of the $\mathbf{s} < 0$ deformations of the free massless boson theory.

5.2 Generalised coordinate transformations: the $\mathbf{s} < 0$ case

To gain precise information about the spectrum, it turns out to be particularly convenient to first extend the current setup to the $\mathbf{s} < 0$ cases. We consider another coordinate transformation

$$\boxed{\Psi_\tau^{(s')}\ :\ \mathbb{C} \to \mathbb{C}\ :\ \mathbf{z} \to \mathbf{w} = \Psi_\tau^{(s')}(\mathbf{z})\ , \quad s' = -|\mathbf{s}| < 0\ ,} \tag{5.20}$$

whose associated Jacobian and inverse Jacobian are obtained from (5.2)-(5.3) with the replacement $s \to s'$, which results in

$$\mathcal{J}^{(s')} = \frac{1}{\Delta^{(s')}(\mathbf{w})}\begin{pmatrix} 1 + 2\tau\,\bar{\mathrm{T}}_{s+1}(\mathbf{w}) & -2\tau\,\bar{\Theta}_{s-1}(\mathbf{w}) \\ -2\tau\,\Theta_{s-1}(\mathbf{w}) & 1 + 2\tau\,\mathrm{T}_{s+1}(\mathbf{w}) \end{pmatrix}\ , \tag{5.21}$$

$$\left(\mathcal{J}^{(s')}\right)^{-1} = \begin{pmatrix} 1 + 2\tau\,\mathrm{T}_{s+1}(\mathbf{w}) & 2\tau\,\bar{\Theta}_{s-1}(\mathbf{w}) \\ 2\tau\,\Theta_{s-1}(\mathbf{w}) & 1 + 2\tau\,\bar{\mathrm{T}}_{s+1}(\mathbf{w}) \end{pmatrix}\ , \tag{5.22}$$

where $s = |\mathbf{s}|$ and

$$\Delta^{(s')}(\mathbf{w}) = \det\left[\left(\mathcal{J}^{(s')}\right)^{-1}\right] = \left(1 + 2\tau\,\mathrm{T}_{s+1}(\mathbf{w})\right)\left(1 + 2\tau\,\bar{\mathrm{T}}_{s+1}(\mathbf{w})\right) - 4\tau^2\,\Theta_{s-1}(\mathbf{w})\,\bar{\Theta}_{s-1}(\mathbf{w})\ . \tag{5.23}$$

We arrived to (5.21)-(5.22) using the reflection relations (A.11), which corresponds to the following reflection property at the level of the 1-forms

$$\mathfrak{I}_{s'} = \bar{\mathfrak{I}}_s\ , \quad \bar{\mathfrak{I}}_{s'} = \mathfrak{I}_s\ , \tag{5.24}$$

or, equivalently,

$$\mathcal{I}_{s'} = \bar{\mathcal{I}}_s\ , \quad \bar{\mathcal{I}}_{s'} = \mathcal{I}_s\ . \tag{5.25}$$

Using the continuity equations (4.113), it is easy to verify that (5.22) fulfils again the conditions (5.6), therefore (5.20) is a consistent coordinate transformation on-shell.

Repeating the computations (5.8)-(5.15) using (5.21)-(5.22) one finds that (5.1) become

$$\begin{aligned} \mathrm{T}^{(s')}_{k+1}(\mathbf{z},\tau) &= \frac{\mathrm{T}_{k+1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big) + 2\tau\left[\mathrm{T}_{k+1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)\,\bar{\mathrm{T}}_{s+1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big) - \Theta_{k-1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)\,\bar{\Theta}_{s-1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)\right]}{\Delta^{(s')}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)}\ , \\ \Theta^{(s')}_{k-1}(\mathbf{z},\tau) &= \frac{\Theta_{k-1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big) + 2\tau\left[\Theta_{k-1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)\,\mathrm{T}_{s+1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big) - \mathrm{T}_{k+1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)\,\Theta_{s-1}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)\right]}{\Delta^{(s')}\big(\Psi_\tau^{(s')}(\mathbf{z})\big)}\ , \end{aligned} \tag{5.26}$$

while (5.14)-(5.15) become

$$\begin{aligned}\int \mathcal{H}_k^{(s')}(\mathbf{z},\tau)\, dz \wedge d\bar{z} &= \int \Big[\mathcal{H}_k(\mathbf{w}) - 2\tau\big(\mathcal{T}_{+,k}(\mathbf{w})\,\bar{\mathcal{T}}_{-,s}(\mathbf{w}) + \text{c.c.}\big)\Big]\, dw \wedge d\bar{w} \\ &= \int \mathcal{H}_k(\mathbf{w})\, dw \wedge d\bar{w} + 2\tau \int (\mathfrak{I}_k + \bar{\mathfrak{I}}_k) \wedge (\mathfrak{I}_s - \bar{\mathfrak{I}}_s)\ , \end{aligned} \tag{5.27}$$

$$\begin{aligned}\int \mathcal{P}_k^{(s')}(\mathbf{z},\tau)\, dz \wedge d\bar{z} &= \int \Big[\mathcal{P}_k(\mathbf{w}) - 2\tau\big(\mathcal{T}_{-,k}(\mathbf{w})\,\bar{\mathcal{T}}_{-,s}(\mathbf{w}) - \text{c.c.}\big)\Big]\, dw \wedge d\bar{w} \\ &= \int \mathcal{P}_k(\mathbf{w})\, dw \wedge d\bar{w} + 2\tau \int (\mathfrak{I}_k - \bar{\mathfrak{I}}_k) \wedge (\mathfrak{I}_s - \bar{\mathfrak{I}}_s)\ , \end{aligned} \tag{5.28}$$

which are formally equal to (5.14)-(5.15), except for the sign of τ. However, the positive and negative spin sectors are not simply related by a change of sign in the coupling constant τ as the comparison between (5.14)-(5.15) and (5.27)-(5.28) would naively suggests. By studying in detail the $\mathbf{s} < 0$ deformations of the massless free boson model (see section 5.2.1 below), the difference w.r.t. the $\mathbf{s} = 1$ perturbation, *i.e.* the $\mathrm{T\bar{T}}$, clearly emerges.

5.2.1 The classical Burgers-type equations

In this section, we consider deformations of the massless free boson theory induced by the coordinate transformations (5.20), and we derive the higher conserved currents of the deformed models. As already discussed in section 4.6, the most general level-$\mathbf{k}$ current of the hierarchy can be expressed in the form (4.126). While the structure of the deformed currents does not emerge clearly by working with the general combination (4.126), we observed that the subset (4.124) is analytically much easier to treat since it does not mix with the others. This property allows to obtain compact expressions for the deformed currents which are formally identical to the exact quantum results of section 6.3.

Using (4.124) in (5.21)-(5.22), the coordinate transformations read explicitly

$$\mathcal{J}^{(s')} = \begin{pmatrix} \frac{1}{1-\tau\,(\partial_w\phi)^{s+1}} & 0 \\ 0 & \frac{1}{1-\tau\,(\partial_{\bar{w}}\phi)^{s+1}} \end{pmatrix}\ , \tag{5.29}$$

$$\big(\mathcal{J}^{(s')}\big)^{-1} = \begin{pmatrix} 1-\tau\,(\partial_w\phi)^{s+1} & 0 \\ 0 & 1-\tau\,(\partial_{\bar{w}}\phi)^{s+1} \end{pmatrix}\ . \tag{5.30}$$

Repeating the same computation performed in section 4.6.1, we first express $(\partial_w\phi, \partial_{\bar{w}}\phi)^{\mathrm{T}}$ in terms of $(\partial_z\phi, \partial_{\bar{z}}\phi)^{\mathrm{T}}$, by solving the set of equations

$$\begin{pmatrix} \partial_z\phi \\ \partial_{\bar{z}}\phi \end{pmatrix} = \mathcal{J}^{(s')} \begin{pmatrix} \partial_w\phi \\ \partial_{\bar{w}}\phi \end{pmatrix} \quad \longleftrightarrow \quad \begin{cases} \partial_z\phi = \frac{\partial_w\phi}{1-\tau(\partial_w\phi)^{s+1}} \\ \partial_{\bar{z}}\phi = \frac{\partial_{\bar{w}}\phi}{1-\tau(\partial_{\bar{w}}\phi)^{s+1}} \end{cases} . \tag{5.31}$$

The solutions to (5.31) can be written in terms of generalized hypergeometric functions for any value of $s' = -s$ as

$$\partial_w\phi = \tilde{F}_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_z\phi)^{s+1}}{2}\right)\partial_z\phi\ , \quad \partial_{\bar{w}}\phi = \tilde{F}_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_{\bar{z}}\phi)^{s+1}}{2}\right)\partial_{\bar{z}}\phi\ , \tag{5.32}$$

with

$$\tilde{F}_n(x) = {}_nF_{n-1}\left(\frac{1}{n+1},\dots,\frac{n}{n+1};\frac{2}{n},\dots,\frac{n-1}{n},\frac{n+1}{n};x\right) \quad , \quad (n \in \mathbb{N}-\{0\}) \; . \tag{5.33}$$

Plugging (5.32) into

$$\begin{pmatrix} \partial_w f \\ \partial_{\bar{w}} f \end{pmatrix} = \left(\mathcal{J}^{(s')}\right)^{-1} \begin{pmatrix} \partial_z f \\ \partial_{\bar{z}} f \end{pmatrix} \quad , \quad \forall\, f : \mathbb{C} \to \mathbb{R} \; , \tag{5.34}$$

one finds the differential map

$$\partial_w f = \tilde{F}_s\left(-2\tau \frac{(s+1)^{s+1}}{s^s}\frac{(\partial_z\phi)^{s+1}}{2}\right)\partial_z f \; , \quad \partial_{\bar{w}} f = \tilde{F}_s\left(-2\tau \frac{(s+1)^{s+1}}{s^s}\frac{(\partial_{\bar{z}}\phi)^{s+1}}{2}\right)\partial_{\bar{z}} f \; . \tag{5.35}$$

Combining (5.35) and (5.7) (with $s' \to s$), we can derive the Lax pair of the deformed theory starting from the original one according to the observation 27. One finds

$$\begin{aligned} L^{(s')}(\mathbf{z},\tau) &= \frac{1}{2}\frac{\tilde{F}_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_z\phi)^{s+1}}{2}\right)\partial_z\phi}{1-\tau\left[\tilde{F}_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_z\phi)^{s+1}}{2}\,\partial_z\phi\right)\right]^{s+1}}\mathbf{u}_3 = \frac{\partial_z\phi}{2}\mathbf{u}_3 = L(\mathbf{z}) \; , \\ \bar{L}^{(s')}(\mathbf{z},\tau) &= -\frac{1}{2}\frac{\tilde{F}_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_{\bar{z}}\phi)^{s+1}}{2}\right)\partial_{\bar{z}}\phi}{1-\tau\left[\tilde{F}_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_{\bar{z}}\phi)^{s+1}}{2}\,\partial_{\bar{z}}\phi\right)\right]^{s+1}}\mathbf{u}_3 = -\frac{\partial_{\bar{z}}\phi}{2}\mathbf{u}_3 = \bar{L}(\mathbf{z}) \; , \end{aligned} \tag{5.36}$$

where we simply used the fact that (5.32) is a solution to (5.31). Since the deformed Lax pair coincide with the original one, it follows that the dynamics is unchanged, namely the EoM of the deformed theory is

$$\partial_z\partial_{\bar{z}}\phi = 0 \; , \tag{5.37}$$

which reflects the fact that the $\mathbf{s} < 0$ perturbations of CFTs do not mix the holomorphic and anti-holomorphic derivatives, as already emerged from (5.31). We conclude that the deformed Lagrangian density coincides with the original one

$$\mathcal{L}^{(s')}(\mathbf{z}) = \partial_z\phi\,\partial_{\bar{z}}\phi = \mathcal{L}(\mathbf{z}) \; . \tag{5.38}$$

Using the technique described in section 4.6, we can now derive the deformed currents. Plugging the differential map (5.35) into (5.26), we obtain

$$\begin{aligned} \mathrm{T}^{(s')}_{s+1}(\mathbf{z},\tau) &= -\frac{s}{2\tau\,(s+1)}\left[-1+F_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_z\phi)^{s+1}}{2}\right)\right] \; , \quad \Theta^{(s')}_{s-1}(\mathbf{z},\tau) = 0 \; , \\ \bar{\mathrm{T}}^{(s')}_{s+1}(\mathbf{z},\tau) &= -\frac{s}{2\tau\,(s+1)}\left[-1+F_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\frac{(\partial_{\bar{z}}\phi)^{s+1}}{2}\right)\right] \; , \quad \bar{\Theta}^{(s')}_{s-1}(\mathbf{z},\tau) = 0 \; , \end{aligned} \tag{5.39}$$

with

$$F_n(x) = {}_nF_{n-1}\left(-\frac{1}{n+1},\frac{1}{n+1},\dots,\frac{n-1}{n+1};\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n};x\right) \quad , \quad (n \in \mathbb{N}-\{0\}) \; , \tag{5.40}$$

where we used the following relation between generalised hypergeometric functions[1]

$$F_n(x) = \frac{1}{n}\left(-1 + \frac{(n+1)^{n+2}}{(n+1)^{n+1} + n^n\, x\left(\tilde{F}_n(x)\right)^{n+1}}\right) , \tag{5.41}$$

to trade $\tilde{F}_n(x)$ with $F_n(x)$. From (5.37), it follows that $\phi(\mathbf{z},\tau) = \varphi(z,\tau) + \bar{\varphi}(\bar{z},\tau)$ and therefore $\mathrm{T}^{(s')}_{s+1}(\mathbf{z},\tau)$ and $\bar{\mathrm{T}}^{(s')}_{s+1}(\mathbf{z},\tau)$ depend only on z and $\bar{z}$, respectively.

Observation 31. Using (5.35) in (5.29), the Jacobian can be rewritten in terms of the deformed components (5.39) as

$$\mathcal{J}^{(s')} = \begin{pmatrix} 1 - 2\tau\,\mathrm{T}^{(s')}_{s+1}(\mathbf{z},\tau) & -2\tau\,\bar{\Theta}^{(s')}_{s-1}(\mathbf{z},\tau) \\ -2\tau\,\Theta^{(s')}_{s-1}(\mathbf{z},\tau) & 1 - 2\tau\,\bar{\mathrm{T}}^{(s')}_{s+1}(\mathbf{z},\tau) \end{pmatrix} , \tag{5.42}$$

which confirms what we anticipated in the observation 30, at least in this particular case.

In terms of the combinations (A.17) we can more transparently rewrite the result as

$$\begin{aligned} \mathcal{I}^{(s')}_s(\mathbf{z},\tau) &= \frac{s}{2\tau\,(s+1)}\left[-1 + F_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\,\mathcal{I}_s(\mathbf{z})\right)\right] , \\ \bar{\mathcal{I}}^{(s')}_s(\mathbf{z},\tau) &= \frac{s}{2\tau\,(s+1)}\left[-1 + F_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\,\bar{\mathcal{I}}_s(\mathbf{z})\right)\right] . \end{aligned} \tag{5.43}$$

Quite remarkably, the latter expressions arise as solutions to simple algebraic equations of the form

$$\mathcal{I}^{(s')}_s(\mathbf{z},\tau) = \frac{\mathcal{I}_s(\mathbf{z})}{\left(1 + 2\tau\,\mathcal{I}^{(s')}_s(\mathbf{z},\tau)\right)^s} , \quad \bar{\mathcal{I}}^{(s')}_s(\mathbf{z},\tau) = \frac{\bar{\mathcal{I}}_s(\mathbf{z})}{\left(1 + 2\tau\,\bar{\mathcal{I}}^{(s')}_s(\mathbf{z},\tau)\right)^s} . \tag{5.44}$$

In general, one can show that the combinations $\mathcal{I}^{(s')}_k(\mathbf{z},\tau)$ and $\bar{\mathcal{I}}^{(s')}_k(\mathbf{z},\tau)$ of the generic level-k deformed currents are

$$\begin{aligned} \mathcal{I}^{(s')}_k(\mathbf{z},\tau) = \mathcal{I}_k(\mathbf{z})\left\{1 + \frac{s}{s+1}\left[-1 + F_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\,\mathcal{I}_s(\mathbf{z})\right)\right]\right\}^{-k} , \\ \bar{\mathcal{I}}^{(s')}_k(\mathbf{z},\tau) = \bar{\mathcal{I}}_k(\mathbf{z})\left\{1 + \frac{s}{s+1}\left[-1 + F_s\left(-2\tau\frac{(s+1)^{s+1}}{s^s}\,\bar{\mathcal{I}}_s(\mathbf{z})\right)\right]\right\}^{-k} , \end{aligned} \tag{5.45}$$

and fulfil the following equations

$$\boxed{\mathcal{I}^{(s')}_k(\mathbf{z},\tau) = \frac{\mathcal{I}_k(\mathbf{z})}{\left(1 + 2\tau\,\mathcal{I}^{(s')}_s(\mathbf{z},\tau)\right)^k} , \quad \bar{\mathcal{I}}^{(s')}_k(\mathbf{z},\tau) = \frac{\bar{\mathcal{I}}_k(\mathbf{z})}{\left(1 + 2\tau\,\bar{\mathcal{I}}^{(s')}_s(\mathbf{z},\tau)\right)^k} ,} \tag{5.46}$$

which generalise (5.44).

Before moving to the next section, let us make a few important remarks:

[1] Relation (5.41) can be easily checked expanding both sides in powers of x around $x = 0$.

Observation 32. The net effect of the $\mathbf{s} < 0$ deformations is to generate a $\tau-$dependent flow in the conserved currents of the hierarchy without affecting the dynamics. In fact the Lagrangian associated to the deformed theories coincides with the original one (cf. eq. (5.38)) and the associated canonical Hamiltonian structure is generated by

$$\mathcal{H}^{(s')}(\mathbf{x}) = \mathbb{i}\,\Pi\phi_t + \mathcal{L}^{(s')}(\mathbf{x}) = \frac{1}{4}\phi_x^2 + \Pi^2 \;, \quad \Pi = 2\mathbb{i}\,\phi_t \;. \tag{5.47}$$

The level-1 momentum and Hamiltonian density derived directly from the coordinate transformation[1]

$$\mathcal{P}^{(s')}(\mathbf{x},\tau) = \mathcal{I}_1^{(s')}(\mathbf{x},\tau) - \bar{\mathcal{I}}_1^{(s')}(\mathbf{x},\tau) \;, \quad \mathcal{H}^{(s')}(\mathbf{x},\tau) = \mathcal{I}_1^{(s')}(\mathbf{x},\tau) + \bar{\mathcal{I}}_1^{(s')}(\mathbf{x},\tau) \;, \tag{5.48}$$

and, in particular, the corresponding integrated quantities

$$P^{(s')}(R,\tau) = \int_0^R \mathcal{P}^{(s')}(\mathbf{x},\tau)\,dx \;, \quad E^{(s')}(R,\tau) = \int_0^R \mathcal{H}^{(s')}(\mathbf{x},\tau)\,dx \;, \tag{5.49}$$

do not coincide with the generators of translations in space and time, respectively, which are represented by the momentum and energy of the original theory

$$P^{(s')}(R) = \int_0^R \mathcal{P}^{(s')}(\mathbf{x})\,dx = P(R) \;, \quad E^{(s')}(R) = \int_0^R \mathcal{H}^{(s')}(\mathbf{x})\,dx = E(R) \;. \tag{5.50}$$

where $\mathcal{P}^{(s')}(\mathbf{x}) = -\pi\phi_x$ is the conserved momentum density of the original theory. In general the deformed charges

$$P_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) = \int_0^R \mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(\mathbf{x},\tau)\,dx \;, \quad E_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) = \int_0^R \mathcal{H}_{\mathbf{k}}^{(\mathbf{s})}(\mathbf{x},\tau)\,dx \;, \tag{5.51}$$

are conserved in t, but they evolve the system along "generalised space-time" directions, which differ from the original ones for $\mathbf{k} \neq \mathbf{s}$.

Observation 33. The $\mathrm{T\bar{T}}$ and the $\mathrm{J\bar{T}}$ examples discussed in [28, 72] suggest that, at least formally, the evolution equations for the quantised spectra already emerge at classical level after replacing the classical densities with the corresponding integrated quantities averaged over the volume R:

$$\begin{aligned} \mathcal{I}_k(\mathbf{x}) \longrightarrow \frac{I_k(R)}{R} = \frac{I_k^{(+)}(R)}{R} \quad &, \quad \bar{\mathcal{I}}_k(\mathbf{x}) \longrightarrow \frac{\bar{I}_k(R)}{R} = \frac{I_k^{(-)}(R)}{R} \;, \\ \mathcal{I}_k^{(\mathbf{s})}(\mathbf{x},\tau) \longrightarrow \frac{I_k^{(\mathbf{s})}(R,\tau)}{R} = \frac{I_k^{(\mathbf{s},+)}(R,\tau)}{R} \quad &, \quad \bar{\mathcal{I}}_k^{(\mathbf{s})}(\mathbf{x},\tau) \longrightarrow \frac{\bar{I}_k^{(\mathbf{s})}(R,\tau)}{R} = \frac{I_k^{(\mathbf{s},-)}(R,\tau)}{R} \;. \end{aligned} \tag{5.52}$$

where

$$I_k^{(\mathbf{s})}(R,\tau) = \int_0^R \mathcal{I}_k^{(\mathbf{s})}(\mathbf{x},\tau)\,dx \;, \quad \bar{I}_k^{(\mathbf{s})}(R,\tau) = \int_0^R \bar{\mathcal{I}}_k^{(\mathbf{s})}(\mathbf{x},\tau)\,dx \;, \tag{5.53}$$

[1] When we change the argument of the currents from $\mathbf{z}$ to $\mathbf{x}$, we implicitly move from complex to cartesian coordinates and express everything in terms of ϕ_x and Π.

and the labels $(+)$ and $(-)$ stand for the right and left light-cone component of the conserved charges, respectively (cf. section 3.3). Implementing (5.52) in (5.46) gives

$$\boxed{I_k^{(s',\pm)}(R,\tau) = \frac{R^k\, I_k^{(\pm)}(R)}{\left(R + 2\tau\, I_s^{(s',\pm)}(R,\tau)\right)^k}\ ,} \tag{5.54}$$

which coincides with the CFT quantum result (6.50) of section 6.3.

Observation 34. Although (5.54) were derived for the $\mathbf{s} < 0$ case, it is natural to conjecture that they can be extended also to $\mathbf{s} \geq 0$. From the reflection property (5.25), we find:

$$\boxed{I_k^{(s,\pm)}(R,\tau) = \frac{R^k\, I_k^{(\pm)}(R)}{\left(R + 2\tau\, I_s^{(s,\mp)}(R,\tau)\right)^k}\ ,} \tag{5.55}$$

which again match the $\mathbf{s} > 0$ CFT quantum result quoted in (6.56).

Observation 35. It is straightforward to check that (5.54)-(5.55) are solutions to the evolution equations

$$\begin{aligned} \partial_\tau I_k^{(s,\pm)}(R,\tau) &= 2 I_s^{(s,\mp)}(R,\tau)\, \partial_R I_k^{(s,\pm)}(R,\tau)\ , \quad (s>0)\ , \\ \partial_\tau I_{k'}^{(s',\pm)}(R,\tau) &= 2 I_{s'}^{(s',\pm)}(R,\tau)\, \partial_R I_{k'}^{(s',\pm)}(R,\tau)\ , \quad (s' = -s < 0)\ . \end{aligned} \tag{5.56}$$

In addition,they are also solutions to the more general equations (6.33) (with (6.36)) for the deformed quantum spectrum, which hold also for massive models. In Appendix D.1, we show that indeed equations (6.33) (with (6.36)) reduce to (5.54)-(5.55) in the CFT limit.

Observation 36. The result (5.37) shows that the coordinate transformation (5.29)-(5.30) is, on the plane, an automorphism of the space of classical solutions of the free boson theory.

