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THE ECONOMIC ROLE OF INVESTMENT BENCHMARKS

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Abstract

Investment benchmarks are at the heart of the controversy surrounding the profession of money managers. Benchmarks are hypothetical investment portfolios assembled and managed using simple rules. Starting from the seminal paper of Jensen (1968), empirical literature showed that professionally managed investment funds fail to consistently outperform these hypothetical portfolios. These empirical findings, often interpreted as a shortage of skill in the investment profession, are at odds with the apparent success of the investment management as an industry over the past five decades, and are a subject of an ongoing debate in the academia and among the finance profession.

A less known controversy, which motivated the research for this thesis, is the one that surrounds the investment benchmarks. While the benchmarks have become a vital part of investing since the 1970s, academic research found no microeconomic reasons for their use. On the contrary, when introduced in market models, investment benchmarks were shown to be ineffective in addressing agency frictions between investors and managers (Bhattacharya and Pfleiderer (1985); Admati and Pfleiderer (1997); Stoughton (1993)), to distort market efficiency (Roll (1992); Gomez and Zapatero (2003); Cuoco and Kaniel (2011); Basak and Pavlova (2013); Vayanos and Woolley (2016); Breugem and Buss (2018)), to reduce the investors' welfare (Duarte et al. (2015)) and to cause contagion across markets (Basak and Pavlova (2016)). The disconnect between the theoretical inefficiency and the practical widespread use of benchmarks in the investment industry is not a mere intellectual puzzle, it is an important concern for policy makers because of the central role that financial markets play in the economic activity.

In this thesis, we approach the problems of the skill of investment managers and the economic role of investment benchmarks jointly, as the two controversies are related and are a result of the assumption that investment skill should be manifested via an outperformance with respect to a benchmark. Based on the insights of Bhattacharya and Pfleiderer (1985), Admati and Pfleiderer (1988), Admati and Pfleiderer (1990), Allen (1990) and Ross (2005), we explore an alternative framework where investment managers are not treated as agents entrusted by investors to perform an investment task, but are rather sellers of private information to investors. In this framework all agents, investors and managers alike, have the ability to trade in asset markets and possess some private information. An investor, while trading actively in asset markets on the own account, invests in funds managed by his peers to benefit from their private information. In this framework professional investment industry featuring multiple funds appears because the private information in the economy is decentralized.

The thesis consists of three parts. Part 1 presents a general model of delegated asset management as a market for decentralized information. In this model agents can buy the private information about future payoffs of the risky assets indirectly by acquiring shares in investment portfolios of their peers. In equilibrium, agents trade on their own accounts and invest a part of their wealth in funds. The overall exposure to risky assets for each agent is larger in the presence of funds, compared to an economy where agents trade only on their own private information. As a consequence, more information gets incorporated into the asset prices, lowering the equity premium and enhancing the information efficiency of asset markets.

Part 2 considers in greater detail the transaction between an investor and a manager within the framework developed in Part 1. We show that the main concern for investors is the discovery of the fund managers' types and constraining the discretion of the managers to alter their types in the future. Managers, in turn, are concerned with the marketability of their funds to investors. Both concerns might be addressed by introducing a benchmark portfolio in the transaction between an investor and a manager. We then derive the implications of this setup. In particular, fund portfolios based on private information do not necessarily outperform benchmark portfolios per dollar invested. The structure of an appropriate benchmark might not be uniquely fixed, but is a result of a social agreement.

Part 3 explores an economy where prices fully reveal the private information, and agents might be better off investing directly in benchmarks than paying the fees to active managers. In traditional models of asset trading the in the presence of private information such fully revealing equilibria, though existing theoretically, are not implementable. We show that when the institution of delegation, i.e. the market for funds, is introduced, it might work as a stabilizing force supporting a fully revealing equilibrium. Thus a market for funds could play a role of an implementation mechanism for the aggregation of private information. A tension, however, arises between the need to compensate fund managers and the incentive for investors to get access to the optimal outcome by simply investing in the market portfolio for free. We formulate this problem as a volunteer game, where the market for funds is treated as a public good that benefits the whole economy by making asset prices efficient. The game of privately supplying this public good might have equilibria where a part of agents are paying for the good, and the rest are free-riding by investing in the market benchmark. This provides a rationale for the observed coexistence of active funds and index funds in the asset management industry.

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Introduction

> *... a major industry appears to be built largely on an illusion of skill... Professional investors, including fund managers, fail a basic test of skill: persistent achievement.*
>
> – David Kahneman, "Thinking fast and slow" (2011)

A significant share of global wealth is delegated to investment managers.[1] Yet, as the quote above manifests, the economic research could not settle on a coherent explanation of why the industry of money management exists. The heart of the problem is the question about the performance of investment managers. Empirical studies starting from the seminal paper of Jensen (1968) showed that only a small minority of actively managed funds outperformed randomly selected stocks. Theoretical arguments from the widely quoted book of Malkiel (1973), related to the efficient market hypothesis of Fama (1970), showed that the superior performance of active managers is elusive, and probably does not exist. French (2008) estimated that active mutual funds underperform the benchmark indices by 0,67 % per year[2]. The discouraging evidence notwithstanding, the demand for the services of investment managers appeared to be robust with the money management industry thriving over the past five decades.

The mainstream paradigm of investment industry is based the premise that professional managers provide superior investment performance. With the empirical evidence contradicting this view, theoretical research proposed several explanations for the demand for asset management services. One possibility is to assume that the superior performance exists but its benefits accrue to the managers themselves due to a strong bargaining power, or the agency frictions. Another possibility is that superior performance does not exist, but "investors just cannot do it on their

[1] PwC estimates the total amount of assets under delegation to be 38.9% of total investable wealth in 2015.

[2] See Cremers et al. (2019) for a recent review of the vast empirical literature on investment performance of active managers.

own", because of restricted trading opportunities or behavioral biases. These efforts produced a mosaic of partial explanations that do not integrate in a coherent picture of professional investment management.

In this study we explore an alternative framework where investment managers are treated not as performance-producing agents, but rather as intermediaries who sell information to investors. All market participants in our study trade actively in asset markets and regard investing in a managed fund as a way to acquire an input for their trading activity. The fundamental reason for the professional managers to exist is the decentralization of private information in the economy. In this framework all agents, investors and managers alike, have ability to trade in asset markets and possess some private information. Indeed, there is no need to separate agents into managers and investors, as any agent might play both roles at the same time. No assumption about a priori superior skills is needed either: all agents use the same investing technology and the performance of their portfolios depends on the individual private information.

A joint consideration of the downstream market (for assets) and the upstream market (for investment funds) is naturally embedded in this framework. It allows to study the flows and the aggregation of information in this joint market, and account for the externality that exists because of the leakage of information through the asset prices. On the one hand, this allows to draw the limits on the existence of investment delegation: it would exist as long as the aggregate private information is not fully revealed by the asset prices. On the other hand, it gives the way to understand how the delegation changes the aggregation of private information and affects the asset prices and their efficiency.

An investment industry model based on this paradigm has several key differences with respect to the performance-based approach. Agents, acting as investors, do not in general invest with only one manager, but assemble many complementary inputs by investing in multiple funds. The reason to hire or not a particular manager depends not on his superior performance, but on the private information he possesses. Benchmarks, as we will argue in this study, could be seen as tools that help to structure the fund market in a way that facilitates the utilisation of private information embedded in the investment funds.

0.1 A paradox of investment benchmarks

Benchmarks, which are hypothetical portfolios with a publicly known composition, are used to analyse managers' performance, to set investment objectives, to define investment constraints and to structure the investment manager compensation. In fact, the use of benchmarks is so pervasive in the investment industry that the business of creating and maintaining benchmarks have been experiencing an explosive growth. As of October 2018 MSCI, one of the biggest providers of investment benchmarks, maintained 200000 indices only for the stock market sector, with more than USD 13.9 Trl in equity assets benchmarked to MSCI indices worldwide.

Despite the important role that the investment benchmarks play in the asset management industry, they have a mixed reputation in the financial economics literature. On the one hand, empirical research is being mostly positive about benchmarks. Benchmarks are a crucial tool for building portfolio performance measures. Moreover, benchmarks are a basis of indexing, or passive investing, an approach to investing that eliminates completely the manager's discretion from the delegation relationship. Passive investing enjoys a successful performance record, fuelling further the industry debate about the value added by the active management[3].

On the other hand, investment benchmarks are being consistently criticised in the theoretical literature. It was shown that benchmarks are ineffective in addressing agency frictions between investors and managers(Bhattacharya and Pfleiderer (1985); Admati and Pfleiderer (1997); Stoughton (1993)), distort market efficiency (Roll (1992); Gomez and Zapatero (2003); Cuoco and Kaniel (2011); Basak and Pavlova (2013); Vayanos and Woolley (2016); Breugem and Buss (2018)), reduce the investors' welfare (Duarte et al. (2015)) and provoke contagion across markets (Basak and Pavlova (2016)). Overall, the theoretical literature has found no economic efficiency grounds for the use of benchmarks in investment management.

Historically, investment benchmarks were long considered primarily as a tool to structure the incentives of investment managers. The investor-manager contracts are incomplete because of the asymmetric information about the manager's effort or the skill. The use of benchmarks as a basis for relative performance analysis came after the Wharton report of 1962, along with the changes in the regulation of the performance-related compensation of investment managers in the US. When left to their own devices, managers and investors used to conclude contracts with bonus-like performance sharing, where, in addition to a flat fee, managers were paid a share of investment gains and did not participate in investment losses. The Securities and Exchange Commission and the US Congress have opposed repeatedly these practices as "unfair" to investors and since 1970 restricted the options for contractual compensation for registered investment advisors, requiring the performance sharing, if any, to be symmetric around the performance of a benchmark index. The response of the management industry to this regulatory requirement showed that managers are

[3]SPIVA scorecards, a yearly report published by Standard and Poor's, show that a representative panel of active managers fails consistently to outperform their benchmark indices.

unwilling to share investment losses, even in relative terms. The progressive elimination of the performance fees in the market sectors where benchmarks are easy to construct is one of the main stylized facts about the impact of benchmarks on the investment industry.

This study takes an approach opposite to the view of benchmarks as an incentive tool in agency problem. The historical origins of investment benchmarks indicate that investment benchmarks have a role to play outside the agency setting. In fact, the efforts to build first investable benchmarks were not related to the principal-agent frictions (Sikorsky (1982)). It was rather related to the expansion of the US money managers to the overseas asset markets and the need to coordinate the investment across different portfolios within a firm. Moreover, the theoretical literature has demonstrated that benchmarks as incentive tools are ineffective in the investment management context (Stoughton (1993); Admati and Pfleiderer (1997)).

We thus build a framework of investment delegation without imposing a priori frictions or reasons for opportunistic behaviour. All agents in the economy can freely trade the risky securities and can invest an arbitrary amount of their wealth in multiple funds. All agents have rational preferences and possess private information. The delegation in this setup allows agents to build superior risky exposures by effectively using the benefits of private information available to other agents. The demand for delegation is formed endogenously in the equilibrium, and is driven by the diversity and complementarity of private information. The model allows to study the impact of delegation on the prices and the market efficiency, because delegation and asset trading are interrelated decisions jointly made by the agents.

We then study a role that benchmarks might play in such a model. The intuition behind the introduction of investment benchmarks is the following. In an ideal world where agents have perfect knowledge of the informational characteristics of the fund managers there is no need for benchmarks. When, instead, the knowledge of the quality of the managers' signals is incomplete, there is a scope to introduce a device to signal the manager's type to investors. Such a device could be structured using a portfolio with a publicly known composition. The advantage of this approach is that negative effects of the benchmarks could be weighted against their economic benefits, which makes the model more appealing for policy experiments.

Another advantage of a model of delegation based on the market for private information is the possibility to study the impact of the delegation contracts on the aggregation of private information in the economy. Contrary to the conventional view that asset markets should be inefficient in order for the delegation to exist (Garleanu and Pedersen (2018)), we build an example of an efficient asset market sustained by the presence of delegation. Being informationally efficient, the passive market portfolio is optimal in such a model, and a part of agents may rationally choose to invest in this passive portfolio instead of investing in the funds of informed managers.

0.2 Benchmarks in the theoretical literature

We now review briefly the theoretical literature that featured investment benchmarks. This literature can be roughly divided into two waves, the agency models of the 1980s and the general equilibrium models of the late 1990s and 2000s. The studies of the late 1980s considered benchmarks using the principal-agent setting with moral hazard. The use of the moral hazard partial equilibrium models of delegation was inspired by the standard moral hazard models, such as Stiglitz (1974); Holmstrom (1979). In particular, in other industries, such as agriculture, it was shown that the linear performance sharing rule was second-best efficient Stiglitz (1974). Also, Holmstrom (1979) showed that the principal could improve the linear performance sharing by observing some extra information related to the agent's effort, so it was hoped that the performance of a suitable benchmark could be such a valuable information in the investment delegation problem, and including the benchmark-related term in the compensation contract might be an efficient response to the agency problem in delegation.

Bhattacharya and Pfleiderer (1985) studied a partial equilibrium agency model where the investment delegation was framed as information acquisition (principals acquired fundamentals-related information from agents of heterogenous forecasting ability). They showed that a linear performance sharing rule does not allow to efficiently screen higher-ability agents, neither it does elicit truthful revelation of the information by agents. Though Bhattacharya and Pfleiderer (1985) proposed a benchmark-related compensation rule that allowed a truthful revelation of the signal, this compensation was quadratic in relative performance and not linear as used in the investment industry. The formulation of the problem here excluded the scope for the principal to hire multiple agents, because it was assumed that the agents of identical precision would receive identical signals.

Later Stoughton (1993) and Admati and Pfleiderer (1997) framed delegation as a "full service" problem where managers were in charge of the signal acquisition (reduced to a costly effort) as well as building the portfolio on behalf of the investor. They concluded that the performance sharing rule does not provide portfolio managers with an incentive to perform the effort (the so-called *irrelevance result*), and that including a benchmark in the compensation does not motivate agents to undertake effort, and is generally a distorting factor that leads to suboptimal portfolio choice and does not help to achieve optimal risk-sharing between the agent and the principal.

A series of papers considered nonlinear performance sharing rules with benchmarks, but reached the conclusion that the most widespread nonlinear contracts (the performance-based bonuses) are actually inferior to the simple linear performance-sharing, because the latter, albeit being non-optimal solution, at least aligned risk attitudes of the agent and the principal. In addition, Allen and Gorton (1993) and J. and Gorton (1997) showed that bonus-like contracts attract low-quality managers and incentivize noise-trading by both low and high-quality managers. Roll (1992) explored a quadratic investment objective, eliminating agency frictions and imposing a tracking error constraint with respect to a benchmark on the portfolio manager. He derived an exact composition of the tracking-error optimal portfolio and showed that it is not mean-variance efficient.

The agency literature reached a consensus that the linear and bonus-like performance sharing contracts are not optimal in the delegated portfolio management, and that the benchmarks do not add value.

The next wave of the literature on benchmarks started in the 2000s and was motivated by the growing importance of institutional investors in the markets (Stein (2009)). Institutional investors are associated with benchmark-dependent investment objectives, thus the literature began to study effects of benchmark-dependent utilities on asset pricing in the context of general equilibrium models.

The majority of the early models (Brennan (1993), Gomez and Zapatero (2003)) considered a simple asset pricing model with no information asymmetry or agency frictions, by simply relating investment objectives of some agents to externally imposed benchmarks. The studies suggested that a "benchmark risk" is priced in the market, lowering prices of the off-benchmark securities.

Richer general equilibrium models with asymmetric private signals and nontrivial institution of delegation (including negotiating of delegation contracts, e.g. Kapur and Timmermann (2005) and Cuoco and Kaniel (2011) found that delegated portfolios have larger holdings of the benchmark asset, and the introduction of benchmarks distorts asset prices in equilibrium.

The models of Basak-Pavlova (2013) and Buffa et al. (2014) extended the problem to multiple risky assets and included wealth effects in the utility. In accordance with the previous literature, they found an increased pressure on the prices of the stocks that are constituents of the benchmark index. When the proportion of institutional investors increases, the volatility of the benchmark assets increases.

Several papers studied the problem of contagion due to the benchmarks. Duarte et al. (2015) showed that the trading by benchmark-dependent institutional investors impacts retail investors. Benchmarking incites institutional managers to overinvest in stocks that are highly correlated with their benchmarks. The jumps in the benchmark provoke firesales by the institutional investors. The firesales then propagate to the non-benchmark asset and impact portfolio allocations.

A series of articles on the topic of financialization of commodities (for example, Tang and Xiong (2012), Basak and Pavlova (2016)) pointed that the commodities prices became highly correlated to financial indices due to the inclusion of the commodities futures in the diversified portfolio benchmarks.[4]

The recent literature started to include the informational efficiency aspects in the models of economies with institutional investors. For example,Breugem and Buss (2018) included information acquisition in a rational expectation equilibrium model and jointly determined the portfolio choices and the choices to acquire private information about the fundamentals in equilibrium. They showed that institutional investors acquire less private information, and hence an increase in the proportion of institutional investors in the economy brings a decline in the information efficiency of prices. They also find that higher participation of institutional investors increase price volatility. In contrast to the previous general equilibrium literature, they find that benchmarking might lead to a decline of the prices of the securities that belong to the benchmark, due to their lower informativeness.

In a nutshell, the above literature reached a variety of negative conclusions about the effects of the benchmarks, using a mosaic of modelling approaches. Each model highlighted some particular aspect of the problem and often used a reduced-form formulation of the ingredients of delegation problem. In particular, the majority of the models did not model explicitly the information structure in the economy and the details of the delegation transaction (competition, compensation structure, ..). In a majority of the papers investors are forbidden from investing directly in the asset markets and the benchmarks are exogenous and are imposed as constraints.

0.3 The origins of investment benchmarks

Although stock indices existed since 1896, it was not until the 1970s that the wide adoption of the benchmarks in investment industry began. There were three simultaneous developments that contributed to this shift.

[4]An influential report of OXFAM (Murphy et al. (2012)) questioned the role of investing in the financialization of food prices and the adverse effect the investment funds might have on the supply of food. Though benchmarks were not explicitly mentioned in the report, it had an immediate impact on the benchmark industry, with managers of passive commodity funds switching to "no-grain" versions of the commodity benchmarks.

The first was the Wharton report of 1962 commissioned by the US congress, which analysed the contractual and compensation arrangements of mutual funds. Although the report did not question the quality of financial advise (did not raise the problem of "good" versus "bad" managers), it questioned the fairness of the compensation arrangements of mutual funds and the contracting clauses observed in the industry, in particular the practice of the funds sharing investment gains but not the losses. As a result, Amendment to Investor Act in 1970 prohibited the performance fees for mutual funds unless they were symmetric around an 'appropriate index'. The regulatory prescription thus required sharing relative losses, while investment managers had a rather strong preference for bonus-like performance-related compensation, although no guidance was given as to what can serve as an appropriate index.

The second development was the movement towards rigorous performance measurement and the introduction of relative performance by large investors, such as pension funds. This was in part a consequence of the Wharton report and the regulatory pressure, and in part a result of the technological advances in computing and to development by the academia of an array of performance measurement ratios based on the insights of the modern portfolio theory.

The third development was equally important and highlights the role of fund management firms in the development of investment benchmarks. Note, that until 1968 the stock indices existing in the market were of heterogeneous quality and were not comparable to actual portfolios. In 1962 Capital International (CI), a Geneva subsidiary of a financial services firm Capital Group and a manager of mutual funds, decided to expand investment offer for their investors, creating funds invested in the non-US markets. To assist the company to ".. better understand the.. individual markets and ... reinforce investment management capabilities" (Sikorsky (1982)) this US-based fund manager developed a suite of country stock indices to "represent faithfully the evolution of unmanaged portfolios" invested in the markets outside the US. The main difference between the CI indices and the already existing country stock indices (for example, S&P 500) was the investability achieved through the application of the Laspeyre's concept of weighted average, of the chain linking of the multi-period returns, and of the principles of dynamic management of the index membership. Due to these technical innovations, CI indices were the first benchmarks that could be compared to real-life investment portfolios. The fact that the first investable benchmarks appeared as a coordination device within an investment management firm, points to the possibility that investment benchmarks have a role to play outside the principal-agent relationship.

We conclude that the origins of investment benchmarks are mixed. Undeniably, the agency frictions played a role, and the movement towards "fair compensation" of fund managers was an important driver for the adoption of benchmarks. On the other hand, the technological origin of the first benchmarks indices as a tool to enhance portfolio management capabilities within investment firms indicated that benchmarks have a role to play outside the principal-agent relationship.

Since the 1970s the money management industry witnessed a progressive elimination of performance fees in the majority of markets. This trend was especially pronounced for funds investing in publicly traded securities where benchmarks are easier to construct.

We conclude this section with a list of several characteristics of the investment benchmarks.

- A benchmark represents more than a mere knowledge of the performance of a portfolio, but also includes the list of constituents and the weights of different constituents within the benchmark portfolio.

- Although the majority of the benchmark indices in the market are built on publicly available information, and although the task of reconstructing any particular index would require almost no resources, the business of creating and maintaining indices is extremely concentrated. A handful of big index providers, including MSCI (who inherited the original CI indices), Standard and Poor's, Russell, STOXX and FTSE supply the majority of the indices in the stock markets. The concentration is explained by strong network effects in the demand for indices: the more a particular index used in the industry, the greater is the demand for it.

- Index providers do not possess any property right on a certain portfolio construction method, but protect their index products by using a copyright, meaning that index providers essentially are selling their brand name.

- Index providers are exempt from any liability related to the use of their indices, including the case where the index provider makes a mistake in computing the index value or the index composition.

- The indices do not exist in isolation but are bundled in families, with a typical family covering all the investable public companies.

Summarizing, it is an empirical fact that benchmark does not exist in isolation, that the demand for benchmarks exist in the market where exist a diverse set of active managers, the benchmarks tend to be privately supplied, and benchmark providers exercise a monopoly power.

Understanding benchmarks means understanding how the relationship between investors and managers work. Benchmarks have characteristics of both information and technology.

The challenge is to build a model where benchmark indices having the above characteristics could appear endogenously.

1

PART 1: Fund management as a market for private information

Delegated asset management plays a major role in modern financial markets. According to PwC, a consultancy, in 2015 the amount of assets under delegation was \$78.7 Trl, or 38.9% of total investable wealth. Yet, despite a solid demand for its products, the added value of the asset management industry is difficult to define. Funds managed by professional money managers do not consistently outperform passive investment benchmarks, casting a doubt whether investors get value in return for the managers' fees.

One difficulty on the way of answering this question is a conceptual gap between the characteristics we assign to agents trading in asset markets and to investors delegating money to asset managers. Asset trading models are populated by rational optimizing agents, possessing private information, forming rational expectations, and trading freely. Models of delegation instead have routinely divided agents into skilled rational managers and uninformed investors, the latter often having difficulties in trading assets and in avoiding behavioral biases. As a consequence, a large body of theoretical literature attributes the investors' decision to delegate investing to managers to the constraints imposed on them, or to behavioral issues, such as aversion to making decisions under ambiguity. Starting from the premises that investors just cannot do it on their own, such models do not provide a coherent framework for the understanding of the role of delegated management. One has to take into account that the two markets, that for the assets and that for delegation, are integrated and populated by the same agents. A challenge is thus to develop an integrated framework where the decision to delegate, rather than being driven by constraints, comes from the characteristics of the asset market and the investing technology.

Important steps in this direction were the recognition that delegated asset management is a vehicle for sharing private information among agents (Admati and Pfleiderer (1988), Admati and Pfleiderer (1990)), and that endogenous formation of delegation occurs in equilibrium (Ross (2005); Garcia and Vanden (2009); Garleanu and Pedersen (2018)). We contribute to this literature by considering joint markets for asset trading and delegation in the case of multiple risky assets and a generic structure of the private information.

We develop a rational model of asset markets where privately informed agents find it optimal to delegate investments one to another. Investment skills in this model are given by a private noisy signal about the future payoffs of the risky assets. We keep the information structure as general as possible, so the agents cannot be clearly ranked from more informed to less informed, as they can specialize in different assets and investment styles. The same continuum of agents interacts in two markets: the downstream asset market where the agents invest directly in multiple risky assets and a riskless asset, and the upstream delegation market where each agent can act both as a manager of the own fund and an investor of any other fund, an individual fund being an optimal risky portfolio given the fund manager's private information.

The model has three stages. At date 0, agents decide what fees to charge for their funds and how much to invest in the funds of the others. At date 1, agents receive their private signals and trade in the asset market. The asset market clearing prices are determined by matching the asset demands from the own-account and the fund portfolios to the noisy asset supply. At date 2, assets pay liquidating dividends, the funds distribute their trading profits, and all agents consume their final wealth. We refer to date 0 as delegation stage, date 1 as asset trading stage, and date 2 as consumption stage.

Under the assumptions of competitive upstream and downstream markets, and assuming the agents form rational expectations, any agent who receives a private signal can be manager in the upstream fund market, irrespective of the precision of the signal. The demand for the funds comes from the complementarity among individual funds, as the noise in the private signals is uncorrelated across fund managers. Thus agents find it optimal to spread the delegated money among all the existing funds. All agents, regardless of their informational characteristics, are also potential fund investors in the upstream market. At asset trading stage, all agents continue to invest directly in assets on their own accounts, notwithstanding the fact that they have already an asset exposure through the funds.

The model yields two main insights. First, the model challenges the conventional wisdom that investing in funds occurs because of the fund's expected output, it's performance. The demand for funds is driven instead by their role as an input in the investment allocation. If the markets for the assets and for the funds are competitive, and if the private information exhibits complementarity, an agent invests in multiple funds to aggregate the information contained in any individual fund. As the aggregated information is superior to any individual signal, the investor problem in the fund market thus resembles more a problem of a firm managing multiple complementary inputs in a productive process (as described in Alchian and Demzets (1972)), than a search for a fund manager with superior performance. As a consequence, any agent possessing private information could become a manager and attract a positive investment. Complementarity in the demand for individual funds also creates a power to charge fund fees above the marginal cost (which we assume to be zero), similar to the case of monopolistic competition in Dixit and Stiglitz (1977).

The second insight related to the effect of delegation on the asset prices. We show that in the presence of delegation and under certain assumptions (zero delegation fees and perfect knowledge of the precisions of the others) the market-clearing prices aggregate all private information, even in the presence of noise traders. It is well known that, in an equilibrium with rational expectations and asymmetric information, trading in asset markets leads to the revelation of private information through equilibrium prices. Because equilibria with a full revelation of private information are problematic (Grossman and Stiglitz (1980)), the rational expectations (REE) models usually assume a source of market inefficiency, such as noise trading, which limits the revelation of information and maintains the incentives for information acquisition. In partially revealing REE models prices are semi-strong efficient, and agents with differentiated information earn differentiated returns based on their own signal and the information made public through prices. In our model noise trading, although being necessary to create the demand for the funds, does not necessarily prevent the full revelation of private information through prices.

The upstream delegation market allows agents to share their private information. Indeed, as was noted by Admati and Pfleiderer (1988), Admati and Pfleiderer (1990), Allen (1990), selling shares of an investment portfolio is an indirect way to sell the signal used to build this portfolio. In this sense, the fund management represents a market for information, which is an input for the downstream asset market. Thus sharing of information in the fund market amplifies the aggregation of private information by the asset market and leads to more informative prices. Though the fully revealing equilibria that we find are obtained under unrealistic assumptions, they highlight the role of the delegation institution as a decentralized mechanism to implement fully revealing allocations.

1.1 Literature review

The first milestone in the understanding of the role of information in the asset markets was reached in the stream of literature on the markets with distributed information. In a competitive economy with rational expectations asset prices aggregate private information and make the economy informationally efficient (Grossman (1976, 1978)). Yet, since the aggregated information becomes available through prices to all agents, the existence of equilibrium and the incentives to acquire private information crucially depend on the presence of noise trading preventing the prices to be fully revealing (Grossman and Stiglitz (1980); Diamond and Verrecchia (1981)). In the noisy REE equilibrium, agents trade on their private information and gain differential returns depending on the precision of their private signal (Hellwig (1980), Admati (1985)).

The second milestone was the important insight that fund managers are nothing but agents possessing private information. Uninformed agents investing in a fund do it to acquire the manager's private information, which they could use as an input in the asset allocation. In their pioneering work in this direction, Admati and Pfleiderer (1988, 1990) pointed that investing in funds and buying investment advise (the signal) directly are essentially equivalent. Under an assumption that the market for private information consists of a single monopolist, they also analyzed the profits of the information seller in the presence of leakage of the information through prices. Admati and Pfleiderer (1990) showed that selling the signal through funds is in general more profitable for the monopolist, although they suggested that in the case of many information sellers this way of information selling would become inefficient because buyers could have difficulties in "unbundling" different signals from the individual fund portfolios.

While the papers of Admati and Pfleiderer (1988, 1990) were focused on the optimal design of the vehicle for selling information, other studies explored the frictions related to the selling of asymmetric information. Bhattacharya and Pfleiderer (1985), in a partial equilibrium context, showed that the incentive to buy the signal may be compromised by the adverse selection problem, when a manager misrepresents the value and the precision of the signal. This paper considered the use of compensation incentives to alleviate this problem and concluded that, while a linear performance sharing is not efficient, a compensation being a quadratic function of the performance deviation from an uninformed benchmark portfolio would induce a truthful revelation. The problem of signal credibility was addressed differently in Allen (1990), where information seller engaged in costly signalling of the information quality, enlisting the help of intermediaries. In the framework of Allen (1990) fund managers are the intermediaries who have perfect knowledge of the signal quality are able to credibly resell the signal.

In the general equilibrium context, the idea of a manager being an outside information seller was used in Garcia and Vanden (2009) and Garleanu and Pedersen (2018) to build fund markets involving multiple fund managers. Garcia and Vanden (2009) explored the issues of competition among managers in the presence of signal complementarity and barriers of entry, and endogenized the information acquisition in the presence of delegation. Garleanu and Pedersen (2018) analysed how the frictions in the asset market and in the fund market affect the equilibrium, and argued that a certain level of inefficiency, or noise, is needed for both in the asset market, and in the fund market to function.

The third milestone was the insight of Ross (2005) that informed agents become managers endogenously in equilibrium. He considered a population of agents who are heterogeneously informed about the payoff of the single risky asset. Instead of trading on their own signals, agents try to sell their signals by offering funds to other agents, and at the same time get exposure to the risky asset by investing in the funds of others. Ross (2005) showed that in equilibrium all agents, including the informed ones, invest in funds. Thus the profession of fund manager appears endogenously. The only viable funds are those managed by the agents with the high precision. Ross (2005) also considered the problem of signal credibility and concluded that it is difficult to separate truly informed agents from uninformed based on the performance of their funds, suggesting that probably an important share of the managers in the industry are uninformed.

The above models, despite each being tailored to investigate a specific question about the market for fund management, have several recurring features. In particular, investing in funds is possible only in the presence of noise trading, as otherwise the private information gets revealed through prices and no investor would pay the fund fees. Noise trading limits the leakage of information through prices and sustains a rational demand for funds. This problem of signal acquisition through funds is a counterpart of the Grossman-Stiglitz paradox related to a direct information acquisition. Another common feature of these models is the increase in the information efficiency of asset prices in the presence of delegation. This feature appears universally when delegation is introduced, although the rationale for the boost in informational efficiency differs depending of the specific model of delegation.

The model in this chapter relates to the above literature. We consider a general equilibrium in the joint market for assets and for funds as. Similar to Ross (2005), we introduce the delegation institution endogenously and do not separate agents into distinct groups such as "households" and "managers", as do Garcia and

Vanden (2009); Garleanu and Pedersen (2018). We do not a priori restrict the entry in the fund market and the ability of agents to invest in multiple funds. As the aforementioned papers, we abstract from agency frictions, by assuming a zero cost of the private signals and a competitive large market for funds and for assets.

Our model differs from the previous literature in several important aspects. First, we consider a richer structure of the market and of the private information. The bulk of the previous literature on joint modelling of delegation and asset markets was based on the extension of the model of Hellwig (1980); Grossman and Stiglitz (1980). This model featured one risky asset and a common signal received by all informed agents. Our model uses the economy of Admati (1985) for the downstream asset market, featuring multiple risky assets, and a generic distribution of private information. Second, we introduce residual uncertainty in the asset payoffs, which affects the demand for funds. The concept of residual uncertainty has to do with the ability of the aggregated private information to predict risky payoffs. Third, we do not restrict agents from trading on their own account, even if they invest in funds. The own-account trading turns out to play an important role in the fine-tuning the overall risky exposure (that is, the sum of the exposure to risky assets coming from investing in the funds and investing in the assets directly). This possibility to fine-tune the risk by own-account trading has profound consequences on the demand for funds, on the viability of the funds with low precision, and on the extent the information gets aggregated in the economy. Thus our results rather than being driven by the cost of the information or the market frictions, are driven by the complementarity of private information, by the ability of market prices to aggregate and reveal decentralized information, and by the monopolistic competition among the managers who possess independent pieces of information.

This study also relates to a more general question of the properties of markets for information, which is used as an input in a downstream market. A model of delegation allows one to investigate the product choice and the profits of information sellers, as a response to the structure of the competition and externalities in the downstream market (Admati and Pfleiderer (1990); Bimpikis et al. (2019)).

Questions of the micro-economic origins of delegation and the effect of the delegation on asset prices were studies disjointly in separate streams of literature. The first stream includes the models of delegation that justify the decision to delegate on the grounds of the superior technology of money managers (as in Berk and Green (2004)). Our model is different from these papers, as we do not equip asset managers with an ad-hoc production technology.

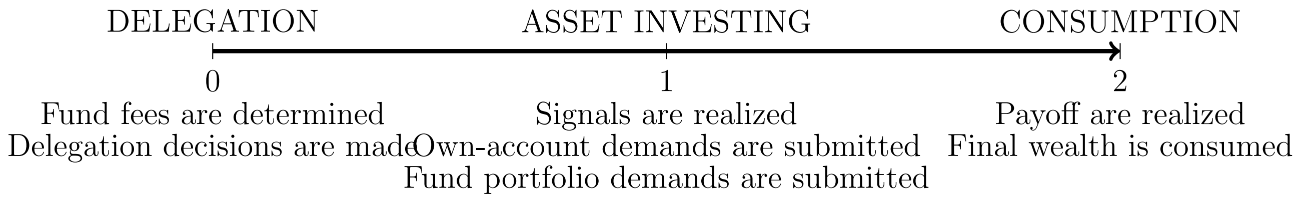
The stream of literature on the effects of delegation on the dynamic of asset markets includes the so-called intermediary asset pricing models of He and Krishnamurthy (2012) and He and Krishnamurthy (2013), who develop a general equilibrium framework of delegation when households have a limited access to the market of risky assets. This framework differs from our study in that it limits the access of any agent to the risky assets, includes agency frictions, and does not incorporate diverse private information. As a result, the intermediary asset pricing models support investment in only one fund, while in our model the presence of diverse private information makes it optimal to invest in allocations of funds. Hence, the importance of the manager's stake in the fund and the role of the constraints on fund capital that plays a crucial role in the intermediary asset pricing approach, is not supported by our model. The intermediary asset pricing models might be more suited to describe the specialist funds, such as hedge funds or structured products, while our model is more suited to represent the market for mutual funds, where the sharing of profits and capital requirements for fund managers are less of concern.

1.2 The model

We extend the asset market structure of Admati (1985), introducing residual uncertainty in the payoffs of risky assets as in Wang (1994), and adding a preliminary stage in the model where agents interact in the delegation market.

Our model has three dates, marking the transactions in the fund market, the trading in the asset market and the final consumption. At date 0, agents meet in the fund market and each agent makes two decisions: how many shares of each available fund to purchase (an "investor" decision), and what fee per share to charge the agents that will invest in his own fund (a "manager" decision). Then, at date 1 all agents observe realizations of their private signals and submit own account demands to the asset market. At the same date the agents perform their duty as managers and use their signals to build fund portfolios and transmit to the asset market the demands relative to the delegated money. At date 2 the final asset payoffs are determined, agents receive the proceedings of their own-account and delegated investments and consume their final wealth.

Figure 1.1: The model timeline



1.2.1 Agents

There is a continuum of agents in the economy, labelled by $m \in [0, 1]$. All agents receive at date 0 the same initial endowment of the consumption good e_0^m. As in Admati (1985), we assume that agents have CARA exponential utility $U = -exp(-\rho x)$. For simplicity we assume that, prior to receiving the private signal, the agents are homogeneous, so $e_0^m = e_0$, and $\rho_m = \rho$.

There is no consumption until the final date 2. Agents invest their entire endowments at date 1, maximizing their expected utilities of the terminal wealth. We assume there are no transaction costs for trading in asset markets.

1.2.2 Asset markets

There is one riskless asset in elastic supply. This asset costs 1 at date 1 and pays a fixed payoff R at date 2. There are N risky assets, labeled $i = 1, .., N$. Risky asset i pays a dividend D_i at date 2. The unconditional distribution of the dividend vector is a joint normal with the mean $\bar{\theta}$ and variance-covariance matrix $V + V_D$. The prices P_i are determined endogenously in the equilibrium at date 1.

Risky assets are in noisy supply z, assumed to come from noise traders. The supply is distributed normally with the mean $\bar{z}$ and covariance U. Random supplies z are independent from random payoffs D.

All assets are traded at no cost. At the end of the period when the payoffs are collected, the assets are liquidated and have no residual value.

We introduce here some notations. Denote by x_B^m is the number of shares of the riskless asset held directly by agent m, and by x_i^m denote the number of shares of risky asset i held directly by agent m. We will refer to the risky portfolio x_i^m as the own-account portfolio of agent m. Then, we denote by $x = \int_0^1 x^m dm$ the aggregate of own-account portfolios of all the agents. Note, that in the absence of delegation the market clearing would imply $x = \int_0^1 x^m = z$.

1.2.3 Information Structure

Each investor knows the unconditional distributions of the random variables (z, D). In addition, agents possess private information about the future payoffs of some or all the assets. Here we do not restrict a priori the information structure, as in Admati (1985). We also assume that there exists a residual uncertainty ($V_D > 0$) in the asset payoffs, following Wang (1994).

The future dividends have a structure:

$$D = \theta + \epsilon_D, \qquad \theta \sim N(\bar{\theta}, V) \qquad \epsilon_D \sim N(0, V_D), \qquad Cov(\theta, \epsilon_D) = 0 \tag{1.1}$$

where all the parameters refer to unconditional distributions. The dividend is a sum of two independent random variables. The first variable θ could be forecasted by collecting private information. This would correspond to the part of the future payoffs depending on the activities already realized by date 1, such as the production, investment, and financing decisions already made by the company. The second variable ϵ_D, which is a pure noise, represents a genuine uncertainty that cannot be predicted before date 2. The analogue of this uncertainty would be an unforeseeable regulatory decision affecting the company's business, a natural disaster, or an unexpected change in the demand.

The majority of existing models of asset markets with distributed private information assumed no residual uncertainty in the asset payoffs (Grossman (1976, 1978); Admati (1985); Ross (2005); Garcia and Vanden (2009)). In these models the aggregate private information perfectly predicts the payoffs. The exception are the model of Grossman and Stiglitz (1980) and its extensions, such as Garleanu and Pedersen (2018), where all informed agents get the same signal and have the same signal error. In such a case the signal error plays a role similar to that of the residual uncertainty.

