Federigo Enriques and the Philosophical Background to the Discussion of Implicit Definitions

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Abstract. Implicit definitions have been much discussed in the history and philosophy of science in relation to logical positivism. Not only have the logical positivists been influential in establishing this notion, but they have addressed the main problems connected with the use of such definitions, in particular the question whether there can be such definitions of basic mathematical concepts, and the problem of delimiting their scope. This paper aims to draw further insights on implicit definitions from the development of this notion from its first occurrence in German language in Enriques's "Principles of Geometry" (1907) to Schlick's *General Theory of Knowledge* (1918). It will be argued that Enriques offers one way to counter some of the classical objections against the early twentieth-century conceptualization of implicit definitions. Enriques did not conflate the distinct notions that had been identified as implicit definitions in the recent history of mathematics, but he tried to offer an account of the process leading to structural definitions. The paper will point out, furthermore, that Enriques's account differs significantly from Schlick's. The scientific interpretations of implicit definitions in Schlick's theory of knowledge depend on the coordination of the terms of abstract mathematical structures with physical realities. By contrast, Enriques addressed the problem of bridging the gap between abstract and concrete terms by identifying patterns within mathematics that provide a clarification of conceptual relations, and so also serve (indirectly) the purposes of applied mathematics.

Keywords: Implicit definitions, modern axiomatics, abstraction, concept formation, Federigo Enriques

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1 Introduction

Implicit definitions or definitions by means of axioms have been much discussed with reference to the mathematical works of Peano and Hilbert, as well as in connection with Schlick's attempt to generalize such a tool to all scientific concepts. According to Schlick, implicit definitions made possible exact thought in geometry in the late nineteenth-century; however, there arose the problem of how such definitions can be applied to physical reality.

A further problem is posed by the usage of implicit definitions in geometry. As Hilbert seemed to suggest on several occasions, especially after his exchanges with Burali-Forti, Padoa, Pieri and other members of the Peano School, the axioms of his system of 1899 are supposed to provide definitions of the basic concepts, implying that their meaning is determined by all and only the relations established by the axioms. Hilbert's rejoinders to Frege's criticisms, however, also suggest that what his axioms actually define are higher-order concepts that can be interpreted by infinitely many systems of basic elements. In other words, there seems to be a meaning shift in this mathematical tradition from an usage of "implicit definition" as referred to some mathematical entities and as referred to what are now understood as mathematical structures (cf. Gabriel 1978).

Giovannini and Schiemer (2019) have shown that both ways of thinking about structural definitions in modern axiomatics have interesting connections with modern debates in analytic philosophy. In particular, the first kind of structural definitions play an important role in the contemporary epistemological debate about analyticity, as well as in the discussion about the status of abstraction principles in neo-logicism. The second kind of definitions is employed in contemporary variants of mathematical structuralism. With reference to the early twentieth century discussion, however, it remains unclear what is supposed to be defined implicitly, and on what grounds philosophers such as Schlick believed that implicit definitions should be employed in a general theory of knowledge.

This paper aims to provide some insights into these issues by taking into account the development of the notion of implicit definition from its first occurrence in German language in Enriques's "Principles of Geometry" (1907) to Schlick's *General Theory of Knowledge* (1918). It will be shown that the main ideas for the development of Enriques's notion had been given a comprehensive articulation in his *Problems of Science*, first published in Italian in 1906 and subsequently in Kurt Grelling's German translation in 1910 and in Katharine Royce's English translation in 1914. The central chapters of this work will provide the background for a better understanding of Enriques's usage in contrast with Schlick's. It is beyond the scope of this paper to discuss other important exchanges between Enriques and members of the Peano school, who proposed various accounts of mathematical definitions. A brief account of Enriques's remarks in his later work *For the History of Logic* (1922), along with further literature, will be given in Section 2, only insofar as it will serve the purpose of contextualizing his view of implicit definition.

The first part of the paper will offer a brief reconstruction of the philosophical background of Enriques's usage, which was motivated in part by the discussion on mathematical definitions originating in the Peano School, and in part by the debate on the psychological origins of spatial notions in the works of Mach, Helmholtz, and Poincaré. Based on this reconstruction, it will be argued that Enriques deliberately put in connections these different lines of debate in the following way. He maintained that implicit definitions in mathematical contexts determine an abstract form or what we call today a class of structures. It will be pointed out that, while Enriques did not have a formal notion of "isomorphism" between structures in the current sense, he did employ some (informal) notion of it in examples such as the metrical geometries that are derivable from projective geometry. When addressing the epistemological implications of this and other kinds of definitions at use in the Peano School, including the so-called definitions "by postulates", "by abstraction" etc., Enriques focused on the question of how such definitions can be achieved starting from more basic forms of cognition. In this connection, Enriques took into consideration various attempts to define the primitive geometric notions in the history of geometry, and pointed out that such attempts culminated with implicit definitions of abstract concepts.

My suggestion is that Enriques's epistemological argumentation offers one way to counter some of the classical objections against the early twentieth-century conceptualization of implicit definitions. Enriques himself did not conflate the distinct notions that had been identified as implicit definitions in the recent history of mathematics, but he tried to offer a general account of the process leading to structural definitions. I will point out, furthermore, that Enriques's account differs significantly from Schlick's. The scientific interpretations of implicit definitions in Schlick's theory of knowledge depend on the coordination of the terms of abstract mathematical structures with physical realities. By contrast, Enriques addressed the problem of bridging the gap between abstract and concrete terms by identifying patterns within mathematics that provide a clarification of conceptual relations, and so also serve (indirectly) the purposes of applied mathematics.

2 Implicit definitions in Enriques's encyclopedia article on the principles of geometry

Implicit definitions are often associated with Hilbert's "Grundlagen der Geometrie" (Foundations of Geometry) of 1899. However, Hilbert himself did not use this term here or elsewhere. It was only gradually that Hilbert's way to characterize axiomatic systems was put in connection with structural definitions, and that these became known as implicit. Hilbert famously said, referring to the "Grundlagen," that "these axioms, are at the same time, definitions of the basic concepts" at the International Congress of Philosophy that took Place in Paris in 1900 (1902, p.71-72).