5.3 s = 0 deformations and JT̄-type models

So far, we described possible extensions of the TT̄ coordinate transformation involving conserved currents of the hierarchy with Lorentz spin $\mathbf{s} \neq 0$. The $\mathbf{s} = 0$ case is somehow special since a spin zero current is not present in any IFT[1]. In this section we restrict to the massless free boson model and consider deformations generated by the spin zero currents generated by the $U(1)_L \times U(1)_R$ symmetry. In complex coordinates $\mathbf{w}$, the chiral components of these currents are

$$\begin{aligned} J_+(\mathbf{z}) &= -2\mathbb{i}\, \mathrm{T}_1(\mathbf{z}) = \mathbb{i}\, \partial_z \phi \quad , \qquad J_-(\mathbf{z}) = -2\mathbb{i}\, \Theta_{-1}(\mathbf{z}) = 0\ , \\ \bar{J}_+(\mathbf{z}) &= 2\mathbb{i}\, \bar{\mathrm{T}}_1(\mathbf{z}) = -\mathbb{i}\, \partial_{\bar{z}} \phi \quad , \qquad \bar{J}_-(\mathbf{z}) = 2\mathbb{i}\, \bar{\Theta}_{-1}(\mathbf{z}) = 0\ , \end{aligned} \tag{5.57}$$

where $\{\mathrm{T}_1(\mathbf{z}), \Theta_{-1}(\mathbf{z}), \bar{\mathrm{T}}_1(\mathbf{z}), \bar{\Theta}_{-1}(\mathbf{z})\}$ correspond to the case $\mathbf{k} = 0$ in (4.124), while the additional factor $\mathbb{i}$ in (5.57) is used to write the holomorphic and anti-holomorphic components of the stress-energy tensor in *Sugawara form* (see [29])

$$\mathrm{T}(\mathbf{z}) = \frac{1}{2}\left(J_+(\mathbf{z})\right)^2\ , \quad \bar{\mathrm{T}}(\mathbf{z}) = \frac{1}{2}\left(\bar{J}_+(\mathbf{z})\right)^2\ . \tag{5.58}$$

[1] In the sine-Gordon model, for example, there is no $\mathbf{s} = 0$ current.

Expressions (5.57) can again be interpreted as components of the closed 1-forms

$$\mathfrak{J}_0 = J_+(\mathbf{w})\, dw + J_-(\mathbf{w})\, d\bar{w} \ , \quad \bar{\mathfrak{J}}_0 = \bar{J}_+(\mathbf{w})\, d\bar{w} + \bar{J}_-(\mathbf{w})\, dw \ . \tag{5.59}$$

The cartesian components of the $U(1)$ currents are related to (5.57) through

$$\begin{aligned} J_1 &= J_+ - J_- \ , \quad J_2 = \mathbb{i}\left(J_+ + J_-\right) \ , \\ \bar{J}_1 &= \bar{J}_+ - \bar{J}_- \ , \quad \bar{J}_2 = -\mathbb{i}\left(\bar{J}_+ + \bar{J}_-\right) \ , \end{aligned} \tag{5.60}$$

and they fulfil the continuity equations

$$\partial_\mu J_\mu(\mathbf{x}) = 0 \ , \quad \partial_\mu \bar{J}_\mu(\mathbf{x}) = 0 \ . \tag{5.61}$$

From (5.61), it follows that the quantities

$$Q(R) = \int_0^R J_2(\mathbf{x})\, dx \ , \quad \bar{Q}(R) = \int_0^R \bar{J}_2(\mathbf{x})\, dx \ , \tag{5.62}$$

are the conserved charges associated to the $U(1)_L \times U(1)_R$ symmetry.

Following the same spirit of the $\mathbf{s} \neq 0$ deformations discussed in sections 5.1 and 5.2, we want to construct a coordinate transformation built out of the currents (5.57). The most general choice giving rise to a well defined coordinate transformation, *i.e.* the associated Hessian matrix is symmetric, is the following $4-$parameters transformation

$$\Psi^{(0)}_{\vec{\tau}} \ : \ \mathbb{C} \to \mathbb{C} \ : \ \mathbf{z} \to \mathbf{w} = \Psi^{(0)}_{\vec{\tau}}(\mathbf{z}) \ , \quad \vec{\tau} = \left(\tau^{(1)}, \ldots, \tau^{(4)}\right) \ , \tag{5.63}$$

with Jacobian and inverse Jacobian given by

$$\mathcal{J}^{(0)} = \frac{1}{\Delta^{(0)}(\mathbf{w})} \begin{pmatrix} 1 + \tau^{(4)}\, \partial_{\bar{w}}\phi & -\tau^{(2)}\, \partial_w \phi \\ -\tau^{(3)}\, \partial_{\bar{w}}\phi & 1 + \tau^{(1)}\, \partial_w \phi \end{pmatrix} \ , \tag{5.64}$$

$$\left(\mathcal{J}^{(0)}\right)^{-1} = \begin{pmatrix} 1 + \tau^{(1)}\, \partial_w \phi & \tau^{(2)}\, \partial_w \phi \\ \tau^{(3)}\, \partial_{\bar{w}}\phi & 1 + \tau^{(4)}\, \partial_{\bar{w}}\phi \end{pmatrix} \ , \tag{5.65}$$

where we defined

$$\Delta^{(0)}(\mathbf{w}) = 1 + \tau^{(1)}\, \partial_w\phi + \tau^{(4)}\, \partial_{\bar{w}}\phi + \left(\tau^{(1)}\tau^{(4)} - \tau^{(2)}\tau^{(3)}\right) \partial_w\phi\, \partial_{\bar{w}}\phi \ . \tag{5.66}$$

Since the general case is quite cumbersome to treat, we restrict our analysis to some particular limits.

$\boxed{\textbf{Case } \tau^{(1)} = \tau^{(4)} = 0 \mid \tau^{(2)} = \tau^{(3)} = -\tau}$

With this choice of the parameters, expressions (5.64)-(5.65) become

$$\mathcal{J}^{(0)} = \frac{1}{1 - \tau^2\, \partial_w\phi\, \partial_{\bar{w}}\phi} \begin{pmatrix} 1 & \tau\, \partial_w\phi \\ \tau\, \partial_{\bar{w}}\phi & 1 \end{pmatrix} \ , \tag{5.67}$$

$$\left(\mathcal{J}^{(0)}\right)^{-1} = \begin{pmatrix} 1 & -\tau\, \partial_w\phi \\ -\tau\, \partial_{\bar{w}}\phi & 1 \end{pmatrix} \ , \tag{5.68}$$

which correspond to the limit $s \to 0$ in (5.2)-(5.3). Following the procedure described in detail in section 4.5, one can easily derive the differential map

$$\partial_w f = \partial_z f - \frac{\tau\,\partial_z\phi}{1+\tau\,\partial_{\bar z}\phi}\partial_{\bar z} f \;, \quad \partial_{\bar w} f = \partial_{\bar z} f - \frac{\tau\,\partial_{\bar z}\phi}{1+\tau\,\partial_z\phi}\partial_z f \;, \quad \forall\, f\,:\,\mathbb{C}\to\mathbb{R}\,, \tag{5.69}$$

from which the deformed EoMs are

$$\partial_z\partial_{\bar z}\phi = \tau\,\frac{\partial_z\phi\,\partial_{\bar z}^2\phi\,(1+\tau\,\partial_z\phi)+\partial_{\bar z}\phi\,\partial_z^2\phi\,(1+\tau\,\partial_{\bar z}\phi)}{1+\tau\,(\partial_z\phi+\partial_{\bar z}\phi)+2\tau^2\,\partial_z\phi\,\partial_{\bar z}\phi}\;. \tag{5.70}$$

Setting $s = 0$ in (5.1) and using (5.69), we can access to the full deformed integrable hierarchy. In particular, we are interested in the expression of the deformed level-0 and level-1 currents. One finds

$$\mathrm{T}_1^{(0)}(\mathbf{z},\tau) = -\frac{1}{2}\frac{\partial_z\phi\,(1+\tau\,\partial_z\phi)}{1+\tau\,(\partial_z\phi+\partial_{\bar z}\phi)}\;, \quad \Theta_{-1}^{(0)}(\mathbf{z},\tau) = -\frac{1}{2}\frac{\tau\,\partial_z\phi\,\partial_{\bar z}\phi}{1+\tau\,(\partial_z\phi+\partial_{\bar z}\phi)}\;, \tag{5.71}$$

and

$$\begin{aligned}\mathrm{T}^{(0)}(\mathbf{z},\tau) &= -\frac{1}{2}\frac{(\partial_z\phi)^2\,(1+\tau\,\partial_z\phi)}{(1+\tau\,\partial_{\bar z}\phi)\,(1+\tau\,(\partial_z\phi+\partial_{\bar z}\phi))}\;,\\ \Theta^{(0)}(\mathbf{z},\tau) &= -\frac{1}{2}\frac{\tau\,\partial_{\bar z}\phi\,(\partial_z\phi)^2}{(1+\tau\,\partial_{\bar z}\phi)\,(1+\tau\,(\partial_z\phi+\partial_{\bar z}\phi))}\;.\end{aligned} \tag{5.72}$$

Observation 37. Plugging (5.69) into (5.67), we notice that the Jacobian can be rewritten in terms of the deformed level-0 currents (5.71) as

$$\mathcal{J}^{(0)} = \begin{pmatrix} 1-2\tau\,\bar\Theta_{-1}^{(0)}(\mathbf{z},\tau) & -2\tau\,\mathrm{T}_1^{(0)}(\mathbf{z},\tau)\\ -2\tau\,\bar{\mathrm{T}}_1^{(0)}(\mathbf{z},\tau) & 1-2\tau\,\Theta_{-1}^{(0)}(\mathbf{z},\tau)\end{pmatrix}\;, \tag{5.73}$$

which again confirms the statement made in observation 30.

Writing (5.71) and (5.72) in terms of the combinations (A.17), one finds the following relations

$$\begin{aligned}\mathcal{I}_0^{(0)}(\mathbf{z},\tau) &= \frac{1}{2}\partial_z\phi = \mathcal{I}_0(\mathbf{z}) \quad , \quad \bar{\mathcal{I}}_0^{(0)}(\mathbf{z},\tau) = \frac{1}{2}\partial_{\bar z}\phi = \bar{\mathcal{I}}_0(\mathbf{z})\;,\\ \mathcal{I}_1^{(0)}(\mathbf{z},\tau) &= \frac{1}{2}\frac{(\partial_z\phi)^2}{1+\tau\,\partial_{\bar z}\phi} = \frac{\mathcal{I}_1(\mathbf{z})}{1+2\tau\,\bar{\mathcal{I}}_0^{(0)}(\mathbf{z},\tau)} \quad , \quad \bar{\mathcal{I}}_1^{(0)}(\mathbf{z},\tau) = \frac{1}{2}\frac{(\partial_{\bar z}\phi)^2}{1+\tau\,\partial_z\phi} = \frac{\bar{\mathcal{I}}_1(\mathbf{z})}{1+2\tau\,\mathcal{I}_0^{(0)}(\mathbf{z},\tau)}\;.\end{aligned} \tag{5.74}$$

Case $\tau^{(1)} = \tau^{(4)} = -\tau \mid \tau^{(2)} = \tau^{(3)} = 0$

With this choice of the parameters, (5.64)-(5.65) become

$$\mathcal{J}^{(0)} = \frac{1}{(1-\tau\,\partial_w\phi)\,(1-\tau\,\partial_{\bar w}\phi)}\begin{pmatrix} 1-\tau\,\partial_{\bar w}\phi & 0\\ 0 & 1-\tau\,\partial_w\phi\end{pmatrix}\;, \tag{5.75}$$

$$\left(\mathcal{J}^{(0)}\right)^{-1} = \begin{pmatrix} 1-\tau\,\partial_w\phi & 0\\ 0 & 1-\tau\,\partial_{\bar w}\phi\end{pmatrix}\;, \tag{5.76}$$

which corresponds to the limit $s' \to 0$ in (5.21)-(5.22). Following the usual procedure, we derive the differential map

$$\partial_w f = \frac{1}{1+\tau\,\partial_z\phi}\,\partial_z f\ , \quad \partial_{\bar{w}} f = \frac{1}{1+\tau\,\partial_{\bar{z}}\phi}\,\partial_{\bar{z}} f\ , \quad f\,:\,\mathbb{C}\to\mathbb{R}\ , \tag{5.77}$$

from which it descends that the deformed EoMs coincide with the undeformed one

$$\partial_z\partial_{\bar{z}}\phi = 0\ , \tag{5.78}$$

consistently with what we found for the $\mathbf{s} < 0$ deformations in section 5.2.1. Setting $s' = 0$ in (5.26) and using (5.77), we can again derive the whole deformed integrable hierarchy, however we restrict our attention to the deformed level-0 currents and level-1 currents, which are

$$\mathrm{T}_1^{(0)}(\mathbf{z},\tau) = -\frac{1}{2}\partial_z\phi \quad , \quad \Theta_{-1}^{(0)}(\mathbf{z},\tau) = 0\ , \tag{5.79}$$

$$\mathrm{T}^{(0)}(\mathbf{z},\tau) = -\frac{1}{2}\frac{(\partial_z\phi)^2}{1+\tau\,\partial_z\phi} \quad , \quad \Theta^{(0)}(\mathbf{z},\tau) = 0\ , \tag{5.80}$$

Observation 38. Plugging (5.77) into (5.75), the Jacobian can be expressed in terms of the deformed level-0 currents (5.79)

$$\mathcal{J}^{(0)} = \begin{pmatrix} 1+2\tau\,\bar{\mathrm{T}}_1(\mathbf{z},\tau) & -2\tau\,\bar{\Theta}_{-1}(\mathbf{z},\tau) \\ -2\tau\,\Theta_{-1}(\mathbf{z},\tau) & 1+2\tau\,\mathrm{T}_1(\mathbf{z},\tau) \end{pmatrix}\ , \tag{5.81}$$

which once again confirms the statement made in observation 30.

Writing (5.79) and (5.80) in terms of the combinations (A.17), one finds

$$\begin{aligned} \mathcal{I}_0^{(0)}(\mathbf{z},\tau) &= \frac{1}{2}\partial_z\phi = \mathcal{I}_0(\mathbf{z}) \quad , \quad \bar{\mathcal{I}}_0^{(0)}(\mathbf{z},\tau) = \frac{1}{2}\partial_{\bar{z}}\phi = \bar{\mathcal{I}}_0(\mathbf{z})\ , \\ \mathcal{I}_1^{(0)}(\mathbf{z},\tau) &= \frac{1}{2}\frac{(\partial_z\phi)^2}{1+\tau\,\partial_z\phi} = \frac{\mathcal{I}_1(\mathbf{z})}{1+2\tau\,\mathcal{I}_0^{(0)}(\mathbf{z},\tau)} \quad , \quad \bar{\mathcal{I}}_1^{(0)}(\mathbf{z},\tau) = \frac{1}{2}\frac{(\partial_{\bar{z}}\phi)^2}{1+\tau\,\partial_{\bar{z}}\phi} = \frac{\bar{\mathcal{I}}_1(\mathbf{z})}{1+2\tau\,\bar{\mathcal{I}}_0^{(0)}(\mathbf{z},\tau)}\ . \end{aligned} \tag{5.82}$$

Case $\tau^{(1)} = \tau^{(3)} = \delta \mid \tau^{(2)} = \tau^{(4)} = \tau$

In this case (5.64)-(5.65) reduce to

$$\begin{aligned} \mathcal{J}^{(0)} &= \begin{pmatrix} 1-\delta\,\partial_z\phi & -\tau\,\partial_z\phi \\ -\delta\,\partial_{\bar{z}}\phi & 1-\tau\,\partial_{\bar{z}}\phi \end{pmatrix}\ , \\ \left(\mathcal{J}^{(0)}\right)^{-1} &= \begin{pmatrix} 1+\delta\,\partial_w\phi & \tau\,\partial_w\phi \\ \delta\,\partial_{\bar{w}}\phi & 1+\tau\,\partial_{\bar{w}}\phi \end{pmatrix}\ , \end{aligned} \tag{5.83}$$

which corresponds to a change of variables $\Psi_{\delta,\tau}^{(0)}$ of the form

$$\boxed{\Psi_{\delta,\tau}^{(0)}\,:\,\mathbb{C}\to\mathbb{C}\,:\,\mathbf{z}\to\mathbf{w} = \Psi_{\delta,\tau}^{(0)}(\mathbf{z}) = \big(z-\delta\phi(\mathbf{z}),\bar{z}-\tau\phi(\mathbf{z})\big)\ .} \tag{5.84}$$

Observation 39. In section 4.7, we observed that the solutions to the T$\bar{\text{T}}$-deformed EoM fulfil the non-linear evolution equation (4.135). Repeating the same derivation, it is easy to see that its extension to the generalised coordinate transformations $\Psi_\tau^{(\mathbf{s})}$ is

$$\partial_\tau \phi_0^{(\mathbf{s})}(\mathbf{z},\tau) + 2\left(\int^{\mathbf{z}} \bar{\mathfrak{I}}_{\mathbf{s}}\right) \partial_z \phi_0^{(\mathbf{s})}(\mathbf{z},\tau) + 2\left(\int^{\mathbf{z}} \mathfrak{I}_{\mathbf{s}}\right) \partial_{\bar{z}} \phi_0^{(\mathbf{s})}(\mathbf{z},\tau) = 0 \; . \tag{5.85}$$

The multi-parameter variant of (5.85) associated to the coordinate transformation (5.64) is, instead

$$\partial_{\tau^{(i)}} \phi_0^{(0)}(\mathbf{z},\vec{\tau}) + \left(\partial_{\tau^{(i)}} z\right) \partial_z \phi_0^{(0)}(\mathbf{z},\vec{\tau}) + \left(\partial_{\tau^{(i)}} \bar{z}\right) \partial_{\bar{z}} \phi_0^{(0)}(\mathbf{z},\vec{\tau}) = 0 \; . \tag{5.86}$$

In general, equations (5.85)-(5.86) cannot be explicitly integrated, however, in the case of the coordinate transformation (5.84), equations (5.86) become a set of inviscid Burgers equations for the function ϕ_0 in the variables $(z,\bar{z},\tau,\delta)$

$$\boxed{\partial_\tau \phi_0(z,\bar{z},\tau,\delta) + \frac{1}{2}\partial_{\bar{z}} \phi_0^2(z,\bar{z},\tau,\delta) = 0 \; , \quad \partial_\delta \phi_0(z,\bar{z},\tau,\delta) + \frac{1}{2}\partial_z \phi_0^2(z,\bar{z},\tau,\delta) = 0 \; ,} \tag{5.87}$$

whose solution can be expressed, in implicit form, as

$$\boxed{\phi_0(z,\bar{z},\tau,\delta) = \phi_0(z-\delta\phi, \bar{z}-\tau\phi) \; .} \tag{5.88}$$

Using the method discussed in sections 4.5, we can write down the deformed EoM, *i.e.*

$$\partial_z \partial_{\bar{z}} \phi = -\frac{\tau\, \partial_z\phi\, \partial_{\bar{z}}^2\phi\,(1-\delta\,\partial_z\phi) + \delta\,\partial_{\bar{z}}\phi\,\partial_z^2\phi\,(1-\tau\,\partial_{\bar{z}}\phi)}{1-\delta\,\partial_z\phi - \tau\,\partial_{\bar{z}}\phi + 2\tau\delta\,\partial_z\phi\,\partial_{\bar{z}}\phi} \; , \tag{5.89}$$

and the tower of deformed higher conserved currents from the original hierarchy. In particular, the components of the deformed $U(1)$ currents (5.57) are

$$\begin{aligned} J_+^{(0)}(\mathbf{z},\tau,\delta) &= \mathbb{i}\,\partial_z\phi\,\frac{1-\delta\,\partial_z\phi}{1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi} \; , \qquad & J_-^{(0)}(\mathbf{z},\tau,\delta) &= -\mathbb{i}\,\frac{\delta\,\partial_z\phi\,\partial_{\bar{z}}\phi}{1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi} \; , \\ \bar{J}_+^{(0)}(\mathbf{z},\tau,\delta) &= -\mathbb{i}\,\partial_{\bar{z}}\phi\,\frac{1-\tau\,\partial_z\phi}{1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi} \; , \qquad & \bar{J}_-^{(0)}(\mathbf{z},\tau,\delta) &= \mathbb{i}\,\frac{\tau\,\partial_z\phi\,\partial_{\bar{z}}\phi}{1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi} \; , \end{aligned} \tag{5.90}$$

while the components of the deformed stress-energy tensor are

$$\begin{aligned} \mathrm{T}^{(0)}(\mathbf{z},\tau,\delta) &= -\frac{1}{2}\,(\partial_z\phi)^2\,\frac{1-\delta\,\partial_z\phi}{(1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi)^2} \; , \quad & \Theta^{(0)}(\mathbf{z},\tau,\delta) &= \frac{1}{2}\,\frac{\delta\,(\partial_z\phi)^2\,\partial_{\bar{z}}\phi}{(1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi)^2} \; , \\ \bar{\mathrm{T}}^{(0)}(\mathbf{z},\tau,\delta) &= -\frac{1}{2}\,(\partial_{\bar{z}}\phi)^2\,\frac{1-\tau\,\partial_{\bar{z}}\phi}{(1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi)^2} \; , \quad & \bar{\Theta}^{(0)}(\mathbf{z},\tau,\delta) &= \frac{1}{2}\,\frac{\tau\,(\partial_{\bar{z}}\phi)^2\,\partial_z\phi}{(1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi)^2} \; . \end{aligned} \tag{5.91}$$

Therefore, the deformed Hamiltonian and momentum density are

$$\begin{aligned} \mathcal{H}^{(0)}(\mathbf{z},\tau,\delta) &= \mathcal{I}_1^{(0)}(\mathbf{z},\tau,\delta) + \bar{\mathcal{I}}_1^{(0)}(\mathbf{z},\tau,\delta) \\ &= -\frac{\partial_z\phi\,\partial_{\bar{z}}\phi\,(\delta\,\partial_z\phi+\tau\,\partial_{\bar{z}}\phi) - (\partial_z\phi)^2(1-\delta\,\partial_z\phi) - (\partial_{\bar{z}}\phi)^2(1-\tau\,\partial_{\bar{z}}\phi)}{2\,(1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi)^2} \; , \end{aligned} \tag{5.92}$$

$$\begin{aligned} \mathcal{P}^{(0)}(\mathbf{z},\tau,\delta) &= \mathcal{I}_1^{(0)}(\mathbf{z},\tau,\delta) - \bar{\mathcal{I}}_1^{(0)}(\mathbf{z},\tau,\delta) \\ &= \frac{(\partial_z\phi+\partial_{\bar{z}}\phi)\,(\partial_z\phi\,(1-\delta\,\partial_z\phi) - \partial_{\bar{z}}\phi\,(1-\tau\,\partial_{\bar{z}}\phi))}{2\,(1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi)^2} \; , \end{aligned} \tag{5.93}$$

and the corresponding deformed Lagrangian[1] is

$$\mathcal{L}^{(0)}(\mathbf{z},\tau,\delta) = \frac{\partial_z\phi\,\partial_{\bar{z}}\phi}{1-\delta\,\partial_z\phi-\tau\,\partial_{\bar{z}}\phi}\ . \tag{5.94}$$

Observation 40. The EoM associated to (5.94) coincide with (5.89) which has been derived directly from the coordinate transformation. Therefore, in this case, the integrals over the space of the level-1 currents (5.92) and (5.93) are the generators of time and space translations, respectively, in the deformed theory.

