Small points on subvarieties of algebraic tori: results and methods

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Abstract. This paper is a survey on some quantitative versions of Bogomolov's conjecture for a torus obtained by the author together with Sinnou David. The proofs are partially sketched, starting with the simplest case of hypersurfaces.

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1 Introduction : from torsion to small points

The former Manin-Mumford conjecture predicts that the set of torsion points of a curve of genus ≥ 2 embedded in its jacobian is finite. More generally, let $\mathbb{G}$ be a semi-abelian variety and V an algebraic subvariety of $\mathbb{G}$, defined over some algebraically closed field K. We say that V is a *torsion subvariety* if V is a translate of a proper subtorus by a torsion point of $\mathbb{G}$. We also denote by $V_{\rm tors}$ the set of torsion points of $\mathbb{G}$ lying on V. Then we have the following generalization of the Manin-Mumford conjecture

Theorem 1.1 *i) If V is not a torsion subvariety, then the set $V_{\rm tors}$ of torsion points of $\mathbb{G}$ lying on V is not Zariski dense.*

ii) The Zariski closure of $V_{\rm tors}$ is a finite union of torsion subvarieties.

The two assertions are clearly equivalent. Theorem 1.1 was proved by Raynaud ([Ray 1983]) when $\mathbb{G}$ is an abelian variety, by Laurent ([Lau 1984]) if $\mathbb{G} = \mathbb{G}_m^n$, and finally by Hindry ([Hin 1988]) in the general situation.

We assume from now on that all varieties are algebraic, defined over $\overline{\mathbb{Q}}$ and geometrically irreducible. Bogomolov ([Bog 1981]) gave the following generalization of the former Manin-Mumford conjecture: a curve $\mathcal{C}$ of genus

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≥ 2 embedded in its jacobian is discrete for the metric induced by the Néron-Tate height. In other words, Bogomolov conjectures that the set of points of "sufficiently small" height on $\mathcal{C}$ is finite, while the former Manin-Mumford conjecture makes a similar assertion on the set of torsion points (which are precisely the points of zero height).

More generally, let $\mathbb{G}$ be a semi-abelian variety and let $\hat{h}$ be a normalized height on $\mathbb{G}(\overline{\mathbb{Q}})$. Hence, $\hat{h}$ is the Neron-Tate height if $\mathbb{G}$ is abelian, and it is the Weil height if $\mathbb{G} = \mathbb{G}_m^n \hookrightarrow \mathbb{P}_n$ (see section 2 for details); in particular $\hat{h}$ is a non-negative function on $\mathbb{G}$ and $\hat{h}(P) = 0$ if and only if P is a torsion point. Given an algebraic subvariety of $\mathbb{G}$, we denote by V^* the complement in V of the Zariski closure of the set of torsion points of V. Therefore, by theorem 1.1, $V \backslash V^* = \overline{V_{\text{tors}}}$ is a finite union of torsion varieties.

Theorem 1.2 *Let V be a subvariety of a semi-abelian variety $\mathbb{G}$. Then:*

i) If V is not a torsion subvariety, then there exists $\theta > 0$ such that the set $V(\theta) = \{P \in V \text{ such that } \hat{h}(P) \leq \theta\}$ is not Zariski dense in V.

ii) V^* is discrete for the metric induced by $\hat{h}$, i.e.

$$\inf\{\hat{h}(P) \text{ such that } P \in V^*\} > 0.$$

It is easy to see that the two assertions are equivalent. In this formulation, theorem 1.2 was proved for $\mathbb{G} = \mathbb{G}_m^n$ by Zhang (see [Zha 1995]). In the abelian case, Ullmo (see [Ull 1998]) proved Bogomolov's original formulation for curves ($\dim(V) = 1$); immediately after Zhang (see [Zha 1998]) prove theorem 1.2. The semi-abelian case was solved by David and Philippon (see [Dav-Phi 2000]).

In this article we describe some quantitative versions of theorem 1.2 for a torus $\mathbb{G} = \mathbb{G}_m^n$ and we sketch proofs of theorems which prove these conjectures "up to an ε".

The plan of the paper is as follows: in section 2 we introduce the normalized height and the essential minimum of an algebraic subvariety of $\mathbb{G}_m^n$. In section 3 we recall the main conjectures and results on the essential minimum. Proofs will be sketched in section 4. Finally, in the last section, we give some more precise conjectures and results on the distribution of points of small heights ("small points").

Contents

2 Heights

Let $\alpha \in \overline{\mathbb{Q}}$ and let K be *any* number field containing α. We denote by $\mathcal{M}_K$ the set of places of K. For $v \in K$, let K_v be the completion of K at v and let $|\cdot|_v$ be the (normalized) absolute value of the place v. Hence

$$|\alpha|_v = |\sigma\alpha|,$$

if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. If v is a non archimedean place associated with the prime ideal $\wp$ over the rational prime, we have

$$|\alpha|_v = p^{-\lambda/e},$$

where e is the ramification index of $\wp$ and λ is the exponent of $\wp$ in the factorization of the ideal (α) in the ring of integers of K. This standard normalization agrees with the product formula

$$\prod_{v\in\mathcal{M}_K} |\alpha|_v^{[K_v:\mathbb{Q}_v]} = 1$$

which holds for any $\alpha \in K^*$.

We define the Weil height of α by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in\mathcal{M}_K} [K_v:\mathbb{Q}_v] \log\max\{|\alpha|_v, 1\}.$$

It is easy to see that this definition does not depend on the field K containing α; hence, it defines a function $h\colon \overline{\mathbb{Q}} \to \mathbb{R}^+$.

More generally, let $\boldsymbol{\alpha} = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}_n(K)$ and let K be any number field containing $\alpha_0, \ldots, \alpha_n$. We define the Weil height of $\boldsymbol{\alpha}$ by:

$$h(\boldsymbol{\alpha}) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in\mathcal{M}_K} [K_v:\mathbb{Q}_v] \log\max\{|\alpha_0|_v, \ldots, |\alpha_n|_v\}.$$

As before, this definition does not depend on the number field K; moreover it does not depend on the projective coordinates of $\boldsymbol{\alpha}$ (by the product formula).

The Weil height of an algebraic number is related to the Mahler measure of a polynomial. Let $f \in \mathbb{C}[x]$ be non-zero; then its Mahler measure is

$$M(f) = \exp \int_0^1 \log\left|f\left(e^{2\pi i t}\right)\right| dt.$$

Let $\alpha_1, \ldots, \alpha_d$ be the roots of f and f_d be its leading coefficient; by Jensen's formula we easily see that

$$M(f) = |f_d| \prod_{j=1}^{d} \max\{|\alpha_j|, 1\}. \tag{1}$$

Let $f \in \mathbb{Z}[x]$ be the minimal polynomial of α over $\mathbb{Z}$ (*i.e.* f is irreducible in $\mathbb{Z}[x]$, $f(\alpha) = 0$ and its leading coefficient is positive); from (1) it is also easy to see that

$$h(\alpha) = \frac{\log M(f)}{[\mathbb{Q}(\alpha):\mathbb{Q}]}. \tag{2}$$

We now consider a torus $\mathbb{G}_m^n$ and we fix the "standard embedding" $\iota\colon \mathbb{G}_m^n \hookrightarrow \mathbb{P}_n$,

$$\iota(x_1, \dots, x_n) = (1 : x_1 : \dots : x_n).$$

This gives the height function $\hat{h}(x_1, \dots, x_n) = h(1 : x_1 : \dots : x_n)$. The following properties hold:

i) the function $\hat{h}$ is a positive function on $\mathbb{G}_m^n(\overline{\mathbb{Q}})$, vanishing only on its torsion points;

ii) $\hat{h}(\boldsymbol{\alpha\beta}) \leq \hat{h}(\boldsymbol{\alpha}) + \hat{h}(\boldsymbol{\beta})$. Moreover, if $\boldsymbol{\zeta}$ is a torsion point, $\hat{h}(\boldsymbol{\zeta\alpha}) = \hat{h}(\boldsymbol{\alpha})$ and if $n \in \mathbb{N}$ then $\hat{h}(\boldsymbol{\alpha}^n) = n\hat{h}(\boldsymbol{\alpha})$;

iii) a subset of $\mathbb{G}_m^n(\overline{\mathbb{Q}})$ of bounded height and bounded degree is finite.