Burali-Forti's presentation at the same Congress gave wider circulation to the various attempts to provide a comprehensive taxonomy of definitions by him and other members of the Peano School.[1] Burali-Forti classified all definitions into nominal, by postulates, and by abstraction. According to this classification only nominal definitions are explicit, in that a symbol is defined in terms of a concept. Even though Burali-Forti did not adopt this

[1]On the development of Burali-Forti's taxonomy from 1894 to 1900 and its connections to works by Peano, Vailati, and Russell, see Mancosu (2016, pp.92–98). On the debate about the definition of equality in the Peano School, see Cantù (2010).

terminology, what became known later as implicit definitions correspond to his characterization of definitions by postulates: "One uses the definitions by postulates for a grouping of objects x, when we do not know or do not want to define them nominally. The group x is defined by postulates by means of the logical relations among the x" (Burali-Forti 1901, p.295). Burali-Forti's examples included Peano's definition of the group of words "whole," "number," "zero," "successor of a number," and of the group of words "point," "segment," as well Pieri's definitions of "point," "movement." Subsequently, Hilbert's "Grundlagen" were taken to offer one the main examples of this way to define the basic concepts of geometry.

The notion of implicit definition was introduced in the German-speaking debate by Enriques in his entry, "Prinzipien der Geometrie" (Principles of Geometry) in the *Encyclopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* (Encyclopaedia of Mathematical Sciences including Their Applications). Enriques was invited to contribute this entry after several exchanges with Felix Klein.[2] As one of the representatives of the Italian School of algebraic geometry, Enriques was significantly influenced by Klein's and Lie's works on group theory and projective geometry. Interestingly, however, it seems that Enriques's exchanges with Klein had been focused mainly on their views on epistemology and mathematical education.[3]

We know from Enriques's correspondence that he had first met with Klein during Klein's trip to Bologna, in 1899. According to Enriques's report, he informed Klein in detail about his plan to write an article on the foundations of geometry, and their discussion had been focused on psychological issues relating to mathematics (see Enriques's letter to Castelnuovo from 28 March 1899, in Bottazzini *et al.*, 1996, p.404). Enriques published his article in 1901 with the title "Sulla spiegazione psicologica dei postulati della geometria" (On the psychological explanation of the postulates of geometry). This paper contains the account of mathematical concept formation that Enriques incorporated later in *Problemi della Scienza*. It is apparent from Enriques's focus on psychology that his inquiry differs substantially from what became known as foundational inquiries in the wake of Hilbert's "Grundlagen." So the question arises as to why Klein charged someone like Enriques with the task of writing such an important entry for the characterization of modern axiomatics. Arguably, Klein might have agreed with Enriques's account in some important respects.

Klein defended the role of spatial intuitions as a starting point for the development of mathematical concepts, where "intuition" is used as an umbrella term for different types of evidence, from the use of diagrams in synthetic geometry to empirical data and experiments. This terminology was quite common among nineteenth-century mathematicians, and did not necessarily relate to the Kantian theory of pure intuitions. On the contrary, Klein sometimes adopted a naturalized notion of intuition in the wake of an "empiricist view" advocated by Helmholtz and Pasch (see esp. Klein 1890, pp.570-72; 1893, pp.298ff.).

[2]The genesis of Enriques's entry can be reconstructed in some detail from Enriques's correspondence (see Ciliberto & Gario 2019, pp.125–126). Enriques reported to Castelnuovo to have gladly accepted the invitation to write the entry on the principles of geometry in 1897. However, several years passed before the appearance of the final version of 1907. During this period Enriques reported to have been able to take into account several works, in particular Klein's lectures on non-Euclidean geometry (in Bottazzini *et al.*, 1996, p.364).

[3]For a detailed discussion of Klein's influence on Enriques, see Giacardi (2010).

The way in which Klein characterized intuitions reflects, nonetheless, the philosophical background of this debate. Intuition in Kantian terminology indicates an immediate mode of cognition, which is directed towards singular objects. Klein identified the main characteristics of intuitions in accord with this philosophical tradition with regard to the particular scope of intuitions. He defended the empiricist view, insofar as he maintained that intuitions, on account of their limitation, are always inexact. I have argued elsewhere that Klein saw the role of intuitions in concept formation as perfectly compatible with an account of the various geometries in terms of instantiations of structures (Biagioli 2020). The argument can be summarized as follows: Because our spatial intuitions are inexact and always restricted to a particular region of space, the introduction of exact postulates into inexact intuitions makes possible a variety of topological spaces that are locally isomorphic to the Euclidean plane. Klein maintained that the mathematical investigation of the different possible forms of space constitutes a precondition for measurement in physics (see esp. Klein 1898). We will see in Section 4 that Enriques argued along similar lines that intuitions provide provisional definitions of geometric concepts, which are replaced in higher geometry by structural definitions.

Both Klein and Enriques held, furthermore, that these epistemological views would prove themselves fruitful when applied to mathematics teaching.[4] In teaching elementary geometry, for example, Enriques's recommendation was to gradually lead secondary school students to overcome the difficulties in the formulation of the principles, and to "situate geometry in the order of sciences by explaining its admirable structure" (in Enriques *et al.* 1900, p.31)

My suggestion is that the philosophical background might shed light on Enriques's article on the principles of geometry of 1907, in particular on the connection between his notion of implicit definition and the epistemological themes of his previous works. Enriques sketched his argument in the introduction of the article by emphasizing a change of perspective in modern foundational inquiries. As he recalled, "axioms" in the Euclidean tradition were supposed to be evident propositions valid in virtue of the meaning of the expressions contained in them. Consequently, Euclidean axioms presupposed the characterization of the basic concepts given in the definitions. The remaining concepts had to be constructed based on the rules stipulated by postulates, which do not necessarily presuppose that the expressions contained in them have been defined. One way to characterize the perspective of modern axiomatics in Enriques's reconstruction is to say that even what counted as axioms in the Euclidean tradition have turned out to imply postulates, that is, as Enriques put it, "propositions expressing the relations assumed between the basic concepts of geometry" (1907, p.7 and note).

For the understanding of axioms in terms of postulates, Enriques referred to two important addresses held at the 1908 edition of the International Congress of Philosophy, in

[4]Enriques in 1900 edited a collective textbook of mathematics for the secondary school titled "Questioni riguardanti la geometria elementare" (Issues concerning Elementary Geometry). As Enriques explained in the Introduction, the approach of the textbook had been inspired by Klein's *Vorträge über ausgewälte Fragen der Elementargeometrie* (Lectures on Selected Issues of Elementary Geometry). Subsequently, Klein wrote an enthusiastic preface to the German Edition of Enriques's, *Lezioni di geometria proiettiva* (Lectures on Projective Geometry) of 1898, which was translated into German in 1903.

