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ORIGINAL PAPER

Solution theory to semilinear stochastic equations of Schrödinger type on curved spaces I: operators with uniformly bounded coefficients

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Abstract

We study the Cauchy problem for Schrödinger type stochastic semilinear partial differential equations with uniformly bounded variable coefficients, depending on the space variables. We give conditions on the coefficients, on the drift and diffusion terms, on the Cauchy data, and on the spectral measure associated with the noise, such that the Cauchy problem admits a unique function-valued mild solution in the sense of Da Prato and Zabczyc.

Keywords Stochastic partial differential equations · Schrödinger equation · Curved space · Function-valued solutions · Variable coefficients · Fundamental solution

Mathematics Subject Classification Primary 35R60 · 60H15; Secondary 35Q40

1 Introduction and main result

In this paper we study the Cauchy problem associated with a semilinear variable coefficients stochastic partial differential equation (SPDE for short, in the sequel) of Schrödinger type, that is,

$$\begin{cases} P(x, \partial_t, \partial_x)u(t,x) = \gamma(t,x,u(t,x)) + \sigma(t,x,u(t,x))\dot{\Xi}(t,x), & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

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where:

- P is a variable coefficients Schrödinger type operator of the form considered, e.g., by Craig (see [16] and the literature mentioned therein), namely,

$$\begin{aligned} P(x, \partial_t, \partial_x) &= i\partial_t + \frac{1}{2}\sum_{j,\ell=1}^{d} \partial_{x_j}\left(a_{j\ell}(x)\partial_{x_\ell}\right) + m_1(x, -i\partial_x) + m_0(x, -i\partial_x) \\ &= i\partial_t + a(x, D_x) + a_1(x, D_x) + m_1(x, D_x) + m_0(x, D_x), \end{aligned} \tag{1.2}$$

  where, having set, as usual, $D_x = -i\partial_x$:
  $a(x,\xi) := -\dfrac{1}{2}\displaystyle\sum_{j,\ell=1}^{d} a_{j\ell}(x)\xi_j\xi_\ell$, $a_{j\ell} = a_{\ell j}$, $j, \ell = 1, \dots, d$, is the *Hamiltonian* of the equation,
  $a_1(x,\xi) := \dfrac{i}{2}\displaystyle\sum_{j,\ell=1}^{d} \partial_{x_j} a_{j\ell}(x)\xi_\ell$, while $m_1(x,\xi)$ comes from a magnetic field and $m_0(x,\xi)$ is a potential term;
- γ and σ are real-valued functions, subject to certain regularity conditions (see Definition 1.8 below), representing, respectively, drift and diffusion;
- Ξ is an $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, centered (i.e. with mean zero), white in time and coloured in space, with *correlation measure* Γ and *spectral measure* $\mathfrak{M}$, i.e. with covariance functional given by

$$\begin{aligned} \mathbb{E}[\Xi(\phi)\Xi(\psi)] &= \int_0^\infty \int_{\mathbb{R}^d} \left(\phi(t) * \tilde{\psi}(t)\right)(x)\, \Gamma(dx)dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} [\mathcal{F}\phi(t)](\xi) \cdot \overline{[\mathcal{F}\psi(t)](\xi)}\, \mathfrak{M}(d\xi)dt, \end{aligned} \tag{1.3}$$

  where $\tilde{\psi}(t,x) := (2\pi)^{-d}\, \overline{\psi(t,-x)}$, $*$ is the convolution operator and Γ is a nonnegative, nonnegative definite, tempered measure on $\mathbb{R}^d$; by [34, Chapter VII, Théorème XVIII], we know that $\mathcal{F}\Gamma = \widehat{\Gamma} = \mathfrak{M}$, with $\mathfrak{M}$ a nonnegative tempered measure on $\mathbb{R}^d$ ($\mathcal{F}$ and $\widehat{\ }$ denote the Fourier transform);
- u is an unknown stochastic process, called *solution* of the Cauchy problem (1.1), in the sense that is going to be specified in Definition 1.1 below.

We observe that the operators of the form (1.2) that we treat in the present paper include those involving Laplace-Beltrami operators associated with a(n asymptotically flat,) non-constant metric $(a_{j\ell}(x))$ on $\mathbb{R}^d$, see [16] and Remark 1.3 below.

Schrödinger's equation, the basic equation of quantum mechanics, is one of the most extensively studied partial differential equation, both from the physical and from the mathematical point of view, in the deterministic case $\sigma \equiv 0$ as well as in the stochastic case $\sigma \not\equiv 0$. We mention just a few references from the huge literature about this topic, related to the model (1.1) within the environments we are interested in.

The problem of existence and uniqueness of the solution of the initial value problem in Sobolev spaces on $\mathbb{R}^d$ for the (deterministic) Schrödinger equation with a power potential, namely,

$$i\partial_t u = \Delta u + \lambda u|u|^{2q}, \tag{1.4}$$

$q > 0$, $\lambda \in \mathbb{R}$, focusing (respectively, defocusing) for $\lambda > 0$ ($\lambda < 0$), linear for $\lambda = 0$, has attracted a lot of interest in the last decades, starting from the paper [12]. Properties of the solutions for (general) non-linear Schrödinger equations, including smoothing effects, have been studied in [9, 28, 29]. Similar problems, linear and non-linear, on Riemannian manifolds, then involving the (non-flat) Laplace-Beltrami operator associated with the underlying metric, have been investigated as well, see, for instance, [10] in the case of compact manifolds (see also [24, 25] for two-dimensional space-periodic cases, that is, on the torus $\mathbb{T}^2$, and their global extensions to $\mathbb{R}^2$), and [26] for the case of asymptotically Euclidean manifolds (a relevant class of non-compact manifolds; see, for instance, [30] for basic definitions and properties). Long-time behaviour for the defocusing non-linear Schrödinger equation is studied in the recent paper [11].

Concerning the stochastic case, several papers in literature are devoted to studying the Schrödinger equation with random potential

$$i\partial_t u = \Delta u + \lambda u|u|^{2q} + u\dot{\Xi}, \tag{1.5}$$

$q > 0, \lambda \in \mathbb{R}$. Equation (1.5) finds applications in nonlinear optics, since it describes a laser ray propagation into a disordered dispersive medium. It is also related to the study of Anderson localization, in relation with (a complex version of) the so-called PAM model, see, for instance, [1, 2, 13]. When the potential depends on x, as it is the case in (1.5), it has an effect on the dynamics of the solution. It has been widely studied in L^p_x-modeled Sobolev spaces, for instance when $\dot{\Xi}$ is a white noise depending only on space (see [24]), or when it is a white in time and coloured in space noise (see [20, 21, 23, 32]), together with the stopping time for the solution. The problem has also been considered in the L^2-based Sobolev space H^1 in [22], where the authors prove existence of a unique solution when the exponent q is small enough. The existence and uniqueness of solutions in weighted (Besov and Sobolev) spaces is studied in [25].