2.1 Hypersurfaces

We have a "natural" definition of height on hypersurfaces since we can extend the Mahler measure to polynomials in several variables. Let $f \in \mathbb{C}[x_1, \dots, x_n]$; we define its Mahler measure as:

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log \left| f\left(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}\right) \right| dt_1 \dots dt_n.$$

Let now K be a number field and let V be an hypersurface in $\mathbb{G}_m^n$ defined over K:

$$V = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ such that } f(\boldsymbol{\alpha}) = 0\}$$

for some polynomial $f \in K[\mathbf{x}]$ (irreducible over $\overline{\mathbb{Q}}[\mathbf{x}]$). We define:

$$\hat{h}(V) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v-adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$.

Remark 2.1 *If $n = 1$, then $V = \{\alpha\}$ and we have*

$$\hat{h}(V) = \hat{h}(\alpha), \tag{3}$$

since $M_v(x - \alpha) = \max\{1, |\alpha|_v\}$ (by (1))

Let V be an arbitrary subvarieties of $\mathbb{G}_m^n$. For $l \in \mathbb{N}$, we define

$$[l]^{-1}V = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ such that } \boldsymbol{\alpha}^l \in V\}$$

and

$$[l]V = \{\boldsymbol{\alpha}^l \text{ such that } \boldsymbol{\alpha} \in V\}.$$

We are interested in relations between the degree[1] and the height of V, $[l]^{-1}V$ and $[l]V$. For the degree we have:

$$\deg([l]^{-1}V) = l^{\operatorname{codim}(V)} \deg(V) \tag{4}$$

and

$$\deg([l]V) = \frac{l^{\dim(V)} \deg(V)}{|\operatorname{Ker}([l]) \cap \operatorname{Stab}(V)|}, \tag{5}$$

where $\operatorname{Stab}(V) = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ such that } \boldsymbol{\alpha}V = V\}$ is the stabilizer of V. The first equality is easily proved, while the second one follows from the first and from

$$[l]^{-1}[l]V = \bigcup_{\omega \in \operatorname{Ker}([l])} \omega V.$$

For further references we remark that

$$|\operatorname{Ker}([l]) \cap \operatorname{Stab}(V)| = l^{\dim \operatorname{Stab}(V)} |\operatorname{Ker}([l]) \cap (\operatorname{Stab}(V)/\operatorname{Stab}(V)^0)|, \tag{6}$$

where $\operatorname{Stab}(V)^0$ is the neutral component of $\operatorname{Stab}(V)$ (*i.e.* its connected component containing $\mathbf{1}$).

Let us suppose that V is an hypersurface. We have

$$\hat{h}([l]^{-1}V) = \hat{h}(V)$$

and

$$\hat{h}([l]V) = \frac{l^n \hat{h}(V)}{|\operatorname{Ker}([l]) \cap \operatorname{Stab}(V)|}.$$

Let $f \in \overline{\mathbb{Q}}[\mathbf{x}]$ be an equation for V. Again, the first equality is clear, since $f(\mathbf{x}^l)$ is an equation for $[l]^{-1}V$ and $M_v(f(\mathbf{x}^l)) = M_v(f)$, while the second equality follows from the first one and from the multiplicativity of M_v.

Let $\|f\|_1$ be the sum of the absolute values of the coefficients of $f \in \mathbb{C}[\mathbf{x}]$ (the "length" of f). Since the maximum of $|f|$ on the product of unit disks is bounded by $\|f\|_1$, we have $M(f) \leq \|f\|_1$. Moreover,

$$\|f\|_1 \leq 2^{d_1 + \cdots + d_n} M(f),$$

[1]The degree of an algebraic set $V \subseteq \mathbb{G}_m^n$ is the degree of its Zariski closure in $\mathbb{P}_n$.

where $d_1, \ldots, d_n$ are the partial degrees of f. If $n = 1$, this follows from (1) and from usual formulas for the coefficients in $\mathbb{C}[x]$, while the general case can be proved by induction on n, see [Mig 1992] for details.

Now, let $\|\cdot\|$ be any norm on $\mathbb{C}[x_1, \ldots, x_n]$ such that

$$\log \|f\| = \log \|f\|_1 + o\big((\deg f)^{1+1/(n-1)}\big) \tag{7}$$

We define an height on hypersurfaces of $\mathbb{G}_m^n$ by choosing the norm $\|\cdot\|$ at the archimedean places. We define

$$h(V) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log H_v(f),$$

where $H_v(f) = M_v(f)$ if v is non archimedean, and $H_v(f) = \|\sigma f\|$ if v is an archimedean place associated with the embedding $\sigma \colon K \hookrightarrow \overline{\mathbb{Q}}$. Then, by the previous discussion,

$$\hat{h}([l]V) = h([l]V) + o\big(\deg([l]V)^{1+1/(n-1)}\big),$$

as $l \to +\infty$. Using the relations between degree and height of V and $[l]V$ we see that

$$\frac{\hat{h}([l]V)\deg(V)}{l \deg([l]V)} = \hat{h}(V)$$

and $\deg([l]V) \leq l^{n-1}\deg(V)$. We have proved:

Remark 2.2 *Let $h(\cdot)$ be any height function on the hypersurfaces of $\mathbb{G}_m^n$ defined at the archimedean places by a norm satisfying* (7)*. Then*

$$\lim_{l \mapsto \infty} \frac{h([l]V)\deg(V)}{l \deg([l]V)} = \hat{h}(V).$$

2.2 Subvarieties of arbitrary dimension

The last remark suggests a "simple" definition of normalized height on subvarieties of $\mathbb{G}_m^n$, alternative to the one commonly used in Arakelov theory. We start by choosing a height on the subvarieties. If V is a d dimensional subvariety and F is its Chow form[2], we can define the height $h(V)$ as the

[2] *i.e.* the irreducible multihomogeneous polynomial $F(\mathbf{u}^1, \ldots, \mathbf{u}^{d-1})$, where $\mathbf{u}^j = (u_0^j, \ldots, u_n^j)$, vanishing precisely if the intersection of V with the hyperplanes of coordinates $\mathbf{u}^1, \ldots, \mathbf{u}^{d-1}$ is non empty.

height of the hypersurface in $\mathbb{G}_m^{(d-1)n}$ defined by $\{F = 0\}$, choosing any reasonable norm (*i.e.* satisfying (7)) at the archimedean places. David and Philippon (see [Dav-Phi 1999]) prove that the limit

$$\hat{h}(V) = \lim_{l \to +\infty} \frac{h([l]V)\deg(V)}{l \deg([l]V)}$$

exists. It is easy to see (compute the Chow form) that this definition of normalized height specializes in the previous ones if V is a point or if V is an hypersurface (see [Dav-Phi 1999]). Moreover:

i) the function $\hat{h}(\cdot)$ is non-negative;

ii) for every $l \in \mathbb{N}$ we have

$$\hat{h}([l]^{-1}V) = l^{\operatorname{codim}(V)-1}\hat{h}(V)$$

and

$$\hat{h}([l]V) = \frac{l^{\dim(V)+1}\hat{h}(V)}{|\operatorname{Ker}([l]) \cap \operatorname{Stab}(V)|}.$$

iii) for every torsion point $\boldsymbol{\zeta}$ we have $\hat{h}(\boldsymbol{\zeta}V) = \hat{h}(V)$.

Using property iii) and ii), we see that a torsion subvariety $V = \boldsymbol{\zeta}H$ has height zero. Indeed, if $\boldsymbol{\zeta}$ is a torsion point and H is a subtorus, then $\hat{h}(\boldsymbol{\zeta}H) = \hat{h}(H)$ and $\hat{h}(H) = \hat{h}([l]H) = l\hat{h}(H)$ for any $l \in \mathbb{N}$ (since $H = [l]H$ and $|\operatorname{Ker}([l]) \cap H| = l^{\dim(H)}$).

Are torsion varieties the only varieties of zero height? The answer is positive; more precisely, this question is equivalent to the multiplicative analogue of the former Bogomolov's conjecture. To see this, let us define the essential minimum $\hat{\mu}^{\mathsf{ess}}(V)$ of a subvariety V as the infimum of the set of $\theta > 0$ such that the subset

$$V(\theta) = \{P \in V \text{ such that } \hat{h}(P) \leq \theta\}$$

is Zariski dense in V. Theorem 1.2 asserts that $\hat{\mu}^{\mathsf{ess}}(V) = 0$ if and only if V is torsion. By a special case of an inequality of Zhang (see [Zha 1995], theorem 5.2.), we also have

$$\hat{\mu}^{\mathsf{ess}}(V) \leq \frac{\hat{h}(V)}{\deg(V)} \leq (\dim(V) + 1)\hat{\mu}^{\mathsf{ess}}(V). \tag{8}$$

Hence $\hat{h}(V) = 0$ if and only if $\hat{\mu}^{\mathsf{ess}}(V) = 0$.