Heidelberg, by Giovanni Vailati and by Hieronymus Georg Zeuthen. As Enriques made clear in the rest of the entry, he took these views to be representative for the approaches of mathematicians including Pasch, Peano and his School, Veronese, Hilbert, and himself.[5] This is not to suggest that Enriques agreed with all the views expressed by these mathematicians. On the contrary, he disagreed with Peano and other members of his school on a number of issues, including the role and status of symbolic logic, Peano's distinction between membership and inclusion, the question whether the characterization of equality should be unique or relative to the type of objects under consideration. In particular, he engaged in a harsh polemic with Burali-Forti in 1921.[6] What Enriques suggested is that, these disagreement notwithstanding, all these different sources converged towards an understanding of the principles of mathematics, in particular geometry, in terms of what members of the Peano school called definitions by postulates and Enriques himself renamed to "implicit definitions". A more detailed account of how the idea originated is found in the third chapter of Enriques's later work *Per la storia della logica* (For the History of Logic). Enriques credited Pasch (1882) with being the first to rephrase the principles of geometry as purely logical relations between primitive, undefined concepts. In Enriques's account, Peano's logical calculus enabled him to translate a modified version of Pasch's principles using the symbols of mathematical logic, even though Enriques believed that the symbolic form might have been the cause of the delayed reception of Peano (1889). In this regard, Enriques wrote:

> As far as we can tell, by memory, the meaning of the logical form had to be regained as a personal conquest perhaps by every critical mathematician of this generation, although a general, more or less indirect, influence by the predecessors cannot be absolutely excluded.
> The acquisition of this "logical meaning" is clearly recognizable, despite some obscurities, in the work of Giuseppe Veronese, "Foundations of Geometry" (Padova, 1891), and subsequently in the investigations on the foundations of projective geometry by Federigo Enriques (1894). Referring back to these investigations and using Peano's symbols, Giovanni Vailati and Mario Pieri were

[5] Enriques did not mention Klein in this connection, arguably because Klein himself did not adopt a worked-out axiomatic approach in his mathematical work. It is worth noting that, nevertheless, Klein had characterized the modern notion of axioms in terms similar to Enriques's, as "demands that introduce exact statements into inexact intuitions" in (1890, p.571).

[6] The polemic started with Enriques's review of the second edition of Burali-Forti's *Logica matematica* (1919), which was followed by an exchange published in *Periodico di matematiche* (Enriques 1919; Burali-Forti and Enriques 1921). Enriques identified Burali-Forti's work as a comprehensive exposition of Peano's system, showing the limits of a symbolism completely detached from natural language and devoid of meaning. Together with Beppo Levi, Enriques defended a "dynamic" conception of logic as consisting of laws ruling over a mental process, and opposed what he took to be "static" representations of it by Peano and Russell, among others. On Enriques's critique of Peano's system, see esp. Enriques 1922, pp.182–196; Lolli 1993. A more fruitful exchange was hampered by the polemical tone used by Enriques, arguably due at least in part to his aversion against Burali-Forti's nationalist political views. It should be noticed, however, that there have been exchanges worth of closer attention between Enriques and other members of the Peano school, in particular Padoa, Mario Pieri, Giovanni Vailati, and especially Giovanni Vacca, as documented by Luciano (2021).

> able to show that the need for the logical form was intended substantially in the same way in the different expositions. And the same can be said about Alessandro Padoa's critical notes on Veronese's work. (Enriques 1922, p.166)

Enriques went on to argue that similar ideas emerged independently in Hilbert's "Grundlagen". What Enriques took to be the common denominator to these works was not so much a view shared by all these mathematicians, as a new way to understand the principles that was implied in some key developments in nineteenth-century geometry.[7]

Enriques claimed that the decisive contribution to the modern understanding of geometrical axioms in terms of postulates came from the development of non-Euclidean geometry by the denial of Euclid's parallel postulate. Enriques recalled that geometries based on non-Euclidean hypotheses had been first investigated by Gauss, Bolyai and Lobachevskii in the 1830s; but it was especially Riemann's merit to have shown the various physical possibilities concerning space, which would differ from ordinary, Euclidean space in his habilitation lecture from 1854 (later published in 1867). In Enriques's account, these developments led to distinguish the physical notion of space from the ordinary or intuitive notion. Furthermore, Enriques emphasized the mediating role of abstract spaces which, in his definition, are "the universal concepts that are derived from the ordinary concept of (intuitive) space by abstracting from or generalization of its properties" (Enriques 1907, p.8). Enriques's examples were Klein's projective system, which included Euclidean and non-Euclidean metrics as special cases, and the non-Archimedean spaces investigated by Veronese and Hilbert. What these examples have in common according to Enriques is that they are based on the construction of different "hierarchies of geometrical concepts" (*Rangordnungen der geometrischen Begriffe*) shedding light on the psychological and physical presuppositions of ordinary (Euclidean) geometry, while opening the door to the investigation of various systems of arbitrary hypotheses (Enriques 1907, p.9).

Enriques introduced the notion of implicit definition in an attempt to identify the logical form required for the above view of geometrical concepts. Our discussion will focus especially on the following passages from Enriques's 1907 article:

> It was recognized that a definition, as well as a proof, is something completely relative, and therefore it became necessary to formulate the primitive concepts, i.e., those concepts that one does not want to define in an actual system but that are logically connected to one another in the definitions, explicitly as such. And since the postulates appeared to be relations between these concepts, then it was decided that they should be given such a form that would make them recognizable, even when one abstracts away from the meanings that one can attach to the concepts on the basis of intuition or experience (and are not used in the logical development). (Enriques 1907, p.10)

[7] It is not easy to say whether and to what extent Enriques's own mathematical activity might have played a role in shaping his notion of implicit definition. Besides the above-mentioned work from 1894, which offered an important contribution to the modern axiomatization of projective geometry, it is straightforward to see that his account of the main steps towards an abstract conception of geometry was at least informed by his own work in that field (see, e.g., Enriques 1922, pp.138–141), even thought it would go beyond the scope of this paper to reconstruct such a connection.

