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Higher Order Polynomial Utility Functions: Advatages in their Use

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Abstract

A long aged debate concerns the use of polynomials as utility functions. The main advantages of the polynomial use within a mostly normative and even prescriptive outlook are discussed. Furthermore, conditions on the polynomial coefficients to guarantee the most common utility function requirements for desired wealth-levels are set out. These results have a more clearly prescriptive flavour and display an additional advantage of using polynomials.

1. Introduction

A debated issue in Decision Theory according to the von Neumann and Morgenstern framework concerns the use of polynomials as utility functions. The discussion takes its origin from the Arrow (1964) and Hicks (1962) criticism against quadratic functions, because of their non decreasing absolute risk-aversion for any wealth level. According to Arrow (1964), an appropriate utility function for a risk-avert individual should display a set of essential properties: positive and decreasing marginal utility, decreasing marginal absolute risk-aversion and increasing marginal relative risk-aversion. Afterwards, other authors (see Levy (1969), Tsiang (1972), Kraus- Litzenberger (1976)) have expressed similar perplexities. Recently, Benishay (1987) has considered a particular fourth order polynomial and afterwards, its conformity to Arrow-Pratt requirements has been shown in Benishay (1992).

The aim of this note is to go one step further and support the use of polynomial utility functions within the framework which leads us to the extended de Finetti model (see Rossi (1994)). The interest of this peculiar theory, as regards our paper, lies in its validity over a closed and bounded interval; feature which is not present in the better known von Neumann-Morgestern theory, that requires consideration at the whole real axis (see von Neumann-Morgenstern, 1944). From the application point of view this assumption is acceptable in many situations. Such is the case when considering amounts of money: in practical situations infinite money amounts do not occur, so it is reasonable to assume that amounts are bounded.

The paper is organised in two parts. The first part lists some advantages of the polynomial use within a mostly normative outlook. The principal one lies in the fact that any sufficiently smooth utility function can be expanded by the Taylor formula, so the polynomials appear to be the most natural approximation for the "true" function. Moreover, the polynomial utility function exhibits the handy peculiarity of expressing the integral in the various formulas of the expected utility of a random amount as a linear combination of the distribution moments. That is in accordance with a preference ordering established in terms of the moments of the distribution (Richter (1960), Scott and Horvarth (1980)). The second part concerns conditions on the third degree polynomial coefficients that guarantee the most common utility function requirements for desired wealth-levels. The use of piece-wise polynomials is also suggested. Constraints for the polynomial coefficients are set out. This part has a more clearly prescriptive flavour and is another advantage of using polynomials.

The paper is organised as follows. Sec. 2 is devoted to explaining the advantages in the polynomial use. Sec. 3 concerns the conditions on the polynomial coefficients to guarantee a set of significant properties. A numerical example is presented in Sec. 4. Sec. 5 deals with the determination of the certainty equivalent of a random amount.

2. Advantages in using polynomials

Assume that the agent utility function u is differentiable until the n-th order. The familiar Taylor formula yields

$$u(x) \cong u(x_0)+u'(x_0)(x-x_0)+\frac{u''(x_0)}{2}(x-x_0)^2+\ldots+\frac{u^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{1}$$

where $u^{(i)}, i = 1,\ldots,n$ is the i-th derivative[1] of u and x_0 is an appropriate point.

Formula (1) displays the main advantage in using a polynomial utility function: in fact, it offers a simple and *local* approximation of the "true" function u around the value x_o. This dependence on the point x_o suggests construction of the approximating utility function by means of *different* "polynomial pieces" defined over *different* compact intervals.

2.1 Expected utility approximations

Usefulness of (1) comes out whenever the aim is the evaluation of the expected utility of a random variable (r.v.) X with finite support. Imposing special values to x_o, different approximating formulas can be set out.

Let $x_0 = \mu$, where μ is the mean of X. Substituting x_0 by μ in (1) and taking the expected value of both sides, we obtain the expected utility of X as a linear combination of the i-th central moments μ_i, $i = 2,\ldots,n$

[1]Formula (1) requires some strong regularity assumptions. This is the case, for example, when the above differentiability conditions are not acceptable because of the "cowardly behaviour" of the agent at the value x_o. Connections between the regularity of the utility function and the concept of non-cowardice has been explored in Montrucchio-Tibiletti (1994). This issue has long been discussed in the literature and goes back to Samuelson (1963). In the latter, the author deals with the "irrational behaviour" of an agent who rejects very small gambles in his favour. Following Segal-Spivak (1990), Montrucchio-Tibiletti (1994) prefer to use the term "non-cowardice". Sufficient conditions to guarantee a non-cowardly behaviour are the Lipschitz continuity of the first derivative u' or the compactness of the random amount support.

