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Analysis of the controllability of bilinear closed quantum systems

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Abstract

In the present dissertation, we discuss the controllability of the bilinear Schrödinger equation appeared in literature after the seminal work on bilinear systems [BMS82] by Ball, Mardsen and Slemrod, then mostly popularized by Beauchard and Laurent with the work [BL10].

In order to facilitate the reading, we present below a brief outline of the manuscript.

Chapter 1: We provide a wide overview about the existing works on the topic and we explain the main outcomes obtained in the thesis.

Chapter 2: We study the global exact controllability of the bilinear Schrödinger equation in order to provide explicit controls and times for the result.

Chapter 3: Given infinitely many bilinear Schrödinger equations, we prove the simultaneous global exact controllability "in projection".

Chapter 4: We consider the bilinear Schrödinger equation on compact graphs. We prove the well-posedness, the global exact controllability and the "energetic controllability".

Appendix A: We show some results about the solvability of the so-called "moment problem".

Appendix B: We exploit some techniques of perturbation theory adopted in the manuscript.

Notation: We collect the main notations used in the thesis in order to avoid misunderstandings and simplify the reading.

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Contents

Chapter 1

Introduction

In non-relativistic quantum mechanics, any pure state of a system is mathematically represented by a wave function ψ in the unit sphere of a Hilbert space $\mathscr{H}$. For $T > 0$, its time evolution is described by a Cauchy problem

$$\begin{cases} i\partial_t \psi(t) = H(t)\psi(t), & t \in (0,T), \\ \psi(0) = \psi^0, \end{cases} \tag{1.1}$$

where $H(t)$ is a time-dependent self-adjoint operator, called Hamiltonian. We aim to describe the evolution of a particle confined in a bounded region and subjected to an external electromagnetic field that plays the role of a control. A standard choice for such a setting is $\mathscr{H} = L^2(\Omega, \mathbb{R})$, where Ω models the spatial domain, and the Hamiltonian $H(t)$ appearing in (1.1) is

$$H(t) = A + u(t)B. \tag{1.2}$$

The influence of the external field is modeled by the second term in (1.2), where the symmetric operator B describes the action of the field and the function u its (time-dependent) intensity. The operator A is the Laplacian equipped with suitable self-adjoint type boundary conditions, *e.g.*

$$\Omega = (0,1), \quad D(A) = H^2((0,1),\mathbb{C}) \cap H^1_0((0,1),\mathbb{C})),$$
$$A\psi = -\Delta\psi, \qquad \forall \psi \in D(A).$$

We call Γ^u_t the unitary propagator generated by $H(t)$ (when it is defined) and the dynamics of the particle is modeled by the so-called bilinear Schrödinger equation

$$\begin{cases} i\partial_t \psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0,T), \\ \psi(0) = \psi^0. \end{cases} \tag{BSE}$$

A natural question of practical implications is whether, given any couple of states, there exists $u \in L^2((0,T),\mathbb{R})$ steering the quantum system from the first state in the second one and how to build explicitly this control function.

The controllability of finite-dimensional quantum systems (*i.e.* modeled by an ordinary differential equation) is currently well-established.
If we consider the problem (BSE) in $\mathbb{C}^N$ such that A and B are $N \times N$ Hermitian matrices and $t \mapsto u(t) \in \mathbb{R}$ is the control, then the controllability of the the problem is linked to the rank of the Lie algebra spanned by A and B (we refer to [AD03] by Albertini and D'Alessandro, [Alt02] by Altafini, [Bro73] by Brockett and [Cor07] by Coron).
Nevertheless, the Lie algebra rank condition can not be used for infinite-dimensional quantum systems (see [Cor07] for further details). This is why different techniques were developed in order to deal with this type of problems.

Regarding the linear Schrödinger equation, the controllability and observability properties are reciprocally dual (which is often referred to the Hilbert Uniqueness Method). One can therefore address the control problem directly or by duality with various techniques: multiplier methods ([Fab92] by Fabre, [Lio83] by Lions, [Mac94] by Machtyngier), microlocal analysis ([BLR92] by Bardos, Lebeau and Rauch, [Bur91] by Burq and [Leb92] by Lebeau), Carleman estimates ([BM08] by Baudouin and Mercado, [LT92] by Lasiecka and Triggiani and [MOR08] by Mercado, Osses and Rosier). For non-linear equations, we refer to the works [DGL06] (by Dehman, Gerard and Lebeau), [LT07] (by Lange and Teismann), [RZ09] (by Rosier and Zhang), [Lau10a] and [Lau10b] (by Laurent).

Well-posedness in $\mathscr{H}$ and non-controllability result.

Even though the linear Schrödinger equation is widely studied in the literature, the bilinear Schrödinger equation can not be approached with the same techniques since it is non-controllable in $D(A)$. We refer to the seminal work on bilinear systems [BMS82] by Ball, Mardsen and Slemrod, where the well-posedness and the non-controllability are provided.
In the case of the bilinear Schrödinger equation, the mentioned work guarantees that if $B : D(A) \to D(A)$ and $u \in L^1((0,T),\mathbb{R})$ with $T > 0$, then (BSE) admits a unique solution

$$\psi \in C((0,T),\mathscr{H}),$$

for any initial state in $\mathscr{H}$. Moreover, let S be the unit sphere in $\mathscr{H}$ and $\Gamma_T^u\psi_0$ be the value at time $T>0$ of the solution of (BSE) with initial state $\psi_0 \in S \cap D(A)$. The set of the attainable states from ψ_0,

$$\left\{\Gamma_T^u\psi_0 : \ T>0, \ u \in L^2((0,T),\mathbb{R})\right\},$$

is contained in a countable union of compact sets. Then, it has dense complement in $S \cap D(A)$. As a consequence, the exact controllability of the bilinear Schrödinger equation can not be achieved in $S\cap D(A)$ with controls $u \in L^2_{loc}((0,\infty),\mathbb{R})$ (see also [Tur00] by Turinici).

Despite this negative result, many authors address the problem with weaker notions of controllability. Indeed, even though this outcome is not guaranteed in $D(A)$, there may exist suitable subspaces of $D(A)$ where the exact controllability can be verified.

Well-posedness in $D(A^{\frac{3}{2}})$.

We start by mentioning Beauchard and Laurent [BL10] who study the bilinear Schrödinger equation in $\mathscr{H} = L^2((0,1),\mathbb{C})$ for A such that

$$D(A) = H^2((0,1),\mathbb{C}) \cap H^1_0((0,1),\mathbb{C}),$$

$$A\psi = -\Delta\psi, \qquad \forall \psi \in D(A).$$

Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a complete orthonormal system of $\mathscr{H}$ composed by eigenfunctions of A and associated to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ ($\lambda_k = \pi^2 k^2$). For $s>0$, they consider the spaces

$$H^s_{(0)} := D(A^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} := \left(\sum_{j=1}^{\infty} |j^s \langle \phi_j, \cdot\rangle_{\mathscr{H}}|^2\right)^{\frac{1}{2}}.$$

In [BL10], Beauchard and Laurent prove the well-posedness of the bilinear Schrödinger equation in $H^3_{(0)}$ when B is a multiplication operator for $\mu \in H^3((0,1),\mathbb{R})$. In particular, for $T>0$, $\psi^0 \in H^3_{(0)}$ and $u \in L^2((0,T),\mathbb{R})$, they provide the existence of a unique mild solution of (BSE) in $H^3_{(0)}$, i.e. $\psi \in C^0([0,T], H^3_{(0)})$ such that

$$\psi(t,x) = e^{-iAt}\psi^0(x) - i\int_0^t e^{-iA(t-s)}(u(s)\mu(x)\psi(s,x))ds, \qquad \forall t \in [0,T].$$

Moreover, for every $R > 0$, there exists $C = C(T, \mu, R) > 0$ such that, if

$$\|u\|_{L^2((0,T),\mathbb{R})} < R,$$

then the solution satisfies, for every $\psi^0 \in H^3_{(0)}$, the following identities

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leq C\|\psi^0\|_{(3)}, \qquad \|\psi(t)\|_{\mathscr{H}} = \|\psi^0\|_{\mathscr{H}} \quad \forall t \in [0,T].$$

The peculiarity of the result is that the well-posedness in $H^3_{(0)}$ is guaranteed even if B does not stabilize $H^3_{(0)}$ due to an hidden regularizing effect. The main hypothesis used in its proof are

$$B : H^2_{(0)} \longrightarrow H^2_{(0)}, \qquad B : H^3_{(0)} \longrightarrow H^3((0,1),\mathbb{C}) \cap H^1_0((0,1),\mathbb{C}).$$

The well-posedness can also be proved thanks to the arguments developed by Kato in [Kat53]. When $u \in BV((0,T),\mathbb{R})$ and $B \in L(H^2_{(0)})$, the mentioned work shows that Γ^u_t stabilizes $H^{s_1}_{(0)}$ for every $s_1 \in [2,4]$. However, in [BL10] the result is provided for a wider class of controls.

Local exact controllability.

- Let $\mathcal{M} \subset \mathscr{H}$ be a normed space and $V \subset \mathcal{M}$ be a neighborhood of $\psi^1 \in \mathcal{M}$. The problem (BSE) is said to be **locally exactly controllable** (Figure 1.1) in V when, for every $\psi^2 \in V$ such that $\|\psi^2\|_{\mathscr{H}} = \|\psi^1\|_{\mathscr{H}}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Gamma^u_T \psi^1 = \psi^2.$$

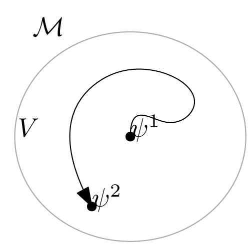


Figure 1.1: The figure represents the dynamics for the local exact controllability driving $\psi^1 \in V$ to $\psi^2 \in V$.