Enriques went on to provide a brief sketch of how the use of such definitions had become widespread in recent foundational inquiries:

> The basis for the logical treatment of geometry according to Pasch lies in the postulates (even though these can be introduced as products of a psychological process that takes off from intuition).
> This idea of rigor has increasingly become a common view in geometrical researches of this kind ever since (see, e.g., Peano, Principii, 1889; Veronese, Fondamenti, 1891; Hilbert, Grundlagen, 1898 etc.).
> According to this view, the postulates appear from an abstract logical standpoint as arbitrary stipulations, and the totality of logical relations, which they express, constitutes a kind of implicit definition of the primitive concepts. (Enriques 1907, p.11)

Finally, Enriques added in a footnote a reference to an article by Giovanni Vacca for Gergonne's earlier usage of "implicit definition" in the following sense:

> This notion of definition is found, as G. Vacca (Riv. di mat. 1899, p.185) noticed, already in J. D. Gergonne (Gerg. Ann. 9 (1818–19), p.1). Consider the following passages from Gergonne's paper: "Propositions, in which the meaning of a term is determined by the knowledge of the meaning of the other terms contained in it, can be called *implicit definitions*, in contrast to the usual definitions, which one can call explicit. ... So, it occurs often that two propositions, which connect two new terms with known terms, determine their meaning." (Enriques 1907, p.10 and note)

These passages have been criticized by Gabriel (1978) as conflating the two different meanings of structural definitions as we have outlined before, that is, as definitions of the basic concepts, on the one hand, and of higher-order concepts, on the other. Gabriel has pointed out, furthermore, that none of these kinds of structural definitions coincides with what Gergonne had called implicit definitions. To mark the difference between these kinds of definitions, Giovannini and Schiemer (2019) have labelled the latter "implicit definitions in the strict sense" and the former "structural definitions."

The following section will address Gabriel's and some related objections against Enriques's usage. I will suggest a more consistent reading of Enriques's argument, according to which the different meanings associated with implicit definitions correspond to different stages in the process of acquisition of geometrical notions. Evidence for such a reading is his emphasis on the provisional character of the various attempts to define the primitive concepts. Enriques in the first of the above quotations called such definitions relative, insofar as their function is to distinguish primitive concepts from derivative ones, and suggested that the only rigorous way to fulfill this task is to acknowledge that the postulates determine only how concepts are related to one another. Given the fact that modern axiomatic systems in Enriques's account are based solely on postulates, his claim amounts to saying that attempts to provide explicit definitions of primitive concepts have been replaced by structural definitions. At the same time, Enriques attached particular importance to the epistemological implications of the intermediate steps towards the formulation of axiomatic systems, in particular the introduction of implicit definitions of concepts. I will draw further evidence for my reading from Enriques's more detailed discussion of implicit definitions in *Problemi della Scienza*.

3 The meaning shift in Enriques's notion

Gabriel's objection against Enriques's considerations is that they involve several meanings associated with implicit definitions. According to Gabriel, these meanings are simply conflated in Enriques's entry, and subsequently in the philosophical reception of Hilbert's axiomatic method.

Gabriel points out, firstly, that Hilbert's axioms differ from Gergonne's implicit definitions. In the type of definitions analyzed by Gergonne, the meaning of the term to be defined is initially unknown, but can be inferred from knowledge of the meanings of the other terms contained in the proposition. Gergonne compared a system of such propositions to a system of equations with one or more unknowns. It followed that in such a system the number of the terms to be defined equals the number of implicit definitions. By contrast, the number of axioms can differ from the number of the basic predicates. Furthermore, an axiomatic system, unlike a system of equations, can have a variety of "solutions," that is, its models (Gabriel 1978, p.420). Despite that, especially in philosophical discussions, implicit definitions came to designate Hilbertian axioms rather than Gergonne's definitions.

Secondly, Gabriel points out that even when referred to Hilbert's axioms, it remained unclear whether implicit definitions provided actual definitions of terms corresponding to the basic concepts or of higher-order concepts. Gabriel maintains that Hilbert might have been influenced by the mathematicians of the Peano School in claiming that his axioms should provide definitions of the basic concepts; however, Hilbert also made clear in response to Frege's criticisms that axiomatic systems in his sense characterized a "scaffolding of concepts" that can be reinterpreted in various ways (Hilbert's letter to Frege from 29 December 1899, in Frege 1980, p.40). Hilbert's rejoinders to Frege suggest that the system of the "Grundlagen" as a whole provided definitions of what Frege called second-order concepts, which can have different instantiations in terms of objects. Gabriel's suggestion is that in the early twentieth-century context, prior to the formalization of structural definitions in model-theoretic terms, Fregean logic offered the best available resources to clarify what axiomatic systems are about. However, Hilbert notoriously refused to make public his exchange with Frege. Hilbert's claims at the Paris Congress had a strong echo in the philosophical discussion of modern axiomatics. Finally, Gabriel points out that after Enriques's article of 1907 it became commonplace to associate Hilbertian systems with the use of implicit definitions, without it being clarified how these were supposed to determine the meaning of the basic concepts of an axiomatic system.

A related objection is that the unclarity surrounding the mathematical usage of implicit definitions affected later attempts to generalize their use to all scientific knowledge, in particular Schlick's *General Theory of Knowledge*. Schlick relied on the use of implicit definitions in Hilbertian axiomatics to articulate a holistic conception of knowledge as a system of interconnected judgments whose concepts get their meaning from their mutual relations within the system. The use of implicit definitions, according to Schlick, showed that pure geometry has nothing to do with intuitive space, insofar as it constitutes a variety of uninterpreted systems. He advocated a form of geometrical conventionalism, according to which the choice among all possible geometries in physics is arbitrary and guided only by the constraint of the univocality of the coordination of systems of symbols to the system of sense experience. He sought to generalize this picture further by showing that all concepts of science are individuated by logical forms. However, there arose the problem of differentiating the formal systems of pure mathematics from scientific knowledge of reality. Schlick contended that scientific concepts, unlike purely mathematical ones, have a content, that is, real qualities satisfying our initially uninterpreted judgments. As Schlick himself pointed out in an important passage, however, the relationship between form and content became problematic. Schlick wrote:

> In implicit definitions we have found a tool that makes possible completely determinate concepts and therefore rigorously exact thought. However, we require for this purpose a radical separation between concepts and intuition, thought and reality. To be sure, we place the two spheres one upon the other, but they appear to be absolutely unconnected, the bridge between them are demolished. (Schlick 1918, p.36)

Friedman emphasizes how Schlick struggled with the problem of clarifying the form/content relation in his theory of 1918 without finding a coherent solution. Friedman's suggestion is that, in trying to address the above issue, Schlick in this phase of his intellectual career was continuously tempted to renounce his holistic conception of meaning and finally advocated an atomistic empiricist conception which views the intuitively given as the ultimate repository of meaning (Friedman 1999, pp.27–28).

Friedman's objection can be rephrased by saying that Schlick's attempt to provide a holistic account of knowledge reflected a shift of meaning in his notion of implicit definition. In saying that implicit definitions "completely determine concepts," Schlick seems to echo Hilbert's claim that his axioms provide definitions of the basic concepts. The way in which Schlick separates pure geometry from reality, however, seems to imply the more standard account of implicit definitions as applying to uninterpreted structures. Friedman's objection amounts to saying that Schlick in his account of scientific concept formation oscillates likewise between atomistic and holistic conceptions of meaning.