$$\mathrm{E}(u(X)) \cong u(\mu)+\frac{u''(\mu)}{2}\mu_2+\frac{u'''(\mu)}{3!}\mu_3+\ldots+\frac{u^{(n)}(\mu)}{n!}\mu_n \tag{2}$$

Let $x_0 = 0$. Taking the expected value of both sides of (2), we obtain the expected utility of X as a linear combination of the i-th moments from the origin m_i, $i = 2,..,n$

$$\mathrm{E}(u(X)) \cong u(0)+\frac{u''(0)}{2}m_2+\frac{u'''(0)}{3!}m_3+\ldots+\frac{u^{(n)}(0)}{n!}m_n \tag{3}$$

Since m_i, $i = 2,...n$ can be expressed in terms of both the central moments and cumulants further approximating formulas can be easily derived. A relevant consequence is that formulas (2) and (3) can be easily transformed each one to the other. Formulas (2) and (3) lead to some spontaneous considerations.

Existence of the n-th order ($n \geq 1$) distribution moment is an essential condition for (2) and (3) (note that some distributions do not have moments of all orders, such is the case, for example, of the Cauchy one). Nevertheless, the assumption of bounded support for X guarantees our object.

Truncation error must be evaluated over the whole interval in which the polynomial is assumed to hold for our purposes. Formulas (2) and (3) can be truncated to the second order term whenever the distribution involved is completely determined by the mean and the variance. In other cases, a polynomial of order higher than the second is suggested for a better clustering around function u.

Formulas (2) and (3) explicitly impose the use of the moments for evaluating the expected value of X. It is worth noting that it is coherent with the Tintner-Allais Principle (see Allais (1953) and Tintner (1941)[2]) stating the following:

...The *whole* probability distribution function over psychological values *must* be considered. This may be captured by some functions of *all* the distribution moments...

Conditions under which the moments uniquely determinate the distribution have received considerable attention in the Probability Theory (the so-called *moment problem*). The main result goes back to the Hausdorff theorem guaranteeing

[2]For use of this principle, in addition to the cited papers, see the Linearized Moment Model set out by Munera (1986), where the principle is reported as quoted.

the statement for every distribution concentrated on some finite interval[3] provided with a complete monotone sequence of moments (see for example Feller, 1971).

3. Choice of polynomial coefficients

Above we have focused our attention on finding polynomial functions which approximate the "true" utility function u, whenever the higher order derivatives of u in x_0 are *known*. Now, a different problem is addressed. In what follows, we assume that the agent utility is fully described by a n-th degree polynomial, the coefficients of which are to be chosen

$$u(x) = a_0(x-x_0)^n + a_1(x-x_0)^{n-1} + a_2(x-x_0)^{n-2} + \ldots + a_n, \quad a_i \in \Re, i = 0,1,\ldots,n \tag{4}$$

where x_0 is a suitable point.

It is worth pointing out that there are circumstances where the utility function *must be necessarily* an n-degree polynomial such as (4). That is the case, for example, when the agent adopts a preference ordering that is established in terms of the first n moments of X (see Markowitz (1959), Richter (1960) and Borch (1969)).

Our investigation is aimed at finding extreme bounds for the polynomial coefficients $a_i, i = 1,\ldots,n$ in order that u be consistent with a set of desirable requirements.

3.1 A Set of Desirable Requirements

Basic requirements that we will consider involve (*a*) the signs of the higher order derivatives and (*b*) the absolute and relative risk-aversion coefficients and the prudence coefficient.

3.1.1 Signs of the higher order derivatives

Conditions on the higher order derivative signs of a utility function have long been discussed in the literature both from an empirical and a theoretical point of view. The crucial point is that they determine the preference towards the distribution moments, as shown by formula (2). Considerable attention has been devoted to the third derivative (see Arditti (1967), Kraus and Litzenberger (1976), Simkowitz and Beedles (1978)). These Authors support the thesis that a risk-avert agent is inclined to a positive third moment, or in other terms the third derivative of the utility function underlying his behaviour is positive. Specifically, a risk-avert investor appears willing to give up some return in order

[3]For distributions which are not concentrated on some finite interval, additional restrictions on the rate of growth of the moments of even order are needed (see for example, the Carleman theorem reported in Feller, 1971).