3 Quantitative results.

We are interested in lower bounds for the essential minimum of a non-torsion subvariety V of $\mathbb{G}$. These lower bounds will depend on some geometric invariants of V, for instance its degree. Moreover, if we do not make any further geometric assumption on the variety, they must also depend on its field of definition ("arithmetic case"). Indeed, let H be any subgroup of $\mathbb{G}_m^n$ and let $\boldsymbol{\alpha}_n$ be a sequence of non-torsion points whose heights converge to zero (for instance, $\boldsymbol{\alpha}_n = (2^{1/n}, \dots, 2^{1/n})$). Then, the varieties $V_n = H\boldsymbol{\alpha}_n$ have fixed degree $\deg(H)$ and essential minimum $\hat{\mu}^{\mathsf{ess}}(V_n) \leq \hat{h}(\boldsymbol{\alpha}_n) \to 0$. In spite of that, if we also assume that V is not a translate of a proper subgroup (even by a point of infinite order), then Bombieri and Zannier ([Bom-Zan 1995]) proved that the essential minimum of V can be bounded from below only in terms of degree of V ("geometric case"). In the sequel, we formulate some sharp conjectures and we describe more recent results in the arithmetic and in the geometric case.

The problem of finding sharp lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ for subvarieties of $\mathbb{G}_m^n$ is a generalization of a famous problem of Lehmer. Let α be a non-zero algebraic number of degree d which is not a root of unity. Lehmer (see [Leh 1933]) asks whether there exists an absolute constant $c > 0$ such that

$$h(\alpha) \geq \frac{c}{d}.$$

This should be the best possible lower bound for the height (without any further assumption on α), since $h(2^{1/n}) = (\log 2)/n$.

Lehmer's conjecture is still open, but a celebrated result of Dobrowolski ([Dob 1979]) shows that it is almost true:

Theorem 3.1 (Dobrowolski) *For an algebraic number α of degree $d \geq 2$ which is not a root of unity, we have*

$$\hat{h}(\alpha) \geq \frac{d}{1200}\left(\frac{\log\log d}{\log d}\right)^3.$$

Our aim is to generalize Lehmer's problem and Dobrowolski's theorem (originally stated on $\mathbb{G}_m$) to $\mathbb{G}_m^n$. Let V be a subvariety of $\mathbb{G}_m^n$ and let K be a subfield of $\overline{\mathbb{Q}}$. We denote by $\overline{V}^K$ the algebraic set

$$\bigcup_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)} \sigma V$$

and we remark that $\deg\left(\overline{V}^K\right) = [LK : K]\deg(V)$, where L is the field of definition of V. Let us define the "obstruction index" $\omega_K(V)$ of V over K as the minimum of $\deg\left(\overline{Z}^K\right)$ where Z is an hypersurface containing V.

For instance, if $V = \{\alpha\} \subseteq \mathbb{G}_m$ we have $\omega_K(\alpha) = [K(\alpha) : K]$ and, if $V = \{\boldsymbol{\alpha}\} \subseteq \mathbb{G}_m^n$,

$$\omega_K(V) \leq n[K(\boldsymbol{\alpha}) : K]^{1/n} \tag{9}$$

by a linear algebra argument. More generally, if Z is any subvariety of $\mathbb{G}_m^n$ containing V,

$$\omega_K(V) \leq n \deg\left(\overline{Z}^K\right)^{1/\mathrm{codim}(Z)} \tag{10}$$

by a result of Chardin ([Cha 1988]).

It turns out that $\omega_{\mathbb{Q}}(V)$, and not the degree of $\overline{V}^{\mathbb{Q}}$, is the right invariant to formulate the sharpest conjectures on $\hat{\mu}^{\mathsf{ess}}(V)$ in the "arithmetic case". Similarly, $\omega_{\overline{\mathbb{Q}}}(V)$, and not $\deg V$, is the right invariant in the "geometric case".

3.1 Arithmetic case

We propose the following conjecture, which generalizes Lehmer's one:

Conjecture 3.2 *Let V be a subvariety of $\mathbb{G}_m^n$ and assume that V is not contained in any torsion subvariety. Then, there exist a constant $c(n)$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)}.$$

We remark that a 0-dimensional variety $V = \{\boldsymbol{\alpha}\}$ is contained in a torsion subvariety if and only if $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent (indeed a subtorus of $\mathbb{G}_m^n$ is contained in a subtorus of codimension 1, which has equation $x_1^{\lambda_1} \ldots x_n^{\lambda_n} = \pm 1$ for some integers $\lambda_1, \ldots, \lambda_n$).

In [Amo-Dav 1999] (case $\dim V = 0$), [Amo-Dav 2000] (case $\mathrm{codim} V = 1$) and [Amo-Dav 2001] (general case) the following analogue of Dobrowolski theorem on $\mathbb{G}_m^n$ is proved:

Theorem 3.3 *Let V be a subvariety of $\mathbb{G}_m^n$ of codimension k and assume that V is not contained in any torsion subvariety. Then there exists two positive constants $c(n)$ and $\kappa(k)$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)} \left(\log 3\omega_{\mathbb{Q}}(V)\right)^{-\kappa(k)}.$$

This theorem sometimes produces lower bounds for the height of algebraic numbers which are even stronger than what is expected by Lehmer's conjecture. Let $\alpha_1, \ldots, \alpha_n$ multiplicatively independent algebraic numbers of height $\leq h$, lying in a number field of degree D. Let $V = \{\boldsymbol{\alpha}\}$. Then $\hat{\mu}^{\mathsf{ess}}(V) \leq h$ and, by (9),

$$\omega_{\mathbb{Q}}(V) \leq nD^{1/n}.$$

Thus, by theorem 3.3,

$$h \geq \frac{c(n)}{D^{1/n}} (\log 3D)^{-\kappa(n)}.$$

for some $c(n) > 0$.

3.2 Geometric case

Assuming that the subvariety V is not a translate of a subgroup, we now look for lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ which do not depend on the field of definition of V. Then we have ([Amo-Dav 2003]) the following conjecture which is the analogue of conjecture 3.2.

Conjecture 3.4 *Let V be a subvariety of $\mathbb{G}_m^n$ and assume that V is not contained in any translate of a proper subgroup. Then, there exists a positive constant $c(n)$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\overline{\mathbb{Q}}}(V)}.$$

In the same paper the following analogue of theorem 3.3 is proved :

Theorem 3.5 *Let V be an irreducible subvariety of $\mathbb{G}_m^n$ of codimension k and assume that V is not contained in any translate of a proper subgroup. Then there exist two positive constants $c(n)$ and $\lambda(k)$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\overline{\mathbb{Q}}}(V)} \left(\log 3\omega_{\overline{\mathbb{Q}}}(V)\right)^{-\lambda(k)}.$$

4 Methods

We now describe the methods of the proofs of theorems 3.3 and 3.5. By contradiction, we assume that the essential minimum is sufficiently small. We then follow the usual steps of a transcendence proof: interpolation (construction of an auxiliary function), extrapolation, zero estimates and conclusion. For subvariety of codimension $k > 1$, we need a rather technical extra step (descent argument). At the end of the proof, we obtain a contradiction which shows that the assumption on the essential minimum was false.

As usual, we introduce some parameters which depend on the obstruction index of V (and on the dimension n). A parameter L which bounds from above the degree of the auxiliary function, T which bounds from below the multiplicity of the auxiliary function on V, and k parameters $N_1, \ldots, N_k$ which control the set on which we shall extrapolate.

To simplify our exposition, we start with the case where V is an hypersurface. In this simpler case, the zero estimate are trivial in both arithmetic and geometric case, since the varieties have codimension 1. Moreover we can conclude without the descent argument.

Afterwards, we consider varieties of arbitrary dimension, with an extra assumption which allows us to avoid again the descent. In this case the zero estimates we need are variants of Philippon's zero estimate (see [Phi 1986] and [Phi 1996]).

Finally (section 4.3) we give a sketch of the final step of the proofs of theorems 3.3 and 3.5 in the more general situation.