The Schrödinger's SPDEs studied in the papers mentioned above involve the flat, Euclidean Laplacian. We are not aware of any systematic treatment, in literature, of stochastic semilinear Schrödinger's equation on a non-flat, non-compact environment. We start such analysis in this paper, by considering the case of operators with coefficients which are smooth and uniformly bounded with respect to the space variables, paving the way to the study of analog operators defined on (classes of) Riemannian non-compact manifolds. Both such more general situation, as well as the cases involving coefficients of low regularity in time, and/or admitting a polynomial growth with respect to the space variable $x \in \mathbb{R}^n$, will be the subject of forthcoming papers. Notice that the Cauchy problems associated with (1.4) and (1.5) are special cases of the general model problem (1.1).

To give meaning to (1.1) we rewrite it formally in its corresponding integral form and look for *mild solutions*, that is, stochastic processes $u(t)$ satisfying an integral equation of the form

$$\begin{aligned} u(t) = S(t)u_0 - i\int_0^t S(t-s)\gamma(s,u(s))ds \\ -i\int_0^t S(t-s)\sigma(s,u(s))d\Xi(s), \quad \forall t\in[0,T_0],\ 0<T_0\leq T,\ x\in\mathbb{R}^d, \end{aligned} \tag{1.6}$$

where $S(t)$ is the propagator of the evolution operator P, that is, a family of operators depending on the parameter $t \in [0, T_0]$ such that, for every $t \in [0, T_0]$, it holds $P(x, \partial_t, \partial_x) \circ S(t) = 0$ and $S(0) = Id$. Note that the first integral in (1.6) is of deterministic type, while the second is a stochastic integral.

In this paper, to make sense of the stochastic integral appearing in (1.6), we focus on the Da Prato-Zabczyk approach (see [19]). Namely, we associate to the random noise a Brownian motion valued in an appropriately chosen Hilbert space $\mathcal{H}$, and then define the stochastic integral as an infinite sum of Itô integrals with respect to one-dimensional Brownian motions. Consequently, the solutions of the Cauchy problem (1.1) that we construct involve $\mathcal{H}$-valued random functions, as described in the next Definition 1.1 (for a short recap about the stochastic integration with respect to a cylindrical Wiener process, appearing in (1.7) below, see the Appendix).

Definition 1.1 For a given separable Hilbert space $\mathcal{H}$, we call *(mild) function-valued solution to* (1.1) an $L^2(\Omega, \mathcal{H})$-family of random elements $u(t)$, satisfying the stochastic integral equation

$$u(t) = S(t)u_0 - i\int_0^t S(t-s)\gamma(s, u(s))ds - i\int_0^t S(t-s)\sigma(s, u(s))dW_s, \qquad (1.7)$$

for all $t \in [0, T_0]$, $x \in \mathbb{R}^d$, where $T_0 \in (0, T]$ is a suitable time horizon, $v_0(t) := S(t)u_0 \in \mathcal{H}$ for $t \in [0, T_0]$, $S(t)$ is the propagator of P provided by Theorem 1.6, γ and σ are nonlinear operators defined by the so-called Nemytskii operators associated with the functions γ and σ in (1.1), and W is an $\mathcal{H}$-valued cylindrical Wiener process associated with Ξ.

An alternative approach would be the one by Walsh and Dalang, see [14, 17, 35], who make sense of the stochastic integral by considering it as a stochastic integral with respect to a martingale measure derived from the random noise $\dot{\Xi}$ (see the concluding Remark 2.3).

In literature, the existence of a unique solution to the Cauchy problem for an SPDE driven by a non-white in space noise is often stated under suitable conditions on the coefficients and a *compatibility condition* between the noise Ξ and the equation in (1.1), expressed in terms of integrals with respect to the spectral measure $\mathfrak{M}$ associated with the noise. Recently, we studied classes of SPDEs with (t, x)-depending unbounded coefficients, admitting, at most, a polynomial growth as $|x| \to \infty$. We dealt both with hyperbolic and parabolic type operators, constructing a solution theory for the associated Cauchy problems, see [4, 7]. Our research on SPDEs will now continue with the investigation of semilinear Schrödinger type equations with variable coefficients, paving the way to consider them in the environment of (suitable classes of) non-compact Riemannian manifolds. A basic example of such geometric settings is $\mathbb{R}^d$ equipped with an asymptotically flat metric (see Remark 1.3 below), then involving a *non-flat* Laplacian. Already in the deterministic case $\sigma \equiv 0$, the associated solution theories are far from being straightforward (see, e.g., [16, 26, 27, 31, 36]).

In this paper we consider an operator P of the form (1.2) with uniformly bounded coefficients. Since we plan to continue our analysis considering also the case of potential terms with polynomial growth, we adopt since now a unified treatment, employing the class $S^{m,\mu}(\mathbb{R}^d)$, $m, \mu \in \mathbb{R}$, of symbols of order (m, μ), given by the set of all functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying, for every $\alpha, \beta \in \mathbb{Z}^n_+$, the global estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \le C_{\alpha,\beta}\langle x\rangle^{m-|\beta|}\langle \xi\rangle^{\mu-|\alpha|}, \quad (x, \xi) \in \mathbb{R}^{2d},$$

for suitable constants $C_{\alpha\beta} > 0$, where $\langle y\rangle = \sqrt{1+|y|^2}$, $y \in \mathbb{R}^d$. Recall that, with any symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, it is associated a pseudodifferential operator

$$[\mathrm{Op}(a)u](x) = [a(\cdot, D)u](x) = (2\pi)^{-d}\int e^{\mathrm{i}x\xi}a(x, \xi)\hat{u}(\xi)d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d), \qquad (1.8)$$

linear and continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself, extended by duality to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself (see, e.g., the introductory sections of [4–7, 15], for details about the associated calculus). Moreover, $\mathrm{Op}(a)$ acts continuously from $H^{r,\rho}(\mathbb{R}^d)$ to $H^{r-m,\rho-\mu}(\mathbb{R}^d)$, where, given $r, \rho \in \mathbb{R}$, the *weighted Sobolev* spaces $H^{r,\rho}(\mathbb{R}^d)$ (also known as Sobolev-Kato spaces) are defined by

$$H^{r,\rho}(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d) |\, \mathrm{Op}(\lambda_{r,\rho})u \in L^2(\mathbb{R}^d)\}, \quad \lambda_{r,\rho}(x,\xi) = \langle x\rangle^r \langle \xi\rangle^\rho.$$

Here we assume the following hypotheses on the operator P (see [16]):

(1) the *Hamiltonian* satisfies $a \in S^{0,2}(\mathbb{R}^d)$;
(2) the *lower order metric terms* satisfy $a_1 \in S^{-1,1}(\mathbb{R}^d)$;
(3) the *magnetic field term* satisfies $m_1 \in S^{0,1}(\mathbb{R}^d)$ and is real-valued;
(4) the *potential* satisfies $m_0 \in S^{0,0}(\mathbb{R}^d)$;
(5) a satisfies, for all $x, \xi \in \mathbb{R}^d$, $C^{-1}|\xi|^2 \leq a(x,\xi) \leq C|\xi|^2$.