to participate in what amounts to a lottery for larger gains. Recently, Kimball (1991) has formalised this concept. An individual is said to be prudent if the so-called coefficient of absolute prudence $\phi(x) = -u'''(x)/u''(x)$ is positive. This assumption, coupled with the negativeness of u'', implies the positiveness of u'''. The importance of the fourth derivative sign has been recently recognised by Eeckhoudt *et al.*(1994), since it could determine a different behaviour of a rational agent in problems of comparative static. For higher moments, Scott and Horvarth (1980) proved that the signs of the consecutive derivatives of the investor's utility function alternate from positive to negative, implying that rational investors prefer the odd moments and dislike the even moments of the distribution[4].

3.1.2 Relative and Absolute Risk-Aversion and Absolute Prudence Coefficients

Almost all the common utility functions exhibit decreasing absolute risk-aversion (DARA) and increasing relative risk-aversion (IRRA) coefficients. These two properties will be fulfilled in the following.

It is worth noting that since DARA assumption implies the positiveness of u''', therefore a risk-avert agent endowed with a DARA utility function is also prudent, in the above sense[5].

3.1.3 Further decision maker's requirements

Another consequence of the presence of a linear combination of moments is that we can identify our u and its coefficients by means of a comparison of different uncertainty situations. This idea was firstly proposed by Diale (1993) and developed in a pre-print reading by Rossi (1991). These comparisons help us to clarify our attitude towards risk, by considering our attitude towards the various moments. In this way some bounds for coefficients are achieved and can be added to the ones proposed in this section. The final result is a set of feasible choices for coefficients quite like the domain of a mathematical programming problem. If the set is non-empty, the requirements we imposed are compatible and not contradictory. Increasing the degree of the polynomial gives us space for stronger requirements. We have then the opportunity to impose other requirements we think should be respected, such as some smoothness ones, thus completing the structure of the mathematical programming problem. This feature is, in our opinion, interesting under a prescriptive point of view.

[4]In line with these results, multidimensional asset pricing models have been also set out (see, for example, Homaifar and Graddy (1988))

[5]In fact, if the derivative of the absolute risk aversion $(-u''/u')$ is negative, then $u''' > 0$.

4. A Numerical Example

In what follows, for explanatory purposes we will deal with a third-degree polynomial to which we shall impose the commonly assumed conditions 4.1.1 and 4.1.2. Let

$$u(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3, \quad x \in [s,t] \quad s,t \in \Re, \ a_i \in \Re, i = 0,1,2,3 \tag{5}$$

be the utility function of the agent[6]. Straightforward calculations lead to

$u'(x) = 3a_0 x^2 + 2a_1 x + a_2$

$u''(x) = 6a_0 x + 2a_1$

$u'''(x) = 6a_0$

It is worthwhile noting that:

- coefficient a_3 is irrelevant and
- because of u is determinate up to a positive linear affine transformation, one of $a_i, i = 0,1,2$ can be defined arbitrarily. Subsequently, the choice is aimed to obtain computational facilities.

Derivatives signs

Imposing $u'(x) > 0$, $u''(x) < 0$, $u'''(x) > 0$, the following **necessary and sufficient** conditions come out:

$$\text{if } a_2 > \frac{a_1^2}{3a_0}: \quad a_0 > 0 \text{ and } t < -\frac{a_1}{3a_0}, \tag{6}$$

$$\text{if } a_2 \le \frac{a_1^2}{3a_0}: \quad a_0 > 0 \text{ and } \frac{-a_1 + \sqrt{a_1^2 - 3a_0 a_2}}{3a_0} \le s \text{ or}$$

$$a_0 > 0 \text{ and } t \le -\frac{a_1 + \sqrt{a_1^2 - 3a_0 a_2}}{3a_0}, \tag{7}$$

Decreasing Absolute Risk Aversion (DARA) and Increasing Relative Risk Aversion (IRRA)

Imposing the decrease of the *absolute risk aversion coefficient, i.e.*,

$$\frac{d}{dx}\left(-\frac{u''(x)}{u'(x)}\right) < 0.$$

we obtain $9a_0^2 x^2 + 6a_0 a_1 x - \left(3a_0 a_2 - 2a_1^2\right) < 0$.

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[6]For sake of simplicity, we use directly variable x. Analogue computations could be carried out using variables like $(x - x_0)$, for appropriate value of x_0.