Notice that, while the deformed action is mapped exactly into the undeformed one under (5.84), the integral of (5.92) transforms with an additional term

$$\int \mathcal{H}^{(0)}(\mathbf{z},\tau,\delta)\,dz\wedge d\bar{z} = \int \left[\mathcal{H}(\mathbf{w}) + \mathbb{i}\,(\delta-\tau)\big(J_+(\mathbf{w})\,\bar{\text{T}}_2(\mathbf{w}) - \text{c.c.}\big)\right]\,dw\wedge d\bar{w}\ .$$

In order to unambiguously identify the perturbing operator – at least at the first perturbative order – we must Legendre transform (5.92). First of all, we switch from complex $\mathbf{z}$ to cartesian coordinates $\mathbf{x} = (x^1,x^2) = (x,t)$ according to the convention (A.1). Then, inverting the Legendre map one finds

$$\begin{aligned}\Pi &= \mathbb{i}\,\frac{\partial\mathcal{L}^{(0)}(\mathbf{x},\tau,\delta)}{\partial\phi_t}\ ,\\ \phi_t &= \mathbb{i}\,\frac{\big(1+2\,\Pi\,(\tau-\delta)\big)\big(-2+(\tau+\delta)\,\phi_x\big)+2\sqrt{\big(1+2\,\Pi\,(\tau-\delta)\big)\big(1-\tau\,\phi_x\big)\big(1-\delta\,\phi_x\big)}}{(\tau-\delta)\big(1+2\,\Pi\,(\tau-\delta)\big)}\ .\end{aligned} \tag{5.95}$$

Plugging (5.95) in (5.92)-(5.93), or equivalently, performing the canonical Legendre transformation, one gets

$$\begin{aligned}\mathcal{H}^{(0)}(\mathbf{x},\tau,\delta) &= \mathcal{I}_1^{(0)}(\mathbf{x},\tau,\delta)+\bar{\mathcal{I}}_1^{(0)}(\mathbf{x},\tau,\delta) = \mathbb{i}\,\Pi\,\phi_t + \mathcal{L}^{(0)}(\mathbf{x},\tau,\delta)\\ &= -\frac{\big(1+\Pi\,(\tau-\delta)\big)\big(-2+(\tau+\delta)\,\phi_x\big)+2\sqrt{\big(1+2\,\Pi\,(\tau-\delta)\big)\big(1-\tau\,\phi_x\big)\big(1-\delta\,\phi_x\big)}}{(\tau-\delta)^2}\ ,\end{aligned} \tag{5.96}$$

and

$$\mathcal{P}^{(0)}(\mathbf{x},\tau,\delta) = \mathcal{I}_1^{(0)}(\mathbf{x},\tau,\delta)-\bar{\mathcal{I}}_1^{(0)}(\mathbf{x},\tau,\delta) = -\Pi\,\phi_x = \mathcal{P}(\mathbf{x})\ . \tag{5.97}$$

which implies that the momentum density is unaffected by the deformation. Finally, expanding (5.96) at the first order in τ and δ, we can identify the perturbing operator at first order

$$\mathcal{H}^{(0)}(\mathbf{x},\tau,\delta) \underset{\substack{\tau\to 0\\ \delta\to 0}}{\sim} \mathcal{H}(\mathbf{x}) + 2\left(\tau\,J_2(\mathbf{x})\,\bar{\text{T}}(\mathbf{x}) + \delta\,\bar{J}_2(\mathbf{x})\,\text{T}(\mathbf{x})\right) + \mathcal{O}(\tau\delta)\ , \tag{5.98}$$

[1] A straightforward way to obtain the Lagrangian (5.94) from the Hamiltonian (5.92), is to start from a formal series expansion of $\mathcal{L}^{(0)}(\mathbf{z},\tau,\delta)$ around $\tau=\delta=0$ and fix the unknown coefficients by matching the Legendre transformation of $\mathcal{L}^{(0)}(\mathbf{z},\tau,\delta)$ with (5.92).

where $\mathcal{H}(\mathbf{x}) = \mathcal{I}_1(\mathbf{x}) + \bar{\mathcal{I}}_1(\mathbf{x}) = \Pi^2 + \frac{1}{4}\,\phi_x^2$ is the undeformed Hamiltonian and

$$\mathcal{I}_1(\mathbf{x}) = -\mathrm{T}(\mathbf{x}) = \frac{1}{8}\,(2\,\Pi - \phi_x)^2 \ , \quad \bar{\mathcal{I}}_1(\mathbf{x}) = -\bar{\mathrm{T}}(\mathbf{x}) = \frac{1}{8}\,(2\,\Pi + \phi_x)^2 \ , \tag{5.99}$$

are the holomorphic and anti-holomorphic components of the undeformed stress-energy tensor, respectively. In (5.98), the cartesian components of the undeformed chiral currents $J_\mu(\mathbf{x})$ follows from the definition (5.60)

$$\begin{aligned} J_1(\mathbf{x}) &= J_+(\mathbf{x}) - J_-(\mathbf{x}) = \frac{\mathbb{i}}{2}\,(-2\Pi + \phi_x) \ , \\ J_2(\mathbf{x}) &= \mathbb{i}\left(J_+(\mathbf{x}) + J_-(\mathbf{x})\right) = \frac{1}{2}\,(2\Pi - \phi_x) \ , \\ \bar{J}_1(\mathbf{x}) &= \bar{J}_+(\mathbf{x}) - \bar{J}_-(\mathbf{x}) = \frac{\mathbb{i}}{2}\,(-2\Pi - \phi_x) \ , \\ \bar{J}_2(\mathbf{x}) &= -\mathbb{i}\left(\bar{J}_+(\mathbf{x}) + \bar{J}_-(\mathbf{x})\right) = \frac{1}{2}\,(-2\Pi - \phi_x) \ . \end{aligned} \tag{5.100}$$

In a similar way, we denote the cartesian components of the deformed chiral currents as

$$\begin{aligned} J_1^{(0)}(\mathbf{x},\tau,\delta) = J_+^{(0)}(\mathbf{x},\tau,\delta) - J_-^{(0)}(\mathbf{x},\tau,\delta) \ , \quad J_2^{(0)}(\mathbf{x},\tau,\delta) = \mathbb{i}\left(J_+^{(0)}(\mathbf{x},\tau,\delta) + J_-^{(0)}(\mathbf{x},\tau,\delta)\right) \ , \\ \bar{J}_1^{(0)}(\mathbf{x},\tau,\delta) = \bar{J}_+^{(0)}(\mathbf{x},\tau,\delta) - \bar{J}_-^{(0)}(\mathbf{x},\tau,\delta) \ , \quad \bar{J}_2^{(0)}(\mathbf{x},\tau,\delta) = -\mathbb{i}\left(\bar{J}_+^{(0)}(\mathbf{x},\tau,\delta) + \bar{J}_-^{(0)}(\mathbf{x},\tau,\delta)\right) \ , \end{aligned} \tag{5.101}$$

which again fulfil the continuity equations

$$\partial_\mu J_\mu^{(0)}(\mathbf{x},\tau,\delta) = 0 \ , \quad \partial_\mu \bar{J}_\mu^{(0)}(\mathbf{x},\tau,\delta) = 0 \ . \tag{5.102}$$

From (5.102), it follows that the quantities

$$Q(R,\tau,\delta) = \int_0^R J_2^{(0)}(\mathbf{x},\tau,\delta)\,dx \ , \quad \bar{Q}(R,\tau,\delta) = \int_0^R \bar{J}_2^{(0)}(\mathbf{x},\tau,\delta)\,dx \ . \tag{5.103}$$

are the conserved charges associated to the $U(1)_L \times U(1)_R$ symmetry in the deformed theory.

Case $\tau^{(1)} = \tau^{(3)} = 0 \mid \tau^{(2)} = \tau^{(4)} = \tau$

This case can be easily retrieved from the previous one by sending $\delta \to 0$. It corresponds to the coordinate transformation associated to the J$\bar{\mathrm{T}}$ deformation (see [28, 29]). First we notice that, setting $\delta = 0$ in (5.90)-(5.91), the deformation preserves the Sugawara construction for the holomorphic sector

$$\mathrm{T}^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{z},\tau) \equiv \mathrm{T}^{(0)}(\mathbf{z},\tau,0) = -\frac{1}{2}\,\frac{(\partial_z\phi)^2}{(1-\tau\,\partial_{\bar{z}}\phi)^2} = \frac{1}{2}\left(J_+^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{z},\tau)\right)^2 \ , \quad J_+^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{z},\tau) = J_+^{(0)}(\mathbf{z},\tau,0) \ , \tag{5.104}$$

but this is not true for the anti-holomorphic sector. Then, we observe that the Lagrangian (5.94) reduces to

$$\mathcal{L}^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{z},\tau) \equiv \mathcal{L}^{(0)}(\mathbf{z},\tau,0) = \frac{\partial_z\phi\,\partial_{\bar{z}}\phi}{1-\tau\,\partial_{\bar{z}}\phi} \ , \tag{5.105}$$

and the corresponding Legendre transformed Hamiltonian (5.96) becomes

$$\mathcal{H}^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) \equiv \mathcal{H}^{(0)}(\mathbf{x},\tau,0) = \mathcal{P}(\mathbf{x}) + \frac{2}{\tau^2}\left(\left(1+\tau\,J_2(\mathbf{x})\right) - \mathcal{S}^{(\mathrm{J}\bar{\mathrm{T}})}\right) \ , \tag{5.106}$$

where

$$\mathcal{S}^{(\mathrm{J}\bar{\mathrm{T}})} = \sqrt{\left(1+\tau\, J_2(\mathbf{x})\right)^2 - 2\tau^2\,\bar{\mathcal{I}}_1(\mathbf{x})}\,. \tag{5.107}$$

Writing (5.106) as $\mathcal{H}^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) = \mathcal{I}_1^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) + \bar{\mathcal{I}}_1^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau)$ with

$$\begin{aligned}
\mathcal{I}_1^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) &= -\left(\mathrm{T}_2^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) + \Theta_0^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau)\right) = \mathcal{P}(\mathbf{x}) + \frac{1}{\tau^2}\left(\left(1+\tau\, J_2(\mathbf{x})\right) - \mathcal{S}^{(\mathrm{J}\bar{\mathrm{T}})}\right)\,, \\
\bar{\mathcal{I}}_1^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) &= -\left(\bar{\mathrm{T}}_2^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) + \bar{\Theta}_0^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau)\right) = \frac{1}{\tau^2}\left(\left(1+\tau\, J_2(\mathbf{x})\right) - \mathcal{S}^{(\mathrm{J}\bar{\mathrm{T}})}\right)\,,
\end{aligned} \tag{5.108}$$

we find that

$$J_2^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau) = -\frac{1}{\tau}\left(1+\mathcal{S}^{(\mathrm{J}\bar{\mathrm{T}})}\right) = J_2(\mathbf{x}) - \tau\,\bar{\mathcal{I}}_1^{(\mathrm{J}\bar{\mathrm{T}})}(\mathbf{x},\tau)\,. \tag{5.109}$$

As already discussed in section 2.3.4 for the T$\bar{\text{T}}$ deformation of a generic bosonic theory, we argue that also for the J$\bar{\text{T}}$ deformation of the massless boson field theory the energy density of the right and left movers (5.108) has the same formal expression of the quantised spectrum obtained in [30] (cf. (6.89)), where the classical densities are replaced by the corresponding integrated quantities averaged over the volume (up to rescalings). In addition, also the deformation of the chiral current density (5.109) admits a straightforward generalisation at the quantum level (cf. (6.91)). Analogous considerations apply to the case $\tau^{(1)} = \tau^{(1)} = \tau$, $\tau^{(1)} = \tau^{(4)} = 0$, which corresponds to the T$\bar{\text{J}}$ deformation.

Case $\tau^{(1)} = \tau^{(2)} = \tau^{(3)} = \tau^{(4)} = \tau$

All the equations (5.83)-(5.96) can be obtained setting $\delta = \tau$. In this case, the square root in (5.96) disappears, and the Hamiltonian takes the simple form

$$\mathcal{H}^{(0)}(\mathbf{x},\tau) \equiv \mathcal{H}^{(0)}(\mathbf{x},\tau,\tau) = \frac{\mathcal{H}(\mathbf{x}) + \tau\,\mathcal{P}(\mathbf{x})\left(J_2(\mathbf{x}) + \bar{J}_2(\mathbf{x})\right) + \tau^2\,\mathcal{P}^2(\mathbf{x})}{1+\tau\left(J_2(\mathbf{x}) - \bar{J}_2(\mathbf{x})\right)}\,. \tag{5.110}$$

Again, we split the Hamiltonian as $\mathcal{H}^{(0)}(\mathbf{x},\tau) = \mathcal{I}_1^{(0)}(\mathbf{x},\tau) + \bar{\mathcal{I}}_1^{(0)}(\mathbf{x},\tau)$ with

$$\mathcal{I}_1^{(0)}(\mathbf{x},\tau) = -\left(\mathrm{T}^{(0)}(\mathbf{x},\tau) + \Theta^{(0)}(\mathbf{x},\tau)\right) = \frac{\mathcal{I}_1(\mathbf{x}) + \tau\, J_2(\mathbf{x})\,\mathcal{P}(\mathbf{x}) + \frac{\tau^2}{2}\mathcal{P}^2(\mathbf{x})}{1+\tau\left(J_2(\mathbf{x}) - \bar{J}_2(\mathbf{x})\right)}\,,$$

$$\bar{\mathcal{I}}_1^{(0)}(\mathbf{x},\tau) = -\left(\bar{\mathrm{T}}^{(0)}(\mathbf{x},\tau) + \bar{\Theta}^{(0)}(\mathbf{x},\tau)\right) = \frac{\bar{\mathcal{I}}_1(\mathbf{x}) + \tau\, \bar{J}_2(\mathbf{x})\,\mathcal{P}(\mathbf{x}) + \frac{\tau^2}{2}\mathcal{P}^2(\mathbf{x})}{1+\tau\left(J_2(\mathbf{x}) - \bar{J}_2(\mathbf{x})\right)}\,. \tag{5.111}$$

Finally, the deformed chiral currents fulfil

$$J_2^{(0)}(\mathbf{x},\tau) = J_2(\mathbf{x}) + \tau\,\mathcal{P}(\mathbf{x})\,, \quad \bar{J}_2^{(0)}(\mathbf{x},\tau) = \bar{J}_2(\mathbf{x}) + \tau\,\mathcal{P}(\mathbf{x})\,. \tag{5.112}$$

In section 6.5 we will argue that the quantum version of this perturbation (cf. (6.95)-(6.97)) can be obtained by introducing two different scattering factors in the NLIEs (3.28).

5.4 Deformed classical solutions

The method described in section 4.7 to obtain solutions to the $\mathrm{T}\bar{\mathrm{T}}-$ deformed EoMs from the corresponding original solutions can be straightforwardly applied to the case of the generalised coordinate transformations discussed in this chapter.

For the $\mathbf{s} > 0$ case, we compute the relation between the sets of coordinates $\mathbf{w}$ and $\mathbf{z}$ by integrating (5.3)

$$\begin{cases} \frac{\partial z^{(s)}(\mathbf{w})}{\partial w} = 1 + 2\tau\,\bar{\Theta}_{s-1}(\mathbf{w})\,, \\ \frac{\partial z^{(s)}(\mathbf{w})}{\partial \bar{w}} = 2\tau\,\bar{\mathrm{T}}_{s+1}(\mathbf{w})\,, \end{cases} \qquad \begin{cases} \frac{\partial \bar{z}^{(s)}(\mathbf{w})}{\partial w} = 2\tau\,\mathrm{T}_{s+1}(\mathbf{w})\,, \\ \frac{\partial \bar{z}^{(s)}(\mathbf{w})}{\partial \bar{w}} = 1 + 2\tau\,\Theta_{s-1}(\mathbf{w})\,, \end{cases} \tag{5.113}$$

where we denoted $\mathbf{z}^{(s)}(\mathbf{w}) = (z^{(s)}(\mathbf{w}), \bar{z}^{(s)}(\mathbf{w})) = \left(\Psi_\tau^{(s)}\right)^{-1}(\mathbf{w})$. The components of the higher charges $\mathrm{T}_{s+1}(\mathbf{w})$, $\Theta_{s-1}(\mathbf{w})$ along with their complex conjugates are implicitly evaluated on a specific field configuration $\phi_0(\mathbf{w})$ solution to the original EoMs. Inverting the relation $\mathbf{z}^{(s)} = \left(\Psi_\tau^{(s)}\right)^{-1}(\mathbf{w})$, we find the deformed solution as

$$\phi_0^{(s)}(\mathbf{z},\tau) = \phi_0\left(\Psi_\tau^{(s)}(\mathbf{z})\right)\ . \tag{5.114}$$

Analogously, for the $\mathbf{s} \leq 0$ case we should integrate (5.22)

$$\begin{cases} \frac{\partial z^{(s')}(\mathbf{w})}{\partial w} = 1 + 2\tau\,\mathrm{T}_{s+1}(\mathbf{w})\,, \\ \frac{\partial z^{(s')}(\mathbf{w})}{\partial \bar{w}} = 2\tau\,\Theta_{s-1}(\mathbf{w})\,, \end{cases} \qquad \begin{cases} \frac{\partial \bar{z}^{(s')}(\mathbf{w})}{\partial w} = 2\tau\,\bar{\Theta}_{s-1}(\mathbf{w})\,, \\ \frac{\partial \bar{z}^{(s')}(\mathbf{w})}{\partial \bar{w}} = 1 + 2\tau\,\bar{\mathrm{T}}_{s+1}(\mathbf{w})\,, \end{cases} \tag{5.115}$$

and repeat the same procedure described above.

The purpose of this section is to highlight the difference between the $\mathrm{T}\bar{\mathrm{T}}$ perturbation and those corresponding to spins $\mathbf{s} \neq 1$, through the study of a simple example. Since perturbations with higher spins should break explicitly Lorentz symmetry, it is convenient to start from a solution of the Laplace equation which is particularly symmetric under rotations, *i.e.* the "*spiral staircase*" solution of [73]:

$$\phi_0(\mathbf{w}) = d\,\log\left(\frac{w+\xi}{\bar{w}+\bar{\xi}}\right)\,, \quad (\xi,\bar{\xi} \in \mathbb{C}\ ,\ d \in \mathbb{R})\ , \tag{5.116}$$

where w and $\bar{w}$ are complex conjugated variables. Using

$$\partial_{\bar{w}}\frac{1}{w+\xi} = -\pi\,\delta(w+\xi)\ , \quad \partial_w\frac{1}{\bar{w}+\bar{\xi}} = -\pi\,\delta(\bar{w}+\bar{\xi})\ , \tag{5.117}$$

one can show that (5.116) is indeed solution to the undeformed EoM, *i.e.* $\partial_w\partial_{\bar{w}}\phi = 0$.

For sake of brevity, we will only consider perturbations involving the set of charges (4.125), since the deformations associated to (4.124) can be obtained from the results described here through a simple redefinition of the coupling τ. Integrating (5.113) using the set of charges (4.125) evaluated on the solution (5.116) we obtain

$$z^{(s)}(\mathbf{w}) = w + \tau\,\frac{\Gamma\left(\frac{s+1}{2}\right)^2}{s}\,\frac{d^2}{(\bar{w}+\bar{\xi})^s}\ , \quad \bar{z}^{(s)}(\mathbf{w}) = \bar{w} + \tau\,\frac{\Gamma\left(\frac{s+1}{2}\right)^2}{s}\,\frac{d^2}{(w+\xi)^s}\ . \tag{5.118}$$

The latter equations cannot be explicitly inverted, for generic spin $s > 1$, as $\mathbf{w} = \Phi_{\tau}^{(s)}(\mathbf{z})$. However, for $s = 1$, *i.e.* the $\mathrm{T}\bar{\mathrm{T}}$ deformation, (5.118) can be written as

$$z + \xi = (w + \xi)\left[1 + \tau \frac{d^2}{(w+\xi)(\bar{w}+\bar{\xi})}\right] , \quad \bar{z} + \bar{\xi} = (\bar{w} + \bar{\xi})\left[1 + \tau \frac{d^2}{(w+\xi)(\bar{w}+\bar{\xi})}\right] , \tag{5.119}$$

with $\mathbf{z} = \mathbf{z}^{(1)}(\mathbf{w})$, from which

$$\phi_0(\mathbf{w}) = d \, \log\left(\frac{w+\xi}{\bar{w}+\bar{\xi}}\right) = d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) = \phi_0(\mathbf{z}) \, . \tag{5.120}$$

Therefore, the solution (5.116) is a fixed point of the $\mathrm{T}\bar{\mathrm{T}}$ flow. To clearly see how the $\mathbf{s} \neq 1$ perturbations affect the solution (5.116), we shall restrict to the $\mathbf{s} < 0$ perturbations, where the deformed solutions can be found explicitly. Integrating (5.115) using the set of charges (4.125), again evaluated on (5.116), we find

$$z^{(s')}(\mathbf{w}) = w + \tau \frac{\Gamma\left(\frac{s+1}{2}\right)^2}{s} \frac{d^2}{(w+\xi)^s} , \quad \bar{z}^{(s')}(\mathbf{w}) = \bar{w} + \tau \frac{\Gamma\left(\frac{s+1}{2}\right)^2}{s} \frac{d^2}{(\bar{w}+\bar{\xi})^s} \, . \tag{5.121}$$

Comparing (5.118) with (5.121) we see that the difference between the $\mathbf{s} > 0$ and $\mathbf{s} < 0$ perturbations lies in the substitution $w + \xi \leftrightarrow \bar{w} + \bar{\xi}$ in the term proportional to τ, which implies that, for $\mathbf{s} < 0$, there is no mixing between holomorphic and anti-holomorphic components.