4.1 Hypersurfaces

Let V be an hypersurface in $\mathbb{G}_m^n$ and assume that V is non-torsion (arithmetic case) or that V is not a translate of a subgroup (geometric case). Let

$$K = \begin{cases} \mathbb{Q} & \text{(arithmetic case);} \\ \overline{\mathbb{Q}} & \text{(geometric case)} \end{cases}$$

and define $\omega = \deg\left(\overline{V}^K\right)$. Let also[3]

$$N \approx \frac{(\log\omega)^2}{\log\log\omega};$$

$$T \approx \begin{cases} \dfrac{\log\omega}{\log\log\omega} & \text{(arithmetic case);} \\ \dfrac{N\log\omega}{\log\log\omega} & \text{(geometric case)} \end{cases}$$

and $L \approx T^2\omega$.

We suppose that the essential minimum of V is small:

$$\hat{\mu}^{\mathsf{ess}}(V) \ll \begin{cases} \dfrac{\log\omega}{NL} & \text{(arithmetic case);} \\ \dfrac{\log\omega}{L} & \text{(geometric case).} \end{cases} \tag{11}$$

4.1.1 Interpolation

In the arithmetic case the auxiliary function is a polynomial with <u>rational</u> integer coefficients, degree $\leq L$, vanishing on V with multiplicity at least T and of "small" height[4].

Let $S \subseteq \mathbb{G}_m^n(\overline{\mathbb{Q}})$ and define $\Lambda(S)$ to be the vector space of polynomials with rational coefficients, degree $\leq L$, vanishing on S with multiplicity $\geq T$. Let us assume that S is a set of bounded height and that $\Lambda = \Lambda(S)$ is a vector space of non-negative finite dimension. Using Bombieri-Vaaler's version of Siegel's lemma ([Bom-Vaa 1983]) we can prove that there exists a non-zero $F \in \Lambda$ with integer coefficients satisfying

$$h(F) \leq r((T+n)\log(L+1) + L\sup_{\boldsymbol{\alpha}\in S} h(\boldsymbol{\alpha}))\ ,$$

where

$$r = \frac{\binom{L+n}{n} - \dim\Lambda(S)}{\dim\Lambda(S)}.$$

[3]The symbols $\approx$, $\ll$ and $\gg$ have the following meaning: $A \approx B$ if and only if $A = c(n)B$ with $c(n) > 0$. The constant $c(n)$ is assumed to be sufficiently large (or small) in such a way that the forthcoming assumptions are verified. Similarly, $A \ll B$ (or $B \gg A$) if and only if $A \leq c(n)B$ where $c(n) > 0$ has the same meaning as before.

[4]Let $F \in \overline{\mathbb{Q}}[\mathbf{x}]$ be a polynomial of n variables and of total degree d. Let $N = \binom{d+n}{n}$. The height $h(F)$ of F is the Weil height of the vector $\underline{f} \in \mathbb{P}_{N-1}(\overline{\mathbb{Q}})$ of its coefficients.

Let $\theta > \hat{\mu}^{\mathsf{ess}}(V)$ and let $V(\theta)$ be the set of points on V of height $\leq \theta$. Then $\Lambda(V) = \Lambda(V(\theta))$ (since $V(\theta)$ is Zariski dense on V) and

$$\dim \Lambda(V) = \binom{L - T\omega + n}{n}.$$

By the previous application of Siegel's lemma, there exists a non-zero polynomial $F \in \mathbb{Z}[x_1, \ldots, x_n]$ of degree at most L, vanishing on V with multiplicity at least T, and of height

$$h(F) \leq r\big((T+n)\log(L+1) + L\theta\big),$$

where

$$r = \frac{\binom{L+n}{n} - \binom{L-T\omega+n}{n}}{\binom{L-T\omega+n}{n}} \ll \frac{T\omega}{L}$$

By our choice of parameters (and by the assumption on $\hat{\mu}^{\mathsf{ess}}(V)$) we have $\hat{h}(F) \ll \log \omega$.

The construction in the geometric case is similar, but we need to avoid any dependence on the field of definition of V. For this, we use an absolute Siegel's lemma (see [Dav-Phi 1999], lemma 4.7) which is a consequence of Zhang's inequality (see [Zha 1995], theorem 5.2). The auxiliary function is now a non-zero polynomial F with algebraic integers coefficients, vanishing on V as before, whose degree and height satisfy the same conditions as before.

4.1.2 Extrapolation

In the arithmetic case we extrapolate on some "multiples" of V. Let $\wp$ be the set of prime numbers p such that $N/2 \leq p \leq N$. Using a variant of Dobrowolski's main lemma ([Dob 1979], lemma 1) and a density argument, we prove that our auxiliary function must vanish on $[p]V$ for all $p \in \wp$.

Indeed, if F does not vanish on some $[p]V$, then there exists $\boldsymbol{\alpha} \in V$ of height $\leq \hat{\mu}^{\mathsf{ess}}(V) + \varepsilon$ such that $F(\boldsymbol{\alpha}^p) \neq 0$. A generalization of Dobrowolski's main lemma gives:

$$|F(\boldsymbol{\alpha}^p)|_v \leq p^{-T} \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\}^{pL}$$

for any $v \mid p$. Using the product formula, we obtain

$$\begin{aligned} 0 &\leq -T\log p + h(F) + n\log(L+1) + pLh(\boldsymbol{\alpha}) \\ &\leq -T\log(N/2) + h(F) + n\log(L+1) + NL(\hat{\mu}^{\mathsf{ess}}(V) + \varepsilon) \end{aligned}$$

which is a contradiction[5], since

$$T\log(N/2) - h(F) - n\log(L+1) \gg \log\omega$$

by the choice of the parameters and by the estimate on $h(F)$, and

$$NL(\hat{\mu}^{\mathsf{ess}}(V)+\varepsilon) \ll \log\omega$$

by the assumption on $\hat{\mu}^{\mathsf{ess}}(V)$.

In the geometric case, we extrapolate on some translate of V, and we show that the auxiliary function must vanish on $\boldsymbol{\zeta}V$ for all p-torsion points $\boldsymbol{\zeta}$ and for all $p \in \wp$ (where $\wp$ is the same set as before).

Indeed, let $p \in \wp$; if F does not vanish on $\boldsymbol{\zeta}V$ for some p-torsion point $\boldsymbol{\zeta}$, then there exists $\boldsymbol{\alpha} \in V$ of height $\leq \hat{\mu}^{\mathsf{ess}}(V)+\varepsilon$ such that $F(\boldsymbol{\zeta\alpha}) \neq 0$. The inequality $|1-\zeta|_v \leq p^{-1/p}$, which holds for a p-root of unity ζ and a place $v \mid p$, gives

$$|F(\boldsymbol{\alpha})|_v \leq p^{-T/p}\max\{1, |\alpha_1|_v, \dots, |\alpha_n|_v\}^L$$

for any $v \mid p$. Using the product formula, we obtain

$$\begin{aligned}
0 &\leq -T\frac{\log p}{p} + h(F) + n\log(L+1) + Lh(\boldsymbol{\alpha}) \\
&\leq -\frac{T}{N}\log(N/2) + h(F) + n\log(L+1) + L(\hat{\mu}^{\mathsf{ess}}(V)+\varepsilon)
\end{aligned}$$

which is a contradiction again (see note 5), since

$$\frac{T}{N}\log(N/2) - h(F) - n\log(L+1) \gg \log\omega$$

by the choice of the parameters and by the estimate on $h(F)$, and

$$L(\hat{\mu}^{\mathsf{ess}}(V)+\varepsilon) \ll \log\omega$$

by the assumption on $\hat{\mu}^{\mathsf{ess}}(V)$.

4.1.3 Zero estimate and conclusion

Let us consider the arithmetic case first.