Agents receive private signals only about the first component of the payoffs, θ:

$$s^m = \theta + \epsilon^m \tag{1.2}$$

where m labels different agents, and the signal noise is distributed as a multivariate normal: $\epsilon^m \sim N(0, \Sigma^m)$. The errors ϵ^m are independent on the other random variables in the model, and are independent across investors:

$$Cov(\epsilon^m, \epsilon_D) = 0, \qquad Cov(\epsilon^m, \theta) = 0, \qquad Cov(\epsilon^m, \epsilon'_m) = 0 \tag{1.3}$$

We assume, as in Admati (1985), that agents, in aggregate, have the full information about the variable θ:

$$\int_0^1 s_m dm = \theta \tag{1.4}$$

Following Admati (1985), we do not require any given agent to have signals for all the risky assets, so matrices Σ_m^{-1} could have zero entries on the diagonal. The matrix Σ_m of an individual investor can thus be singular, and its inverse Σ_m^{-1} might not be positive definite. For a unique equilibrium solution to exist, it is required that a sufficient fraction of investors have a "full" signal, that is receive private signals on all the assets with nonsingular error covariance. The inverse covariance matrices of signal errors Σ_m^{-1} are called precision matrices. The average of the precision matrices in the economy is called $Q = \int \Sigma_m^{-1} dm$. We assume that Q is nonsingular and is known to all agents.

The following result about the private errors covariance matrices was assumed to hold in Admati (1985): the average of the signal errors weighted by the corresponding precision matrices is zero.

$$\int_0^1 \Sigma_m^{-1} \epsilon^m dm = 0 \tag{1.5}$$

In this paper we will use a generalized proposition:

$$\int_0^1 N_m \epsilon^m dm = 0 \tag{1.6}$$

for any matrix-valued coefficients N_m that are bounded. This would follow from the Law of Large Numbers (see Chung (1974), section 5.4), as the random variables under the integral are independent with uniformly bounded variances.

According to the rational expectation hypothesis, agents also use the observed asset prices P to make inferences about the payoff component θ. The conditional distribution of the payoffs, given the signal and the publicly observed price, is gaussian.

Following Admati (1985) we consider a linear price functional that depends on the random variables that are realized at date 1(though not observed by all agents):

$$P = A_0 + \int_0^1 A_{1m} s^m dm - A_2 z = A_0 + A_1 \theta - A_2 z \tag{1.7}$$

where the second equality follows from (1.2) and (1.6).

1.2.4 Equilibrium without delegation

Before introducing the fund market, we briefly review the equilibrium in the asset market without delegation, and introduce some useful notations. The asset market equilibrium extends the result of Admati (1985) by adding the residual uncertainty.

We begin by deriving an optimal portfolio of agent m when there is no delegation in the economy.

Agent m maximizes:

$$\max_{x_B^m, x^m} \mathbb{E}(U^m(x^m(D - PR)) \mid s^m, P), \tag{1.8}$$

where (s^m, P) is an information set available to agent m. The maximization is subject to the budget constraints (note that throughout the chapter we use matrix notations and omit the sign of transposition):

$$x_B^m + x^m P = e_0 \tag{1.9}$$

The solution to this optimization problem has a usual mean-variance form:

$$x_0^m = \frac{1}{\rho}(V_m + V_D)^{-1}(\mathbb{E}(\theta \mid s^m, P) - PR) \tag{1.10}$$

where we define the conditional covariance and mean of the payoff component θ as:

$$V_m = Var(\theta \mid s^m, P) = \Sigma_m^{-1} + V^{-1} + A_1 A_2^{-1} U^{-1} A_2^{-1} A_1 \tag{1.11}$$

and

$$\begin{aligned} \theta_c^m &= E(\theta \mid s^m, P) = V_m(\Sigma_m^{-1} s^m + V^{-1}\bar{\theta} + A_1 A_2^{-1} U^{-1} A_2^{-1} P - \\ &- A_1 A_2^{-1} U^{-1} A_2^{-1} A_0 + A_1 A_2^{-1} U^{-1} \bar{z}) \end{aligned} \tag{1.12}$$

We also obtain here a useful representation of the mean-variance portfolio x_0^m as a combination of a speculative bet on the private signal s^m and an uninformed exposure to risky assets.

Let's denote by V_u the conditional variance of an uninformed investor, and by x_0^u the optimal portfolio of an uninformed investor:

$$x_0^u = \frac{1}{\rho}(V_u + V_D)^{-1}(\mathbb{E}(\theta \mid P) - PR) \tag{1.13}$$

Then one represent the optimal portfolio (3.5) of an informed investor with the signal s^m and precision Σ_m^{-1} as:

$$x_0^m = O_m(s^m - PR) + E_m x_0^u \tag{1.14}$$

where

$$O_m = \frac{1}{\rho}(V_m + V_D)^{-1} V_m \Sigma_m^{-1}, \qquad E_m = (1 + V_m^{-1} V_D)^{-1}(1 + V_u^{-1} V_D) \tag{1.15}$$

The first term in the decomposition is interpreted as a bet on the private signal. The second, following Vives (2007), represents a market-making activity, i.e. exploiting discrepancies between the public information and the fundamental value. Formula (3.5) is a useful representation, allowing to separate the uninformed part of a portfolio from the part containing the signal.

Note, that the expression O_m in (1.15) is equal to the precision Σ_m^{-1} if there is no residual uncertainty ($V_D = 0$). It is natural for the optimal portfolio to be proportional to the signal precision, which represents the confidence of the agent in the signal. Agents with higher precisions take larger bets on their signals. In the presence of residual uncertainty, however, the bet on the signal is not identical to the signal precision Σ_m^{-1} because betting on the signal exposes the agent to the residual uncertainty. Thus, in the presence of residual uncertainty the quantity O_m has a meaning similar to precision, as it generalizes the notion of a confidence in one's signal. We will often refer to O_m, which in general is a nonlinear monotone function of the signal precision, as simply a precision in this sense. The second term in (1.14) is proportional to an uninformed portfolio. Note, that this term is exactly equal to the uninformed portfolio (1.13) when there is no residual uncertainty. And, given that E^m is a decreasing function of the agent's conditional precision V_m^{-1}, and thus is a decreasing function of the signal precision Σ_m^{-1}, one can conclude that agents with higher precision will participate less intensely in the market-making activity in the presence of residual uncertainty. Note also, that from the point of view of the peers of agent m, the uninformed part of his portfolio is known, as long as agents in the economy have perfect knowledge of the precisions of others. This means that any agent could, if needed, build a synthetic exposure to the signal of agent m, by buying one share of his fund x_0^m and selling short $-E^m x_0^u$ of assets directly.

1.2.5 The market for funds

The market for funds is an institution that governs investing in portfolios of other agents without revealing neither their private signals, nor their portfolio composition. Any agent can hire as many "managers" as he chooses to, can delegate any arbitrary share of his own wealth, and can invest the residual in the assets directly. At the same time, the same agent can set up his own fund and accept the delegated money.

Investment funds are the products exchanged in the delegation market. We assume that a fund is a promise to return a payoff of a portfolio of risky assets conditional on the fund manager's private signal, corresponding to $x_0^{m'}$ in (1.14). Running a fund does not impose any costs on the manager. The signals are given exogenously in our model, and their cost is sunk; there is no setup fee to establish a fund. With the zero cost assumption the supply of fund shares is perfectly elastic. This corresponds to the case of open-ended investment funds. We ensure that at any time the total net holdings of the funds are equal to the supply of the risky assets minus the net holdings of direct investment accounts.

To keep the delegation problem simple, we abstract here from the contract enforcement issues. This assumption is in line with the competitive nature of the economy, as agents are too small to influence individually asset prices and to have incentives to behave strategically. Moreover, as long as we assume that managing a fund requires no additional investment or effort, that agents are perfectly informed about the signal precisions of others, and that the fund fee does not contain a performance sharing component, there is no ground for the moral hazard or adverse selection issues. When taking the limit of the fund fees going to zero, a manager becomes indifferent if to accept an additional investment in his fund. We assume that in this case managers would continue to accept the delegated money. We rule out the possibility of managers investing the delegated money in the funds managed by others (no delegation of the delegated money).

When agent m delegates to a fund manager m', he chooses the number of shares of the fund to buy, which we denote by $\gamma^{mm'}$ and will refer to as delegation quantities. Agents are allowed to delegate any fraction of their wealth to a fund, implying $\gamma^{mm'} \in \mathbb{R}_+$, $\forall m'$. As any private signal is informative, restricting the delegation quantities to be nonnegative does not represent a restriction, as no agent would bet against genuine information.

We make an important assumption about the vector of delegation quantities of an agent m, requiring $\gamma^{mm'}$ to be non-atomic and smoothly distributed across all available funds. This assumption is motivated by the properties of the informational structure, in particular by the fact that signal errors of different agents $\epsilon_{m'}$ are uncorrelated. Thus a smooth fund allocation would allow to completely diversify the signal risk. To give the intuition, consider the number of shares of risky assets held by an investor m indirectly through funds, and use the definitions (1.2, 3.5):

$$\int_{m' \neq m} \gamma^{mm'} x_0^{m'} dm' = (\theta - PR) \int \gamma^{mm'} O_{m'} dm' + \int \gamma^{mm'} O_{m'} \epsilon^{m'} dm' + x_0^u \int \gamma^{mm'} E_{m'} dm' \tag{1.16}$$

The second term represents the aggregated signal error, and it vanishes if the function $\gamma(m')$ is bounded and nonnegative, using the property (1.6). It is straightforward to see that, in the case of zero fund fees ,the optimal fund allocations will necessarily satisfy the smoothness assumption, because adding one more fund to the allocation brings the benefit of diversifying the signal risk. We argue that this conclusion will hold even in the presence of nonzero fund fees, because the complementarity of private information will ensure that the indifference curve representing preferences for the funds will never cross the axes. Thus, the constraint on possible values of delegation quantities $\gamma^{mm'}$ essentially embeds the diversity-loving preferences for funds.

Funds are complementary products because spreading the investment across multiple funds diversifies the signal risk. In the conventional models of monopolistic competition the product complementarity is explicitly introduced in the consumers' utility function (e.g. Dixit and Stiglitz (1977)). In our case the demand for funds will not contain a term representing the aggregate signal risk, as we assume it to be always diversified away. The diversity-loving preferences for funds in our setup are represented by the constraints on the delegation quantities $\gamma^{mm'}$. This setup is different from the model ofRoss (2005). He uses a model with a finite number of funds and computes the derived utility function averaging over the signal errors explicitly without imposing a restriction on possible values of $\gamma^{mm'}$ as we do. As a result, the derived utility function in his study depends explicitly on the aggregate signal risk. Due to the difficulties involved in handling expectations over an infinity of individual signal errors, we prefer to deal with a simpler version of the derived utility function, where the term representing the signal risk is eliminated, and the complementarity is handled via an optimization constraint.

The assumption of complete diversification of signal risk is analogous to the diversification of idiosyncratic risk in asset portfolios. As was shown by Stiglitz (1989), when choosing a portfolio of risky securities, agents would diversify across all available securities. Thus, any security introduced in the market will be added to the optimal portfolio. The observable limited variety of risky investments in the real markets comes rather from the finite supply due to, for example, fixed costs of quoting the security on the public market. As such fixed costs are absent in our model of the fund market, the demand will exist for all the funds that could be potentially introduced by informed agents.

We turn now to the assumptions about the pricing of the funds. Fund managers set the fund fees. No explicit bargaining is introduced between the agents. We limit the fund pricing schemes to a simple linear uniform fee k^m per share of the fund. Then the profit from fund management is equal to:

$$\Pi^m(k^m) = k^m \int_{m' \neq m} \gamma^{m'm} dm' \tag{1.17}$$

We assume that fund managers are paid at the final date 2, and add the fee to their final wealth.

Extensive literature is dedicated to the issues of moral hazard in fund management, where managers exploit the sources of profits that are not explicitly bargained for, or cannot be ruled out in the investment agreements (Mahoney (2004)). We assume in this model that no such sources of profit exist. Given the model assumptions, managers do not have a possibility to shirk or to lie about their signal precisions. Given the absence of performance sharing, managers cannot profit by altering the structure of their funds. The competition among the managers does not provide incentives to distort their signals, as it would affect the complementarity between the funds.

Finally, we make several assumptions about the knowledge requirements for building equilibria in this economy. In order for the fund market to be viable, agents should be able to build expectations of the payoff from the delegated money. When agents have perfect knowledge of informational characteristics of all their peers, given by the set of precision matrices $\{\Sigma_m^{-1}\}$, the calculation of the fund payoff expectation is straightforward. Note, however, that the delegation market may function with much weaker knowledge requirements. The minimal knowledge needed for delegation is for agents to know the average precision of private information in the economy $Q = \int \Sigma_m^{-1} dm$. In this case agents cannot discriminate among managers, but have still an incentive to invest in the funds by building a symmetric fund allocation, i.e. by essentially treating all funds as having the average precision Q. An interesting possibility is for agents to have probabilistic knowledge about the others' precisions. We leave this case for future research. Different knowledge requirements would produce different equilibria in the joint markets for the funds and for the assets.

1.3 Equilibrium definitions

An equilibrium in the joint market for assets and for funds has a complex structure, with every agent playing simultaneously the roles of a trader, a fund investor and a fund manager. In this section we prepare the ground for the derivation of the equilibrium and introduce several concepts that will be useful to give a structure to the agents' behavior in equilibrium. In particular, we will give intuition about how the exposure to funds modifies the agent's wealth and risky allocation, how the freedom to trade on the own account affects the decision to delegate, how agents deal with the signal risk they assume via funds, and how agents separate their decision of investing in the funds of others from the decision to price their own funds.

For the sake of clarity in exposition, we consider here several separate partial equilibrium problems:

1. Partial equilibrium at the asset trading stage, when investors take the delegation decisions as exogenous.

2. The problem of optimal exposure to the funds given optimal own-account decisions at the asset trading stage, and treating the fund fees as exogenous.

1.3.1 The effect of delegation on wealth

The final wealth of agent m is composed of the return on the risk-free investment, the payoff of the own account risky portfolio, the payoff from delegated money minus the fees due, and the profit from managing his own fund:

$$W_2^m = x_B^m R + x^m D + \int_0^1 \gamma_{mm'} x_0^{m'} dm' D - \int \gamma^{mm'} k^{m'} dm' + k^m \int \gamma^{m'm} dm' \quad (1.18)$$

In what follows we will write the integral over the continuum of the invested funds simply as $\int_0^1 (..) dm'$, omitting the mention $m \neq m'$. Since no investor "is large" in the economy the exclusion of one point in the interval (0,1) does not change the value of the integral.

Using the agent's budget constraint $x_B^m + (x^m + \int_{m' \neq m} \gamma_{m'}^m x_0^{m'} dm')P = e_0$ the final wealth becomes:

$$W_2^m = e_0 R + x^m (D - PR) + \int_0^1 \gamma_{mm'} x_0^{m'} dm' (D - PR) - \int \gamma^{mm'} k^{m'} dm' + k^m \int \gamma^{m'm} dm' \quad (1.19)$$

At the asset trading stage the first and the two last terms in the wealth, related to the initial endowment and the fund fees, are already fixed, thus the own account optimal portfolio will be independent of the management fees.

Using the structure of the fund portfolio $x_0^{m'}$ (1.14), one can rewrite the wealth as follows (we omit at this point the constant terms that are irrelevant for the optimization at the asset trading stage):

$$W_2^m \equiv \tilde{x}^m (D - PR) + (\theta - PR) N^m (D - PR) + \int \gamma^{mm'} \epsilon^{m'} O^{m'} dm' (D - PR) \quad (1.20)$$

where we introduce the following notations:

$$\tilde{x}^m = x^m + \int \gamma^{mm'} E^{m'} dm' x_0^u \quad (1.21)$$

$$N^m = \int \gamma^{mm'} O_{m'} dm' \tag{1.22}$$

The first term the final wealth (1.20) is linear in the risky asset payoffs, being composed of the return on the own account portfolio x^m and the return from an "uninformed" part of the managers' portfolios, proportional to x_0^u. The structure of the linear exposure to the asset risk, denoted by $\tilde{x}^m$ and given by ex-expression (1.21) suggest the following proposition.

Proposition 1. *The own-account trading allows fund investors to offset the uninformed exposure contained in the fund portfolios.*

The proof of proposition follows from the fact that, for exogenously given delegation quantities $\gamma^{mm'}$ the final wealth (1.20) depends on the own-account portfolio x^m only through $\tilde{x}^m$. Thus, the optimal own-account exposure will always have the form $x^{m*} = \tilde{x}^{m*} - \int \gamma^{mm'} E^{m'} dm' x_0^u$. That is, the agent investing in funds will always offset the uninformed exposure to risky assets coming from the funds. **Q.E.D.**

The offset of the uninformed fund exposure might be understood as follows. Investing $\gamma^{mm'}$ in funds and subtracting the quantity $\int \gamma^{mm'} E^{m'} dm' x_0^u$ from the own-account portfolio the agent builds a synthetic exposure to the signal-related part of the fund portfolio. If the own-account trading is not prohibited, any uninformed exposure within a fund will be then irrelevant to fund investors. This is logical if we recall that funds in our model is are vehicles for selling the signals.

This feature of the own-account trading seems to be an insignificant mechanical shift in the risky allocation, but turns out to have profound consequences for the structure of the fund market. In the absence of own-account trading, any additional share of the fund exposes investor to extra uninformed risk, and thus the optimal demand for the fund shares would be a result of the trade-off between the desire of investor to get exposure to the private information, and his tolerance to the uninformed risk that is bundled together with the private signal in the fund portfolio. Consequently, the funds with higher precision become more valuable because, given the structure of the funds portfolio (1.14), they allow to get a larger exposure to the signal per unit of uninformed exposure. This is the case in the model of Ross (2005), where agents do not trade in assets directly, and thus are constrained to invest only in the funds with the highest precision. In the words of Ross, the low-precision funds are not viable. The proposition shows that this non-viability is the consequence of the model constrain on own-account trading. If this constraint is absent, agents have incentives to invest in all funds, because adding an additional fund helps to diversify the signal risk.

Matrix N^m (1.22) appears in the portion of the final wealth (1.20) quadratic in the payoff variables. This portion of the wealth represents a payoff from the agent's exposure to the noise-free private information via funds. We will refer to matrix N^m as a *delegation exposure*. The effect of the delegation exposure on the wealth of an agent is given by the following proposition:

Proposition 2. *Investing in a fund transforms the risky exposure and the final wealth of an agent in the same way as would have done an increase in the precision of his private information.*

To illustrate this proposition we consider a simple problem of peer-to-peer delegation where one agent with the signal precision Σ^{-1} invests in a fund based on the signal $\tilde{s} = \theta + \tilde{\epsilon}$ with precision $\bar{\Sigma}^{-1}$. We denote by $\bar{\gamma}$ the amount of shares of the fund bought, and assume for simplicity that there is no residual uncertainty ($V_D = 0$) and the fund has zero fees ($\bar{k} = 0$).

Fund investor knows the structure of the fund portfolio (1.14), which under the assumption of no residual uncertainty simplifies to:

$$\bar{x} = \frac{1}{\rho}\bar{\Sigma}^{-1}(\bar{s} - PR) + x_0^u \tag{1.23}$$

where x_0^u denotes the portfolio of uninformed agents.

The end-of-period wealth of the fund investor is:

$$W_2^m = e_0 R + (x + \bar{\gamma} x_0^u)(\theta - PR) + \frac{\bar{\gamma}}{\rho}(\tilde{s} - PR)\tilde{\Sigma}^{-1}(\theta - PR) \tag{1.24}$$

In appendix (1.8) we show that the optimal own-account portfolio x and the optimal delegation quantity $\bar{\gamma}$ have the following form:

$$x^* = \frac{1}{\rho}\Sigma^{-1}(s - PR) + (1 - \bar{\gamma})x_0^u, \tag{1.25}$$

$$\bar{\gamma}^* = 1, \quad if\, \bar{\Sigma} < \infty \tag{1.26}$$

The total exposure to risky assets for the delegating agent is composed of the optimal own-account portfolio and the risky exposure via the fund:

$$Exposure = x^* + \bar{\gamma}^* \bar{x} = \frac{1}{\rho}\Sigma^{-1}(s - PR) + \frac{1}{\rho}\bar{\Sigma}^{-1}(\bar{s} - PR) + x_0^u \tag{1.27}$$

It is easy to show that this is an optimal own-account portfolio of a hypothetical agent who would have observed both signals $(s, \bar{s})$ and would benefit from the precision of the combined signal $(\Sigma^{-1} + \bar{\Sigma}^{-1})$. Thus, delegating to funds has the effect of augmenting the precision of the agents, in the sense that it allows agents to construct risky exposures corresponding to signal precisions that are higher than the precision of their own signals. **Q.E.D.**

We introduce the following definition:

Definition 1. *Implied conditional precision G_m is the precision that could be associated with the risky exposure of agent m when the agent invests in funds.*

Note, that in the case of peer-to-peer delegation, presented in appendix (1.8) we showed that the agent with the precision of the own signal equal to Σ^{-1} who invests an arbitrary amount of shares $\tilde{\gamma}$ in the fund with precision $\tilde{\Sigma}^{-1}$ has the implied precision given by: $G = V^{-1} + \Sigma^{-1} + 2(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2)\tilde{\Sigma}^{-1}$. When the delegation quantity is optimal $\tilde{\gamma}^* = 1$, the implied precision reaches its maximum $G^* = V^{-1} + \Sigma^{-1} + \tilde{\Sigma}^{-1}$. Note, however, that even suboptimal amounts of fund investment would increase the agent's conditional precision, provided that $(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2) > 0$.

The exact form of implied precision in the case of investing in multiple funds will be given in the next section. We will show that, in general, implied conditional precision G_m is a function of the delegation exposure N_m. The consequence of proposition 2 is that agents view the problem of optimal fund investing as a problem of reaching maximal possible implied conditional precision. Agents use the available funds to build fund allocations that produce the optimal delegation exposure N_m^* giving the maximal implied conditional precision G_m^*.

1.3.2 Diversification of the signal risk

The third term in the final wealth (1.20) represents an aggregated signal error of the fund managers. In appendix 1.8 we showed that, when this term is present, the benefit from delegation is necessarily bounded. In the simple case of peer-to-peer investing the boost in the conditional precision, given by $2(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2)\tilde{\Sigma}^{-1}$, is a non-monotonous function of the delegation quantity $\tilde{\gamma}$. Increasing this quantity above 1 is suboptimal, because beyond this point the marginal penalty coming from the signal risk $\tilde{\epsilon}$ outweighs the marginal benefit from the exposure to the noise-free part of the signal.

It is easy to show that when an agent invests in several funds simultaneously, the signal risk of the individual funds is partially compensated. We make an assumption that in equilibrium all agents choose delegation strategies $\gamma^{mm'}$ such as the signal risk is completely diversified away:

$$\int \gamma^{mm'} \epsilon^{m'} O^{m'} dm' = 0 \tag{1.28}$$

The constraint of perfect diversification of the signal risk (1.28) requires that the random variables $\gamma^{mm'} \epsilon^{m'} O^{m'}$ have zero means and uniformly bounded variances (Chung (1974), theorem 5.4.1).

$$\mathbb{E}(\gamma^{mm'} \epsilon^{m'} O^{m'}) = \gamma^{mm'} \mathbb{E}(\epsilon^{m'}) O^{m'} = 0, \quad if\ \gamma^{mm'} \geq 0 \tag{1.29}$$

$$Var(\gamma^{mm'} \epsilon^{m'} O^{m'}) = (\gamma^{mm'})^2 O^{m'} \Sigma^{m'} O^{m'} < \infty, \quad \mid \gamma^{mm'} \mid < \infty \tag{1.30}$$

In other words, to satisfy (1.28) the delegation quantities $\gamma^{mm'}$ should be non-negative and finite. The latter requirement implies that delegation should be spread across a continuum of funds, instead of being concentrated on a portion or a countable set of available funds. It is self-evident that the requirement (1.28) will be always satisfied when delegation fees are the same for all funds. In such a case adding an additional fund always gives a benefit of increased diversification of signal risk without changing the total cost of delegation, thus smooth and bounded fund allocations will be the only optimal strategies for fund investors. We would argue that this assumption will not be restrictive even in the case of arbitrary configuration of fund fees. As far as there is a continuum supply of potential funds, it is not possible to have an equilibrium fee configuration where fund investors concentrate only on a portion of available funds. Managers who are left out would always find it worthwhile to align their fees with the rest of the fund market to attract investors.

The smoothness assumption (1.28) allows one to greatly simplify the calculation of the expected utility function by discarding the third term in (1.20). Technically, it means that the derived utility function (i.e. the ex-ante utility function optimized by agents at date 0, where the optimal own-account trading decision at date 1 is already taken into account) will not contain an explicit trade-off between the total delegation fee and the diversification of signal errors. Rather the complementarity among the funds will be embedded in the constraint (1.28). This will require some extra attention at date 0 optimization.

1.3.3 Definition of the equilibrium at asset trading stage

At date 1 agent m maximizes:

$$\max_{\{x^m\}} \mathbb{E}(U^m(W_2^m) \mid s^m, P), \tag{1.31}$$

where (s^m, P) is an information set available to the investor m, and x^m is the agent's own-account portfolio. The wealth W_2 is given by (1.20).

We showed in proposition 1 that the optimal risky portfolio x^m will contain a compensating term $-\int \gamma^{mm'} E^{m'} dm' x_0^u$, which offsets the uninformed exposure to risky assets coming from the fund portfolios. From the point of view of a fund investor, the uninformed portion of the fund x_0^u increases the agent's overall risk without contributing to the increase of the agent's implied conditional precision. Thus the agent would always eliminate this exposure through own-account trading.

The own-account portfolio of agent m will also contain a part corresponding to a bet on his private signal s^m, unless the price P fully reveals all the private information in the economy. This means that agents investing in funds will at the same continue to trade on their own signals. Information structure where individual signal errors can be diversified does not in general lead to binary "delegate-all-or-nothing" decisions, since some own-account investing is optimal even when one has already delegated some wealth to the funds.

The market clearing condition in the presence of the own-account trading and the investment in funds has the following form:

$$\int_0^1 (x^m + \int_{m' \neq m} \gamma^{mm'} x_0^{m'} dm') dm = z \tag{1.32}$$

where x^m are own-account portfolios, and x_0^m are fund portfolios given by (1.14). Using the fact that in a CARA-normal model the optimal own-account portfolios and the fund portfolios depend linearly on the payoff random variables, the price will be a linear functional of the aggregate private information $\int_0^1 s^m dm = \theta$ and the risky supply z:

$$P = A_0 + A_1\theta - A_2 z \tag{1.33}$$

Let us define the equilibrium in asset markets at date 1, assuming that agents have perfect knowledge of informational characteristics of all their peers.

Definition 2. *Equilibrium in asset markets, given the public knowledge about the informational characteristics of all the agents and the exogenously fixed delegation quantities $\gamma^{mm'}$, is a competitive equilibrium with rational expectations, where prices (1.33) aggregate private information coming from the optimal own-account portfolios and from the funds, which together satisfy the market-clearing condition (1.32).*

The equilibrium price will have the form (1.33) with the coefficients A_0, A_1, A_2 depending on the delegation quantities:

$$A_0 = A_0(\gamma^{mm'}), \quad A_1 = A_1(\gamma^{mm'}), \quad A_2 = A_2(\gamma^{mm'}) \tag{1.34}$$

The own-account portfolio will have the structure:

$$x^m = \alpha(\gamma^{mm'}, O^{m'})(s^m - PR) - \int \gamma^{mm'} E^{m'} dm' x_0^u \tag{1.35}$$

where $O^{m'}$ and Em' are functions of signal precisions $\Sigma_{m'}^{-1}$ given by (1.15), and α denotes a functional form of the dependence, which will be fixed in the equilibrium.

We would like to highlight three features of the asset trading equilibrium that are evident without performing further calculations. First, the noise in the risky asset supplies z is crucial for the existence of equilibrium. Without noise traders the prices would be fully revealing of the aggregate signal, which would contradict the rationale for investing in funds: if agents could deduce the aggregate signal from prices at no cost, they will not participate in the fund market in the first place. Second, in the presence of the fund market the asset market will necessarily aggregate more information than when it functions in isolation. In a joint market for funds and assets any agent will have higher implied conditional precision, which will translate in the increased intensity of trading on information. Third, the aggregated private information is not necessarily revealed through prices. When the prices are noisy $A_2 \neq 0$, agents can not perfectly reconstruct the payoff component θ even if the prices aggregates the full information about it: $A_0 = 0$, $A_1 = \frac{1}{R}$.

1.3.4 Definition of the equilibrium at delegation stage

To determine the optimal demand for the available funds and the optimal fees for the own funds, agents will use a derived utility function. The derived utility is the expected ex-ante utility of the final wealth computed at date 0 by taking into account the optimal own-account trading decisions at date 1. In the derived utility function the profit from selling the shares of the own fund and the benefit from investing in the funds of others will be separated, as illustrated by the following proposition:

Proposition 3. *At delegation stage the optimization problem of any agent splits into two independent sub-problems:*

1. *The investor problem, to find an optimal demands for delegation $\gamma^{mm'}(\{k^{m'}\})$ taking as given the fees of other funds $k^{m'}$:*

$$max_{\gamma^{mm'}} e^{\rho \int \gamma^{mm'} k^{m'} dm'} \mathbb{E}(e^{-\rho x^{m*}(\gamma^{mm'})(D-PR) - \rho \int_0^1 \gamma^{mm'} x_0^{m'} dm' (D-PR)} \mid s^m, P) \tag{1.36}$$

   where $x^{m*}(\gamma^{mm'})$ is the optimal own-account portfolio of agent m, given the delegation quantities $\gamma^{mm'}$.

2. *The manager problem, to set a fee for the own fund, given the demand of other agents for the fund:*

$$max_{k^m} k^m \int \gamma^{m'm}(k^m, k^{-m}) dm' \tag{1.37}$$

where k^{-m} stands for the average fee of all other fund managers.

The proof of Proposition 1 is straightforward, given the structure of the final wealth of agent m (1.19) and the CARA utilities. **Q.E.D.**

According to this proposition, every agent in the fund market behaves as a fund investor and as a fund manager at the same time. This follows from the property of the CARA preferences, as well as from the model assumptions ruling out a possibility to gain profits from fund management in ways other than selling of the signal.

Given the perfectly elastic supply of fund shares, individual solution to the investor problem (1.36) will be aggregated to provide the demand function for the funds. Then the equilibrium in the fund market will be determined by the pricing decisions of fund managers, given the aggregate demand structure. In their pricing decisions managers will take into account the complementarity between the own pricing decision and the average fee in the market. Each manager will take into account the average fees charged by the others as an exogenous variable, and the equilibrium fees will be a result of a Nash equilibrium in the pricing decisions.

Definition 3. *Equilibrium in the fund market, provided the public knowledge of the informational characteristics of all the agents, is given by the optimal delegation quantities $\gamma^{mm'}(\{k^{m'}\})$ and a collection of fees $\{k^{m'}\}$, which constitute a pure strategy Nash equilibrium for the continuum of managers $m' \in (0, 1)$. Each fee is optimal in the sense of (1.37).*

We make no a priori assumptions about the set of fees k^m. In the case of publicly known signal precisions the only a priori restriction on the fees appears for the uninformed agents, who will not be able to attract demand for their funds at a positive fee. Instead, they might become managers with nonzero fee in the cases when the individual precisions are not publicly known, limiting fund investors, unable to discriminate among individual managers, to use symmetric delegation strategies.

1.4 Equilibrium in the asset market

To find the general equilibrium we work the problem backwards, starting from date 1. Assuming a given delegation quantities $\gamma^{mm'}$, we find the partial equilibrium at the asset trading stage and give the expressions for the optimal own-account portfolio of investor m and the market clearing prices P as a function of exogenously given delegation quantities.

1.4.1 The structure of the expected utility

As we argued above, the problem of investing in funds could be reformulated as a problem of selecting a delegation exposure N^m that gives the highest implied conditional precision. The notion of implied conditional precision G^m was introduced in definition 1; it plays a role of the agent's inverse conditional variance of the payoff component θ given the private signal, and the signal indirectly possessed via delegation.

Proposition 4. *The implied conditional precision G^m of agent m who invests in a fund allocation $\gamma^{mm'}$ satisfying the diversification constraint (1.28) is given by the following expression:*

$$G^m = V_m^{-1} + 2N^m - N^m V_D N^m \tag{1.38}$$

where V_m^{-1} is the conditional precision of the agent in the absence of delegation, N^m is the delegation exposure of agent m, given by (1.22), and V_D is the variance matrix of the residual uncertainty component of the risky asset payoffs.

The proof of the proposition is in appendix 1.9. **Q.E.D.**

Proposition 4 holds for any fund allocation $\gamma^{mm'}$, not necessarily for the optimal one. In particular, when agent m does not delegate ($\gamma^{mm'} = 0$), the implied conditional precision is the same as the inverse conditional covariance V_m^{-1}.

It is obvious from proposition 4 that the effect of delegation on the implied conditional precision, and, consequently, on the agent's utility is not monotone in the presence of residual uncertainty $V_D > 0$. Intuitively, the effect of delegation can be understood as a sum of a benefit and a penalty. The benefit, equal to the additional term $2N_m > 0$, increases the precision due to access to the manager's signals, while the penalty $-N_m V_D N_m$ comes from the fact that, in order to obtain an additional benefit from the information contained in the signals, one has to increase the exposure to risky assets, and hence assume a greater exposure to the residual risk ϵ_D. This means that in the presence of residual uncertainty the optimal delegation exposure N_m will be finite, and the utility has a satiation point. It is also obvious that at the optimum $det(G^m) \geq det(V_m^{-1})$: the optimal amount of delegation should always improve precision.

To make the meaning of the implied precision even more straightforward, recall that investment in funds is a form of information sharing (Admati and Pfleiderer

(1988, 1990)). Investment in funds enlarges the agent's information set, as would do a direct acquisition of the signal. As a result, the precision of the agent improves the following sense. The agent's exposure to the risky assets when delegating optimally would be identical to that of a person who observes the signal within the fund and hence updates the conditional precision to G^m.

The structure of the implied precision (1.38) provides an insight about the demand in the fund market. Given that the effect of delegation is that of improving the agent's precision, the demand for funds will exist as long as an additional share of a fund improves the implied conditional precision G^m. One then should expect the demand for funds to be potentially unlimited in the absence of residual uncertainty, because the implied precision is unbounded from above. In the presence of the residual uncertainty the demand for funds will exhibit a satiation point because of the penalty term $-N_m V_D N_m$.

Note also, that the implied conditional precision G^m is superior to the own conditional precision of agents V_m^{-1}, even if all agents have the same signal precision to begin with. The boost in the individual conditional precision due to delegation comes not from an exposure to more precise signals, but from aggregating the signals of the funds in which the agent invests. Thus, even if all private signals in the economy have the same precision, the aggregated signal has much higher precision. The only exception is the case when no complementarity among signals exist, as when all agents have the same signal error. This is the case in the models of Grossman and Stiglitz (1980) and Garleanu and Pedersen (2018).

Before deriving the expected utility at the asset trading stage we introduce several useful notations. Let's denote by t^m the expected exposure of agent m to the risky assets. This exposure is composed of the own-account portfolio and the expected risky exposure of the fund allocation:

$$t^m = x^m + \int \gamma^{mm'} E_{m'} dm' x_0^u + N^m(\theta_c^m - PR) \tag{1.39}$$

where θ_c^m is the vector of the expected payoff of assets conditional on the information set of investor m (1.12), and N^m, given by formula (1.22), defines the risky exposure due to delegation.

For conveniency, we also introduce the following notations:

$$L_m = 1 - N_m V_D, \tag{1.40}$$

$$I = \int e^{-\frac{1}{2}(\theta-\theta_c^m)V_m^{-1}(\theta-\theta_c^m)} d\theta \tag{1.41}$$

$$J(G_m) = \int e^{-\frac{1}{2}(\theta-\theta_c^m)G_m(\theta-\theta_c^m)}d\theta \tag{1.42}$$

Note that quantities G^m, L_m, and $J(G^m)$ depend on the delegation quantities $\gamma^{mm'}$ through the delegation exposure matrix N^m. In the limit of no delegation there is no risky exposure ($N^m = 0$), the implied conditional precision $G^m = V_m^{-1}$, $L^m = 1$ and $J(G^m) = I$.

Proposition 5. *Assuming perfect knowledge of agents' precisions, the conditional expected utility of final wealth (1.20) of an agent m who has delegated fixed quantities $\gamma^{mm'}$ is given by:*

$$\begin{aligned}\log(-\mathbb{E}(U \mid s^m, P)) &= \log(\frac{I}{J(G_m)}) + \frac{1}{2}\rho^2 t^m V_D t^m - \rho t^m(\theta_c^m - PR) + \\ &+ \frac{\rho^2}{2}(t^m L_m^T + (\theta_c^m - PR)N^m)G_m^{-1}(L_m t^m + N^m(\theta_c^m - PR))\end{aligned} \tag{1.43}$$

The proof of the proposition is obtained by direct calculation of conditional expectations (see appendix 1.9 for details). **Q.E.D.**

In the limit of no delegation the expected utility (1.43) reduces to the familiar mean-variance form:

$$\log(-\mathbb{E}(U \mid s^m, P))_{\gamma=0} = \frac{\rho^2}{2}x^m(V_D + V_m)x^m - \rho x^m(\theta_c^m - PR) \tag{1.44}$$

To get more insight on how delegation affects the expected utility, let's consider the case of no residual uncertainty. In this case $L^m = 1$, and the matrix G^m simplifies to $G^m(V_D = 0) = V_m^{-1} + 2N_m$. The expression (1.43) becomes:

$$\begin{aligned}\log(-\mathbb{E}(U \mid s^m, P))_{V_D=0} &= \log(\frac{I}{J(G_m)}) - \rho\tilde{x}^m(\theta_c^m - PR) + \\ &+ \frac{\rho^2}{2}(\tilde{x}^m + (\theta_c^m - PR)N^m)G_m^{-1}(\tilde{x}^m + N^m(\theta_c^m - PR))\end{aligned} \tag{1.45}$$

Note, that the structure of the logarithm of the expected utility (1.45) is quadratic in the asset demand $\tilde{x}^m$, similar to the quadratic structure of the mean-variance utility (1.44). Matrix G^m plays in (1.45) the role similar to that of conditional precision of asset payoffs V_m^{-1} plays in (1.44).

1.4.2 Optimal own-account portfolios

Proposition 6. *Given the exogenous delegation quantities $\gamma^{mm'}$, an asset trading equilibrium exists. Optimal own account risky portfolios have the structure:*

$$x^m = \frac{1}{\rho}(V_D + L_m^T G_m^{-1} L_m)^{-1} L_m G_m^{-1} V_m^{-1}(\theta_c^m - PR) - \int \gamma^{mm'} E_{m'} dm' x_0^u \tag{1.46}$$

The market clearing price coefficients (A_0, A_1, A_2) are given by a solution to the following system of algebraic equations:

$$A_2^{-1}A_1 = \frac{\gamma}{\rho} I_3 - A_2^{-1} A_1 I_4 \tag{1.47}$$

$$\frac{R}{\rho}(I_3 + I_1(V^{-1} + A_1 A_2^{-1} U^{-1} A_2^{-1} A_1) + \frac{\rho}{1+\gamma} A_2^{-1} A_1 I_2) A_2 =$$
$$= 1 + \frac{1}{\rho} I_1 V_D A_1 A_2^{-1} U^{-1} - \frac{A_1 A_2^{-1}}{1+\gamma} I_2 A_1 A_2^{-1} U^{-1} \tag{1.48}$$

$$\frac{R}{\rho}(I_3 + I_1(V^{-1} + A_1 A_2^{-1} U^{-1} A_2^{-1} A_1) + \frac{\rho}{1+\gamma} A_2^{-1} A_1 I_2) A_0 =$$
$$= (\frac{1}{\rho} I_1 - \frac{A_1 A_2^{-1}}{1+\gamma} I_2 V_D)(V^{-1}\bar{\theta} + A_1 A_2^{-1} U^{-1} \bar{z}) \tag{1.49}$$

The integrals definition is:

$$\begin{aligned} I_1 &= \int_0^1 (1 + V_m^{-1} V_D)^{-1} dm \\ I_2 &= \int_0^1 \gamma^m (1 + V_D V_m^{-1})^{-1} dm \\ I_3 &= \int_0^1 (1 + V_m^{-1} V_D)^{-1} \Sigma_m^{-1} dm \\ I_4 &= \int_0^1 \gamma^m (1 + V_D V_m^{-1})^{-1} \Sigma_m^{-1} dm \end{aligned} \tag{1.50}$$

The proof of the proposition follows from the straightforward maximization of (1.43) and substituting the expression for optimal portfolios (1.46) into the market clearing condition (see the appendix (1.10) for details). **Q.E.D.**

Conditions (1.48) and (1.49) are solved easily if the solution for the (1.47) is available. Yet, since the integrals I_1,I_2, I_3 and I_4 depend non-linearly on $A_2^{-1}A_1$, the closed solution is not possible. The integrals depend on $A_2^{-1}A_1$ via the conditional precision V_m^{-1} defined by (1.11). Note, that if $V_D = 0$ the integrals do not depend on the price parameters anymore and the equilibrium solution $A_2^{-1}A_1 = \int_0^1 \Sigma_m^{-1} dm = Q$ follows.