Another important outcome proved by Beauchard and Laurent in [BL10] is the local exact controllability. They show that if B is a multiplication operator for a function $\mu \in H^3((0,1),\mathbb{R})$ such that there exists $C > 0$ implying

$$|\langle \phi_j, \mu\phi_1\rangle_{\mathscr{H}}| \geq \frac{C}{j^3}, \quad \forall j \in \mathbb{N}, \tag{1.3}$$

then the bilinear Schrödinger equation is locally exactly controllable in a neighborhood of the first eigenfunction of A in $H^3_{(0)}$.
Heuristically speaking, the condition (1.3) quantifies how much the operator B mixes the eigenfunctions of A. In the current work, we adopt similar assumptions which also appear in other recent manuscripts.

An important aspect of their work is that they popularize a set of techniques that are widely used in literature for this type of results. In particular, they prove that the local exact controllability is equivalent to the controllability of the linearized system in a neighborhood of the first eigenfunction of A. It corresponds to the solvability of a "moment problem"

$$x_k = \int_0^T e^{i(\lambda_k-\lambda_1)s}u(s)ds, \qquad \forall k \in \mathbb{N},\ \{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C}) \tag{1.4}$$

for $u \in L^2((0,T),\mathbb{R})$ and $T > 0$ large enough. In the proof, the validity of the gap condition

$$\inf_{k\neq l} |\lambda_k - \lambda_l| > 0$$

is crucial and it allows to use classical results of solvability of moment problems as Ingham's Theorem and Haraux's Theorem.

For the sake of completeness, we refer to the works [Bea05], [Bea08] and [BC06] for other local exact controllability results. Therefore, the controllability proved by Beauchard and Laurent belongs to the classical framework of local controllability results for non-linear systems, proved with fixed point arguments as [CC09], [Ros97], [RZ96], [Zha99] and [Zua93].

Global approximate controllability.

- We say that the problem (BSE) is **globally approximately controllable** (Figure 1.2) in a normed space $\mathcal{M} \subset \mathscr{H}$ if, for any $\psi^1, \psi^2 \in \mathcal{M}$ such that $\|\psi^2\|_{\mathscr{H}} = \|\psi^1\|_{\mathscr{H}}$ and for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\|\Gamma_T^u\psi^1 - \psi^2\|_{\mathcal{M}} < \epsilon.$$

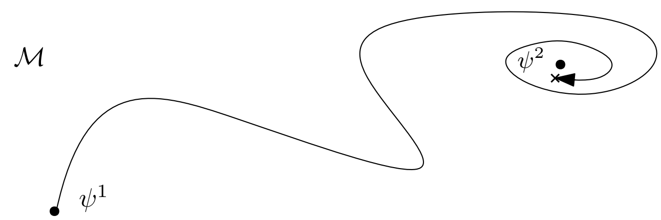


Figure 1.2: The figure represents the dynamics for the global approximate controllability driving $\psi^1 \in \mathcal{M}$ close to $\psi^2 \in \mathcal{M}$.

Let us consider $N \in \mathbb{N}$ symmetric operators $\{B_j\}_{j\leq N}$ in a Hilbert space $\mathscr{H}$, the functions $\{u_j\}_{j\leq N} \subset L^2((0,T),\mathbb{R})$ and a self adjoint operator A. Results of global approximate controllability for dynamics generated by Hamiltonians as

$$A + \sum_{j\leq N} u_j(t)B_j$$

are vastly studied in literature and the first examples that we present are [BGRS15] and [BCMS12] where adiabatic techniques are adopted.
The global approximate controllability is provided by Lyapunov techniques in [Mir09], [Ner09], [Ner10] and [NN12], while by Lie-Galerking arguments in [BCCS12], [BCS14] and [CMSB09].
The most useful for our purpose is the work [BdCC13] by Boussaïd, Caponigro and Chambrion, where Lie-Galerking arguments are adopted in order to verify the global approximate controllability in $D(|A|^{\frac{s}{2}})$ for some $s > 0$.
The main assumption considered in [BdCC13] (common for this type of results) is the so-called "non-degenerate chain of connectedness". Let $N = 1$. Heuristically speaking, the condition requires that $\{\lambda_j\}_{j\in\mathbb{N}}$ (the eigenvalues of A) are non-resonant (all gaps are different) and B_1 "sufficiently couples" the eigenstates.

Technically, the assumption requires that the following hypotheses are satisfied. Let $\mathcal{N}$ be the subset of $\mathbb{N}^2$ given by all the couples (k_1, k_2) such that $\langle \phi_{k_1}, B_1\phi_{k_2}\rangle_{\mathscr{H}} \neq 0$. We assume that

$$\lambda_j \neq \lambda_k$$

for every $(j,k) \in \mathcal{N}$ such that $j \neq k$ (resonant eigenvalues are not coupled by B_1). Let S be a subset of $\mathcal{N}$ such that the graph of vertices the elements of $\mathbb{N}$ and whose edges are the elements of S is connected (see Figure 1.3). The problem admits a "non-degenerate chain of connectedness" if, for every

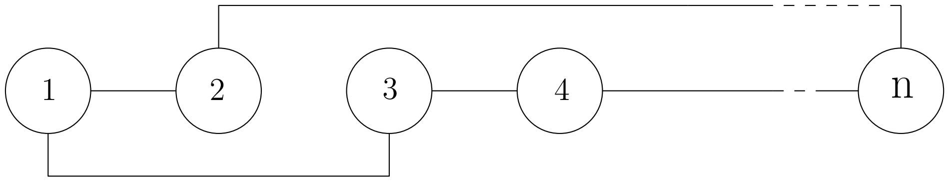


Figure 1.3: Each vertex of the graph represents an eigenstate of A. An edge links two vertices $j,k\in\mathbb{N}$ if and only if $\langle\phi_j,B_1\phi_k\rangle_{\mathscr{H}}\neq 0$.

$(j_1,j_2)\in S$ and every $(k_1,k_2)\in\mathcal{N}$ different from (j_1,j_2) and (j_2,j_1), there holds

$$|\lambda_{j_1}-\lambda_{j_2}|\neq|\lambda_{k_1}-\lambda_{k_2}|.$$

In [BdCC13], Boussaïd, Caponigro and Chambrion show that for $N=1$ and in presence of a non-degenerate chain of connectedness, if $B_1\in L(D(|A|^{\frac{s_1}{2}}))$ with $s_1>0$, then the problem is globally approximately controllable in $D(|A|^{\frac{s}{2}})$ for $s\in[0,s_1)$.

In the present work, we refer to this result and we adopt perturbation theory techniques in order to exhibit a non-degenerate chain of connectedness.

Simultaneous local and global exact controllability.

- Each type of controllability is said to be **simultaneous** (*e.g.* Figure 1.4) when it is simultaneously satisfied with the same control between more couples of states.

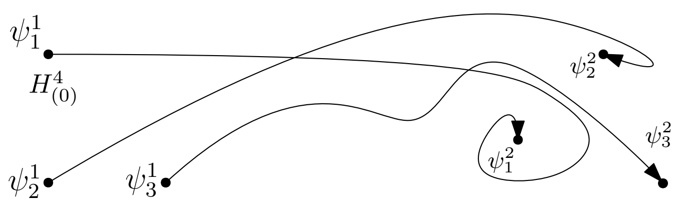


Figure 1.4: The figure shows the dynamics driving $\{\psi_k^1\}_{k\leq 3}\subset\mathcal{M}$ in $\{\psi_k^2\}_{k\leq 3}\subset\mathcal{M}$ obtained by the simultaneous global exact controllability.

Relevant results of simultaneous local exact controllability are provided by Morancey in [Mor14]. Let $\mathscr{H}=L^2((0,1),\mathbb{C})$, $N\in\{2,3\}$ and B be a multiplication operator for a function $\mu \in H^3((0,1),\mathbb{R})$ such that there exists $C>0$ such that

$$|\langle\phi_j,\mu\phi_k\rangle_{\mathscr{H}}| \geq \frac{C}{j^3}, \qquad \forall j\in\mathbb{N},\ k\leq N \tag{1.5}$$

(a similar condition to (1.3)). Morancey proves in [Mor14] the simultaneous local exact controllability in $H^3_{(0)}$ for N bilinear Schrödinger equations when μ satisfies (1.5) and

$$\begin{cases}\langle\phi_1,\mu\phi_1\rangle_{\mathscr{H}} \neq \langle\phi_2,\mu\phi_2\rangle_{\mathscr{H}}, & \text{if } N=2,\\ 5\langle\phi_1,\mu\phi_1\rangle_{\mathscr{H}} - 8\langle\phi_2,\mu\phi_2\rangle_{\mathscr{H}} + 3\langle\phi_3,\mu\phi_3\rangle_{\mathscr{H}} \neq 0, & \text{if } N=3.\end{cases} \tag{1.6}$$

In other words, Morancey proves that there exists a suitable neighborhood $V\subset (H^3_{(0)})^N$ of $\{\phi_j\}_{j\leq N}$ such that, for every $T>0$ and $\{\psi_j\}_{j\leq N}\in V$ with $\|\psi_j\|_{\mathscr{H}}=1$ for $j\leq N$, there exists $u\in L^2((0,T),\mathbb{R})$ such that

$$\psi_j = \Gamma^u_T\phi_j, \qquad 1\leq j\leq N.$$

In the work, the author adopts the "Coron's return method" but also the technique already presented by Beauchard and Laurent in [BL10].

In [MN15], Morancey and Nersesyan extend the previous result and achieve the simultaneous global exact controllability of any finite number of (BSE).
Let $N\in\mathbb{N}$. They prove the existence of Q, a residual subset of $H^4((0,1),\mathbb{R})$ (a countable intersection of dense open subsets of $H^4((0,1),\mathbb{R})$), such that for every multiplication operator B for $\mu\in Q$, the simultaneous global exact controllability is verified in $H^4_{(0)}$ for N bilinear Schrödinger equations.