Since holomorphic and anti-holomorphic parts are completely decoupled, we can integrate explicitly (5.121) for w and $\bar{w}$. The result is

$$\begin{aligned} w\left(\mathbf{z}^{(s')}\right) &= z^{(s')} + \frac{s}{1+s}\left[-1 + F_s\left(\left(\frac{1+s}{s}\right)^{1+s} \Gamma\left(\frac{1+s}{2}\right)^2 \frac{\tau \, d^2}{\left(z^{(s')}+\xi\right)^{1+s}}\right)\right]\left(z^{(s')}+\xi\right) , \\ \bar{w}\left(\mathbf{z}^{(s')}\right) &= \bar{z}^{(s')} + \frac{s}{1+s}\left[-1 + F_s\left(\left(\frac{1+s}{s}\right)^{1+s} \Gamma\left(\frac{1+s}{2}\right)^2 \frac{\tau \, d^2}{\left(\bar{z}^{(s')}+\bar{\xi}\right)^{1+s}}\right)\right]\left(\bar{z}^{(s')}+\bar{\xi}\right) , \end{aligned} \tag{5.122}$$

Finally, the deformed solutions are recovered by plugging (5.122) into (5.116)

$$\boxed{\phi^{(s')}(\mathbf{z}, \tau) = d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) + d \, \log\left(\frac{1 + s \, F_s\left(\left(\frac{1+s}{s}\right)^{1+s} \Gamma\left(\frac{1+s}{2}\right)^2 \frac{\tau \, d^2}{(z+\xi)^{1+s}}\right)}{1 + s \, F_s\left(\left(\frac{1+s}{s}\right)^{1+s} \Gamma\left(\frac{1+s}{2}\right)^2 \frac{\tau \, d^2}{(\bar{z}+\bar{\xi})^{1+s}}\right)}\right) .} \tag{5.123}$$

Let us discuss in detail the case $\mathbf{s} = -1$ of (5.123),

$$\phi^{(-1)}(\mathbf{z}, \tau) = d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) + d \, \log\left(\frac{1 + \sqrt{1 - \frac{4\tau \, d^2}{(z+\xi)^2}}}{1 + \sqrt{1 - \frac{4\tau \, d^2}{(\bar{z}+\bar{\xi})^2}}}\right) . \tag{5.124}$$

We observe that, as soon as the perturbation is switched on, a pair of square root branch points appears at $z = \pm 2d\sqrt{\tau} - \xi$. Considering for simplicity $\xi, \bar{\xi} \in \mathbb{R}$, they form a branch cut on the real axis of the complex plane of z

$$\mathcal{C} = \left(-2d\sqrt{\tau} - \xi; +2d\sqrt{\tau} - \xi\right) \; , \tag{5.125}$$

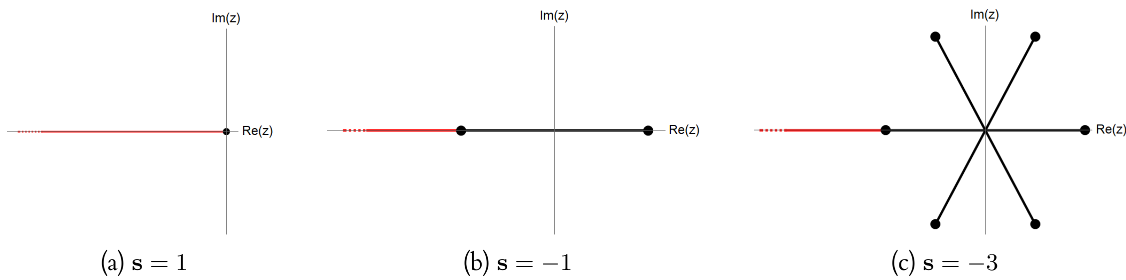


Figure 5.1: Analytic structure of the deformed solution (5.123) in the complex plane of z for different values of $\mathbf{s}$, with $\xi = \bar{\xi} = 0$. The black lines correspond to the square root branch cuts connecting the singularities (5.130), while the red lines correspond to the logarithmic cuts.

i.e. the black line in Figure 5.1b. Instead, the logarithmic singularity of the undeformed solution (5.116) at $w = -\xi$ cancels out with the singularities coming from the additional term in (5.124). Therefore, the logarithmic cut of (5.116) which runs, in our convention, along the real axis from $w = -\infty$ to $w = -\xi$, now connects $z = -\infty$ on the first sheet to $z = -\infty$ on the secondary branches reached by passing through $\mathcal{C}$ (see the red line in Figure 5.1b). This implies that the behaviour of (5.124) at $z = \infty$ is different according to the choice of the branch. On the first sheet one has

$$\phi^{(-1)}(\mathbf{z}, \tau) \underset{\substack{\mathbf{z}\to\infty \\ \text{(1-th sheet)}}}{\sim} d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) + \mathcal{O}(z^0) \,, \tag{5.126}$$

while, on the second sheet, flipping the $+$ sign in front of the square roots in (5.124) into a $-$ sign one finds

$$\phi^{(-1)}(\mathbf{z}, \tau) \underset{\substack{\mathbf{z}\to\infty \\ \text{(2-nd sheet)}}}{\sim} -d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) + \mathcal{O}(z^0) \,. \tag{5.127}$$

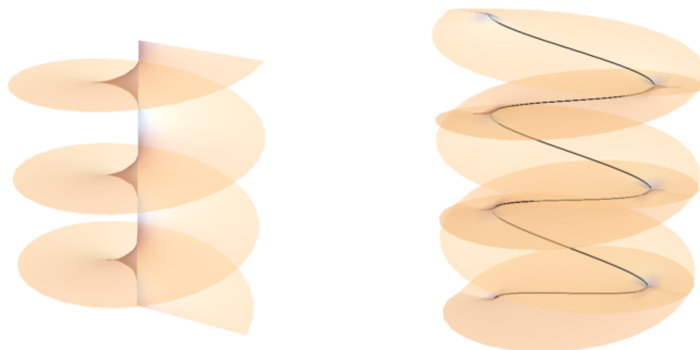
In Figure 5.2, is represented the Riemann surface of the solution (5.124) (Figure 5.2b) together with that of the bare solution (Figure 5.2a), which coincides with the $\mathrm{T}\bar{\mathrm{T}}$ deformed solution. Notice that the analytic structure of (5.124) can be read out from the implicit map (5.121). In fact, for $\mathbf{s} = -1$, equation (5.121) reduces to the Zhukovsky transformation

$$z + \xi = (w+\xi) + \tau \, \frac{d}{(w+\xi)} \,, \quad \bar{z} + \bar{\xi} = (\bar{w}+\bar{\xi}) + \tau \, \frac{d}{(\bar{w}+\bar{\xi})} \,, \tag{5.128}$$

from which we see that $z = \infty$ on the first sheet is mapped into $w = \infty$, while $z = \infty$ on the second sheet is mapped into $w = -\xi$. Moreover (5.128) captures the large z behaviour of the solution. In fact,

$$d \, \log\left(\frac{w+\xi}{\bar{w}+\bar{\xi}}\right) \underset{w\to\infty}{\sim} d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) \,, \quad d \, \log\left(\frac{w+\xi}{\bar{w}+\bar{\xi}}\right) \underset{w\to-\xi}{\sim} -d \, \log\left(\frac{z+\xi}{\bar{z}+\bar{\xi}}\right) \,. \tag{5.129}$$

Let us now consider the generic solution (5.123). The hypergeometric functions appearing in (5.123) are of the form ${}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; x)$, with coefficients $\{a_i\}_{i=1}^{p+1}$, $\{b_j\}_{j=1}^{p} \in \mathbb{Q}$. Generally, these hypergeometric functions have branch points at $x = \infty$ and $x = 1$, which in our case are mapped



(a) (b)

Figure 5.2: Riemann surface of the deformed solution $\phi^{(\mathbf{s})}(\mathbf{z},\tau)$, $(\xi=\bar{\xi}=0)$, in the complex plane of z for $\mathbf{s}=1$ (a), and $\mathbf{s}=-1$ (b).

into $z=0$ and $z=x_n$, respectively, with

$$x_n=\frac{1+s}{s}\left[d\,\Gamma\left(\frac{1+s}{2}\right)\right]^{\frac{2}{1+s}}\tau^{\frac{1}{1+s}}e^{\frac{2\pi\mathrm{i}}{1+s}n}-\xi\,,\quad(n=0,\ldots,s)\,. \tag{5.130}$$

The branch points (5.130) are all of square root type. In conclusion, roughly speaking, starting from a rotational-symmetric solution, the perturbations with $\mathbf{s}<0$ have explicitly broken the original $U(1)$ symmetry down to a discrete $\mathbb{Z}_{2s}$.

Chapter 6

Generalised phase factors

This final chapter is devoted to the quantisation of the classical deformations associated to the generalised coordinate transformations introduced in chapter 5. The first step consists in the identification of the phase factors which generate the modification of the $S-$matrix. Subsequently, following the strategy adopted in the $\mathrm{T}\bar{\mathrm{T}}$ case, we derive the evolution equations for the conserved charges of the hierarchy in the framework of the NLIE.

6.1 Identification of the phase factors

Although we arrived to (5.54) by considering a particular model – a massless scalar field theory – and only a very specific set of conserved currents, in the following we assume the general validity of (5.54) and of (5.55), obtained from (5.54) using the reflection property (5.25). Naturally, in order to put this argument on a more solid foundation, it would be very important to extend the proof of (5.54) to the set of conserved currents (4.126) and also to massive theories.

Consider first the quantum version of (5.54), where the holomorphic and anti-holomorphic sectors are not coupled together by the interaction. Then, the level-k Hamiltonian and momentum operators factorise as

$$\hat{E}_k^{(s')}(R,\tau) \quad = \quad \hat{I}_k^{(s',+)}(R,\tau)\otimes \mathbb{1} + \mathbb{1}\otimes \hat{I}_k^{(s',-)}(R,\tau)\,, \tag{6.1}$$
$$\hat{P}_k^{(s')}(R,\tau) \quad = \quad \hat{I}_k^{(s',+)}(R,\tau)\otimes \mathbb{1} - \mathbb{1}\otimes \hat{I}_k^{(s',-)}(R,\tau)\,, \tag{6.2}$$

and their action on a generic multi-particle state

$$|N^{(+)},N^{(-)}\rangle_\tau = |N^{(+)}\rangle_\tau \otimes |N^{(-)}\rangle_\tau = |\theta_1^{(+)},\theta_2^{(+)},\dots,\theta_{N^{(+)}}^{(+)}\rangle_\tau \otimes |\theta_1^{(-)},\theta_2^{(-)},\dots,\theta_{N^{(-)}}^{(-)}\rangle_\tau \ , \tag{6.3}$$

is determined by [1]

$$\hat{I}_k^{(s',\pm)}(R,\tau)\,|N^{(\pm)}\rangle_\tau = \frac{\hat{\gamma}_k}{2}\left(\sum_{i=1}^{N^{(\pm)}} e^{\pm k\theta_i^{(\pm)}}\right)|N^{(\pm)}\rangle_\tau \ , \tag{6.4}$$

[1] We have adopted here the convention of [9], where the single particle energy and momentum for right $(+)$ and left $(-)$ movers are parametrised as $(\frac{\hat{m}}{2}e^{\pm\theta}, \pm\frac{\hat{m}}{2}e^{\pm\theta})$.

in the large R limit. Notice that, in the massless boson theory under consideration, there is only one species of elementary excitations and the set of rapidities $\{\theta_i^{(\pm)}\}$ completely characterises an asymptotic quantum state. In addition

$$\hat{I}_k^{(\pm)}(R)\,|N^{(\pm)}\rangle_0 \;=\; \left(\frac{\hat{\gamma}_k}{2}\right)\left(\frac{2}{\hat{m}}\right)^k \sum_{i=1}^{N^{(\pm)}} \left(\frac{2\pi n_i^{(\pm)}}{R}\right)^k |N^{(\pm)}\rangle_0 \;, \quad \left(\{n_i^{(\pm)}\} \in \mathbb{Z}^+\right) \;, \tag{6.5}$$

where $\hat{m} = \hat{\gamma}_1$ and we used the fact that the original theory is free. Assuming that, at least in the large R limit the deformed charges are commuting

$$[\hat{I}_k^{(s',\pm)}(R,\tau), \hat{I}_{k'}^{(s',\pm)}(R,\tau)] = 0 \;, \tag{6.6}$$

then, from (5.54) it follows

$${}_0\langle N^{(\pm)}|\,\hat{I}_k^{(s',\pm)}(R,\tau)\left(R + 2\tau \hat{I}_s^{(s',\pm)}(R,\tau)\right)^k |N^{(\pm)}\rangle_\tau = {}_0\langle N^{(\pm)}|\,\hat{I}_k^{(\pm)}(R)\,|N^{(\pm)}\rangle_\tau \; R^k, \tag{6.7}$$

and using (6.4)-(6.5) one has

$$\left(\sum_{i=1}^{N^{(\pm)}} e^{\pm k\theta_i^{(\pm)}}\right)\left(R + 2\tau\,\frac{\hat{\gamma}_s}{2}\sum_{j=1}^{N^{(\pm)}} e^{\mp s'\theta_j^{(\pm)}}\right)^k = \left(\frac{2}{\hat{m}}\right)^k \sum_{i=1}^{N^{(\pm)}} \left(2\pi n_i^{(\pm)}\right)^k \;, \quad (\forall k \in \mathbb{Z}) \;. \tag{6.8}$$

The only consistent solutions to (6.8) are

$$\pm\, R\frac{\hat{m}}{2} e^{\pm\theta_i^{(\pm)}} \pm \tau\, \frac{\hat{m}}{2}\hat{\gamma}_{s'} \sum_{j=1}^{N^{(\pm)}} e^{\pm\left(\theta_i^{(\pm)} - s'\theta_j^{(\pm)}\right)} = \pm 2\pi n_i^{(\pm)} \;, \quad (i = 1, 2, \ldots, N^{(\pm)}) \;, \tag{6.9}$$

i.e. the asymptotic Bethe Ansatz (BA) equations for our models. The two body scattering amplitudes involving right- and left-movers are

$$\delta^{(s')}_{(\pm,\mp)}(\theta,\theta') = 0 \;, \quad \delta^{(s')}_{(\pm,\pm)}(\theta,\theta') = \pm\tau\,\frac{\hat{m}}{2}\hat{\gamma}_{s'}\, e^{\pm(\theta - s'\,\theta')} \;, \quad (s' = -|\mathbf{s}| < 0) \;. \tag{6.10}$$

Similarly, starting from (5.55) we find

$$\delta^{(s)}_{(\pm,\mp)}(\theta,\theta') = \pm\tau\,\frac{\hat{m}}{2}\hat{\gamma}_s\, e^{\pm(\theta - s\,\theta')} \;, \quad \delta^{(s)}_{(\pm,\pm)}(\theta,\theta') = 0 \;, \quad (s = |\mathbf{s}| > 0) \;. \tag{6.11}$$

The arguments presented in this section, strongly support the idea that the classical theories introduced in chapter 5 through a field dependent coordinate transformation can be consistently quantised within the exact $S-$matrix approach through the introduction of specific Lorentz-breaking phase factors. The natural generalisation of (6.10) and (6.11) to a massive field theory is

$$\delta^{(\mathbf{s})}(\theta,\theta') = \tau\, m\gamma_{\mathbf{s}} \sinh(\theta - \mathbf{s}\,\theta') \;, \tag{6.12}$$

with asymptotic BA equations

$$R\,m\sinh(\theta_i) + \tau\, m\gamma_s \sum_{j=1}^{N} \sinh(\theta_i - \mathbf{s}\,\theta_j) = 2\pi n_i \;, \quad (n_i \in \mathbb{Z} \;,\; i = 1, 2, \ldots, N) \;. \tag{6.13}$$

Summing over all the rapidities we see that, apart for $\mathbf{s} = 1$, the kinetic total momentum is not quantised, *i.e.*

$$P(R,\tau) = \mathcal{P}_1^{(\mathbf{s})}(R,\tau) = \sum_{i=1}^{N} m \sinh(\theta_i) \neq \frac{2\pi}{R} n \;, \quad n = \sum_{i=1}^{N} n_i \;, \quad (\mathbf{s} \neq 1) \;. \tag{6.14}$$

Since translational invariance is not broken at classical level, this result suggests that the deformed momentum P do not coincide, in general, with the generator of space translations $\check{P}$, in agreement with the discussion of section 5.2.1. From (6.13) it follows that the natural definition of a quantised momentum is, in the large R limit

$$\check{P}(R) = P(R,\tau) + \frac{\tau}{R}\left[P(R,\tau)\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau) - E(R,\tau)\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\right] = \frac{2\pi}{R} n \;, \quad (n \in \mathbb{Z}) \;, \tag{6.15}$$

with

$$\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})} = \sum_{i=1}^{N} \gamma_s \cosh(\mathbf{k}\,\theta_i) \;, \quad \mathcal{P}_{\mathbf{k}}^{(\mathbf{s})} = \sum_{i=1}^{N} \gamma_s \sinh(\mathbf{k}\,\theta_i) \;, \quad E = \mathcal{E}_1^{(\mathbf{s})} \;. \tag{6.16}$$

Relation (6.15) can also be written as

$$\check{P} = \check{P}^{(+)} - \check{P}^{(-)} \;, \quad \check{P}^{(\pm)} = I_1^{(\mathbf{s},\pm)} + \frac{2\tau}{R} I_1^{(\mathbf{s},\pm)} I_{\mathbf{s}}^{(\mathbf{s},\mp)} \;. \tag{6.17}$$

Observation 41. The quantisation of P on a circle would be preserved by taking symmetrised versions of the scattering phases (6.10)-(6.11) or (6.12). In the massive case, for example, the combination

$$\delta'^{(\tilde{\mathbf{s}},\mathbf{s})}(\theta,\theta') = \frac{\tau}{2} m\gamma_{\mathbf{s}} \left(\sinh(\tilde{\mathbf{s}}\,\theta - \mathbf{s}\,\theta') + \sinh(\mathbf{s}\,\theta - \tilde{\mathbf{s}}\,\theta')\right) \;, \tag{6.18}$$

with $\tilde{\mathbf{s}} = 1$ guarantees the quantisation of P. This is not, however, the phase factor that our classical analysis suggests for the spectrum of the deformed charges (5.51). There are concrete evidences [74] in support of the fact that the phase factors (6.18), with arbitrary integers $\tilde{\mathbf{s}}$ and $\mathbf{s}$, correspond instead to the deformations generated by operators of the form

$$T_{\mathbf{s}+1}\bar{T}_{\tilde{\mathbf{s}}+1} + \bar{T}_{\mathbf{s}+1}T_{\tilde{\mathbf{s}}+1} - \Theta_{\mathbf{s}-1}\bar{\Theta}_{\tilde{\mathbf{s}}-1} - \bar{\Theta}_{\mathbf{s}-1}\Theta_{\tilde{\mathbf{s}}-1} \;, \tag{6.19}$$

which have been discussed in [32].

Observation 42. The phase factors

$$\tilde{\delta}^{(\mathbf{s})}(\theta,\theta') = \tau\, m\gamma_{\mathbf{s}} \sinh(\mathbf{s}\,\theta - \theta') \;, \tag{6.20}$$

appears to be related, instead, to the spectrum of the corresponding mirror deformed Hamiltonians[1] of the models under consideration [74].

[1] See [75] for a rigorous definition of mirror theory in a similar, non relativistic invariant, Bethe Ansatz context

6.2 Generalised spectral flow equations

Motivated by the results of section 6.1, we are led to conjecture that the classical deformations arising from the generalised coordinate transformations described in sections 5.1 and 5.2, correspond to a modification of the original S−matrix by a phase factor

$$S(\theta,\theta') \to S(\theta,\theta')\, f(\theta,\theta') \;, \quad f(\theta,\theta') = \exp\left(\mathbb{i} \sum_{\mathrm{s}} \delta^{(\mathrm{s})}(\theta,\theta') \right) \;, \tag{6.21}$$

where

$$\delta^{(\mathrm{s})}(\theta,\theta') = \tau^{(\mathrm{s})}\gamma_1\gamma_{\mathrm{s}} \sinh(\theta - \mathrm{s}\,\theta') \;, \tag{6.22}$$

with $\{\tau^{(\mathrm{s})}\}$ independent coupling parameters and

$$\gamma_{\mathrm{s}} = \frac{(2\pi m)^s}{C_s} \;, \quad \gamma_1 = m \;, \tag{6.23}$$

Observation 43. The function $f(\theta)$ do not fulfil (3.43), hence it is not a CDD factor. Moreover it explicitly breaks the Lorentz invariance of the original theory, contrary to the family of CDD factors (1.21) considered in [1].

In principle the sum in (6.22) runs over all the positive and negative odd integers. However, since the whole analysis can be straightforwardly analytically extended to arbitrary values of s, in the following we shall relax this constraint, at least to include the case $\mathrm{s} = 0$ and the set of non-local conserved charges [76]. The phase factor (6.22) leads to multi-parameter deformations of the original QFT spectrum which, due to the intrinsic non-linearity of the problem, can effectively be studied only on the case-by-case basis. A detailed analytic and numerical study of specific multi-parameter deformations of a massive QFT, such as the sine-Gordon model, is a very challenging long-term objective. Most of the checks have been performed considering CFTs deformed (explicitly at leading order) by a single irrelevant composite field. For simplicity, we restrict the analysis to phase factors with a single non-vanishing irrelevant coupling. Setting $\tau \equiv \tau^{(\mathrm{s})}$ in (6.22) and performing the transformation

$$S(\theta,\theta') \to S(\theta,\theta')\, e^{\mathbb{i}\delta^{(\mathrm{s})}(\theta,\theta')} \;, \tag{6.24}$$

the kernel appearing in the NLIE (3.15) gets modified as

$$\mathcal{K}(\theta-\theta') \to \mathcal{K}(\theta-\theta') + \frac{1}{2\pi}\partial_\theta \delta^{(\mathrm{s})}(\theta - \mathrm{s}\,\theta') = \mathcal{K}(\theta-\theta') + \tau\, m \frac{\gamma_{\mathrm{s}}}{2\pi} \cosh(\theta - \mathrm{s}\,\theta') \;. \tag{6.25}$$

Inserting (6.25) in (3.15), after simple manipulations, we find that the deformed version of $f_\nu(\theta)$ fulfils (3.15) with

$$\nu = \nu\big(\mathcal{R}_0^{(\mathrm{s})}, \alpha_0 \,|\, \theta - \theta_0^{(\mathrm{s})}\big) \;, \tag{6.26}$$

where $\mathcal{R}_0^{(\mathrm{s})}$ and $\theta_0^{(\mathrm{s})}$ are defined through

$$\mathcal{R}_0^{(\mathrm{s})} \cosh\left(\theta_0^{(\mathrm{s})}\right) = R + \tau\, \mathcal{E}_{\mathrm{s}}^{(\mathrm{s})}(R,\tau) \;, \quad \mathcal{R}_0^{(\mathrm{s})} \sinh\left(\theta_0^{(\mathrm{s})}\right) = \tau\, \mathcal{P}_{\mathrm{s}}^{(\mathrm{s})}(R,\tau) \;, \tag{6.27}$$

with $\mathcal{R}_0 \equiv \mathcal{R}_0^{(1)}$ and $\theta_0 \equiv \theta_0^{(1)}$. Equations (6.27) imply

$$\boxed{\left(\mathcal{R}_0^{(\mathbf{s})}\right)^2 = \left(R + \tau\, \mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\right)^2 - \left(\tau\, \mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\right)^2 \ .} \tag{6.28}$$

The quantities $\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)$ and $\mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)$ denote the $\mathbf{k}$-th higher conserved charges of the theory deformed with the $\mathbf{s}$-th perturbation

$$\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) = I_{\mathbf{k}}^{(\mathbf{s},+)}(R,\tau) + I_{\mathbf{k}}^{(\mathbf{s},-)}(R,\tau)\ , \quad \mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) = I_{\mathbf{k}}^{(\mathbf{s},+)}(R,\tau) - I_{\mathbf{k}}^{(\mathbf{s},-)}(R,\tau)\ , \tag{6.29}$$

and $I_{\mathbf{k}}^{(\mathbf{s},\pm)}$ are again defined through (3.23)-(3.24) but with the deformed driving term (6.26). Formula (6.26) shows that the solutions of the deformed NLIE are modified simply by a redefinition of the length R and by a rapidity shift which, in contrast to the analogous formula (3.45) found in the $\mathrm{T\bar{T}}$ context, here involve the generic level$-\mathbf{s}$ charges.