Closed-form expressions are possible to obtain in a special case when $V_D = 0$. Then the integrals (I_1, I_2, I_3, I_4) do not depend on the price parameters anymore and the solution $A_2^{-1}A_1 = \int_0^1 \Sigma_m^{-1} dm = Q$ follows. In this case prices depend only on the aggregate delegation quantities $\gamma = \int\int \gamma^{mm'} dm dm'$. In appendix 1.11 we provide the exact expressions that allow one to study the behavior of the coefficients of prices as functions of γ.

Delegation affects the own-account risky portfolios in two ways. First, a compensating term $-\int \gamma^{mm'} E_{m'} dm' x_0^u$ is added to offset the risky exposure of the funds that is unrelated to the managers' private signals. What might seem as a pure technicality, the possibility to offload uninformed risky exposure has a profound effect on the equilibrium in the fund market. It allows investors to get a larger exposure to the funds, and makes it optimal to fully diversify away the signal risk. Without the possibility to trade on the own account when delegating, as in Ross (2005), investors limit their exposure to the funds by the amount of uninformed risk they are able to tolerate. Indeed, in Ross (2005) investors have to limit their fund allocation by investing in the funds of the most precise managers, because these allow to get a higher exposure to the private signals per unit of uninformed exposure. As a consequence, in the model of **??** some residual signal risk remained in the fund allocations and the funds of less precise managers were not viable. In our model investors are not deterred by the uninformed risk they take on when investing in fund portfolios, because they can always offset it using the own-account trading.

If one would redefine the fund structure 1.14 and eliminate the uninformed exposure of the fund portfolios entirely, it would not alter the equilibrium. The only effect of the redefinition of the fund structure would be a disappearance of the offset term from the own-account investing. No material quantity defining the asset and fund markets, such as the demand for delegation, or the equilibrium prices, would be affected. If anything, redefining the fund portfolio by allowing only a signal-related part would make the act of selling private information more straightforward. However, after eliminating the uninformed part from the fund portfolio (the part proportional to x_0^u), the portfolio will not be optimal conditional on the manager's information set. The suboptimality of the fund portfolio would be at odds with the common belief that a fund manager should deliver an optimal portfolio. Such obligation makes no sense in our model: the true mission of the manager is not to deliver an efficient portfolio to investors, but to faithfully reflect his unique private signal in the fund. Indeed, in the model of Admati and Pfleiderer (1988), who pioneered the concept of indirect sale of information via funds, fund portfolios were simply proportional to the signal $x_0^m = s^m$.

The second way in which the own account portfolios are modified by delegation relates to trading on the agent's own signal. The first term in (1.46) represents a bet on an agent's private signal. When the agent invests in funds, this bet gets smaller, but does not necessarily vanish. It is easy to see that there exists a special case of delegation exposure, corresponding to the choice of $\gamma^{mm'}$ such that $N_m = \frac{1}{\rho} V_D^{-1}$, implying $L_m = 0$, when the own account portfolio is not sensitive anymore to the private signal s^m. Such delegation exposure is optimal, for example, if the fund fees are negligible. It is easy to see that this choice of delegation quantities leads to aggregation of the full private information. In such a case prices P become sufficient statistic for the pair (s^m, P). Indeed, if this is the case, it follows from (1.43) that the expected risky exposure $\tilde{x}$ disappears from the expectation and the optimum is simply:

$$\tilde{x}^m = V_D^{-1}(\theta_c^u - PR) \tag{1.51}$$

From the definition of $\tilde{x}^m$ it follows that in this case:

$$x^m = -\int \gamma^{mm'} E_{m'} dm' x_0^u \tag{1.52}$$

the own-account portfolio is insensitive to the agent's own private signal. The market clearing price in this case is found from:

$$-\int x^m dm + V_D^{-1}(\theta - PR) + \int \gamma^{mm'} E_{m'} dm' x_0^u = z$$

which implies:

$$P = \frac{1}{R}\theta - \frac{\rho}{R} V_D z \tag{1.53}$$

The market-clearing price here is exactly the same as one would obtain if the dividend component θ is publicly known and the dividend D has probability distribution with the mean θ and the variance V_D. In such a case the full information about the realization of θ is aggregated in the individual risky allocations without the information being publicly revealed.

1.4.3 Aggregation of private information

The discussion in the previous section brings us to the important interpretation of delegation as a mechanism that amplifies information aggregation in asset markets.

Considering funds as vehicles to share private information, as pointed in Admati and Pfleiderer (1988); Ross (2005); Garcia and Vanden (2009), it is immediately clear that investing in funds will aggregate and incorporate in the prices more information than does the own-account trading alone.

The observation of the increase in the informativeness of prices, and, consequently, the decrease in the equity premium as a result of delegation, was a consistent finding across a wide array of delegation models. It was reported in the literature on performance-based contracts in asset management (see, for example, Kapur and Timmermann (2005); Basak and Pavlova (2013)), as well as in the models of delegation with rational expectations and information acquisition (for example, Ross (2005); Garcia and Vanden (2009); Garleanu and Pedersen (2018)).

The explanations of the phenomenon of increased informational efficiency due to delegation vary depending on the assumptions of individual models. In many cases the increased information quality in the market was explained by the replacement of less informed investors by more informed managers. In some cases the increase in the price informativeness followed from the increase in the incentives for information acquisition, in particular because in the presence of delegation information generates additional profits in the form of fund fees.

We give here an alternative explanation of the phenomenon. In our model the main channel responsible for the increase in the price informativeness is related to the increase in the aggregate risk-bearing capacity of agents due to delegation. Indeed, in our model the total amount of private information is fixed, all agents become managers, and everybody is free to trade in asset markets. Thus prices could not become more informative due to an external supply of additional information, or because the low-precision agents stop trading. Rather, we explain the increase in the price informativeness by the fact that each signal gets transmitted to the asset market with greater trading intensity. Because of the trading in fund portfolios, the total amount of trading according to each private signal is much greater than the own-account bet by the signal owner, which is limited by the individual risk aversion. Moreover, due to the aggregation of private information, an individual agent has much higher implied precision and is able to tolerate larger exposure to risky assets. Hence the total exposure to the individual signal is larger than it would be without delegation.

We will continue the discussion on the full revelation of private information in the next section where we will derive additional conditions that should be satisfied for the delegation to be able to aggregate fully the private information. For now, we only note that, if these conditions are not met, agents will always continue to trade on their own signals, irrespective of the amount they delegate.

1.4.4 The derived utility

Before turning to equilibrium in the fund market, we provide the form of the derived utility that agents use at the date 0.

Proposition 7. *At the delegation stage the agent's derived utility (that is, the ex-ante utility at date 0 taking into account the optimal own-account trading decision at date 1) has the following form:*

$$U^m = -\frac{\sqrt{\det(G_m^{-1})}}{I^m} e^{-\rho(\theta_c^m - PR)M^m(\theta_c^m - PR) + \rho\int k^{m'}\gamma^{mm'}dm' - \rho k^m \int_0^1 \gamma^{m'm}dm'} \tag{1.54}$$

where G^m is given by (1.38), and M^m does not depend on the delegation quantities $\gamma^{mm'}$:

$$M^m = V_m^{-1} + V_D^{-1} \tag{1.55}$$

The proposition is derived by substituting the expression for the optimal own-account portfolio (1.46) into the expected utility given by expression (1.43) and adding the terms corresponding to the fund fees paid and received. **Q.E.D**.

A remarkable feature of the derived utility is that the benefits of delegation for agent m are fully defined by the matrix G^m, which has an interpretation of implied conditional precision. Recall, that all the expressions relative to delegation are derived using the assumption of complete diversification of signal risk (1.28). Hence the derived utility (1.54) does not contain the explicit trade-off between the reduction of the signal risk and the cost of adding an additional unit of a fund. The complementarity of different funds is rather embedded in the condition of non-negativity and finiteness of the delegation quantities $\gamma^{mm'}$. Thus, as the delegation stage, except in some special cases, one has to perform a constrained maximization of the derived utility function, because an unconstrained optimization will not reflect the variety-loving preferences for funds.

1.5 Costless delegation and full aggregation of private information

We now turn to the construction of equilibrium at the delegation stage. Before considering the complete equilibrium, we solve a partial equilibrium problem where fund fees are forced to be zero. It will allow us to introduce useful concepts and notations, and get some important insights about the demand for delegation.

1.5.1 Demand for delegation in case of zero fees

When there are no fund fees, the investor optimization problem can be solved by computing a first-order condition from the derived utility function (1.54), because the constraint of bounded and positive $\gamma^{mm'}$ will be automatically satisfied. Since there are only benefits and no costs to spreading the delegation across more funds, any concentrated fund allocation would be suboptimal.

The problem of finding the optimal fund allocation in the case of zero fund fees simplifies to:

$$maxdet(G_m)_{\gamma^m_{m'}} = maxdet_{\gamma^{mm'}}(V_m^{-1} + 2\rho N_m - \rho^2 N_m V_D N_m) \tag{1.56}$$

The first order conditions with respect to the delegation quantities $\gamma^{mm'}$ will have the following structure:

$$-tr(G_m^{-1} O_{m'} L_m) = 0 \tag{1.57}$$

where $O_{m'} = \frac{dN_m}{d\gamma^m_{m'}}$ is given by (1.15), and L_m is defined by (1.40).

We state the following proposition:

Proposition 8. *An optimal fund allocation, satisfying (1.57) can be obtained by choosing the individual delegation quantities* $\gamma^{mm'}$ *such as:*

$$N_m = \int_0^1 \gamma^{mm'} O_{m'} dm' = \frac{1}{\rho} V_D^{-1} \tag{1.58}$$

This allocation will be an optimum because all the entries of the matrix under the trace in (1.57) will become zero and the condition will be satisfied. To demonstrate the proposition we rely on the fact that, given infinitely many optimization variables $\gamma^m(m')$, $m' \in [0, 1]$, one could find a function $\gamma(m')$ such that the $k(k+1)/2$ equalities in (1.58) are satisfied[1]. Indeed, there might be many alternative configurations $\gamma(m')$ that achieve this result.

A formal solution to (1.58) can be obtained in the form of a generalized inverse of the matrix O_{mL} where m is an index labelling investors, and L is the index counting $k(k+1)/2$ entries of a symmetric matrix O for a fixed m. To rewrite (1.58) in matrix notations, let's call Γ a row of variables $\gamma(m)$, and B a column formed from the entries of the matrix $\frac{1}{\rho} V_D^{-1}$ (capital latin letters represent the index labelling all the

[1] The number of equalities follows from the fact that N_m is a symmetric matrix.

distinct pairs (i,j) of a k-dimensional matrix). In these notations one can reformulate (1.58) as:

$$\Gamma_m \cdot O_L^m = B_L \tag{1.59}$$

The above is an underdetermined system of linear equations for the vector Γ. A formal solution to (1.59) is given with the help of the generalized inverse of matrix O_L^m, denoted by $O^+ = (O^T O)^{-1} O^T$ (the superscript T denotes a transposed matrix):

$$\Gamma = B^T (O^T O)^{-1} O^T + f^T (I - OO^+) \tag{1.60}$$

where the second term is a member of the kernel of matrix O and vector function $f(m)$ is an arbitrary function.

Note, that we have already shown that the delegation solution (1.58) produces an equilibrium in asset markets where the full private information is aggregated, and where own-account portfolios do not contain private signals. It follows that when delegation fees are forced to be zero, the fund market represents a mechanism that in equilibrium allow to fully aggregate private information. Conversely, it is clear that when nonzero fees are introduced, the information aggregation will remain partial, because with the fees the demand for delegation will be lower.

The full revelation of private information could be also prevented by causes unrelated to the fees. Observing the structure of (1.58), it is clear that, as a system of linear equations, it will be overspecified and admit no solution if matrices $O_{m'}$ of different managers are not diverse enough. For the solution to exist, one needs at least $n(n+1)/2$ distinct matrices $O_{m'}$. Let's call a category of managers the set of funds where every manager has the same precision of the private signal, and, as a result, the same $O^{m'}$. If there happen to be too few categories of managers in the economy, it is not possible to solve the system for arbitrary V_D. The optimal delegation quantities will be then determined by the first order condition (1.57), and the resulting fund allocation will not lead to the full aggregation of private information.

In the next section we illustrate the solution concept in the case of a zero-fee delegation when the asset market contains only one risky asset.

1.5.2 Delegation with zero fees in the case of one risky asset

In the case of one risky asset the solution to (1.60) is obtained by noting that index L has just one possible value, so O is just a row vector:

$$O_m = O(m) = \frac{1}{\rho} \frac{\sigma_m^{-2} v_m}{v_m + v_D}, \qquad B = \frac{1}{\rho} v_D^{-1}$$

where $v_m^{-1} = \sigma_m^{-2} + v_u^{-1}$, v_u^{-1} is the inverse conditional variance of an uninformed investor.

Let's denote the constant:

$$H_2 = (O^T O) = \frac{1}{\rho^2} \int_0^1 \frac{v_{m'}^2 \sigma_{m'}^{-4}}{(v_{m'} + v_D)^2} dm'$$

Then the generalized inverse $O^+ = (O^T O)^{-1} O^T$ is a scalar function:

$$O_m^+ = \frac{\rho}{H_2} \frac{v_m \sigma_{m'}^{-2}}{v_m + v_D}$$

The solution (1.60) in the case of one risky asset is:

$$\gamma^{mm'} = \frac{1}{H_2} \frac{v_{m'} \sigma_{m'}^{-2}}{v_{m'} + v_D} v_D^{-1} + f(m') - \frac{1}{H_2} (\int_0^1 f(n) \frac{v_n \sigma_n^{-2}}{v_n + v_D} dn) \frac{v_{m'} \sigma_{m'}^{-2}}{v_{m' + v_D}} \tag{1.61}$$

Function $f(m')$ is arbitrary, and reflects the ambiguity of the solution. Every investor is free to pick his own function $f^m(m')$. If we assume that all investors choose the same functional form of $\gamma(m')$, the amounts of delegation will not depend on the delegating agent m. One possible fully aggregating equilibrium results if agents demand identical fund allocations $\gamma^{mm'} = \gamma(m')$, irrespective of their own signal precision.

In general, the solution (1.61) depends on the manager's precision $\sigma_{m'}^{-1}$. Note, however, that the function f can be always chosen such that the delegated amounts $\gamma^{mm'}$ become independent of the manager's precisions:

$$\gamma(m') = \gamma = \frac{1}{\rho v_D} \frac{1}{\int_0^1 \frac{v_n \sigma_n^{-2}}{v_m + v_D} dn}$$

The above particular solution can be obtained from the general solution above by choosing $w(m')$ of the following form:

$$f(m') = \frac{1}{v_D H_1}$$

Assuming investors have sufficient information about the population distribution of private precisions to evaluate the integral $H_1 = \int_0^1 \frac{v_n \sigma_n^{-2}}{v_m + v_D} dn$, there is no need to know individual precisions to extract the most value from delegation in the case of one risky asset. Just delegating the same amount to each manager is enough to achieve the optimal result. The fact that a simple average over the funds achieves the first best result is quite remarkable, meaning no further improvements such as investment in learning precisions, would be justified in an economy with one risky asset.

A similar conclusion, namely that the full aggregation of private information is easy to achieve with just one source of ucertainty, occurs frequently in models of information aggregation and social learning.[2] Indeed, simple averaging across the continuum of private signals will usually be enough to get rid of the signal risk. With this in mind, one would rather have to explain why in some cases, namely for asset markets with multiple risky assets, the private information *could not* be fully aggregated by taking a simple average of funds.

1.5.3 Delegation with zero fees in the case of two risky assets

Assume there are two risky assets in the economy and that variance matrices V, V_D and Σ_m are diagonal. Assume also, that the economy has two possible precision categories: Σ_A^{-1} and Σ_B^{-1}. Agents $m \in (0, 1-\omega)$ belong to category A. We assume the precision of category A to be a diagonal matrix $\Sigma_A^{-1} = \sigma_A^{-1}\mathbb{1}$. Agents $m' \in (1-\omega, 1)$ belong to category B, specializing exclusively in asset 2: $(\Sigma_B^{-1})_{11} = 0$. The funds portfolios have mean-variance structure: $x^m = O^A(\Sigma_A^{-1})(s^m - PR)$ for $m \in (0, 1-\omega)$ and $x^m = O^B(\Sigma_B^{-1})(s^m - PR)$ for $m \in (0, 1-\omega)$, where O_A and O_B are 2x2 diagonal matrices. Note, that portfolios of the funds in the same category are not identical, because the signal noise is individual for each fund.

Let's denote by γ^A, γ^B the demands for shares of the funds of category A and B, respectively. The solution to (1.58) in this case is:

$$\omega\gamma^A = \frac{1}{\rho O_{A11}} V_{D11}^{-1}, \tag{1.62}$$

$$(1-\omega)\gamma^B = \frac{1}{\rho O_{B22}}(V_{D22}^{-1} - V_{D11}^{-1}\frac{O_{A22}}{O_{A11}}) \tag{1.63}$$

Because the delegation amount γ^B is not allowed to be negative, the full revelation of private information in the economy is possible only if $V_{D22}^{-1} > V_{D11}^{-1}$, i.e. if investors of category B choose to specialize in the asset with a lower residual uncertainty. This highlights the fact that the demand for funds does not depend exclusively on the funds' precisions $O_A(\Sigma_A^{-1})$, $O_B(\Sigma_B^{-1})$, as the amount of residual uncertainty in the assets' payoffs does also play an important role. High precision of a manager's signal does not necessarily motivate a large investment in his fund for two reasons. First, a fund with high precision of signal invests more in the asset, so investor needs less

[2] I am grateful to Ignacio Monzon for pointing out this fact.

shares of the fund to get a desired informed exposure to the asset. Second, an asset with a good precision of the private information might be too risky to hold in large amounts because of the residual uncertainty. Lastly, the demand for an individual fund depends also on the mass of the funds in the category. More managers fall into a category, less will be the individual amount of delegation to each of them.

The diversity of the precision matrices (which manager specializes in which asset) and the relative residual uncertainty of the assets matter more for investors than the magnitude of the precision of an individual manager. The relative irrelevance of fund precisions is in agreement with the findings in the case of one risky asset considered above, where it was possible to delegate optimally just by spreading the investment equally across the managers.

The demand for delegation in the case of multiple risky assets has important differences compared to the case of one risky asset. Even with a simple diagonal structure of variances, the solution (1.60) does not always exist, and, as a result, a fully aggregating fund allocation might not exist if the information structure is not diverse enough.

1.6 Equilibrium in the delegation market

We have already mentioned that the derived utility function (1.54) does not characterize the optimization problem of a fund investor completely, because we imposed a constraint that delegation quantities $\gamma^{mm'}$ should be nonnegative and bounded. To handle the difficulty of dealing with an implicit constraint, we make a change of variables and split the problem into two tasks: finding an optimal delegation exposure N_m, and then finding the quantities $\gamma^{mm'}$ producing the optimal delegation exposure at the lowest cost. We call this approach a two-step optimization procedure.

1.6.1 The two-step procedure

The solution procedure to the problem of fund investor (1.36) might be formulated, noting that an investor cares not about the individual quantities $\gamma^{mm'}$, but about the aggregate delegation exposure N_m and the total cost of delegation $\int_0^1 k^{m'}\gamma^{mm'}dm'$. Therefore, the problem of finding the optimal fund allocation can be framed as a two-step procedure, where the main decision variables are the components of the delegation exposure N_m. The individual delegation densities $\gamma^{mm'}$ are then found from the requirement that the desired delegation exposure be achieved at the lowest cost.

Start with the investor optimization problem, which can be written as:

$$min_{\gamma^{mm'}} - \frac{1}{2} \ln \det(G_m) + \rho \int_0^1 k^{m'} \gamma^{mm'} dm' \tag{1.64}$$

The implied precision G_m depends on $\gamma^{mm'}$ only through the integral $N_m = \int O_{m'} \gamma^{mm'} dm'$. In (1.60) we gave a general formula to reconstruct the individual delegation quantities $\gamma^{mm'}$ from the integral.

Adopting concatenated matrix notations, where capital latin labels $L, K, ..$ denote distinct pairs of indices (i, j), a matrix $(N_m)_{ij}$ becomes a vector $(N_m)_L$ for fixed m. If the index m runs over the entire set of manager labels, N_{mL} is a matrix. Similarly, the collection of precision matrices O^m_{ij} becomes a single matrix O_{mL}. For any fixed m, a quantity $\gamma^{mm'}$ becomes a vector in infinite-dimensional space: $\Gamma_{m'}$. For brevity we use below the Einstein summation notations, when a summation is implied over all the values of an index that is repeated twice in a single term. The definition of the aggregate delegation exposure for a manager m in concatenated matrix notations becomes:

$$\Gamma_{m'} O_{m'L} = N_L \tag{1.65}$$

Here we omit the fixed manager label m, and do not write the symbol for integration, as it is implicitly assumed when indices m' repeat.

The inversion of (1.65) gives:

$$\Gamma_{m'} = N_L O^+_{Lm'} + f_{m''}(\delta_{m''m'} - O_{m''L} O^+_{Lm'}) \tag{1.66}$$

where $f_{m''}$ is an infinite-dimensional vector, and $O^+_{Lm'} = (O^T O)^{-1}_{LK} O^T_{Km'}$ is a generalized inverse of matrix $O_{Lm'}$. Any choice of $f_{m''}$ gives the same aggregate exposure N, but in general at different cost:

$$Cost = \Gamma_{m'} k^{m'} = N_L O^+_{Lm'} k^{m'} + f^{m''}(\delta^{m'm''} - O^+_{m'L} O^T_{Lm''}) k_{m'} \tag{1.67}$$

Below we outline the two-step procedure that fund investors might use in order to solve the optimization problem:

1. Step 1: for every delegation exposure N_L, find a form of the arbitrary function $f_{m'}$ that gives the lowest aggregate cost. This will give an expression for Γ as a unique function of N_L.

2. Step 2: insert the exact form of the individual delegation exposure found in Step 1 into the investor's optimization problem (1.64). Solve the optimization problem using the components N_L as decision variables.

First, we give the solution to Step 1. Note that the matrix $(\delta^{mm'} - O^+_{mL}O^T_{Lm'})$ is a symmetric matrix. Thus, if we choose $f_{m'} = -\eta k^{m'}$, $\eta > 0$, the second term in (1.66) will be strictly negative, represenitng a negative adjustment to the aggregate delegation cost. The scaling factor η is to be fixed such that all individual delegation quantities are nonnegative.

The minimal cost Γ is then given by:

$$\Gamma^{min}_{m'} = N_L O^+_{Lm'} - \eta k^{m''}(\delta^{m''m'} - O_{m''L}O^+_{Lm'}) \tag{1.68}$$

Second, we can perform Step 2. Using the minimum cost individual delegation quantities (1.68), the investor's optimization problem (1.64) can be written as:

$$min_{N_L} \quad -\frac{1}{2}\ln\det(G_m) + \rho k^{m'} O^+_{m'L} N^m_L - \rho\eta k^{m'}(\delta^{m'm''} - O^+_{mL}O^T_{Lm''})k^{m''} \tag{1.69}$$

To find the optimal exposure N^{m*}_L, let's write and solve the first-order conditions for the problem (1.69). We introduce the following definition:

Definition 4. *Average precision-weighted cost Ω_L represents an average fee weighted by the generalized inverse of the managers' precisions $O^+_{m'L}$:*

$$\Omega_L = \frac{\partial}{\partial N_{mL}}(\rho \int k^{m'} O^+_{m'L} N_{mL})dm' = \rho \int k^{m'} O^+_{m'L} dm' \tag{1.70}$$

Going back from concatenated matrix notations to ordinary matrix notations, $\Omega_L = \Omega_{ij}$ is a $N \times N$ matrix. It has a complex dependence on precisions, but, importantly, it does not depend on the delegation quantities $\gamma^{mm'}$ and thus on the delegation exposure N^m.

Now we are able to write down the full expression for the first order conditions of the problem (1.69). In ordinary matrix notations it reads:

$$-\rho(G^{-1}_m)^T + \frac{\rho^2}{2}(V_D N_m G^{-1}_m)^T + \frac{\rho^2}{2}(G^{-1}_m N_m V^D)^T + \Omega = 0 \tag{1.71}$$

One can conveniently rewrite the above condition via matrix L_m, defined in (1.40). Note, that since G_m is a symmetric matrix, we used $(G^{-1})^T = G^{-1}$:

$$-\frac{\rho}{2}(G^{-1})^T L - \frac{\rho}{2} L^T (G^{-1})^T + \Omega = 0 \tag{1.72}$$

A solution to the investor's optimal delegation problem is given by the following proposition.

Proposition 9. *Optimal delegation exposure of investor m is given by the following conditions:*

$$N_m^* = \frac{1}{\rho} V_D^{-1} (1 - \frac{1}{\rho} \Omega G_m^*) \tag{1.73}$$

where optimal implied precision G_m^* satisfies the following matrix equation:

$$G_m = V_m^{-1} + V_D^{-1} - \frac{1}{\rho^2} G_m \Omega V_D^{-1} \Omega G_m \tag{1.74}$$

To derive the proposition, one notes that a formal solution to the first order condition (1.72) is given by:

$$L_m = \frac{1}{\rho} \Omega G_m \tag{1.75}$$

Then, using the definitions (1.38) and (1.40) one can represent G_m in terms of L_m as:

$$G_m = V_m^{-1} + V_D^{-1} - L_m^T V_D^{-1} L_m \tag{1.76}$$

Substituting (1.75) into (1.76) one gets the nonlinear matrix equation for (1.74).

The optimal delegation exposure (1.73) can be obtained easily by inverting (1.40) that satisfies (1.75). **Q.E.D**

Proposition 9 shows that optimal delegation exposure N_m^* and implied precision G_m^* decrease when delegation fees increase. The full aggregation of private information is possible only if $\Omega = 0$, which implies $L_m = 0$ and corresponds to the no-fees solution (1.58).

The condition (1.74) is a quadratic matrix equation, known as the Algebraic Riccati Equation (ARE). We solve it using the standard technique, as outlined, for example, in Kucera (1973). Before deriving a general form of the solution we specify two useful limits where the solution to (1.74) can be obtained immediately.

1. $\Omega = 0$, which corresponds to zero fund fees. As was shown above, it results in the full aggregation of private information. Here the (1.73) and (1.74) admit a trivial solution:

$$G_m^* = V_m^{-1} + V_D^{-1}, \quad N_m^* = \frac{1}{\rho} V_D^{-1} \tag{1.77}$$

2. $V_D = 0$, which corresponds to zero residual uncertainty. The solution is obtained trivially from (1.74) by noting that $V_D = 0$ implies $L = 1$, and thus:

$$G_m^* = \rho\Omega^{-1}, \quad N_m^* = \frac{1}{2}\Omega^{-1} - \frac{1}{2\rho}V_m^{-1} \tag{1.78}$$

Note, that the implied conditional precision G_m^* is bounded when delegation fees are nonzero, because the delegation exposure N^{m*} becomes bounded in the presence of fees. It is easy to see from (1.78) that agents with high conditional precision $V_m^{-1} > \rho\Omega^{-1}$ will not invest in funds. For these agents the fund fees are too high and outweigh the benefits of delegation.

An ARE equation may possess a variety of solutions, including no solution and infinitely many solutions. Under certain regularity conditions for matrices V_D, Ω, V_m, a solution can be shown to exist. Since a nonlinear equation (1.74) admits in general several solutions, we would be interested in those compatible with the limiting cases considered above.

A general solution to equation (1.74) is given in the following proposition.

Proposition 10. *The optimal implied precision G_m^* has the following form:*

$$G_m^* = Y\tilde{D}^{-1}Y^{-1}(V_m^{-1} + V_D^{-1}) \tag{1.79}$$

where $\tilde{D}$ is a diagonal matrix containing eigenvalues of the Hamiltonian M of dimension ($2n \times 2n$):

$$M = \begin{bmatrix} -\frac{1}{2}\mathbb{1} & -\frac{1}{\rho^2}\Omega V_D^{-1}\Omega \\ V_m^{-1} + V_D^{-1} & -\frac{1}{2}\mathbb{1} \end{bmatrix} \tag{1.80}$$

And Y is a matrix the columns of which are the lower parts of the eigenvectors of matrix M:

$$Ma_i = \lambda_i a_i, \quad i = 1, .., 2n$$
$$a_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

The above structure of matrix M implies that the eigenvalues λ will have the structure:

$$\lambda_{\pm i} = \pm\sqrt{\frac{1}{4} + \xi_i}, \quad i = 1, .., n \tag{1.81}$$

where $\xi = \lambda^2 - \frac{1}{4}$, are the eigenvalues of ($n \times n$) matrix H:

$$H = \Omega V_D^{-1}\Omega(V_m^{-1} + V_D^{-1}) \tag{1.82}$$

Proof of proposition is given in appendix 1.12. **Q.E.D.**

In the following sections we illustrate the solution to (1.73) and (1.74) in the case of one risky asset and two risky assets.

1.6.2 Optimal delegation demand with one risky asset

Here we apply the general solution procedure outlined in proposition 1.12 in the simplest case of one risky asset. For one risky asset, the determinant in (1.69) can be omitted, and the concatenated matrix notation coincides with the regular matrix notation. Thus, one can solve the problem directly, without applying the solution technique of the matrix Riccati equation. Still, for the sake of exercise, we follow the matrix procedure to derive the solution.

Let's denote the one-dimensional matrix G by g . Similarly, denote by small latin and greek letters the other variables: b stands for B, q stands for Q, ω for Ω, v_D for V_D. By definition:

$$b^2 = \frac{\omega^2}{v_D \rho^2}, \quad q = \frac{1}{v_m} + \frac{1}{v_D}$$

The matrix M for one risky asset has the form:

$$M = \begin{bmatrix} -\frac{1}{2} & -b^2 \\ -q & \frac{1}{2} \end{bmatrix}$$

The characteristic equation $-(\frac{1}{4} - \lambda^2) - b^2 q = 0$ gives the solution:

$$\lambda_{\pm} = \pm\sqrt{b^2 q + \frac{1}{4}} \tag{1.83}$$

The general solution for g, according to (1.133) is:

$$g^* = q(\frac{1}{2} - \lambda_-)^{-1} = \frac{q}{\frac{1}{2} + \sqrt{b^2 q + \frac{1}{4}}} \tag{1.84}$$

Rewriting explicitly b and q in terms of v_D, ω and v_m gives:

$$g^* = \frac{v_m^{-1} + v_D^{-1}}{\frac{1}{2} + \sqrt{\frac{\omega^2}{\rho^2} v_D^{-1}(v_m^{-1} + v_D^{-1}) + \frac{1}{4}}} \tag{1.85}$$

The optimal solution for delegation exposure n^* is:

$$n^* = \frac{1 - \frac{1}{\rho}\omega g^*}{v_D \rho} = \frac{1}{\rho v_D} - \frac{\omega(v_m^{-1} + v_D^{-1})}{\rho^2 v_D(\frac{1}{2} + \sqrt{\frac{\omega^2}{\rho^2} v_D^{-1}(v_m^{-1} + v_D^{-1}) + \frac{1}{4}})} \tag{1.86}$$

The two limits discussed above give:

$$v_D = 0 \;\rightarrow\; g^*_m = \rho\omega^{-1}, \qquad n^*_m = \frac{1}{2\rho\omega} - \frac{1}{2\rho v_m}$$

$$\omega = 0 \;\rightarrow\; g^*_m = (v_m^{-1} + v_D^{-1}), \quad n^*_m = \frac{1}{\rho} v_D^{-1}$$

The second limit ($\omega = 0$) was already discussed above in detail. We discuss now the optimal solutions for the first limit ($v_D = 0$). With no residual uncertainty, the optimal matrix $g^*_m = \rho\omega^{-1}$ is the same for all agents who choose to delegate, irrespective of their own signal precisions. The delegation exposure n^*_m, however, is not the same for different agents and depends on the agent's own precision via v_m^{-1}. Agents with higher precision delegate less. An agent with a precision σ^{-1} will stop delegating if the cost variable ω is equal to $\omega = \frac{1}{2} v_m = \frac{1}{2}(v_u + \sigma)$. Hence for any given ω, only a portion of agents with the precisions smaller than $\sigma_{max} = 2\omega - v_u$ will invest in funds.

The total delegation exposure over the continuum of agents is:

$$\int_0^1 n^*_m dm = \frac{1}{\rho}(1 - \frac{1}{\rho}\omega \int_0^1 g^*_m dm) v_D^{-1}$$

The solution in terms of individual delegation quantities in a market with one risky asset is obtained from (1.68) as:

$$\gamma^{mm'} = \frac{n^*_m o_{m'}}{\int_0^1 o^2_{m'} dm'} - \eta \int_0^1 k^{m''} \left(\delta^{m''m'} - \frac{o_{m''} o_{m'}}{\int_0^1 o^2_{m''} dm''} \right) dm'' \tag{1.87}$$

Or, using the definition of ω (1.70), one can rewrite the delegation quantities as:

$$\gamma^{mm'} = \frac{(n^*_m + \frac{\eta}{\rho}\omega \int_0^1 o^2_{m''} dm'') o_{m'}}{\int_0^1 o^2_{m'} dm'} - \eta k^{m'} \tag{1.88}$$

The (1.70) for one risky asset simplifies to:

$$\omega = \rho \frac{\int_0^1 k^{m'} o_{m'} dm'}{\int_0^1 o^2_{m'} dm'} \tag{1.89}$$

The optimal delegation quantities $\gamma^{mm'}$ grow with the manager's precision $o^{m'}$ and the depend on delegation fees through the aggregate cost quantity ω, as well as through the individual delegation fee $k^{m'}$. The coefficient η determines the level of penalty that investors impose on managers who quote fees higher than the average (or, more precisely, higher than the precision-weighted average). The penalty parameter has a special role in shaping the competition between managers. Indeed, when $\eta = 0$ the demand for funds is not sensitive to individual fees, providing managers with unbounded incentives to increase individual fees. With a penalty $\eta \neq 0$, the demand is comprised of an inelastic term and a term sensitive to the individual fee, providing a balance between the incentive to increase the fee to profit from the inelastic part of the demand and to lower the fee to mitigate the penalty.

1.6.3 The equilibrium fee structure

One can easily solve the manager's problem of finding optimal fee k^m (1.37), given the general form of the investor's optimal delegation quantities (1.68). The problem becomes:

$$max_{k^m} \quad k^m \int N_L^{m'} dm' O^+_{Lm} - \eta k_m^2 + \eta \int k^{m'} O_{m'L} O +_{Lm} dm' k^m \tag{1.90}$$

Note, that $N_L^{m'}$ depends on the fees only through the aggregates, and thus is not sensitive to an individual fee k^m.

The first order condition gives:

$$\int N_L^{m'} dm' O^+_{Lm} - 2\eta k_m + \eta \int k^{m'} O_{m'L} O^+_{Lm} dm' = 0 \tag{1.91}$$

It gives a linear equation for k^m, which admits the following solution:

$$k^{m*} = \frac{1}{2\eta} \int_0^1 (N_L^{m'} + \eta k^{m'} O_{m'L}) dm' O^+_{Lm} \tag{1.92}$$

Expression (1.92) represents the condition for the Nash equilibrium, as the optimal fee of manager m depends on the average fee of other managers. The optimal fees are proportional to the generalized inverse of the manager's precision matrix: $k^{m*} = a_L O^+_{Lm}$. Note that the scaling coefficient $a_L = \frac{1}{2\eta} \int_0^1 (N_L^{m'} + \eta k^{m'} O_{m'L}) dm'$ is the same for all managers.

The fee structure (1.92) exhibits a strategic complementarity in the managers' pricing decisions: the higher the average precision-weighted fee Ω, the higher is the fee an individual managers demands. Conversely, higher individual fees k^m drive up the average precision-weighted fee Ω. The fees will grow exponentially, however, because the demand is reduced when the aggregate fee gets higher.

To find the equilibrium, we have to derive the equilibrium condition for the precision-weighted average cost:

$$a_L = \rho \int_0^1 k^m O_{mL} dm \tag{1.93}$$

Proposition 11. *In equilibrium the average precision-weighted fee is proportional to the aggregate delegation exposure:*

$$a_L^* = \frac{\rho}{\eta} \int_0^1 N_L^{*m'} dm' \tag{1.94}$$

To derive the proposition, we multiply both sides of (1.92) by O_m and integrate over m. **Q.E.D.**

Substituting the solution for N^* (1.73, 1.133) into (1.94), we derive the explicit condition on a_L that one has to solve in order to obtain the equilibrium:

$$a_L^* = \frac{1}{\eta}(V_D^{-1} - \frac{1}{\rho} V_D^{-1} \Omega(a) \int_0^1 Y \tilde{D}^{-1}(a) Y^{-1} (V_u^{-1} + \Sigma_{m'}^{-1} + V_D^{-1}) dm') \tag{1.95}$$

The terms Ω and $\tilde{D}$ depend on a_L. While the dependence of $\Omega(a)$ is linear $\Omega_L = (O^T O)^{-1}_{LK} a_K$, the diagonal matrix $\tilde{D}(a)$ depends on the aggregate fee in a nonlinear way, as the components of the matrix are functions of eigenvalues. In what follows, we will solve (1.94) explicitly in some dimensions, but cannot provide a general closed-form solution.

One important qualitative result that follows from (1.94) is the existence of nonzero delegation fees in equilibrium. Indeed, the only way to get zero fees is to drive the aggregate delegation exposure $\int N_m^* dm$ to zero, which means that all agents stop delegating. In order to investigate the properties of the equilibrium quantities and their dependence on the economy parameters, we derive explicit solutions for the economies with one risky asset and two risky assets.

1.6.4 Equilibrium fees in the case of one risky asset

In the case of one risky asset we use small letters instead of capital letters to represent matrices: o_m for O_m, ω for Ω, ecc.. The precision-weighted average cost is denoted as $a = \rho \int_0^1 k^{m'} o_{m'} dm'$ and is a scalar. Thus we have:

$$\omega = \frac{a}{\int_0^1 o_{m'}^2 dm'}, \quad k^{m'} = \frac{a o_{m'}}{\rho \int_0^1 o_{m''}^2 dm''} = \frac{1}{2\eta} \int_0^1 (n^{m''} + \eta k^{m''} o_{m''}) dm' \frac{o_{m'}}{\int_0^1 o_{m''}^2 dm''}$$

The condition (1.94), using the solution (1.85) becomes:

$$a = \frac{1}{\eta}v_D^{-1} - \frac{1}{\rho\eta}v_D^{-1}\frac{a}{\int_0^1 o_{m'}^2 dm'}\int_0^1 \frac{v_u^{-1}+\sigma_{m'}^{-1}+v_D^{-1}}{\frac{1}{2}+\sqrt{\frac{a^2}{\rho^2}(\int_0^1 o_{m''}^2 dm'')^{-2}v_D^{-1}(v_u^{-1}+\sigma_{m'}^{-1}+v_D^{-1})+\frac{1}{4}}}dm' \tag{1.96}$$

To solve (1.96) one needs to specify a distribution of the precisions in the economy. Below we will consider two cases:

1. Uniform precisions: $\sigma_m = \sigma$, and
2. Heterogeneous precisions distributed uniformly between σ_{min} and σ_{max}.