My suggestion is that Enriques escapes these objections, insofar as he offered a dynamical account of how implicit definitions emerged. Let us begin with Enriques's considerations about the introduction of the term "implicit definition."

Enriques's reference to Gergonne was taken from a column of Peano's *Rivista di matematica* dedicated to the forerunners of mathematical logic. Vacca in his entry on Gergonne referred, more specifically, to the origins of logical notations: Whereas some symbols are introduced by ordinary (explicit) definitions as abbreviations of other symbols, Gergonne called implicit definitions a different way to introduce new symbols by implying their meaning in a proposition. Vacca's emphasis in his report was on the need to enrich further the language of the exact sciences allowing a maximum of freedom (Vacca 1899, p.186). Subsequently, Padoa included implicit definitions in Gergonne's sense in his "Introduction logique à une théorie déductive quelconque" (Logical introduction to any deductive theory) of 1900.

Enriques in *Problemi della Scienza* was the first to adopt implicit definitions in a broader sense, as a way to formalize basic concepts that might be familiar from past experiences. While Enriques largely relied on the Peano School for his classification of definitions, we have seen that he took a different stance on the scope and the status of logic. Notably, Enriques maintained that the possibility of symbolic or mathematical logic has to be grounded in the processes of thought.[8]

Enriques maintained, furthermore, that such an investigation should look at the relevant processes at work in the history of the exact sciences. In particular Enriques emphasized the fact that the recent history of geometry culminated with the study of abstract fields, such as non-Euclidean and non-Archimedean spaces, that exist only as "logical constructions" (edifizii logici), without referring to objects in reality (Enriques 1906, p.109). In Enriques's account, the new conception of geometry emerged with the systematic use of transfer principles by Julius Plücker and Felix Klein, and was developed further in the axiomatic approaches of Pasch, Peano, Veronese, Hilbert (*ibid.*). It is noteworthy that these are the same authors that are mentioned in Enriques's entry of 1907 in connection with the introduction of implicit definitions in geometry. In *Problemi della Scienza* though, Enriques is much clearer on the fact that what is being determined by an axiomatic system is a structure rather than the meaning of the basic concepts. Consequently, such concepts are designated by abstract terms. He wrote:

> A geometrical theory may be regarded as a system of logical relations, holding for certain concepts designated by the words "point," "line," etc. We may attribute to these words an abstract and indeterminate meaning, thus regarding them as the symbols of unknown concepts, but such as formally satisfy the fundamental propositions which express geometrical relations. It is then allowable to decide at will, by some convention, the meaning of our symbols, provided this be done in a way that will fulfil the formal conditions already stated. In this way we obtain an infinite number of possible concrete interpretations of abstract geometrical theories. (Enriques 1906, p.109)

Enriques went on to argue that from the standpoint of modern axiomatics there is a clear distinction between the above concepts, that are assumed as primitives, and the derivative

[8]Enriques's conception of logic reflects the fact that modern mathematical logic coexisted until the 1920s with logic understood as a philosophical discipline addressing epistemological topics such as the delimitation of objective from subjective knowledge and the systematization of scientific methodologies. See Haaparanta 2009, pp.235–243 for a general account of the relations between logic, epistemology and psychology in the late nineteenth century and in the first decades of the twentieth century. As we will see in Section 4, Enriques adopted a peculiar stance in this debate by offering an account of the psychological processes involved in geometry informed by experimental physiology, but also guided by knowledge of higher mathematics.

concepts: only the latter allow for a definition such that the term that is being defined can be replaced by the terms used to define it. Following Burali-Forti's classification, Enriques called these definitions nominal. Enriques emphasized that only these definitions are logical. Such definitions, for example, of a "point" as an object without an extension or of a "line" as a tightly stretched thread can be called definitions only in a psychological meaning, a name, as Enriques put it, "which shows their office in recalling certain images, and in suggesting the notion of their relations" (Enriques 1906, p.113).

Enriques introduced the notion of implicit definitions of concepts in an attempt to characterize the different stages of a process leading from the use of psychological definitions of the basic concepts of geometry to the recognition that the words for such concepts have an abstract meaning. Enriques explained further his terminology by dividing all definitions into nominal and real, where the meaning established by real definitions is determined not formally but by a series of observations and experiences. Enriques divided real definitions into "concrete" definitions naming an object present to the senses, and "descriptions" of psychological processes that can be learned by somebody else. The psychological definitions of the basic concepts of elementary geometry, along with all descriptions of complex and abstract ideas, belong to this second kind of real definitions. However, this mode of definition is always uncertain in a twofold way: it leaves the meaning of the terms undetermined, because a concrete definition is not available; but it also makes it impossible to verify whether the description produces the same psychological processes in different people. Correspondingly, the sciences are faced both with the underdetermination of their basic concepts and with the underdetermination of the deductive theory concerning them. Enriques maintained that as far as geometry is concerned, the underdetermination of the theory has been mastered in modern axiomatics by replacing psychological attempts to provide (explicit) definitions of the basic concepts with implicit definitions by means of a system of postulates. He wrote:

> When we say that: "The postulates represent the logical relations of the fundamental concepts A, B, C ..." we mean that they take on a general or abstract form, such that they remain intelligible even when cut off from all mental images of these concepts, so that we retain only the fact that A, B, C .... are obtained, for example, by the union of certain undefined elements etc. (Enriques 1906, p.115)

This passage clearly suggests that what is being characterized in terms of postulates in Enriques's account is what he called here a "general or abstract form." In this connection Enriques emphasized that the meaning of the basic concepts themselves is determined only relative to the formal development of the theory in the following claim: "*The sum total of the postulates is to be regarded as forming the implicit definition of the given concepts, in so far as is needful for the theory founded upon them*" (*ibid.*).