Putting together (6.29) and the relation between the deformed and the original charges (see equation (3.47))

$$I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau) = e^{\pm \mathbf{k}\theta_0^{(\mathbf{s})}}\, I_{\mathbf{k}}^{(\pm)}\left(\mathcal{R}_0^{(\mathbf{s})}\right)\ , \tag{6.30}$$

one finds the same Lorentz-type transformation (3.48) derived for the $\mathrm{T\bar{T}}-$deformed charges

$$\begin{pmatrix} \mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) \\ \mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) \end{pmatrix} = \begin{pmatrix} \cosh\left(\mathbf{k}\theta_0^{(\mathbf{s})}\right) & \sinh\left(\mathbf{k}\theta_0^{(\mathbf{s})}\right) \\ \sinh\left(\mathbf{k}\theta_0^{(\mathbf{s})}\right) & \cosh\left(\mathbf{k}\theta_0^{(\mathbf{s})}\right) \end{pmatrix} \begin{pmatrix} \mathcal{E}_{\mathbf{k}}\left(\mathcal{R}_0^{(\mathbf{s})}\right) \\ \mathcal{P}_{\mathbf{k}}\left(\mathcal{R}_0^{(\mathbf{s})}\right) \end{pmatrix}\ . \tag{6.31}$$

which implies

$$\boxed{\left(\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)\right)^2 - \left(\mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)\right)^2 = \left(\mathcal{E}_{\mathbf{k}}\left(\mathcal{R}_0^{(\mathbf{s})}\right)\right)^2 - \left(\mathcal{P}_{\mathbf{k}}\left(\mathcal{R}_0^{(\mathbf{s})}\right)\right)^2\ ,} \tag{6.32}$$

To find the generalisations of the flow equations (3.56) describing the evolution of the level$-\mathbf{k}$ charges under the $\mathrm{T\bar{T}}$ deformation, we follow the same strategy of section 3.4. First, we differentiate both sides of (6.30) w.r.t. τ at fixed $\mathcal{R}_0^{(\mathbf{s})}$

$$\partial_\tau I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau) + R'\partial_R I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau) = \pm\mathbf{k}(\theta_0^{(\mathbf{s})})' I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau)\ . \tag{6.33}$$

then we determine $\partial_\tau R = R'$ and $\partial_\tau \theta_0^{(\mathbf{s})} = (\theta_0^{(\mathbf{s})})'$. To do that, we rewrite (6.27) as

$$\begin{cases} \mathcal{R}_0^{(\mathbf{s})} = R\, e^{-\theta_0^{(\mathbf{s})}} + 2\tau\, e^{(\mathbf{s}-1)\theta_0^{(\mathbf{s})}}\, I_{\mathbf{s}}^{(+)}\left(\mathcal{R}_0^{(\mathbf{s})}\right) \\ \mathcal{R}_0^{(\mathbf{s})} = R\, e^{\theta_0^{(\mathbf{s})}} + 2\tau\, e^{-(\mathbf{s}-1)\theta_0^{(\mathbf{s})}}\, I_{\mathbf{s}}^{(-)}\left(\mathcal{R}_0^{(\mathbf{s})}\right) \end{cases}, \tag{6.34}$$

then we differentiate both equations in (6.34) w.r.t. τ which gives the following set of algebraic equations for R' and $(\theta_0^{(\mathbf{s})})'$

$$\begin{cases} 0 = R' e^{-\theta_0^{(\mathbf{s})}} - R\, e^{-\theta_0^{(\mathbf{s})}}(\theta_0^{(\mathbf{s})})' + 2e^{(\mathbf{s}-1)\theta_0^{(\mathbf{s})}} I_{\mathbf{s}}^{(+)}\left(\mathcal{R}_0^{(\mathbf{s})}\right) + 2\tau e^{(\mathbf{s}-1)\theta_0^{(\mathbf{s})}}(\mathbf{s}-1)\,(\theta_0^{(\mathbf{s})})'\, I_{\mathbf{s}}^{(+)}\left(\mathcal{R}_0^{(\mathbf{s})}\right) \\ 0 = R' e^{\theta_0^{(\mathbf{s})}} + R\, e^{\theta_0^{(\mathbf{s})}}(\theta_0^{(\mathbf{s})})' + 2e^{-(\mathbf{s}-1)\theta_0^{(\mathbf{s})}} I_{\mathbf{s}}^{(-)}\left(\mathcal{R}_0^{(\mathbf{s})}\right) - 2\tau e^{-(\mathbf{s}-1)\theta_0^{(\mathbf{s})}}(\mathbf{s}-1)\,(\theta_0^{(\mathbf{s})})'\, I_{\mathbf{s}}^{(-)}\left(\mathcal{R}_0^{(\mathbf{s})}\right) \end{cases}, \tag{6.35}$$

whose solution is

$$\begin{aligned}\partial_\tau R = R' &= -\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau) + \frac{(\mathbf{s}-1)\tau\left(\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\right)^2}{(\mathbf{s}-1)\tau\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-R}\,,\\ \partial_\tau\theta_0^{(\mathbf{s})} = (\theta_0^{(\mathbf{s})})' &= -\frac{\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)}{(\mathbf{s}-1)\tau\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-R}\,.\end{aligned} \tag{6.36}$$

The equations for the level$-\mathbf{k}$ total energy and momentum are then

$$\boxed{\begin{cases}\partial_\tau\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)-\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\partial_R\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) = -\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\dfrac{(\mathbf{s}-1)\tau\,\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\,\partial_R\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)+\mathbf{k}\mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)}{(\mathbf{s}-1)\tau\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-R}\\ \partial_\tau\mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)-\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\partial_R\mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau) = -\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\dfrac{(\mathbf{s}-1)\tau\,\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\,\partial_R\mathcal{P}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)+\mathbf{k}\mathcal{E}_{\mathbf{k}}^{(\mathbf{s})}(R,\tau)}{(\mathbf{s}-1)\tau\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-R}\end{cases}.} \tag{6.37}$$

Contrary to (3.57), the equations for the energy and momentum, *i.e.* $E(R,\tau) = \mathcal{E}_1^{(\mathbf{s})}(R,\tau)$ and $P(R,\tau) = \mathcal{P}_1^{(\mathbf{s})}(R,\tau)$, obtained setting $\mathbf{k} = 1$ in (6.37), are intrinsically coupled to the evolution equations for $\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)$ and $\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)$

$$\boxed{\begin{cases}\partial_\tau\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\partial_R\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau) = -\left(\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\right)^2\dfrac{(\mathbf{s}-1)\tau\,\partial_R\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)+\mathbf{s}}{(\mathbf{s}-1)\tau\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-R}\\ \partial_\tau\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\partial_R\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau) = -\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\dfrac{(\mathbf{s}-1)\tau\,\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\,\partial_R\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)+\mathbf{s}\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)}{(\mathbf{s}-1)\tau\,\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)-R}\end{cases}.} \tag{6.38}$$

Observation 44. As already observed in section 6.1 when we discussed the large R limit, here again we find that the deformed momentum P, defined through (6.37) is not always quantized for $\tau \neq 0$, as it flows according to the complicated non-linear equations (6.37)-(6.38). One can argue (see for example [77]) that a quantised object is

$$\check{P}(R) = \frac{1}{2\pi R}\left(\int_{\mathcal{C}_1} d\theta\, p(\theta)\log\left(1+e^{-f_\nu(\theta)}\right) - \int_{\mathcal{C}_2} d\theta\, p(\theta)\log\left(1+e^{f_\nu(\theta)}\right)\right)\,, \tag{6.39}$$

with

$$p(\theta) = \partial_\theta\nu\left(\mathcal{R}_0^{(\mathbf{s})},\alpha_0\,|\,\theta-\theta_0^{(\mathbf{s})}\right) = -im\mathcal{R}_0^{(\mathbf{s})}\cosh\left(\theta-\theta_0^{(\mathbf{s})}\right)\,. \tag{6.40}$$

Using (6.27) and (6.40) in (6.39), we find

$$\check{P} = P(R) = P(R,\tau) + \frac{\tau}{R}\left(P(R,\tau)\mathcal{E}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau) - E(R,\tau)\mathcal{P}_{\mathbf{s}}^{(\mathbf{s})}(R,\tau)\right)\,, \tag{6.41}$$

which coincides with (6.15) obtained in section 6.1 from BA considerations.

6.3 The CFT limit

In this section, we discuss the CFT limit of the NLIE derived in section 6.2. In particular, we explicitly compute the expressions of the charges of the deformed hierarchy. We refer the reader to section 3.3 for the derivation of the CFT NLIEs obtained as a massless limit of the sine-Gordon NLIE.

In the CFT limit, the τ−dependent phase factor (6.22) (with $\tau \equiv \tau^{(\mathbf{s})}$) splits, for $\mathbf{s} > 0$, into

$$\delta^{(s)}_{(\pm,\mp)}(\theta,\theta') = \pm\tau\,\tfrac{\hat{m}}{2}\hat{\gamma}_s\,e^{\pm(\theta - s\,\theta')}\;, \quad \delta^{(s)}_{(\pm,\pm)}(\theta,\theta') = 0\;, \tag{6.42}$$

with

$$\hat{\gamma}_{\mathbf{s}} = \frac{(2\pi\hat{m})^s}{C_s}\;, \quad \hat{\gamma}_1 = \hat{m}\;, \tag{6.43}$$

breaking conformal invariance by explicitly introducing a coupling between the right- and the left-mover sectors. The resulting NLIE is identical to (3.28) with driving term $\nu^{(\pm)}(\mathcal{R}_0^{(s,\pm)}, \alpha_0\,|\,\theta)$, where we set

$$\mathcal{R}_0^{(s,\pm)} = R + 2\tau\, I_s^{(s,\mp)}(R,\tau)\;. \tag{6.44}$$

For $\mathbf{s} < 0$, instead, the two chiral sectors remain decoupled

$$\delta^{(s')}_{(\pm,\mp)}(\theta,\theta') = 0\;, \quad \delta^{(s')}_{(\pm,\pm)}(\theta,\theta') = \pm\tau\,\frac{\hat{m}}{2}\hat{\gamma}_{s'}\,e^{\pm(\theta - s'\,\theta')}\;, \quad (s' = -|\mathbf{s}| < 0)\;, \tag{6.45}$$

and, due to the reflection property $I_s^{(s',\pm)} = I_{s'}^{(s',\mp)}$, the corresponding NLIE is again (3.28) with driving term $\nu^{(\pm)}(\mathcal{R}_0^{(s',\mp)}, \alpha_0\,|\,\theta)$. In turn, the length redefinition (6.44) implies

$$I_k^{(s,\pm)}(R,\tau) = I_k^{(\pm)}\left(\mathcal{R}_0^{(s,\pm)}\right)\;, \quad (s > 0)\;, \tag{6.46}$$
$$I_k^{(s',\pm)}(R,\tau) = I_k^{(\pm)}\left(\mathcal{R}_0^{(s',\mp)}\right)\;, \quad (s' = -s < 0)\;, \tag{6.47}$$

which are solutions to the evolution equations (5.56), deduced at classical level. In appendix D.1 it is shown that equations (5.56) indeed are contained in the more general equations (6.33).

Observation 45. Using the scaling property of the CFT charges (3.30), we can eliminate from (5.56) the derivative w.r.t. R of the charges, obtaining further simplified expressions

$$\begin{aligned}
\partial_\tau I_k^{(s,\pm)} &= -\frac{2k\left(R - 2\tau\,(s-1)\,I_s^{(s,\pm)}\right)I_s^{(s,\pm)}I_k^{(s,\mp)}}{2\tau\left(R - 2\tau\,(s^2-1)\,I_s^{(s,+)}\right)I_s^{(s,-)} + R\left(R + 2\tau I_s^{(s,+)}\right)}\;, \quad (s > 0)\\
\partial_\tau I_k^{(s',\pm)} &= -\frac{2k\,I_k^{(s',\pm)}\,I_s^{(s',\pm)}}{R + 2\tau\,(s+1)\,I_s^{(s',\pm)}}\;, \quad (s' = -s < 0)\;.
\end{aligned} \tag{6.48}$$

Notice that, setting $k = s = 1$ in the first equation of (6.48) we get

$$\partial_\tau I_1^{(\pm)} = \frac{-2\,I_1^{(+)}I_1^{(-)}}{R + 2\tau\left(I_1^{(+)} + I_1^{(-)}\right)}\;, \tag{6.49}$$

which matches with the T$\bar{\text{T}}$ result quoted in [78].

The next step is to explicitly solve the algebraic equations (6.46)-(6.47). Afterwards, we will consider the $\mathbf{s} \to 0$ limit starting from the massive case.

Deformations with $\mathbf{s} < 0$

Since the generic $k-$th charges of a CFT scale as[1] R^{-k} according to (3.30) and the corresponding deformed charges $I_k^{(s',\pm)}(R,\tau)$ fulfil the same equation with $R \to \mathcal{R}_0^{(s',\pm)}$, then

$$I_k^{(s',\pm)}(R,\tau) = I_{k'}^{(s',\mp)}(R,\tau) = \frac{R^k\, I_{k'}^{(\mp)}(R)}{\left(\mathcal{R}_0^{(s',\pm)}\right)^k} = \frac{2\pi\, a_{k'}^{(\mp)}}{\left(\mathcal{R}_0^{(s',\pm)}\right)^k}\,, \tag{6.50}$$

where $a_k^{(\pm)} = a_{k'}^{(\mp)}$, $k' = -k,\ s' = -\mathrm{s}$, and $\mathcal{R}_0^{(s',\pm)}$ is defined in (6.44).

In order to find the solution to (6.50) for generic k, one must first solve (6.50) for $k = s$ ($k' = s'$). In this case, the solution can be reconstructed perturbatively as

$$I_s^{(s',\pm)}(R,\tau) = I_{s'}^{(s',\mp)}(R,\tau) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \binom{j(1+s)+(s-1)}{j} (2\tau)^j\, \frac{\left(2\pi\, a_s^{(\pm)}\right)^{j+1}}{R^{j(1+s)+s}}\,. \tag{6.51}$$

This expression can be resummed for various values of s as

- $\mathrm{s} = -1:$

$$I_1^{(-1,\pm)}(R,\tau) = \frac{R}{4\tau}\left[-1 + \sqrt{1 + 8\tau\, \frac{2\pi\, a_1^{(\pm)}}{R^2}}\,\right]\,. \tag{6.52}$$

  Both the classical (see the $s' = -s = -1$ case in (5.43)) and quantum results suggest that the leading perturbing operator corresponds to the Lorentz breaking operator typically appearing in effective field theories for discrete lattice models [80].

- $\mathrm{s} = -2:$

$$I_2^{(-2,\pm)}(R,\tau) = \frac{4R}{6\tau}\, \sinh\left[\frac{1}{3}\,\mathrm{arcsinh}\left(\frac{3\sqrt{3}}{2}\left(2\tau\,\frac{2\pi\, a_2^{(\pm)}}{R^3}\right)^{1/2}\right)\right]. \tag{6.53}$$

For generic spin $\mathrm{s} < 0$ the result can be written in terms of a single generalised hypergeometric function:

$$\boxed{I_s^{(s',\pm)}(R,\tau) = I_{s'}^{(s',\mp)}(R,\tau) = \frac{s\,R}{2\tau\,(1+s)}\left[-1 + \, F_s\left(-2\tau\, \frac{(1+s)^{1+s}}{s^s}\, \frac{2\pi\, a_s^{(\pm)}}{R^{1+s}}\right)\right]}\,, \tag{6.54}$$

where $F_s(x)$ is the same hypergeometric function defined in (5.40). The total momentum and energy are

$$E(R) = I_1^{(s',+)}(R,\tau) + I_1^{(s',-)}(R,\tau)\,, \quad P(R) = I_1^{(s',+)}(R,\tau) - I_1^{(s',-)}(R,\tau)\,, \tag{6.55}$$

where $I_1^{(s',\pm)}(R,\tau)$ are obtained by solving (6.50) with $k' = -k = -1$ using (6.54).

Observation 46. Even spin charges do not, in general, correspond to local conserved currents in the sine-Gordon model. They can occasionally emerge from the set of non-local charges, at specific rational values of β^2. The results concerning the exact quantum spectrum can be smoothly deformed in s, therefore they formally also describe deformations of the sine-Gordon model by non-local

[1]Separately, the NLIEs in (3.28), with generic parameters β and $\alpha_0^{(\pm)}$ can be also associated to the quantum KdV theory, as extensively discussed in [52, 79]. The coefficients a_k for $k = 3, 5$ can be recovered from [52].

currents [81]. Moreover, there are many integrable systems with extended symmetries where even spin charges appear. The sign of the corresponding eigenvalues depends on the internal flavor of the specific soliton configuration considered. Since the flow equations (6.36) (with (6.38)), should properly describe the evolution of the spectrum driven by analogous deformations in a very wide class of systems, perturbations by currents with s even, may lead to interesting quantum gravity toy models where the effective sign of the perturbing parameter τ depends on the specific state under consideration.

Deformations with s > 0

In the case s > 0, the left- and right-mover sectors are now coupled and the solution to the generalised Burgers equations become equivalent to the set of equations

$$I_k^{(s,\pm)}(R,\tau) = \frac{2\pi\, a_k^{(\pm)}}{\left(R + 2\tau\, I_s^{(s,\mp)}(R,\tau)\right)^k} \; . \tag{6.56}$$

While, for models with $\mathcal{P}_k^{(s)} = I_k^{(s,+)}(R,\tau) - I_k^{(s,-)}(R,\tau) = 0$ the left- and right-mover sectors are completely symmetric and the relations (6.56) formally reduce to the equations (6.50), for $\mathcal{P}_k^{(s)} \neq 0$ the solution to (6.56) for $k = s$ are

$$\begin{aligned} I_s^{(s,\pm)}(R,\tau) &= \frac{2\pi\, a_s^{(\pm)}}{R^s}\Bigg[1 + 2\pi\, a_s^{(\mp)} \sum_{k=1}^{\infty}\sum_{l=0}^{k-1} \frac{1}{l+1} \begin{pmatrix} (k-l)s + l - 1 \\ l \end{pmatrix} \\ &\times \begin{pmatrix} (l+1)s + k - l - 1 \\ k - l \end{pmatrix} \left(2\pi\, a_s^{(\pm)}\right)^l \left(2\pi\, a_s^{(\mp)}\right)^{k-l-1} \frac{(-1)^k (2\tau)^k}{R^{(s+1)k}}\Bigg] \; . \end{aligned} \tag{6.57}$$

We were not able to find a general compact expression of (6.57), except for the already known (s $= 1$) T$\bar{\text{T}}$-related result

$$\begin{aligned} \mathcal{E}_1^{(1)}(R,\tau) &= \frac{R}{2\tau}\left(-1 + \sqrt{1 + \frac{4\tau}{R}\left(\frac{2\pi\left(a_1^{(+)} + a_1^{(-)}\right)}{R}\right) + \frac{4\tau^2}{R^2}\left(\frac{2\pi\left(a_1^{(+)} - a_1^{(-)}\right)}{R}\right)^2}\right) \; , \\ \mathcal{P}_1^{(1)}(R) &= \frac{2\pi\left(a_1^{(+)} - a_1^{(-)}\right)}{R} \; . \end{aligned} \tag{6.58}$$

Deformations with s $\to 0$

As anticipated, we shall perform the limit s $\to 0$ starting from the massive case. Sending $\gamma_s \to 0$ and simoultaneously rescaling τ, we can recast the driving term in the form

$$\nu(R, \alpha_0 \,|\, \theta) = 2\pi\mathbb{i}\, \alpha_0 - \mathbb{i}\, mR\sinh(\theta) + \mathbb{i}\tau\mathcal{G}\, \cosh(\theta) \; , \tag{6.59}$$

where

$$\mathcal{G} = -\frac{1}{2}\left(\int_{\mathcal{C}_1} \frac{d\theta}{2\pi\mathbb{i}} \log\left(1 + e^{-f_\nu(\theta)}\right) - \int_{\mathcal{C}_2} \frac{d\theta}{2\pi\mathbb{i}} \log\left(1 + e^{f_\nu(\theta)}\right)\right) \; . \tag{6.60}$$

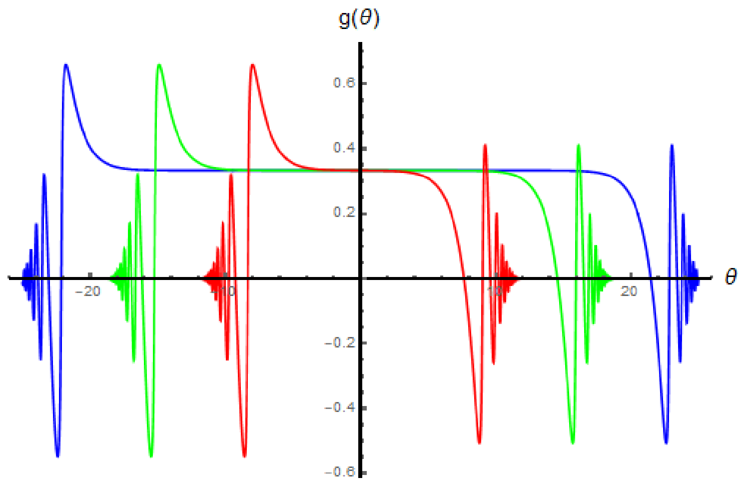


Figure 6.1: The formation of the plateau in $g(\theta) = -\frac{1}{2\pi}\,\mathrm{Im}\left[\log\left(1+e^{-f_\nu(\theta+\mathrm{i}0^+)}\right)\right]$ as $m \to 0$ at $\beta^2 = 1/2$, $\alpha_0 = 1/6$. For this choice of the parameters, the height of the plateau of $g(\theta)$ is $1/3$.

Notice that, in the $m \to 0$ limit, the leading contribution to $\mathcal{G}$ in (3.31) is coming from a large plateau of the integrand $g(\theta)$ (see figure 6.1) of height

$$-\,\mathrm{i} f_\nu(\theta) \underset{m\to 0}{\sim} 4\pi\alpha_0\beta^2\;, \tag{6.61}$$

and width growing as $\sim 2\log(mR/2)$. In conclusion, we have

$$\mathcal{G} \underset{m\to 0}{\sim} \log\left(\frac{mR}{2}\right) 2\alpha_0\beta^2\;. \tag{6.62}$$

The emergence of the plateau can be deduced analytically from the NLIE (see for example [6] for a related discussion in the TBA context). Therefore, at fixed finite R, after a further rescaling $\tau \to \tau\pi/\log(mR/2)$, as m tends to zero, we obtain for the driving terms in the NLIEs

$$\nu^{(\pm)} = \nu^{(\pm)}(\mathcal{R}_0^{(\pm)}, \alpha_0\,|\,\theta) = 2\pi\mathrm{i}\,\alpha_0 \mp \mathrm{i}\,\frac{\hat{m}}{2}\,e^{\pm\theta}\mathcal{R}_0^{(\pm)}\;, \tag{6.63}$$

with

$$\mathcal{R}_0^{(\pm)} = R \mp \tau Q_0\;, \quad Q_0 = 2\pi\alpha_0\beta^2\;. \tag{6.64}$$

Thus, perturbing the theory with the phase factor (6.22) with $\mathrm{s} = 0$ is equivalent, in an appropriate scaling limit, to a constant shift of the volume R. At fixed normalisation of the deformation parameter τ, the shift turns out to be directly proportional to the topological charge Q_0. We see, from (6.63) that the left- and right-mover sectors remain decoupled also at $\tau \neq 0$. However, starting directly from the CFT limit, one can argue that there exist less trivial ways to couple the two sectors. The most general variant involves four different coupling constants $\{\tau^{(a|b)}\}_{(a,b)\in\{+,-\}}$

$$\mathcal{R}_0^{(\pm)} = R \mp \left(\tau^{(\pm|+)}Q_0^{(+)} + \tau^{(\pm|-)}Q_0^{(-)}\right)\;, \quad Q_0^{(\pm)} = 2\pi\alpha_0^{(\pm)}\beta^2\;. \tag{6.65}$$

Notice that in (6.65) we allowed the possibility for two different topological charges $Q_0^{(\pm)}$, associated to the $U(1)_R \times U(1)_L$ symmetry of the $c = 1$ free (compactified) boson model, corresponding to the CFT limit of the sine-Gordon theory.