Constant precisions

Start with the first case. In the case of constant precisions equation (1.96) can be reduced to a cubic equation:

$$-\frac{\eta^2}{\rho}v_D a^3 + 2\frac{\eta}{\rho}a^2 + (\frac{v_u^{-1}+\sigma^{-1}}{\rho} + o^2\eta v_D)a - o^2 = 0 \tag{1.97}$$

The easiest way to derive (1.97), is to use the definition $g = v_m^{-1} + 2\rho n - \rho^2 n^2 v_D$ instead of $g^* = w\frac{v_u^{-1}+\sigma^{-1}+v_D^{-1}}{\frac{1}{2}+\sqrt{\frac{a^2}{\rho^2}o^{-4}v_D^{-1}(v_u^{-1}+\sigma^{-1}+v_D^{-1})+\frac{1}{4}}}$, and substitute a for $\frac{\eta}{\rho}n$. where variable o can be expressed via v_D, σ, v_u as:

$$o = \frac{1}{\rho}\frac{v_D^{-1}\sigma^{-1}}{v_D^{-1}+v_u^{-1}+\sigma^{-1}} = \frac{1}{\rho}\frac{(v_u^{-1}+\sigma^{-1})^{-1}\sigma^{-1}}{(v_u^{-1}+\sigma^{-1})^{-1}+v_D}$$

We are interested in the behavior of a as a function of (v_D, σ, η, v_u). Since the explicit form of the cubic roots of (1.97) is quite complicated, we start by deriving an explicit form of a^* in the case of $v_D = 0$, and then present a graphical analysis of the numeric solution in the case of $v_D \neq 0$.

For $v_D = 0$, equation (1.97) reduces to a quadratic equation for a, the solution to which reads:

$$a^*(v_D = 0) = \frac{1}{4\eta}v_m^{-1}\left(\sqrt{1+\frac{8}{\rho}\sigma^{-2}\eta v_m^2} - 1\right) \tag{1.98}$$

where we used the expression $o = \frac{1}{2}\sigma^{-1}$.

The solution (1.98) shows that, as expected, the average fee a is nonzero when the penalty coefficient $\eta > 0$ is introduced. Thus price competition among managers does not lead to the Bertrand result when a penalty is imposed by investors. When there is no penalty, the inelastic demand for individual funds would lead managers to bid up the fees up to a point where the total delegation exposure will be zero $n^* = 0$. Such behaviour happens when $\eta = 0$. When the penalty is introduced, the managers increasing their fees above the average have to balance the benefit from the increase of the fee with the loss from the penalty. Conversely, if the average fee happens to be too high, there is a marginal incentive for managers to lower fees to earn some income from the penalty term, up to a point when it balances exactly the loss from the inelastic part of the demand. The optimal delegation exposure in equilibrium is:

$$n^* = \frac{1}{4\rho} v_m^{-1} \left(\sqrt{1 + \frac{8}{\rho} \sigma^{-2} \eta v_m^2} - 1 \right) \tag{1.99}$$

Out of the equilibrium the cost that investor m has to pay for delegation has the following form:

$$Inv.\, cost^m = \int_0^1 \gamma^{mm'} k^{m'} dm' = n^* \bar{k} o - \eta(\int_0^1 k_{m'}^2 dm' - \bar{k})^2 \tag{1.100}$$

The penalty ($\eta > 0$) allows investors to reduce the total cost of delegating by the quantity proportional to the dispersion of fees across managers. Yet, as can be seen from substituting the equilibrium expressions for n^* and $k^m = \bar{k} = \frac{a}{\rho o}$, the penalty term disappears in equilibrium, and the cost expression becomes:

$$Inv.\, cost^m = \frac{1}{2\rho} - \frac{v_m^{-2}}{8\eta\sigma^{-2}} \left(\sqrt{1 + \frac{8}{\rho} \sigma^{-2} \eta v_m^2} - 1 \right) \tag{1.101}$$

Let's plot the average fee, the total delegation exposure n^*, and the total cost investors have to pay for different values of η to appreciate the impact of the penalty on these quantities.

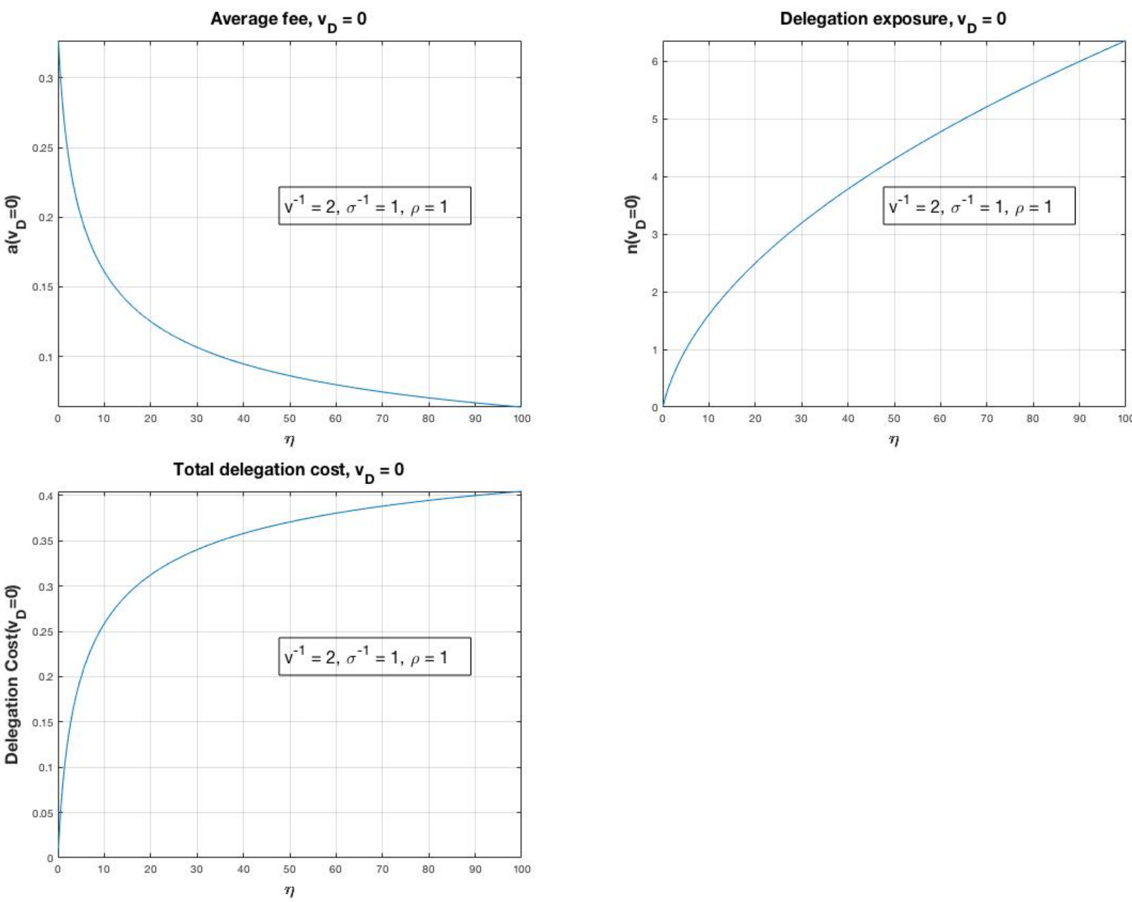


Figure 1.2: Effect of the penalty on delegation in equilibrium.

As is evident from Fig.1.2, the penalty coefficient η plays an important role in equilibrium. There are two limiting cases: $\eta = 0$ and $\eta = \infty$. When $\eta = 0$ (no penalty), the demand for the funds is inelastic in individual fees, and managers increase the fees to a point where delegation becomes too expensive for investors. As is the case in the tragedy of the commons, individual incentives do not take into account the aggregate effect of the fees on the demand. In the other limiting case, $\eta \to \infty$, the demand for individual funds becomes infinitely elastic with respect to the changes in the fund's fee. As a result, a perfect competition ensues and the equilibrium fee goes to zero. At the same time, the delegation exposure goes to infinity, consistent with the no-fee demand derived earlier. The equilibrium reveals fully the private information. (Yet, a difference exists with the no-fee case. The total fee that investors pay managers is nonzero even for infinite demand for funds. Hence managers capture some value from selling funds. But the aggregate fee is finite and is divided between an infinite number of managers.). For intermediate values of η the penalty acts as an equilibrating force, preventing managers from increasing fees excessively, but also allowing nonzero profits.

Investors have a clear incentive to increase the parameter η as much as possible. Yet, it is obvious from (1.68) that large values of η would violate the constraint of nonzero delegation quantities γ. In equilibrium, individual fees have a structure that forces the penalty term in the demand for delegation (1.68) to vanish, so one does not have to worry about $\gamma^{mm'}$ going negative in equilibrium. Off equilibrium, however, the application of a punishment with large η is not optimal for an investor, as it might exclude from his delegating portfolio a substantial mass of funds, thus not allowing to diversify the idiosyncratic signal error. As a consequence, the off-equilibrium penalty strategy cannot be credibly implemented. Since in equilibrium here is no mechanism to determine the optimal value η, its value will likely depend on the initial dispersion of the fees in the economy, or on the investor's expectations of the dispersion.

Let's examine the effect of other parameters (ρ, σ^{-1}, and v_u^{-1}) on the equilibrium. Below we plot the impact of risk aversion ρ on the equilibrium quantities:

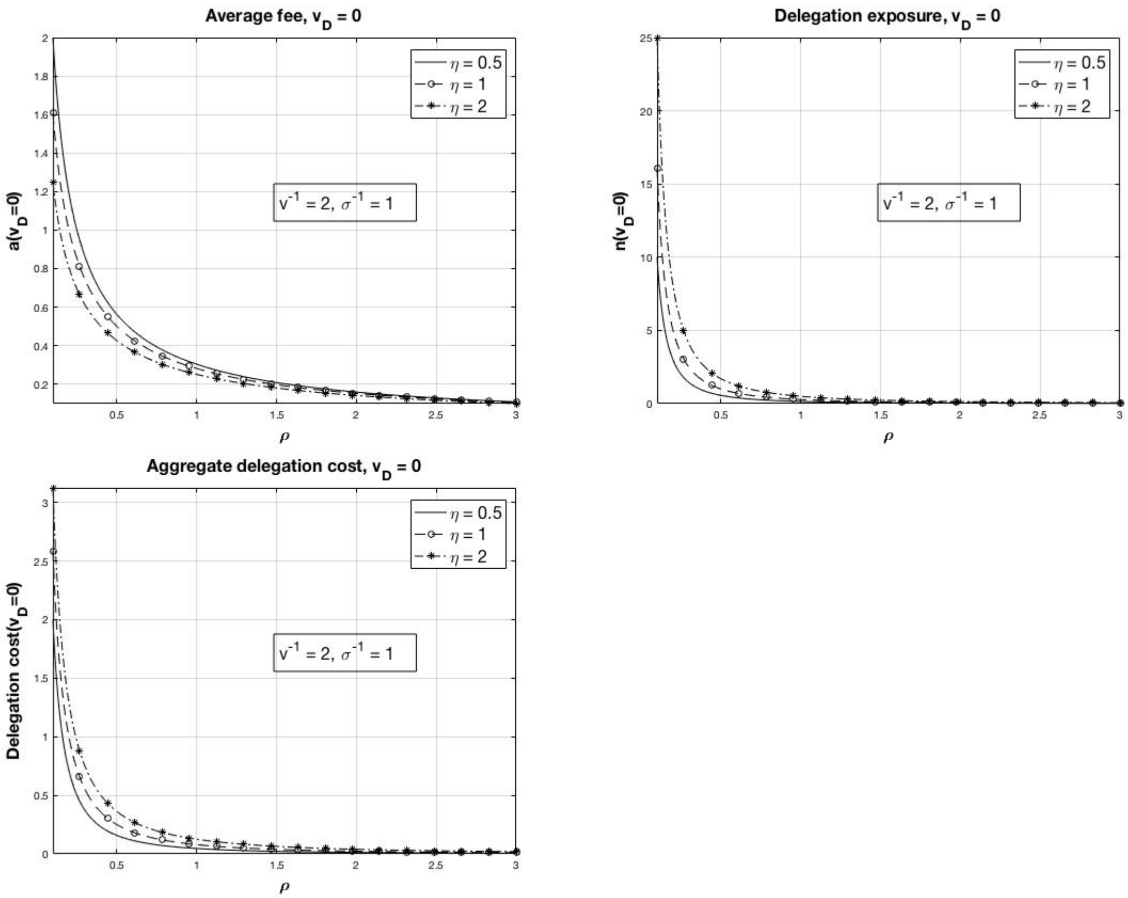


Figure 1.3: Effect of the risk aversion on delegation in equilibrium.

One sees from Fig.1.3 that high risk aversion suppresses delegation. Despite the fact that the average fee is a decreasing function of risk aversion, investors delegate less as their risk aversion grows. The reluctance to delegate at high risk aversion creates a counterintuitive situation: risk averse investors could have benefited from delegation at low fees; after all, when all idiosyncratic risk is gone, the gains from delegation are surely positive. But highly risk averse investors are not willing to pay for it. It seems controversial, because when there is no residual uncertainty ($v_D = 0$) one could use delegation to get exposure to the perfect signal without uncertainty, so why does risk aversion still matter? In turns out that, even if all information is aggregated in a delegation portfolio, there still remains an uncertainty about the value of the future gain (or, better, of the future marginal gain), because investors cannot observe the value of the signal. Thus they face a risk that the fee might be higher than their future gain.

Finally, we discuss the dependence of the equilibrium on the variable σ^{-1}, the precision of the agents' private information. The average fee and the delegation exposure grow with σ^{-1} almost linearly (except for some small values of the precision). Asymptotically, when $\sigma^{-1} >> v_u^{-1}$, the equilibrium becomes:

$$a_{v_D=0} \rightarrow \frac{\sigma^{-1}}{4\eta}(\sqrt{1+\frac{8\eta}{\rho}}-1), \quad n_{v_D=0} \rightarrow \frac{\sigma^{-1}}{4\rho}(\sqrt{1+\frac{8\eta}{\rho}}-1)$$

The delegation cost grows even more quickly with the precision σ^{-1}. Asymptotically, $\sigma^{-1} >> v_u^{-1}$, the cost grows as $const * \sigma^{-2}$. When the precision gets smaller, the cost decreases, approaching zero when there is no private information.

Now, we look at the full solution of (1.97). We plot the equilibrium average fee a^*, the equilibrium delegation exposure n^* and the total cost of delegation for investors. First, we fix the other variables and show the behavior of these quantities with v_D, and after compare the dependence on ρ, η and σ^{-1} for several values of v_D.

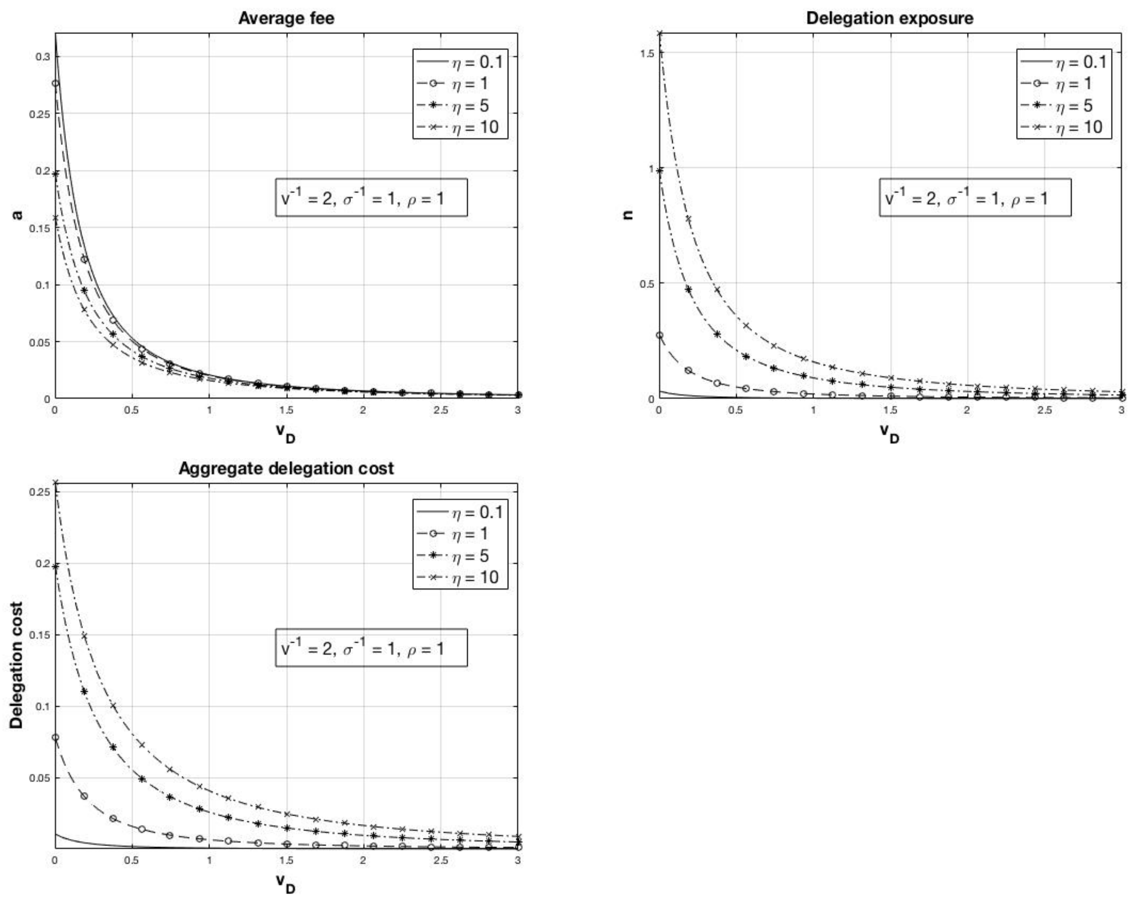


Figure 1.4: Effect of the residual uncertainty on delegation in equilibrium.

It is clear from the plots on Fig.1.4 that the presence of residual uncertainty greatly reduces the levels of delegation. For a modest value of $v_D = 0.5$, which is comparable to other variance parameters v_m and σ used to produce the plots, equilibrium delegation exposure n^* is only a fraction of the level it has when $v_D = 0$. The delegation cost reduction is even more drastic, it is from 3 to 6 times smaller than the reference fee at $v_D = 0$. Interestingly, a variation of the penalty parameter produces significant changes in the delegation exposure and in the aggregate delegation cost, but the differences become almost indistinguishable for the average fee. The sensitivity of the delegation exposure to the fee $\frac{\partial n}{\partial a}$ is large when v_D, and in equilibrium even small changes in the average fee produce large effect on the delegation demand. So, when one varies η, it produces small changes in the average fee a , but much larger changes in n and in the total delegation cost. As a consequence, even if the average fee appears to be insensitive to the penalty parameter η in the presence of aggregate uncertainty, there are still small differences that are manifested through other delegation quantities.

Heterogeneous precisions

We examine now the case when precisions are heterogeneous and distributed uniformly within an interval $\sigma_m^{-1} \in [\sigma_{min}^{-1}, \sigma_{max}^{-1}]$. Without loss of generality, we assume that $\sigma_{min}^{-1} = 0$. Also, we assume that managers do not make inferences from prices, and their portfolio structure o_m depends only on $v^{-1} + \sigma_m^{-1}$. To simplify notations, denote the precision σ^{-1} as x, and σ_{max}^{-1} as x_{max}.

Using the optimal delegation exposure in 1 dimension (1.86), one can write the equation for the average fee a as:

$$a = \frac{\rho}{\eta}\int_0^1 n_m^* dm = \frac{\rho}{\eta}\int_0^{x(\omega)} n_m^*(x)dx \tag{1.102}$$

Note, that we perform integration not up to x_{max}, but up to $x(\omega) = min(x_{max}, \frac{\rho}{\omega} - v_u^{-1})$, taking into account the fact that each investor stops delegating when the aggregate fee coefficient ω reaches a critical level for this investor. The higher the own precision, the lower the critical fee. It is straightforward to derive a bound on the precision from the identity $n_m^* = 0$, yielding a solution $\omega_{critical} = \rho v_m$.

After a straightforward integration the equilibrium condition becomes:

$$\begin{aligned} a &= \frac{\rho}{\eta}\int_0^1 n_m^* dm = \frac{x(\omega)}{\eta v_D x_{max}} + \frac{\rho}{2\omega\eta}\frac{x(\omega)}{x_{max}} - \\ &- \frac{2}{3\eta v_D^{1/2} x_{max}}\left(v_D^{-1} + v^{-1} + \frac{\rho^2 v_D}{4\omega^2}\right)^{3/2}\left(\left(1 + \frac{x(\omega)}{v_u^{-1} + v_D^{-1} + \frac{\rho^2 v_D}{4\omega^2}}\right)^{3/2} - 1\right) \end{aligned} \tag{1.103}$$

It gives an equation for ω, provided that:

$$\omega = \frac{a}{\int_0^1 o_m^2 dm}, \quad x(\omega) = min(x_{max}, \frac{\rho}{\omega} - v_u^{-1})$$

where the constant $\int_0^1 o_m^2 dm$ in the case of uniformly distributed precisions takes the form:

$$\int_0^1 o_m^2 dm = \frac{1}{\rho^2 v_D^2}\left(\frac{2+z}{1+z} - \frac{2}{z}\ln(1+z)\right) \tag{1.104}$$

with $z = \frac{x_{max}}{v_D^{-1}+v^{-1}}$. The condition (1.104) is nonlinear. In the case $v_D = 0$, it simplifies to:

$$\frac{\eta}{3\rho^2}\omega = \frac{\frac{\rho}{\omega} - v_u^{-1}}{2x_{max}^3}\left(\frac{\rho}{\omega} - v_u^{-1}\right), \quad if\ \frac{\rho}{\omega} - v_u^{-1} < x_{max} \tag{1.105}$$

$$\frac{\eta}{3\rho^2}\omega = \frac{x_{max}}{2x_{max}^3}\left(\frac{\rho}{\omega} - v_u^{-1}\right), \quad if\ \frac{\rho}{\omega} - v_u^{-1} > x_{max} \tag{1.106}$$

The condition (1.105) defines an equilibrium where only a part of investors delegates, while (1.106) corresponds to the solution where the fees are low enough for all investors to delegate. The only nonnegative solution to the condition (1.106) is:

$$\omega = \frac{3\rho^2}{2\eta}\left(-\frac{v_u^{-1}}{2x_{max}^2} + \sqrt{\frac{v_u^{-2}}{4x_{max}^4} + \frac{2\eta}{3\rho}}\right)$$

The above solution represents an equilibrium if the following condition on the economy parameters is met:

$$\left(\frac{v_u^{-1}}{x_{max}}\right)^2 > 1 - \frac{2\eta}{3\rho}$$

Thus in equilibrium everybody delegates either when $\frac{v_u^{-1}}{x_{max}}$ is greater than 1 (the maximal signal precision is lower than the precision of the unconditional payoff distribution), or when the economy parameters satisfy $\frac{2\eta}{3\rho} > 1$. The latter condition means that in some economies the penalties for deviation from the average fee η are quite high, or the risk aversion ρ is quite low, ensuring that in equilibrium all investors delegate.

1.6.5 Equilibrium fees in the case of two risky assets

In the case of two risky assets the closed-form solution to the investor optimization problem can be found by working out the general-form expressions (1.133) and

(1.130). Contrary to the case of one risky asset, here several solutions to (1.74) exist. One can show that they can be represented in an explicitly symmetric matrix form as follows:

$$G = -\frac{1}{2}(BB^T)^{-1} + (B^T)^{-1}XB^{-1} \tag{1.107}$$

where $B = \Omega(V_D)^{1/2}$, and matrix X satisfies the following simple algebraic Riccati equation:

$$XX = \frac{1}{4}\mathbb{1} + B^T(V_m^{-1} + V_D^{-1})B \tag{1.108}$$

In appendix 1.13 we construct explicit solutions of a simple matrix equation $XX = Q$ and show how to select a maximal symmetric solution, which is positive definite and the difference between this solution and any other symmetric solution is positive definite.

Using the results of appendix 1.13 one can show that the maximal solution for (1.108) in two dimensions has the form:

$$X^* = \frac{(\lambda_1 - \lambda_2)}{D}\left(\frac{1}{4}\mathbb{1} + B^T(V_m^{-1} + V_D^{-1})B + \begin{bmatrix} \lambda_1\lambda_2 & 0 \\ 0 & \lambda_1\lambda_2 \end{bmatrix}\right) \tag{1.109}$$

where:

$$\lambda_{1,2} = \sqrt{\frac{tr(\frac{1}{4} + B^T(V_m^{-1} + V_D^{-1})B) \pm D}{2}},$$

and

$$D = \sqrt{tr(B^T(V_m^{-1} + V_D^{-1})B)^2 - 4\det B^T(V_m^{-1} + V_D^{-1})B}$$

The corresponding solution for the implied conditional precision G, using (1.107), is:

$$G_m^* = a_m\Omega^{-1}V_D\Omega^{-1} + b_m(V_m^{-1} + V_D^{-1}) \tag{1.110}$$

where

$$a_m = \left(\frac{(\lambda_1 - \lambda_2)}{D}(\frac{1}{4} + \lambda_1\lambda_2) - \frac{1}{2}\right), \quad b_m = \left(\frac{(\lambda_1 - \lambda_2)}{D}(\frac{1}{4} + \lambda_1\lambda_2) - \frac{1}{2}\right)$$

and the optimal delegation exposure for investor m is:

$$N_m^* = \frac{1}{\rho}V_D^{-1} - a_m\Omega^{-1} - \frac{b_m}{\rho^2}V_D^{-1}\Omega(V_m^{-1} + V_D^{-1})$$

To derive the optimal delegation quantities $\gamma^{mm'}$ and the equilibrium in the delegation market one has first to specify an explicit form of the distribution of agents' precision. It will allow to compute explicitly the quantities such as O_m^+, which involves integration over the continuum of funds.

We explore here a configuration of the information structure where several categories of precisions exist. It is a generalization of the example of constant precisions to the case of multiple risky assets. Indeed, while with one risky asset it was enough to have a single category of precisions $\sigma_m^{-1} = \sigma^{-1}$ to obtain first-best fund allocations, in the case of two risky assets one category would be quite limiting. For example, we showed above that with only one category of precisions there is no possibility to construct a fully aggregating solution in the case of zero fees. One needs at least $n(n+1)/2$ categories of precisions for such a solution to exist.

Consider, as a toy example, an economy with three categories of precisions given by three matrices Σ_i^{-1}, $i = 1, 2, 3$. Let's denote the proportions of agents having precision Σ_i^{-1} as f_i, where $f_1 + f_2 + f_3 = 1$.

$$\Sigma_1^{-1} = \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_2^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix}, \quad \Sigma_3^{-1} = \frac{\sigma_3^{-1}}{1+\sigma_3^{-1}} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \tag{1.111}$$

The above choice of precision matrices can be interpreted as follows. Agents in category 1 have private information about only the first risky asset, agents in category 2 about the second. The third category agents specialize in both assets, seeking information on a common factor that drives the payoffs of both assets. Off-diagonal terms in Σ_3^{-1} are nonzero because the signal errors are correlated, as the signals for both assets are derived from the same forecast of the common factor. Also, the infinite precision in the factor forecast σ_3^{-1} does not lead to an infinite precision in forecasting the payoffs.

Note, that despite the fact that agents in category 1 and 2 get each a signal only on one asset, their fund portfolios might in general invest in both assets. The correlation in the "discoverable" uncertainty, represented by covariance V is not a sufficient condition for agents of categories 1 and 2 to invest in both assets, neither it is a necessary condition. What matters, is the presence of the residual uncertainty. Indeed, if $V_D = 0$, managers in categories 1 and 2 will invest only in one asset in which they specialize, even if the asset is correlated to other assets in the economy. Instead, if there is a residual uncertainty, $V_D \neq 0$, and even if it is diagonal, managers will start investing in a second asset. Inclusion of the non-core asset in the fund portfolio might look at first as an act of passive investing for the sake of diversification. It is not. Recall, we assumed that managers do not implement the passive part of their portfolios, because investors can easily construct such exposures by themselves. The inclusion of the second asset has instead informational grounds. Bayesian updating makes agents to update their beliefs on the assets they have no private information about, because the payoffs are correlated to the payoff of the assets of their private signal ($cov(\theta_2, s_1) \neq 0$). Why then in the case without residual uncertainty the signal-related portfolio does not contain asset 2? It turns out that in the absence of residual uncertainty the benefit of extending the forecast to an additional asset is precisely offset by the correlation between the "extended" forecast and the forecast of the original asset. To understand how the situation changes when one introduces the residual uncertainty, let's consider two distinct cases:

1. **Correlated assets $V_{12} \neq 0$ and uncorrelated residual uncertainty** $(V_D)_{12} = 0$.

   The update of the forecast $\theta^c \sim V_1 \Sigma_1^{-1} \begin{bmatrix} s_1 \\ 0 \end{bmatrix}$ reads:

$$\theta_1^c = \frac{V_{11}\sigma_1^{-1}}{V_{11}\sigma_1^{-1} + 1} s_1, \quad \theta_2^c = \frac{V_{12}\sigma_1^{-1}}{V_{11}\sigma_1^{-1} + 1} s_1$$

   where s_1 is the signal on asset 1. Asset 2, despite the absence of the private signal, has its forecast updated if $V_{12} \neq 0$. The exposure of this agent to asset 2 is given by the second component of:

$$(V_D + V_1)^{-1} V_1 \Sigma_1^{-1} \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = \begin{pmatrix} \frac{\sigma_1^{-1}(\det(\mathrm{V}) + V_{11}\,\mathrm{V}_{\mathrm{D}22})}{A} & 0 \\ \frac{\sigma_1^{-1}(V_{12}\,\mathrm{V}_{\mathrm{D}11})}{A} & 0 \end{pmatrix} \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_1^{-1}(\det(\mathrm{V}) + V_{11}\,\mathrm{V}_{\mathrm{D}22})}{A} s_1 \\ \frac{\sigma_1^{-1}(V_{12}\,\mathrm{V}_{\mathrm{D}11})}{A} s_1 \end{bmatrix}$$

   where $A = \det(\mathrm{V}) + \det(\mathrm{V_D}) + V_{11}\,\mathrm{V}_{\mathrm{D}22} + V_{22}\,\mathrm{V}_{\mathrm{D}11} + V_{11}\det(\mathrm{V_D})\sigma_1^{-1} + \mathrm{V}_{\mathrm{D}11}\det(\mathrm{V})\sigma_1^{-1}$.

   It follows that investment in asset 2 is positive if the signal on the first asset is positive, and if the two assets are positively correlated. Such situation is quite unusual from the point of view of diversification, because the correlation normally reduces the exposure to the asset. The exposure is nonzero only if $(V_D)_{11} > 0$, and grows with V_{D11}. At the same time, exposure to asset 1 is dependent on V_{D22}. We interpret it as follows: residual uncertainty motivates extending the signal bet on other assets, especially if these are correlated with the original asset and have less severe residual uncertainty. The ratio of the first asset to the second in the portfolio will be $\frac{\det(V) + V_{11}V_{D22}}{V_{12}V_{D11}}$. If, for example, V_{D11} is quite large, and V_{D22} is small, then the exposure to the second asset will be relatively much more important than to the first asset. Indeed, the second asset, being much less volatile, serves as a proxy to get an exposure to the first asset. An agent who possesses a signal on the first asset prefers to speculate on it using the second asset that is safer. Such use of a proxy asset might be a known phenomena in trading on private information. For the moment, until we find the relevant references, we will refer to it as signal tunnelling.

2. **Uncorrelated assets $V_{12} = 0$ and correlated residual uncertainty $(V_D)_{12} \neq 0$.**

   Here we explore a configuration where the discoverable components of assets' payoffs θ are uncorrelated, so that $V_{12} = 0$. Then the forecast for the first asset does not generate a forecast for the second asset:

$$\theta^c \sim V_1 \Sigma_1^{-1} \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{V_{11}\sigma_1^{-1}}{V_{11}\sigma_1^{-1}+1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{V_{11}\sigma_1^{-1}}{V_{11}\sigma_1^{-1}+1} s_1 & 0 \end{bmatrix}$$

   The exposure to the assets is:

$$(V_D + V_1)^{-1} V_1 \Sigma_1^{-1} \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = \begin{pmatrix} \frac{V_{11}\sigma_1^{-1}(V_{22}+\mathrm{V}_{\mathrm{D}22})}{B} & 0 \\ -\frac{V_{11}\,\mathrm{V}_{\mathrm{D}12}\sigma_1^{-1}}{B} & 0 \end{pmatrix} \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{V_{11}\sigma_1^{-1}(V_{22}+\mathrm{V}_{\mathrm{D}22})}{B} s_1 \\ -\frac{V_{11}\,\mathrm{V}_{\mathrm{D}12}\sigma_1^{-1}}{B} s_1 \end{bmatrix}$$

   where

   $B = \det(V) + V_{11}\,\mathrm{V}_{\mathrm{D}22} + V_{22}\,\mathrm{V}_{\mathrm{D}11} + \det(V_D) + V_{11}\,V_{22}\,\mathrm{V}_{\mathrm{D}11}\sigma_1^{-1} + V_{11}\det(V_D)\sigma_1^{-1}$

   Here the exposure to asset 2 is negative if the signal for asset 1 is positive and the residual uncertainty components of the assets are positively correlated. The exposure is large if V_{11} is large, yet the ratio of the exposure to the two assets does not depend on V_{11}, but is instead $-\frac{(V_{22}+V_{D22})}{V_{D12}}$. In what looks more like a signal hedging, asset 2 is added to the portfolio to partially hedge the position in asset 1.

When both $V_{12} \neq 0$ and $V_1 1 \neq 0$ the two effects: signal tunnelling and signal hedging, are present at the same time.

1.7 Conclusion

In this Part we developed a tractable framework for the analysis of the structure of a joint market for the assets and for the funds. Our model extends the literature on fund management in the presence of asymmetric private information (Ross (2005), Garcia and Vanden (2009), Garleanu and Pedersen (2018)) by introducing multiple risky assets and heterogeneously informed agents.

Delegation in our model is motivated by the presence of diverse private information and represents an indirect information sharing. Unlike the vast majority of the models of fund management, we do not introduce separate groups of managers and investors and let the institution of delegation to form endogenously in equilibrium. In this respect our model is similar to Ross (2005), but we obtain a qualitatively different equilibrium structure of the fund market compared to his model, because we impose no restriction on the simultaneous investing in funds and own trading in the risky assets.

We show that, when agents of heterogeneous quality of private information are present, more informed agents might invest in funds of less informed ones as long the the signal errors of the agents are not perfectly correlated. Any agent who decides to invest in funds, will favor fund allocations including all available funds, where the signal risk is completely diversified. Thus the behavior of a fund investor in our model is different from the common assumption that investors should invest only in the funds with the highest precision, which are assumed to provide a better investment performance. Rather, fund investor sees individual funds as complementary inputs to the risky allocation, and investing in several funds produces always a superior performance than just selecting the highest precision manager.

It is optimal for fund investors to offset any uninformed exposure to risky assets present in the funds, which is possible when fund investors are allowed to trade on their own account. We show that own-account trading is an important factor in building fully aggregating fund allocations. Investor building such as allocation will fully diversify the signal risk, and will achieve the ex-post risky exposure and wealth of someone with the perfect knowledge of all private information.

Private information might be fully aggregated despite the presence of noise traders. Our model shows that it is possible to build fund allocations that fully reveal private information dispersed in the economy, meaning that the indirect sale of information via funds is not inherently suboptimal to the direct sale of signals, contrary to the intuition of Admati and Pfleiderer (1990). Investors are capable of unbundling the signals contained in the funds and assemble optimal allocations, exactly s they would do if they would acquired the signal and assembled the risky asset portfolio of assets. But no competitive fully aggregating equilibrium exists when the access to the private information of others is costly. .

Price informativeness depends on the aggregated trading intensity of agents (Vives (2007)). Thus the prices become more informationally efficient when the delegation is introduced, because it results in the increased willingness of agents to hold risky assets. The rationale for the increase in the price informativeness in our model differs from the previous literature, because it does not involve a substitution of uninformed investors by informed managers, is not driven by the increased incentives to acquire information, or is not explained by the risk-sharing effect of performance-based contracts.

Fund fees are nonzero in equilibrium, even in the case of infinitely large and competitive market for funds. The power to charge nonzero fees comes from the strategic complementarity in the managers' pricing decisions.

1.8 Appendix 1.A: Peer-to-peer Delegation

Here we analyse a partial equilibrium problem of one agent delegating investment to another one. We assume that the agent considering delegation is informed about the precision matrix $\tilde{\Sigma}$ of the other investor, but cannot observe his signal $\tilde{s}$:

$$\tilde{s} = \theta + \tilde{\epsilon}$$

Conditional expectation of the signal of the peer agent is equal to the conditional expectation of the payoff, given that the signal errors are independent across agents:

$$E(\tilde{s} \mid s, P) = E(\theta + \tilde{\epsilon} \mid s, P) = E(\theta \mid s, P)$$

Assuming the absence of the residual risk ($V_D = 0$), the structure of the fund portfolio is:

$$\tilde{x} = \frac{1}{\rho}\tilde{\Sigma}^{-1}(\tilde{s} - PR) + x_0^u \tag{1.112}$$

where x_0^u denotes a portfolio of uninformed agent given by (1.13).

The end-of-period wealth of the agent is:

$$W_2 = e_0 R + (x + \tilde{\gamma} x_0^u)(\theta - PR) + \frac{\tilde{\gamma}}{\rho}(\tilde{s} - PR)\tilde{\Sigma}^{-1}(\theta - PR) \tag{1.113}$$

The full expression for the expected conditional utility is:

$$\begin{aligned} E(-e^{-\rho W_2} \mid s, P) &= -\int e^{-\rho(x+\tilde{\gamma}x_0^u)(\theta-PR)-\tilde{\gamma}(\theta-PR)\tilde{\Sigma}^{-1}(\theta-PR)-\tilde{\gamma}\tilde{\epsilon}\tilde{\Sigma}^{-1}(\theta-PR)} \times \\ &\times \; e^{-\frac{1}{2}(\theta-\theta_c)V_c^{-1}(\theta-\theta_c)-\frac{1}{2}\tilde{\epsilon}\tilde{\Sigma}^{-1}\tilde{\epsilon}} d\theta d\tilde{\epsilon} \end{aligned} \tag{1.114}$$

where θ_c and V_c are expectation and variance of the risky payoffs conditional on the information set (P, s), given by the formulae (1.12, 1.11).

Integrating over $\tilde{\epsilon}$ one gets:[3]

$$E(-e^{-\rho W_2} \mid s, P) = -\int e^{-\rho(x+\tilde{\gamma}x_0^u)(\theta-PR)-(\tilde{\gamma}-\frac{1}{2}\tilde{\gamma}^2)(\theta-PR)\tilde{\Sigma}^{-1}(\theta-PR)} \times \times e^{-\frac{1}{2}(\theta-\theta_c)V_c^{-1}(\theta-\theta_c^m)}d\theta \quad (1.115)$$

The integration over θ gives:

$$E(U \mid s^m, P) = -\frac{J(\tilde{\gamma})}{I}e^{-t(\theta_c-PR)+(\tilde{\gamma}-\frac{1}{2}\tilde{\gamma}^2)(\theta_c-PR)\tilde{\Sigma}^{-1}(\theta_c-PR)+\frac{1}{2}t(V_c^{-1}+2(\tilde{\gamma}-\frac{1}{2}\tilde{\gamma}^2)\tilde{\Sigma}^{-1})^{-1}t} \quad (1.116)$$

where

$$t = \rho(x + \tilde{\gamma}x_0^u) + 2(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2)\tilde{\Sigma}^{-1}(\theta_c - PR)$$

The resulting utility structure suggests that there is a gain in precision by delegating due to the improvement of the conditional precision $V_c^{-1} \to V_c^{-1} + 2(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2)\tilde{\Sigma}^{-1}$.