As Enriques recognized more pointedly in *Per la storia della logica*, the abstract form of this kind of definitions raises the question of whether the basic concepts are really being defined. Enriques's answer is twofold. In the first place, he acknowledged the possibility of defining equivalent systems by verifying "in every single case, whether two systems of entities satisfying the system can be put into a one-to-one correspondence such that the properties of the one are translated in perfectly homologous properties of the other, so that they appear to be equal from an abstract point of view, in the frame of mind under consideration" (Enriques 1922, p.198). To mention a classical example from abstract geometry, Enriques considered how the definition of projective spaces allows one to put the different metrical spaces into correspondence while preserving congruence relations. Enriques deemed a system of postulate "complete", if it satisfies the above condition or (in current terminology) it defines a class of structures up to isomorphism.[9]

Secondly, Enriques pointed out that implicit definitions call for a concrete interpretation for the possibility of applying mathematical theories to different fields of knowledge. Returning to the example of projective and metrical geometries, the implicit definition of space characterizes it in geometrical terms alone; but it can also be applied to real objects, by adding fundamental concepts and postulates linking them to the properly geometrical ones, and including geometry in the more comprehensive system of physics (Enriques 1922, p.199). In general, the interpretation presupposes that some objects are assumed as given, either in external or in psychological and social reality. Examples of physically given objects in Enriques's sense include quantities that are being defined implicitly by the equations determining their mutual relations; the mediating term, that is, what allows one to attribute a physical meaning to the deductive theory is measurement. To mention a different example from social sciences, Enriques contended that a right, e.g., of property, can be considered implicitly defined by all relevant articles of the legal code, along with the complex of social relations involving them.

It is clear from all these examples that the basic concepts themselves are relative to the formal system adopted for their determination and subject to change. This is apparent in the case of property, that is defined differently in different Civil Codes. Enriques emphasized that circumstances can lead to redefine basic physical concepts as well. He mentioned for example the fact that there have been different ways to define temperature starting with the observation that an increase of temperature is proportional to the dilatation of some bodies. The recognition that proportionality is not always the case led to a more precise way to define temperature relative to gas thermometers, because of their agreement amongst themselves. The recognition that even gas thermometers can disagree, as measurement grows more precise, led to the introduction of absolute temperature as an ideal standard of measure. Enriques stated more generally about the deductive theory of measurement:

> Now the development of this theory by the aid of experiment comes to modify our equations, and hence to correct progressively the implicit definition, which therefore *expresses at every moment of the progress of the theory, the most advanced synthesis of the data thus far obtained.* (Enriques 1906, p.102)

These examples show that Enriques's generalization of the notion of implicit definition significantly departs from Schlick's on the characterization of the ultimate data, as well

[9] I take the above quote, together with Enriques's geometrical examples, to show an informal notion of isomorphism at work in his characterization of equivalent systems.

as on the form/content relation, and therefore is not subject to the same objections. It remains open to debate whether there is a conflation of different meanings of implicit definition in Enriques's account of modern axiomatics, as suggested by Gabriel.

Regarding the origins of the notion, it is noteworthy that Enriques pointed out more explicitly elsewhere that the multiple interpretability of concepts marks a decisive step away from Gergonne's characterization in analogy with systems of equations towards the characterization of systems of postulates in modern geometry. He nevertheless credited Gergonne with having emphasized the significance of the principle of duality, which offered one of the first examples of the transfer principles at work in abstract geometry (see Gergonne 1825–1826; Enriques 1922, pp.134–141).

Summing up my considerations so far, I believe that the above passages provide enough evidence that Enriques was aware of the different meanings associated with implicit definitions. Enriques deliberately took this terminology from a discussion within the Peano School about the origins of logical notations and used it to address what he considered to be a more fundamental epistemological issue concerning the characterization of abstract mathematical concepts. It will be argued in the following that the solution proposed by Enriques depends on his account of how abstract concepts are obtained. This is condensed in the above claim that the abstract form given to the concepts by the use of postulates is obtained by doing abstraction from all image associated with them and retaining only the relational structure connecting them as undefined elements. Enriques accounted for this notion of abstraction in the terms of his psychological logic, by tracing back set-theoretic operations such union, division etc. to basic psychological processes. As a result of abstraction, what is being determined is the relational structure itself. This is the procedure that corresponds to what Burali-Forti had called definition by postulates. By using a later terminology, one can also say that implicit definitions in Enriques's sense find a precise formulation in terms of structural definitions. At the same time, he contended that the meaning of the basic concepts is determined relative to the structure defined by the postulates, along with all its formal consequences and possible instantiations.

4 Enriques's account of mathematical concept formation

Enriques sought to present the taxonomy of definitions that had been investigated in the Peano School in connection with the late nineteenth-century debate on the origins of spatial representations. Notably, he referred to a line of argument that goes back to Helmholtz, according to which the free mobility of solid bodies is a precondition for the development of our geometrical abilities (Helmholtz 1867, p.447). Similar arguments are found in Mach (1886, p.100) and Poincaré (1902, p.61). Enriques summarized the outcome of these inquiries by saying that: "The geometrical concept of space is the abstract of the various physiological spaces that are possible in relation to a moving observer" (Enriques 1906, p.205).

Enriques's explanation of how such a concept of space emerges sheds further light on his psychological understanding of logic, especially on the fact that his starting point is the articulation of modern geometry rather than a properly empirical investigation of the genesis of spatial concepts. The problem addressed by Helmholtz was to determine the role of the different types of sensations in the formation of spatial representation. Enriques emphasizes, however, that even the interpretation of Helmholtz's and others' experimental results presupposes "an analysis of spatial concepts as only the mathematician can carry out" (Enriques 1901, p.76).[10] Enriques bore in mind the nineteenth-century distinction between metrical and projective geometry, and referred, more specifically, to the following suggestion by Klein:

> The partition into metrical geometry and projective geometry is to be regarded, not as arbitrary or determined only by the nature of the mathematical methods, but as corresponding to the actual genesis of our spatial intuition, whereby as a matter of fact mechanical experiences (i.e., the motion of solid bodies) combine themselves with experiences of visual space (concerning the various projections of the intuited objects). (Klein 1898, p.593)

On this basis, Klein used a projective model of non-Euclidean geometry to provide a mathematical interpretation of free mobility as a precondition for the possibility of measurement including Euclidean and non-Euclidean metrics as special cases.

Enriques gave a more systematic articulation to Klein's claims by arguing that the basic concepts of geometry can be classified according to the corresponding group of sense impressions. As Klein had suggested, Enriques associated basic projective concepts such as straight lines and planes with visual impression, and basic metrical concepts, in particular distance, to special tactile-muscular sensations localized especially in the hands. In addition to this, Enriques maintained that the two main branches of nineteenth-century geometry as characterized by Klein presuppose a common ground in the theory of continuity.[11] The basic concept of the latter theory according to Enriques is the concept of line, and is associated with a more general kind of tactile-muscular sensations involving larger portions of skin, including the retina. Enriques sided with Helmholtz in advocating an empiricist explanation of binocular vision as the outcome of normal fusion of the two retinal images.[12] Enriques assumed, furthermore, that the formation of an image on the retina is equivalent to a central plane projection of an object, and that the orientation of

[10]Enriques's approach is consistent with his interpretation of modern philosophy as a scientific endeavor informed by how the sciences present themselves in their latest developments. The case of geometry is emblematic, because a plurality of possible hypotheses emerged from the nineteenth-century investigations of the abstract forms of space. Enriques related the idea of scientific philosophy to a neo-Kantian methodology, according to which science as a "fact" is a necessary starting point for the investigation of the preconditions for scientific inquiries. He departed from the neo-Kantian epistemology, insofar as he contended that the study of the fact of science should be aided also by the results of psychology (Enriques 1906, p.52). On Enriques's relation to neo-Kantianism, see Ferrari 2014.