In other words, let $U(\mathscr{H})$ be the space of the unitary operators on $\mathscr{H}$. For every $(\psi_i^1,...,\psi_i^N)$, $(\psi_f^1,...,\psi_f^N)\subset H^4_{(0)}$ unitarily equivalent, *i.e.* there exists $\widehat{\Gamma}\in U(\mathscr{H})$ such that $\psi_i^1=\widehat{\Gamma}\psi_f^j$ for every $j\leq N$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that

$$\psi_f^k = \Gamma^u_T\psi_i^k, \qquad 1\leq k\leq N.$$

In this work, the Coron's return method and the technique from Beauchard and Laurent [BL10] lead to the simultaneous local exact controllability of N bilinear Schrödinger equations. The result is gathered with the simultaneous global approximate controllability proved by Lyapunov techniques.

1.1 Main results

Explicit times and controls for the global exact controllability.

Let $\Omega = (0,1)$, B a bounded symmetric operator and A such that

$$D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})),$$
$$A\psi = -\Delta\psi, \qquad \forall \psi \in D(A).$$

In Chapter 2, we study the global exact controllability of the bilinear Schrödinger equation. Even though this result is well-established (it can be deduced from [MN15] by Morancey and Nersesyan), most of the existing works prove the existence of controls and times without providing them explicitly. For this reason, we ensure the global exact controllability with particular techniques which allow to precise those elements.

First, for any couple of eigenfunctions ϕ_j and ϕ_k, for $k \in \mathbb{N}$ such that

$$m^2 - k^2 \neq k^2 - l^2, \qquad \forall m, l \in \mathbb{N},$$

we exhibit controls and times such that the relative dynamics of (BSE) drives ϕ_j close to ϕ_k as much desired with respect to the $H^3_{(0)}-$norm.
Second, we show a neighborhood of ϕ_k in $H^3_{(0)}$ where the local exact controllability is satisfied in a given time.
Third, by gathering the two previous results, we define a dynamics steering any eigenstate of A in any other in an explicit time.
In conclusion, we generalize the result for every $k \in \mathbb{N}$.

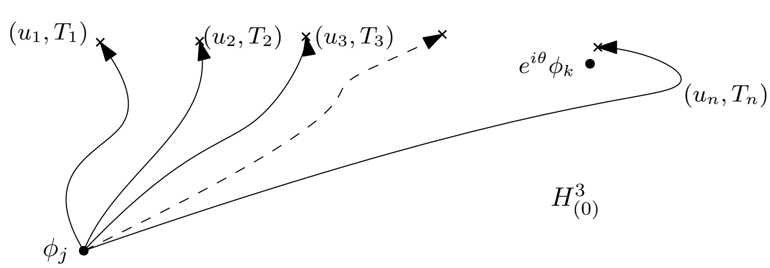
In more technical terms, we prove the following outcomes.

- For any ϕ_j and ϕ_k for $k \in \mathbb{N}$ such that

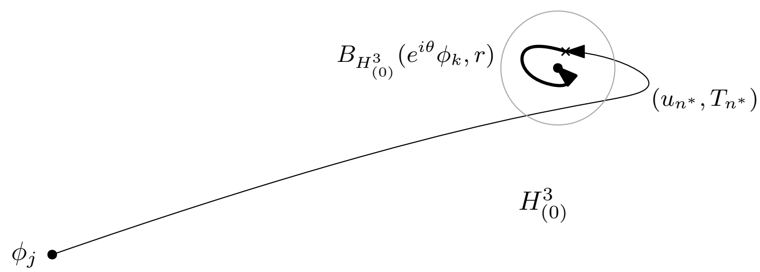
  $$m^2 - k^2 \neq k^2 - l^2, \qquad \forall m, l \in \mathbb{N},$$

  we construct a sequence of control functions $\{u_n\}_{n\in\mathbb{N}}$ and a sequence of positive times $\{T_n\}_{n\in\mathbb{N}}$ such that

  $$\exists\ \theta \in \mathbb{R} \quad : \quad \lim_{n\to\infty} \|\Gamma^{u_n}_{T_n}\phi_j - e^{i\theta}\phi_k\|_{(3)} = 0.$$



- We provide a neighborhood of ϕ_k with a suitable radius $r > 0$ where the local exact controllability is satisfied and so that there exists $n^* \in \mathbb{N}$ such that
$$\|\Gamma^{u_{n^*}}_{T_{n^*}}\phi_j - e^{i\theta}\phi_k\|_{(3)} < r.$$
By gathering the two results, we explicit $\tilde{T} > 0$ so that there exists $u \in L^2\big((0,\tilde{T}),\mathbb{R}\big)$ such that
$$\Gamma^u_{\tilde{T}}\Gamma^{u_{n^*}}_{T_{n^*}}\phi_j = e^{i\theta}\phi_k.$$



- In conclusion, we generalize the result for every $k \in \mathbb{N}$.

In Chapter 2, we also treat the example of $B : \psi \mapsto x^2\psi$. We define a control and a time such that the dynamics of (BSE) drives the second eigenstate ϕ_2 in the first ϕ_1. For
$$u(t) = (2,38 \cdot 10^{185})^{-1}\cos(3\pi^2 t), \qquad T = (2,38 \cdot 10^{185})\frac{9\pi^3}{8},$$
there exists $\theta \in \mathbb{R}$ so that $\big\|e^{i\theta}\phi_1 - \Gamma^u_T\phi_2\big\|_{(3)} \le 2.4 \cdot 10^{-6}$. In addition, there exists $\tilde{u} \in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that
$$\Gamma^u_T\Gamma^{\tilde{u}}_{\frac{4}{\pi}}\phi_2 = e^{i\theta}\phi_1.$$
The provided dynamics steers ϕ_2 in ϕ_1 (up to a phase) in a time of $T + \frac{4}{\pi}$ and the initial state approaches the target up to a well-defined distance with an explicit control.

The achieved result is far from being optimal since the aim of the chapter is to show the techniques which can be used in order to achieve the result. However, our intention is to optimize the provided estimates in later works.

Simultaneous global exact controllability in projection.

In Chapter 3, we consider the same problem of Chapter 2 and we study the simultaneous controllability (Figure 1.4) for infinitely many bilinear Schrödinger equations. In particular, we provide explicit conditions in B implying the simultaneous global exact controllability "in projection".

The meaning of controllability in projection is the following. Let Π be an orthogonal projector mapping $\mathscr{H}$ in a suitable subspace of $\mathscr{H}$. The problem (BSE) is globally exactly controllable in projection in $H^3_{(0)}$ with respect to Π when, for every $\psi^1, \psi^2 \in H^3_{(0)}$ such that $\|\psi^1\|_{\mathscr{H}} = \|\psi^2\|_{\mathscr{H}}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Pi\ \psi^2 = \Pi\ \Gamma^u_T \psi^1.$$

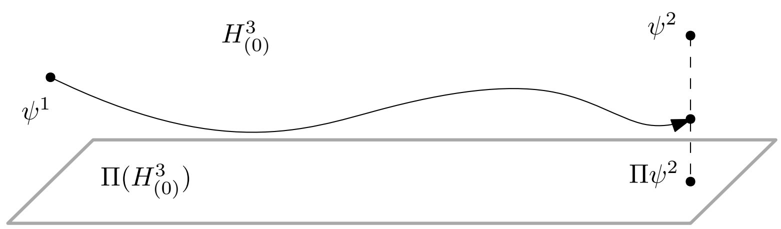


Figure 1.5: Controllability in projection: the dynamics drives ψ^1 in a state sharing the same projection of the state ψ^2 in $\Pi(H^3_{(0)})$.

The simultaneous global exact controllability in projection of infinitely many (BSE) in $H^3_{(0)}$ follows the same idea when we consider infinite couples of states in $H^3_{(0)}$ with same norms.

In more technical terms, we consider

$$\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}, \qquad \mathscr{H}_N(\Psi) := \operatorname{span}\{\psi_j : j \leq N\},$$

and $\pi_N(\Psi)$ the orthogonal projector onto $\mathscr{H}_N(\Psi)$. We prove that the following result is valid under suitable assumptions on B and Ψ.
Let $\{\psi^1_j\}_{j\in\mathbb{N}}$, $\{\psi^2_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be unitarily equivalent. For any $N \in \mathbb{N}$, there exist $T > 0$ and a control function $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi)\ \psi^2_j = \pi_N(\Psi)\ \Gamma^u_T \psi^1_j, \qquad j \in \mathbb{N}.$$

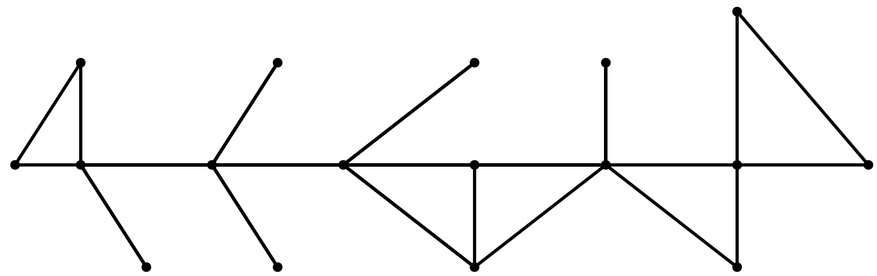


Figure 1.6: Example of compact graph

When $\Psi = \Psi^2$, we have

$$\begin{cases} \Gamma_T^u \psi_j^1 = \psi_j^2, & j \leq N, \\ \pi_N(\Psi^2)\ \Gamma_T^u \psi_j^1 = \pi_N(\Psi^2)\psi_j^2, & j > N. \end{cases}$$

The result implies the simultaneous global exact controllability (without projecting) of N bilinear Schrödinger equations. As we mentioned before, a similar outcome is ensured by Morancey and Nersesyan in [MN15].
However, we provide a novelty since we exhibit explicit conditions in B implying the validity of the result.