The first order conditions of the problem of maximization of (1.116) read:

$$\frac{\partial}{\partial t} : \quad (\theta_c - PR) - (V_c^{-1} + 2(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2)\tilde{\Sigma}^{-1})^{-1}t = 0 \quad (1.117)$$

$$\frac{\partial}{\partial \tilde{\gamma}} : \quad (1-\tilde{\gamma})tr\left(V_c^{-1}\tilde{\Sigma} + 2(\tilde{\gamma} - \frac{1}{2}\tilde{\gamma}^2)\right)^{-1} = 0$$

The first condition gives the following structure of the own-account portfolio:

$$x^* = \frac{1}{\rho}\Sigma^{-1}(s - PR) + (1-\tilde{\gamma})x_0^u \quad (1.118)$$

The solution to the second condition in (1.117) is:

$$\begin{aligned} \tilde{\gamma} &= 0, \quad if \quad \tilde{\Sigma} = \infty \\ \tilde{\gamma} &= 1, \quad otherwise \end{aligned} \quad (1.119)$$

The surprising result is that an agent with a good own precision of the private signal (i.e. a high Σ^{-1}) will be willing to buy one share of a peer portfolio with arbitrary precision $\tilde{\Sigma}^{-1} > 0$.

[3] The explicit integration over $\tilde{epsilon}$ gives the following result:

$$\int e^{-\tilde{\gamma}\tilde{\epsilon}\tilde{\Sigma}^{-1}(\theta-PR)-\frac{1}{2}\tilde{\epsilon}\tilde{\Sigma}^{-1}\tilde{\epsilon}}d\tilde{\epsilon} = e^{\frac{1}{2}(\gamma^m)^2(\theta-PR)\tilde{\Sigma}^{-1}(\theta-PR)}$$

1.9 Appendix 1.B: Solution to the optimization problem at the asset trading stage

Under the assumption of fully diversified signal risk the final wealth (1.20) simplifies to:

$$W_2^m = \tilde{x}^m(D - PR) + (\theta - PR)N^m(D - PR) \tag{1.120}$$

To compute the expected utility $-\mathbb{E}(e^{-\rho W_2^m} \mid s^m, P)$ we first take an integral over the variable ϵ_D. The part of the expectation containing the residual uncertainty is integrated using the formula:

$$\mathbb{E}_{\epsilon_D}(e^{-\rho(\tilde{x}^m + (\theta - PR)N^m)\epsilon_D}) = \mathbb{E}_{\epsilon_D}(e^{h\epsilon_D}) = e^{\frac{1}{2}hV_D h} \tag{1.121}$$

where $h = -\rho(\tilde{x}^m + (\theta - PR)N^m$.

The effect of conditioning on the information set (s^m, P) is the Bayesian update in the probability distribution of the payoff component θ. The integration over θ, z, ϵ^m is reduced to taking an expectation over a conditional random variable $\theta \mid s^m, P$ with conditional distribution $N(\theta_c^m, V^m)$, where the mean and the variance are given by (1.12 , 1.11). Inserting the result in the expectation one obtains:

$$\begin{aligned} -\mathbb{E}(e^{-\rho W_2^m} \mid s^m, P) &= \frac{1}{I}\int e^{-\rho\tilde{x}^m(\theta - PR) - \rho(\theta - PR)Nm(\theta - PR) + \frac{1}{2}\rho^2(\tilde{x}^m + N^m(\theta - PR))V_D\tilde{x}^m + N^m(\theta - PR)} \times \\ &\times \; e^{-\frac{1}{2}(\theta - \theta_c^m)V_m^{-1}(\theta - \theta_c^m)} d\theta \end{aligned} \tag{1.122}$$

where $I = \int e^{-\frac{1}{2}(\theta - \theta_c^m)V_m^{-1}(\theta - \theta_c^m)} d\theta$. After simplifying the expression in the exponent one gets:

$$-\mathbb{E}(e^{-\rho W_2^m} \mid s^m, P) = \frac{1}{I} e^{\frac{1}{2}\rho^2\tilde{x}^m V_D \tilde{x}^m - \rho\tilde{x}^m(\theta_c^m - PR)} \int e^{(-\rho\tilde{x}^m L_m^T - (\theta_c^m - PR)N^m)(\theta - \theta_c^m) - \frac{1}{2}(\theta - \theta_c^m)G^m(\theta - \theta_c^m)} d\theta \tag{1.123}$$

where matrix L_m is just a compact notation for the matrix expression $L_m = (1 - V_D N^m)$ and the quantity G^m is:

$$G_m = V_m^{-1} + 2N^m - N^m V_D N^m \tag{1.124}$$

The above quantity is the implied conditional precision introduced in proposition 4, which represents a boost in the conditional precision of an agent who invests in an allocation of funds $\gamma^{mm'}$ resulting a delegation exposure N^m, defined in (1.22).

It is easy to see that the integral (1.123) could be transformed by a change of variables in the expectation over a random variable having normal distribution with the mean θ_c^m and variance G^m:

$$\frac{1}{I}\int e^{t(\theta-\theta_c^m)-\frac{1}{2}(\theta-\theta_c^m)G^m(\theta-\theta_c^m)}d\theta = \frac{J(G^m)}{I}e^{\frac{1}{2}t(G^m)^{-1}t}$$

where $J(G^m) = \int e^{-\frac{1}{2}(\theta-\theta_c^m)G^m(\theta-\theta_c^m)}d\theta$.

Using the above formula and simplifying the expression one obtains the conditional expectation of the logarithm of the agent's utility in closed form:

$$\begin{aligned}\log(-\mathbb{E}(U \mid s^m, P)) &= \log(\frac{I}{J(G_m)}) + \frac{1}{2}\rho^2 t^m V_D t^m - \rho t^m(\theta_c^m - PR) + \\ &+ \frac{\rho^2}{2}(t^m L_m^T + (\theta_c^m - PR)N^m)G_m^{-1}(L_m t^m + N^m(\theta_c^m - PR))\end{aligned} \quad (1.125)$$

where $t^m = \tilde{x}^m + N^m(\theta_c^m - PR)$.

1.10 Appendix 1.C: Optimization solution at asset trading stage

The appendix is in progress.

1.11 Appendix 1.D: Equilibrium prices in the case of no residual uncertainty

The appendix is in progress.

1.12 Appendix 1.E: General solution to Algebraic Riccati Equation

Following Kucera (1973), a general form of a matrix Riccati equation is:

$$-GBB^TG + GA + A^TG + Q = 0 \quad (1.126)$$

In our case,

$$B = \frac{1}{\rho}\Omega(V_D^{-1})^{1/2}, \quad A = -\frac{1}{2}\mathbb{1}, \quad Q = V_m^{-1} + V_D^{-1}$$

A matrix Riccati equation may possess a variety of solutions, including no solution and infinitely many solutions. Under certain regularity conditions for matrices A, B, Q, a solution can be constructed in the following way.

Define a matrix M of dimensions (2n x 2n), where n is the number of risky assets:

$$M = \begin{bmatrix} A & -BB^T \\ Q & -A \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbb{1} & -BB^T \\ -Q & \frac{1}{2}\mathbb{1} \end{bmatrix} \tag{1.127}$$

Consider eigenvalue problem for matrix M:

$$Ma_i = \lambda_i a_i, \quad i = 1, .., 2n$$

$$a_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

Theorem 1 from Kucera (1973) defines a general solution of Riccati equation as:

$$G = Y \cdot X^{-1} \tag{1.128}$$

where X, Y are matrices of dimension (n x 2n) constructed from the upper and lower portions of eigenvectors a_i:

$$X = [x_1, .., x_{2n}], \quad Y = [y_1, .., y_{2n}]$$

From the definition of eigenvalue problem one has:

$$\begin{bmatrix} -\frac{1}{2}\mathbb{1} & -BB^T \\ -Q & \frac{1}{2}\mathbb{1} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \lambda_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i = 1, .., 2n$$

It follows:

$$-Qx_i + (\frac{1}{2} - \lambda_i)\mathbb{1}y_i = 0$$

$$x_i = Q^{-1}(\frac{1}{2} - \lambda_i)\mathbb{1}y_i$$

or, in matrix notations:

$$X = Q^{-1}Y\tilde{D} \tag{1.129}$$

where $\tilde{D}$ is (2n x 2n) matrix:

$$\tilde{D} = \begin{pmatrix} \frac{1}{2}-\lambda_1 & 0 & \dots & 0 \\ 0 & \frac{1}{2}-\lambda_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \frac{1}{2}-\lambda_{2n} \end{pmatrix} \tag{1.130}$$

$$X^{-1} = \tilde{D}^{-1}Y^{-1}Q \tag{1.131}$$

and where Y^{-1} is a right-inverse of Y: $YY^{-1} = \mathbb{1}$, and Y, Y^{-1} should be defined such as X^{-1} exists, and, in addition, satisfy the following condition:

$$-Z\tilde{D}^{-1}Z\tilde{D}Z + Z = \mathbb{1} \tag{1.132}$$

where $Z = Y^{-1}Y$.

The solution is:

$$G = Y\tilde{D}^{-1}Y^{-1}Q \tag{1.133}$$

It is easy to check now that the Riccati equation (1.126) is satisfied with (1.133).

$$-Y\tilde{D}^{-1}Y^{-1}QBB^TY\tilde{D}^{-1}Y^{-1}Q - Y\tilde{D}^{-1}Y^{-1}Q + Q = 0$$

$$-Y\tilde{D}^{-1}Y^{-1}QBB^TY\tilde{D}^{-1}Y^{-1} - Y\tilde{D}^{-1}Y^{-1} + \mathbb{1} = 0$$

From the eigenvalue problem one has:

$$X = -BB^TYD = Q^{-1}Y\tilde{D}$$

where

$$D = \begin{pmatrix} \frac{1}{\frac{1}{2}+\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\frac{1}{2}+\lambda_2} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \frac{1}{\frac{1}{2}+\lambda_n} \end{pmatrix}$$

and one has $D^{-1} = \mathbb{1} - \tilde{D}$.

$$Y\tilde{D}^{-1}Y^{-1}Y(\mathbb{1} - \tilde{D})Y^{-1}Q - Y\tilde{D}^{-1}Y^{-1}Q + Q = 0$$

$$-Y\tilde{D}^{-1}Y^{-1}Y\tilde{D}Y^{-1} + \mathbb{1} = 0$$

From here we obtain the condition, replacing $Y^{-1}Y$ by Z. Note, that Z^{-1} does not exist. By construction, Z is a projector: $ZZ = Z$.

Now we have all elements to derive solutions for the optimal G^* in arbitrary dimension. Before doing so, we study the two limiting cases, $\Omega = 0$ and $V_D = 0$.

1/ When $\Omega = 0$, a no-fee solution is easily obtained by putting $B = 0$. Matrix M has two roots with multiplicity n each: $\lambda + = 1/2$ and $\lambda_- = -1/2$. Matrix $\tilde{D}$ has the form:

$$\tilde{D} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

is not invertible.

Let's denote by y_+, x_+ components of eigenvectors corresponding to eigenvalue λ_+, and y_-, x_- components of eigenvectors corresponding to eigenvalue λ_-.

From the eigenvalue problem with $B = 0$ it follows that:

$$x_+ = 0, \quad x_- - arbitrary, \quad y_+ - arbitrary, \quad y_- = Qx_-$$

$$X = [0, x_-], \quad Y = [y_+, Qx_-]$$

We cannot define X^{-1} via Y^{-1} in this case, because $\tilde{D}^{-1}$ does not exist. In general:

$$X^{-1} = \begin{bmatrix} b \\ x_-^{-1} \end{bmatrix}$$

We have also to require that $y_+ = 0$, otherwise the solution $YX^{-1} = Q + y_+ b$ does not satisfy (1.126) for $B = 0$.

2/ The second case $V_D = 0$ cannot be solved using a Riccati equation (**??**), because the matrices B and Q become infinite in the limit. To solve it, we note that from $L = \Omega G = 1$ one immediately gets : $G^* = \Omega^{-1}$.

Then, one cannot use L to reconstruct N. But from the definition of G via N: $G = V_m^{-1} + 2N_m$, one gets the solution for $N^* = \frac{1}{2}(\Omega^{-1} - V_m^{-1})$.

Now, we give the form of the characteristic equation of matrix M and specify the structure of its eigenvalues.

$$\det(M - \lambda \mathbb{1}) = 0$$

We can reduce the determinant of (2n x 2n) matrix to a determinant of a smaller, (n x n) matrix using Schur's lemma:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\cdot\det(A - BD^{-1}C)$$

$$\det\begin{bmatrix} (-\frac{1}{2}-\lambda)\mathbb{1} & -\Omega V_D^{-1}\Omega \\ -(V_m^{-1}+V_D^{-1}) & (\frac{1}{2}-\lambda)\mathbb{1} \end{bmatrix} = \det\left((\lambda^2 - \frac{1}{4})\mathbb{1} - \Omega V_D^{-1}\Omega(V_m^{-1}+V_D^{-1})\right) = 0$$

Defining $\xi = \lambda^2 - \frac{1}{4}$, and $H = \Omega V_D^{-1}\Omega(V_m^{-1}+V_D^{-1})$ one has the following eigenvalue problem for the matrix of dimension (n x n):

$$\det(H - \xi\mathbb{1}) = 0, \quad H = \Omega V_D^{-1}\Omega(V_m^{-1}+V_D^{-1}) \tag{1.134}$$

The above structure of matrix M implies that eigenvalues λ will have the form:

$$\lambda_{\pm i} = \pm\sqrt{\frac{1}{4}+\xi_i}, \quad i = 1,..,n \tag{1.135}$$

The matrix of eigenvectors Y and the matrix $\tilde{D}$ are then naturally represented as:

$$Y = [Y_+, Y_-], \quad \tilde{D} = \begin{bmatrix} \tilde{D}_+ & 0 \\ 0 & \tilde{D}_- \end{bmatrix} = \begin{bmatrix} (\frac{1}{2}-\lambda_{+i}) & 0 \\ 0 & (\frac{1}{2}+\lambda_{+i}) \end{bmatrix}$$

An explicit calculation shows that in order to satisfy the requirement $-Z\tilde{D}^{-1}Z\tilde{D}Z + Z = 0$ one has to restrict Y^{-1} to be:

$$Y^{-1} = \begin{bmatrix} 0 \\ Y_-^{-1} \end{bmatrix}$$

The above condition will ensure that:

$$Y\tilde{D}^{-1}Y^{-1} = Y_-\tilde{D}_-^{-1}Y_-^{-1}$$

In the last expression, all matrices are quadratic (n x n) and the inverse of $Y_-^{-1}Y_-$ exists.

To determine the eigenvalues ξ, one has to solve the corresponding eigenvalue problem, which requires the specification of the matrix Ω. We will do it explicitly in dimensions 1 and 2.

1.13 Appendix 1.F: Solutions to Algebraic Riccati Equation in two dimensions

Just for the sake of exercise we explore all the solutions of a Riccati equation:

$$XX = Q$$

The objective is to understand how to pick from the multiple solutions the only one that is positive definite and "bigger" than other solutions, in the sense that the difference between the maximal solution and any other one is positive definite.

We start, as usual, by writing down the Hamiltonian for $XX = Q$:

$$H = \begin{bmatrix} 0 & -\mathbb{1} \\ -Q & 0 \end{bmatrix}$$

H has four distinct eigenvalues that satisfy:

$$\lambda^4 - tr(Q)\lambda^2 + \det(Q) = 0$$

And the eigenvector for any particular λ_i has a form:

$$H \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \lambda_i \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad \begin{matrix} v_i = -\lambda_i u_i & v_i = -\lambda_i u_i \\ Qu_i = -\lambda_i v_i & Qu_i = \lambda_i^2 u_i \end{matrix}$$

So, the lower part of eigenvector u_i is itself an eigenvector of matrix Q with an eigenvalue λ_i^2.

The four eigenvalues are:

$$(\lambda_1^+ = \lambda_1, \lambda_1^- = -\lambda_1, \lambda_2^+ = \lambda_2, \lambda_2^- = -\lambda_2)$$

where

$$\lambda_1^2 = \frac{tr(Q) + \sqrt{tr(Q)^2 - 4\det(Q)}}{2}, \quad \lambda_2^2 = \frac{tr(Q) - \sqrt{tr(Q)^2 - 4\det(Q)}}{2}$$

Eigenvector u is a two-dimensional column and has a form:

$$(u_i)_1 = -\frac{Q_{12}a}{Q_{11} - \lambda_i^2}, \quad (u_i)_2 = a$$

Let's now lost all the four full eigenvectors of H. Note, that each is parametrized by an arbitrary scalar a, b, c, d.

$$\begin{bmatrix} u_1^+ \\ v_1^+ \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_1^2}a \\ a \\ \lambda_1\frac{Q_{12}}{Q_{11}-\lambda_1^2}a \\ -\lambda_1 a \end{bmatrix}, \quad \begin{bmatrix} u_1^- \\ v_1^- \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_1^2}b \\ b \\ -\lambda_1\frac{Q_{12}}{Q_{11}-\lambda_1^2}b \\ \lambda_1 b \end{bmatrix}, \quad \begin{bmatrix} u_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_2^2}c \\ c \\ \lambda_2\frac{Q_{12}}{Q_{11}-\lambda_2^2}c \\ -\lambda_2 c \end{bmatrix}, \quad \begin{bmatrix} u_2^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_2^2}d \\ d \\ -\lambda_2\frac{Q_{12}}{Q_{11}-\lambda_2^2}d \\ \lambda_2 d \end{bmatrix}$$

Now, we construct and compare four solutions: $(1,2),(1,-2),(-1,2),(-1,-2)$. Start with $(1,2)$.

$$\begin{bmatrix} u_{12} \\ v_{12} \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_1^2}a & -\frac{Q_{12}}{Q_{11}-\lambda_2^2}c \\ a & c \\ \lambda_1\frac{Q_{12}}{Q_{11}-\lambda_1^2}a & \lambda_2\frac{Q_{12}}{Q_{11}-\lambda_2^2}c \\ -\lambda_1 a & -\lambda_2 c \end{bmatrix}$$

$$u_{12}^{-1} =$$

$$X_{12} = v_{12}u_{12}^{-1} = \frac{(\lambda_2-\lambda_1)}{D}\begin{pmatrix} Q_{11}+\lambda_1\lambda_2 & Q_{12} \\ Q_{12} & Q_{22}+\lambda_1\lambda_2 \end{pmatrix}$$

where $D = \sqrt{tr(Q)^2 - 4\det(Q)}$

$$\det(X_{12}) = \lambda_1\lambda_2$$

But X_{12} is not positive definite because $(X_{12})_{11} < 0$ (because $\lambda_2 < \lambda_1$). So, X_{12} is not a maximal solution.

Now try (1, -2).

$$\begin{bmatrix} u_{1,-2} \\ v_{1,-2} \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_1^2}a & -\frac{Q_{12}}{Q_{11}-\lambda_2^2}d \\ a & d \\ \lambda_1\frac{Q_{12}}{Q_{11}-\lambda_1^2}a & -\lambda_2\frac{Q_{12}}{Q_{11}-\lambda_2^2}d \\ -\lambda_1 a & \lambda_2 d \end{bmatrix}$$

$$X_{1,-2} = v_{1,-2}u_{1,-2}^{-1} = \frac{(\lambda_2+\lambda_1)}{D}\begin{pmatrix} -Q_{11}+\lambda_1\lambda_2 & -Q_{12} \\ -Q_{12} & -Q_{22}+\lambda_1\lambda_2 \end{pmatrix}$$

where $D = \sqrt{tr(Q)^2 - 4\det(Q)}$

The solution $X_{1,-2}$ is not positive definite, because the two diagonal terms cannot be both positive simultaneously.

Now try (-1, 2).

$$\begin{bmatrix} u_{-1,2} \\ v_{-1,2} \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_1^2}b & -\frac{Q_{12}}{Q_{11}-\lambda_2^2}c \\ b & c \\ -\lambda_1 \frac{Q_{12}}{Q_{11}-\lambda_1^2}b & \lambda_2 \frac{Q_{12}}{Q_{11}-\lambda_2^2}c \\ \lambda_1 b & -\lambda_2 c \end{bmatrix}$$

$$X_{-1,2} = v_{-1,2}u_{-1,2}^{-1} = \frac{(\lambda_2 + \lambda_1)}{D} \begin{pmatrix} Q_{11} - \lambda_1\lambda_2 & Q_{12} \\ Q_{12} & Q_{22} - \lambda_1\lambda_2 \end{pmatrix}$$

where $D = \sqrt{tr(Q)^2 - 4\det(Q)}$ Idem, one of the diagonal elements of $X_{(1,2)}$ is always negative, so the solution is not positive definite.

The last solution: $X_{(-1,-2)}$:

$$\begin{bmatrix} u_{-1,2} \\ v_{-1,2} \end{bmatrix} = \begin{bmatrix} -\frac{Q_{12}}{Q_{11}-\lambda_1^2}b & -\frac{Q_{12}}{Q_{11}-\lambda_2^2}d \\ b & d \\ -\lambda_1 \frac{Q_{12}}{Q_{11}-\lambda_1^2}b & -\lambda_2 \frac{Q_{12}}{Q_{11}-\lambda_2^2}d \\ \lambda_1 b & \lambda_2 d \end{bmatrix}$$

$$X_{-1,-2} = v_{-1,-2}u_{-1,-2}^{-1} = \frac{(\lambda_1 - \lambda_2)}{D} \begin{pmatrix} Q_{11} + \lambda_1\lambda_2 & Q_{12} \\ Q_{12} & Q_{22} + \lambda_1\lambda_2 \end{pmatrix}$$

By construction, $\lambda_1 > \lambda_2$, and the solution $X_{-1,-2}$ has positive determinant and positive diagonal elements, so it is the maximal solution. To check, compute the differences: $(X_{-1,-2} - X_{-1,2})$ ecc and verify that these are positive definite.

2

PART 2: Knowledge asymmetry in the fund market

The model of a joint market for assets and funds in Part 1 is built on a rather stringent assumption: the structure of every fund portfolio is required to be fixed (1.14) and known to all agents in the economy. Such knowledge is possible only if agents know perfectly the informational characteristics of all private signals and are able to tell which covariance matrix of the signal error belongs to which manager. In real markets agents do not possess such an overwhelming quantity of knowledge, and thus they will find it difficult to form expectations about the payoffs of individual funds. In addition, the impossibility of observing the fund composition leads to the problem of enforcement: the fund structure, even if agreed upon by investor and manager, is problematic to enforce when investors cannot observe the actual fund portfolios. Thus real markets equilibria will differ from the ideal equilibrium in the joint market for securities and funds described in Part 1. In Part 2 we relax the assumption of perfect knowledge and investigate how the demand for funds could be sustained when agents have only a limited knowledge about each other's private information.

In is known that the market for assets might reach equilibrium without the common knowledge of the informational characteristics of the agents, even in presence of diverse private information. Agents, if they refrain from using rational expectations, or are not allowed to observe prices, might be able to coordinate their demands using public information only, such as the commonly known unconditional distribution of the securities payoffs. A somewhat greater amount of common knowledge is needed in the models with rational expectations (Hellwig (1980); Admati (1985)): in order to deduce the informational content of asset prices the agents need to know at least the average precision of private signals in the economy.

When the fund market is introduced, the transaction between an investor and a manager is usually deemed possible if the pattern of the manager's behaviour is known, i.e. when it is clear how a manager will incorporate the private signal in the fund portfolio. That is why the models of the market for funds often assume the maximal knowledge, when every investor knows with certainty the precision matrix of every manager (Ross (2005); Garcia and Vanden (2009); Garleanu and Pedersen (2018)). Without the knowledge about the signal precision of a manager the investor cannot evaluate the expected utility of investing in the manager's fund, and thus would not have a basis for the investing decision. It turns out, however, that investing in funds remains possible even in the absence of detailed knowledge about funds.

In this Part we will show that the fund market might be viable with the minimal knowledge requirements. Since investors in our model care about building fund allocations rather that about selecting one manager, it is sufficient to know the behaviour of managers in aggregate, without having to be able to distinguish one manager from another. The simplest example of an aggregate is a symmetric allocation of funds, which amounts to investing in the average fund portfolio. Building such an allocation requires from investor a mere knowledge of the average precision of the private information in the economy, which is exactly the requirement needed to sustain asset trading in the economy with rational expectations. Although the minimal common knowledge turns out to be sufficient to sustain the market for funds, the level of investment in the funds will be lower than in the case of perfectly known precisions, and agents will gain less benefit from delegating.

To realize fully the surplus from investment delegation, investors are motivated to learn informational characteristics of their peers. A portion of the extra surplus might be also captured by managers, either in the form of additional profits coming from the increase in the demand for funds, or from higher fees a manager might command after the revelation of the precisions to investors. As a consequence, both investors and managers will have incentives to increase and refine the common knowledge about signal precisions.

Several strategies are adopted in the markets to learn and signal fund characteristics related to the value-adding private information. Agents spend resources on marketing their funds and on performing due diligence of the funds of others. Investors pay third parties to categorize the existing funds and create performance labels (such as Morningstar's star awards). An alternative strategy is to assign to each fund a suitable benchmark portfolio with a benchmark composition being common knowledge between an investor and a manager. We show in this study how the latter rule helps shaping the investors' expectations about the future performance of the fund portfolio, and, importantly, how it creates a piece of common knowledge between an investor and a manager, which serves as the basis for the the enforcement of their transaction. Based on the possibility to use benchmarks as a substitute for learning the fund precisions, we suggest that the primary role of investment benchmarks is a social device that facilitates the diffusion of common knowledge between agents and helps to extract more gains from the investor-manager transaction.

Being a social institution used to separate managers in types, benchmarks do not exist in isolation. Different benchmarks are needed to signal distinct types of private information, which explains the great variety of existing benchmark indices. Although a particular benchmark should have certain properties to be more or less reliably associated to a certain type of funds, its portfolio structure is not entirely fixed, but is a result of a social agreement. We show that, within the framework where funds are vehicles for private information, the arbitrariness in the benchmark structure does not affect neither the demand for funds, nor the equilibrium fees charged by fund managers. The intuition is that the demand for funds and the fund fees depend only on the characteristics of private information, and are not affected by the addition to the fund portfolio of a "piece" with a publicly known composition.

Investor-manager transactions with asymmetric knowledge have been a subject of extensive research. The main focus in the literature, which we review later in this Part, was on the agency frictions resulting from the imperfect knowledge of managers' types and unobservability of their actions. The asymmetric knowledge was shown to motivate strategic behavior of managers who tend in this case to undersupply effort and misrepresent the quality of their services.

Our analysis differs from the agency literature, because, instead of focusing on the possibility for strategic behavior, we focus on how investors navigate the complexity of the fund market. Using the terminology from industrial organisation, we consider the problem of horizontal differentiation of managers (identifying managers as belonging to distinct types, without one type being superior to the other), instead of vertical differentiation (separating "good" managers from "bad"). The horizontal differentiation was largely absent from the previous literature because, due to tractability concerns, the models were often restricted to contain only one risky asset (Ross (2005)), or to introduce only one type of informed managers in the economy (Garcia and Vanden (2009)), where the horizontal differentiation was not possible.

We derive the implications of the investment delegation in presence of private information for the task of performance measurement. According to the common premise in investing, a superior information should manifest itself via an abnormal performance of the informed portfolio. The abnormal performance is often defined with respect to a benchmark, and thus is sensitive to the choice of the benchmark portfolio composition. We show that in the presence of private information an informed portfolio does not necessarily outperform any uninformed benchmark. The absence of outperformance does not indicate though that informed managers do not add value: fund portfolios are valued by investors as inputs to their overall allocation of risky assets rather than sources of individual performance. Delegation benefits investors mainly by increasing their implied conditional precision (introduced by definition 1 in Part 1), which enables them to invest more in the risky securities and realize greater expected total gains. Thus, when it comes to performance analysis, investors should focus on detecting the presence of a private signal, rather than on ascertaining the persistence of outperformance of the fund with respect to a benchmark, as such outperformance is problematic because of the signal error and of the eventual residual uncertainty in asset returns.

Our findings are consistent with the literature on performance measurement in the presence of private information (Admati and Ross (1985); Grinblatt and Titman (1989)), in that the relative performance measures often fail to capture the presence of private information, and the benefit to investor from delegation is manifested via the gross payoff of the fund. We extend their results by adding the residual uncertainty, and by explicitly introducing a benchmark portfolio in the performance analysis. The sensitivity of the fund performance evaluation to the choice of the benchmarks is also consistent with the empirical literature (for example, Lehman and Modest (1987)).

Another model that provides a rational explanation to the absence of abnormal performance in the fund market and introduces a benchmark index is Berk and Green (2004). They use a reduced-form modelling of investment skill with no explicit trading in securities, and featuring risk-neutral agents, which is different from our setup. The explanation of the absence of abnormal performance in Berk and Green (2004) is based on the scale diseconomies in producing investment performance, on the monopoly power of managers in their own markets, and on the freedom of managers to alter the investment portfolios unilaterally. Instead, our results derive from the complementarity among funds that makes them compete monopolistically.

2.1 Knowledge requirements and viability of the fund market

The endogenous creation of the institution of delegation presented in the model in Part 1 is based on the overwhelming knowledge requirements: the precisions of all private signals in the economy should be common knowledge. Unlike trading in securities, where publicly observed prices are generally sufficient to coordinate actions of agents and to reach equilibrium, investing in individual funds is possible only when investors could form beliefs about the future payoffs of their investments. As a fund's payoff depends on the information of the manager and on his actions, a minimal amount of knowledge required for the investor-manager transaction includes the informational characteristics of the manager's private signal and the knowledge of the pattern of the manager's behaviour given his information and compatible with his incentives. Without this knowledge, an investor cannot take a rational action.

To avoid confusion, in what follows we will use the term *asymmetric information* to denote the existence of private signals about the assets payoffs, while we will call *asymmetric knowledge* the situation where investors do not have deterministic knowledge about the manager's precisions. While asymmetric information is the fuel of investment delegation, as investors buy fund shares precisely because of the diversity of the private information, asymmetric knowledge is often the factor compromising delegation. Investors limit their levels of investment in funds because asymmetric knowledge usually gives managers an incentive to behave strategically, or simply because investors cannot evaluate the benefit of their investment in funds.

Asymmetric knowledge problem is not unique to investment delegation. Indeed, the principal-agent frictions due to asymmetric knowledge are present in other industries that were a subject of extensive academic research, such as sharecropping in agriculture. For example, the seminal paper of Holmstrom (1979) established the theoretical framework to treat the problem of moral hazard in agency setting and include. Holmstrom (1982) considered the case when the delegated task is split between several agents who act as a team. These classical models of moral hazard suggested that the asymmetric knowledge frictions could be mitigated by sharing the outcome between the agent and the principal, and by conditioning the agent's compensation on the signals correlated with the effort.

There is a vast theoretical literature addressing the adverse effects of asymmetric knowledge in investment delegation (see, for example, a comprehensive review of Stracca (2006)). An important stream of literature considers the problem of moral hazard, where managers pursue own rational incentives in detriment to the outcome for investors. Moral hazard models are often applied to delegated investments because actions of the managers are unobservable and provide a scope for strategic behavior, such as undersupplying effort (Buffa et al. (2014)), or pursuing objectives different from that of the investor's.

The compensation sharing agreements were considered in the literature on delegated investment management as a tool to alleviate the asymmetric knowledge problem. Yet, the findings from the traditional agency models were shown to be not applicable to money managers. The performance sharing compensation in the investment delegation tends to generate strategic behavior by managers, who find ways to "undo" the incentives created by performance sharing (Stoughton (1993); Admati and Pfleiderer (1997); J. and Gorton (1997); Das and Sundaram (2001)).

Indeed, investment managers are well-known for not sharing any investment losses with investors, which prevents them from internalizing the externality generated by asymmetric knowledge. Performance-based compensation, when allowed by regulation, tended to take a form of a performance bonus. A nonlinear bonus produces quite different incentives compared to a linear sharing rule (Starks (1987)). The historical analysis shows that, when unregulated, investor-manager contracts tended to include a bonus related to the fund performance. Once the performance based compensation were required by 1940 and 1970 regulation to be strictly symmetric around an index, managers progressively discontinued the use performance sharing fee and opted instead for a flat fee for their services. The evolution of the compensation of investment managers is thus at odds with the theoretical result stating that the performance sharing and the use of the signals related to managers' effort (the benchmark index performance for example) improves the level of effort supplied. Even if investors would abandon the idea to use performance-sharing as an incentive, and would focus instead on performance monitoring, for example comparing performance to a benchmark, the result will be suboptimal. The contracts between investors and managers stipulating a limit on tracking error produce incentives for managers to hold suboptimal portfolios (Roll (1992); Brennan (1993)). Moreover, the choice of the benchmark in this case will be important for the final result, with a wise choice of benchmark being the optimal mean-variance portfolio. Yet, to make such a choice investor would need to possess a lot of knowledge about the distribution of the assets payoffs. It is unclear why, having all this knowledge, one would delegate in the first place.

Another stream of literature on signalling focused on the case where the effort is not relevant, and the outcome depends on the agent's type, which is unobservable by the principal. Agents of the inferior type thus are keen to misrepresent their type to get hired and to get rewarded as the agents of superior type. Such adverse selection problem might disrupt the market for goods (G. (1978)). Credible signalling by agents usually takes a form of costly commitment tied to the outcome, such as, once again, performance sharing, co-investing with the principal, or participating in losses of the principal (franchise). The success of signalling strategies depends on the existence of separating equilibria in the signalling game. If only pooling equilibria are possible, the principal might invest in learning the agents' types.

Credible signalling of the manager's type was explored in Bhattacharya and Pfleiderer (1985); Allen (1990). This literature is closer to our model, because the scope for moral hazard is limited when, as in our model, the own private signal is obtained at no cost, and the profit of the strategic behavior is negligible because of the assumption of a large market. Bhattacharya and Pfleiderer (1985) showed that, while a linear performance sharing is not effective in separating more informed managers from less informed, a compensation based on a quadratic deviation from the benchmark index could attract managers of the superior type. Allen (1990) showed that credibility of the signal might be supported by the own-account bet of the signal seller on his signal. The main impediment for the signalling strategy is the risk aversion of the seller, so there might be a scope for an intermediary who has perfect knowledge about the quality of the signal and has lower risk aversion than the seller, to step in and assume the risk of the signalling commitment upon compensation. Allen (1990) calls this leasing of risk tolerance the origins of financial intermediation. More recently, in the model of Ross (2005) which is the most relevant to our study, explored the possibility of separating agents by types in the context of investment delegation. Ross concluded that pooling equilibria were to prevail, and there are probably many professional managers in the investment industry who are truly uninformed.

We show in this study that, while learning the manager type and signalling have a scope in fund market, there exists a demand for delegation even without signalling. Contrary to the majority of agency models with adverse selection, where a principal's problem is to pick one agent, in our setup agents invest in all available funds. Hence the investment outcome depends on the average characteristics of funds, rather than on that of individual managers. As a result, even in the case when no knowledge about the types of individual managers is available, there always exists a possibility to delegate symmetrically, investing the same amount with every manager, provided that the average precision of private information is known. Such minimal knowledge requirement is the same minimal amount of knowledge that guarantees the existence of equilibrium in asset markets with private information without delegation (Admati (1985)). Indeed, the knowledge of the average quality of the private signals in the economy is exactly the information that allow agents to form rational expectations of the informational content of the asset prices.

In this Part we present a case of symmetric delegation demand and show that investors might consider investing in funds even if they do not know precisions of individual managers. We further show that symmetric delegation provides inferior outcome for investors, compared to the delegation with known precisions. Investors are able to evaluate the benefit from learning individual precisions, and will be inclined to incur expenses to learn precisions, such as paying due diligence costs, remunerating of third parties for certifying the managers' quality, or, eventually, paying a fee to license a benchmark index.

2.2 Investment delegation with minimal knowledge requirements

Here we derive the equilibrium in the absence of public information about the signal errors of individual agents (i.e. neither the signal realizations s^m, nor the signal error covariances Σ^m are known). Funds are thus portfolios with unknown composition and unknown structure. As a result, investors are not able to form expectations about the performance and risk of individual funds. Despite of the uncertainty, delegation can be made meaningful as long as certain aggregate characteristics of broad fund allocations could be defined.

The minimal knowledge requirement turns out to be the knowledge of the average precision of private information, which is enough to deduce the probability distribution of a symmetric allocations of funds. The solution for individual delegation quantities $\gamma^m_{m'}$ will be symmetric, i.e. the delegation quantities $\gamma^{mm'}$ will not depend on the precisions of individual managers m'. We show below how to derive such solution if the fund structure is assumed to be identical to the own-account portfolio of the manager:

$$\nu^{m'} = x^{m'} \tag{2.1}$$

The above definition of the fund structure is different from the definition used in Part 1. There we assumed that a manager m has to deliver to investors the performance of a no-delegation portfolio (1.14). The two portfolios, x^m and x^m_0, are, in general, different, because x^m_0 is optimal given the private information of agent m in an economy with no delegation, while x^m is optimal in the presence of delegation. In particular, the direct investment in securities x^m offsets the uninformed risks that agent m assumes via the fund allocation. Yet, in the present case of minimal knowledge requirements, the two definitions of fund structure can be shown to lead to the same demand for delegation. We chose to define the fund portfolio as (2.1) to make it more explicit that no additional knowledge about the economy is needed to introduce a viable market for funds, beyond the standard knowledge required to build a rational expectation equilibrium in asset market.

To formulate the agent's optimization problem one needs to evaluate the terminal wealth given by (1.19), where the fund structure $x^{m'}_0$ is replaced with $x^{m'}$. In the absence of any information to discriminate among the available funds $x^{m'}$ agents will look for a symmetric delegation solution:

$$\gamma^{mm'} = \gamma^m, \qquad \forall \quad m' \tag{2.2}$$

That is, the agent m chooses to buy the same amount of units of every fund m'.

Proposition 12. *When signal precisions of individual managers are unknown, the optimization problem admits a symmetric solution (2.2) and the expression for the final wealth (1.19) simplifies to:*

$$W_1 = e_0 R + x^m (D - PR) + \gamma^m \frac{z}{1+\gamma}(D - PR) \tag{2.3}$$

The proof rests on the fact that a symmetric delegation to a continuum of available funds is the same as investing in the aggregate active portfolio x:

$$\int_{m' \neq m} \gamma^m x^{m'} dm' = \gamma^m \left(\int_0^1 x^{m'} dm'\right) = \gamma^m \cdot x \tag{2.4}$$

Due to the market clearing (1.32), in the case of symmetric delegation (2.2), the random variable x has the same probability distribution as the stochastic asset supply z:

$$z = \int_0^1 (x^m + \int_0^1 \gamma^m x^{m'} dm') dm = x + \int_0^1 \gamma^m dm \cdot x = x(1+\gamma) \tag{2.5}$$

where the aggregate number of shares invested in the funds is:

$$\gamma = \int_0^1 \gamma^m dm \tag{2.6}$$

In what follows, we will also refer to the parameter γ as the total amount or the total level of delegation in the economy. **Q.E.D.**

Note, that the symmetric solution does not require the number of shares delegated to be the same for all agents, in general $\gamma^m \neq \gamma^{m'}$. So, the symmetric solution allows for variability of the delegation quantities γ^m across investors.