6.4 Further deformations involving the topological charge

A further, natural extension of the quantum models studied in section 6.3, corresponds to scattering phase factors of the form

$$\delta^{(\tilde{\mathbf{s}},\mathbf{s})}(\theta,\theta') = \tau\,\gamma_{\tilde{\mathbf{s}}}\gamma_{\mathbf{s}}\sinh(\tilde{\mathbf{s}}\,\theta - \mathbf{s}\,\theta')\;. \tag{6.66}$$

Deforming the kernel according to (6.25) and using (6.66) instead of (6.22), the driving term of (3.15) becomes

$$\nu = \nu(R,\alpha_0\,|\,\theta) - \mathbb{i}\tau\tilde{\mathbf{s}}\,\gamma_{\tilde{\mathbf{s}}}\Big(e^{\tilde{\mathbf{s}}\,\theta} I_{\mathbf{s}}^{(\tilde{\mathbf{s}},\mathbf{s},-)}(R,\tau) - e^{-\tilde{\mathbf{s}}\,\theta} I_{\mathbf{s}}^{(\tilde{\mathbf{s}},\mathbf{s},+)}(R,\tau)\Big)\;, \tag{6.67}$$

where $I_{\mathbf{k}}^{(\tilde{\mathbf{s}},\mathbf{s},\pm)}(R,\tau)$ are defined through (3.30) and (3.31) with the deformed driving term (6.67). The cases with $\tilde{\mathbf{s}} = \pm 1$, have been extensively discussed in the previous sections, they all correspond to gravity-like theories, where the effect of the perturbation can be re-absorbed into a redefinition of the volume R plus a shift in the rapidity θ, according to (6.26). From equation (6.67), we see that for generic values of $\tilde{\mathbf{s}} \neq \pm 1$ it is no longer possible to re-absorb the perturbation in the same fashion. However, an interesting possibility is recovered in the scaling limit $\{\tilde{\mathbf{s}},\gamma_{\tilde{\mathbf{s}}},m\} \to \{0,0,0\}$, such that the product $\tilde{\mathbf{s}}\gamma_{\tilde{\mathbf{s}}}$ remains finite. In the following we shall set $\tilde{\mathbf{s}}\gamma_{\tilde{\mathbf{s}}} = 2\pi$. This situation corresponds to the standard massless limit where, in the original ($\tau = 0$) NLIE, the right- $(+)$ and the left-mover $(-)$ sectors are completely decoupled, while a residual interaction between the two sectors is still present at $\tau \neq 0$. In fact, the deformed versions of $f_{\nu^{(\pm)}}(\theta)$ fulfil (3.28) with

$$\nu^{(\pm)} = \nu^{(\pm)}\big(R,\alpha_0^{(\pm)} - \alpha^{(\mathbf{s},\pm)}\,|\,\theta\big)\;,\quad \alpha^{(\mathbf{s},\pm)} = \Big(\tau^{(\pm)} I_{\mathbf{s}}^{(\mathbf{s},-)}(R,\vec{\tau}) - \tau^{(\mp)} I_{\mathbf{s}}^{(\mathbf{s},+)}(R,\vec{\tau})\Big)\;, \tag{6.68}$$

where the charges $I_{\mathbf{k}}^{(\mathbf{s},\pm)}$ are defined through (3.31) with driving term (6.68), and $\{\tau^{(-)},\tau^{(+)}\} = \vec{\tau}$ are two coupling parameters defined as

$$\tau^{(\pm)} = \tau e^{\pm\sigma}\;,\quad \pm\sigma = \lim_{\substack{\theta\to\pm\infty\\ \tilde{\mathbf{s}}\to 0^+}} \tilde{\mathbf{s}}\,\theta\;. \tag{6.69}$$

Therefore, the contributions of the non-trivial interaction can be formally re-absorbed in a redefinition of the vacuum parameters $\alpha_0^{(\pm)}$. In turn, this affects the value of the effective central charges as

$$c^{(\mathbf{s},\pm)} = c_0^{(\pm)} + 24\beta^2(\alpha_0^{(\pm)})^2 - 24\beta^2\left((\alpha_0^{(\pm)})^2 - \alpha^{(\mathbf{s},\pm)}\right)^2\;,\quad c_0^{(\pm)} = c - 24\beta^2(\alpha_0^{(\pm)})^2\;. \tag{6.70}$$

Considering the formal identification $Q_0^{(\pm)} = 2\pi\alpha_0^{(\pm)}\beta^2$ made in (6.64), the redefinition (6.68) corresponds to the following dressing of the topological charges

$$Q^{(\mathbf{s},\pm)}(R,\vec{\tau}) = Q_0^{(\pm)} - \frac{\kappa}{4\pi}\Big(\tau^{(-)} I_{\mathbf{s}}^{(\mathbf{s},-)}(R,\vec{\tau}) - \tau^{(+)} I_{\mathbf{s}}^{(\mathbf{s},+)}(R,\vec{\tau})\Big)\;, \tag{6.71}$$

with

$$\kappa = 8\pi^2\beta^2\;. \tag{6.72}$$

Let us focus on the $\mathbf{s} = 1$ case. The equations for the spectrum are

$$I_1^{(\pm)}(R,\vec{\tau}) = \frac{2\pi}{R}\left(n^{(\pm)} - \frac{c^{(\pm)}}{24}\right)\;,\quad \left(I_1^{(\pm)} = I_1^{(1,\pm)}\;,\; c^{(\pm)} = c^{(1,\pm)}\right)\;, \tag{6.73}$$

which can be solved exactly for $I_1^{(\pm)}(R,\vec{\tau})$ at any values of the parameters $\tau^{(\pm)}$. However, since the general analytic expressions for $I_1^{(\pm)}$ are very cumbersome, we will restrict the discussion to specific scaling limits:

$\boxed{\textbf{Case } \tau^{(+)} = \tau^{(-)} = \tau}$

$$I_1^{(+)}(R,\tau) - I_1^{(-)}(R,\tau) = P(R) = \frac{2\pi}{R}\left(h^{(+)} - h^{(-)}\right) , \tag{6.74}$$

$$I_1^{(+)}(R,\tau) = I_1^{(+)}(R) + \tau\frac{Q_0^{(\pm)}}{R}P(R) + \tau^2\frac{\kappa}{8\pi R}P^2(R) , \tag{6.75}$$

with $h^{(\pm)} = h_0^{(\pm)} + n^{(\pm)}$ and (6.71) becomes

$$Q^{(\pm)}(R,\tau) = Q_0^{(\pm)} + \tau\frac{\kappa}{4\pi}P(R) . \tag{6.76}$$

Surprisingly, with the replacement

$$I_1^{(\pm)}(R,\tau) = \frac{2\pi}{R}\left(h^{(\pm)}(\tau) - \frac{c}{24}\right) , \tag{6.77}$$

equations (6.74) and (6.75) lead to exact expressions for the conformal dimensions $h^{(\pm)}(\tau)$, which match precisely the form of the (left) conformal dimension in the $\mathrm{J\bar{T}}$ model, as recently shown in [31]:

$$h^{(\pm)}(\tau) = h^{(\pm)} + \tau\frac{Q_0^{(\pm)}}{2\pi}P + \tau^2\frac{\kappa}{16\pi^2}P^2 . \tag{6.78}$$

As we will shortly see, this is the first instance among many that link the phase factor (6.66) with $\mathrm{s} = 1$, in the scaling limit (6.69) to the quantum $\mathrm{J\bar{T}}$ model.

$\boxed{\textbf{Case } \tau^{(+)} = 0 \mid \tau^{(-)} = \tau}$

the two sectors $(\pm)$ are completely decoupled

$$\begin{aligned} I_1^{(\pm)}(R,\tau) &= I_1^{(\pm)}(R) \pm \tau\frac{Q_0^{(\pm)}}{R}I_1^{(\pm)}(R,\tau) + \tau^2\frac{\kappa}{8\pi R}\left(I_1^{(\pm)}(R,\tau)\right)^2 , & (6.79)\\ Q^{(\pm)}(R,\tau) &= Q_0^{(\pm)} \pm \tau\frac{\kappa}{4\pi}I_1^{(\pm)}(R,\tau) , & (6.80)\end{aligned}$$

and the solutions to (6.79) are

$$I_1^{(\pm)}(R,\tau) = \frac{4\pi}{\kappa\tau^2}\left(R \mp \tau Q_0^{(\pm)} - \sqrt{\left(R \mp \tau Q_0^{(\pm)}\right)^2 - \tau^2\kappa\left(h^{(\pm)} - \frac{c}{24}\right)}\right) . \tag{6.81}$$

We notice that (6.79) can be rewritten as

$$I_1^{(\pm)}(R,\tau) = \frac{2\pi\left(h^{(\pm)} - \frac{c}{24}\right)}{R - \tau\left(\pm Q_0^{(\pm)} + \tau\frac{\kappa}{8\pi}I_1^{(\pm)}(R,\tau)\right)} , \tag{6.82}$$

which suggests that, in this limit, the perturbation has a dual geometric description, since it can be interpreted as a redefinition of the length R. The dual deformation of the NLIE corresponds to a deformed version of $f_{\nu^{(\pm)}}(\theta)$ which fulfils (3.28) with

$$\nu^{(\pm)} = \nu^{(\pm)}\left(\mathcal{R}_0^{(\pm)}, \alpha_0 \,\middle|\, \theta\right) , \quad \mathcal{R}_0^{(\pm)} = R - \tau\left(\pm Q_0^{(\pm)} + \tau\frac{\kappa}{8\pi}I_1^{(\pm)}(R,\tau)\right) . \tag{6.83}$$

Expressions (6.81) are trivially solutions of two decoupled Burgers-like equations

$$\partial_\tau I_1^{(\pm)}(R,\tau) \pm Q^{(\pm)}(R,\tau)\,\partial_R I_1^{(\pm)}(R,\tau) = 0\,. \tag{6.84}$$

Therefore, $I_1^{(\pm)}(R,\tau)$ fulfils a Burgers-type equation analogous to (2.47), where $Q^{(\pm)}(R,\tau)$ play the role of velocities.

6.5 The quantum J$\bar{\mathrm{T}}$ model

Formula (6.81) strongly resembles the expression of the right-movers energy of a J$\bar{\mathrm{T}}$−deformed CFT derived in [29,30]. However, as we shall show below, the J$\bar{\mathrm{T}}$ deformation originates from an asymmetry in the $(\pm)$ sectors. For $\mathrm{s}=1$, a possible asymmetric generalisation of (6.68) with four free parameters $\vec{\tau} = \{\tau^{(a|b)}\}_{(a,b)\in\{+,-\}}$ is

$$\begin{aligned} \nu^{(\pm)} &= \nu^{(\pm)}\big(R, \alpha_0^{(\pm)} - \alpha^{(\pm)}(\vec{\tau})\,|\,\theta\big)\,, \\ \alpha^{(\pm)}(\vec{\tau}) &= \Big(\tau^{(\pm|-)} I_s^{(-)}(R,\vec{\tau}) - \tau^{(\mp|+)} I_s^{(+)}(R,\vec{\tau})\Big)\,, \end{aligned} \tag{6.85}$$

where we also allowed for the possibility of having two different initial values of the twist parameters $\alpha_0^{(\pm)}$ in the two sectors. From (6.85), the J$\bar{\mathrm{T}}$ model is recovered with the choice

$$\tau^{(\pm|-)} = 0\,, \quad \tau^{(\mp|+)} = \tau\,, \quad \alpha_0^{(+)} = \alpha_0^{(-)} = \alpha_0\,. \tag{6.86}$$

Correspondingly, relations (6.85) become

$$\nu^{(\pm)} = \nu^{(\mathrm{J\bar{T}},\pm)}\left(R, \alpha_0 - \alpha^{(\mathrm{J\bar{T}},\pm)}\,\middle|\,\theta\right)\,, \quad \alpha^{(\mathrm{J\bar{T}},\pm)} = -\tau\, I^{(\mathrm{J\bar{T}},+)}(R,\tau)\,. \tag{6.87}$$

Observation 47. Alternatively, the $(+)$ sector can be equivalently described with a redefinition of the length R as

$$\nu^{(+)} = \nu^{(+)}\left(\mathcal{R}^{\mathrm{J\bar{T}}}, \alpha_0\,|\,\theta\right)\,, \quad \mathcal{R}^{\mathrm{J\bar{T}}} = R - \tau\left(Q_0 + \tau\frac{\kappa}{8\pi}\, I^{(\mathrm{J\bar{T}},+)}(R,\tau)\right)\,. \tag{6.88}$$

The right-moving solution in (6.81) and the topological charge become

$$I^{(\mathrm{J\bar{T}},+)}(R,\tau) = \frac{4\pi}{\kappa\tau^2}\left(R - \tau Q_0 - \sqrt{(R-\tau Q_0)^2 - \tau^2\kappa\left(h^{(+)} - \frac{c}{24}\right)}\right)\,, \tag{6.89}$$

$$I^{(\mathrm{J\bar{T}},+)}(R,\tau) - I^{(\mathrm{J\bar{T}},-)}(R,\tau) = P(R) = \frac{2\pi}{R}\left(h^{(+)} - h^{(-)}\right)\,, \tag{6.90}$$

$$Q^{(+)}(R,\tau) = Q^{(\mathrm{J\bar{T}},+)}(R,\tau) = Q_0 + \tau\,\frac{\kappa}{4\pi}\, I^{(\mathrm{J\bar{T}},+)}(R,\tau)\,. \tag{6.91}$$

Therefore, $Q_0 = 2\pi\alpha_0\beta^2$ and $\kappa = 8\pi^2\beta^2$ have been again consistently identified with the topological charge and the chiral anomaly, respectively. The results described here are in full agreement with [30] and the classical results presented in section 5.3.

6.6 A simple example involving a pair of scattering phase factors

As already discussed, in principle one may introduce several scattering phase factors to deform the NLIEs. In this section we will consider a particular combination of $\mathbf{s} \to 0$ and $\tilde{\mathbf{s}} \to 0$ scattering factors which allows us to match with the classical results (5.111)-(5.112). We consider a double deformation made of a length redefinition (6.65) with $\tau^{(\pm|+)} = \mp\tau$ and $\tau^{(\pm|-)} = \pm\tau$, together with a shift of the twist parameter (6.85) with $\tau^{(\mp|+)} = +\tau$ and $\tau^{(\pm|-)} = +\tau$. The corresponding deformed driving term is then

$$\nu^{(\pm)} = \nu^{(\pm)}\left(\mathcal{R}_0^{(\pm)}, \alpha_0^{(\pm)} - \alpha^{(\pm)}(R,\tau)\,\middle|\,\theta\right) \ , \tag{6.92}$$

with

$$\begin{aligned} \alpha^{(\pm)}(R,\tau) &= -\tau\, P(R) \ , \\ \mathcal{R}_0^{(\pm)} &= R + \tau\left(Q^{(+)}(R,\tau) - Q^{(-)}(R,\tau)\right) = R + \tau\left(Q_0^{(+)} - Q_0^{(-)}\right) \ , \end{aligned} \tag{6.93}$$

where in the last equality we used the fact that $Q^{(\pm)}(R,\tau) = 2\pi\,\beta^2\left(\alpha_0^{(\pm)} - \alpha^{(\pm)}(R,\tau)\right)$. Since the central charges are affected by the deformation as

$$c^{(\pm)}(R,\tau) = c + 24\beta^2\left(\alpha_0^{(\pm)}\right)^2 - 24\beta^2\left(\alpha_0^{(\pm)} - \alpha^{(\pm)}(R,\tau)\right)^2 \ , \tag{6.94}$$

one finds

$$I_1^{(\pm)}(R,\tau) = \frac{2\pi}{\mathcal{R}_0^{(\pm)}}\left(n^{(\pm)} - \frac{c^{(\pm)}(R,\tau)}{24}\right) = \frac{R\, I_1^{(\pm)}(R) + \tau\, Q_0^{(\pm)} P(R) + \frac{\kappa}{8\pi}\,\tau^2\, P^2(R)}{R + \tau\left(Q_0^{(+)} - Q_0^{(-)}\right)} \ , \tag{6.95}$$

$$I_1^{(+)}(R,\tau) - I_1^{(-)}(R,\tau) = P(R) \ , \tag{6.96}$$

$$Q^{(\pm)}(R,\tau) = Q_0^{(\pm)} + \tau\frac{\kappa}{4\pi}\, P(R) \ , \tag{6.97}$$

which match exactly with the results (5.111)-(5.112) obtained at the classical level for the corresponding densities.

Chapter 7

Conclusions and outlook

The purpose of this thesis is to review some of the recent results concerning irrelevant deformations of 2−dimensional QFTs, including the publications [24, 43, 46] by the present author. In the first part of the work, we described the content of [24, 46], where a thorough investigation of the effect of the $\mathrm{T\bar{T}}$ deformation on the classical integrable setup was performed. The second part of the thesis contains, instead, the results of [43], in which an infinite family of irrelevant deformations – including $\mathrm{T\bar{T}}$ and $\mathrm{J\bar{T}}$ as particular representatives – was introduced and studied both at classical and quantum level.
Many new interesting physical phenomena have emerged from the study of this relatively new research field. However, there are still important long-standing open problems associated with irrelevant perturbations which certainly deserve further investigation.

- **integrable structure:** from the point of view of pure integrability, these exactly solvable deformations furnish a large variety of integrable models which would be interesting to address using rigorous methods such as *Hirota*, *Bäcklund* and *Inverse Scattering*. For example, they would enable a systematic analysis of multi-soliton solutions.

- **generalised CDD deformations:** As largely discussed in this dissertation, irrelevant deformations of IQFTs are related to the inclusion of phase factors in the $S-$matrix of the original theory. In particular, the most general family of CDD factors is given by (1.21), which can be more conveniently rewritten in the representation [1]

$$f(\theta) = \prod_{p=1}^{N} \frac{B_p - \mathbb{i}\sinh(\theta)}{B_p + \mathbb{i}\sinh(\theta)} \, , \tag{7.1}$$

  where $\cup_N \{B_p\}_{p=1}^N$ constitute a set of parameters alternative to $\{\tau^{(s)}\}$ in (1.21). It would be important to perform a systematic numerical exploration of the finite-size spectrum of a simple model deformed through (7.1), to understand how the UV behaviour is affected.

- **Correlation functions:** Most of the exact quantum results obtained in this field concern the spectrum. It would be nice to continue the study, along the lines of [19], of correlation functions of local operators which contain off-shell information on the theory. These studies might shed some light on the properties of these models in the deep UV regime and clarify issues related to non-locality and entanglement entropy [82]. An important step in this direction was made in [19], where exact evolution equations for correlation functions were proposed. It would be nice to check the correctness of the proposal of [19] using, for example, a perturbative approach.

- **Extension to higher dimensions:** The $T\bar{T}$ deformation and its higher-spin variants are all defined in $2-$dimensions. It is natural to look for consistent generalisations in higher dimensions which retain some of the special features displayed by the $T\bar{T}$ operator. For example, the factorisation property and a universal flow equation for the finite-size spectrum. An interesting proposal, based on holography-related arguments, appeared in [83]. Starting from a completely different perspective, the authors of [24] noted that the classical Maxwell Born-Infeld theory of non-linear electrodynamics in 4 dimensions, arises as a deformation of the Maxwell theory (without matter fields) generated by the square root of the determinant of the Hilbert stress-energy tensor (see appendix E.1). It is currently unclear whether or not it exists a connection between the ideas of [83] and [24]. Although the observation made in [24] involves a specific model of non-linear field theory which enjoys unique features (see, for example [84] and references therein), it would be beautiful to find an exact flow equation for its finite-volume spectrum, or at least for the Casimir energy.

Appendix A

A.1 Space-time conventions

Throughout the thesis, we deal with classical field theories defined on a $2-$dimensional flat Euclidean space-time. Depending on the situation, we parametrise it using cartesian or complex coordinates. We consider the two sets of cartesian coordinates $\mathbf{x} = (x^1, x^2)$ and $\mathbf{y} = (y^1, y^2)$ related to the sets of complex coordinates $\mathbf{z} = (z, \bar{z})$ and $\mathbf{w} = (w, \bar{w})$, respectively, through

$$\begin{cases} z = x^1 + \mathbb{i}\, x^2 \\ \bar{z} = x^1 - \mathbb{i}\, x^2 \end{cases} , \quad \begin{cases} w = y^1 + \mathbb{i}\, y^2 \\ \bar{w} = y^1 - \mathbb{i}\, y^2 \end{cases} , \tag{A.1}$$

The line element ds^2 in cartesian and complex coordinates reads

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dz\, d\bar{z} \ , \quad ds^2 = (dy^1)^2 + (dy^2)^2 = dw\, d\bar{w} \ , \tag{A.2}$$

since the basis of the cotangent space $d\mathbf{z} = (dz, d\bar{z})$, $d\mathbf{w} = (dw, d\bar{w})$, $d\mathbf{x} = (dx^1, dx^2)$ and $d\mathbf{y} = (dy^1, dy^2)$ transform as

$$\begin{cases} dz = dx^1 + \mathbb{i}\, dx^2 \\ d\bar{z} = dx^1 - \mathbb{i}\, dx^2 \end{cases} , \quad \begin{cases} dw = dy^1 + \mathbb{i}\, dy^2 \\ d\bar{w} = dy^1 - \mathbb{i}\, dy^2 \end{cases} . \tag{A.3}$$

Instead, the basis of the tangent space $\partial_{\mathbf{z}} = (\partial_z, \partial_{\bar{z}})$, $\partial_{\mathbf{w}} = (\partial_w, \partial_{\bar{w}})$, $\partial_{\mathbf{x}} = (\partial_{x^1}, \partial_{x^2})$ and $\partial_{\mathbf{y}} = (\partial_{y^1}, \partial_{y^2})$ transform as

$$\begin{cases} \partial_z = \frac{1}{2}\left(\partial_{x^1} - \mathbb{i}\, \partial_{x^2}\right) \\ \partial_{\bar{z}} = \frac{1}{2}\left(\partial_{x^1} + \mathbb{i}\, \partial_{x^2}\right) \end{cases} , \quad \begin{cases} \partial_w = \frac{1}{2}\left(\partial_{y^1} - \mathbb{i}\, \partial_{y^2}\right) \\ \partial_{\bar{w}} = \frac{1}{2}\left(\partial_{y^1} + \mathbb{i}\, \partial_{y^2}\right) \end{cases} . \tag{A.4}$$

Eventually, we will move from Euclidean to Minkowski signature by means of a Wick rotation

$$x^2 \equiv -\mathbb{i}\, t_x \ , \quad y^2 \equiv -\mathbb{i}\, t_y \ , \quad (t \in \mathbb{R}) \ , \tag{A.5}$$

where $\mathbf{x}_{\mathrm{M}} = (x, t_x)$, $\mathbf{y}_{\mathrm{M}} = (y, t_y)$, $(x \equiv x^1 \ , \ y \equiv y^1)$, are the cartesian coordinates in the $2-$dimensional Minkowski space-time. In some cases, we will switch from cartesian $\mathbf{x}_{\mathrm{M}} = (x, t_x)$, $\mathbf{y}_{\mathrm{M}} = (y, t_y)$ to light-cone coordinates $\tilde{\mathbf{x}}_{\mathrm{M}} = (x^+, x^-)$, $\tilde{\mathbf{y}}_{\mathrm{M}} = (y^+, y^-)$ respectively, according to

$$x^{\pm} = x \pm t_x \ , \quad y^{\pm} = y \pm t_y \ , \tag{A.6}$$

which are obtained from (A.1) by applying the Wick rotation (A.5).