Since V is not torsion, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the varieties $[p]V$ and $\sigma[q]V$ are distinct if p and q are distinct primes. Moreover, if we avoid some

[5] If we choose properly the implicit constants in the parameters.

exceptional primes, $[p]V \neq \sigma[p]V$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma V \neq V$. A generalization of a combinatorial lemma of Dobrowolski ([Dob 1979] lemma 3) shows that the number of these primes is $\leq (\log r)/2$, where r is the degree over $\mathbb{Q}$ of the field of definition of V. Moreover, we have

$$\deg([p]V) = p^{n-1-\dim \mathrm{Stab}(V)} \deg(V),$$

if p does not divide the index $\lambda = [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$ (see (5) and (6)). Again the number of these exceptional primes is bounded by

$$\frac{\log \lambda}{\log 2} \leq \frac{\log \deg \mathrm{Stab}(V)}{\log 2} \leq \frac{n \log \deg(V)}{\log 2} .$$

Let $E(V)$ be the set of all the previous exceptional primes. Then

$$\mathrm{Card}(E(V)) \ll \log \deg \left(\overline{V}^{\mathbb{Q}}\right) = \log \omega, \tag{12}$$

and therefore is negligible[6]. Using the Prime Number Theorem we obtain:

$$\begin{aligned} L \geq \deg(F) \geq \deg & \left(\bigcup_{p \in \wp \setminus E(V)} \overline{[p]V}^{\mathbb{Q}} \right) \\ & \geq \sum_{p \in \wp \setminus E(V)} p^{n-1-\dim \mathrm{Stab}(V)} \deg \left(\overline{V}^{\mathbb{Q}}\right) \qquad (13) \\ & \gg \frac{N^{n-\dim \mathrm{Stab}(V)} \omega}{\log N} \geq \frac{N\omega}{\log N}. \end{aligned}$$

Since

$$L \approx T^2 \omega \approx \frac{(\log \omega)^2 \omega}{(\log \log \omega)^2}$$

and

$$\frac{N\omega}{\log N} \approx \frac{(\log \omega)^2 \omega}{(\log \log \omega)^2}$$

we get a contradiction (see note 5). Thus, our assumption on $\hat{\mu}^{\mathsf{ess}}(V)$ is false and we have

$$\hat{\mu}^{\mathsf{ess}}(V) \gg \frac{\log \omega}{NL} \gg \frac{1}{\omega} \left(\frac{\log \log \omega}{\log \omega} \right)^3 .$$

[6] Because $\mathrm{Card}(\wp) \approx N/\log N \approx (\log \omega / \log \log \omega)^2$ is much bigger than $\log \omega$.

Remark 4.1 *Let* $s = \dim \mathrm{Stab}(V)$*; if* V *is not a translate of a subgroup, then* $n - s > 1$ *and we can improve the error term in the previous lower bound by choosing* $N \approx \left((\log\omega)^2/\log\log\omega\right)^{1/(n-s)}$*. We obtain*

$$\hat{\mu}^{\mathsf{ess}}(V) \gg \frac{\log\omega}{NL} \gg \frac{1}{\omega} \times \frac{(\log\log\omega)^{2+1/(n-s)}}{(\log\omega)^{1+2/(n-s)}}.$$

Unfortunately, there is now a technical problem with the exceptional set $E(V)$*, since its cardinality could now exceeds*

$$\mathrm{Card}(\wp) \approx \frac{N}{\log N} \approx \frac{(\log\omega)^{2/(n-s)}}{(\log\log\omega)^{1+1/(n-s)}}.$$

A generalisation of an inductive argument of Rausch ([Rau 1985]) allows us to avoid this problem (see [Amo-Dav 2000] for details).

We now consider the geometric case.

Let us set again $\lambda = [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$, $s = \dim \mathrm{Stab}(V)$ and let $E(V)$ be the set of exceptional primes dividing λ. Then, if p_1, $p_2 \in \wp\backslash E(V)$ are two primes not dividing λ and if $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ are two torsion points of order exactly p_1 and p_2, then $\boldsymbol{\zeta}_1 V = \boldsymbol{\zeta}_2 V$ if and only if $\boldsymbol{\zeta}_1 \equiv \boldsymbol{\zeta}_2 \mod \mathrm{Stab}(V)^0$. Moreover, for $p_1 \neq p_2$ we have $\boldsymbol{\zeta}_1 \equiv \boldsymbol{\zeta}_2 \mod \mathrm{Stab}(V)^0$ if and only if $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2 \in \mathrm{Stab}(V)^0$ (by Bezout's theorem). Since the cardinality of the set of p-torsion points in $\mathrm{Stab}(V)^0$ is p^s, we have

$$\begin{aligned} L \geq \deg(F) &\geq \deg\left(\bigcup_{p\in\wp\backslash E(V)} \bigcup_{\boldsymbol{\zeta}\in\mathrm{Ker}[p]} \boldsymbol{\zeta} V\right) \\ &= \left(1 + \sum_{p\in\wp\backslash E(V)} (p^{n-s} - 1)\right) \deg(V). \end{aligned}$$

As in the arithmetic case, the set $E(V)$ has cardinality less than

$$\mathrm{Card}(E(V)) \leq \frac{\log\lambda}{\log 2} \leq \frac{\log\deg \mathrm{Stab}(V)}{\log 2} \leq \frac{n \log\deg V}{\log 2} \ll \log\omega \qquad (14)$$

and therefore is again negligible (see note 6). Hence, by the Prime Number Theorem,

$$L \gg \frac{N^{n-s+1}\omega}{\log N}$$

We have that $s < n-1$, <u>since</u> V <u>is</u> <u>not</u> <u>a</u> <u>translate</u> <u>of</u> <u>a</u> <u>subgroup</u>; hence $n-s+1 \geq 3$ and we deduce that

$$L \gg \frac{N^3\omega}{\log\log\omega}.$$

By the choice of the parameters, we have:

$$L \approx T^2\omega \approx \frac{N^2(\log\omega)^2\omega}{(\log\log\omega)^2} \approx \frac{N^3\omega}{\log\log\omega};$$

which gives again a contradiction (see note 5). Again our assumption on $\hat{\mu}^{\mathsf{ess}}(V)$ is false, hence

$$\hat{\mu}^{\mathsf{ess}}(V) \gg \frac{\log\omega}{L} \gg \frac{1}{\omega}\frac{(\log\log\omega)^4}{(\log\omega)^5}.$$

Remark 4.2 *It could be possible to improve the error term in the previous lower bound. Choosing* $N \approx \big((\log\omega)^2/\log\log\omega\big)^{1/(n-s-1)}$ *we would obtain*

$$\hat{\mu}^{\mathsf{ess}}(V) \gg \frac{1}{\omega} \times \frac{(\log\log\omega)^{2+2/(n-s-1)}}{(\log\omega)^{1+4/(n-s-1)}}.$$

Unfortunately, as in the arithmetic case (c.f. remark 4.1), this choice of N *gives a technical problem with the cardinality of the exceptional set* $E(V)$*.*

4.2 Varieties of arbitrary dimension

Let V be a subvariety of $\mathbb{G}_m^n$ of codimension k and let us assume that V is not contained in any torsion subvariety (arithmetic case) or that V is not contained in any proper translate of a subgroup (geometric case). Let, as before,

$$K = \begin{cases} \mathbb{Q} & \text{(arithmetic case);} \\ \overline{\mathbb{Q}} & \text{(geometric case)} \end{cases}$$

and $\omega = \omega_K(V)$. We also define, as for hypersurfaces, the exceptional set of a subvariety Z of $\mathbb{G}_m^n$ as the set of primes p such that

$$p \mid [\mathrm{Stab}(Z) : \mathrm{Stab}(Z)^0]$$

or[7] $[p]Z = \sigma[p]Z$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ such that $\sigma Z \neq Z$. As for hypersurfaces

$$\mathrm{Card}(E(Z)) \ll \log\deg\left(\overline{Z}^K\right)$$

(*c.f.* (12) and (14)).

To simplify the arguments we make for the moment the following additional assumption on V:

Hypothesis 4.3 *There exists an hypersurface Z_0 containing V such that*

$$\omega_K([l]V) = \deg\left(\overline{[l]Z_0}^K\right)$$

for all positive integer l.

Let $l \in \mathbb{N}$ and assume that its prime factors do not belong to the exceptional set $E(Z_0)$. Then,

$$\omega_{\mathbb{Q}}([l]V) = \deg\left([l]\overline{Z_0}^{\mathbb{Q}}\right) \geq \deg\left(\overline{Z_0}^{\mathbb{Q}}\right) = \omega_{\mathbb{Q}}(V) \tag{15}$$

in the arithmetic case (see the discussion at the beginning of section 4.1.3), and

$$\begin{aligned}\omega_{\overline{\mathbb{Q}}}([l]V) &= \deg([l]Z_0)\\ &= l^{\dim(Z_0)-\dim(\mathrm{Stab}(Z_0))}\deg(Z_0) \geq l\deg(Z_0) = l\,\omega_{\overline{\mathbb{Q}}}(V),\end{aligned} \tag{16}$$

in the geometric case (see (5) and (6)), since Z_0 is not a translate of a subgroup.