Another goal of the chapter is to prove the simultaneous local exact controllability in projection up to phases for any $T > 0$. To this aim, we use different techniques from the Coron's return method usually adopted for those types of results, *e.g.* [Mor14] and [MN15].

Bilinear Schrödinger equation on graphs structures.

In Chapter 4, we consider the bilinear Schrödinger equation in $\Omega = \mathscr{G}$ a compact graph structure (*e.g.* Figure 1.6). Considering (BSE) on such a complex structure is useful when one has to study the dynamics of wave packets on graph type model. The use of graph theory in condensed matter physics, pioneered by the work of many chemical and physical graph theorists, is today well-established and gaining even more popularity after the recent discovery of graphene. Other important applications appear in condensed matter physics, statistical physics, quantum electrodynamics, electrical networks and vibrational problems.
Let us recall here the basic features that define the notion of compact graph.

- We call graph $\mathscr{G}$ a set of points (vertices) connected by a set of segments (edges).

- A graph $\mathscr{G}$ is metric when it is equipped with a metric structure (see [BK13, $Definition$ 1.3.1]).
- A metric graph $\mathscr{G}$ with a finite number of edges of finite length is said to be compact.

We study the controllability of the bilinear Schrödinger equation in $\mathscr{H} = L^2(\mathscr{G},\mathbb{C})$ for B a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$. The operator A is a Laplacian and the domain of A is composed by functions satisfying Dirichlet or Neumann type boundary conditions in those vertices that are connected with only one edge (external vertices).
In the remaining ones (internal vertices), we impose the "Neumann-Kirchhoff" boundary conditions. In particular, a function f satisfies Neumann-Kirchhoff boundary conditions in an internal vertex v when

$$\begin{cases} f \text{ is continuous in } v, \\ \sum_{e\in N(v)} \frac{df}{dx_e}(v) = 0, \end{cases}$$

for $N(v)$ the set of edges containing v. The derivatives are assumed to be taken in the directions away from the vertex (outgoing directions).

Our purpose is to prove the controllability of the bilinear Schrödinger equation in

$$H^s_{\mathscr{G}} := D(A^{\frac{s}{s}})$$

for suitable $s > 0$. A peculiarity of the problem is that $\{\lambda_k\}_{k\in\mathbb{N}}$, the ordered eigenvalues of A, do not satisfy the following gap condition

$$\inf_{k\neq l} |\lambda_k - \lambda_l| > 0.$$

We only know that there exist $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$|\lambda_{k+\mathcal{M}} - \lambda_k| \geq \mathcal{M}\delta, \qquad \forall k \in \mathbb{N}.$$

For this reason, the common techniques adopted for proving the local exact controllability results can not be directly applied.

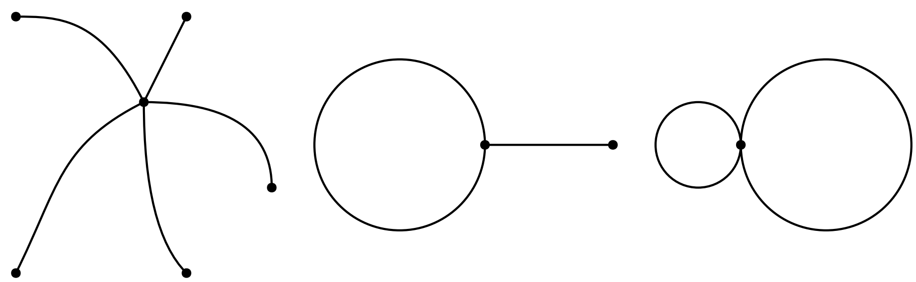


Figure 1.7: Star graph, tadpole graph and double-ring graph

Well-posedness and global exact controllability: Let $\mathscr{G}$ be such that, for suitable $\epsilon > 0$, there exists $C > 0$ such that

$$(1.7) \qquad |\lambda_{k+1} - \lambda_k| \geq \frac{C}{k^\epsilon}, \qquad \forall k \in \mathbb{N}.$$

The well-posedness of the bilinear Schrödinger equation is guaranteed in $H^{3+d(\epsilon)}_{\mathscr{G}}$ when $u \in L^2((0,T),\mathbb{R})$ with specific $d(\epsilon) \geq 0$ depending on ϵ (under suitable assumptions on B).
A crucial part of the proof is the interpolation features that we prove for the Sobolev spaces $H^s_{\mathscr{G}}$ as

$$(1.8) \qquad H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2} \quad \text{for} \quad s_1 \in \mathbb{N} \cup \{0\},\ s_2 \in [0, 1/2).$$

According to the choice of boundary conditions, stronger relations can be satisfied.

When the hypotheses of the well-posedness are verified and B satisfies a similar condition to (1.3), we prove the global exact controllability of the bilinear Schrödinger equation in $H^{3+d(\epsilon)}_{\mathscr{G}}$.

By using diophantine approximation techniques and the Roth's Theorem [Rot56], we show some types of graphs such that the spectral assumptions (1.7) are satisfied, *e.g.* star graphs, tadpole graphs and double-ring graphs (Figure 1.7). We present examples of B and $\mathscr{G}$ verifying the remaining hypotheses of the global exact controllability.

Contemporaneous global exact controllability: An interesting application of the previous result is the following. Let $\mathscr{G} = \{I_j\}_{j\leq N}$ be a set of bounded intervals of lengths $\{L_j\}_{j\leq N}$ for $N \in \mathbb{N}$ and $\Gamma_t^{u,j}$ be the unitary propagator generated by

$$A|_{L^2(I_j)} + uB|_{L^2(I_j)}.$$

When the global exact controllability is verified for the introduced graph $\mathscr{G}$, we have the "contemporaneous global exact controllability", *i.e.* for $\{\psi_j^1\}_{j\leq N}$, $\{\psi_j^2\}_{j\leq N}$ such that

$$\psi_j^1, \psi_j^2 \in H_{I_j}^s := D\Big(A|_{L^2(I_j)}^{s/2}\Big), \qquad \|\psi_j^1\|_{L^2(I_j,\mathbb{C})} = \|\psi_j^2\|_{L^2(I_j,\mathbb{C})}, \qquad \forall j \leq N,$$

there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Gamma_T^{u,j}\psi_j^1 = \psi_j^2, \qquad \forall j \leq N.$$

Heuristically speaking, the contemporaneous controllability allows to control functions belonging to different Sobolev's space at the same time (Figure 1.8). The result is different from the simultaneous global exact controllability which considers sequences of functions belonging to the same Sobolev's space.

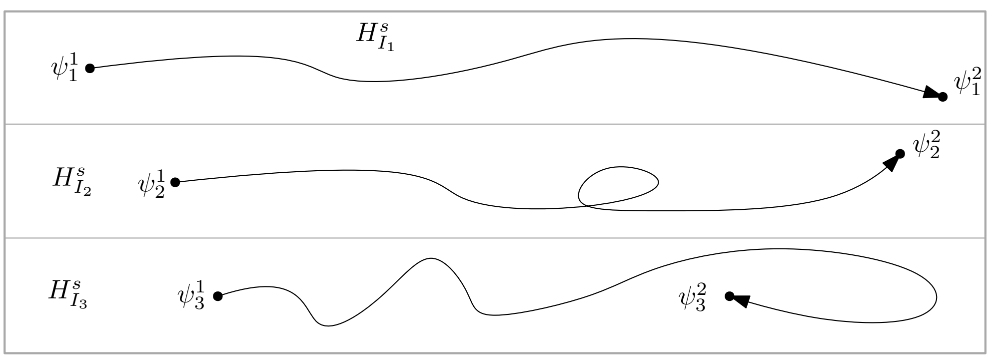


Figure 1.8: Example of contemporaneous global exact controllability.

We prove that if all the ratios L_k/L_j are algebraic irrational numbers, then the required spectral assumptions are verified and, under suitable assumptions on B, the bilinear Schrödinger equation is contemporaneously globally exactly controllable.

Energetic controllability: When $\mathscr{G}$ is a complex graph, it is not always possible to verify the spectral hypothesis of the global exact controllability.

In those situations, we study the "energetic controllability", *i.e.* the existence of a subset $\{\varphi_j\}_{j\in\mathbb{N}}$ of the eigenstates of A (corresponding to a set of eigenvalues $\{\mu_j\}_{j\in\mathbb{N}}$) such that, for every $\varphi_j, \varphi_k \in \{\varphi_l\}_{l\in\mathbb{N}}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Gamma_T^u \varphi_j = \varphi_k.$$

If $\{\mu_j\}_{j\in\mathbb{N}}$ corresponds to the set of the eigenvalues of A (not repeated with their multiplicity), then the problem is said to be "fully energetically controllable".