A.2 Local conserved currents

Local translational and rotational symmetry of the theory reflects, according to the Noether's theorem, into the existence of a conserved current, *i.e.* the stress-energy tensor $\boldsymbol{T} \in \mathcal{F}$. In IFTs there exists an infinite set of local conserved currents $\{\boldsymbol{T}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbb{Z}} \in \mathcal{F}$, which do not descend from a Noetherian symmetry. The symbol $\mathbf{s}$ is the so-called *rank* or *Lorentz spin* and, in general, it takes values in $\mathbb{Z}$. Local currents are generically polynomials in the field and its derivatives and $|\mathbf{s}|$ counts the total number of ∂_z derivatives minus the total number of $\partial_{\bar{z}}$ derivatives. Notice that the stress-energy tensor represents the rank-1 current of the hierarchy, *i.e.* $\boldsymbol{T} \equiv \boldsymbol{T}_1$. Formally, $\boldsymbol{T}_{\mathbf{s}}$ is a rank-2 tensor and its components in the set of coordinates $\mathbf{x}$ and $\mathbf{z}$ are related through

$$\begin{aligned}
(\boldsymbol{T}_{\mathbf{s}})_{11} &= (\boldsymbol{T}_{\mathbf{s}})_{zz} + (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}} + (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}} + (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z} \ , \\
(\boldsymbol{T}_{\mathbf{s}})_{12} &= \mathbb{i}\left[(\boldsymbol{T}_{\mathbf{s}})_{zz} - (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}} - (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}} + (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z}\right] , \\
(\boldsymbol{T}_{\mathbf{s}})_{21} &= \mathbb{i}\left[(\boldsymbol{T}_{\mathbf{s}})_{zz} - (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}} + (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}} - (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z}\right] , \\
(\boldsymbol{T}_{\mathbf{s}})_{22} &= -(\boldsymbol{T}_{\mathbf{s}})_{zz} - (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}} + (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}} + (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z} \ ,
\end{aligned} \tag{A.7}$$

where $(\boldsymbol{T}_{\mathbf{s}})_{x^\mu x^\nu} = (\boldsymbol{T}_{\mathbf{s}})_{\mu\nu}$, $((\mu,\nu) \in \{1,2\})$, and $\{(\boldsymbol{T}_{\mathbf{s}})_{zz}, (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}}, (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}}, (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z}\}$ denote the chiral components, which are normalized according to the standard CFT convention [85]

$$\mathbf{T}_{\mathbf{s}+1} = -2\pi\ (\boldsymbol{T}_{\mathbf{s}})_{zz} \ , \quad \bar{\mathbf{T}}_{\mathbf{s}+1} = -2\pi\ (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}} \ , \quad \boldsymbol{\Theta}_{\mathbf{s}-1} = 2\pi\ (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}} \ , \quad \bar{\boldsymbol{\Theta}}_{\mathbf{s}-1} = 2\pi\ (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z} \ . \tag{A.8}$$

In line with the conventions of [24, 43], throughout the text we mainly use the rescaled chiral components

$$\begin{aligned}
\mathrm{T}_{\mathbf{s}+1} = -(\boldsymbol{T}_{\mathbf{s}})_{zz} &= \frac{\mathbf{T}_{\mathbf{s}+1}}{2\pi} \quad , \qquad \bar{\mathrm{T}}_{\mathbf{s}+1} = -(\boldsymbol{T}_{\mathbf{s}})_{\bar{z}\bar{z}} = \frac{\bar{\mathbf{T}}_{\mathbf{s}+1}}{2\pi} \ , \\
\Theta_{\mathbf{s}-1} = (\boldsymbol{T}_{\mathbf{s}})_{z\bar{z}} &= \frac{\boldsymbol{\Theta}_{\mathbf{s}-1}}{2\pi} \quad , \qquad \bar{\Theta}_{\mathbf{s}-1} = (\boldsymbol{T}_{\mathbf{s}})_{\bar{z}z} = \frac{\bar{\boldsymbol{\Theta}}_{\mathbf{s}-1}}{2\pi} \ .
\end{aligned} \tag{A.9}$$

As already mentioned at the beginning of this appendix, $\mathbf{s}$ can be both a positive and a negative integer. However, assuming that the theory is invariant under parity transformation, the currents are symmetric under the exchange $\mathbf{s} \leftrightarrow -\mathbf{s}$. For this reason, we split the set $\{\mathbf{s}\} = \{s'\} \cup \{0\} \cup \{s\}$ into a positive ($\{s\}$) and a negative subset ($\{s'\}$) using the notation

$$s = |\mathbf{s}| > 0 \ , \quad s' = -|\mathbf{s}| < 0 \ , \quad (\forall\, \mathbf{s} \neq 0) \ . \tag{A.10}$$

The case $\mathbf{s} = 0$ is quite special and therefore it is considered separately. Starting from the level$-s$ rescaled chiral components $\{\mathrm{T}_{s+1}, \Theta_{s-1}\}$, the corresponding level$-s'$ components $\{\mathrm{T}_{s'+1}, \Theta_{s'-1}\}$ are obtained through the following reflection

$$\boxed{\mathrm{T}_{s'+1} = \bar{\Theta}_{s-1} \ , \quad \Theta_{s'-1} = \bar{\mathrm{T}}_{s+1} \ .} \tag{A.11}$$

The conservation of $\boldsymbol{T}_{\mathbf{s}}$ is represented by the continuity equations, namely

$$\boxed{\partial_\mu \boldsymbol{T}^{\mu\nu}_{\mathbf{s}}(\mathbf{x}) = 0 \ , \quad (\nu = 1,2) \ ,} \tag{A.12}$$

in cartesian coordinates $\mathbf{x}$ with $\partial_\mu = \frac{\partial}{\partial x^\mu}$, and

$$\boxed{\partial_{\bar{z}} \mathrm{T}_{\mathbf{s}+1}(\mathbf{z}) = \partial_z \Theta_{\mathbf{s}-1}(\mathbf{z}) \ , \quad \partial_z \bar{\mathrm{T}}_{\mathbf{s}+1}(\mathbf{z}) = \partial_{\bar{z}} \bar{\Theta}_{\mathbf{s}-1}(\mathbf{z}) \ ,} \tag{A.13}$$

in complex coordinates $\mathbf{z}$, using the rescaled components (A.9). From (A.12), it follows that the quantities

$$Q^\mu_{\mathbf{s}} = \int_C \boldsymbol{T}^{2\mu}_{\mathbf{s}}(\mathbf{x})\, dx^1 \ , \quad (\mu = 1, 2) \ , \tag{A.14}$$

do not depend on x^2 if we assume that $\boldsymbol{T}^{1\mu}_{\mathbf{s}}(\mathbf{x})$ vanish at the boundary of the domain $C \subseteq \mathbb{R}$. Since $Q^\mu_{\mathbf{s}}$ do not evolve in time, we refer to them as *conserved charges*. Using the standard convention for $\boldsymbol{T}_{\mathbf{s}}$ in cartesian coordinates, we have

$$\boldsymbol{T}^{21}_{\mathbf{s}} = \mathbb{i}\, \mathcal{P}_{\mathbf{s}} \ , \quad \boldsymbol{T}^{22}_{\mathbf{s}} = -\mathcal{H}_{\mathbf{s}} \ , \tag{A.15}$$

where $\mathcal{H}_{\mathbf{s}}(\mathbf{x})$ and $\mathcal{P}_{\mathbf{s}}(\mathbf{x})$ are the level$-\mathbf{s}$ Hamiltonian and momentum density, respectively. In terms of the rescaled chiral components, they are written as

$$\boxed{\mathcal{P}_{\mathbf{s}} = \mathcal{I}_{\mathbf{s}} - \bar{\mathcal{I}}_{\mathbf{s}} \ , \quad \mathcal{H}_{\mathbf{s}} = \mathcal{I}_{\mathbf{s}} + \bar{\mathcal{I}}_{\mathbf{s}} \ ,} \tag{A.16}$$

where we defined

$$\boxed{\mathcal{I}_{\mathbf{s}} = -\left(\mathrm{T}_{\mathbf{s}+1} + \Theta_{\mathbf{s}-1}\right) \ , \quad \bar{\mathcal{I}}_{\mathbf{s}} = -\left(\bar{\mathrm{T}}_{\mathbf{s}+1} + \bar{\Theta}_{\mathbf{s}-1}\right) \ .} \tag{A.17}$$

Therefore

$$Q^1_{\mathbf{s}} = \mathbb{i} \int_C \mathcal{P}_{\mathbf{s}}(\mathbf{x})\, dx^1 = \mathbb{i}\, P_{\mathbf{s}} = \mathbb{i}\left(I_{\mathbf{s}} - \bar{I}_{\mathbf{s}}\right) \ , \quad Q^2_{\mathbf{s}} = -\int_C \mathcal{H}_{\mathbf{s}}(\mathbf{x})\, dx^1 = -E_{\mathbf{s}} = -\left(I_{\mathbf{s}} + \bar{I}_{\mathbf{s}}\right) \ , \tag{A.18}$$

where $E_{\mathbf{s}}$ and $P_{\mathbf{s}}$ are the level$-\mathbf{s}$ total energy and momentum, integrated over the domain C, and analogously

$$I_{\mathbf{s}} = \int_C \mathcal{I}_{\mathbf{s}}(\mathbf{x})\, dx^1 \ , \quad \bar{I}_{\mathbf{s}} = \int_C \bar{\mathcal{I}}_{\mathbf{s}}(\mathbf{x})\, dx^1 \ . \tag{A.19}$$

Appendix B

B.1 Partition function of a $T\bar{T}-$deformed Yang-Mills theory

In this appendix we review a result of [46], in which the *partition function* and *heat kernel* for a $T\bar{T}-$deformed Yang-Mills theory were conjectured. In support of the conjecture, several consistency checks are provided.

The partition function of a Yang-Mills theory with generic gauge group G on a $2-$dimensional orientable manifold $\mathcal{M}$ with genus p and metric $g_{\mu\nu}$ is [86–89]

$$Z^{\mathcal{M}}(A) = \int \mathcal{D}\mathcal{A}_\mu \, e^{-\frac{1}{4\tilde{g}^2}\int_{\mathcal{M}} dx^2 \sqrt{g}\,\mathrm{Tr}[F^a_{\mu\nu}F^{\mu\nu}_a]} = \sum_{\mathcal{R}} d_{\mathcal{R}}^{2-2p} e^{-\frac{\tilde{g}^2}{2}A\,\mathbf{C}_2(\mathcal{R})} \,, \tag{B.1}$$

where

- $\tilde{g}$ is the Yang-Mills coupling constant;
- A is the total area of $\mathcal{M}$;
- the index $\mathcal{R}$ in the sum $\sum_{\mathcal{R}}$ runs over all the equivalence classes of irreducible representations $\mathcal{R}$ of the gauge group G;
- $d_{\mathcal{R}}$ is the dimension of $\mathcal{R}$;
- $\mathbf{C}_2(\mathcal{R})$ is the quadratic Casimir in the representation $\mathcal{R}$.

The generalization of (B.1) to a manifold with genus p and n boundaries corresponds to the so-called heat kernel:

$$Z^{\mathcal{M}}(g_1,\dots,g_n|A) = \sum_{\mathcal{R}} d_{\mathcal{R}}^{2-2p-n} \chi_{\mathcal{R}}(g_1)\dots\chi_{\mathcal{R}}(g_n) e^{-\frac{\tilde{g}^2}{2}A\,\mathbf{C}_2(\mathcal{R})} \,, \tag{B.2}$$

where g_i are the Wilson loops evaluated along the boundaries, and $\chi_{\mathcal{R}}$ denotes the Weyl character of the representation $\mathcal{R}$.

According to (2.138), it is natural to conjecture that the $T\bar{T}$ contribution to the partition function and heat kernel consists in a simple redefinition of the eigenvalues of the quadratic Casimir operator

$$\boxed{\mathbf{C}_2(\mathcal{R}) \to \mathbf{C}_2(\mathcal{R},\tau) = \frac{\mathbf{C}_2(\mathcal{R})}{1-\tau\frac{\tilde{g}^2}{2}\mathbf{C}_2(\mathcal{R})} \,.} \tag{B.3}$$

Since (B.2) depends only on the surface area A of the manifold, the deformed version $Z^{\mathcal{M}}(g_1,\ldots,g_n;\tau|A)$ fulfils

$$\boxed{-\partial_\tau Z^{\mathcal{M}}(g_1,\ldots,g_n;\tau|A) = A\,\partial_A^2 Z^{\mathcal{M}}(g_1,\ldots,g_n;\tau|A)\ .} \tag{B.4}$$

With the prescription (B.3), all the diffusion-type relations introduced in [18] (see also [20,23]) for the partition functions on various geometries are automatically fulfilled:

- **Cylinder**: The cylinder partition function $Z^{\text{Cyl}}(g_1,g_2|A)$ corresponds to the $n=2$, $p=0$ case of (B.2). Setting $A=RL$, and implementing the prescription (B.3), $Z^{\text{Cyl}}(g_1,g_2;\tau|A)$ trivially satisfies Cardy's equation:

$$-\partial_\tau Z^{\text{Cyl}}(g_1,g_2;\tau|A) = (\partial_L - 1/L)\partial_R Z^{\text{Cyl}}(g_1,g_2;\tau|A)\ . \tag{B.5}$$

- **Torus**: The partition function on the torus, $Z^{\text{T}}(A)$ corresponds to the $n=0$, $p=1$ case of (B.2) with $A = L_1L_2' - L_2L_1'$, while the consistency equation for the deformed partition function is:

$$-\partial_\tau Z^{\text{T}}(\tau|A) = \left[\partial_{L_1}\partial_{L_2'} - \partial_{L_2}\partial_{L_1'} - \frac{1}{A}\left(L_1\partial_{L_1} + L_1'\partial_{L_1'} + L_2\partial_{L_2} + L_2'\partial_{L_2'}\right)\right] Z^{\text{T}}(\tau|A)\ . \tag{B.6}$$

- **Disk and Cone**: In the case of a disk, or more in general of a cone with opening angle $\mathcal{X}$, the deformed partition function $Z^{\text{Cone}}(g_1;\tau|A)$ corresponding to $n=1$, $p=0$ and area $A=\frac{1}{2}\mathcal{X}R^2$ satisfies

$$-\partial_\tau Z^{\text{Cone}}(g_1;\tau|A) = \frac{1}{R}\mathcal{X}\partial_{\mathcal{X}}\left(\frac{1}{\mathcal{X}}\partial_R Z^{\text{Cone}}(g_1;\tau|A)\right)\ . \tag{B.7}$$

The modification (B.3) in (B.2) is expected to hold in general for any value of p and n, possibly leading to a consistent deformation of the whole Yang-Mills setup. The authors of [90] investigated thoroughly the partition function of T$\bar{\text{T}}-$deformed Yang-Mills on a sphere, showing that the Douglas-Kazakov phase transition still persists for a range of values of τ.

Appendix C

C.1 Short review on surfaces embedded in $\mathbb{R}^3$

In this appendix we briefly review the basics of the classical theory of surfaces embedded in $\mathbb{R}^3$ following the standard constructive approach which can be found, for example, in [91].

Let us consider a surface Σ and the vector-valued function

$$\mathbf{r} \,:\, \mathbb{R}^2 \to \mathbb{R}^3 \,:\, \mathbf{z} = (z^1, z^2) \in \mathbb{R}^2 \to \mathbf{r}(\mathbf{z}) = \big(r_1(\mathbf{z}), r_2(\mathbf{z}), r_3(\mathbf{z})\big) \in \mathbb{R}^3 \;, \tag{C.1}$$

which defines the embedding in $\mathbb{R}^3$. The vectors

$$\mathbf{r}_\mu = \frac{\partial}{\partial z^\mu}\mathbf{r} \;, \quad (\mu = 1, 2) \;, \tag{C.2}$$

span the tangent plane $T_P\Sigma$ to the surface at any non-critical point $P \in \Sigma$.[1] For simplicity, we disregard the presence of critical points, hence we assume that $\mathbf{r}_1(\mathbf{z}) \neq \mathbf{r}_2(\mathbf{z}) \;,\; \forall\, \mathbf{z} \in \mathrm{dom}(\mathbf{r})$. The basis of the tangent plane can be promoted to a basis σ of $\mathbb{R}^3$ by adding to (C.2) the unit normal vector $\mathbf{n}$

$$\sigma = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}\} \;, \qquad \mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} \;. \tag{C.3}$$

The surface Σ inherits a metric structure from the ambient space $\mathbb{R}^3$ and its line element, also known as *first fundamental quadratic form*, is

$$\mathrm{I} \equiv ds^2 = d\mathbf{r} \cdot d\mathbf{r} = g_{\mu\nu}\, dz^\mu dz^\nu \;, \quad g_{\mu\nu} = \mathbf{r}_\mu \cdot \mathbf{r}_\nu \;, \tag{C.4}$$

and the tensor $g_{\mu\nu}$ is called *first fundamental tensor* or *metric tensor* of the surface Σ. Then, we define the so-called *second fundamental quadratic form* as

$$\mathrm{II} = -d\mathbf{r} \cdot d\mathbf{n} = d_{\mu\nu}\, dz^\mu dz^\nu \;, \quad d_{\mu\nu} = \left(\frac{\partial}{\partial z^\mu}\mathbf{r}_\nu\right) \cdot \mathbf{n} \;, \tag{C.5}$$

where the tensor $d_{\mu\nu}$ describes the projection of the vectors $\frac{\partial}{\partial z^\mu}\mathbf{r}_\nu$ on the normal direction at each point $P \in \Sigma$ and measures how much the surface curves away from the tangent space in an infinitesimal interval around P. According to the classical theorem by Bonnet [63], these two objects

[1] We say that $P_c \in \Sigma$ is a critical point if $\mathbf{r}_1(\mathbf{z}_c) = \mathbf{r}_2(\mathbf{z}_c)$, where $\mathbf{z}_c$ is such that $\mathbf{r}(\mathbf{z}_c) = P_c$.

uniquely define, up to isometries, the embedding of a surface in a flat 3−dimensional space.

From $g_{\mu\nu}$ and $d_{\mu\nu}$, we define a new object

$$s_\mu^\nu = d_{\mu\rho}\, g^{\rho\nu} \ , \quad g_{\mu\rho}\, g^{\rho\nu} = \delta_\mu^\nu \ , \tag{C.6}$$

known as *shape* or *Weingarten operator*, whose eigenvalues κ_1 and κ_2 are the *principal curvatures* of the surface Σ. The latter quantities are geometric invariants, meaning that they do not change under reparametrisations of the surface. Usually, κ_1 and κ_2 are combined into the *Gauss* and *mean curvatures*

$$K = \kappa_1\kappa_2 = \det\left[s_\mu^\nu\right] \ , \quad H = \frac{\kappa_1+\kappa_2}{2} = \frac{1}{2}s_\mu^\mu \ . \tag{C.7}$$

The tensors $g_{\mu\nu}$ and $d_{\mu\nu}$ determine the dynamics of the frame σ along the surface, which is encoded in the *Gauss equations*

$$\frac{\partial}{\partial z^\mu}\mathbf{r}_\nu = \Gamma^\rho_{\mu\nu}\,\mathbf{r}_\rho + d_{\mu\nu}\,\mathbf{n} \ , \tag{C.8}$$

and the *Weingarten equations*

$$\frac{\partial}{\partial z^\mu}\mathbf{n} = s_\mu^\nu\,\mathbf{r}_\nu \ , \tag{C.9}$$

where we introduced the *Christoffel symbols*

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial}{\partial z^\nu}g_{\mu\sigma} + \frac{\partial}{\partial z^\mu}g_{\nu\sigma} - \frac{\partial}{\partial z^\sigma}g_{\mu\nu}\right) \ . \tag{C.10}$$

Observe that the Gauss–Weingarten equations can be recast into the following linear system

$$\frac{\partial}{\partial z^\mu}\sigma = U_\mu\,\sigma \ , \tag{C.11}$$

with[1]

$$U_1 = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & d_{11} \\ \Gamma^1_{12} & \Gamma^2_{12} & d_{12} \\ -s_1^1 & -s_1^2 & 0 \end{pmatrix} \ , \qquad U_2 = \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & d_{12} \\ \Gamma^1_{22} & \Gamma^2_{22} & d_{22} \\ -s_2^1 & -s_2^2 & 0 \end{pmatrix} \ . \tag{C.12}$$

These structural equations are subject to a set of compatibility conditions called *Gauss-Mainardi-Codazzi* (GMC) system, which takes the form of a ZCR for the matrices $\{U_\mu\}_{\mu=1}^2$

$$\partial_2 U_1 - \partial_1 U_2 + [U_1, U_2] = 0 \ . \tag{C.13}$$

However, notice that the matrices $\{U_\mu\}_{\mu=1}^2$ do not form a Lax pair in the usual sense, since no spectral parameter is present and they do not belong to a semi-simple Lie algebra. Working out this general construction for the sine-Gordon model, we will show how to build a proper Lax pair out of $\{U_\mu\}_{\mu=1}^2$.

[1] Note that $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ and $d_{\mu\nu} = d_{\nu\mu}$.

As a first example, let us consider a pseudo-spherical surface, *i.e.* the Gauss curvature is $K = -\mu^2 < 0$, $(\mu \in \mathbb{R})$. In this case, one can choose as parametric curves the asymptotic lines, for which $d_{11} = d_{22} = 0$. Setting $\Delta^2 = \det[g_{\mu\nu}]$, we see that

$$K = -\frac{d_{12}^2}{\Delta^2} . \tag{C.14}$$

After some manipulations [91], it can be shown that in this case the GMC system implies

$$\Gamma^1_{12} = \Gamma^2_{12} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial z^2}(g_{11}) = \frac{\partial}{\partial z^1}(g_{22}) = 0 . \tag{C.15}$$

Defining the angle ω between the parametric lines as

$$\cos\omega = \frac{g_{12}}{\sqrt{g_{11}g_{22}}} , \quad \sin\omega = \frac{\Delta}{\sqrt{g_{11}g_{22}}} , \tag{C.16}$$

we have the following expression for the fundamental forms

$$\mathrm{I} = g_{11}\left(dz^1\right)^2 + 2\sqrt{g_{11}g_{22}}\,\cos\omega\, dz^1 dz^2 + g_{22}\left(dz^2\right)^2 , \quad \mathrm{II} = 2\mu\sqrt{g_{11}g_{22}}\,\sin\omega\, dz^1 dz^2 . \tag{C.17}$$

Now, given the (anti-)holomorphicity of g_{11} and g_{22}, we can define the rescaled variables $z'^\mu = \sqrt{g_{\mu\mu}}\, z^\mu$ (no summation over repeated indices here) in terms of which one has[1]

$$\mathrm{I} = \left(dz'^1\right)^2 - 2\cos\omega\, dz'^1 dz'^2 + \left(dz'^2\right)^2 , \quad \mathrm{II} = 2\mu\sin\omega\, dz'^1 dz'^2 . \tag{C.18}$$

It is possible to show that the GMC system (C.13) reduces to the sine-Gordon equation

$$\frac{\partial}{\partial z'^1}\frac{\partial}{\partial z'^2}\omega = \mu^2 \sin\omega . \tag{C.19}$$

Let us now consider the matrices $\{U_\mu\}_{\mu=1}^2$

$$U_1 = \begin{pmatrix} \omega_1 \cot\omega & -\omega_1 \csc\omega & 0 \\ 0 & 0 & \mu\sin\omega \\ \mu\cot\omega & -\mu\csc\omega & 0 \end{pmatrix} , \quad U_2 = \begin{pmatrix} 0 & 0 & \mu\sin\omega \\ -\omega_2 \csc\omega & \omega_2 \cot\omega & 0 \\ -\mu\csc\omega & \mu\cot\omega & 0 \end{pmatrix} , \tag{C.20}$$

where $\omega_\mu = \frac{\partial}{\partial z^\mu}\omega$. As already mentioned, the matrices (C.20) do not form a Lax pair since they do not belong to a semi-simple Lie algebra – in this case we would expect $\mathfrak{su}(2)$ due to the appearence of the sine-Gordon EoMs – and do not contain a spectral parameter. We can fix these apparent problems with the following considerations. First we notice that the triple $\sigma = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{n}\}$ is not orthonormal. However, the rotation

$$\sigma \longrightarrow \tilde{\sigma} = M\sigma , \quad M = \begin{pmatrix} 1 & 0 & 0 \\ -\cot\omega & \csc\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{C.21}$$

[1] This corresponds to a parametrisation of the surface by arc-length along the asymptotic lines.

which corresponds to a gauge transformation on the matrices $\{U_\mu\}_{\mu=1}^{2}$

$$U_\mu \;\longrightarrow\; \tilde{U} = \left(\partial_\mu M\right) M^{-1} + M U_\mu M^{-1} \,, \tag{C.22}$$

leaves the compatibility equation – the sine-Gordon equation – invariant and maps (C.20) into

$$\tilde{U}_1 = \begin{pmatrix} 0 & -\omega_1 & 0 \\ \omega_1 & 0 & \mu \\ 0 & -\mu & 0 \end{pmatrix} \,, \quad \tilde{U}_2 = \begin{pmatrix} 0 & 0 & \mu\sin\omega \\ 0 & 0 & -\mu\cos\omega \\ -\mu\sin\omega & \mu\cos\omega & 0 \end{pmatrix} \,, \tag{C.23}$$

which belong to the $\mathfrak{su}(2)$ algebra. Finally, the spectral parameter can be recovered by noticing that the sine-Gordon equation is invariant under the following transformation

$$\left(z'^1, z'^2, \mu\right) = \left(\alpha\tilde{z}^1, \beta\tilde{z}^2, \frac{1}{\sqrt{\alpha\beta}}m\right) \,, \quad \left(\forall\, \alpha, \beta \in \mathbb{C}\right) . \tag{C.24}$$

Choosing $\alpha = \sqrt{2}m$, $\beta = \sqrt{2}\frac{m}{\lambda^2}$ and writing $\omega = \beta\phi$, we obtain

$$\mathrm{I} = 2m^2\left(\left(d\tilde{z}^1\right)^2 - \frac{2}{\lambda^2}\cos\left(\beta\phi\right) d\tilde{z}^1 d\tilde{z}^2 + \frac{1}{\lambda^4}\left(d\tilde{z}^2\right)^2\right) \,, \quad \mathrm{II} = 2\sqrt{2}\frac{m^2}{\lambda}\sin\left(\beta\phi\right) d\tilde{z}^1 d\tilde{z}^2 \,, \tag{C.25}$$

which coincide with the quadratic forms (4.55)-(4.56).