Let $N_1, \ldots, N_k$ be some parameters such that $(\log\omega)^{1+\varepsilon} \ll N_j$ and $\log N_j \approx \log\log\omega$. Let also

$$T \approx \begin{cases} \left(\dfrac{\log\omega}{\log\log\omega}\right)^k & \text{(arithmetic case);}\\[2ex] N_1\ldots N_k\left(\dfrac{\log\omega}{\log\log\omega}\right)^k & \text{(geometric case)}\end{cases}$$

and

$$L \approx T^2\omega.$$

To simplify the notations, we denote by $\wp_j$ the set of primes $p \notin E(Z_0)$ such that $N_j/2 \leq p \leq N_j$ and we let $\wp_j' = \wp_j \cup \{1\}$. The cardinality of

[7] The following condition is empty in the geometric case.

$E(Z_0)$ is $\ll \log \omega$ and therefore it is negligible (since $(\log \omega)^{1+\varepsilon} \ll N_j$); we have:

$$\operatorname{Card}(\wp_j) \approx \frac{N_j}{\log N_j} \approx \frac{N_j}{\log \log \omega}. \tag{17}$$

We finally suppose by contradiction, that the essential minimum of V is small:

$$\hat{\mu}^{\mathsf{ess}}(V) \ll \begin{cases} \dfrac{\log \omega}{N_1 \cdots N_k L} & \text{(arithmetic case);} \\ \dfrac{\log \omega}{L} & \text{(geometric case).} \end{cases} \tag{18}$$

4.2.1 Interpolation

As in section 4.1, we construct a non-zero polynomial in n variables having rational integer coefficients (arithmetic case) or algebraic integer coefficients (geometric case), of degree at most L, vanishing on V with multiplicity at least T and of height

$$h(F) \ll \log \omega.$$

4.2.2 Extrapolation

We repeat the extrapolation process k times: we show that F must vanish on

$$[p_1 \cdots p_k]V$$

for $(p_1, \ldots, p_k) \in \wp'_1 \times \cdots \times \wp'_k$ (arithmetic case) or on

$$\boldsymbol{\zeta}_1 \cdots \boldsymbol{\zeta}_k V$$

for $(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k) \in \operatorname{Ker}[p_1] \times \cdots \times \operatorname{Ker}[p_k]$ and for $(p_1, \ldots, p_k) \in \wp_1 \times \cdots \times \wp_k$ (geometric case).

4.2.3 Zero estimate and conclusion

Let us consider the arithmetic case first.

A variant of Philippon's zero estimate shows that there exist two integers r and k' with $k' \leq r \leq k$ and a variety Z of codimension k', containing $[p_{r+1} \cdots p_{k'+1}]V$ for some $(p_{r+1}, \ldots, p_{k'+1}) \in \wp'_{r+1} \times \cdots \times \wp'_{k'+1}$, such that

$$\deg \left(\bigcup_{p \in \wp_r} \overline{[p]Z}^{\mathbb{Q}} \right) \leq (N_1 \ldots N_{r-1} L)^{k'}.$$

As in (13),

$$\deg\left(\bigcup_{p\in\wp_r\setminus E(Z)}\overline{[p]Z}^{\mathbb{Q}}\right)\geq\left(\sum_{p\in\wp_r\setminus E(Z)}1\right)\deg\left(\overline{Z}^{\mathbb{Q}}\right)\gg \mathrm{Card}(\wp_r)\deg\left(\overline{Z}^{\mathbb{Q}}\right),$$

since the cardinality of $E(Z)$ is negligible.

Let $l=p_{r+1}\cdots p_{k'+1}$; since $[l]V\subseteq Z$, using (10), the two last displayed inequalities and the estimate (17) we obtain:

$$\begin{aligned}\omega_{\mathbb{Q}}([l]V)&\leq n\deg\left(\overline{Z}^{\mathbb{Q}}\right)^{1/k'}\\&\ll \mathrm{Card}(\wp_r)^{-1/k'}N_1\dots N_{r-1}L\\&\ll\left(\frac{\log\log\omega}{N_r}\right)^{1/k'}N_1\dots N_{r-1}\left(\frac{\log\omega}{\log\log\omega}\right)^{2k}\omega.\end{aligned}$$

Therefore, using $1/k\leq 1/k'\leq 1$,

$$\omega_{\mathbb{Q}}([l]V)\leq\frac{CN_1\dots N_{r-1}(\log\omega)^{2k}}{N_r^{1/k}}\,\omega,\tag{19}$$

for some positive constant $C=C(n)$. We choose the parameters $N_1,\dots,N_k$ increasing rapidly in such a way that

$$\frac{CN_1\dots N_{r-1}(\log\omega)^{2k}}{N_r^{1/k}}<1$$

for $r=1,\dots,k$. Then, by (19),

$$\omega_{\mathbb{Q}}([l]V)<\omega=\omega_{\mathbb{Q}}(V)$$

which contradicts inequality (15). Therefore, the inequality (18) is false and we must have

$$\hat{\mu}^{\mathsf{ess}}(V)\gg\frac{\log\omega}{N_1\cdots N_kL}\gg\frac{1}{\omega}(\log\omega)^{-\kappa(k)},$$

for some $\kappa(k)$.

We now consider the geometric case.

Another variant of Philippon's zero estimate shows that there exist two integers r and k' with $k' \leq r \leq k$ and a family of subvarieties $\{Z_p\}_{p \in \wp_r}$ of codimension k' such that for all p the subvariety Z_p contains some translate of V and

$$\deg\left(\bigcup_{\boldsymbol{\zeta}_1 \in \mathrm{Ker}[p_1]} \cdots \bigcup_{\boldsymbol{\zeta}_{r-1} \in \mathrm{Ker}[p_{r-1}]} \bigcup_{p \in \wp_r} \bigcup_{\boldsymbol{\zeta} \in \mathrm{Ker}[p]} \boldsymbol{\zeta}_1 \cdots \boldsymbol{\zeta}_{r-1} \boldsymbol{\zeta} Z_p \right) \leq L^{k'}.$$

for some $(p_1, \ldots, p_{r-1}) \in \wp_1 \times \cdots \times \wp_{r-1}$. Let $l_0 = p_1 \cdots p_{r-1}$; we remark that, for $p \in \wp_r$,

$$\bigcup_{\boldsymbol{\zeta}_1 \in \mathrm{Ker}[p_1]} \cdots \bigcup_{\boldsymbol{\zeta}_{r-1} \in \mathrm{Ker}[p_{r-1}]} \bigcup_{\boldsymbol{\zeta} \in \mathrm{Ker}[p]} \boldsymbol{\zeta}_1 \cdots \boldsymbol{\zeta}_{r-1} \boldsymbol{\zeta} Z_p = [l_0 p]^{-1} [l_0 p] Z_p.$$

and this last variety has degree $(l_0 p)^{k'} \deg([l_0 p] Z_p)$ by equation (4). Let $p_r \in \wp_r$ such that $(l_0 p_r)^{k'} \deg([l_0 p_r] Z_{p_r})$ is minimal and define $l = l_0 p_r$ and $Z = Z_{p_r}$. If the varieties $([l_0 p]^{-1} [l_0 p] Z_p)_{p \in \wp_r}$ had only "few" common components, then we would have

$$L^{k'} \gg \sum_{p \in \wp_r} (l_0 p)^{k'} \deg([l_0 p] Z_p) \geq \mathrm{Card}(\wp_r) l^{k'} \deg([l] Z),$$

and

$$\deg([l] Z) \ll \frac{(L/l)^{k'}}{\mathrm{Card}(\wp_r)} \tag{20}$$

We prove a bound which is very close to the heuristic estimate (20), namely

$$\deg([l] Z) \ll \frac{(L/l)^{k'} \log \omega}{\mathrm{Card}(\wp_r)}. \tag{21}$$

If p, $q \in \wp_r$ define $p \sim q$ if there exist $\boldsymbol{\alpha} \in \ker[l_0 p]$ and $\boldsymbol{\beta} \in \ker[l_0 q]$ such that $\boldsymbol{\alpha} Z_p = \boldsymbol{\beta} Z_q$. Let $C_1, \ldots, C_s$ be the equivalence classes of this relation. Then

$$\sum_{j=1}^{s} \deg\left(\bigcup_{p \in C_j} [l_0 p]^{-1} [l_0 p] Z_p \right) \leq L^{k'}. \tag{22}$$