[11]Klein referred in particular to the characterizations of continuity given by Dedekind and Weierstraß, as well as to the study of non-Archimedean continua by Veronese. In his article of 1901, Enriques mentioned, furthermore, his own work, "Sulle ipotesi che permettono l'introduzione delle coordinate in una varietà a più dimensioni" (On the Hypotheses that Allow the Introduction of Coordinates on a Multidimensional Manifold) of 1898.

[12]Enriques's account of the basic facts concerning vision is rooted in the nineteenth-century debate on the implications of experimental physiology. Whereas Helmholtz held that the two retinas produce two sets of sensations that we have to learn to refer to a single object, Ewald Hering and others (including

the eye for any position of the line of vision is constant. He considered geometric concepts to be the result of complexes of associations. He maintained that the concept of line, as all spatial concepts, is acquired by a combination of actual images with genetic ones. He identified the actual image of the line as a limiting surface, and the genetic image of it as a succession of points. Similarly, Enriques stated that a point can be represented both genetically, as part of a line, or as an actual physical point.

These are the kinds of psychological definitions of basic geometric concepts that Enriques contrasted with the logical definitions of the derivative terms in his taxonomy. His explanation starts from these notions to show how geometrical postulates can be introduced in a stepwise process of idealization. The first postulate of the line in Enriques's account states that between any two points in it there is always an intermediate point. However, this is contradicted by the fact that to our sense perception such a point is distinguishable from the others only up to a certain threshold of precision. Enriques pointed out that one can nonetheless experience an increase of precision, for example in the perception of the same group of points from a different angle. As Enriques put it: "This conceived extension of experience, then, leads us to think that it may be possible to place a point between A and B, on a given line, even if this is not immediately evident to the senses. However this extension has in fact a limit that is soon reached" (1906, p.217). A further step that is required for the indefinite intercalation of the intermediate point between two given points in Enriques's account is the comparison between two lines of varying extension, e.g., represented by elastic threads stretched to different degrees of tension. The formulation of the postulate corresponds to the assumption of an ideal one-to-one correspondence between the points of the lines. Enriques's conclusion was that: "This indefinite possibility of interpolation is then a necessary property of the concept of a line, in so far as that concept represents the ideal product of the combination and abstraction of all the empirically attainable ideas of the genesis of lines" (p.218). He went on to describe how the second postulate of the line establishing its continuity can be abstracted from the association of different ways to represent a point, in its genetic image as the generating element of the line, and in its actual image as a limiting point. These two different images according to Enriques correspond to the two formulations of the principle of continuity given by Dedekind and Weierstrass, respectively.

Enriques examined the visual representations he associated with projective geometry in a similar way. The concept of a straight line, in his account, has an actual image, as projected into a point when seen by one eye through the visual center; it has a genetic image, as a line which if not passing through the visual center, has rectilinear projections. Enriques argued that the association of these two ideas lies at the basis of the postulate that two distinct points determine a straight line. Given two points A and B viewed from

Helmholtz's former teacher Johannes Müller) supposed the two retinas to be anatomically connected with each other. Helmholtz called the latter approaches "nativist" as opposed to the "empiricist" approach advocated by him (Helmholtz 1867, p.456). A related issue was whether the two– and three-dimensionality of space is primarily given or acquired. For a comprehensive reconstruction of the empiricism/nativism debate in nineteenth-century physiology, see Turner (1994). Enriques emphasized that the interpretation of experiments was open to debate, and sided with the empiricists, mainly because associationism seemed to confirm his mathematical interpretation (Enriques 1901, pp.78–81).

A, Enriques identified the visual ray AB as the series of points that is the image of B; such a series is seen as a straight line from a point external to AB, and as a point from B. It follows that AB is undistinguishable from BA (AB = BA). Another point C of the straight line, viewed from A, gives the same image. Therefore, by definition, Enriques established the equality of the visual rays AB and AC (AB = AC). Considering another point D and combining the former equalities (AB = AC = CA = CD), he obtained: AB = CD. This shows that the straight line is uniquely determined by any pair of its points (Enriques 1901, p.87; 1906 p.224, p.).

Finally, Enriques maintained that ordinary metrical geometry unites under the same concept of straight line the visual and the tactile images of it in virtue of our experiencing a physical symmetry of optical and mechanical phenomena. Enriques maintained that the psychological necessity of the ordinary intuition of space depends on the assumption that such a symmetry must take place on account of what he called a "projective-metric" association, that is, a reiterated combination of visual and tactile representations. However, he pointed out that alternative hypotheses cannot be excluded a priori. To make this point clear, Enriques focused on the example of the projective-metric association in the conception of parallel lines as equidistant straight lines in a plane. The parallel postulate derives from the fact that such an association makes us perceive as single a straight line whose rays are optically parallel to another given straight line in both its directions. However, this tends to obscure the possibility of non-Euclidean hypotheses. Enriques's supposition was that someone lacking the experience of the projective-metric association, on the contrary, might find it easier to conceive a non-Euclidean space. He also reported to have tested his conjecture on a blind from birth. However, Enriques recognized that this single test did not provide sufficient evidence, considering that the answers given might have been influenced by leading questions.

Enriques thought that the main evidence in support of his view came from the history of non-Euclidean geometry, which gave particular impulse to the modern way of considering geometries as systems of postulates. Enriques traced back the modern approach to the recognition, by mathematicians such as Saccheri and Lambert, that the received definitions of the basic concepts tacitly presupposed hypotheses that were completely different from logical axioms. In particular, Enriques referred to Saccheri's criticism of Borrelli's definition of parallel lines as equidistant straight lines as a "fallacy of complex definition" in *Euclides ab omni naevo vindicatus* (Euclid Vindicated from Every Blemish) of 1733. Such a fallacy consists in adopting definitions that attribute different properties to the thing to be defined without having verified their compatibility. In Borrelli's definition, parallels are identified both as sets of points (on the same plane) equidistant to a given straight line and as straight lines. In order to avoid this kind of fallacy, Saccheri required that all real definitions should be replaced by demonstrations showing the existence (based on previously admitted postulates) or possibility of constructing what is being defined.