Remark 1.2 (i) In the sequel, we will often omit the base spaces $\mathbb{R}^d$, $\mathbb{R}^{2d}$, etc., from the notation.
(ii) The symbol spaces $S^{m,\mu}$ are denoted by $S^{\mu,m}(1,0)$ in [16], where it is remarked that the ellipticity condition (5), together with the other hypotheses on a and a_1, implies that the matrix $(a_{j\ell})$ is invertible, as well as that the Riemannian metric given by the matrix $(a_{j\ell})^{-1} = (a^{j\ell}) = \mathfrak{a}$ is asymptotically flat.

Remark 1.3 In view of the hypotheses on $\mathfrak{a}$, our analyis actually covers the case

$$P(x, \partial_t, \partial_x) = i\partial_t + \frac{1}{2}\Delta_a + \widetilde{m}_1(x, \partial_x) + m_0(x, \partial_x),$$

where $\widetilde{m}_1 \in S^{0,1}$, $m_0 \in S^{0,0}$, and

$$\Delta_a = \sqrt{\det(\mathfrak{a})} \sum_{j,\ell=1}^{d} \partial_{x_j} \left[\sqrt{\det(\mathfrak{a})^{-1}}\, a^{j\ell}\, \partial_{x_\ell}\right]$$

is the Laplace-Beltrami operator associated with $\mathfrak{a}$. Indeed, (1), (2) and (5) imply that $a_{j\ell} \in S^{0,0}$ and $\det(\mathfrak{a}) \geq c > 0$. In turn, this implies $\det(\mathfrak{a}), \det(\mathfrak{a})^{-1}, a^{j\ell}, \sqrt{\det(\mathfrak{a})}, \sqrt{\det(\mathfrak{a})^{-1}} \in S^{0,0}$ and

$$\begin{aligned}
\Delta_a &= \sqrt{\det(\mathfrak{a})} \sum_{j,\ell=1}^{d} \partial_{x_j} \left[\sqrt{\det(\mathfrak{a})^{-1}}\, a^{j\ell}\, \partial_{x_\ell}\right] \\
&= \sum_{j,\ell=1}^{d} a^{j\ell}\partial_{x_j}\partial_{x_\ell} + \sum_{j,\ell=1}^{d} \sqrt{\det(\mathfrak{a})} \left\{\partial_{x_j}\left[\sqrt{\det(\mathfrak{a})^{-1}}\, a^{j\ell}\right]\right\} \partial_{x_\ell} \\
&= \sum_{j,\ell=1}^{d} a^{j\ell}\partial_{x_j}\partial_{x_\ell} \quad \mathrm{mod}\ \mathrm{Op}(S^{-1,1}).
\end{aligned}$$

Then, the *non-selfadjoint terms* can be included into $\widetilde{m}_1 \in S^{0,1} \supset S^{-1,1}$, see [16, p.XX-4].

To state the main result of the present paper we need to introduce a subclass of the Sobolev-Kato spaces and a class of Lipschitz functions (the latter, analogous to those appearing in [4]).

Definition 1.4 Given $z \in \mathbb{N}$, $\zeta \in \mathbb{R}$, set $\mathcal{H}_{z,\zeta}(\mathbb{R}^d) := \bigcap_{j=0}^{z} H^{z-j,j+\zeta}(\mathbb{R}^d)$. The space $\mathcal{H}_{z,\zeta}(\mathbb{R}^d)$ is endowed with the norm

$$\|u\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)} := \sum_{j=0}^{z} \|u\|_{H^{z-j,j+\zeta}(\mathbb{R}^d)}. \tag{1.9}$$

By the immersion properties of the Sobolev-Kato spaces, we immediately see that $H^{z,z+\zeta}(\mathbb{R}^d) \subset \mathcal{H}_{z,\zeta}(\mathbb{R}^d) \subset H^{z,\zeta}(\mathbb{R}^d)$.

Remark 1.5 (i) Since every space $H^{r,\rho}$ with $r \geq 0$ and $\rho > d/2$ is an algebra (see [3, Proposition 2.2]), also $\mathcal{H}_{z,\zeta}$ is an algebra for $\zeta > d/2$ (see Example 1.10 below).

(ii) The Hilbert spaces based on the norm (1.9) for an arbitrary $\zeta \in \mathbb{N}$ are mentioned in [16, Page XX-12], where, in particular, the *unweighted* Sobolev spaces $H^{0,\rho}$ is denoted, as usual, by H^ρ, and $\mathcal{H}_{r,0}$, the space of spatial moments up to order $r \in \mathbb{N}$, is denoted by W^r.

In the linear deterministic case, that is, for $\sigma = \gamma \equiv 0$, the existence of a unique solution to the Cauchy problem (1.1) and the evolution of its solution has been fully described.

Theorem 1.6 [16, Page XX-12] *Under the assumptions (1)-(5), the solution $u(t)$ to the Cauchy problem (1.1) with $\sigma = \gamma \equiv 0$ satisfies the estimate*

$$\|u(t)\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)} \leq e^{C_{z,\zeta}t} \|u_0\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)}, \quad t \in [0, T_0],$$

for $T_0 \in (0, T]$ and a positive constant $C_{z,\zeta}$ depending only on $z, \zeta \in \mathbb{N}$.

Remark 1.7 As a consequence of Theorem 1.6, the *propagator* S (or, equivalently, the fundamental solution) of P defines continuous maps $S(t) : \mathcal{H}_{z,\zeta} \to \mathcal{H}_{z,\zeta}$, whose norms can be bounded by $e^{C_{z,\zeta}t}$, $t \in [0, T_0]$, $z, \zeta \in \mathbb{N}$.

Let us now define the Lipschitz class where we are going to take the nonlinear terms.