If $a_2 > \frac{a_1^2}{3a_0}$: $$a_0 > 0 \text{ and } \frac{-a_1 - \sqrt{3a_0a_2 - a_1^2}}{3a_0} < s \le t < \frac{-a_1 + \sqrt{3a_0a_2 - a_1^2}}{3a_0} \quad (8)$$

If $a_2 \le \frac{a_1^2}{3a_0}$, DARA is not achieved. Coupling (8) with conditions (6)-(7) we yield

$$\frac{-a_1 - \sqrt{3a_0a_2 - a_1^2}}{3a_0} < s \le t < -\frac{a_1}{3a_0} \text{ where } a_0 > 0,\ a_2 > \frac{a_1^2}{3a_0}. \quad (9)$$

That derives from the fact that *only* condition (6) is consistent with the assumption of the increasing absolute risk aversion coefficient. Note that $a_2 > 0$.

Impose, now, the increase of the *relative risk aversion coefficient, i.e.*,

$$\frac{d}{dx}\left(-\frac{u''(x)}{u'(x)}x\right) > 0.$$

Substituting the derivative expressions, we obtain $3a_0a_1x^2 + 6a_0a_2x + a_1a_2 < 0$.

We can distinguish two cases:

i) Let $a_1 < 0$. Therefore, we yield

$$s > \frac{3a_0a_2 + \sqrt{3a_0a_2\left(3a_0a_2 - a_1^2\right)}}{-3a_0a_1} \text{ or } t < \frac{3a_0a_2 - \sqrt{3a_0a_2\left(3a_0a_2 - a_1^2\right)}}{-3a_0a_1}. \quad (10)$$

Note that

$$\frac{-a_1 - \sqrt{3a_0a_2 - a_1^2}}{3a_0} < \frac{3a_0a_2 - \sqrt{3a_0a_2\left(3a_0a_2 - a_1^2\right)}}{-3a_0a_1} < \frac{-a_1}{3a_0} < \frac{3a_0a_2 + \sqrt{3a_0a_2\left(3a_0a_2 - a_1^2\right)}}{-3a_0a_1}$$

Combining (9) with (10) we obtain the final condition

$$\frac{-a_1 - \sqrt{3a_0a_2 - a_1^2}}{3a_0} < s \le t < \frac{3a_0a_2 - \sqrt{3a_0a_2\left(3a_0a_2 - a_1^2\right)}}{-3a_0a_1} \text{ where } a_0 > 0,\ a_1 < 0, a_2 > \frac{a_1^2}{3a_0}.$$

A special case

As mentioned before a_0 can be arbitrarily chosen, for sake of simplicity we fix $a_0 = 1/3$. The above conditions become

$$-a_1 - \sqrt{a_2 - a_1^2} < s \le t < \frac{a_2 - \sqrt{a_2\left(a_2 - a_1^2\right)}}{-a_1} \text{ where } a_1 < 0,\ a_2 > a_1^2. \quad (11)$$

Above we have provided conditions on the support $[s,t]$ in terms of the polynomial parameters a_i. Analogously, we can find conditions on each parameter a_i involving the remaining ones and the values of s and t.

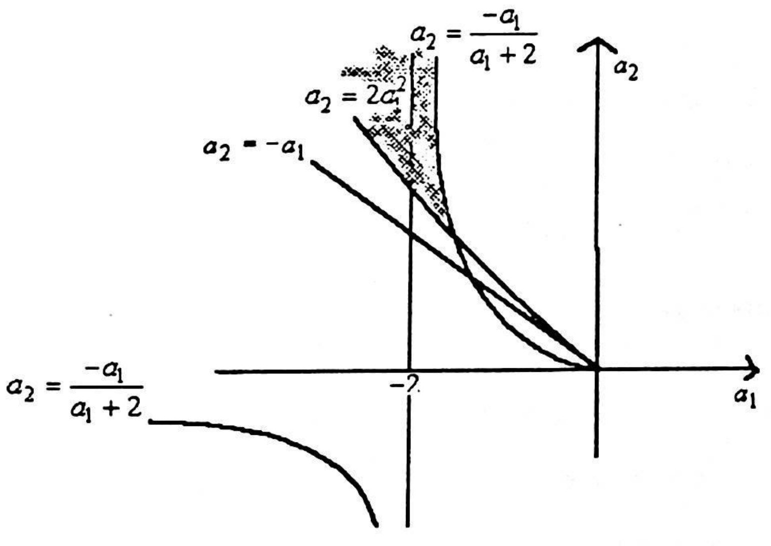
For explanatory purposes, let us assume $0 \le s < t$. Conditions (11) can be rewritten as the union of the solutions of the following systems

$$\begin{cases} a_1 < 0 \\ a_2 > a_1^2 \\ a_1 \ge -s \\ a_2 > -ta_1 \\ a_2(a_1+2t) < -t^2 a_1 \end{cases} \quad \text{and} \quad \begin{cases} a_1 < 0 \\ a_2 > a_1^2 \\ a_1 < -s \\ a_2 > 2a_1^2 + 2sa_1 + s^2 \\ a_2 > -ta_1 \\ a_2(a_1+2t) < -t^2 a_1 \end{cases}$$