Thus we showed that the minimal knowledge about the asset market structure and the average precision of private information is sufficient to form expectations about the behaviour of a simple fund allocation (2.2). The symmetric allocation shares an important property with the full-knowledge fund allocations built using known individual precisions: it diversifies completely the signal risk of individual managers. In the following section we derive several other properties of the symmetric allocation and show that it is in general suboptimal from the point of view of the full private information, and inferior to the allocation built with the knowledge of individual precisions.

2.2.1 Properties of the symmetric fund allocation

The agent's optimization problem consists in splitting the investment between the own conditional portfolio x^m and the market portfolio z:

$$\max_{x^m,\gamma^m} \mathbb{E}(U^m(x^m(D-PR)+\gamma^m \frac{z}{1+\gamma}(D-PR)) \mid s^m, P) \tag{2.7}$$

The market portfolio z is unobserved, but the investor can still perform the optimization using the conditional distribution of the variables θ and z that are related to the observables s^m and P via the linear expressions (1.2) and (1.33):

$$z = A_2^{-1}A_0 + A_2^{-1}A_1\theta - A_2^{-1}P \tag{2.8}$$

The result does not change in the presence of the delegation structure[1].

In an economy with no delegation ($\gamma^m = 0$), the optimal allocations have a usual mean-variance form (3.5). If delegation is allowed, the quadratic structure of uncertainty in the terminal wealth allows to derive a closed-form expression for the expected utility (2.7).

Recall, that in the case of known precisions we introduced several useful definitions. Matrix N_m (1.22), referred in our study as delegation exposure, represents an indirect exposure to risky assets via delegation; matrix G_m (1.38), referred to as implied conditional precision, has the meaning of the conditional precision of agent m, modified by his risky delegation exposure N_m. The symmetric solution could also be conveniently expressed via these quantities, together with the quantities V_m, θ_c^m and L_m introduced in Part 1 (1.12, 1.11, 1.21, 1.40).

Proposition 13. *CARA-expected conditional utility of the terminal wealth (2.3) in the case of a symmetric fund allocation is given by:*

$$\mathbb{E}(U \mid s^m, P) = \frac{I}{J(G_m)} e^{\frac{1}{2}\rho^2 \tilde{x}^m V_D \tilde{x}^m - \rho \tilde{x}^m(\theta_c^m - PR) + \frac{1}{2}\tilde{\tilde{x}}^m L_m^T G_m^{-1} L_m \tilde{\tilde{x}}^m} \tag{2.9}$$

where the boost in implied precision coming from delegation depends on matrix N_m:

$$N_m = 2\frac{\gamma^m \rho}{1+\gamma} A_2^{-1} A_1 \tag{2.10}$$

[1]The coefficients of the price functional A_i do depend on the aggregate levels of delegation, but since the aggregate levels cannot be affected by actions of one investor, the investor treats the coefficients A_i as constants in the portfolio optimization problem.

`Proof:` To compute the expectation in (2.7) we first take the integral over variable ϵ_D. The relevant part of the expectation in (2.7) is:

$$const \int e^{-\rho(x^m+\gamma^m \frac{z}{1+\gamma})\epsilon_D - \frac{1}{2}\epsilon_D V_D^{-1}\epsilon_D} d\epsilon_D = \mathbb{E}_{\epsilon_D}(e^{t\epsilon_D}) = e^{\frac{1}{2}tV_D t} \tag{2.11}$$

where t is the negative of investor's risky exposure $t = -\rho(x^m + \gamma^m \frac{z}{1+\gamma})$.

Using the relation (2.8) one can eliminate the variable z and reduce the expectation to an integral over the variable θ:

$$\begin{aligned} \mathbb{E}(U \mid s^m, P) &= const \int e^{\frac{1}{2}(\rho\tilde{x}^m + \frac{1}{2}N_m(\theta-\theta_c^m))V_D(\rho\tilde{x}^m+\frac{1}{2}N_m(\theta-\theta_c^m) - \rho\tilde{x}^m(\theta-PR)} \times \\ &= \times e^{-\frac{1}{2}(\theta-\theta_c^m)N_m(\theta-PR)-\frac{1}{2}(\theta-\theta_c^m)V_m^{-1}(\theta-\theta_c^m)} d\theta \end{aligned} \tag{2.12}$$

Here several quantities are the same as in the previously studied case of known precisions: the implied conditional precision G_m is given by (1.38), the conditional expectation of the dividend θ_c^m is given by (1.12) and the conditional variance matrix V_m is given by (1.11).

The variable $\tilde{x}^m$ has the meaning of the conditional expected risky exposure: $\tilde{x}^m = x^m + \gamma^m \frac{z_c^m}{1+\gamma}$, and z_c^m is the conditional expectation of the risky stock supplies z: $z_c^m = A_2^{-1}(A_0 - P) + A_2^{-1}A_1\theta_c^m$.

After simplifying the expression in the exponent one gets:

$$\mathbb{E}(U \mid s^m, P) = const e^{\frac{1}{2}\rho^2\tilde{x}^m V_D \tilde{x}^m - \rho\tilde{x}^m(\theta_c^m - PR)} \int e^{(-\rho\tilde{x}^m L_m^T - \frac{1}{2}(\theta_c^m - PR)N_m)(\theta-\theta_c^m) - \frac{1}{2}(\theta-\theta_c^m)G_m(\theta-\theta_c^m)} d\theta \tag{2.13}$$

Taking the integral over $d\theta$ and simplifying the expression one obtains the conditional expectation of investor's utility (2.9) in a closed form. **Q.E.D**

The derived utility (2.9) is a mirror of the result (1.43) in Part 1. The main difference is that the delegation exposure N_m in the symmetric case is limited to the structure (2.10) possessing only one degree of freedom, while in the case of known precisions agents had more flexibility in shaping their delegation exposure N_m. As a result, the delegation exposure built with the knowledge of precisions allows the investors to extract more value from delegation.

The expression for the optimal value of $\tilde{x}^m$ and equilibrium prices have the same form in the symmetric case as in proposition 6. For simplicity, assume that the fund fees are zero. Then the optimization problem of fund investor is given by (1.56), where the only optimization variable is γ^m. The only first order condition provides the equation to find the optimal delegation quantity γ^m:

$$-tr(G_m^{-1} A_2^{-1} A_1 L_m) = 0 \tag{2.14}$$

The variable γ^m enters (2.14) via the quantities G^m and L_m, which depend on N^m given by (2.10). We do not have a closed form solution to problem (2.14), unless the residual uncertainty is eliminated ($V_D = 0$). But without the residual uncertainty in the asset payoffs the delegation incentives become unbounded and lead to infinite individual and aggregate delegation $\gamma^{m*} = \gamma^* \to \infty$.

When the residual uncertainty is present, the optimal solution will be bounded, because matrix G_m given by (1.56) is required to be positive definite. Hence investors will choose a finite amount of delegation in the presence of the residual uncertainty, the individual level of delegation depending on the private precision on each agent.

To summarize, when allowing delegation in economies with residual uncertainty and with no common knowledge about individual precisions, agents delegate symmetrically (i.e., buy the same number of shares of every existing fund, which amounts to investing in the market portfolio). The level of symmetric delegation exposure depends on the investor's precision, and is inferior to the level of delegation in the economies with no residual uncertainty.

2.2.2 Fully revealing symmetric allocations

Without the residual uncertainty in the economy, the first order condition of the delegation problem (2.14) is trivial, because $L_m = 1$ is independent on the optimization variable γ^m. Hence the only way to have a vanishing trace $G_m^{-1} A_2^{-1} A_1$, is to require the implied conditional precision G_m to approach infinity. A full aggregation of private information will follow, which, in the absence of residual uncertainty, will imply that the securities payoffs become certain. In such a case the delegation incentives become unbounded leading to an infinite individual and aggregate delegation $\gamma^{m*} = \gamma^* \to \infty$.

Although the divergent equilibrium given by a symmetric costless delegation without residual uncertainty has no practical relevance, it illustrates an important feature of the aggregation of private information. If a naive symmetric fund allocation is sufficient to build a fully revealing solution, no value could be added by acquiring more precise information about individual funds. In other words, in an economy where the full information about the securities' payoffs is dispersed in the economy, a symmetric allocation could achieve the first best, and incentives to learn individual precisions are absent.

There is another case when a symmetric allocation is sufficient to build a first-best delegation exposure: when there is only one risky security in the economy. With only one risky asset the optimization problem (1.56) becomes:

$$max_{\gamma^m} v_m^{-1} + 2a\gamma^m - a^2 v_D (\gamma^m)^2 \tag{2.15}$$

The solution is the same for all investors, regardless of their private conditional precision v_m^{-1}:

$$(\gamma^m)^* = a^{-1} v_D^{-1}, \qquad a = \frac{\rho}{1+\gamma} A_1 A_2^{-1} \tag{2.16}$$

Here, as in the case of the symmetric solution with no residual uncertainty, the pattern of delegation is uniform, all investors have the same solution for the optimal amount of delegation irrespective of their own precisions. As a result, in the case of one risky asset, delegation is equally appealing to investors of all precisions, informed as well as uninformed.

The implied precision for any investor then becomes:

$$G_m^* = v_m^{-1} + v_D^{-1}$$

In the case of one risky asset simply delegating the same quantity to each fund produces a fully revealing outcome, notwithstanding the presence of the residual uncertainty and the presence of noise traders.

In the cases when a symmetric fund allocation is sufficient to reveal the full private information, learning precisions of individual managers will change only the bargaining power between the managers and the investors, without creating added value in the economy. On the contrary, when learning the managers' precisions gives rise to more optimal fund allocations, there exist extra gains that could be split between investors and managers. In the remaining part of this section we are concerned with such cases, where learning managers' precisions will give investors access to a better technology of investing, a more efficient way of using private information embedded in the funds.

The majority of literature on delegation has limited the analysis to only one risky asset (Ross (2005); Garleanu and Pedersen (2018)), assumed the absence of residual uncertainty (Ross (2005), Admati (1985)) or imposed a constant precision for all managers (Garcia and Vanden (2009)). In such cases, as we argued above, there is a limited scope to spend resources on learning. Thus, the potential for the market models to embed learning strategies and tools, such as investment benchmarks, has been so far largely unexplored.

2.3 Investment benchmarks as a mechanism to signal precisions

To extract the maximal surplus from delegation agents need to have some knowledge of the informational characteristics of individual managers. When the structure of the market and the private information is sufficiently complex, investors are confronted with two learning tasks: (i) identification of the manager's type, and (ii) estimating the magnitude of the managers' precisions. The notion of the type is similar to the concept of horizontal quality in models of industrial production of heterogenous goods. Horizontal quality means that all consumers do not rank the goods the same way from the lowest quality to the highest. An example is the location of a shop, which does not have a universal value to all consumers, but depends on the distance of the shop to their home. Similarly, managers in general are not universally comparable. They might specialize in different securities and follow various investment styles. In mathematical terms it means that, given two precision matrices, one does not necessarily dominate the other, unless their difference is positive definite.

Roughly speaking, identifying a manager's type is akin to assigning the manager to a category defined by a certain structure of covariance matrix of the signal error:

$$\Sigma_m = a_m^I \Sigma_I$$

where I labels the categories of managers. With a continuum of agents, we will often assume that there is a finite number of types I, and an infinite number of managers within each type. Within a particular type I, managers can be ranked by the value of their precision scale a_m^I, from the lowest to the highest, as in the case of vertical product differentiation. We will show that any desired delegation exposure could be built by investors given only the knowledge of the types of the managers and the average precision scales $\bar{a}_I$ within each type.

Learning managers' types is achieved by many means in the markets. Regulators require fund managers to indicate their investment styles and to list the securities they might invest in; investors run due diligence to know the investment strategy of potential managers; and third party data providers maintain classifications of funds, such as the Morningstar's style box. Finally, managers could self-declare their type by picking an investment benchmark. Among all the learning and signalling tools in the market for funds, the use of investment benchmarks developed in a veritable institution.

The role of institution, as argue O'Driscoll and Rizzo (1996), is in transmitting the knowledge among the market participants. The information about the managers' types is the kind of knowledge that could make sense to transmit via an institution, as it represents an information about a persistent pattern of manager's behaviour. Besides, the characteristics of the private information might be subject to uncertainty, for example might change in time. It is also plausible that managers simply cannot communicate or even articulate the exact precision they possess. As argue O'Driscoll and Rizzo (1996), there is a scope for promoting and following a set of rules that would remain static in time, and direct individual behavior in a pattern.

We argue in this section that the primary role of investment benchmarks might be that of an institution providing a stable basis for the transactions between investors and managers. The benchmark institution appeared endogenously, when, after the 1970s, the financial services industry diversified its product range, with the fund promoters building fund complexes (Frankel and Laby (2015)), and offering investors the exposure to securities outside the domestic market (Sikorsky (1982)). The fact that the supply of benchmarks is private, and the business of benchmark calculation and diffusion is profitable, indicates that benchmark providers, alongside with consultants and providers of fund data, captured a portion of the gains in utility that investors derive from learning the managers' types. The main distinction of the benchmarks from the other strategies to learn precisions is the explicit creation of a common knowledge in the investor-manager transaction. Such common knowledge is important because, according to the principles of the contract law, a transaction between the parties will be enforced only if it was based on the common understanding, the meeting of minds. The manager's obligations relative to a benchmark, such as trading only the securities included in the benchmark portfolio, or respecting the maximal level of risk relative to the benchmark, represent enforceable promises.

2.3.1 Utility gains from learning precisions

Here we argue that both investors and managers have incentives to establish an institution transmitting the knowledge of investment types. The amount of resources investors would be willing to spend on learning the manager's characteristics is bounded from above by the maximal gain in utility that exists between the symmetric allocation and the allocation with known precisions.

Proposition 14. *In the economies with more than one risky asset, with at least several distinct categories of managers, and in the presence of residual uncertainty, the social gain from learning the managers' precisions exists.*

`Proof:` One can compute the maximal gain in utility from learning the peers' precisions as a logarithm of ratios of utilities corresponding to the symmetric allocation and the allocation with known precisions. Below we evaluate the gain in utility as a certainty equivalent:

$$c = -\frac{1}{\rho}\ln\frac{E(U_m)_{known\,precisions}}{E(U_m)_{symmetric}} = -\frac{1}{\rho}\ln det(G_m(V_m^{-1}+V_D^{-1})^{-1}) \qquad (2.17)$$

Given that the maximal possible value of G_m is $V_m^{-1}+V_D^{-1}$, the logarithm in the expression is negative, meaning that the gain from learning the precisions is positive, as long as $\det(G_m)$ is smaller than $\det(V_m^{-1}+V_D^{-1})$. As we've shown in the previous section, this condition is met when there is at least one risky asset, several categories of managers, and the residual uncertainty. **Q.E.D.**

An institution based on voluntary transmission of knowledge from managers to investors could not be workable if managers have incentives to misrepresent their characteristics. A lot of empirical and theoretical literature, as well as concerns voiced by investors and regulation authorities, highlight strong incentives for opportunistic behaviour of managers. Truthful revelation of private information is problematic to establish because less informed managers tend to exaggerate the quality of their information. There is a consensus in the literature that signalling schemes based on performance sharing between investor and manager (the so-called performance fees) do not lead to separating equilibria and fail to attract high quality managers (Bhattacharya and Pfleiderer (1985); Ross (2005)).

We argue here that, contrary to the conventional wisdom about the opportunistic signalling behaviour, managers could have incentives to reveal a part of their informational characteristics, namely, their type. When investors could learn the types correctly, they would be able to invest more across the funds belonging to all types, thus bringing higher profits to all the managers.

To support the above intuition, we give below an example of a simple economy with two risky assets and two categories of managers having the average precision $Q = \omega\Sigma_1^{-1} + (1-\omega)\Sigma_2^{-1}$. We calculate the delegation levels and the profits of both categories of managers with two assumptions about the common knowledge: the minimal knowledge when only the matrix Q is known, and the knowledge of the precision matrices Σ_1^{-1} and Σ_2^{-1}. For simplicity, we assume that the fund fees are fixed in both cases and equal to k. Then it is enough to show that the demand for funds of type 1 and type 2, γ_1 and γ_2 is greater than the symmetric demand for funds $\gamma_1 > \gamma$, and $\gamma_2 > \gamma$.

To prove the possibility of truthful type revelation one should consider a game when managers have a possibility to misrepresent their type and earn the profits of investors of the opposite type and to demonstrate that a Nash equilibrium of truthful type revelation exists. We leave this topic for future research.

To the contrary, the self-revelation of the precision scale within a type (i.e. whether the precision is high or low compared to other funds of the same type) is problematic, because the "vertical" information would not increase the overall levels of delegation by investors, but will be used by investors to reduce delegation costs by discriminating better among the managers of a certain type. Thus, within one type, managers would not truthfully reveal their precision scale.

In principle, investors could attempt to solve the problem of estimation of the precision scales using past performances (Admati and Ross (1985)). For example, if a manager m has precision Σ_m^{-1} and his fund portfolio is given by (1.14), then performance of the fund m over the period t will be given by:

$$\begin{aligned} p_t^m &= (s_t^m - P_t R) O_m (D_t - P_t R) + E_m x_0^u (D_t - P_t R) = \\ &= (\theta_t - P_t R) O_m (\theta_t - P_t R) + E_m x_0^u (\theta_t - P_t R) + \delta_t(\epsilon_{Dt}, \epsilon_t^m) \end{aligned} \tag{2.18}$$

A regression of the fund performance on the past realizations of the variable $(\theta - PR)$ could in principle reveal the full precision matrix. The task is even easier if the category of the manager is known: $\Sigma_m = a_{mI}\Sigma_I^{-1}$: the only slope coefficient to estimate is the precision scale a_{mI}.

Active efforts to reveal precision scales are taking place in the fund market constantly. Investors collect past performances of the funds, invest in the services of third parties, such as investment consultants, establish fund rankings and recommended lists. There is no clear consensus if such activity adds value, as little evidence of the persistence in past performance (Carhart (1997)) is difficult to reconcile with the demand for the performance analysis. The search for a, alternative performance analysis approach relevant in the context of diverse private information would be a step forward and was advocated by Admati and Ross (1985); Grinblatt and Titman (1989).

Adding to the existing literature, we would like to highlight two more reasons why persistent investment performance might be not easy to detect by the traditional performance measures or by the regressions like (2.18). First, as we will argue below, a fund portfolio need not be an optimal portfolio with respect to any particular information set, as long as the primary role of the fund is to truthfully incorporate the manager's private signal. The second reason is that the precision matrix, which is a mathematical abstraction, does not exist in reality, and thus cannot be perfectly formulated even by the manager himself, especially if we allow for the inevitable changes in time of the manager's environment and knowledge. Thus, some drift in the precisions is inevitable from one period to another, which further complicates the performance comparison.

In the next section we formally introduce a mechanism of sharing the precision type between a manager and an investor with the help of benchmark portfolios.

2.3.2 The role of benchmarks in horizontal differentiation of funds

The problem of learning the manager's type might be formulated as a problem of credibly communicating the type covariance matrix Σ_I^{-1} to investors. As we commented above, the precision matrix is a mathematical abstraction that is useful in market models but could not be formulated and publicly announced by the manager. Thus managers have found alternative ways of signalling their type.

One way to signal the type is to provide investors with a description of the investment strategy, the list of securities on the manager's watch list, ecc. Such an approach might potentially transmit rich information to investors, but its drawback is in the subjectivity of the description of each manager (managers in the same category might describe differently the same investment approach), and in the subjectivity of interpretation by investors.

Another way to reveal the type is to rely on a third party to classify managers based on the publicly available information on their style and investment universe. Many investors rely on market data providers and consultants to learn about the fund types. Yet, it seems that the service of revealing the fund type by a third party is always bundled with a service of revealing the precision scales within a class. In other words, it is possible that a service of revealing only types is not profitable and should be complemented with a more lucrative business of selling recommendations about the "best-in-class" managers to pick.

Finally, since the 1970s, a new institution to signal the managers' types have evolved. Instead of relying on managers' own description of their type, families of benchmark portfolios were introduced to represent the type. A benchmark portfolio is a portfolio with a fixed composition, which is made a common knowledge to all market participants. Each benchmark portfolio is restricted to a subset of available securities (the benchmark's investment universe), so that together a family of benchmarks covers all the securities in the market. A manager with precision $a_{mI}\Sigma_I^{-1}$ picks securities from the investment universe of benchmark B_I and promises to deliver the performance of a portfolio:

$$x_m^B = a_{mI}\Sigma_I^{-1}(s^m - PR) + B_I \tag{2.19}$$

which would have a certain variability around the benchmark portfolio B_I, measured, for example, with the help of a tracking error or downside risk limit. Thus, instead of communicating the type's precision matrix, the manager communicates how his investment approach would on average deviate from the appropriate benchmark portfolio.

For example, in the toy model of two risky assets considered in Part 1, where managers belonged to three categories given by precision matrices (1.111), three benchmark portfolios could be introduced to signal their types:

$$B_1 = b_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_2 = b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_3 = b_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.20}$$

If asset 1 is interpreted as large-capitalisation stock, and asset 2 as small cap stock, then the manager types will be interpreted, accordingly, as a large-cap manager, a small cap manager and a manager holding a broad portfolio.

The promise (2.19) is a credible commitment, because a manager could be sued for the breach of contract or for negligence if ex-post his portfolio would be shown to contain off-benchmark securities or to have contained bets incompatible with the tracking error announced (see, for example, Perold et al. (2003) for the account of the lawsuit Unilever Superannuation Fund v Merrill Lynch for negligence on the basis that the fund had 8% deviation from the benchmark performance given the contractual limit of 3%).

Note, that the portfolio definition (2.19) does not prevent uninformed managers with $a_{mI} = 0$ from selecting a benchmark and marketing their funds. The ex-post deviation from the benchmark will be either negligible, which is consistent with the phenomenon of closet indexing (Petajisto (2013)), or will come from random deviations of the fund portfolio from the benchmark, if the manager tries to imitate a signal s^m with a pure noise.

The above definition (2.19) did not fix the exact composition of the benchmark portfolio B_I. In the market, such a decision is made by the index provider, an independent third party. Often the benchmark indices are built by including all existing shares of the securities in the benchmark's investment universe, although some benchmark indices have historically used a different approach to the securities weighting (notably, the Dow Jones Industrial Average, which is one of the oldest market indices).

Proposition 15. *The weighting of securities inside a benchmark portfolio is irrelevant for the levels of delegation.*

`Proof`: The proof of the proposition relies on the ability of fund investors to offset the exposure to any uninformed portion of the fund portfolio via own account trading. Indeed, if the composition of portfolio B_I is perfectly known to investor, his own account trading x^m will adjust such as to keep $\tilde{x}$ the sum of the own account positions and the fixed exposure of the manager's positions, on the mean-variance frontier defined by the investor's information set, as in (1.21):

$$\tilde{x}^m = x^m + \gamma_I B_I \tag{2.21}$$

As a consequence, the delegation quantities γ_I and the equilibrium fees in the economy will not change if the composition of the benchmark indices changes in a predictable way. **Q.E.D.**

As we assumed that the main role of investment managers is to sell their signals, any addition to the fund of a fixed-composition portfolio unrelated to the signal does not affect the value of the manager's services, and, hence, leaves the demand for delegation and the managers' profits unchanged.

The definition of benchmarks as signalling tools for the managers' types is consistent with the observed facts about the benchmark uses. Benchmark indices are not created in isolation: every index provider offers a family of indices, providing collectively a coverage of the whole market of securities. Every time a new investment style appears, a new index is created to offer a benchmark for this investment style. The qualities of the indices are primarily the transparency, stability of their composition, sound governance rules in the index maintenance, and broad recognition of the indices in the investing community (think of creating the common knowledge). The weighting of the securities within a benchmark is not a quality criteria or a competitive edge for index providers. Indeed, all major index providers use almost identical principles of index composition. These stylized facts are consistent with our hypothesis about the role of benchmark indices.

We painted a rough picture of how the knowledge of managers' characteristics could be transmitted through benchmarks. There exist other ways to transmit the detailed knowledge about individual funds (description of the investment process, professional certifications of the fund analysts credentials, etc). The question of whether benchmarks represent a better or an optimal way to convey this information is difficult to answer. According to the intuition of O'Driscoll and Rizzo (1996), in order to justify an institution as optimal, one would need to know the "..very information whose discovery is the object of that institution". As O'Driscoll and Rizzo (1996) put it: "Were individuals to know enough to rationalize the rule, they would generally know enough to abandon it."

2.4 Performance measurement in the presence of private information

In this section we explore how the performance of different managers compares one to another and to the benchmarks in the context of an economy with private information, and in the presence of residual uncertainty.

Measuring investment performance in the presence of asymmetric information presents distinct challenges, as highlighted by Admati and Ross (1985). The manager and the observer/investor do not share the same information set, hence it is not possible to detect superior information by the conventional risk-return measures, such as unconditional Sharpe ratio. Portfolios of informed managers will not lie on the mean-variance frontier of uninformed (or differently informed) investors. Grinblatt and Titman (1989) highlighted that the widespread measure of abnormal performance, the Jensen's alpha, fails to detect the superior information in portfolios employing market timing.

We would like to add to the above another difficulty. As we showed in previous sections, the fund portfolio need not be an optimal portfolio with respect to any particular information set. The only essential requirement for the fund is to truthfully incorporate the manager's private signal. For example, Admati and Pfleiderer (1990) study indirect sale of private information via a fund assuming that the fund composition does not have mean-variance structure but is instead equal to the signal itself. Investor m could always account for the particular structure of the fund portfolio m' by adjusting his decision variables $\gamma^{mm'}$. If the structure of the fund is known to the investor, it's not important whether the fund portfolio satisfies any risk-return objectives, as long as it faithfully incorporates the signal. A suboptimal portfolio structure makes it difficult though to associate the performance of the fund to any measure of abnormal return.

Admati and Ross (1985) proposed to assess the informational content of portfolios directly by estimating the components of the precision matrix $\Sigma^{-}1$. Such estimation could be done in principle by regressing ex-post portfolio returns against realized values of (θ, P), similar to the example we used in the previous section (2.18). If the quadratic terms in the regression are significant, then one might conclude that private information is present in the portfolio. Admati and Ross (1985) show that performance of any two portfolios is not comparable if they have arbitrary precision matrices. If two portfolios could be compared, that is, if the difference of the precision matrices is positive definite, the performance of the portfolio with superior information is greater ex-post than the performance of the one with inferior information. In such a case there exists a possibility of vertical differentiation of managers within the same type, in complement to the horizontal differentiation into types that we discussed in the previous section.

We extend the approach of Admati and Ross (1985) to the economy with the presence of residual uncertainty, and compare the performance of fund portfolios to fixed-composition benchmark portfolios.

Contrary to their result, we find that portfolios with superior information are not guaranteed to have superior ex-post performance as standalone investments in the presence of residual uncertainty ϵ_D. A superior precision of private information will lead to larger bets on the signal, and, as a consequence, to more important exposure to the residual uncertainty. The absence of abnormal performance does not diminish though the value of the informed portfolio as an input into the fund allocation; it's just complicates the task of marketing the fund.

2.4.1 Ex-post performance of a fund portfolio

Let's parametrize a generalized portfolio in the following way:

$$x = G_1(\bar{\theta} - PR) + G_2(\theta - PR) + G_3\epsilon^m \tag{2.22}$$

The above general parametrization covers all meaningful portfolios that appear in an economy with asymmetric information:

- Conditional portfolios of informed investors (the funds) x_0^m:

$$G_1 = \frac{1}{\rho}E^m,\ G_2 = \frac{1}{\rho}O^m,\ G_3 = \frac{1}{\rho}O^m$$

- Portfolios corresponding to the aggregate signal of informed investors, which fully reveals the variable θ:

$$G_1 = 0,\ G_2 = \frac{1}{\rho}V_D^{-1},\ G_3 = 0$$

- Unconditional portfolios of uninformed investors x_0^u:

$$G_1 = (V + V_D)^{-1},\ G_2 = G_3 = 0$$

The term G_3 represents the signal error that is present in portfolios that do not aggregate all the available private information. Such a term is absent in the portfolios that diversify completely the signal risk, or in the unconditional uninformed portfolio that does not contain any private information. If the term G_3 is absent and the term $G_2 \neq 0$, then the portfolio diversifies away the idiosyncratic signal signal risk and aggregates all the available information.

Below we derive expected utility of the generalized portfolio (2.22) for CARA preferences, conditional on the values of (θ, P).

$$
\begin{aligned}
\mathbb{E}(U(x) \mid \theta, P) &= -\mathbb{E}(e^{-\rho x(\theta - PR + \epsilon_D)} \mid \theta, P) = \qquad (2.23)\\
&= -\underbrace{e^{\frac{\rho^2}{2}(\bar\theta - PR)G_1\tilde V_D G_1(\bar\theta - PR)}}_{(1)} \times \underbrace{e^{-\rho(\bar\theta - PR)G_1(1-\rho\tilde V_D(G_2 - \rho G_3\Sigma^m G_3))(\theta - PR)}}_{(2)} \times \\
&\times \underbrace{e^{-\rho(\theta - PR)(G_2 - \frac{1}{2}\rho G_3\Sigma^m G_3 - \frac{1}{2}\rho(G_2 - \rho G_3\Sigma^m G_3)\tilde V_D(G_2 - \rho G_3\Sigma^m G_3))(\theta - PR)}}_{(3)} \times \\
&\times \underbrace{\frac{det(\tilde V_D)}{\det(V_D)}}_{(4)}
\end{aligned}
$$

where $\tilde V_D^{-1} = V_D^{-1} - \rho^2 G_3 \Sigma^m G_3$.

The four terms in (2.23) can be interpreted as follows:

1. The first term has a strictly negative contribution to the utility related to the portfolio term G_1.

2. The second contribution to the utility is also related to the term G_1 and has an ambiguous sign. Intuitively, sometimes a "wrong" portfolio can appear "right" for some specific realizations of θ ("right for the wrong reasons"). So, the term $G_1 \neq 0$ introduces suboptimality through contribution (1), and ambiguity in the comparison via contribution (2).

3. The third contribution comes from the aggregate signal G_2 and also depends on the signal error G_3. If the error is absent $G_3 = 0$, the term in the exponent simplifies to $-\frac{1}{2}\rho^2(\theta - PR)G_2 V_D G_2(\theta - PR)$. The optimal value of $G_2 = \frac{1}{\rho}V_D^{-1}$ gives the greatest contribution to the utility, which corresponds to a portfolio conditional on the aggregate private information.

4. The fourth term reduces utility when the errors G_3 are present. So, the contribution of G_3 - error term, is always negative for the utility.

From the analysis of the decomposition (2.23) one could reach several tentative conclusions. First, any informed investors has inferior performance compared to the aggregate fund allocation, since individual informed portfolios contain the term G_3 that is detrimental to the utility. Second, contrary to the result of Admati and Ross (1985) that the informational edge is sufficient to compare portfolios (the more signal is pooled, the better), we find that some portfolios that do not contain signal risk (and, hence pool all the signal) might still be suboptimal and will make ambiguous yardsticks for performance comparison, because, notably, of their term G_1. Third, the unconditional portfolio of uninformed investors x_0^u, while being an attractive candidate for a benchmark against which to measure informational content of other portfolios, does not provide a unique result for performance comparison because the term (2) in (2.23) has ambiguous sign.

2.4.2 Extraction of precision matrices from realized performances

As shown in Admati and Ross (1985), the performance measurement goal in an economy with asymmetric information is to assess a persistent pattern in portfolio payoffs as reliably as possible and associate each portfolio with a certain precision matrix. The knowledge of individual precision matrices, as we showed in Part 1, allows to build optimal fund allocations when delegation is introduced.

Suppose that one has collected detailed information about portfolio performance, as well as the asset payoffs and prices, during several rounds of asset trading $t = 1, ..., T$. The collected data allows to formulate a regression problem to detect the precision matrices of individual portfolios. Possible regression routines were discussed in Admati and Ross (1985) in the case when aggregate uncertainty was absent. Below we extend their result to the case where aggregate uncertainty in the assets' payoffs is present ($\epsilon^D \neq 0$).

For a portfolio that has a structure (2.22), the realized performance during the round t can be written as:

$$\begin{aligned} perf_t &= (\bar{\theta} - P_t R) G_1 (\theta_t - P_t R + \epsilon_t^D) + \\ &+ (\theta_t - P_t R) G_2 (\theta_t - P_t R + \epsilon_t^D) + \\ &+ \epsilon_t^m G_3 (\theta_t - P_t R + \epsilon_t^D) \end{aligned} \tag{2.24}$$

When $G_3 = 0$, the expression above represents a system of linear equations, where the components of G_1 and G_2 could be retrieved if one has enough observation points t. With $G_3 \neq 0$ one has a regression with the error term $\epsilon_t^m G_3 (\theta_t - P_t R + \epsilon_t^D)$ to build estimates $\hat{G}_1$ and $\hat{G}^2$.[2]

To lay out the linear specification of the regression and the properties of the regression estimates, let's introduce the following notations:

$$
\begin{aligned}
x_t^1 &= \theta_t - P_t R, && (2.25)\\
x_t^2 &= \bar{\theta} - P_t R, && (2.26)\\
x_t^3 &= \theta_t - P_t R + \epsilon_t^D, && (2.27)\\
\nu_t &= \epsilon_t^m G_3 x_t^3 && (2.28)
\end{aligned}
$$

Now, one can rewrite (2.24) as a linear regression with the regressors defined in the following way:

$$z_{ij}^1 = x_i^2 \cdot x_j^3, \quad z_{ij}^2 = x_i^1 \cdot x_j^3, \quad i, j = 1, ..., N \tag{2.29}$$

That is, individual regressors will be x^a and their products.

$$perf_t = tr(G_1 z_t^1) + tr(G_2 z_t^2) + \nu_t \tag{2.30}$$

The linear regression could be performed using the standard OLS techniques, since the error term satisfies the exogeneity condition:

$$\mathbb{E}(\nu_t \mid z_t^1, z_t^2) = 0 \tag{2.31}$$

given that ϵ^m is conditionally independent on θ and ϵ^D.

The property (2.31) implies that unconditional expectation of the error term is zero. It also implies the orthogonality condition:

$$\mathbb{E}(\nu_t z_t^1) = 0, \quad \mathbb{E}(\nu_t z_t^2) = 0 \tag{2.32}$$

$$\mathbb{E}(\nu_t z_t) = \mathbb{E}(\mathbb{E}(\nu_t z_t \mid z_t)) = 0$$

The residuals ν_t are not homoskedastic since the residuals conditional variance changes with t:

$$\mathbb{E}((\nu_t^m)^2 \mid x_t^1, x_t^2, x_t^3) = G_{3ij} x_{tj}^3 G_{3kl} x_{tl}^3 \Sigma_{ik}^m \tag{2.33}$$

Thus one has to use the tools for heteroskedastic OLS estimation, and, in addition, to account that the regressors z_{ij}^1 and z_{ij}^2 are in general correlated.

[2] The value of the matrix G_3 itself cannot be retrieved from regression, unless it is known, for example, that $G_3 = G_2$

There is a caveat in implementing the regression routine discussed above. It is built on the assumption that the split of the past dividend D_t in components θ_t and ϵ_t^D is known for all t. Generally, the components of D_t are not observed separately. Below we discuss possible strategies to retrieve such past information.

Let's try a reverse exercise and ask if, knowing only D_t and past portfolio performances, one could reconstruct the vector θ_t for every t. The equivalent is to ask if one can reconstruct from the past performances how much private information was incorporated into the market prices.

There are two ways to approach the problem. The first is to argue that, if the market prices were equilibrium prices at each round t, and if one knew the exact relation between the prices and θ, then the vector of θ could be deduced from prices or from the market clearing condition directly. For example, if we know that portfolios have the structure $x^m = G_2^m(\theta - PR) + G_3^m \epsilon^m$, then the market clears as:

$$\int_0^1 G_2^m dm(\theta - PR)dm = z, \quad \theta = PR - (\int_0^1 G_2^m dm)^{-1} z \tag{2.34}$$

Such a reverse-engineering of the predictable payoff θ will not be possible, however, if the supplies z are noisy and unobservable, or if the prices are not equilibrium prices.

In principle, one should be able to reconstruct the private information from performance realizations because only what is reflected in investors' portfolios can be defined as private information. If something was not reflected in the portfolio composition, it is as if it did not exist; the private signal exists as long as somebody acts on it.

The exercise of reconstruction is different from a regression in time. Retrieving private information necessitates cross-sectional, instead of time, tools. We would like to argue here that in order to retrieve θ from the past data, the minimal requirement is to know the full distribution of precisions in the economy. Knowing the full distribution of precisions is different from knowing which investor has which precision, so the cross-sectional study may give us the reconstruction of θ even if we do not know the individual G_2^m matrices.

In particular, one can define a cross-sectional random variable:

$$C = (\theta - PR)G_2^m(\theta + \epsilon^D - PR) + \epsilon^m(\theta + \epsilon^D - PR)$$

The sources of cross-sectional variability are G_2^m and ϵ^m. Then, one can compute N moments of the distribution of this variable using the distribution function of precisions F(Q).

In conclusion, the assumption that superior performance should follow from superior information is not appropriate in the presence of asymmetric information and of the residual uncertainty. An informed portfolio should be rather judged by the quality of its input, i.e. the presence of an authentic private signal, than by its performance as a standalone investment. However, because of the traditional focus of the investment industry on measuring the superior performance, the efforts to create a coherent framework to account for asymmetric private information in performance measurement has been so far limited.

2.4.3 Benchmarks in performance measurement

In a world of asymmetric information investors do not agree on an optimal portfolio. The economy is a collection of separate CAPM models with each agent having an own optimal portfolio. Portfolios of peers, al long as they contain unknown signals, are not on the agent's mean-variance frontier and the market portfolio is in general not efficient. The only portfolio with a commonly known composition is the portfolio of unformed investors. In several delegation models a benchmark was defined in such a way: as a portfolio of an uninformed investor (Garleanu and Pedersen (2018)). In markets with frictions, this portfolio serves as a low-cost fund of inferior quality that some categories of agents prefer. The transaction cost-based paradigm for the introduction of benchmarks is different from our approach. With a multitude of complex phenomena related to the use of benchmarks, the cost-saving and the knowledge-transmitting functions of benchmarks might be both present and the relative importance of one of these functions might change depending on the context. For example, in Garleanu and Pedersen (2018) agents possessed perfect knowledge of the informational characteristics of managers, so there was no need for a benchmark to signal the managers' type.

The notion of a common benchmark is natural in the context of models with public-only information, but it is at odds with the concept of abnormal performance. When private information is introduced in a model, the notion of a unique benchmark loses its conventional meaning. Below we will introduce a benchmark in the task of performance measurement using the definition given in the previous section, where we call a benchmark any unmanaged portfolio with publicly known composition (2.19) that is agreed between the investor and the manager.

Introduction of a benchmark facilitates the estimation of precision matrix from the regression in the following ways. First, it eliminates from the fund structure any dependence on the unconditional distribution moments of the payoff components θ and ϵ_D. Thus, investors and managers do not have to worry that they have the same interpretation of public information. The payoff difference of the fund portfolio and the benchmark portfolio becomes:

$$perf_t - B_{tI}(D_t - P_tR) = a_{mI}(\theta_t - P_tR)\Sigma_I^{-1}(D_t - P_tR) + a_{mI}\epsilon_t^m\Sigma_I^{-1}(D_t - P_tR) \quad (2.35)$$

The matrix Σ_I^{-1} is fixed by the choice of the benchmark portfolio B_I, thus the regression will contain only one unknown parameter to estimate: a_{mI}. Communicating a narrow benchmark means that private signal of the manager is limited to a subset of assets, which further facilitates the estimation task.

The expected performance difference, conditional on the observed values (θ_t, P_t) in (2.35) is positive for any realization of θ and P:

$$\mathbb{E}(perf_t - B_{tI}(D_t - P_tR) \mid \theta_t, P_t) = a_{mI}(\theta_t - P_tR)\Sigma_I^{-1}(\theta_t - P_tR) \quad (2.36)$$

We showed, however, using (2.23), that an informed portfolio will not necessarily generate a higher utility than the benchmark, as it might assume too much residual risk.