Definition 1.8 We say that a measurable function $g : [0, T] \times \mathbb{R}^d \times \mathbb{C} \longrightarrow \mathbb{C}$ belongs to the space $\mathcal{L}ip(z, \zeta)$ for some $z \in \mathbb{N}$, $\zeta \in [0, +\infty)$, if there exists a real-valued and non-negative function $C_t = C(t) \in C([0, T])$ such that:

- $\forall v \in \mathcal{H}_{z,\zeta}(\mathbb{R}^d)$, $t \in [0, T]$, we have $\|g(t, \cdot, v)\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)} \leq C(t)\left[1 + \|v\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)}\right]$;
- $\forall v_1, v_2 \in \mathcal{H}_{z,\zeta}(\mathbb{R}^d)$, $t \in [0, T]$, we have

$$\|g(t, \cdot, v_1) - g(t, \cdot, v_2)\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)} \leq C(t)\|v_1 - v_2\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)}.$$

If the properties here above are true only for $v, v_1, v_2 \in U$, with U a suitable open subset of $\mathcal{H}_{z,\zeta}(\mathbb{R}^d)$, then we say that $g \in \mathcal{L}ip_{\text{loc}}(z, \zeta)$. In applications, U is typically a small enough neighbourhood of the initial data.

The main result of the present paper is the subsequent Theorem 1.9.

Theorem 1.9 *Let us consider the Cauchy problem* (1.1) *for a Schrödinger type operator* (1.2) *under assumptions (1)–(5) on page 5, and suppose $u_0 \in \mathcal{H}_{z,\zeta}(\mathbb{R}^d)$, $z, \zeta \in \mathbb{N}$. Assume that $\gamma, \sigma \in \mathcal{L}ip_{\text{loc}}(z, \zeta)$ in some open subset $U \subset \mathcal{H}_{z,\zeta}(\mathbb{R}^d)$ with $u_0 \in U$, and*

$$\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) < \infty, \tag{1.10}$$

where $\mathfrak{M}$ is the spectral measure associated with the noise Ξ, see (1.3). Then, there exists a time horizon $0 < T_0 \le T$ such that the Cauchy problem (1.1) admits a unique solution $u \in L^2([0, T_0] \times \Omega, \mathcal{H}_{z,\zeta}(\mathbb{R}^d))$ satisfying (1.7) for all $t \in [0, T_0]$, where $S(t)$ is the propagator of P, the first integral in (1.7) makes sense as a Bochner integral, the second one as a stochastic integral of the $\mathcal{H}_{z,\zeta}(\mathbb{R}^d)$-valued stochastic process $S(t-*)\sigma(*, u(*))$ with respect to a cylindrical Wiener process associated with the stochastic noise Ξ.

Example 1.10 When $\zeta > d/2$, the function $g(t, x, u) = u^n$, $n \in \mathbb{N}$, is an admissible nonlinearity for the Cauchy problem (1.1). In fact, $g \in \mathcal{L}ip_{\text{loc}}(z, \zeta)$, when $z \in \mathbb{N}$, $\zeta > d/2$, since, if $v \in \mathcal{H}_{z,\zeta}$ is such that $\|v\|_{\mathcal{H}_{z,\zeta}} \le R$, then

$$\|g(t, x, v)\|_{\mathcal{H}_{z,\zeta}} = \|v^n\|_{\mathcal{H}_{z,\zeta}} \le \widetilde{C} R^{n-1} \|v\|_{\mathcal{H}_{z,\zeta}} \le \left(\widetilde{C} R^{n-1}\right) \left[1 + \|v\|_{\mathcal{H}_{z,\zeta}}\right]. \tag{1.11}$$

Indeed, $r = z - j \ge 0$ and $\rho = \zeta + j > d/2$, $j = 0, \dots, z$, and $\|v\|_{\mathcal{H}_{z,\zeta}} \le R \Rightarrow \|v\|_{H^{r,\rho}} \le R$, imply, for the algebra properties of the weighted Sobolev spaces,

$$\|v^n\|_{H^{r,\rho}} \le C_{nr\rho} \|v^n\|_{H^{nr,\rho}} \le C_{nr\rho} \|v\|^n_{H^{r,\rho}} \le C_{nr\rho} R^{n-1} \|v\|_{H^{r,\rho}},$$

and (1.11) immediately follows, by the definition (1.9) of the $\mathcal{H}_{z,\zeta}$ norm. The second requirement in Definition 1.8 follows by similar considerations, applying the Mean Value Theorem on g.

Comparing our result with [20, 21, 23, 32], we observe that there the focus is on the flat Schrödinger operator $P = i\partial_t - \triangle$, the noise is a real-valued Gaussian process, and the solution takes values in some L^p-modeled Sobolev spaces, not necessarily of Hilbert type. Conversely, here we deal with the Schrödinger operator $P = i\partial_t - \triangle_{\mathfrak{g}}$, associated with an asymptotically flat metric $\mathfrak{g}$, we allow the noise to be a distribution-valued Gaussian process, and we look for solutions in certain weighted L^2-modeled Sobolev spaces. Moreover, the assumptions in [20, 21, 23, 32] require that the noise is of Hilbert–Schmidt (or radonifying) type, while here we provide a condition on the noise (precisely, on its spectral measure) so that such property holds true, in analogy with the approach we followed in [4, 7], inspired by [33].

The next Sect. 2 is devoted to proving our main result. For the convenience of the reader, we included a short Appendix, where we recall some basic elements of the stochastic integration that we need. Such materials have appeared, in slightly different forms, e.g. in [4, 7].

2 Construction of function-valued solutions for semilinear Schrödinger type SPDEs

To give meaning to the third term in the right-hand side of (1.7) as a stochastic integral with respect to a cylindrical Wiener process on a suitable Hilbert space, we need to understand the noise Ξ in terms of the associated so-called Cameron–Martin space (see [33]). The latter is defined as

$$\mathcal{H}_\Xi = \{\widehat{\varphi\mathfrak{M}} \colon \varphi \in L^2_{\mathfrak{M},s}(\mathbb{R}^d)\},$$

where $L^2_{\mathfrak{M},s}(\mathbb{R}^d)$ is the space of symmetric functions in $L^2_{\mathfrak{M}}(\mathbb{R}^d)$ (i.e., $\check{\varphi}(x) = \varphi(-x) = \varphi(x)$, $x \in \mathbb{R}^d$, and $\displaystyle\int_{\mathbb{R}^d} |\varphi(x)|^2 \, \mathfrak{M}(dx) < \infty$). Of course, $\mathcal{H}_\Xi \subset \mathcal{S}'(\mathbb{R}^d)$. Moreover, [33, Propostition 2.1] shows that $\mathcal{H}_\Xi$, endowed with the inner product $\langle \widehat{\varphi\mathfrak{M}}, \widehat{\psi\mathfrak{M}} \rangle_{\mathcal{H}_\Xi} :=$