In the special case where $s = 0$ and $t = 1$, we obtain the unique system

$$\begin{cases} a_1 < 0 \\ a_2 > 2a_1^2 \\ a_2 > -a_1 \\ a_2(a_1+2) < -a_1 \end{cases}$$

The solution is represented on the plane (a_1, a_2) by the following darkened area:



ii) Let $a_1 \ge 0$. By (9) it results $s < t < 0$. Simple calculations lead to if $a_1 > 0$ $\quad \dfrac{-a_1 - \sqrt{3a_0 a_2 - a_1^2}}{3a_0} < s \le t < \dfrac{-3a_0 a_2 + \sqrt{3a_0 a_2 (3a_0 a_2 - a_1^2)}}{3a_0 a_1}$,

if $a_1 = 0$ $\quad \dfrac{-a_1 - \sqrt{3a_0 a_2 - a_1^2}}{3a_0} < s \le t < 0$, where $a_0 > 0$, $a_2 > \dfrac{a_1^2}{3a_0}$.

For $a_0 = 1/3$, above becomes

if $a_1 > 0$: $\quad -a_1 - \sqrt{a_2 - a_1^2} < s \le t < \dfrac{-a_2 + \sqrt{a_2(a_2 - a_1^2)}}{a_1}$,

if $a_1 = 0$: $\quad -a_1 - \sqrt{a_2 - a_1^2} < s \le t < 0$.

These conditions are equivalent to the union of the solutions of the following four systems

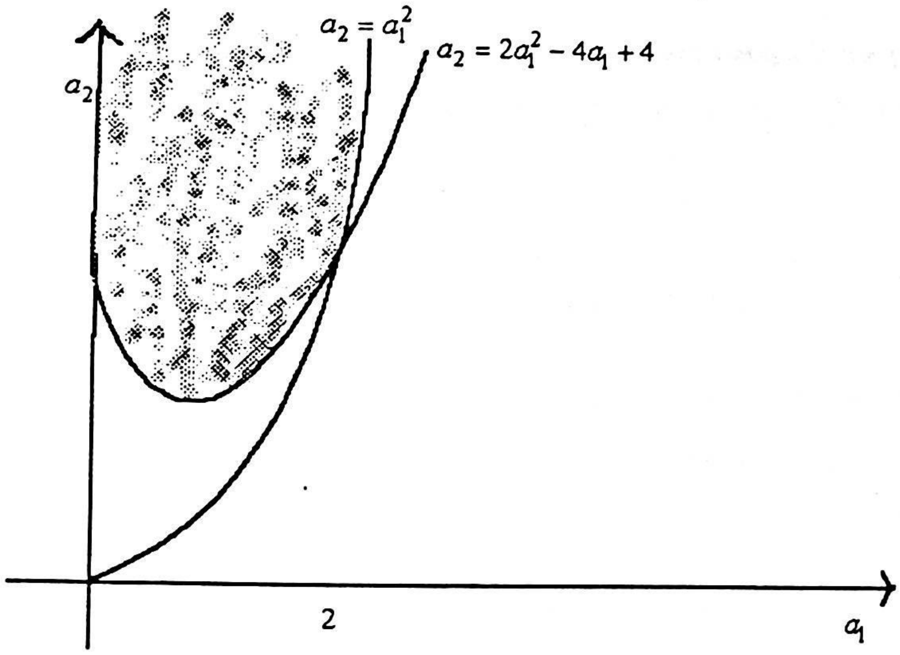
$$\begin{cases} a_2 > a_1^2 \\ a_1 \ge -s \\ a_2 < -ta_1 \end{cases} \quad ; \quad \begin{cases} a_2 > a_1^2 \\ a_1 \ge -s \\ a_2 \ge -ta_1 \\ a_1 < -2t \\ a_2 > \dfrac{-t^2 a_1}{a_1 + 2t} \end{cases}$$

$$\begin{cases} a_2 > a_1^2 \\ a_1 > -s \\ a_2 < -ta_1 \\ a_2 > 2a_1^2 + 2sa_1 + s^2 \end{cases} \quad ; \quad \begin{cases} a_2 > a_1^2 \\ a_1 < -s \\ a_2 \ge -ta_1 \\ a_1 < -2t \\ a_2 > \dfrac{-t^2 a_1}{a_1 + 2t} \\ a_2 > 2a_1^2 + 2sa_1 + s^2 \end{cases}$$