2.4.4 Comparison of performance in terms of rates of return

The performance analysis considered so far in our study was based on the portfolio payoff, while the conventional practice is to compare rates of return of investments, i.e. returns generated per dollar invested. The latter are easier to construct and compare, as they do not depend on the size of investments. Note, however, that rates of return do not produce the same assessment of the value added as payoffs, especially in the case of asymmetric information. To give an intuition, consider the case of one risky asset. An informed manager will have higher precision of his conditional distribution about the asset's payoff and thus will make a larger bet on the asset, generating superior expected payoff. In relative terms, however, both the manager and any uninformed investor invest in the same asset and have identical relative returns per dollar invested. For this reason some studies advocate the use of gross fund payoffs in performance measurement (Berk and van Binsbergen (2015)), to correctly assess the value added by superior asset timing.

Below we contribute to this line of reasoning by showing that, while a fund portfolio defined as (2.19) has larger expected payoff than its benchmark, their performance comparison in relative terms is ambiguous.

We formulate the problem of relative return comparison in a slightly more general way, as a problem of performance comparison between two fund portfolios of the same type Σ^{-1}:

$$x_i = a_i \Sigma^{-1}(\theta - P) + B, \quad i = 1, 2, \quad a_1 > a_2 \tag{2.37}$$

For simplicity, we omit the error terms from the portfolio composition, and assume that there is no residual uncertainty ($\epsilon_D = 0$).

The difference in gross payoffs is given by:

$$\Delta = (a_1 - a_2)(\theta - P)\Sigma^{-1}(\theta - P) \tag{2.38}$$

Fund 1 with higher precision generates larger payoff.

The comparison in relative terms is as follows:

$$\begin{aligned} \Delta_{rel} &= \frac{a_1 \theta \Sigma^{-1}(\theta - P) + \theta B}{a_1 P \Sigma^{-1}(\theta - P) + PB} - \frac{a_2 \theta \Sigma^{-1}(\theta - P) + \theta B}{a_2 P \Sigma^{-1}(\theta - P) + PB} = \\ &= (a_1 - a_2)\left(\frac{\theta \Sigma^{-1}(\theta - P)}{P \Sigma^{-1}(\theta - P)} - \frac{\theta B}{PB}\right) \end{aligned} \tag{2.39}$$

Relative performance of two fund managers of the same type depends on the relative performance of a manager of their type with respect to the relative performance of a benchmark. One can immediately see that in the case of one risky asset the difference in relative performance is zero, irrespective of the structure of the benchmark B.

Let's now rewrite the relative performance of the manager and of the benchmark in relative terms, defining:

$$r_i = \frac{\theta_i - P_i}{P_I}, \quad \frac{\theta \Sigma^{-1}(\theta - P)}{P \Sigma^{-1}(\theta - P)} = w_1 r_1 + w_2 r_2, \quad \frac{\theta B}{PB} = b_1 r_1 + b_2 r_2 \tag{2.40}$$

where

For simplicity, let's assume that there are two risky assets, and all the matrix Σ^{-1} is a unity matrix.

$$w_i = \frac{(\theta_i - P_i)P_i}{P_1(\theta_1 - P_1) + P_2(\theta_2 - P_2)}, \quad b_i = \frac{B_i P_i}{B_1 P_1 + B_2 P_2}, \quad i = 1, 2 \tag{2.41}$$

One can show with a simple algebraic manipulations that the difference (2.39) can be reduced to:

$$\Delta_{rel} = (a_1 - a_2)\frac{(r_1 - r_2)(r_1 b_2 P_1^2 - r_2 b_1 P_2^2)}{r_1 P_1^2 + r_2 P_2^2} \tag{2.42}$$

The sign of Δ_{rel} depends on the composition of the benchmark. If the benchmark structure is such that the asset that happened to have higher return was largely overweighted, then the relative return of the benchmark will be higher than the relative return of a perfectly informed portfolio. As a consequence, a portfolio with higher precision a_1 could underperform a portfolio with lower precision a_2. Thus, a biased benchmark structure may lead to incorrect conclusion that the investment skill is absent. Such benchmark misspecification argument was used by some recent studies (reviewed, for example in Cremers et al. (2013)), to critique the earlier empirical performance studies of fund performance where benchmarks were constructed using factor models. As factor models happen to be disproportionately concentrated on small capitalization stocks, the funds would tend to underperform because managers overweight large stocks where the precision of their private information is greater.

The example above is an idealized case, and in reality the comparison of relative performance is complicated by the presence of idiosyncratic signal errors, and, eventually, by the residual uncertainty. In a situation where the structure of the existing benchmarks is an outcome of a trial-and-error decisions made in the past and held in place because they became popular and easily recognizable by investors, there is little hope that the structure of each benchmark is perfectly aligned with the structure of private information of the managers it represents.

Clearly, more efforts are needed towards the coherent performance measurement in the presence of asymmetric information and in constructing non-performance approaches to assessing the value of private information contained in managed portfolios.

2.5 Fund portfolio design and competition

The model in Part 1, as well as our study of the demand for delegation under asymmetric knowledge, were based on the assumption that fund managers do not behave opportunistically and do not deviate from the portfolio structure expected by investors, such as (3.5) or (2.19). There is no way to justify the assumption of the absence of strategic behavior even under complete knowledge of managers' precisions, because the private signal itself is unobservable and there could not exist an enforcement mechanism to guarantee that manager incorporates the signal in the fund truthfully. As we mentioned earlier, the use of benchmarks in the investor-manager contracts allow to enforce a partial compliance, which is possible if the fund structure and fund returns are observed ex-post. Thus managers might be restricted to hold only the securities in the benchmark portfolio and do not deviate from the performance of the portfolio beyond a certain level.

Here we review the role of the design of fund portfolios in the literature. Many models of delegation assumed that a manager might alter the fund portfolio. In particular, the studies that considered the effect of the performance-based compensation on the managers' behavior kept the freedom for managers to respond to incentives by altering fund portfolio or by misrepresenting the signal in order to maximize profits (Bhattacharya and Pfleiderer (1985)). In other studies the managers' flexibility in structuring portfolios was essential to ensure an optimal cost-benefit outcome to all investors, as for example in the model of Berk and Green (2004), where managers reallocate the invested money between alpha-generating investments and an index fund to avoid welfare losses from scale diseconomies. The effect of the flexibility of the fund portfolio design is most probably mixed and model-dependent, depending on the balance between the potential for managers to improve investment outcomes and the detrimental effect of the strategic behavior.

In the literature on information markets, which is closely related to our model of fund management, the design of informational products plays a major role (Bimpikis et al. (2019)). The profits of the information seller depend crucially on how the information is used by the buyers. For example, if the buyers of information compete among themselves in a downstream market, the type and the intensity of the competition will affect the product design choice of the information seller. In addition, information externalities, such as leakage of information through public prices, may induce the information seller to modify the product design in order to limit the leakage. Finally, the information seller might use price discrimination, 2-part tariffs and combine it with the product differentiation in order to maximize the profits (Admati and Pfleiderer (1988, 1990)). All the above insights were obtained using models where the information seller was a monopolist, and where information buyers did not have doubts about the information quality.

When the quality of information is not known to the buyers, the problem of the seller is not how to optimally design the informational product to extract more of the consumer surplus, but rather how to credibly signal the information quality (Allen (1990)) in order to create the demand for the information. The signalling task is incompatible with the strategic behaviour of managers assumed in the models of fund competition (Glebkin and Makarov (2012)), where managers ".. recognize that other managers do not follow some predetermined investment rules, but rather respond strategically to others' investment behavior". When signalling and commitment to the signalled quality are of the major importance, fund design will be not altered deliberately.

In the monopolistic competition setting the power of one seller to manipulate the usage of information through the information product design is limited, as altering the information content might diminish the benefits from the complementarity of the product to those of other sellers. Thus, our assumption of the managers complying with a certain structure of the fund might be plausible under the incomplete knowledge about managers' information and in large markets for funds. In a way, the fund structure based on benchmarks xmB is a design choice that increases the total use of the information by investors.

Our model thus differs from the models that allow strategic competition among managers, as in our case the product design determines the characteristics of the competition among managers, and not the other way around. Consequently, our setup allows to investigate the impact of a certain choice for information product design (the fund structure) on the levels of delegation and on the market efficiency.

Importantly, it will not be possible to assign a benchmark when the fund structure can be altered deliberately. An example is the difficulty to represent the sector of hedge funds, or the private equity funds. These funds are notorious for having an uncertain composition and commanding performance fees, which suggests that their managers seem more similar to agents exercizing an effort as in (Stiglitz (1974)), than to information sellers. As a result, it is problematic to introduce benchmarks in a strategic competition where managers do not have fixed types. And vice versa, strategic competition is limited if there are benchmarks, because managers have to structure their portfolios around benchmarks.

How does fund competition look in the presence of benchmarks? In our model the only dimension along which funds compete is the fund fee, and benchmarks seem to create a separate pools of competition for each type of managers. Due to complementarity between the managers' signals the demand for individual funds is inelastic, which allows nonzero profits notwithstanding a zero signal cost. In the extreme case, managers might gain a profit without having a signal, for example when investors use a symmetric fund allocations. Further study is needed to analyse the details of fund competition in our model, this is a work in progress.

3

PART 3: Investment delegation as a mechanism for aggregating private information

In Part 3 we show that the delegation institution introduced in Part 1 can be used as a mechanism to implement a fully revealing rational expectation equilibrium in an economy with asymmetrically informed agents.

The motivation for this study comes from the discomforting fact that the rational demand for delegation exists only in the presence of noise trading. Without the inefficiency in the asset markets there is no incentive to pay managers for their private information, as it would become available to everybody via market prices. Noise trading, which entered asset market models as an auxiliary hypothesis (De Long et al. (1990)), seems to have become a foundation for many essential phenomena in asset markets such as limits of arbitrage that prevent asset prices to converge to fundamental asset values.

There are conceptual difficulties with noise trading that might obscure the analysis. Although noise trading could be in part related to rational activities of some market participants, such as rebalancing or hedging, in many cases it hides behavioral biases that the model does not attempt to address explicitly. Because noise traders do not behave rationally, they are not assigned a precise identity or a utility function. Thus the welfare of these market participants cannot be assessed, which makes welfare comparisons across model equilibria ambiguous. Also, the asset supplies being random in the presence of noise trading, it is difficult to define an observable market portfolio.

One of the most famous paradoxes involving noise trading is the Grossman-

Stiglitz paradox (Grossman and Stiglitz (1980)) about the impossibility to provide incentives for information acquisition in the absence of the noise trading. Less known, there exists another related problem with a wide set of models with privately informed trading: the implementation problem. When agents are already privately informed (having acquired information at a zero cost, or considering their information cost as sunk) they may fail to trade on their private signals in the absence of noise trading, expecting the prices to be better predictors of the future payoffs than their signals (Diamond and Verrecchia (1981)). Thus the noise is necessary for prices to aggregate the information, and at the same time limits the aggregation by preventing fully revealing equilibria. The implementation paradox is present for some categories of the informational structures, with so-called exclusivity of private information, which happens to be the case for the delegation model we used in Part 1.

The introduction of noise trading eliminates the implementation problem, because agents cannot infer the future assets' payoff from its price. It allows, however, for partially revealing equilibria only, preventing markets to be informationally efficient. Garleanu and Pedersen (2018) even coined a term "efficiently inefficient" to highlight that some level of inefficiency in asset markets is necessary to support the existence of equilibrium with privately informed trading and delegation.

The situation where a social dilemma prevents the markets from reaching efficiency provides an opportunity to form an institution helping to coordinate the behaviour. We explore here how the introduction of the delegation institution might serve as a mechanism to incorporate private information into asset prices when it is impossible to achieve informational efficiency through decentralised asset trading alone. The intuition is that, when trading on a private signal directly in the absence of noise trading, an agent has to suffer a loss in utility due to the signal error. When, instead, the trading in assets is done using the fund market, neither the manager acting on the signal, nor the fund investors bears the signal risk. The manager acts on behalf of others and thus is insensitive to the signal error, and the investors are able to diversify the signal risk completely investing simultaneously in many funds.

If the market for funds were to be viable in the absence of noise trading, it would lead to informationally efficient asset prices. Moreover, the market portfolio in such a case would become investable and would be optimal. The efficiency of the market portfolio would help to explain benchmarking, i.e. investing directly in a fixed-composition portfolio, the phenomenon observed in the market. The viability of the delegation in such a case could be problematic because investing in the market portfolio, equally attractive, will be accessible to everybody at a zero cost. We address the tension between paying for the service of fund managers and free-riding with a market portfolio by building a *public good game*.

Our analysis has so far reached several tentative conclusions. The markets where fund management is introduced will be in general more robust as implementation mechanisms for fully revealing equilibria than asset trading alone. A simple public good game outcome might be compatible with a simultaneous existence of delegation and benchmarking in the asset markets, although more thorough analysis and a consideration of fund management costs are needed to extract meaningful predictions about the relative importance of delegation and benchmarking.

3.1 Implementation problem for fully revealing equilibria

The very act of trading on private information transmits the information through prices to other market participants. Because the information revealed through prices is of higher quality than any individual signal (Grossman (1976)), the information transmission feature of asymmetrically informed markets is generally beneficial for market participants. In these economies the private information might be in principal fully incorporated in prices through decentralized trading. (The equilibria with full revelation of private information are called FRREE, the term stands for fully revealing rational expectation equilibrium). An extensive literature on the existence of competitive rational expectation equilibria established equivalence between the equilibrium in a FRREE economy and a Walrasian equilibrium in an artificial economy where all private information is made public (Grossman (1978, 1981); Anderson and Sonnenschein (1982)).

A formal existence of fully revealing equilibria does not guarantee, however, that they are implementable through decentralized or even center-mediated trading mechanisms (see Vives (2007), section. 3.1.2). The main trading mechanism explored in the implementation literature is a direct-revelation game where agents communicate their signals to the center and the latter assigns individual allocations based on the revealed signals. If one can prove that all agents in this game will have rational incentives to truthfully reveal their signals, then the FRREE equilibrium is said to be implementable Blume and Easley (1990).

Implementation of a revelation game breaks down, however, if agents have incentives to lie about their signals. Thus a truthful revelation might not be a Nash equilibrium. It was shown that the condition preventing the strategic behavior is the requirement of so-called non-exclusivity of information Postlewaite and Schmeidler (1986); Blume and Easley (1990). There are two versions of the non-exclusivity condition. The first Postlewaite and Schmeidler (1986) requires that the random state should be fully revealed *before* the center asks the agent m about his signal. The second Blume and Easley (1990) requires that all other agents in aggregate are able to recover the signal of trader m from their private information. The two definitions convey the notion of "informational smallness", but they do not coincide.

Compare two following information structures for a continuum of investors labeled by $m \in (0, 1)$ as follows:

1. *Info 1*: $s^m = s = \theta + \epsilon$, i.e. all informed agents have the same signal, and
2. *Info 2*: $s^m = \theta + \epsilon^m, \quad cov(\epsilon^m, \epsilon^{m'}) = 0, \quad \int_0^1 s^m dm = \theta$, i.e. all signal errors are independent and the aggregated signal reveals the state θ.

Both *Info 1* and *Info 2* satisfy the first definition of "smallness", that is, no individual trader is material for the revelation of full information. However, the *Info 1* differs from *Info 2* in the sense of the second definition of non-exclusivity. An agent in *Info 1* simply cannot lie about her signal to the others, because other agents in aggregate already know the signal with certainty. Instead, with *Info 2* , there is no way to recover the realization of the signal s^m knowing all other signals, because individual signals are not perfectly correlated.

The question is whether the economies with *Info 1* and *Info 2* information structures are implementable in the sense of direct-revelation games. The FRREE in *Info 1* economy is clearly implementable, both through a revelation game and via a decentralized auction. Such information structure was used in the model of Grossman and Stiglitz (1980). Since the total private information in this case is the same as the individual information, informed agents will act on their signals and submit demand functions that depend on it. Their trading will ensure that the information s is reflected in prices. As highlighted by Diamond and Verrecchia (1981), in such an economy only a transmission of information exists, but there is no aggregation.

Instead, implementation of equilibrium in the economy with *Info 2* is not feasible. In the context of a revelation game with *Info 2*, even if no investor has rational incentives to lie, no agent will have a rational incentive to tell the truth about her signal either, because individual expected utilities will not depend on the private signals. No punishment can be implemented for a lie, because the peers cannot detect a lie (the second definition of "informational smallness" is violated).

A revelation game might seem an abstraction, but there actually exist real examples of a failure to aggregate private information through a center-mediated mechanism because of the information exclusivity. Gambetta (1993) (pp.220-225) gives an example of the collapse of the market of radio-dispatched taxis in Palermo in the 1980s, when taxi drivers deceived the operators by quoting too short pickup times in order to get the rides. Only a special system of checks where taxi drivers could respond to a call and quote a pickup time only when they were parked at a designated parking lot and observed by their peers, restored the functioning of the market. Using the terminology of revelation games, the new allocation system worked because the private information became non-exclusive.

In asset markets with exclusivity of private information agents will have strong incentives to learn from the price $P(\theta)$, and, given that the price is a sufficient statistic for any pair (P, s^m), $\forall m$, individual signals will disappear from the traders' demand functions. The indifference to individual signals compromises a "strong", rationally supported implementation of FRREE for *Info 2* economy. For example, the proof of Laffont (1985) of the existence of fully revealing FRREE will not hold if one eliminates the assumption of nonzero sensitivity of trader's utility to their private signal.

Anderson and Sonnenschein (1982) proposed an implementation via an implicit coordination of individual models that agents may refine on the basis of empirical observations. But they conclude that the information an agent possesses will not be transmitted to the price unless it alters his demand function. As a consequence, REE is only implementable in the model of Anderson and Sonnenschein (1982) when asset supplies are noisy.

Summarizing, an implementable equilibrium in an economy with asymmetric information crucially depends on the informational structure. When prices are expected to aggregate the full private information, implementation of a fully revealing equilibrium might not be possible, because agents become insensitive to their private signals.

Note, that the implementation difficulty is different from the Grossman-Stiglitz paradox, since the information is given to traders exogenously. Here the prices fail to be informationally efficient not because the agents have no incentive to acquire information, but because they fail to trade on the information they already possess, hoping to profit from a more precise information contained in the equilibrium price.

The standard remedy for the implementation failure in the asymmetric information economy is to introduce noisy asset supplies, which makes it impossible to fully reveal the aggregated private information via prices and thus preserves the incentive to trade on individual signals. Thus many findings about rational exectation equilibrium are obtained exclusively in the noisy REE setup Admati (1985); Vives (2007).

3.2 Costless delegation as an implementation mechanism

We argue here that the introduction of a fund market with zero fees represents a feasible decentralized mechanism for implementing a rational expectation equilibrium without the need to resort to noise trading. In addition, in the economies with costless delegation the equilibrium prices will fully reveal all private information.

The central issue facing economies with the information structure of *Info 2* type is that it is not rational to act on the individual signal s^m, because conditioning on price $P(\theta)$ is available, and the price is a sufficient statistic for the pair $(s^m, P(\theta))$. Nobody is willing to accept a loss in utility by holding an allocation that contains the signal error ϵ^m.

Delegation institution avoids the problem by dissociating the agents' trading actions in the asset market from their utility. Agents, acting as managers, would submit demand schedules to the asset market based on their individual signals, because they trade on behalf of investors and do not experience a loss in utility due to the signal error. Investors, who invest in all available funds, do not assume signal risk because they diversify across managers. In both of their roles, as investors, and as managers, agents act rationally. As a result, individual signals get incorporated into the asset prices.

As we showed in Part 1, in an ideal setup with perfect knowledge of all economy parameters, including the agents' signal precisions, the delegation mechanism results in the same equilibrium price as an economy where the private information is publicly disseminated, notwithstanding the presence of noise trading.

The implementation via delegation is "weak" in the sense that there is no self-enforcing feature that would motivate traders to strongly prefer delegation to direct trading. Once agents know that mono-signal funds are operating, they would infer that the prices are fully revealing and will be indifferent between investing in funds at zero fee or reverse-engineering the full information directly from price and investing in assets directly.

Yet, delegation has several important advantages over other implementation mechanisms, such as revelation games or trading schemes allowing direct trading only:

1. even if one investor decides to skip delegation and invests in the markets directly extracting the perfect signal from the price, the private signal of this investor will not be lost, because he will continue to submit demands based on his signal on behalf of the investors of his fund. An individual signal is lost only if the individual chooses to free-ride, and if nobody invests in the fund of this

individual.

2. when knowledge of the structure of the economy is not perfect, delegation might be strictly preferable to direct investing,

3. delegation allows to avoid explicit communication of the signal to the center for aggregation. As noted by Hayek in his seminal paper on the use of knowledge in economy Hayek (1945), a part of private information might be in a non-communicable form.

4. when delegation fees are introduced, asymmetrically informed traders might actually profit from their information by selling investment funds for a fee. Thus, there might be an endogenous incentive to acquire information in an economy with delegation, even without noise trading.

Thus delegation sustains meaningful asset trading in the presence of private information, results in informationally efficient market prices and, if delegation fees are introduced, allows privately informed agents to extract value from their information by offering investment funds to investors. In an economy where full revelation of private information will follow, agents cannot profit from their information by direct investing. The only way to extract value from the information is to sell a service based on it, an investment fund.

In order for the delegation to be viable, however, one has to resolve the tension between the need to remunerate managers for their role in making the prices efficient, and the possibility to free-ride on the managers' services by investing directly in the efficient market portfolio.

In what follows, we develop a more formal setup to study delegation with nonzero fees, including separate stages for marketing the funds, concluding binding commitments to invest in funds, and trading in assets.

3.3 Structure of the delegation mechanism

We assume that a delegation arrangement consists of three stages: pre-contracting stage, delegation stage and asset trading stage.

During the pre-contracting stage privately informed agents hold portfolios conditionally optimal with respect to their own private signals and refrain from deducing the signals of others from asset prices. During this stage they act as managers who are launching their funds with their own "seed" money and building the track records for their funds. We assume, as we did in Part 1 and 2, that signals are obtained at no cost, so the only loss that agents suffer during the pre-contracting stage is the loss in utility by ignoring the informational content of prices. We also assume that during the pre-contracting stage the uninformed agents are free to condition on prices. At the end of the pre-contracting stage, which could last several "rounds", there is a collection of data accumulated about the performances of different funds. Based on the performance data any agent will be in principle able to deduce the precision of the signal from the fund's performance record, using, for example, a regression routine described in Part 2.

The delegation stage begins with agents approaching their peers and offering to invest in their funds for a fee during the subsequent asset trading round. Fund managers make a promise to invest the delegated money strictly according to the same signal that was used for the fund during the pre-contracting stage (that is, fund managers will continue to refrain from inferring the signals of others from the prices). Investors decide what portion of their own wealth to delegate, and what portion to invest directly in asset markets. At the delegation stage none of the agents is able to observe the next period signals or the future prices of the risky assets. Investor and manager sign contracts to safeguard the promises.

The asset trading stage begins by revealing the private signals. Then managers submit the fund demands according to their signals and all agents submit demands for their own accounts. When trading for their own accounts, agents are free to condition on price. It is possible that some agents will not be willing to participate in the delegation stage, hoping to free-ride on the aggregate private information during the market session without having to pay a fee to the managers. Although the delegation arrangement will not eliminate the information free-riding completely, an equilibrium with aggregation of private information is possible when at least some of the agents decide to delegate.

There are several problems that may perturb such an arrangement:

- Managers might not respect their promises to incorporate the private signals in the fund portfolios. They might prefer either to ignore their signals, or, to try to guess the signals of others. In both cases the drift in the fund's approach is a problem for investors, because their decision to delegate to a particular manager is based on the manager's precision and on the way it is related to the precisions of others. The possibility of the "style drift" is real and investors should plan for this. Though there is no clear incentive for the manager to deviate from the performance promise, the investor might add to the contract some observable performance metrics enhancing the compliance. First, there might be a penalty for performing exactly like the market portfolio (or other suitable benchmark portfolio) to avoid the ”closet-indexing” phenomenon[1]. Second, there might be a penalty on too large deviations from the benchmark (e.g. using the tracking error limit).

- It is important to specify the outside option of the game, that is, what happens when nobody agrees to delegate. The first possibility is the unimplementable equilibrium, where the market breaks down because everybody tries to guess the aggregate information and fails to act on the own signal. The scenario of market failure does not have a unique expected payoff, because everybody submits the order for the market portfolio at any price (so no price schedules can be coordinated). The resulting price is arbitrary. One possibility is to associate with the market failure a payoff equivalent to investing in the riskless asset.

  The second scenario is to assume that, when the delegation breaks down, informed investors ”forget” about their signals (for example, discontinue signal licenses). Then the agents in the economy become all uninformed and a CAPM equilibrium follows. One problem with the CAPM outside option is that, from the welfare point of view, the informational efficiency is not always paired with Pareto efficiency. It might happen that too much information is revealed, eliminating insurance opportunities (Laffont (1985)). The exact welfare comparison of the equilibria with and without the aggregated private information will also depend on the assumption about the supply of the riskless asset. We will assume for the rest of this Part that the outside option gives a payoff inferior to that of the fully revealing equilibrium. We then address the welfare analysis in appendix 3.A.

Given the difficulties outlined above, we proceed by assuming that managers comply with their investment mandate. We assume, without the loss of generality, that there is a threshold delegation level $\gamma*$ that managers require in order to operate their funds. If an investor considers delegation, given that the current delegation level is below $\gamma*$, signing or not the contract with the managers gives the same outcome. Signing the contract when $\gamma > \gamma*$ is not convenient, because the investor is sure that managers will incorporate the information in prices and he will opt for free-riding on the information. One may argue that the only non-trivial decision is at $\gamma = \gamma*$.

[1]Another possibility is a relative performance bonus.

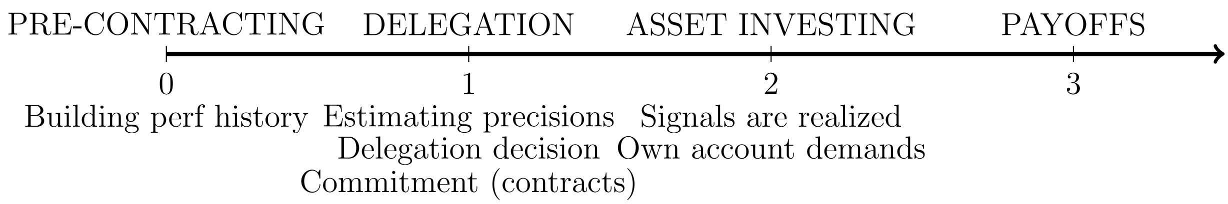


Figure 3.1: The stages of the delegation game

3.4 Equilibrium at the pre-contracting stage

To create the demand for delegation, informed investors refrain from conditioning on the equilibrium price for a certain amount of investment "rounds" and invest their wealth in the mean-variance efficient portfolios conditioned only on their private signals. We assume that there is a continuum of investors labeled by $m \in (0,1)$, each possessing a private signal $s^m = \theta + \epsilon^m$ with the signal error variance $\Sigma_m = Var(s^m \mid \theta)$. We also assume that a portion ω of the continuum of investors has no private signal $\Sigma^{-1} = 0$. These investors do not plan to sell their services as fund managers in the future, and thus do not impose on themselves the constraint of not extracting the signals of others from the prices. In this section we look at the equilibrium during the pre-contracting stage and discuss how investors evaluate performance records of the peers' funds.

We assume that there is a residual uncertainty in the market, as the assets' payoffs D are not perfectly determined by the aggregate private information:

$$D = \theta + \epsilon_D, \quad \epsilon_D \sim N(0, V_D) \tag{3.1}$$

The equilibrium is given by the optimal portfolios of informed investors x_0^m conditioned on their private signals only, by the portfolios of uninformed investors x^u that condition on the market prices, and the equilibrium prices. We assume that all investors have the same CARA preferences and share the knowledge of unconditional probability distribution of risky payoffs $D \sim N(\bar{\theta}, V + V_D)$:

We introduce the following definitions. The expected value of θ conditional on a private signal s^m:

$$\theta^m = \mathbb{E}(\theta \mid s^m) = (V^{-1} + \Sigma_m^{-1})^{-1}(\Sigma^{-1}s^m + V^{-1}\bar{\theta}) \tag{3.2}$$

The expected value of θ conditional on price P:

$$\theta(P) = \mathbb{E}(\theta \mid P) = A_1^{-1}(PR - A_0) \tag{3.3}$$

where A_i are coefficients of the linear price functional:

$$P = A_0 + A_1\theta \tag{3.4}$$

The equilibrium asset demands then are defined as follows:

$$x_0^m = \frac{1}{\rho}E^m(\bar{\theta} - PR) + \frac{1}{\rho}O^m(s^m - PR) \tag{3.5}$$

$$x_u = \frac{1}{\rho}V_D^{-1}(\theta(P) - PR) \tag{3.6}$$

where $E^m = V^{-1}\left(1 + V_D(V^{-1} + \Sigma_m^{-1})\right)^{-1}$, $O^m = \Sigma_m^{-1}\left(1 + V_D(V^{-1} + \Sigma_m^{-1})\right)^{-1}$. The equilibrium price is found from the market clearing condition:

$$z = \bar{E}(\bar{\theta} - PR) + \bar{O}(\theta - PR) + \frac{\omega}{\rho}V_D^{-1}(\theta - PR) \tag{3.7}$$

which gives:

$$A_0 = (\bar{O} + \bar{E} + \omega V_D^{-1})^{-1}(\bar{E}\bar{\theta} - \rho z) \tag{3.8}$$

$$A_1 = (\bar{O} + \bar{E} + \omega V_D^{-1})^{-1}(\bar{O} + \omega V_D^{-1}) \tag{3.9}$$

where $\bar{E} = \int_0^{1-\omega} E^m dm$, and $\bar{O} = \int_0^{1-\omega} O^m dm$.

During the pre-contracting stage informed agents are supplying a public good, because the market clears with informationally efficient prices. The benefit that uninformed investors rceive from the trading of informed fund managers can be measured by comparing expected utilities of the uninformed unconditional portfolio $x_u^0 = \frac{1}{\rho}V^{-1}(\bar{\theta} - PR)$ to the conditional uninformed portfolio (3.6) in the presence of informed trading.

3.5 Delegation game

During the contracting stage fund managers offer investors to buy shares in the funds for a fee. Investors who choose to buy the shares will have to sign binding contracts with the managers, thus securing an indirect access to the full private information. Investors who choose not to invest in funds will have an opportunity to wait until the trading stage when the asset prices will reveal the full information, and build an informationally efficient asset allocation by themselves.

If the fund fees are zero, the two alternatives (delegating and free-riding) will be equally attractive. If one considers informational efficiency of the market to be a public good, one would say that investors can supply the public good by delegating to managers at no cost. Though the choice to delegate does not strictly dominate the choice to free-ride at the individual level, it is unlikely that the public good will not be supplied in the case of zero fund fees. If investors are undecided between delegating and free-riding, they could just make their choice randomly, for example flipping a coin. The public good will be supplied even if a fraction of investors end up delegating, as long as all the funds continue to operate. Using the terminology of the public good games, the strategy to supply the public good, though not dominant, is reasonable if fund fees are zero.

With nonzero fees $k^m \neq 0$, the free-riding alternative would be a rational choice for an individual investor, but will be suboptimal from the collective point of view. If no individual will be willing to pay for the funds, the supply of the public good will be disrupted. The situation fits the description of a social dilemma, where individual incentives go against what is the best for the group. The contracting stage thus turns into a strategic game with multiple players, where everyone's decision to supply the public good will depend on the expectations about the decisions of others.

The common intuition in economics is that in a situation of a social dilemma the public good will be undersupplied and the Nash equilibrium will be Pareto-inefficient. Yet, as highlighted by Archetti and Scheuring (2012), the common intuition might be misleading, because it is built on one particular model of the public good, namely the linear contribution model. In the linear model the value of the public good grows linearly with individual contributions. The dominant equilibrium in such game is when everyone chooses to free-ride and the public good is undersupplied or not supplied at all. The outcome might be quite different, however, for nonlinear models of public goods, which fit better the pure public good representing market efficiency. With nonlinear models there are Nash equilibria when the public good is supplied at the optimal level (Palfrey and Rosenthal (1983)). An extensive theoretical and experimental literature employed nonlinear public good models to explain the phenomenon of private provision of public goods without recurring to behavioural explanations, such as "warm glow", or other-regarding preferences. According to the literature on nonlinear games, "incentive-compatible institutions" might exist and the production of the public good may be feasible (benefit exceeds the cost), dominant, or reasonable.

3.5.1 Private provision of public goods

Here we briefly review the main features of a public good game where the good in question is provided only when the level of contributions meets a certain threshold.

This form of the public good production function is known in the literature under the names of a discrete or threshold public good (Palfrey and Rosenthal (1983)), or a volunteer's dilemma (Diekmann (1985), Bliss and Nalebuff (1984)).

A typical threshold game involves multiple players who have to choose between two strategies: contribute to the public good paying a cost c, or to defect. The public good, without the loss of generality, is assumed to bring utility 1 to each player, regardless of the individual contribution. The good is provided only if the threshold of w contributions is met. Assume for the moment that there is a finite number N of players and denote the actual number of contributions by n.

In order to characterise the equilibria in a threshold game, several other game features have to be specified: the refund of the contribution if the good is not provided, the possible set of strategies (for example, if mixed strategies are allowed), and the possibility for the players to communicate. The possibility to get a refund could have material consequences for the provision of the public good, because it eliminates the "fear" to lose the contribution (van de Kragt et al. (1983)). The possibility to use mixed strategies turns out to be important, because it allows to build equilibria where the good is provided in a symmetric way without dividing the group in pure "contributors" and pure "defectors". The possibility to communicate might be crucial to allow the implementation of asymmetric equilibria, where certain players volunteer in front of the others, or are publicly designated by the group to contribute.

The main takeaway from the literature on threshold public goods is that these games possess multiple equilibria, and in many of the equlibria the public good is provided. Thus, even if the opportunity to free-ride is attractive individually, the strategic motives to cooperate are strong enough to ensure the provision of a threshold public good. The main intuition for the existence of a good equilibrium is as follows. If the number of contributing players is observable, then in equilibrium exactly w investors will contribute. Near the equilibrium the choices are strategic substitutes: if a player thinks that too many of the peers will free-ride, he prefers to contribute; and if he thinks too many players will contribute, he should free-ride (Camerer and Fehr (2006)). Thus there exist equilibria with exactly w contributors, which ensures the provision of the public good, and at the same time results in Pareto-efficiency, because the exact minimum of contributions is met. Would there be $n > w$ contributions, the public good will be still provided, but the contributions above the threshold would be wasted.

The implementation of an asymmetric Pareto-efficient equilibria in a threshold game requires some mechanism to designate the contributors. There are two main options to do so: relying on the heterogeneity of players, or allowing communication. If the players differ in their characteristics, such as wealth or the cost of the provision of the good (Bergstrom et al. (1985), Palfrey and Rosenthal (1983)), the volunteers could be naturally singled out as the wealthiest players (Palfrey and Rosenthal (1983)), or the least-cost contributors (Bergstrom et al. (1985), Bliss and Nalebuff (1984)). Otherwise, if the players are all alike, there is a possibility to allow the group to communicate and decide collectively who will be the volunteer (van de Kragt et al. (1983)). This way each designated volunteer will be aware of the criticality of his contribution to the provision of the public good, and the dominant strategy of the designated volunteers will be to contribute.

The assumptions of heterogeneity of the players or of the possibility to communicate are not always met. For example, for big groups (and, possibly, infinite groups) communication might be practically infeasible, or unreliable because of even a tiny probability of the presence of irrational individuals in the group. Irrational volunteers wight miss the contribution even if it is in their individual interest, and thus break down the supply of the good. The bigger the group the greater the probability of having at least one such individual present. The assumption of heterogeneous costs is not relevant to our model where the cost of he provision of the public good, in our case the fund fees, is the same for everyone. Thus is it important to consider possible equilibria in mixed strategies, where no individual chooses a pure strategy to "only free-ride" or to "only contribute".

Equilibria in mixed strategies are realized when player "m'" decides to contribute to the public good with a certain probability $q_{m'}$. If the equilibrium value of the probability is the same for everyone ($q^*_{m'} = q$), the resulting equilibrium is symmetric, meaning there is no need to sort players into volunteers and defectors, and no need to introduce communication. The mixed equilibria were shown to be inferior from the point of view of Pareto-efficiency (Palfrey and Rosenthal (1983)), but they often result in the provision of the public good.

To conclude, there is a fair chance that the delegation game, where investors choose between paying managers to produce informationally efficient prices, and free-riding by investing directly in risky assets at efficient prices, will result in the provision of the public good. The informationally efficient prices will be enjoyed by the delegating and the free-riding investors alike.

3.5.2 Is delegation a threshold game?

Armed with the intuition that informationally efficient markets could be implemented even if individuals have incentives to free-ride, we have to consider carefully if the the delegation game is indeed a threshold public good game. The game at the delegation stage is obviously more complex than a basic threshold game. Managers have strategic motives in the definition of the fees, making the cost of the public good an endogenous variable. Moreover, investors and managers are not separate groups, so each player simultaneously chooses his strategy as an investor and as a manager. To move forward, we begin by defining a simplified delegation game, where the fees are fixed exogenously, so only the individual decisions of investors matter. We would later transform the game to endogenize the fees. Let's assume for the moment that there is a finite number N of investors. We would then take the limit $N \to \infty$.

To define the delegation game with exogenous fees as a threshold public good game, we need to specify its ingredients: the characteristics of the public good, the payoff from the public good, the cost of the contribution to the public good, the threshold for the provision of the good, as well as the features like the possibility of the refund, the introduction of communication and the use of mixed strategies.

It is fairly straightforward to see that the informational efficiency of prices satisfy the properties of a public good: it is non-rival, meaning that several people could trade on efficient prices simultaneously, and is non-excludable, meaning that once the prices are efficient there is no way to prevent non-contributors from trading on these prices. The payoff from the public good is the same for all investors, because in the absence of noise all agents hold the same informationally efficient allocation, irrespective of their decision to contribute (i.e. delegate) or free-ride (i.e. invest directly in assets). If the public good is not supplied, it is plausible to assume that every investor gets a payoff of zero, which would correspond to a failure of the markets to clear.

All players in the game are aware of the fact that buying the funds is akin to contributing to the public good, in the sense that the contribution does not bring any private benefits beyond the value of the public good of informational efficiency. The cost of an individual contribution to the public good is the total fee that an investor pays to fund managers. Given the linear structure of the fund fees, the fee for investing in one fund m' is equal to the fee per share $k^{m'}$ multiplied by the optimal number of shares of the fund to buy $\gamma^{mm'}$, where m labels investors. In the absence of noisy trading, all investors will have the same demand for funds, because they would stop trading on their private information for the personal account. If the prices are efficient, one cannot achieve a superior allocation to that of an uninformed investor. (Yet, the heterogeneous private information will still matter for the players when they consider their strategy as managers.) Consequently, the fee is independent on m:

$$f = \int_0^1 \gamma^{m'} f^{m'} dm' \tag{3.10}$$

Thus, all players in the public good game have the same cost f of supplying the public good. The game thus differs from the volunteer's dilemma of Bliss and Nalebuff (1984) where the costs were heterogeneous, leading to an equilibrium where the least cost providers volunteered to contribute to the public good.

The minimal level of contribution needed to provide the public good is the minimal total fee that should be paid to fund managers. Since in our model there are neither information acquisition costs, nor fixed fund management costs, there is no obvious candidate for the threshold. One could not rule out a priori that the managers will accept managing the funds for only one investor. In the case of a finite number of players such an assumption would lead to a threshold of just one player, with the minimal contribution being equal to f. Below we will investigate the game properties with an exogenous threshold of $w * f$. Although a manager would not lose money by serving just one client, there might be a strategic incentive to raise the minimal number of required contributions above one, given that, if the public good is provided in the equilibrium, the number of contributions will be around the threshold. The threshold here gives another source of pricing power to managers, who can threaten to withhold the public good to extract more contributions.