$\langle\varphi,\psi\rangle_{L^2_{\mathfrak{M},s}(\mathbb{R}^d)}$ and the corresponding norm $||\widehat{\varphi\mathfrak{M}}||^2_{\mathcal{H}_\Xi} = ||\varphi||^2_{L^2_{\mathfrak{M},s}(\mathbb{R}^d)}$, is a real separable Hilbert space, canonically associated with Ξ. The noise Ξ can be interpreted as a cylindrical Wiener process W on $(\mathcal{H}_\Xi, \langle\cdot,\cdot\rangle_{\mathcal{H}_\Xi})$ which takes values in any Hilbert space $\mathcal{H}$ such that the embedding $\mathcal{H}_\Xi \hookrightarrow \mathcal{H}$ is an Hilbert–Schmidt map. It follows that, to ensure that the stochastic integral appearing in (1.7) makes sense, it will be enough to verify that $S(t-\cdot)\sigma(\cdot,u(\cdot)) \in L^2([0,t]\times\Omega; L_2(\mathcal{H}_\Xi,\mathcal{H}))$ for a suitable separable Hilbert space $\mathcal{H}$, where $L_2(A,B)$ denotes the space of Hilbert–Schmidt maps from the Hilbert space A to the Hilbert space B.

To prove Theorem 1.9, the key result is the next Lemma 2.1. This is a variant of a result due, in its original form, given for wave-type equations, to Peszat [33].

Lemma 2.1 *Let $\sigma \in \mathcal{L}ip(z,\zeta)$, $z,\zeta \in \mathbb{N}$, and let $S(t)$ be the propagator of P provided by Theorem 1.6. If the spectral measure satisfies (1.10), then, for every $w \in \mathcal{H}_{z,\zeta}(\mathbb{R}^d)$, the operator*

$$\Phi(t,s) =: \psi \mapsto S(t-s)\sigma(s,\cdot,w)\psi$$

belongs to the class $L_2(\mathcal{H}_\Xi, \mathcal{H}_{z,\zeta}(\mathbb{R}^d))$, and its Hilbert–Schmidt norm satisfies the estimate

$$\|\Phi(t,s)\|^2_{L_2(\mathcal{H}_\Xi,\mathcal{H}_{z,\zeta}(\mathbb{R}^d))} \lesssim C_{t,s}\left[1+\|w\|_{\mathcal{H}_{z,\zeta}(\mathbb{R}^d)}\right]^2 \int_{\mathbb{R}^d}\mathfrak{M}(d\xi), \tag{2.1}$$

for $C_{t,s} = e^{2C_{z,\zeta}t}C_s^2$, where C_s is the continuous function appearing in Definition 1.8 and $C_{z,\zeta}$ is the constant appearing in Theorem 1.6.

Remark 2.2 Lemma 2.1 means that, under suitable assumptions on σ, the multiplication operator $\mathcal{H}_\Xi \ni \psi \mapsto S(t-s)\sigma(s,u)\cdot\psi$ acts as an Hilbert–Schmidt operator from $\mathcal{H}_\Xi$ to $\mathcal{H} = \mathcal{H}_{z,\zeta}$. As a consequence, the stochastic noise Ξ defines a cylindrical Wiener process on $(\mathcal{H}_\Xi, \langle\cdot,\cdot\rangle_{\mathcal{H}_\Xi})$ with values in $\mathcal{H}_{z,\zeta}$. It follows that the stochastic integral in the right-hand side of (1.7) can be seen as a well-defined stochastic integral with respect to a cylindrical Wiener process on $(\mathcal{H}_\Xi, \langle\cdot,\cdot\rangle_{\mathcal{H}_\Xi})$ which takes values in $\mathcal{H}_{z,\zeta}$.

Proof Let us fix an orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ in $L^2_{\mathfrak{M},s}$, and the corresponding orthonormal basis $\{e_j\}_{j\in\mathbb{N}} = \{\widehat{f_j\mathfrak{M}}\}_{j\in\mathbb{N}}$ of $\mathcal{H}_\Xi$. Using the definition of Hilbert–Schmidt norm and the continuity of the map $S(t) : \mathcal{H}_{z,\zeta} \longrightarrow \mathcal{H}_{z,\zeta}$, we compute

$$\begin{aligned}
&||\Phi(t,s)||^2_{L_2(\mathcal{H}_\Xi,\mathcal{H}_{z,\zeta})}\\
&\quad= \sum_{j\in\mathbb{N}} ||S(t-s)\sigma(s,\cdot,w)\widehat{f_j\mathfrak{M}}||^2_{\mathcal{H}_{z,\zeta}} \le e^{2C_{z,\zeta}t}\sum_{j\in\mathbb{N}}||\sigma(s,\cdot,w)\widehat{f_j\mathfrak{M}}||^2_{\mathcal{H}_{z,\zeta}}\\
&\quad\lesssim e^{2C_{z,\zeta}t}\sum_{j\in\mathbb{N}}\sum_{k=0}^{r}||\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\widehat{f_j\mathfrak{M}}||^2_{L^2}\\
&\quad= e^{2C_{z,\zeta}t}(2\pi)^{-d}\sum_{k=0}^{r}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^d}\left|\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\widehat{f_j\mathfrak{M}}\right)\right|^2(\xi)d\xi,
\end{aligned} \tag{2.2}$$

where $C_{z,\zeta}$ is the costant appearing in Theorem 1.6. Now, since the Fourier transform of a product is $(2\pi)^{-d}$ times the convolution of the Fourier transforms, and by definition of $L^2_{\mathfrak{M},s}$ we have $f_j(-x) = f_j(x)$, using the fact that $\{f_j\}$ is an orthonormal system in $L^2_{\mathfrak{M}}$ and Bessel's inequality, we get

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}} \left|\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\widehat{f_j\mathfrak{M}}\right)\right|^2(\xi)\\
&\quad= (2\pi)^{-d}\sum_{j\in\mathbb{N}} |\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right) * \widehat{\widehat{f_j\mathfrak{M}}}|^2(\xi)\\
&\quad= \sum_{j\in\mathbb{N}} |\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right) * f_j\mathfrak{M}|^2(\xi)\\
&\quad= \sum_{j\in\mathbb{N}} \left|\int_{\mathbb{R}^d}\left[\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right)\right](\xi-\eta)f_j(\eta)\mathfrak{M}(d\eta)\right|^2\\
&\quad\le \int_{\mathbb{R}^d}\left|\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right)\right|^2(\xi-\eta)\mathfrak{M}(d\eta).
\end{aligned}
$$