Chapter 2

Construction of the control function for the global exact controllability

Let us consider the Hilbert space $\mathscr{H} = L^2((0,1),\mathbb{C})$. We denote

$$\langle \psi_1, \psi_2 \rangle := \langle \psi_1, \psi_2 \rangle_{\mathscr{H}} = \int_0^1 \overline{\psi_1}(x)\psi_2(x)dx, \qquad \forall \psi_1, \psi_2 \in \mathscr{H}$$

and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. In $\mathscr{H} = L^2((0,1),\mathbb{C})$, we consider the problem (BSE)

$$\begin{cases} i\partial_t \psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0,T), \\ \psi(0) = \psi^0, \end{cases} \tag{2.1}$$

for $T > 0$ and $A = -\Delta$ the Laplacian equipped with Dirichlet type boundary conditions, *i.e.*

$$D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})).$$

Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis composed by eigenfunctions of A associated to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ ($\lambda_k = \pi^2 k^2$) and

$$\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j. \tag{2.2}$$

We define the following spaces for $s > 0$

$$h^s(\mathbb{C}) = \Big\{ \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{C} \Big| \sum_{j=1}^{\infty} |j^s x_j|^2 < \infty \Big\}, \qquad \| \cdot \|_{h^s} = \left(\sum_{k=1}^{\infty} |k^s \cdot |^2 \right)^{\frac{1}{2}},$$

$$H^s_{(0)} = H^s_{(0)}((0,1),\mathbb{C}) := D(A^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} = \left(\sum_{k=1}^{\infty} |k^s\langle\phi_k,\cdot\rangle|^2\right)^{\frac{1}{2}}.$$

Let $H^s := H^s((0,1),\mathbb{C})$ and $H^s_0 := H^s_0((0,1),\mathbb{C})$. We introduce the following notation for $s > 0$

$$|\!|\!|\cdot|\!|\!| := |\!|\!|\cdot|\!|\!|_{L(\mathscr{H},\mathscr{H})}, \qquad |\!|\!|\cdot|\!|\!|_{(s)} := |\!|\!|\cdot|\!|\!|_{L(H^s_{(0)},H^s_{(0)})},$$

$$|\!|\!|\cdot|\!|\!|_3 := |\!|\!|\cdot|\!|\!|_{L(H^3_{(0)},H^3\cap H^1_0)}.$$

In the current chapter, we consider the space $H^3 \cap H^1_0$ equipped with the norm $\|\cdot\|_{H^3\cap H^1_0} = \sqrt{\sum_{j=1}^3 \|\partial_x^j \cdot\|^2}$.

Assumptions (I)**.** The bounded operator B satisfies the following conditions.

1. For every $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for every $j \in \mathbb{N}$,
$$|\langle\phi_j, B\phi_j\rangle| \geq \frac{C_k}{j^3}.$$
2. $Ran(B|_{D(A)}) \subseteq D(A)$ and $Ran(B|_{H^3_{(0)}}) \subseteq H^3 \cap H^1_0$.

Remark 2.1. *If a bounded operator B satisfies Assumptions I, then $B \in L(H^2_{(0)}, H^2_{(0)})$. Indeed, B is closed in $\mathscr{H}$ and for every $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ such that $u_n \overset{\mathscr{H}}{\longrightarrow} u$ and $Bu_n \overset{\mathscr{H}}{\longrightarrow} v$, there holds $Bu = v$. Now, for every $\{u_n\}_{n\in\mathbb{N}} \subset H^2_{(0)}$ such that $u_n \overset{H^2_{(0)}}{\longrightarrow} u$ and $Bu_n \overset{H^2_{(0)}}{\longrightarrow} v$, the convergences with respect to the $\mathscr{H}$-norm are implied and then $Bu = v$. Hence the operator B is closed in $H^2_{(0)}$ and*
$$B \in L(H^2_{(0)}, H^2_{(0)}).$$
The same argument implies that $B \in L(H^3_{(0)}, H^3 \cap H^1_0)$.

Let us define $B_{j,k} := \langle\phi_j, B\phi_k\rangle$ and

$$b := |\!|\!|B|\!|\!|_{(2)}^6 \, |\!|\!|B|\!|\!| \, |\!|\!|B|\!|\!|_3^{16} \max\left\{ |\!|\!|B|\!|\!|, |\!|\!|B|\!|\!|_3 \right\}$$

only depending on the operator B. Now, $\{B_{j,k}\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$ for every $k \in \mathbb{N}$ and $\{B_{j,k}\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ for every $j \in \mathbb{N}$. For every $k, j \in \mathbb{N}$, $n \in \mathbb{N}$, we denote

$$E(j,k) := e^{\frac{6\,|\!|\!|B|\!|\!|_{(2)}}{|B_{j,k}|}} |k^2 - j^2|^5 C_k^{-16} k^{24} |B_{j,k}|^{-7} \max\{j,k\}^{24},$$

$$u_n(t) := \frac{\cos\big((k^2-j^2)\pi^2 t\big)}{n}, \qquad C' := \sup_{(l,m)\in\Lambda'}\left\{\left|\sin\left(\pi\frac{|l^2-m^2|}{|k^2-j^2|}\right)\right|^{-1}\right\},$$

$$\Lambda' := \big\{(l,m)\in\mathbb{N}^2 :\ \{l,m\}\cap\{j,k\}\neq\emptyset,\ |l^2-m^2|\leq\frac{3}{2}|k^2-j^2|,$$
$$|l^2-m^2|\neq|k^2-j^2|,\ \langle\phi_l,B\phi_m\rangle\neq 0\big\},$$
$$T^* := \frac{\pi}{|B_{k,j}|}.$$

We present the main result of the chapter in the following theorem.

Theorem 2.2. *Let $j,n\in\mathbb{N}$ and $k\in\mathbb{N}$ be such that $k\neq j$ and*

$$m^2-k^2\neq k^2-l^2, \qquad \forall m,l\in\mathbb{N},\ m,l\neq k. \tag{2.3}$$

Let B satisfy Assumptions I. If $n\geq 6^{42}\pi^{12}b\,(1+C')E(j,k)$, then there exists $\theta\in\mathbb{R}$ such that

$$\left\|\Gamma^{u_n}_{nT^*}\phi_j - e^{i\theta}\phi_k\right\|_{(3)} \leq C_k^2(6^2k^3 \,|||\, B \,|||\, ^2_3)^{-1}$$

for C_k defined in Assumptions I. Moreover, there exists $u\in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that

$$\|u\|_{L^2((0,\frac{4}{\pi}),\mathbb{R})} \leq \frac{C_k}{3 \,|||\, B \,|||\, ^2_3 k^3}, \qquad \Gamma^u_{\frac{4}{\pi}}\Gamma^{u_n}_{nT^*}\phi_j = e^{i\theta}\phi_k.$$

Proof. See Paragraph 2.5. □

Examples of values $k\in\mathbb{N}$ satisfying the relation (2.34) are the ones such that $k\leq 3$. However, the result of Theorem 2.2 can be generalized for every $k\in\mathbb{N}$ as it is showed in the following paragraph.

Remark. *The result of Theorem 2.2 is not optimal. The aim of the work is to show how to proceed for this type of problems and we present an approach that can be used in order to establish times and controls for the global exact controllability in $H^3_{(0)}$.*
The purpose of Theorem 2.2 is to exhibit readable results for generic operators B and levels j, k. For any specific choice of B, j and k, it is possible to retrace the proof in order to obtain sharper bounds by using stronger estimates. We briefly treat the example of $B:\psi\to x^2\psi$, $j=2$ and $k=1$ in Paragraph 2.7. In addition, we present in Paragraph 2.6 how to compute and remove the phase appearing in the target state, even though this is not particularly relevant from a physical point of view.

Remark 2.3. *In the proof of Theorem 2.2, the choice of the control function u comes from the techniques developed in [Cha12]. Similar results for other $\frac{2\pi}{|\lambda_k-\lambda_j|}$−periodic controls are valid from the theory exposed in [Cha12].*

2.1 Time reversibility

An important feature of the bilinear Schrödinger equation is the time reversibility. If we substitute t with $T-t$ for $T>0$ in the bilinear Schrödinger equation (BSE), then we obtain

$$\begin{cases} i\partial_t\Gamma^u_{T-t}\psi^0 = -A\Gamma^u_{T-t}\psi^0 - u(T-t)B\Gamma^u_{T-t}\psi^0, & t\in(0,T),\\ \Gamma^u_{T-0}\psi^0 = \Gamma^u_T\psi^0 = \psi^1. \end{cases}$$

We define $\widetilde{\Gamma}^{\widetilde{u}}_t$ such that $\Gamma^u_{T-t}\psi^0 = \widetilde{\Gamma}^{\widetilde{u}}_t\psi^1$ for $\widetilde{u}(t) := u(T-t)$ and

$$\begin{cases} i\partial_t\widetilde{\Gamma}^{\widetilde{u}}_t\psi^1 = (-A-\widetilde{u}(t)B)\widetilde{\Gamma}^{\widetilde{u}}_t\psi^1, & t\in(0,T),\\ \widetilde{\Gamma}^{\widetilde{u}}_0\psi^0 = \psi^1. \end{cases} \tag{2.4}$$

Thanks to $\psi^0 = \widetilde{\Gamma}^{\widetilde{u}}_T\Gamma^u_T\psi^0$ and $\psi^1 = \Gamma^u_T\widetilde{\Gamma}^{\widetilde{u}}_T\psi^1$, it follows

$$\widetilde{\Gamma}^{\widetilde{u}}_T = (\Gamma^u_T)^{-1} = (\Gamma^u_T)^*.$$

The operator $\widetilde{\Gamma}^{\widetilde{u}}_t$ describes the reversed dynamics of Γ^u_t and represents the propagator of (2.4) generated by the Hamiltonian $(-A-\widetilde{u}(t)B)$.