It is well known that Saccheri was one of the first to prove theorems of non-Euclidean geometries in an attempt to derive a contradiction from hypotheses incompatible with the parallel postulate. Vailati emphasized the logical interest of Saccheri's criticism of the above definition of parallel lines. Vailati read Saccheri's argument in connection with modern axiomatics, as foreshadowing the view that the existence of the objects to be defined depends on the consistency of the hypotheses that have been admitted as laying at the basis of geometry (Vailati 1903, p.18).

Enriques reported to have learned from Vailati about Saccheri's conviction that the acquiescence of many geometers in the above definition had caused a delay in the development of the theory of parallel lines. Furthermore, Enriques followed Vailati in taking a complex definition as "an implicit way of postulating the existence of an entity, the concept of which is derived from several combined conceptual constructions" (Enriques 1906, p.201). Enriques supplemented this reading with his own psychological explanation. His suggestion was that the combination of properties in the definition of parallel lines remained unnoticed until Saccheri because of its psychological origin in the projective-metric association. Modern axiomatics in Enriques's account avoids the fallacy of complex definition by assuming the basic concepts as non-defined and by formulating all hypotheses different from logical axioms in terms of postulates (p.163).

The above argument substantiates Enriques's considerations about implicit definitions by offering a concrete example of how real definitions are replaced in modern axiomatics with what the Peano School called definitions by postulates.

Furthermore, his account of concept formation allowed him address the epistemological implications of modern axiomatics from a new perspective. Enriques emphasized a parallel between the mathematical and the epistemological implications of transfer principles in the wake of Plücker. On the one hand, the characteristic indeterminacy of the basic concepts of geometry opened the door to the study of a variety of forms of space. The mathematical concept of space is thus an abstract concept. Enriques wrote:

> The concept of space, in its mathematical acceptation, represents the sum total of the (geometrical) relations that exist among points, considered apart from the particular sensations connected with the image of a point. Space is thus conceived as a *manifold of any elements whatever*, to which we give the name "points" simply because they occur in certain ordered relations that are fitted to represent with considerable approximation the relations of place existing among very minute bodies (physical points). (Enriques 1906, p.185)

In accord with his account of the formation of the basic concepts, Enriques here suggests that even the word "point" is a provisional way to refer to the elements of space in analogy with physical points; as a matter of fact, it is the abstract structure of space (as a "manifold of any elements whatever") that provides the best available approximate representation of physical relations of place in modern physics.

On the other hand, Enriques contended that the recognition of the indeterminacy of the basic geometrical concepts opened the door to a generalization of the very notion of physical space. Enriques's argument is a follows:

> When we follow Plücker's principle into the realm of geometrical applications, it seems natural to compare with physical space, some other varieties of elements in which the properties expressed by the translation of the postulates of

> geometry are only partly satisfied.
> Thus a series of abstract spaces arises, for which different geometries are valid. They must however be constructed on a common basis.
> For example, we can conceive of a series of non-Euclidean spaces, in which the postulate about the parallels is not satisfied, and this series, which depends upon the value of a certain constant, improperly called a curvature, now falls under the concept of a space generalized from the concept of ordinary space. (Enriques 1906, p.185)

In order to illustrate how such a series of non-Euclidean spaces can be conceived, Enriques especially referred to the Clifford-Klein forms of space that are locally isometric to the Euclidean plane. These forms according to Enriques: "undertake to represent possible physical constitutions of space that are radically different for an observer who is limited to the narrowness of our experience, and for one whose limits are very decidedly widened" (Enriques 1906, p.197). Enriques also referred to Klein for the view that on account of the fact that measurements are always limited to a region of space, the geometrical representation of spatial relations is an approximation that can be made more or less precise. Therefore, Enriques maintained that even the choice of the basic concepts of geometry, when considered as a part of physics, is open to revision.

To conclude, Enriques pointed out that the generalization of the abstract concept of space can and has been carried out further solely for mathematical purposes as well. He considered, for example, the hypotheses of Archimedean and non-Archimedean spaces to be physically equivalent. He attached particular importance to this example, because non-Archimedean geometry in Enriques's account gave an interesting illustration of the arbitrary character of the postulates. With regard to this case, Enriques considered geometrical conventionalism to be correct, but only in a more restricted sense than Poincaré's.

5 Concluding remarks

Whereas implicit definitions in Schlick's account presupposed a sharp separation between concepts and intuitions, as well as abstract and concrete scientific domains, Enriques followed Klein in advocating a characteristic interrelation of both aspects of knowledge in the very formation of mathematical concepts. Mathematical abstraction in such a view departs from ordinary spatial intuition (understood psychologically, as the association of tactile and visual sensations), insofar as it introduces the notion of a variety of forms of space and increasingly higher standards of precision in the representation of spatial relations. Klein presented his view in his Evanston lectures by saying that abstraction in mathematics turns "naïve" intuitions into "refined" ones, where "the naïve intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact" (Klein 1894, p.42). Typically, as in the Clifford-Klein problem of space, such a development culminates in the axiomatic characterization of abstract concepts that have different mathematical and physical instantiations.

In a similar vein, Enriques offered an account of how psychological notions have been replaced in modern axiomatics with abstract concepts. As a result of this development, what is being defined by postulates, strictly speaking, are general forms or structures, and in this sense implicit definitions are the same as structural definitions. In Enriques's usage, however, implicit definitions of the basic concepts of geometry indicate that the meaning of the abstract terms designating them in an axiomatic system is determined relative to the formal implications of the theory, along with its possible instantiations in concrete scientific domains. Enriques also gave a fully general interpretation of the notion of implicit definition by identifying the basic concepts of all disciplines as abstract concepts, whose meaning is determined in part by a formal theory and in part by empirical circumstances. In the examples from the theory of measurement considered by him the basic concepts become increasingly abstract or can be modified as improved standards of precision are being introduced (see Section 3). Enriques emphasized that conceptual changes are conceivable even at the basic level, and that it was conceivable that non-Euclidean hypotheses would find applications in physics.

I have pointed out that Enriques's argumentation is not vulnerable to the charge of conflating incompatible notions under the label "implicit definitions," insofar as he emphasized the epistemological rather than strictly logical scope of his notion. In mathematics as well as in nonmathematical domains the aim of implicit definitions in Enriques's sense is to shed light on the abstraction process at work in the formulation of theories. I believe that this notion of implicit definition, although less familiar to the contemporary reader than Schlick's, deserves closer attention especially for Enriques's insights into conceptual and scientific change.

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