Let $j \in \{1, \ldots, s\}$; we remark that the varieties Z_p $(p \in C_j)$ are translates of each other. Therefore $S_j = \mathrm{Stab}(Z_p)$ and $d_j = \deg(Z_p)$ only depend on the equivalence classe C_j of p. We denote by $\tilde{C}_j$ the set of primes $p \in C_j$

dividing $[S_j : S_j^0]$. Then (see the proof of Corollary 4.4 of [Amo-Dav 2001] at page 366 for details),

$$\deg\left(\bigcup_{p\in C_j}[l_0p]^{-1}[l_0p]Z_p\right) \gg \max\{\mathrm{Card}(C_j\backslash\tilde{C}_j),1\}\, l^{k'}\deg([l]Z). \tag{23}$$

By the obvious inequality $\max\{x-y,1\} \geq x/(2y)$, which holds for $x \geq 0$ and $y \geq 1$, we have[8]

$$\max\{\mathrm{Card}(C_j\backslash\tilde{C}_j),1\} \geq \frac{\mathrm{Card}(C_j)}{2\max\{\mathrm{Card}(\tilde{C}_j),1\}} \tag{24}$$

Since (22) implies in particular $d_j \leq L^{k'}$, the number of exceptional primes inside C_j s bounded by

$$\mathrm{Card}(\tilde{C}_j) \leq \frac{\log[S_j : S_j^0]}{\log 2} \leq \frac{n\log\deg S_j}{\log 2} \leq \frac{n\log d_j}{\log 2} \ll \log\omega. \tag{25}$$

From (23), (24) and (25) we obtain:

$$\deg\left(\bigcup_{p\in C_j}[l_0p]^{-1}[l_0p]Z_p\right) \gg \frac{\mathrm{Card}(C_j)}{\log\omega}\, l^{k'}\deg([l]Z).$$

Therefore, using (22),

$$L^{k'} \gg \sum_{j=1}^{s}\frac{\mathrm{Card}(C_j)}{\log\omega}\, l^{k'}\deg([l]Z) = \frac{\mathrm{Card}(\wp_r)}{\log\omega}\, l^{k'}\deg([l]Z).$$

This conclude the proof of (21).

Now, using (10), (21), (17) and $1/k \leq 1/k' \leq 1$, we obtain

$$\omega_{\overline{\mathbb{Q}}}([l]V) \leq n\deg([l]Z)^{1/k'} \ll \frac{L}{l}\frac{(\log\omega)(\log\log\omega)}{N_r^{1/k}}.$$

By our choice of the parameters,

$$\frac{L}{l} = \frac{Ll}{l^2} \approx \frac{T^2 l\omega}{(N_1\cdots N_r)^2} \approx \frac{(N_1\cdots N_k)^2(\log\omega)^{2k} l\,\omega}{(N_1\cdots N_r)^2(\log\log\omega)^{2k}}$$

$$\approx (N_{r+1}\cdots N_k)^2\left(\frac{\log\omega}{\log\log\omega}\right)^{2k} l\,\omega.$$

[8] Although this lower bound looks crude, it is essentially optimal when $\tilde{C}_j = C_j$.

Hence,

$$\omega_{\overline{\mathbb{Q}}}([l]V) \leq \frac{C(N_{r+1}\cdots N_k)^2(\log\omega)^{2k+1}}{N_r^{1/k'}}\, l\omega \tag{26}$$

for some positive constant $C = C(n)$. We choose the parameters $N_1, \ldots, N_k$ decreasing rapidly in such a way that

$$\frac{C(N_{r+1}\cdots N_k)^2(\log\omega)^{2k+1}}{N_r^{1/k}} < 1$$

for $r = 1, \ldots, k$. Then, by (26),

$$\omega_{\overline{\mathbb{Q}}}([l]V) < l\,\omega = l\,\omega_{\overline{\mathbb{Q}}}(V)$$

which contradicts inequality (16). Therefore (18) is false and we have

$$\hat{\mu}^{\mathsf{ess}}(V) \gg \frac{\log\omega}{L} \gg \frac{1}{\omega}(\log\omega)^{-\lambda(k)},$$

for some $\lambda(k)$.

4.3 Descent

We now remove the assumption 4.3. Without it, we need an extra step to conclude the proofs of theorems 3.3 and 3.5, which we briefly sketch in the arithmetic case (see [Amo-Dav 1999], §5.2 and [Amo-Dav 2004a], §3 for details when $\dim(V) = 0$).

Let us assume by contradiction that (18) holds. Then, the argument of the previous section shows that there exists a positive integer l such that

$$\omega_{\mathbb{Q}}([l]V) < \omega_{\mathbb{Q}}(V).$$

Moreover, we can assume that the prime factors of l are not in a *fixed* exceptional set of cardinality $\ll \log\omega$.

Let Z_l be an hypersurfaces containing $[l]V$ such that

$$\deg\left(\overline{Z_l}^{\mathbb{Q}}\right) = \omega_{\mathbb{Q}}([l]V).$$

Let also Z_l' be any component containing V of the algebraic set $[l]^{-1}Z_l$. Hence, $\omega_{\mathbb{Q}}(V) \leq \deg\left(\overline{Z_l'}^{\mathbb{Q}}\right)$ and

$$\deg\left(\overline{[l]Z_l'}^{\mathbb{Q}}\right) \leq \deg\left(\overline{Z_l}^{\mathbb{Q}}\right) = \omega_{\mathbb{Q}}([l]V).$$

Moreover, *if we could choose l with no prime factors in $E(Z_l')$*, we would have

$$\deg\left(\overline{[l]Z_l'}^{\mathbb{Q}}\right) \geq \deg\left(\overline{Z_l'}^{\mathbb{Q}}\right)$$

(see the discussion at the beginning of section 4.1.3) and therefore

$$\omega_{\mathbb{Q}}(V) \leq \omega_{\mathbb{Q}}([l]V),$$

a contradiction.

In practice, we cannot guarantee the existence of such an integer l, since the obstruction variety Z_l' is given *at the end* of the construction and we do not have any control *a priori* on its exceptional set.

To avoid this problem, we repeat several times the transcendence construction, assuming that the essential minimum of V is sufficiently small. By a rather complicated induction we obtain a variety $V' = [l_0]V$ and a positive integer l with

$$\omega_{\mathbb{Q}}([l]V') \leq \varepsilon\, \omega_{\mathbb{Q}}(V') \tag{27}$$

for some $\varepsilon = \varepsilon(k) \in (0, 1]$, such that the following assertion holds. There exist two subvarieties Z', Z of $\mathbb{G}_m^n$ of the same codimension k' (eventually > 1) such that

$$V' \subseteq Z', \qquad [l]Z' \subseteq Z \quad \text{and} \quad p \mid l \Rightarrow p \notin E(Z') \tag{28}$$

and

$$n \deg\left(\overline{Z}^{\mathbb{Q}}\right)^{1/k'} < \varepsilon^{-1}\, \omega_{\mathbb{Q}}([l]V'). \tag{29}$$

From (28) we have (using (10))

$$\omega_{\mathbb{Q}}(V') \leq n \deg\left(\overline{Z'}^{\mathbb{Q}}\right)^{1/k'}, \qquad \deg\left(\overline{[l]Z'}^{\mathbb{Q}}\right) \leq \deg\left(\overline{Z}^{\mathbb{Q}}\right)$$

and

$$\deg\left(\overline{[l]Z'}^{\mathbb{Q}}\right) \geq \deg\left(\overline{Z'}^{\mathbb{Q}}\right).$$

From the last three displayed inequalities and from (29) and (27) we obtain

$$\omega_{\mathbb{Q}}(V') \leq n \deg\left(\overline{Z}^{\mathbb{Q}}\right)^{1/k'} < \varepsilon^{-1}\, \omega_{\mathbb{Q}}([l]V') \leq \omega_{\mathbb{Q}}(V').$$

This contradiction shows that the essential minimum of V cannot be too small, and concludes the proof of the arithmetic theorem 3.3.

The proof of the geometric theorem 3.5 also needs a descent step, which is very similar to the previous one (see [Amo-Dav 2003], §5.2 for details).

5 Further quantitative results: small points.

5.1 Arithmetic case

Let V be a non-torsion subvariety of $\mathbb{G}_m^n$ and define

$$V^* = V \backslash \bigcup_{\substack{B \subseteq V \\ B \text{ torsion}}} B.$$

By the former Manin-Mumford conjecture, V^* is a Zariski-open set, since $V \backslash V^*$ is a finite union of translates of subgroups.