Thanks to the time reversibility, Theorem 2.2 can be generalized for every $k\in\mathbb{N}$ by defining, for every ϕ_j and ϕ_k, a dynamics steering ϕ_j in ϕ_k and passing from the state ϕ_1. Indeed, the theorem is also valid for the reversed dynamics and there exist $\theta_1,\theta_2\in\mathbb{R}$, $T_1,T_2>0$ and $u_1\in L^2((0,T_1),\mathbb{R})$, $u_2\in L^2((0,T_2),\mathbb{R})$ such that

$$e^{i\theta_1}\Gamma^{u_1}_{T_1}\phi_j = \phi_1 = e^{i\theta_2}\widetilde{\Gamma}^{u_2}_{T_2}\phi_k \qquad\Longrightarrow\qquad \Gamma^{\widetilde{u}_2}_{T_2}\Gamma^{u_1}_{T_1}\phi_j = e^{i(\theta_2-\theta_1)}\phi_k$$

for $\widetilde{u}_2(\cdot) = u_2(T_2-\cdot)$. We resume this result in the following corollary. To this purpose, we temporarily redefine the notation introduced in the previous paragraph by adding the dependence from the parameters $j,k\in\mathbb{N}$ as follows. Let us define $T^*_{j,k} := \frac{\pi}{|B_{k,j}|}$ and

$$u_{n;j,k}(t) := \frac{\cos\big((k^2-j^2)\pi^2 t\big)}{n}, \qquad C'(j,k) := \sup_{(l,m)\in\Lambda'(j,k)}\left\{\left|\sin\left(\pi\frac{|l^2-m^2|}{|k^2-j^2|}\right)\right|^{-1}\right\},$$

$$\Lambda'(j,k) := \Big\{(l,m)\in\mathbb{N}^2:\ \{l,m\}\cap\{j,k\}\neq\emptyset,\ |l^2-m^2|\leq\frac{3}{2}|k^2-j^2|,$$
$$|l^2-m^2|\neq|k^2-j^2|,\ \langle\phi_l,B\phi_m\rangle\neq 0\Big\}.$$

Corollary 2.4. *Let $j,k,n_1,n_2\in\mathbb{N}$ be such that $k\neq j$ and let B satisfy Assumptions I. If*

$$n_1\geq 6^{42}\pi^{12}b\ (1+C'(j,1))E(j,1),\qquad n_2\geq 6^{42}\pi^{12}b\ (1+C'(1,k))E(1,k),$$

then there exist $u_1,u_2\in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that, for

$$\widetilde{u}_{n_2;1,k}(\cdot)=u_{n_2;1,k}(n_2T^*_{1,k}-\cdot),$$

there holds

$$\Gamma^{\widetilde{u}_{n_2;1,k}}_{n_2T^*_{1,k}}\,\Gamma^{u_2}_{\frac{4}{\pi}}\,\Gamma^{u_1}_{\frac{4}{\pi}}\,\Gamma^{u_{n_1;j,1}}_{n_1T^*_{j,1}}\phi_j=e^{i\theta}\phi_k.$$

2.2 Well-posedness

As mentioned in the introduction, Beauchard and Laurent prove in [BL10] the well-posedness of (BSE) in $H^3_{(0)}$ when B is a multiplication operator for a suitable function $\mu\in H^3((0,1),\mathbb{R})$. Let the Cauchy problem in $\mathscr{H}$

$$\begin{cases} i\partial_t\psi(t)=A\psi(t)+u(t)\mu\psi(t), & t\in(0,T),\\ \psi(0)=\psi^0. \end{cases}\tag{2.5}$$

Proposition 2.5. $[BL10, Lemma\ 1;\ Proposition\ 2]$
1) *Let $T>0$ and $\widetilde{f}\in L^2((0,T),H^1_0\cap H^3)$. The function $G:t\mapsto\int_0^t e^{iAs}\widetilde{f}(s)ds$ belongs to $C^0([0,T],H^3_{(0)})$. Moreover,*

$$\|G\|_{L^\infty((0,T),H^3_{(0)})}\leq c_1(T)\|\widetilde{f}\|_{L^2((0,T),H^3\cap H^1_{(0)})},$$

where the constant $c_1(T)$ is uniformly bounded with T in bounded intervals.
2) *Let $\mu\in H^3((0,1),\mathbb{R})$, $T>0$, $\psi^0\in H^3_{(0)}$ and $u\in L^2((0,T),\mathbb{R})$. There exists a unique mild solution of (2.5) in $H^3_{(0)}$, i.e. $\psi\in C^0([0,T],H^3_{(0)})$ such that*

$$\psi(t,x)=e^{-iAt}\psi^0(x)-i\int_0^t e^{-iA(t-s)}(u(s)\mu(x)\psi(s,x))ds,\qquad\forall t\in[0,T].\tag{2.6}$$

Moreover, for every $R>0$, there exists $C=C(T,\mu,R)>0$ such that, for every $\psi^0\in H^3_{(0)}$, if

$$\|u\|_{L^2((0,T),\mathbb{R})}<R,$$

then the solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})}\leq C\|\psi^0\|_{(3)},\qquad\|\psi(t)\|_{\mathscr{H}}=\|\psi^0\|_{\mathscr{H}}\quad\forall t\in[0,T].$$

Remark 2.6. *The outcome of Proposition* 2.5 *is not only valid for multiplication operators. Indeed, the same proofs of* [BL10, Lemma 1] *and* [BL10, Proposition 2] *lead to the well-posedness of* (BSE)*, also when* B *is a bounded symmetric operator such that*

$$B \in L(H^3_{(0)}, H^3 \cap H^1_0), \qquad B \in L(H^2_{(0)}).$$

The only difference in the proof is that one has to substitute μ *with* B *and* $\|\mu\|_{H^3}$ *with* $\|B\|_{L(H^3_{(0)}, H^3 \cap H^1_0)}$*. In Proposition* 4.11 *(Chapter* 4.2*), we extend this result by considering domains that are compact graphs.*

2.3 Local exact controllability in $H^3_{(0)}$

Now, we rephrase the so-called "Generalized Inverse Function Theorem".

Proposition 2.7. [Lue69, Theorem 1; p. 240] *Let* $F : X \to Y$ *be a differentiable map between two Banach spaces* X *and* Y*. Let* $x_0 \in X$ *be such that the linear differential map* $d_{x_0}F : T_{x_0}X \to T_{f(x_0)}Y$ *is surjective. There exists a neighborhood* V *of* $F(x_0)$ *in* Y *(i.e. a ball centered in* $F(x_0)$*) such that, for each* $y \in V$*, there exists* $x \in X$ *such that*

$$F(x) = y.$$

Let us provide a brief proof of the local exact controllability in $H^3_{(0)}$ by rephrasing the existing results of local controllability as [Bea05], [BL10], [Mor14] and [MN15]. Our purpose is to introduce the tools that we use in the proof of Theorem 2.2. For $\psi \in H^3_{(0)}$ and $\epsilon > 0$, we define

$$\widetilde{B}_{H^3_{(0)}}(\psi, \epsilon) := \big\{\widetilde{\psi} \in H^3_{(0)} \big|\ \|\widetilde{\psi}\| = \|\psi\|,\ \|\widetilde{\psi} - \psi\|_{(3)} < \epsilon\big\}.$$

Theorem 2.8. *Let* B *satisfy Assumptions I. For every* $l \in \mathbb{N}$ *such that*

$$m^2 - l^2 \neq l^2 - n^2, \qquad \forall m, n \in \mathbb{N},\ m, n \neq l, \tag{2.7}$$

there exist $T > 0$ *and* $\epsilon > 0$ *such that, for every* $\psi \in \widetilde{B}_{H^3_{(0)}}(\phi_l(T), \epsilon)$*, there exists a control function* $u \in L^2((0, T), \mathbb{R})$ *such that*

$$\psi = \Gamma^u_T \phi_l.$$

Proof. First, the local exact controllability is equivalent to the local surjectivity of the map

$$\Gamma^{(\cdot)}_T \phi_l : u \in L^2((0, T), \mathbb{R}) \longmapsto \Gamma^u_T \phi_l \in H^3_{(0)}$$

for $T > 0$. Second, we consider the decomposition

$$\Gamma_t^u \phi_l = \sum_{k=1}^{\infty} \phi_k(t)\langle \phi_k(t), \Gamma_t^u \phi_l \rangle$$

and the map $\alpha_l(u) = \{\alpha_{k,l}(u)\}_{k\in\mathbb{N}}$ such that

$$\alpha_{k,l}(u) = \langle \phi_k(T), \Gamma_T^u \phi_l \rangle, \quad k \in \mathbb{N}.$$

We know that $\Gamma_T^u \phi_l \in H^3_{(0)}$ for every $u \in L^2((0,T),\mathbb{R})$ and then $\alpha_l(u) \in h^3(\mathbb{C})$ for every $u \in L^2((0,T),\mathbb{R})$. The local existence of the control function is equivalent to prove the local surjectivity of

$$\alpha_l : L^2((0,T),\mathbb{R}) \longrightarrow Q := \{\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in h^3(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}$$

for $T > 0$ large enough. To this end, we use the Generalized Inverse Function Theorem (Proposition 2.7) and we study the surjectivity of the Fréchet derivative of α_l, $\gamma_l(v) := (d_u\alpha_l(0)) \cdot v$, the sequence with elements

$$\begin{aligned}\gamma_{k,l}(v) :&= \left\langle \phi_k(T), -i\int_0^T e^{-iA(T-s)} v(s) B e^{-iAs}\phi_l ds \right\rangle \\ &= -i\int_0^T v(s) e^{i(\lambda_k-\lambda_l)s} ds B_{k,l}, \qquad k \in \mathbb{N},\end{aligned}$$

for $B_{k,j} = \langle \phi_k, B\phi_j \rangle = \langle B\phi_k, \phi_j \rangle = \overline{B_{j,k}}$. We identify the space where γ_l takes value by considering that $\langle \mathbf{x}, \mathbf{x} \rangle_{\ell^2} = \|\mathbf{x}\|^2_{\ell^2} = 1$ for every $\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in Q$. Let $\mathbf{x_t} : (0,\epsilon) \to Q$ be a smooth curve for $\epsilon > 0$ such that

$$\mathbf{x_0} = \alpha_l(0) = \delta_l = \{\delta_{k,l}\}_{k\in\mathbb{N}}, \qquad \Big(\frac{d}{dt}\mathbf{x_t}\Big)(t=0) = \mathbf{v} = \{v_l\}_{l\in\mathbb{N}}.$$

We notice that

$$0 = \frac{d\langle \mathbf{x_t}, \mathbf{x_t}\rangle_{\ell^2}}{dt}(0) = \langle \mathbf{v}, \delta_l \rangle_{\ell^2} + \langle \delta_l, \mathbf{v} \rangle_{\ell^2} = 2\Re(v_l),$$

which implies $iv_l \in \mathbb{R}$ and then

$$\gamma_l : L^2((0,T),\mathbb{R}) \longrightarrow T_{\delta_l}Q = \{\{x_k\}_{k\in\mathbb{N}} \in h^3(\mathbb{C}) \mid ix_l \in \mathbb{R}\}.$$