Substituting this estimate in (2.2), and using the fact that the operator $\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}$ acts continuously on Sobolev–Kato spaces, we obtain:

$$
\begin{aligned}
&||\Phi(t,s)||^2_{L_2(\mathcal{H}_\Xi,\mathcal{H}_{z,\zeta})}\\
&\quad\lesssim e^{2C_{z,\zeta}t}(2\pi)^{-d}\sum_{k=0}^{z}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right)\right|^2(\xi-\eta)\mathfrak{M}(d\eta)d\xi\\
&\quad= e^{2C_{z,\zeta}t}(2\pi)^{-d}\sum_{k=0}^{z}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right)\right|^2(\theta)\mathfrak{M}(d\eta)d\theta\\
&\quad= e^{2C_{z,\zeta}t}(2\pi)^{-d}\sum_{k=0}^{z}\left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\int_{\mathbb{R}^d}\left|\mathcal{F}\left(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\right)\right|^2(\theta)d\theta\\
&\quad= e^{2C_{z,\zeta}t}(2\pi)^{-d}\left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\sum_{k=0}^{z}\|\mathcal{F}(\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w))\|^2_{L^2}\\
&\quad\lesssim e^{2C_{z,\zeta}t}\left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\sum_{k=0}^{z}\|\langle\cdot\rangle^{z-k}\langle D\rangle^{k+\zeta}\sigma(s,\cdot,w)\|^2_{L^2}\\
&\quad= e^{2C_{z,\zeta}t}\left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\sum_{k=0}^{z}\|\sigma(s,\cdot,w)\|^2_{H^{z-k,k+\zeta}}\\
&\quad\lesssim e^{2C_{z,\zeta}t}\|\sigma(s,\cdot,w)\|^2_{\mathcal{H}_{z,\zeta}}\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\\
&\quad\le e^{2C_{z,\zeta}t}C_s^2\left(1+\|w\|_{\mathcal{H}_{z,\zeta}}\right)^2\int_{\mathbb{R}^d}\mathfrak{M}(d\eta),
\end{aligned}
$$

where C_s appears in Definition 1.8 and is a continuous function with respect to $s\in[0,T]$. □

We are now ready to prove our main result. The upcoming proof of Theorem 1.9 relies on the mapping properties of the fundamental solution $S(t)$ on the $\mathcal{H}_{z,\zeta}$ spaces, illustrated in Remark 1.7, and on a fixed point scheme, which works, in view of the key estimate (2.1), under assumption (1.10).

Proof Inserting the right-hand side $f(s,u(s))=\gamma(s,u(s))+\sigma(s,u(s))\dot{\Xi}(s)$ in the solution

$$
u(t)=S(t)u_0-i\int_0^t S(t-s)f(s,u(s))ds
$$

of the associated deterministic Cauchy problem with data f, u_0, and operator P, we can formally construct the "mild solution" u to (1.1), namely

$$u(t) = S(t)u_0 - i\int_0^t S(t-s)\gamma(s,u(s))ds - i\int_0^t S(t-s)\sigma(s,u(s))\dot{\Xi}(s)ds.$$

By the linear deterministic theory in [16], we know that

$$v_0(t) := S(t)u_0 \in C([0,T],\mathcal{H}_{z,\zeta}), \tag{2.3}$$

since, by hypotheses, $u_0 \in \mathcal{H}_{z,\zeta}$. Let us now consider the map $u \mapsto \mathcal{T}u$ on $L^2([0,T_0] \times \Omega, \mathcal{H}_{z,\zeta})$, $T_0 \in (0,T]$ small enough, defined as

$$\begin{aligned}\mathcal{T}u(t) &:= v_0(t) - i\int_0^t S(t-s)\gamma(s,u(s))ds - i\int_0^t S(t-s)\sigma(s,u(s))dW_s\\ &:= v_0(t) + \mathcal{T}_1u(t) + \mathcal{T}_2u(t), \quad t \in [0,T_0],\end{aligned} \tag{2.4}$$

where we remark that the last integral on the right-hand side makes sense as the stochastic integral of the stochastic process $S(t-*)\sigma(*,\cdot,u(*,\cdot)) \in L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$ with respect to the cylindrical Wiener process $\{W_t(h)\}_{t\in[0,T],h\in\mathcal{H}_{z,\zeta}}$, associated with the noise $\Xi(t)$, and that by Lemma 2.1 the cylindrical Wiener process is well-defined and takes values in $\mathcal{H}_{z,\zeta}$.

We initially work, for simplicity, under the stronger assumption $\gamma, \sigma \in \mathcal{L}ip(z,\zeta)$.

We want to apply Banach's fixed point Theorem to show the existence of a unique mild solution $u \in L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$ of (1.1) satisfying $u = \mathcal{T}u$, that is (1.7), for a suitable $0 < T_0 \le T$. To this aim, we have to check that

$$\mathcal{T} \colon L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta}) \longrightarrow L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$$

is a well-defined and Lipschitz continuous map on $L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$, and actually a strict contraction, if we take a small enough $T_0 \in (0,T]$.

We first check that $\mathcal{T}u \in L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$ for every $u \in L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$. We have:

- $v_0 \in C([0,T_0],\mathcal{H}_{z,\zeta}) \hookrightarrow L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$;
- $\mathcal{T}_1u$ is in $L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$; indeed, the function $s \mapsto -iS(t-s)\gamma(s,\cdot,u(s))$ takes values in $L^2(\Omega,\mathcal{H}_{z,\zeta})$, and $\mathcal{T}_1u(t)$ is defined as its Bochner integral on $[0,t]$; now, using Bochner integrals properties, continuity of $S(t-s)$ on $\mathcal{H}_{z,\zeta}$ spaces, and the assumption $\gamma \in \mathcal{L}ip(z,\zeta)$, we get

$$\begin{aligned}&\|\mathcal{T}_1u\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}\\ &= \mathbb{E}\left[\int_0^{T_0}\|\mathcal{T}_1u(t)\|^2_{\mathcal{H}_{z,\zeta}}dt\right] = \int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t S(t-s)[\gamma(s,\cdot,u(s))]ds\right\|^2_{\mathcal{H}_{z,\zeta}}\right]dt\\ &\le \int_0^{T_0}\int_0^t \mathbb{E}\left[\|S(t-s)[\gamma(s,\cdot,u(s))]\|^2_{\mathcal{H}_{z,\zeta}}\right]dsdt\\ &\lesssim \int_0^{T_0}\int_0^t e^{2C_{z,\zeta}(t-s)}\,\mathbb{E}\left[\|\gamma(s,\cdot,u(s))\|^2_{\mathcal{H}_{z,\zeta}}\right]dsdt\\ &\le \int_0^{T_0}\int_0^t e^{2C_{z,\zeta}(t-s)}C_s^2\,\mathbb{E}\left[(1+\|u(s)\|_{\mathcal{H}_{z,\zeta}})^2\right]dsdt\\ &\le T_0\left(\max_{0\le s\le t\le T_0} e^{2C_{z,\zeta}(t-s)}C_s^2\right)\int_0^{T_0}\mathbb{E}\left[(1+\|u(s)\|_{\mathcal{H}_{z,\zeta}})^2\right]ds\\ &\lesssim T_0C_{T_0}\left[T_0 + \|u\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}\right] < \infty,\end{aligned} \tag{2.5}$$