As mentioned in the introduction, an equivalent version of theorem 1.2 says that the height on $V^*(\overline{\mathbb{Q}})$ is bounded from below by a positive quantity:

$$\hat{\mu}^*(V) = \inf_{\boldsymbol{\alpha} \in V^*} \hat{h}(\boldsymbol{\alpha}) > 0.$$

More precisely, let us assume that V is the intersection of hypersurfaces $Z_1, \ldots, Z_r$ such that $\deg \overline{Z_j}^{\mathbb{Q}} \leq \delta$ and $h(Z_j) \leq h$. Let also $D = \deg \overline{V}^{\mathbb{Q}}$. Then, we have the following lower bounds:

$$\begin{array}{ll} \hat{\mu}^*(V) \geq \exp_n(-D) & \text{[Bom-Zan 1995]} \\ \hat{\mu}^*(V) \geq \exp(-c\delta e^h) & \text{[Sch 1996]} \\ \hat{\mu}^*(V) \geq (-D \log D)^{4^{\dim(V)+1}} & \text{[Dav-Phi 1999]} \end{array}$$

Remark that obviously $\hat{\mu}^*(V) \leq \hat{\mu}^{\mathsf{ess}}(V)$. Hence one could hope, in analogy to conjecture 3.2, that

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)}$$

for some constant $c(n) > 0$. This lower bound is false, as the following example shows. Let α_k be a sequence of algebraic numbers whose height is positive and tends to zero as $k \to +\infty$. Let us consider

$$V_k = \{(\alpha_k, x_2, x_3) \in \mathbb{G}_m^3 \text{ such that } \alpha_k^2 + \alpha_k^3 - x_2 - x_3 = 0\}.$$

One checks that V_k is not torsion, the height of $\boldsymbol{\alpha}_k = (\alpha_k, \alpha_k^2, \alpha_k^3) \in V_k \backslash V_k^*$ tends to zero and $\omega_{\mathbb{Q}}(V) \leq 3$, since $V_k \subseteq \{x_1^2 + x_1^3 - x_2 - x - 3 = 0\}$.

We therefore introduce another quantity depending on the ideal of definition of V and on its field of definition. Let K be any subfield of $\overline{\mathbb{Q}}$; we let $\delta_K(V)$ be the minimum integer δ such that V is the intersection of hypersurfaces $Z_1, \ldots, Z_r$ with $\deg \overline{Z_j}^K \leq \delta$. Then

$$\left(\deg \overline{V}^K\right)^{1/\operatorname{codim}(V)} \leq \delta_K(V) \leq \deg \overline{V}^K.$$

and both lower and upper bounds can be attained: therefore $\delta_K(V)$ is still more precise than $\deg \overline{V}^K$.

We formulate the following conjecture:

Conjecture 5.1 *Let V be a non-torsion subvariety of $\mathbb{G}_m^n$; then there exists a constant $c(n) > 0$ such that*

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(V)} \; . \tag{30}$$

A more optimistic version of this conjecture (see [Amo-Dav 2004a], conjecture 1.3) predicts that inequality (30) still holds if we replace $\delta_{\mathbb{Q}}(V)$ by $\delta_{\mathbb{Q}^{\mathrm{ab}}}(V)$, where $\mathbb{Q}^{\mathrm{ab}}$ is the union of all the cyclotomic fields. In the direction of conjecture 5.1, we obtain the following result (see op. cit., théorème 1.4).

Theorem 5.2 *Let V be a non-torsion subvariety of $\mathbb{G}_m^n$; then there exist two positive constants $c(n)$ and $\kappa(n)$ such that*

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(V)} (\log 3\delta_{\mathbb{Q}}(V))^{-\kappa(n)}.$$

In the case $n = 2$ the proof of theorem 5.2 can be considerably simplified and the result can be improved; Pontreau ([Pon 2004]) obtains:

Theorem 5.3 *Let V be a non-torsion curve of $\mathbb{G}_m^2$ and let $\boldsymbol{\alpha} \in V$ be a non-torsion point. Let also $D = \deg \overline{V}^{\mathbb{Q}}$. Then*

$$\hat{h}(\boldsymbol{\alpha}) \geq \frac{1.2 \times 10^{-16}}{D} \times \frac{(\log\log(D+15))^{11}}{(\log(D+15))^{13}}.$$

Theorem 5.2 follows by an inductive argument from a "semi-relative" version of theorem 3.3: a quite simple generalization of the method of the proof of theorem 3.3 shows that:

Theorem 5.4 *Let $\boldsymbol{\alpha} \in \mathbb{G}_m^n$ et let K be a cyclotomic extension. Let also $\boldsymbol{\alpha} \in \mathbb{G}_m^n$. Then there exist three positive constants $c(n)$, $\kappa(n)$ and $\lambda(n)$ such that if*

$$\hat{h}(\boldsymbol{\alpha}) < \frac{c(n)}{\omega_K(\boldsymbol{\alpha})} \Big(\log(3[K:\mathbb{Q}]\omega_K(\boldsymbol{\alpha})) \Big)^{-\kappa(n)},$$

then $\boldsymbol{\alpha}$ belongs to a torsion subvariety $B = \boldsymbol{\zeta} H$ such that

$$\deg \Big(\overline{B}^K \Big)^{1/\mathrm{codim}(B)} \leq c(n)^{-1} \, \omega_K(\boldsymbol{\alpha}) \, \Big(\log(3[K:\mathbb{Q}]\omega_K(\boldsymbol{\alpha})) \Big)^{\mu(n)}.$$

Hopefully, the factors $[K : \mathbb{Q}]$ in the previous formulas can be removed. This would allow to obtain a generalization in several variables of the main theorem of [Amo-Zan 2000]. This would also imply a proof "up to an $\varepsilon > 0$" of the full conjecture 1.3 of [Amo-Dav 2004a] (conjecture 5.1 with $\delta_{\mathbb{Q}^{\mathrm{ab}}}$ instead of $\delta_{\mathbb{Q}}$).

5.2 Geometric case

Let V be a subvariety of $\mathbb{G}_m^n$ which is not contained in a union of translates of subgroups and define, as in [Bom-Zan 1995],

$$V^0 = V \backslash \bigcup_{B \subseteq V} B.$$

where the union is now on the set of translates B of subgroups of dimension 1. Again $V \backslash V^0$ is an open set (see [Bom-Zan 1995] and [Sch 1996]); Bombieri and Zannier prove that, outside a finite set, the height on V^0 is bounded from below by a positive quantity depending only on the ideal of definition of V and not on its field of definition. More precisely, Schmidt [Sch 1996] proves that the set of points $\boldsymbol{\alpha} \in V^0$ such that $\hat{h}(\boldsymbol{\alpha}) < q(V)^{-1}$ is finite, of cardinality $\leq q(V)$, where

$$q(V) = \exp\left(n^{\delta_{\overline{\mathbb{Q}}}(V)^n}\right).$$

David and Philippon (see [Dav-Phi 1999]) improve this result, finding a polynomial bound:

$$q(V) = (\deg(V) \log \deg(V))^{4^{\dim(V)}}.$$

Let, for $\theta > 0$,

$$V^0(\theta) = \{\boldsymbol{\alpha} \in V^0 \ \text{ such that } \ h(\boldsymbol{\alpha}) < \theta\}$$

and

$$\hat{\mu}^0(V) = \inf\{\theta > 0 \ \text{ such that } \ \mathrm{Card}(V^0(\theta)) = \infty\}.$$

Again, $\hat{\mu}^0(V) \leq \hat{\mu}^{\mathsf{ess}}(V)$ and, as a consequence of the previous results, $\hat{\mu}^0(V) \geq q(V)$.

As in the arithmetic case, we can conjecture a very precise lower bound for $\hat{\mu}^0(V)$ and we can prove it "up to an $\varepsilon > 0$" (see [Amo-Dav 2004b]).

Conjecture 5.5 *Let V be a subvariety of $\mathbb{G}_m^n$ which is not contained in a union of translates of subgroups; then there exists a constant $c(n) > 0$ such that*

$$\hat{\mu}^0(V) \geq \frac{c(n)}{\delta_{\overline{\mathbb{Q}}}(V)} .$$

This conjecture can be proved "up to an ε":

Theorem 5.6 *Let V be as before. Then there exist two positive constants $c(n)$ and $\lambda(n)$ such that*

$$\hat{\mu}^0(V) \geq \frac{c(n)}{\delta_{\overline{\mathbb{Q}}}(V)} (\log 3\delta_{\overline{\mathbb{Q}}}(V))^{-\lambda(n)}.$$

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