The surjectivity of γ_l in $T_{\delta_l}Q$ consists in proving the solvability of the moment problem

$$\frac{x_k}{B_{k,l}} = -i\int_0^T u(s) e^{i(\lambda_k-\lambda_l)s} ds. \tag{2.8}$$

As B is symmetric, we have $B_{l,l} \in \mathbb{R}$ and $i\big(x_l/B_{l,l}\big) \in \mathbb{R}$. Moreover, the sequence $\{x_k B_{k,l}^{-1}\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ since $\{x_k\}_{k\in\mathbb{N}} \in h^3(\mathbb{C})$. Thanks to the relation (2.7), for every $k, j \in \mathbb{N}$ with $k, j \neq l$, we know that

$$\lambda_k - \lambda_l = \pi^2(k^2 - l^2) \neq \pi^2(l^2 - j^2) = \lambda_l - \lambda_j.$$

The solvability of (2.8) for $u \in L^2((0,T),\mathbb{R})$ is guaranteed by Remark A.8, which follows from Ingham's Theorem (Proposition A.5, Appendix A.1) for

$$T > \frac{2\pi}{\mathscr{G}}, \qquad \mathscr{G} := \pi^2.$$

In particular, for X defined in Remark A.8, the map $\gamma_l : X \longrightarrow T_{\delta_l}Q$ is an homeomorphism. Thus, $\gamma_l : L^2((0,T),\mathbb{R}) \to T_{\delta_l}Q$ is surjective in $T_{\delta_l}Q$ for T large enough and the proof is achieved thanks to the Generalized Inverse Function Theorem (Proposition 2.7), which provides the local surjectivity of the map α_l in Q at the same time T. $\square$

2.3.1 Local exact controllability neighborhood estimate

For any given eigenfunction, we explicit a neighborhood where the local exact controllability is verified in a specific time by using the following lemma.

Lemma 2.9. *Let V be an open subset of a Banach space X. Let $g : V \to Y$ and $h : V \to Y$ be two maps from V to a Banach space Y such that*

- *the application g is an homeomorphism from V to an open set $g(V) \subseteq Y$;*
- *there exists $M > 0$ such that, for every $x, y \in V$,*

$$\|g(x) - g(y)\|_Y \geq M\|x - y\|_X;$$

- *the map h is a Lipschitz application for a constant $k < M$, i.e. for each $x, y \in V$,*

$$\|h(x) - h(y)\|_Y \leq k\|x - y\|_X.$$

The map $f = g + h$ is an homeomorphism from V to the open set $f(V) \subseteq Y$.

Proof. The map f is injective since, for every $x_1, x_2 \in V$,

$$\|f(x_1) - f(x_2)\|_Y \geq \|g(x_1) - g(x_2)\|_Y - \|h(x_1) - h(x_2)\|_Y \geq (M - k)\|x_1 - x_2\|_X.$$

For any $x \in \mathscr{B}$ with $\mathscr{B}$ a Banach space and $r > 0$, we denote $\overline{B}_{\mathscr{B}}(x,r)$ the closed ball in $\mathscr{B}$ of center x and radius r, while $B_{\mathscr{B}}(x,r)$ is the open ball in $\mathscr{B}$ of center x and radius r.

In order to prove that $f(V)$ is open in Y and f is a homeomorphism of V in $f(V)$, it is sufficient to prove that f is an open map or that the image under f of every open ball contained in V and center $x \in V$ contains an open ball of center $f(x)$. As every open ball contains a closed ball with the same center and positive radius, for every $x \in V$ and $r > 0$ such that

$$\overline{B}_X(x,r) \subseteq V,$$

we prove that

$$f(\overline{B}_X(x,r)) \supseteq B_Y(f(x),\tilde{r})$$

for a suitable $\tilde{r} > 0$. Let $x \in V$ and $r > 0$ such that $\overline{B}_X(x,r) \subset V$. As g is an homeomorphism, $g(V)$ is an open set of Y containing $g(x)$. Then, there exists $r_1 > 0$ such that

$$\overline{B}_Y(g(x),r_1) \subseteq g(V).$$

Let $y \in Y$ such that

$$\|y - f(x)\|_Y \leq (M-k)\inf\left(\frac{r_1}{M}, r\right).$$

For every $\tilde{x} \in V$ verifying $\|\tilde{x} - x\|_X \leq \inf\left(\frac{r_1}{M}, r\right)$, we have

$$\begin{aligned}\|y - h(\tilde{x}) - g(x)\|_Y &\leq \|y - h(x) - g(x)\|_Y + \|h(x) - h(\tilde{x})\|_Y \\ &\leq (M-k)\inf\left(\frac{r_1}{M}, r\right) + k\inf\left(\frac{r_1}{M}, r\right) \\ &\leq M\inf\left(\frac{r_1}{M}, r\right) \leq r_1.\end{aligned}$$

Hence, $y - h(\tilde{x}) \in \overline{B}_Y(g(x),r_1) \subseteq g(V)$ and we can consider $g^{-1}(y - h(\tilde{x}))$. The domain of the map

$$\tilde{x} \longmapsto \varphi_y(x) := g^{-1}(y - h(\tilde{x}))$$

contains $\overline{B}_X\Big(x, \inf\left(\frac{r_1}{M}, r\right)\Big)$. For every couple (x_1, x_2) in this domain,

$$\begin{aligned}\|\varphi_y(x_1) - \varphi_y(x_2)\|_X &\leq \|g^{-1}(y - h(x_1)) - g^{-1}(y - h(x_2))\|_X \\ &\leq M^{-1}\|h(x_1) - h(x_2)\|_Y \leq \frac{k}{M}\|x_1 - x_2\|_X.\end{aligned}$$

The map φ_y is Lipschitz for the constant $\frac{k}{M} < 1$ non-depending on y and maps $\overline{B}_X\Big(x, \inf\big(\frac{r_1}{M}, r\big)\Big)$ in itself since, for every $\tilde{x} \in \overline{B}_X\Big(x, \inf\big(\frac{r_1}{M}, r\big)\Big)$,

$$\begin{aligned}\|\varphi_y(\tilde{x}) - x\|_X &= \|g^{-1}(y - h(\tilde{x})) - g^{-1}(g(x))\|_X \\ &\leq M^{-1}\|y - h(\tilde{x}) - g(x)\|_Y \leq \inf\Big(\frac{r_1}{M}, r\Big).\end{aligned}$$

The ball $\overline{B}_X\Big(x, \inf\big(\frac{r_1}{M}, r\big)\Big)$ is complete and the Fixed-Point Theorem leads to the existence of $\tilde{x}$ in this ball such that

$$\varphi_y(\tilde{x}) = g^{-1}(y - h(\tilde{x})) = \tilde{x}, \quad \Longrightarrow \quad f(\tilde{x}) = g(\tilde{x}) + h(\tilde{x}) = y.$$

The point $y \in f\big(\overline{B}_X(x, \inf(r_1/M, r))\big)$, which implies that $y \in f\big(\overline{B}_X(x, r)\big)$. Now, $f\big(\overline{B}_X(x, r)\big) \supseteq \overline{B}_Y\big(f(x), (M-k)\inf\big(r_1/M, r\big)\big)$ and then $f\big(\overline{B}_X(x, r)\big) \supseteq B_Y\big(f(x), (M-k)\inf\big(r_1/M, r\big)\big)$. ☐

For $\psi \in H^3_{(0)}$ and $r > 0$, we recall the following definition

$$\widetilde{B}_{H^3_{(0)}}(\psi, r) := \{\tilde{\psi} \in H^3_{(0)} \;\big|\; \|\psi\| = \|\tilde{\psi}\|,\ \|\tilde{\psi} - \psi\|_{(3)} \leq r\}.$$

Proposition 2.10. *Let B satisfy Assumptions I. Let $l \in \mathbb{N}$ be such that*

$$m^2 - l^2 \neq l^2 - n^2, \qquad \forall m, n \in \mathbb{N},\ m, n \neq l \tag{2.9}$$

and C_l be defined in Assumptions I. For every

$$\psi \in \widetilde{B}_{H^3_{(0)}}\left(\phi_l\Big(\frac{4}{\pi}\Big), \frac{C_l^2}{6^2 l^3 \,|\!|\!|\, B \,|\!|\!|_3^2}\right),$$

there exists a control function $u \in L^2((0, 4/\pi), \mathbb{R})$ such that

$$\|u\|_{L^2((0,4/\pi),\mathbb{R})} \leq \frac{C_l}{3 \,|\!|\!|\, B \,|\!|\!|_3^2 l^3}, \qquad \psi = \Gamma^u_{\frac{4}{\pi}}\phi_l.$$

Proof. Let us define the following notation

$$|\!|\!| \cdot |\!|\!|_{L(L^2((0,T),\mathbb{R}), H^3_{(0)})} = |\!|\!| \cdot |\!|\!|_{(L^2_t, H^3_x)}, \quad |\!|\!| \cdot |\!|\!|_{L(H^3_{(0)}, L^2((0,T),\mathbb{R}))} = |\!|\!| \cdot |\!|\!|_{(H^3_x, L^2_t)},$$

$$\|\cdot\|_{L^\infty((0,T),H^3_{(0)})} = \|\cdot\|_{L^\infty_t H^3_x}, \quad \|\cdot\|_{L^2((0,T),\mathbb{R})} = \|\cdot\|_2.$$

Let $T > \frac{2\pi}{\mathscr{G}}$ for $\mathscr{G} = \pi^2$ (as in the proof of Theorem 2.8). We consider the space X defined in Remark $A.8$ (Appendix $A.1$) and equipped with the

L^2-norm. The local exact controllability provided in the proof of Theorem 2.8 is equivalent to the local surjectivity of the map

$$A_l(\cdot) := \Gamma_T^{(\cdot)}\phi_l : L^2((0,T),\mathbb{R}) \to \{\psi \in H^3_{(0)} \ : \ \|\psi\|_{\mathscr{H}} = 1\}$$

such that

$$A_l(u) = e^{-i\lambda_l T}\phi_l - i\int_0^T e^{-iA(T-s)}u(s)B\Gamma_s^u\phi_l ds.$$

Indeed, due to the proof of Theorem 2.8 (that refers to to Remark A.8), the map

$$F_l(\cdot) : X \to \{\psi \in H^3_{(0)} : \{\langle\phi_j(T),\psi\rangle\}_{j\in\mathbb{N}} \in T_{\delta_l}Q\} = \{\psi \in H^3_{(0)} : i\langle\phi_l(T),\psi\rangle \in \mathbb{R}\}$$

such that $F_l(u) := \big((d_vA_l(v=0))\cdot u\big)$ is an homeomorphism, which implies the local surjectivity of A_l thanks to the Generalized Inverse Function Theorem (Proposition 2.7). We estimate the radius of a neighborhood where the map A_l is surjective with Lemma 2.9. The proof is composed by the following steps.