Finally, as another interesting example of integrable model associated to embedded surfaces, let us briefly discuss a surface with constant mean curvature. In this case one can choose conformal coordinates, in which the fundamental forms simplify to

$$\mathrm{I} = \frac{2}{H^2} e^{\omega} dz^1 dz^2 \,, \quad \mathrm{II} = \frac{1}{H}\left[A_1\left(dz^1\right)^2 + 2e^{\omega} dz^1 dz^2 + A_2\left(dz^2\right)^2\right] . \tag{C.26}$$

Some simple computations show that the GCM equations are equivalent to the system

$$\frac{\partial}{\partial z^1}\frac{\partial}{\partial z^2}\omega = e^{\omega} - A_1 A_2\, e^{-\omega} \,, \quad \frac{\partial}{\partial z^2}A_1 = \frac{\partial}{\partial z^1}A_2 = 0 \,, \tag{C.27}$$

which is known as *modified sinh-Gordon equation*. Its Gauss curvature is

$$K = H^2\left(1 - A_1 A_2\, e^{-2\omega}\right) . \tag{C.28}$$

Rescaling the field as $\omega \to \omega + 2\log H$, the functions $\{A_i\}_{i=1}^{2}$ as $A_i \to H A_i$ and sending $H \to 0$, yields a minimal surface and reduces the GMC system to the Liouville equation

$$\frac{\partial}{\partial z^1}\frac{\partial}{\partial z^2}\omega = K e^{\omega} \,, \quad K = -A_1 A_2\, e^{-2\omega} . \tag{C.29}$$

C.2 Computation of the fundamental quadratic forms from sine-Gordon ZCR

While in appendix C.1 we presented the derivation of soliton equations starting from the basic geometric data of some particular surface, here we follow the reverse path and explicitly show how to obtain the forms (4.55)-(4.56) starting from the sine-Gordon ZCR (4.53)-(4.54). First of all we need to find a basis of $\mathfrak{su}\,(2)$ w.r.t. the Killing form

$$(a,b)_K = \mathrm{Tr}\,\left[\mathbf{Ad}(a)\,\mathbf{Ad}(b)\right]\ ,\quad \left(a,b\in\mathfrak{su}(2)\right)\ . \tag{C.30}$$

In the adjoint representation one has $\mathbf{t}^i = \mathbf{Ad}\left(\mathbf{u}^i\right)$, $(i=1,2,3)$, with

$$\mathbf{t}^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,\quad \mathbf{t}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\ ,\quad \mathbf{t}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\ , \tag{C.31}$$

and

$$\left(\mathbf{t}^i,\mathbf{t}^j\right)_K = -2\delta^{ij}\ . \tag{C.32}$$

The orthonormal basis is easily found to be

$$\mathbf{e}^i = \frac{\mathbb{i}}{\sqrt{2}}\mathbf{u}^i\ , \tag{C.33}$$

and we see that for a pair of matrices A and B belonging to the 2−dimensional representation of $\mathfrak{su}(2)$, one has

$$(A,B)_K = 4\ \mathrm{Tr}\ [AB]\ . \tag{C.34}$$

To compute the metric tensor $g_{\mu\nu}$ we need the partial derivatives of r (4.50). One finds

$$\frac{\partial r}{\partial z^\mu} = \Phi^{-1}\frac{\partial}{\partial\lambda}\left(L_\mu\Phi\right) - \Phi^{-1}L_\mu\Phi\Phi^{-1}\frac{\partial}{\partial\lambda}\Phi = \Phi^{-1}\frac{\partial L_\mu}{\partial\lambda}\Phi\ . \tag{C.35}$$

where in the first equality we used the linear system $\partial_\mu\Phi = L_\mu\Phi$, from which $g_{\mu\nu}$ results

$$g_{\mu\nu} = \left(\frac{\partial r}{\partial z^\mu},\frac{\partial r}{\partial z^\nu}\right)_K = 4\,\mathrm{Tr}\left[\frac{\partial r}{\partial z^\mu}\frac{\partial r}{\partial z^\nu}\right] = 4\,\mathrm{Tr}\left[\frac{\partial L_\mu}{\partial\lambda}\frac{\partial L_\nu}{\partial\lambda}\right]\ . \tag{C.36}$$

Inserting the expressions (4.53)-(4.54) we obtain

$$g_{\mu\nu} = 2m^2\begin{pmatrix} 1 & -\frac{1}{\lambda^2}\cos\left(\beta\phi\right) \\ -\frac{1}{\lambda^2}\cos\left(\beta\phi\right) & \frac{1}{\lambda^4} \end{pmatrix}_{\mu\nu}\ . \tag{C.37}$$

To compute the second fundamental tensor $d_{\mu\nu}$, instead, we need the second derivatives of r and the unit normal vector. One finds

$$\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z^\nu}r = \Phi^{-1}\left(\frac{\partial}{\partial z^\nu}\frac{\partial L_\mu}{\partial\lambda} + \left[\frac{\partial L_\mu}{\partial\lambda},L_\nu\right]\right)\Phi\ . \tag{C.38}$$

while the matrix version of the unit normal is

$$n = \sum_{i=1}^{3} n_i\mathbf{t}^i = \frac{1}{2\sqrt{2}}\frac{\left[\frac{\partial r}{\partial z^1},\frac{\partial r}{\partial z^2}\right]}{\sqrt{\det\left[\left[\frac{\partial r}{\partial z^1},\frac{\partial r}{\partial z^2}\right]\right]}}\ . \tag{C.39}$$

where

$$\det\left[\left[\frac{\partial r}{\partial z^1},\frac{\partial r}{\partial z^2}\right]\right] = \left(\frac{m^2}{2\lambda^2}\sin\left(\beta\phi\right)\right)^2\ . \tag{C.40}$$

Putting all together, the second fundamental tensor results

$$\begin{aligned} d_{\mu\nu} &= \left(\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z^\nu} r, n\right)_K \\ &= \frac{1}{\sqrt{2}}\frac{\lambda^2}{m^2 \sin(\beta\phi)} \mathrm{Tr}\left[\left[\frac{\partial L_1}{\partial \lambda}, \frac{\partial L_2}{\partial \lambda}\right]\left(\frac{\partial}{\partial z^\nu}\frac{\partial L_\mu}{\partial \lambda} + \left[\frac{\partial L_\mu}{\partial \lambda}, L_\nu\right]\right)\right] . \end{aligned} \tag{C.41}$$

and using (4.53)-(4.54) it explicitly reads

$$d_{\mu\nu} = \frac{\sqrt{2} m^2}{\lambda} \sin(\beta\phi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{C.42}$$

Appendix D

D.1 Spectral flow equations in the CFT limit

In this appendix we prove that the general Burgers equations (6.33), which we report here for convenience

$$\partial_\tau I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau) + R' \partial_R I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau) = \pm \mathbf{k} \theta_0' I_{\mathbf{k}}^{(\mathbf{s},\pm)}(R,\tau) \ , \tag{D.1}$$

with R' and θ_0' defined through (6.36), reduce to

$$\partial_\tau I_k^{(s,\pm)}(R,\tau) \ = \ 2 I_s^{(s,\mp)}(R,\tau)\, \partial_R I_k^{(s,\pm)}(R,\tau) \ , \quad (s>0) \ , \tag{D.2}$$

$$\partial_\tau I_{k'}^{(s',\pm)}(R,\tau) \ = \ 2 I_{s'}^{(s',\pm)}(R,\tau)\, \partial_R I_{k'}^{(s',\pm)}(R,\tau) \ , \quad (s' = -s < 0) \ , \tag{D.3}$$

in the CFT limit.

Let us begin with the $\mathbf{s} < 0$ case. From the implicit relation (6.47)

$$I_{k'}^{(s',\pm)}(R,\tau) = I_{k'}^{(s',\pm)} \left(\mathcal{R}_0^{(s',\mp)} \right) \ , \tag{D.4}$$

using (6.44), we have

$$I_{k'}^{(s',\pm)}(R,\tau) = \frac{2\pi a_{k'}}{\left(R + 2\tau I_{s'}^{(s',\pm)}(R,\tau) \right)^k} \ . \tag{D.5}$$

Differentiating (D.5) w.r.t. R for $k' = s'$ ($k = s$), we first express $I_{s'}^{(s',\pm)}(R,\tau)$ as a function of $\partial_R I_{s'}^{(s',\pm)}(R,\tau)$ as

$$I_{s'}^{(s',\pm)}(R,\tau) = -\frac{R\, \partial_R I_{s'}^{(s',\pm)}(R,\tau)}{2\tau s(1+s)\, \partial_R I_{s'}^{(s',\pm)}(R,\tau)} \ . \tag{D.6}$$

Then, differentiating again (D.5) w.r.t. R for generic k' (k) and using (D.6), we express $I_{k'}^{(s',\pm)}(R,\tau)$ as a function of $\partial_R I_{s'}^{(s',\pm)}(R,\tau)$ and $\partial_R I_{k'}^{(s',\pm)}(R,\tau)$ as follows

$$I_{k'}^{(s',\pm)}(R,\tau) = -\frac{s}{k} \frac{R\, \partial_R I_{k'}^{(s',\pm)}(R,\tau)}{2\tau s(1+s)\, \partial_R I_{s'}^{(s',\pm)}(R,\tau)} \ . \tag{D.7}$$

Finally, using (D.7), it is a matter of simple algebraic manipulation to show that

$$- R' \partial_R I_{k'}^{(s',\pm)}(R,\tau) + k' \theta_0' I_{k'}^{(s',\pm)}(R,\tau) = 2 I_{s'}^{(s',\pm)}(R,\tau)\, \partial_R I_{k'}^{(s',\pm)}(R,\tau) \ , \tag{D.8}$$

which proves that (6.33) for $\mathbf{k} = k'$ and $\mathbf{s} = s'$ reduce to (D.3).

Considering the $\mathbf{s} > 0$ case, from the implicit relation (6.46)

$$I_k^{(s,\pm)}(R,\tau) = I_k^{(s,\pm)}\left(\mathcal{R}_0^{(s,\pm)}\right) \ , \tag{D.9}$$

using (6.44), we have

$$I_k^{(s,\pm)}(R,\tau) = \frac{2\pi a_k}{\left(R + 2\tau I_s^{(s,\mp)}(R,\tau)\right)^k} \ . \tag{D.10}$$

Repeating the same procedure as in the $\mathbf{s} < 0$ case, we express $I_k^{(s,\pm)}(R,\tau)$ as a function of $\partial_R I_k^{(s,\pm)}$ and $\partial_R I_s^{(s,\pm)}$ as follows

$$I_k^{(s,\pm)} = -\frac{s}{k}\frac{R\,\partial_R I_k^{(s,\pm)}\left(2\tau s\,\partial_R I_s^{(s,\pm)} - 2\tau\partial_R I_k^{(s,\mp)} + s\right)}{s^2\left(1 + 2\tau\,\partial_R I_s^{(s,+)} + 2\tau\,\partial_R I_s^{(s,-)}\right) + 4\left(s^2-1\right)\tau^2\,\partial_R I_s^{(s,+)}\,\partial_R I_s^{(s,-)}} \ , \tag{D.11}$$

Again, using (D.11), one can show that the following relation holds

$$-R'\partial_R I_k^{(s,\pm)}(R,\tau) + k\theta_0' I_k^{(s,\pm)}(R,\tau) = 2I_s^{(s,\mp)}(R,\tau)\,\partial_R I_k^{(s,\pm)}(R,\tau) \ , \tag{D.12}$$

which proves that (6.33) for $\mathbf{k} = k$ and $\mathbf{s} = s$ reduce to (D.2).

Appendix E

E.1 Maxwell-Born-Infeld electrodynamics in $4-$dimensions

Two-photon plane wave scattering in $4-$dimensional Maxwell-Born-Infeld (MBI) electrodynamics was considered by Schrödinger and others in pre-QED times (see, for example, [84] for a nice historical review on the early period of non-linear electrodynamics theories). Later on, in [92, 93] it was shown that the scattering of two plane waves in MBI electrodynamics can be mapped onto a specific solution of the 2D bosonic Born-Infeld EoMs, the $N = 2$ model in equations (2.116). In particular, it is extremely suggestive that the resulting phase-shift can be nicely interpreted as being the classical analog of the $\mathrm{T}\bar{\mathrm{T}}$-related scattering phase. Compare, for example, the results of [92, 93] with the discussion about the classical origin of the time delay in [94].

Motivated by these observations, in [46] $4-$dimensional MBI theory of electrodynamics was investigated and interestingly it was shown that it shares a lot of common aspects with the $2-$dimensional bosonic models studied in section 2.3. In particular, in this appendix we show that it arises as a deformation of the $4-$dimensional Maxwell theory induced by the square root of the determinant of the Hilbert stress-energy tensor.

Consider the MBI Lagrangian in $4-$dimensions minimally coupled to a generic background metric $g_{\mu\nu}$ as

$$\mathcal{L}_g^{\mathrm{MBI}}(\mathbf{x},\tau) = \frac{-\sqrt{|\det[g_{\mu\nu}]|} + \sqrt{\det\left[g_{\mu\nu} + \sqrt{2\tau}\mathcal{F}_{\mu\nu}\right]}}{2\tau} \;, \quad ((\mu,\nu) \in \{1,2,3,4\}) \;, \tag{E.1}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu(\mathbf{x}) - \partial_\nu \mathcal{A}_\mu(\mathbf{x})$ is the field strength associated to the abelian gauge field $\mathcal{A}_\mu$, in accordance with the notation of section 2.3.5. Restricting for simplicity to Euclidean space-time, *i.e.* $g_{\mu\nu} = \delta_{\mu\nu}$ and $\mathcal{L}_\delta^{\mathrm{MBI}} = \mathcal{L}^{\mathrm{MBI}}$, one has

$$\mathcal{L}^{\mathrm{MBI}}(\mathbf{x},\tau) = \frac{-1 + \sqrt{1 - \tau\,\mathrm{Tr}\left[F^2\right] + \frac{\tau^2}{4}\left(\mathrm{Tr}[F\widetilde{F}]\right)^2}}{2\tau} \;, \tag{E.2}$$

where F and $\tilde{F}$ denote the matrices with elements $\mathcal{F}_{\mu\nu}$ and $\tilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}$, respectively, $\tilde{\mathcal{F}}_{\mu\nu}$ being the Hodge dual field strength. Expanding (E.2) in powers of τ around $\tau = 0$

$$\begin{aligned}\mathcal{L}^{\text{MBI}}(\mathbf{x},\tau) &= -\frac{1}{4}\text{Tr}[F^2] + \frac{\tau}{16}\left(\text{Tr}[F^2]^2 - 4\text{Tr}[F^4]\right) + \mathcal{O}(\tau^2) \\ &= \mathcal{L}^{\text{M}} + \tau\sqrt{\det[T^{\text{M}}]} + \mathcal{O}(\tau^2)\ ,\end{aligned} \tag{E.3}$$

one recognizes that the order $\mathcal{O}(\tau^0)$ coincides with the Maxwell Lagrangian

$$\mathcal{L}^{\text{M}}(\mathbf{x}) = \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{1}{4}\text{Tr}[F^2]\ , \tag{E.4}$$

while the $\mathcal{O}(\tau)$ contribution is related to the determinant of the Hilbert stress-energy tensor of the Maxwell theory $T^{\text{M}}(\mathbf{x})$, which can be computed from the Noether theorem by adding the Belinfante-Rosenfeld improvement to make it symmetric and gauge invariant (cf. 2.3.5), *i.e.*

$$(T^{\text{M}})^{\mu\nu} \equiv \frac{\partial\mathcal{L}^{\text{M}}}{\partial\left(\partial_\mu\mathcal{A}_\rho\right)}F^{\nu\rho} - \eta^{\mu\nu}\mathcal{L}^{\text{M}} = F^{\mu\rho}F^{\nu\rho} - \eta^{\mu\nu}\mathcal{L}^{\text{M}}\ . \tag{E.5}$$

Formula (E.3) hints that $\mathcal{L}^{\text{MBI}}$ may arise from a deformation of Maxwell electrodynamics effected by the operator $\mathcal{O} \equiv \sqrt{\det[T^{\text{MBI}}]}$ according to the flow equation

$$\boxed{\partial_\tau\mathcal{L}^{\text{MBI}} = \sqrt{\det[T^{\text{MBI}}]}\ ,} \tag{E.6}$$

where T^{MBI} is the Hilbert stress-energy tensor associated to the MBI Lagrangian. Using the general definition

$$(T^{\text{MBI}})^{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\delta\mathcal{L}_g^{\text{MBI}}}{\delta g_{\mu\nu}}\ ,\ \sqrt{g} \equiv \sqrt{|\det[g_{\mu\nu}]|}\ , \tag{E.7}$$

it is possible to show that, in euclidean spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$), the following relation holds

$$\mathcal{O} = \frac{-1 + \mathcal{S}^{\text{MBI}}(\mathbf{x},\tau) - 2\tau\,\mathcal{L}^{\text{M}}(\mathbf{x})}{2\tau^2\mathcal{S}^{\text{MBI}}(\mathbf{x},\tau)} = \partial_\tau\mathcal{L}^{\text{MBI}}\ ,\ \mathcal{S}^{\text{MBI}}(\mathbf{x},\tau) \equiv \sqrt{\det\left[\eta_{\mu\nu} + \sqrt{2\tau}F_{\mu\nu}\right]}\ , \tag{E.8}$$

thus proving the validity of (E.6).

As noticed in section 2.3.5, the presence of an internal symmetry (in the current case the $U(1)$ gauge symmetry) makes the definition of the stress-energy tensor ambiguous. As already appears at the perturbative level in (E.3), here the symmetric and gauge invariant Hilbert stress-energy tensor seems to be the natural choice to get the MBI Lagrangian as a deformation of the Maxwell electrodynamics. However let us point out that there is no reason to rule out a priori a deformation induced by the Noether stress-energy tensor, which is neither symmetric nor gauge invariant. Moreover, the two stress energy tensors might give rise to completely different deformations. In fact, it is not known if relation (2.132), or equivalently the factorisation property, holds in dimension $d > 2$.

Interestingly, solving perturbatively the flow equation (E.6) using as initial condition

$$\mathcal{L}^{\text{M},V}(\mathbf{x}) = \mathcal{L}^{\text{M}}(\mathbf{x}) + V\ , \tag{E.9}$$

where V is a potential depending on the fields $\mathcal{A}_\mu$ only, one get the following solution

$$\mathcal{L}^{\text{MBI},V}(\mathbf{x},\tau)=\frac{V}{1-\tau V}+\frac{1}{2\tilde{\tau}}\left(-1+\sqrt{\det\left[\delta_{\mu\nu}+\sqrt{2\tilde{\tau}}F_{\mu\nu}\right]}\right)\;, \tag{E.10}$$

where $\tilde{\tau}=\tau(1-\tau V)$ is the same (local) redefinition of the deformation parameter found also in section 2.3.2 in bosonic interacting theories. A posteriori, it is easy to check that $\mathcal{L}^{\text{MBI},V}(\mathbf{x},\tau)$ is indeed solution to (E.6), *i.e.*

$$\sqrt{\det[T^{\text{MBI},V}]}=-\frac{\mathcal{S}^{\text{MBI}}(\mathbf{x},\tilde{\tau})(2\tilde{\tau}\,V-1)-(2\tau\,V-1)\,(1+2\tilde{\tau}\,\mathcal{L}^{\text{M}}(\mathbf{x}))}{2\tilde{\tau}^2\,\mathcal{S}^{\text{MBI}}(\mathbf{x},\bar{\tau})}=\partial_\tau\mathcal{L}^{\text{MBI},V}(\mathbf{x},\tau)\;. \tag{E.11}$$

Following the logic of section 2.3.4, it is interesting to perform a Legendre transformation on $\mathcal{L}^{\text{MBI},V}(\mathbf{x},\tau)$ to get the Hamiltonian density $\mathcal{H}^{\text{MBI},V}(\mathbf{x},\tau)$. Again, using a shorthand notation for the time derivative $\dot{\mathcal{A}}_\mu=\partial_4\mathcal{A}_\mu$, the conjugated momentum is

$$\Pi^i=\frac{\partial\mathcal{L}^{\text{MBI},V}(\mathbf{x},\tau)}{\partial\dot{\mathcal{A}}_i}\;,\quad \Pi^4\equiv 0\;,\quad (i=1,2,3)\;, \tag{E.12}$$

and the Hamiltonian density takes the form

$$\mathcal{H}^{\text{MBI},V}(\mathbf{x},\tau)=\frac{V}{1-\tau V}+\frac{1}{2\tilde{\tau}}\left(-1+\sqrt{1+4\tilde{\tau}\,\mathcal{H}^{\text{M}}(\mathbf{x})+4\tilde{\tau}^2\,|\vec{\mathcal{P}}^{\text{MBI}}(\mathbf{x})|^2}\right)\;, \tag{E.13}$$

where $\mathcal{H}^{\text{M}}(\mathbf{x})=-\frac{1}{2}\Pi_i\Pi^i+\frac{1}{4}\mathcal{F}_{ij}\mathcal{F}^{ij}=-T^{\text{M}}_{44}(\mathbf{x})$ is the Hamiltonian density of the Maxwell theory and $\mathcal{P}^{\text{MBI}}_i(\mathbf{x})=\mathbb{i}\,T^{\text{MBI}}_{4i}(\mathbf{x},\tau)\;,\;(i=1,2,3)$, is the i-th component of the conserved momentum density of the deformed theory. Notice that $\mathcal{H}^{\text{MBI},V}(\mathbf{x},\tau)$ is formally identical to the Hamiltonian density reported in Section (2.3.4) for the $2-$dimensional bosonic theories.

Here we found that a $4-$dimensional theory arises as a deformation induced by a power $1/2$ of the determinant of the stress-energy tensor. This result apparently does not agree with the generalization to higher dimensions proposed in the first version of [18], from which one would expect a power $1/(D-1)=1/3$ instead. Interestingly, notice also that the operator $\sqrt{\det[T^{\text{MBI}}]}$ can be written in this form

$$\sqrt{\det[T^{\text{MBI}}]}=\frac{1}{4}\left(\frac{1}{2}\text{Tr}\,[T^{\text{MBI}}]^2-\text{Tr}\left[(T^{\text{MBI}})^2\right]\right)\;, \tag{E.14}$$

which strongly resembles the generalization of the $\text{T}\bar{\text{T}}$ operator to higher dimensions recently proposed in [83], except for the factor $1/2$ in front of $\text{Tr}\,[T^{\text{MBI}}]^2$ instead of $1/(D-1)=1/3$. Although in this Section we have pointed out many similarities at the classical level between the 4D Maxwell-Born-Infeld model and the $2-$dimensional bosonic theory discussed in section 2.3.4, the situation at the quantum level is in principle much more complicated. However it would be remarkable if a structure similar to the $2-$dimensional case could emerge for the quantised energy spectrum.

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