It is natural to assume that the fund fees paid by investors will be refunded when the total number of contributions falls short of the threshold. The refund is costless to managers, as no part of the fees is used by managers as an upfront expense to provide the funds. It also reflects the fund management practices when the fees paid upfront are reimbursed when the delegation contract is terminated early. The possibility of a refund eliminates the "fear" motive to free-ride, as pointed out by van de Kragt et al. (1983), although it was shown to make little difference in the equilibrium provision of the public good.

Finally, we do not allow group communication before the delegation stage. Such communication could be in principle used to select the volunteers who will be expected to contribute. Such coordination of expectations was shown to be crucial in implementing asymmetric equilibria in public good games, where some players select to contribute, and the others to free-ride. van de Kragt et al. (1983) reported that the possibility of communication led to the provision of a threshold public good in 100% of the game rounds, versus to about 30% when there was no communication prior to the game. When communication was allowed, players used three mechanisms to designate the contributors: lottery, volunteering, and need basis, with the lottery being the most common mechanism. It is difficult though to imagine any of these three mechanisms of communication to be implemented in the fund market, especially if we are interested in large markets with potentially infinite number of players.

In the absence of communication it is important to retain the possibility to play mixed strategies, when any given player m does not elect a pure strategy to contribute or to free-ride, but chooses his action randomly, with a probability q_m to contribute. We allow for such strategies in our game, and will consider two distinct cases: an a priori symmetric case where $q_m = q$, and a case where each player chooses his optimal q_m^* in equilibrium.

3.5.3 A delegation game with fixed fees and exogenous threshold

Below we give the individual payoff matrix for the public good game with exogenous threshold w. The payoff from the public good is equal to 1. The individual cost of contribution to the public good is equal to f. The cost should be less than $f < 1$ in order for the private provision of public good to be feasible. The contributions are refunded if the threshold is not met.

	$n \leq w$	$n < w - 1$	$n = w - 1$
$Contribute$	$1 - f$	0	$1 - f$
$Free - ride$	1	0	0

(3.11)

We consider here the equilibrium in mixed strategies, when each player contributes with probability q. Such equilibria have desirable property of being symmetric, eliminating the need to specify the mechanism of designating the contributors.

We will look for the answers to the following questions concerning the game above:

1. If players in the threshold public good game use mixed strategies, what is the probability that the public good will be provided?

2. How the probability to provide the public good depends on the threshold?

3. Is there an optimal threshold $w*$ where managers maximize their profits?

Let's denote by n_i the number of contributions excluding the contribution of investor i. The expected payoffs for investor i is derived from the expected payoff from contributing $\mathbb{P}(n_i \geq w - 1)(1 - f)$ and the expected payoff from free-riding

$\mathbb{P}(n_i \geq w)$. In equilibrium the investor i is indifferent between the two options, which defines the condition for an optimal contributing probability q:

$$(1-f)\mathbb{P}(n_i \geq w-1) = \mathbb{P}(n_i \geq w) \tag{3.12}$$

Rearranging the expression one gets:

$$f = \frac{\mathbb{P}(n_i = w-1)}{\mathbb{P}(n_i \geq w-1)} \tag{3.13}$$

With the common probability q for an individual contribution, the number of contributions follows a binomial distribution:

$$\mathbb{P}(n_i = w-1) = \frac{(N-1)!}{(w-1)!(N-w)!} q^{w-1}(1-q)^{N-w} \tag{3.14}$$

Substituting (3.14) into (3.13), and simplifying the expression somewhat one obtains:

$$\frac{1}{f} = \sum_{i=w-1}^{N-1} \frac{(w-1)!(N-w)!}{(N-1-i)!(i)!} \left(\frac{q}{1-q}\right)^{i-w+1} \tag{3.15}$$

The indifference condition (3.13) thus defines a function $F(q; w, N)$. The inverse function $Q(f; w, N)$ exists (see Palfrey and Rosenthal (1983)). In the case of arbitrary w it might not be possible to derive the functional form of $Q(f; w, N)$ explicitly. Yet, one could always solve the condition (3.15) numerically.

Once the optimal probability of contribution q^* is specified, one could compute the probability of the provision of the public good as $\mathbb{P}(n \geq w)$.

Let's calculate the probability of the provision of the public good in a simple example of $w = 1$. The game where only one contribution is sufficient to provide the public good is also known as the volunteer's dilemma (Diekmann (1985), Bliss and Nalebuff (1984)). It is easy to derive the optimal probability to contribute $q^*(w = 1)$ directly from the original indifference condition (3.13), noting that the probability in the denominator is equal to 1:

$$f = \mathbb{P}(n_i = 0) = (1-q)^{N-1} \tag{3.16}$$

Therefore,

$$q^*(w = 1) = 1 - f^{\frac{1}{N-1}} \tag{3.17}$$

The probability of the provision of the public good if players use mixed strategies is:

$$\mathbb{P}(m \geq 1) = 1 - (1 - q^*)^N = 1 - f^{\frac{N}{N-1}} \tag{3.18}$$

The probability that the public good will be provided with the threshold of just one contribution, and using mixed strategies only, is thus a decreasing function of the contribution cost f. For large groups the probability is roughly equal to $1 - f$. This result is different from the prediction of the volunteer game of Bliss and Nalebuff (1984), when the probability of the provision of the public good approached 1 when the group size increased to infinity. The crucial difference between our threshold game and their volunteer game is the heterogeneity of the contribution costs in Bliss and Nalebuff (1984), which provides a reason for a waiting subgame where the least cost player will eventually contribute.

In Appendix 3.B we derive several other special case solutions, as well as a general formula easily adaptable for numerical simulations. Here are some results on the behavior of the probability of the provision of public good for different configuration of parameters (w, N, f).

Probability to provide the public good through symmetric mixed strategies decreases monotonically with the cost f (fig. 3.2). The increase of the threshold (in the above example from 1 to 2) does not have a monotonic effect on the probability: the provision of the public good is more likely with just one volunteer if the costs are low, but becomes more likely with two volunteers when the cost per person is high.

To investigate the behavior of the probability as a function of the number of players N, we plot the probability for three values of the threshold: $w = 1, 2, 3$.

The probability to provide public good (fig. 3.3) has a limit for large N. For small w at least, the limit is reached after roughly $N = 100$. The bigger the threshold w, the smaller is the probability of supplying the public good in the limit.

To see how the probability to supply the public good behaves for arbitrary N and w, we plot the numerical solutions obtained by the method in Appendix 3.B.

From the figures 3.4 it is evident that the provision of the public good has high probability in two configurations:

1. too low threshold $w << N$, and
2. too high threshold $w \sim N$.

The two configurations share the same property of criticality, that is a higher likelihood of an individual contribution being critical for the provision of the good.

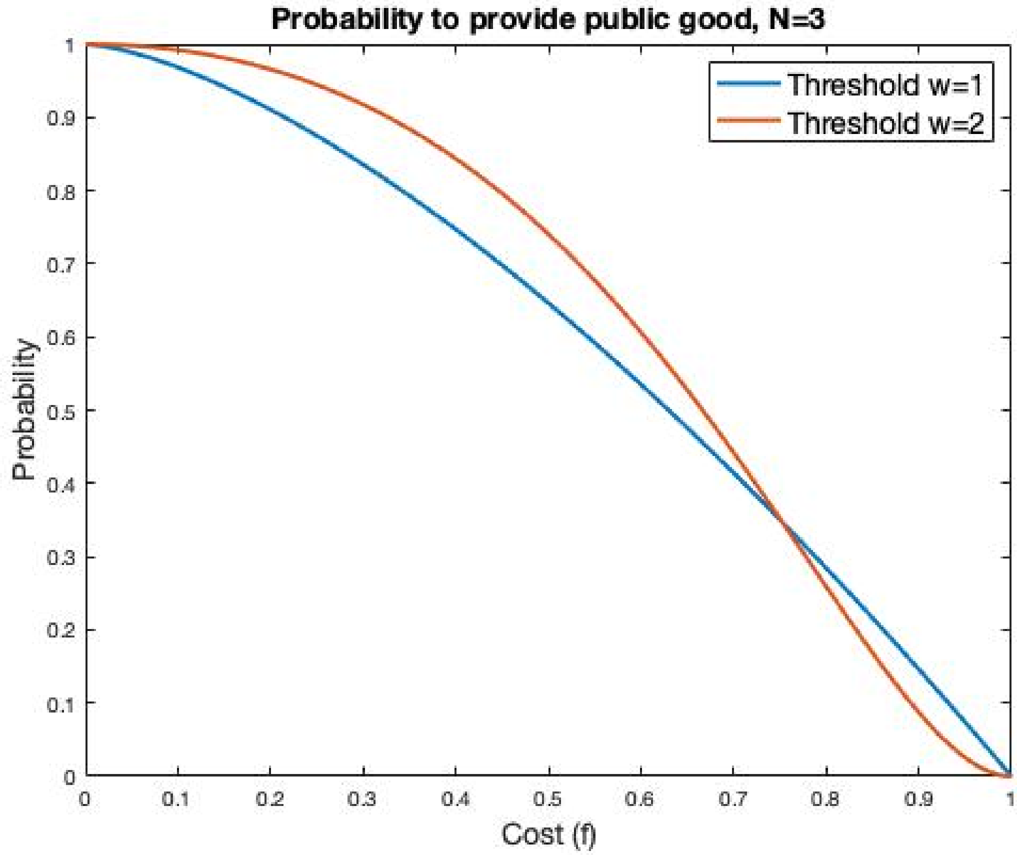


Figure 3.2: Probability to provide public good

Note also that the cost f has a strong effect on the result, with high costs driving the probability to provide public good close to zero for intermediate values of the threshold. For the two favourable configurations of the threshold listed above, however, the good is very likely to be provided even with high costs.

It is important to remind that the threshold game has other asymmetric equilibria in pure strategies, where the public good is provided with certainty. We do not focus on those equilibria here because these would introduce another layer of difficulty, namely specifying the mechanism of appointing contributors.

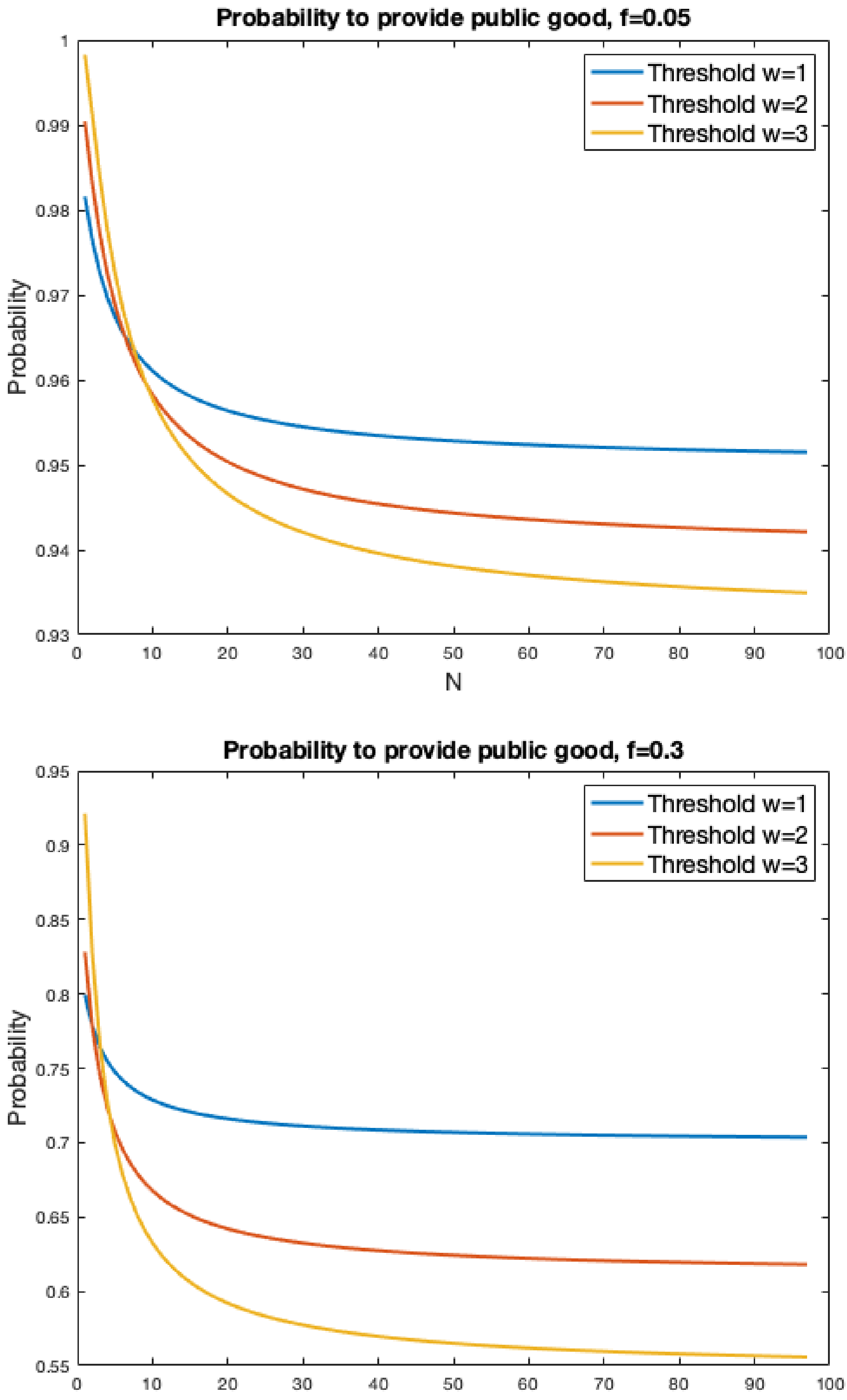


Figure 3.3: Probability to provide public good

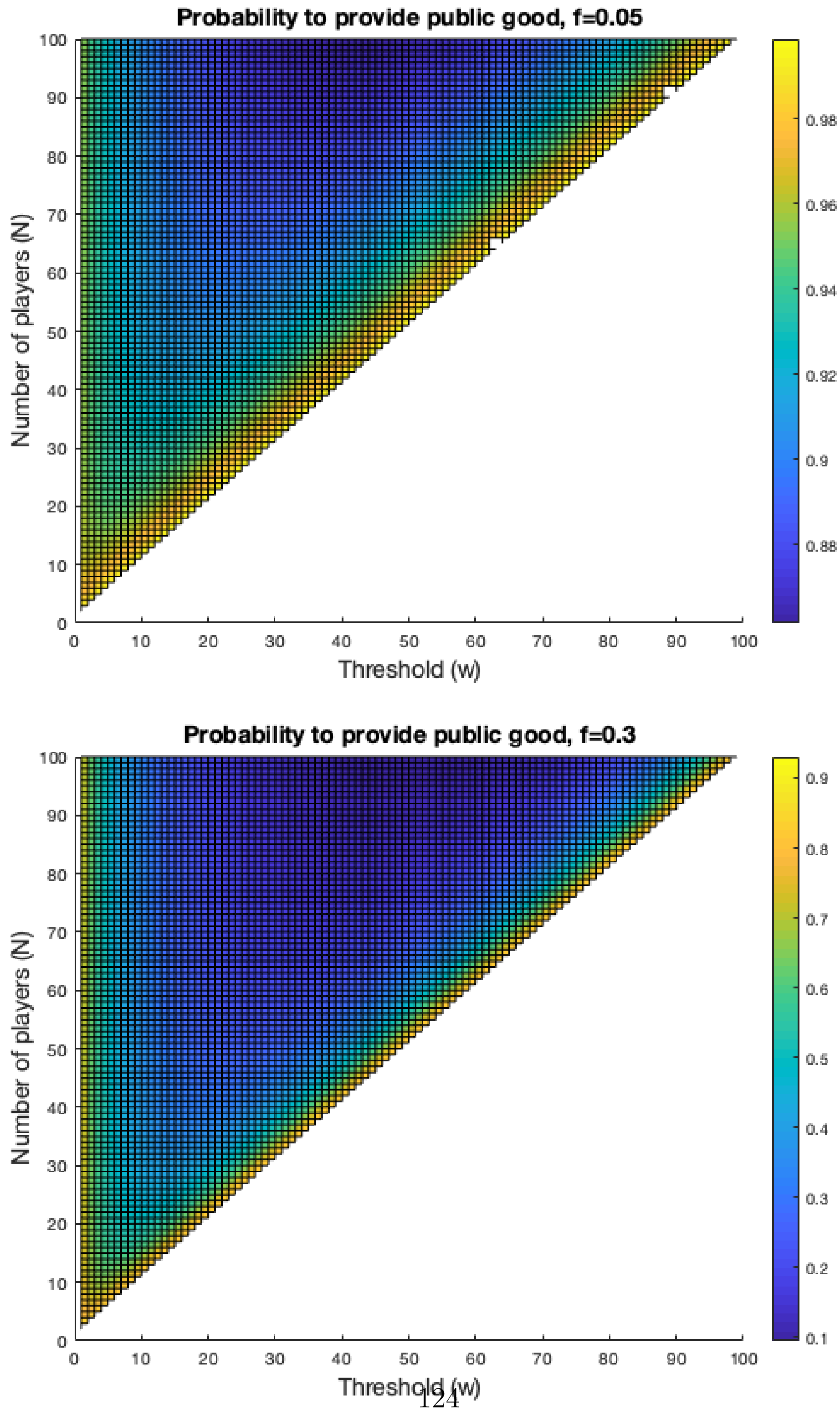


Figure 3.4: Probability to provide public good

3.5.4 A delegation game with exogenous fees and endogenous threshold

The level of the minimal contributions w will be eventually decided by fund managers. Although there is no structural floor for the contributions in the absence of management and information costs, there might be strategic motives for managers to refuse to supply the funds until a certain level of contributions is reached. If a manager expects that in equilibrium the demand for the fund will be around the threshold, it is rational to increase the threshold as much as possible, as long as "good" equilibria, when the public good is provided, are feasible.

Here we consider two possibilities: (i) a homogeneous threshold, and (ii) heterogeneous individual thresholds. The homogeneous threshold (i) could be interpreted as a collective decision of managers, made in order to maximize the total number of contributions to the public good. In the other case (ii) when managers do not coordinate on the same threshold, each of them selects an individual threshold in a subgame. The subgame involves managers anticipating the decisions of other managers and the demand for the funds for each possible threshold. To derive a Nash equilibrium in the subgame one has to specify what will be the demand for the funds when individual thresholds are heterogeneous.

Suppose there are heterogeneous individual thresholds $w_{m'}$. If a number w of investors delegate, only the funds with $w_{m'} < w$ will be active at the asset trading stage. The managers who chose the thresholds $w_{m'} > w$ will close their funds. The managers of the closed funds could choose to trade on their private information for their own accounts. If they do so (or, if investors expect them doing so), then free-riding will become strictly more attractive than investing in the active funds, because the equilibrium price will contain more private information than the active fund allocation. In such a case one would expect everybody to free ride, and the markets will fail to aggregate the private information. Hence, in the case of heterogenous thresholds, the "good" equilibrium is possible only when all funds operate, i.e. when $w \leq w_{m'}^{max}$.

The next step would be to look for an endogenous threshold w^* at which managers get the maximum profit, which we leave for future research.

Summarizing, the exercise of considering delegation in an economy without noise trading inefficiencies highlights the fundamental role of the professionally managed funds, that is making asset prices informationally efficient. It also exposes the tension between the need to remunerate the managers for the provision of the public good (informationally efficient prices), and the incentive to free-ride by forgoing investing in funds and resorting to benchmarking. The nonlinear structure of the public good game in this case leaves a possibility that the good will be provided privately, when at least a portion of agents will volunteer to invest in funds and pay the managers' fees. Such equilibria could be feasible without sophisticated mechanisms of volunteer selection or communication. In particular, several equilibria in mixed strategies described in this Part provide a non-zero probability of provision of the public good and serve as a rough picture of how investing in funds and benchmarking could co-exist.

3.6 Appendix 3.A: On the welfare analysis of the economies with asymmetric information

In economies with asymmetric information one should distinguish between three types of efficiency concepts: ex-ante, interim, and ex-post (Holmstrom and Myerson (1983)).

As shown in Laffont (1985), the rational expectations equilibria, including the fully revealing ones when implementable, might not be Pareto efficient interim and ex-ante.

Suppose the true distribution of risky assets' payoffs is $True = N(\theta, V_D)$ and the publicly known distribution is $False = N(\bar{\theta}, V + V_D)$. The public distribution has higher variance (more uncertainty) and a bias $\theta - \bar{\theta}$. If we solve for equilibrium under the true and the false distributions, all agents will have the same portfolio structure, they will be holding the market portfolio z in both cases. The difference between the two equilibria is the price:

$$P_{False} = \frac{1}{R}(\bar{\theta} - \rho(V + V_D)z) \tag{3.19}$$

$$P_{True} = \frac{1}{R}(\theta - \rho V_D z) \tag{3.20}$$

Let's compute the agent's expected utility in the two equilibria under the true distribution. First, holding the market portfolio with an arbitrary price P one has:

$$\mathbb{E}_{True}(U(z, P)) = -e^{-\rho eR - \rho z(\theta - PR) + \frac{\rho^2}{2} zV_D z} \tag{3.21}$$

Substituting the expressions for P_{False} and P_{True} for the generic pice P one has:

$$\mathbb{E}_{True}(U(z, P_{True})) = -e^{-eR - \frac{\rho^2}{2} zV_D z} \tag{3.22}$$

$$\mathbb{E}_{True}(U(z, P_{False})) = -e^{-eR-\rho z(\theta-\bar{\theta})-\rho^2 zVz-\frac{\rho^2}{2}zV_D z} \tag{3.23}$$

The latter utility (resulting from the False distribution) contains a positive contribution of the extra uncertainty zVz and an ambiguous contribution from the bias $z(\theta - \bar{\theta})$. So, the overall comparison is not conclusive. Moreover, an investor cannot assess the difference between the two possibilities without knowing the true distribution mean θ, which we assume may be learned only at the asset trading stage. So, what measures of utility can investors use at the delegation stage to compare alternatives?

In the discussion above we assumed that the riskless asset was in elastic supply, which should not necessarily be the case. In the absence of noise trading it is easy to introduce a fixed supply condition for the riskless asset and determine the endogenous rate R:

$$\int_0^1 x_B^m dm = 1, \qquad e - zP = 1 \tag{3.24}$$

where e is the endowment (equal for all investors).

The riskless rate will not be the same for informationally efficient and informationally inefficient risky assets' prices.

$$R^{True} = \frac{1}{e-1} z(\theta - \rho V_D z) \tag{3.25}$$

$$R^{False} = \frac{1}{e-1} z(\bar{\theta} - \rho(V + V_D)z) \tag{3.26}$$

The endogenous interest rate is smaller for the economy with greater uncertainty in risky asset prices, thus the economy with inefficient asset prices will have lower return from the riskless asset. Also, the bias in the probability distribution of risky assets will affect the riskless rate: if the bias results in overestimating the mean of risky securities, the riskless rate is larger, and vice versa.

The combined effect of the inefficient prices on investors' utilities in the case of riskless asset in unit supply is the following:

$$\mathbb{E}_{True}(U(z, P, R)) = -e^{-\rho eR-\rho z(\theta-PR)+\frac{\rho^2}{2}zV_D z} = -e^{-\rho z\theta-\rho R+\frac{\rho^2}{2}zV_D z} \tag{3.27}$$

The utility is higher in an economy with a higher endogenous riskless rate R.

If we compare the market with the "False" distribution of risky payoffs and that with the "True" distribution, one has:

$$R_{True} - R_{False} = \frac{1}{e-1}(z(\theta - \bar{\theta}) + \rho z V z) \tag{3.28}$$

So, with endogenous riskless rate the uncertainty zVz penalizes the informationally inefficient economy with the False public probability distribution. Even if the prices of risky assets in this market are lower, the overall amount invested in risky assets is low, and thus the demand for riskless bond is relatively high, depressing the riskless rate.

The bias $\theta - \bar{\theta}$ has, again, an unpredictable sign depending of the realization of θ. Underestimating the probability mean $\theta - \bar{\theta} > 0$ is worse than overestimating it. If the bias is absent, the improvement is the public distribution increases the expected utility.

External effect on utilities

If an agent was offered an ex-ante choice in which economy to live, the one with False public distribution, and the one with True distribution, the decision cannot be based on marginal utility improvements, but is based on a global comparison of expected utilities, measured under the same probability distribution (the True distribution). As was shown before, the expected utility of investing in the market portfolio under the true distribution is the following:

$$\mathbb{E}_{True}(U(z, P, R)) = -e^{-\rho e R - \rho z(\theta - PR) + \frac{\rho^2}{2} z V_D z} = -e^{-\rho z \theta - \rho R + \frac{\rho^2}{2} z V_D z} \tag{3.29}$$

The ratio of utilities is thus:

$$\frac{\mathbb{E}_{True}(z, P_{True}, R_{True})}{\mathbb{E}_{True}(z, P_{False}, R_{False})} = \frac{e^{-\rho e R_{True} + \rho z P_{True} R_{True}}}{e^{-\rho e R_{False} + \rho z P_{False} R_{False}}} \tag{3.30}$$

It is easy to recover the two particular cases we discussed above, when the riskless asset is in elastic supply and in unit supply:

1. Elastic supply $R_{True} = R_{False} = R$.

$$\frac{\mathbb{E}_{True}(z, P_{True}, R)}{\mathbb{E}_{True}(z, P_{False}, R)} = e^{\rho z R(P_{True} - P_{False})} = e^{\rho z(\theta - \bar{\theta}) + \rho^2 z V z} \tag{3.31}$$

Reducing uncertainty in public distribution of risky payoffs is not beneficial in this case, because it leads to higher prices of risky assets and lower returns.

2. Unit supply $e - Pz = 1$:

$$\frac{\mathbb{E}_{True}(z, P_{True}, R_{True})}{\mathbb{E}_{True}(z, P_{False}, R_{False})} = e^{-\rho(R_{True}-R_{False})} = e^{-\frac{\rho}{e-1}(z(\theta-\bar{\theta})+\rho z V z)} \tag{3.32}$$

Reducing the variance of the public probability distribution gives the opposite result compared to the case of elastic supply: it improves agents' utilities. The direction of the bias is also inverted: if for elastic supply of riskless asset it was beneficial to underestimate the true mean, with the unit supply, on the contrary, it underestimation brings a penalty, while overestimation is beneficial.

3.7 Appendix 3.B: Analytical solution for the threshold public game

We derive analytical solution for the indifference condition (3.15) of a threshold public good game, defining the equilibrium probability to contribute when players use mixed strategies. An explicit solution can be easily obtained for $w = 2$ and $w = 3$.

First, recall the indifference condition (3.15):

$$\frac{1}{f} = \sum_{i=w-1}^{N-1} \frac{(w-1)!(N-w)!}{(N-1-i)!(i)!} \left(\frac{q}{1-q}\right)^{i-w+1} \tag{3.33}$$

where q is the probability to contribute to the public good, f is the cost of contributing per person, N is the number of players, and w is the threshold of required contributions.

Recall, that the solutions derived are valid for $N > w$.

First, let's derive the equilibrium probability in the case of $N = 3$, $w = 1, 2$.

The configuration $N = 3, w = 1$ was already solved above, the optimal $q* = 1 - f^{1/(N-1)}$, and the optimal probability of having the public good:

$$\mathbb{P}(m \geq 1) = 1 - f^{N/N-1} = 1 - f^{3/2}$$

Let's solve the case $w = 2$, using two approaches. From the indifference condition 3.13, and 3.15.

$$(1-f)\mathbb{P}(m_i \geq 1) = \mathbb{P}(m_i = 2)$$
$$\mathbb{P}(m_i = 1) = (N-1)q(1-q), \quad \mathbb{P}(m_i = 2) = q^2$$

$$(1-f)(2q(1-q)+q^2)=q^2$$

$$q^* = \frac{2(1-f)}{2-f}$$

Another method is to start formally from:

$$\frac{1}{f} = \sum_{i=w-1}^{N-1} \frac{(w-1)!(N-w)!}{(N-1-i)!(i)!} \left(\frac{q}{1-q}\right)^{i-w+1} \tag{3.34}$$

Let's denote by

$$y = \frac{q}{1-q}, \quad q = \frac{y}{1+y}$$

$$\frac{1}{f} = \sum_{i=w-1}^{N-1} \frac{(w-1)!(N-w)!}{(N-1-i)!(i)!} y^{i-w+1}$$

Redefine $j = i - w + 1$:

$$\frac{1}{f} = (w-1)! \sum_{j=0}^{N-w} \frac{(N-w)!}{(N-w-j)!(j)!(j+1)...(j+w-1)} y^j \tag{3.35}$$

Define:

$$S(y) = \sum_{j=0}^{N-w} \frac{(N-w)!y^j}{(N-w-j)!(j)!} = (1+y)^{N-w}$$

Integrating with respect to y both parts:

$$\int S(y)dy = \sum_{j=0}^{N-w} \frac{(N-w)!y^{j+1}}{(N-w-j)!(j)!(j+1)}$$

$$\int S(y)dy = \int (1+y)^{N-w} dy = \frac{(1+y)^{N-w+1}}{N-w+1} - \frac{1}{N-w+1}$$

The indifference condition (3.35) can be expressed via a (w-1)-times integral of $S(y)$ as follows:

$$\frac{1}{f} = \frac{(w-1)!}{y^{w-1}} \int_{w-1} S(y)dy \tag{3.36}$$

The general form of $w-1$-times integration of $(1+y)^{n-w}$ could be derived in closed form.

For $w=2$ we have:

$$\frac{1}{f} = \frac{(w-1)!}{y^{w-1}} \left(\frac{(1+y)^{N-w+1}}{N-w+1} - \frac{1}{N-w+1} \right)$$

$$\frac{1}{f} = \frac{1}{y} \left(\frac{(1+y)^{N-1}}{N-1} - \frac{1}{N-1} \right) \tag{3.37}$$

Substitute $N=3$:

$$\frac{1}{f} = \frac{1}{y} \left(\frac{(1+y)^{2}}{2} - \frac{1}{2} \right) = \frac{2y+y^2}{2y} = 1 + \frac{y}{2}$$

The optimal:

$$y^* = 2\left(\frac{1}{f} - 1\right)$$

which is the same as q^* obtained above.

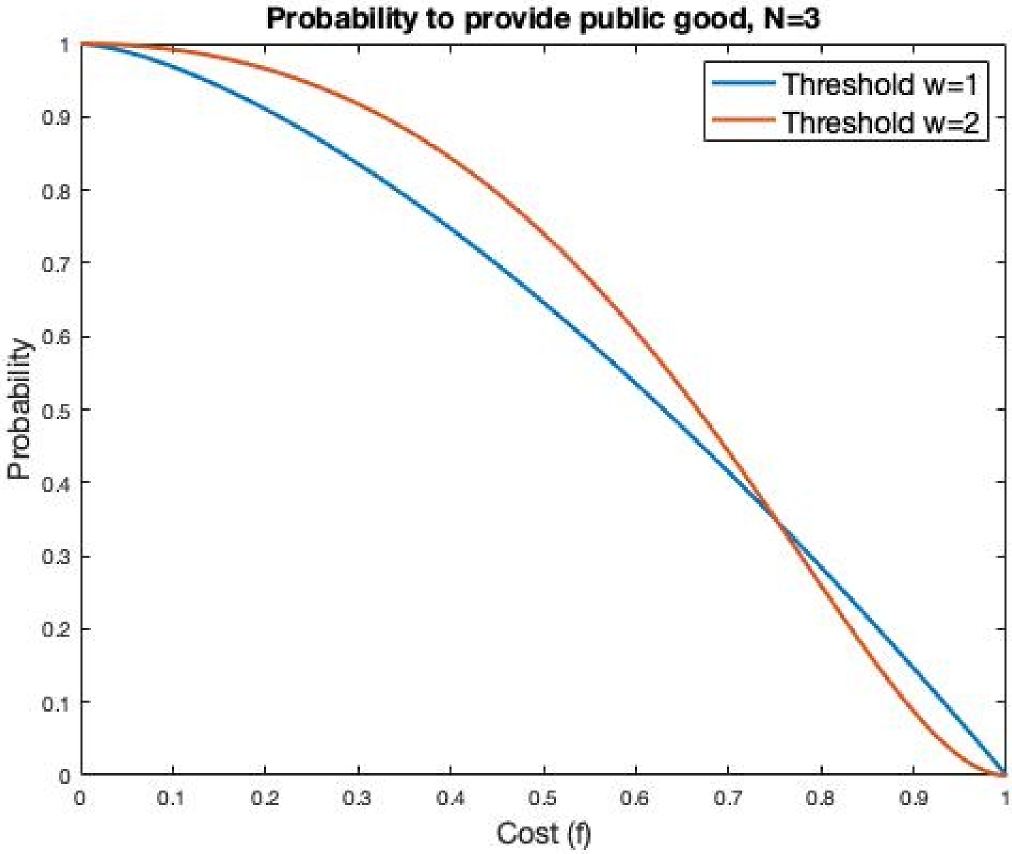
Let's calculate the probability of supplying the public good:

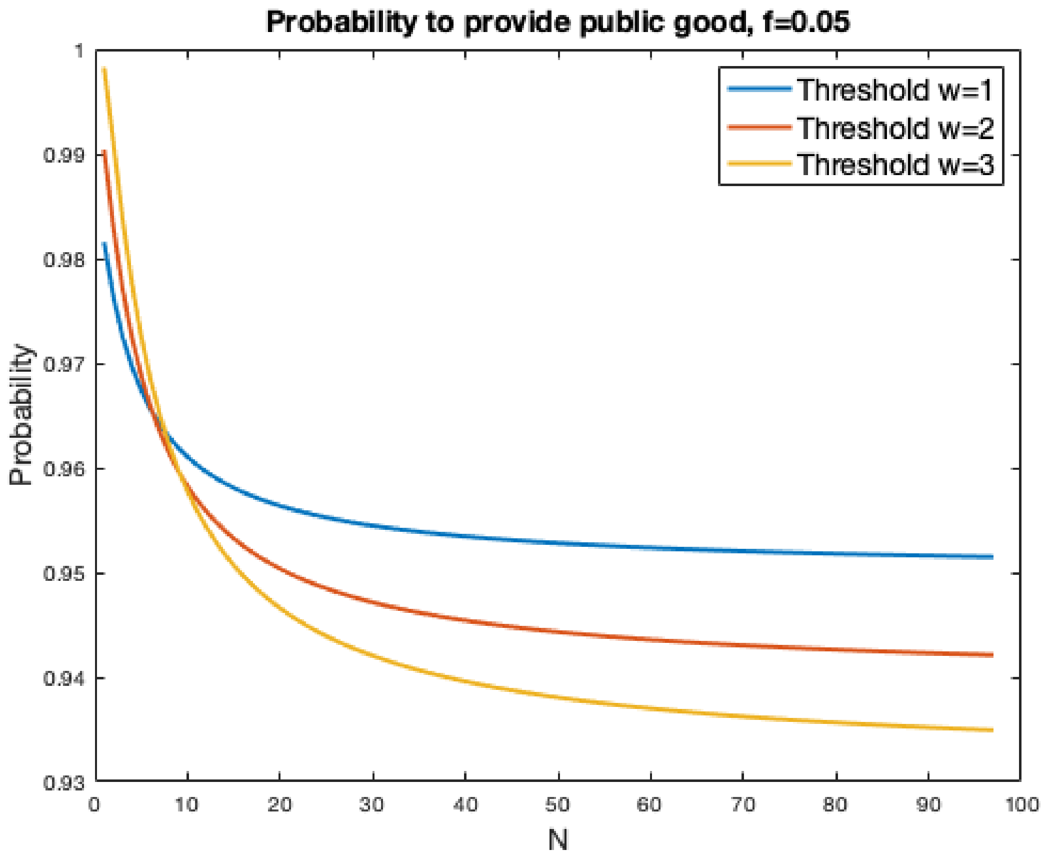
$$\mathbb{P}(m \geq 2) = 3q^{*2}(1-q^*) + q^{*3}$$

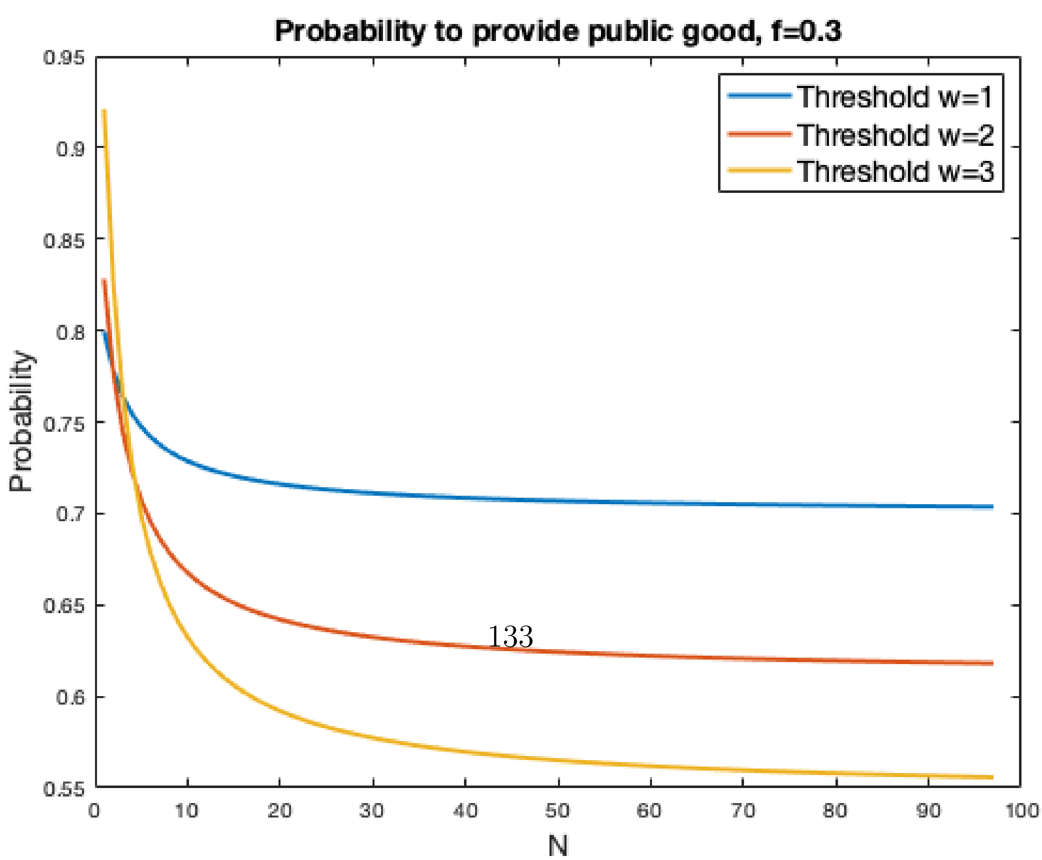
For $w=3$ the condition (3.36), taking the integration two times, will lead to:

$$\frac{1}{f} = \frac{2}{y^2(N-1)(N-2)} \left((1+y)^{N-1} - 1 - (N-1)y \right) \tag{3.38}$$

The probability to provide public good has a limit for large N. For small w at least, the limit is reached after roughly $N=100$. The bigger the threshold w, the smaller is the probability of supplying the public good in the limit.


Probability to provide public good, N=3
Threshold w=1
Threshold w=2
Probability
Cost (f)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Probability to provide public good, f=0.05
Threshold w=1
Threshold w=2
Threshold w=3
Probability
N
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0
10
20
30
40
50
60
70
80
90
100


Probability to provide public good, f=0.3
Threshold w=1
Threshold w=2
Threshold w=3
Probability
N
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
10
20
30
40
50
60
70
80
90
100

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