In the special case where $s = -2$ and $t = -1$, above systems reduce to

$$\begin{cases} a_2 > a_1^2 \\ a_1 \ge 2 \end{cases} \quad ; \quad \begin{cases} a_2 > a_1^2 \\ a_1 < 2 \\ a_2 > 2a_1^2 - 4a_1 + 4 \end{cases}$$

The solution set is provided by the darkened area below



5. Polynomial utility for certainty equivalent

By the Nagumo-Kolmogorov-de Finetti theorem (see de Finetti, 1931 and Hardy-Littlewood-Pólya, 1952, pp. 158-163, where associativity is called quasi-linearity) if a random variable (r.v.) X has a bounded support and is an associative monotonic set function, then its certainty equivalent $\bar{x}$ is an associative and monotonic Chisini mean expressed by

$$\bar{x} = u^{-1} \int_{\Re} u(x) dF(x), \tag{12}$$

where F stands for the distribution function of X. For an extension of the above formula we refer the reader to Rossi (1994).

The antitransform before the integral in formula (12) is not necessary if we wish just to make comparisons having always the same u. If, instead, the aim is addressed at finding the certainty equivalent of a random amount or of its utility, we need the inverse u^{-1} of u. Strictly increasing property over the interval under consideration guarantees its existence and uniqueness. We do not develop the inversion methods as there is an ample literature about them. Just out of curiosity and as we used a third degree polynomial, we shall recall that the first solution of a third order polynomial equation was published by Cardano (1545) and there was a great dispute for attributing about its priority. In the numerical example exploited below we will use just the Cardano formula.

5.1 An example of certainty equivalent calculation

In accordance to the case plotted in Figure 1, we fix $s = 0$ and $t = 1$ and we choose $a_0 = \frac{1}{3}, a_1 = -2, a_2 = 9, a_3 = 0$. The utility function is

$$y = u(x) = \frac{1}{3}x^3 - 2x^2 + 9x.$$

Let us calculate the inverse[7] of u. The equation

$$x^3 - 6x^2 + 18x - 3y = 0$$

is of the type of

$$x^3 + ax^2 + bx + c = 0.$$

Its solution is

$$x = u^{-1}(y) = \frac{1}{3}\left[-a + \sqrt[3]{\frac{1}{2}\left(\sigma + 3\sqrt{-3\Delta}\right)} + \sqrt[3]{\frac{1}{2}\left(\sigma - 3\sqrt{-3\Delta}\right)}\right]$$

where $\sigma = -2a^3 + 9ab - 27c$, $\Delta = -27c^2 + 18abc - 4a^3c - 4b^3 + a^2b^2$, (see for example, Berzolari *et al.* (1932), page 278). Substituting we obtain

$$\sigma = 9(-60 + 9y) \text{ and } \Delta = -243y^2 + 3240y - 11664.$$

Let us consider the r.v. X such that

$$P(X = 0) = \frac{1}{2} \text{ and } P\left(X = \frac{1}{2}\right) = \frac{1}{4} \text{ and } P(X = 1) = \frac{1}{4}.$$

By straightforward calculations, we obtain

$$Eu(X) = 2{,}844 \tag{13}$$

so

$$\bar{x} = u^{-1}[Eu(X)] = 0{,}573. \tag{14}$$

Formulas (13) and (14) highlight two aspects:

- the expected utility has *not* a monetary interpretation: the utility u is defined up to a linear increasing transformation, so the expected utility varies as u varies.
- on the other hand, the certainty equivalent has a significant money explanatory: it provides the price of the lottery X.

Finally, we point out the interesting behaviour of truncation error in the formula (12) as regards. As u is determinated up to a positive linear affine transformation, the truncation error in u is eliminated by the corresponing error in u^{-1}, so

[7]Note that u^{-1} exists, because u is strictly increasing on $[0,1]$.

its numerical evaluation is pointless. Quite to the contrary, truncation is relevant in that using a different degree polynomial as u gives a different shape curve for u and a different certainty equivalent $\bar{x}$

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