where C_{T_0} depends continuously on T_0;
- $\mathcal{T}_2 u$ is in $L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$, in view of the fundamental isometry (A.1), Lemma 2.1, and Fubini's Theorem, which allows to move expectation inside and outside time integrals:

$$\begin{aligned}
&\|\mathcal{T}_2 u\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}\\
&= \mathbb{E}\left[\int_0^{T_0}\|\mathcal{T}_2 u(t)\|^2_{\mathcal{H}_{z,\zeta}}dt\right] = \int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t S(t-s)\sigma(s,\cdot,u(s))dW_s\right\|^2_{\mathcal{H}_{z,\zeta}}\right]dt\\
&= \int_0^{T_0}\int_0^t \mathbb{E}\left[\|S(t-s)\sigma(s,\cdot,u(s))\|^2_{L_2(\mathcal{H}_\Xi,\mathcal{H}_{z,\zeta})}\right]dsdt\\
&\lesssim \int_0^{T_0}\int_0^t \mathbb{E}\left[C_{t,s}\left(1+\|u(s)\|_{\mathcal{H}_{z,\zeta}}\right)^2\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right]dsdt\\
&= \left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\cdot\left(\max_{0\le s\le t\le T_0}C_{t,s}\right)\cdot T_0\cdot\int_0^{T_0}\mathbb{E}\left[\left(1+\|u(s)\|_{\mathcal{H}_{z,\zeta}}\right)^2\right]ds\\
&\lesssim T_0 C_{T_0}\left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\left(T_0+\int_0^{T_0}\mathbb{E}\left[\|u(s)\|^2_{\mathcal{H}_{z,\zeta}}\right]ds\right)\\
&= T_0 C_{T_0}\left(\int_{\mathbb{R}^d}\mathfrak{M}(d\eta)\right)\cdot\left[T_0+\|u\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}\right]<\infty,
\end{aligned}\tag{2.6}$$

with C_{T_0} continuous with respect to T_0.

Now, we show that $\mathcal{T}$ is a contraction for $T_0\in(0,T]$ suitably small. We take $u_1,u_2\in L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$ and compute

$$\begin{aligned}
&\|\mathcal{T}u_1-\mathcal{T}u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}\\
&\le 2\left(\|\mathcal{T}_1 u_1-\mathcal{T}_1 u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}+\|\mathcal{T}_2 u_1-\mathcal{T}_2 u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})}\right)\\
&= 2\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t S(t-s)(\gamma(s,\cdot,u_1(s))-\gamma(s,\cdot,u_2(s)))ds\right\|^2_{\mathcal{H}_{z,\zeta}}\right]dt \qquad (2.7)\\
&\quad +2\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t S(t-s)(\sigma(s,\cdot,u_1(s))-\sigma(s,\cdot,u_2(s)))dW_s\right\|^2_{\mathcal{H}_{z,\zeta}}\right]dt. \qquad (2.8)
\end{aligned}$$

To estimate the term (2.7), we first move inside the integral with respect to s both the expectation and the $\mathcal{H}_{z,\zeta}$-norm. Using then the continuity of $S(t-s)$ on $\mathcal{H}_{z,\zeta}$ spaces and the second requirement in Definition 1.8, we obtain

$$\begin{aligned}
&\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t S(t-s)(\gamma(s,\cdot,u_1(s))-\gamma(s,\cdot,u_2(s)))ds\right\|^2_{\mathcal{H}_{z,\zeta}}\right]dt\\
&\lesssim \int_0^{T_0}\int_0^t \mathbb{E}\left[\|S(t-s)(\gamma(s,\cdot,u_1(s))-\gamma(s,\cdot,u_2(s)))\|^2_{\mathcal{H}_{z,\zeta}}\right]dsdt\\
&\le \int_0^{T_0}\int_0^t e^{2C_{z,\zeta}(t-s)}\mathbb{E}\left[\|\gamma(s,\cdot,u_1(s))-\gamma(s,\cdot,u_2(s))\|^2_{\mathcal{H}_{z,\zeta}}\right]dsdt\\
&\le \int_0^{T_0}\int_0^t e^{2C_{z,\zeta}(t-s)}C_s^2\,\mathbb{E}\left[\|u_1(s)-u_2(s)\|^2_{\mathcal{H}_{z,\zeta}}\right]dsdt
\end{aligned}$$

$$\leq C_{T_0} T_0 \int_0^{T_0} \mathbb{E}\left[\|u_1(s) - u_2(s)\|^2_{\mathcal{H}_{z,\zeta}} \right] ds$$
$$= C_{T_0} T_0 \|u_1 - u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})},$$

with C_{T_0} continuous with respect to T_0. To estimate the term (2.8) we compute as follows:

$$\int_0^{T_0} \mathbb{E}\left[\left\| \int_0^t S(t-s)(\sigma(s,\cdot,u_1(s)) - \sigma(s,\cdot,u_2(s)))dW_s \right\|^2_{\mathcal{H}_{z,\zeta}} \right] dt$$
$$= \int_0^{T_0}\int_0^t \mathbb{E}\left[\|S(t-s)(\sigma(s,\cdot,u_1(s)) - \sigma(s,\cdot,u_2(s)))\|^2_{L_2(\mathcal{H}_\Xi,\mathcal{H}_{z,\zeta})} \right] dsdt$$
$$\lesssim \int_0^{T_0}\int_0^t \mathbb{E}\left[C_{t,s}\|u_1(s) - u_2(s)\|^2_{\mathcal{H}_{z,\zeta}} \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right] dsdt$$
$$\leq \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\eta) \right) \int_0^{T_0}\int_0^t C_{t,s}\mathbb{E}\left[\|u_1(s) - u_2(s)\|^2_{\mathcal{H}_{z,\zeta}} \right] dsdt$$
$$\lesssim C_{T_0} T_0 \|u_1 - u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})} \int_{\mathbb{R}^d} \mathfrak{M}(d\xi),$$

with C_{T_0} continuous with respect to T_0. To pass from the first to the second line we have used the fundamental isometry (A.1), and to pass from the second to the third line we have used the second requirement in Definition 1.8 and the analog of formula (2.1) from Lemma 2.1 with $\sigma(s,\cdot,u_1(s)) - \sigma(s,\cdot,u_2(s))$ in place of $\sigma(s,\cdot,w)$. Summing up, we have proved that

$$\|\mathcal{T}u_1 - \mathcal{T}u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})} \leq C_{T_0} T_0 \left(1 + \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \cdot \|u_1 - u_2\|^2_{L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})},$$

that is, $\mathcal{T}$ is Lipschitz continuous on $L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$. Moreover, in view of the assumption (1.10) and the continuity of C_{T_0} with respect to T_0, we can take $T_0 > 0$ so small that

$$C_{T_0} T_0 \left(1 + \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) < 1,$$

making $\mathcal{T}$ a strict contraction on $L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$, so that it admits a unique fixed point $u = \mathcal{T}u$, $u \in L^2([0,T_0]\times\Omega,\mathcal{H}_{z,\zeta})$, as claimed.

Finally, when $\gamma, \sigma \in \mathcal{L}ip_{\mathrm{loc}}(z,\zeta)$, we first observe that there exists $\mathfrak{B}$, closed ball of radius $R > 0$, centred in u_0, such that $\mathfrak{B} \subset U$. Since, of course,

$$v_0 \in C([0,T_0],\mathcal{H}_{z,\zeta}) \text{ and}$$
$$S(0)u_0 = u_0 \Rightarrow \exists T_0 \in (0,T] \colon \|S(t)u_0 - u_0\|_{\mathcal{H}_{z,\zeta}} \leq \frac{R}{2},\ t \in [0,T_0],$$

by computations similar to those employed to show (2.5) and (2.6), it turns out that, choosing $T_0 \in (0,T]$ suitably small,

$$\|\mathcal{T}_1 u(t)\|_{\mathcal{H}_{z,\zeta}} + \|\mathcal{T}_2 u(t)\|_{\mathcal{H}_{z,\zeta}} \leq \left(T_0 C_{T_0}\right)^{\frac{1}{2}} (1 + \|u_0\|_{\mathcal{H}_{z,\zeta}} + R) \left[\left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right)^{\frac{1}{2}} + 1 \right]$$
$$\leq \frac{R}{2},\ u(t) \in \mathfrak{B}, t \in [0,T_0],$$

so that $\mathcal{T} \colon L^2([0,T_0]\times\Omega,\mathfrak{B}) \to L^2([0,T_0]\times\Omega,\mathfrak{B})$ and it is a strict contraction there.

The proof is complete. □

Remark 2.3 We conclude with a comparison, in the linear case (that is, when drift and diffusion do not depend on u), between the function-valued solution constructed in this paper and the random-field solution u_{rf} of the same linear Cauchy problem that can be constructed, under condition (1.10), following the ideas in [5, 6]. A random-field solution u_{rf} is defined as a map associating to each $(t, x) \in [0, T_0] \times \mathbb{R}^d$ a random variable, and $T_0 > 0$ is the time horizon of the solution. This can be achieved by the properties of the fundamental solution to the operator P from Theorem 1.6. Indeed, random-field solutions are obtained by interpreting the stochastic integral in (1.6) as a stochastic integral with respect to a martingale mesaure, following the Walsh-Dalang approach to stochastic integration. It is well known that in several cases the two approaches to stochastic integration produce "the same solutions", see the comparison paper [18]. The random-field solution u_{rf} turns out to coincide with the function-valued solution u obtained in Theorem 1.9, see [4] for an explicit comparison in the case of hyperbolic SPDEs. The details of the analysis of u_{rf} sketched above will appear elsewhere.

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Appendix A. Stochastic integration

Definition A. 1 Let Q be a self-adjoint, nonnegative definite and bounded linear operator on a separable Hilbert space H. An H-valued stochastic process $W = \{W_t(h); h \in H, t \geq 0\}$ is called a *cylindrical Wiener process on* H on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ if the following conditions are fulfilled:

(1) for any $h \in H$, $\{W_t(h); t \geq 0\}$ is a one-dimensional Brownian motion with variance $t\langle Qh, h\rangle_H$;
(2) for all $s, t \geq 0$ and $g, h \in H$,

$$\mathbb{E}[W_s(g)W_t(h)] = (s \wedge t)\langle Qg, h\rangle_H.$$

If $Q = Id_H$, then W is called a standard cylindrical Wiener process.

Let $\mathscr{F}_t$ be the σ-field generated by the random variables $\{W_t(h); 0 \leq s \leq t, h \in H\}$ and the $\mathbb{P}$-null sets. The predictable σ-field is then the σ-field in $[0, T] \times \Omega$ generated by the sets $\{(s, t] \times A, A \in \mathscr{F}_t, 0 \leq s < t \leq T\}$.

We define H_Q to be the completion of the Hilbert space H endowed with the inner product

$$\langle g, h\rangle_{H_Q} := \langle Qg, h\rangle_H,$$

for $g, h \in H$. In the sequel, we let $\{v_k\}_{k\in\mathbb{N}}$ be a complete orthonormal basis of H_Q. Then, the stochastic integral of a predictable, square-integrable stochastic process with values in H_Q, $u \in L^2([0, T] \times \Omega; H_Q)$, is defined as

$$\int_0^t u(s)dW_s := \sum_{k\in\mathbb{N}} \int_0^t \langle u, v_k \rangle_{H_Q} dW_s(v_k).$$

In fact, the series in the right-hand side converges in $L^2(\Omega, \mathscr{F}, \mathbb{P})$ and its sum does not depend on the chosen orthonormal system $\{v_k\}_{k\in\mathbb{N}}$. Moreover, the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t u(s)dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t \|u(s)\|^2_{H_Q} ds\right]$$

holds true for any $u \in L^2([0, T] \times \Omega; H_Q)$.

This notion of stochastic integral can also be extended to operator-valued integrands. Let $\mathcal{H}$ be a separable Hilbert space and consider $L_2(H_Q, \mathcal{H})$, the space of Hilbert-Schmidt operators from H_Q to $\mathcal{H}$. With this we can define the space of integrable processes (with respect to W) as the set of $\mathscr{F}$-measureable processes in $L^2([0, T] \times \Omega; L_2(H_Q, \mathcal{H}))$. Since one can identify the Hilbert-Schmidt operators in $L_2(H_Q, \mathcal{H})$ with $\mathcal{H} \otimes H_Q^*$, one can define the stochastic integral for any $u \in L^2([0, T] \times \Omega; L_2(H_Q, \mathcal{H}))$ coordinatewise in $\mathcal{H}$. Moreover, it is possible to establish an Itô isometry, namely,

$$\mathbb{E}\left[\left\|\int_0^t u(s)dW_s\right\|^2_{\mathcal{H}}\right] := \int_0^t \mathbb{E}\big[\|u(s)\|^2_{L_2(H_Q,\mathcal{H})}\big]ds. \quad \text{(A.1)}$$

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