- First, X and $\{\psi \in H^3_{(0)} \ : \ i\langle\phi_l(T),\psi\rangle \in \mathbb{R}\}$ are Banach spaces and $F_l : X \to \{\psi \in H^3_{(0)} \ : \ i\langle\phi_l(T),\psi\rangle \in \mathbb{R}\}$ is an homeomorphism for T large enough. We compute a constant $M > 0$ such that

$$\|F_l(u) - F_l(v)\|_{H^3_{(0)}} \geq M\|u - v\|_{L^2((0,T),\mathbb{R})}, \qquad \forall u, v \in X.$$

- Second, we fix $T > 0$ large enough. We provide a neighborhood $U \subset X$ and a constant $M_1 < M$ such that

$$\|(A_l - F_l)(u) - (A_l - F_l)(v)\|_{H^3_{(0)}} \leq M_1\|u - v\|_{L^2((0,T),\mathbb{R})}, \qquad \forall u, v \in U.$$

- Thanks to Lemma 2.9, the map $A_l : U \longrightarrow A_l(U)$ is an homeomorphism.

- From the proof of Lemma 2.9, we deduce that if $U \supset \{u \in X : \|u\|_{L^2((0,T),\mathbb{R})} \leq r\}$ with $r > 0$, then

$$A_l(U) \supset \{\psi \in H^3_{(0)} : \|\psi - \phi_l(T)\|_{H^3_{(0)}} \leq r(M - M_1)\} = \mathcal{M}.$$

- In conclusion, the map A_l is surjective in $\mathcal{M}$.

1) We show a constant $M > 0$ such that

$$\|F_l(v) - F_l(w)\|_{(3)} \geq M\|v - w\|_{L^2}, \qquad \forall v, w \in X.$$

Let us suppose $|||\, B\, |||_3 = 1$. By recalling the proof of Theorem 2.8, we know that the surjectivity of F_l in $H^3_{(0)}$ is equivalent to the surjectivity of γ_l in h^3. For every $\psi \in H^3_{(0)}$, there exist $T > 0$ and $u \in X$ such that

$$\langle \phi_j(T), \psi \rangle = \gamma_{j,l}(u) = \langle \phi_j(T), F_l(u) \rangle \qquad \forall j \in \mathbb{N} \tag{2.10}$$

and

$$F_l^{-1}(\psi) = u. \tag{2.11}$$

For C_l defined in Assumptions I, thanks to Remark $A.8$ (the relation $(A.9)$; Appendix $A.1$), there exists $\widetilde{C}(T) > 0$ such that

$$\begin{aligned}\|F_l^{-1}(\psi)\|_2^2 = \|u\|_2^2 \leq \widetilde{C}(T)^2 \sum_{j=1}^{\infty} \Big|\frac{\gamma_{j,l}(u)}{B_{j,l}}\Big|^2 \leq \frac{\widetilde{C}(T)^2}{C_l^2} \sum_{j=1}^{\infty} |j^3 \gamma_{j,l}(u)|^2 \\ \leq \frac{\widetilde{C}(T)^2}{C_l^2} \|\psi\|_{(3)}^2.\end{aligned}$$

In the last inequality, we used (2.10) and (2.11). For each $\psi, \varphi \in H^3_{(0)}$, there exist $v, w \in X$ such that $\psi = F_l(v)$, $\varphi = F_l(w)$ and

$$\|v - w\|_2 \leq \|F_l^{-1}(\psi - \varphi)\|_2 \leq \,|||\, F_l^{-1}\, |||_{(H^3_x, L^2_t)} \|\psi - \varphi\|_{(3)} \leq \frac{\widetilde{C}(T)}{C_l} \|\psi - \varphi\|_{(3)},$$

which implies

$$\|F_l(v) - F_l(w)\|_{(3)} \geq \frac{C_l}{\widetilde{C}(T)} \|v - w\|_2. \tag{2.12}$$

Then, we choose $M = \frac{C_l}{\widetilde{C}(T)}$.

2) Let $u \in X$ and

$$H_l(u) := -\int_0^T e^{-iA(T-s)}u(s)B\Big(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau\Big)ds.$$

Thanks to the Duhamel's formula,

$$\begin{aligned} A_l(u) = \Gamma^u_T\phi_l &= e^{-i\lambda_l T}\phi_l - i\int_0^T e^{-iA(T-s)}u(s)Be^{-i\lambda_l s}\phi_l ds \\ &- \int_0^T e^{-iA(T-s)}u(s)B\Big(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau\Big)ds \\ &= e^{-i\lambda_l T}\phi_l + F_l(u) + H_l(u). \end{aligned}$$

We exhibit a ball $U \subset X$ with center $u = 0$ such that, for every $u \in U$, the map $A_l : u \in U \mapsto \Gamma^u_T\phi_l \in A_l(U)$ is an homeomorphism thanks to Lemma 2.9 and

$$\|(A_l - F_l)(u) - (A_l - F_l)(v)\|_{H^3_{(0)}} = \|H_l(u) - H_l(v)\|_{H^3_{(0)}}, \qquad \forall u, v \in X.$$

We define U as the neighborhood such that there exists $M_1 \leq M/2$ so that

$$\|H_l(u) - H_l(v)\|_{(3)} \leq M_1\|u - v\|_{L^2}, \qquad \forall u, v \in U.$$

First, we notice that

$$\begin{aligned} H_l(u) - H_l(v) = &-\int_0^T e^{-iA(T-s)}u(s)B\Big(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau\Big)ds \\ &+\int_0^T e^{-iA(T-s)}v(s)B\Big(\int_0^s e^{-iA(s-\tau)}v(\tau)B\Gamma^v_\tau\phi_l d\tau\Big)ds \\ = &-\int_0^T e^{-iA(T-s)}(u(s) - v(s))B\Big(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau\Big)ds \\ &-\int_0^T e^{-iA(T-s)}v(s)B\Big(\int_0^s e^{-iA(s-\tau)}(u(\tau) - v(\tau))B\Gamma^u_\tau\phi_l d\tau\Big)ds \\ &-\int_0^T e^{-iA(T-s)}v(s)B\Big(\int_0^s e^{-iA(s-\tau)}v(\tau)B(\Gamma^u_\tau\phi_l - \Gamma^v_\tau\phi_l)d\tau\Big)ds. \end{aligned}$$

Thanks to Proposition 2.5 and Remark 2.6, there exists a constant $C(T) > 0$ such that, for every $\psi \in H^3 \cap H^1_0$ and $u \in L^2((0,T),\mathbb{R})$,

$$\Big\|\int_0^T e^{-iA(T-s)}u(s)B\psi ds\Big\|_{(3)} \leq C(T)\|u\|_2 \,|||\, B \,|||_3 \|\psi\|_{L^\infty_t H^3_x}. \tag{2.13}$$

Then

(2.14)

$$
\begin{aligned}
\|H_l(u)-H_l(v)\|_{(3)} &\leq C(T)^2\|v-u\|_2 \,|||\, B \,|||\,_3^2(\|u\|_2+\|v\|_2)\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\\
&+C(T)^2\|v\|_2^2 \,|||\, B \,|||\,_3^2\|\Gamma_t^v\phi_l-\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\\
&\leq C(T)^2\|v-u\|_2(\|u\|_2+\|v\|_2)\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\\
&+C(T)^2\|v\|_2^2\|\Gamma_t^v\phi_l-\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}.
\end{aligned}
$$

By using the same technique adopted by (2.14), we obtain

$$
\begin{aligned}
&\|\Gamma_t^v\phi_l-\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\leq\Big\|\int_0^t e^{-iA(t-s)}B(v\Gamma_t^v\phi_l-u\Gamma_t^u\phi_l)\Big\|_{L_t^\infty H_x^3}\\
&\leq C(T)\,|||\, B \,|||\,_3\|v\Gamma_t^v\phi_l-u\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\leq C(T)\|v-u\|_2\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\\
&+C(T)\|v\|_2\|\Gamma_t^v-\Gamma_t^u\|_{L_t^\infty H_x^3}.
\end{aligned}
$$

Let $\mu>1$. If $U=\{u\in X:\ \|u\|_2\leq(\mu C(T))^{-1}\}$, then

$$
\|\Gamma_t^v\phi_l-\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\leq\frac{\mu C(T)}{\mu-1}\|v-u\|_2\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}
$$

for every $u,v\in U$. The relation (2.14) becomes

$$
\begin{aligned}
&\|H_l(u)-H_l(v)\|_{(3)}\leq C(T)^2\|v-u\|_2(\|u\|_2+\|v\|_2)\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\\
&+\frac{\mu}{\mu-1}C^3(T)\|v\|_2^2\|v-u\|_2\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\leq\frac{2}{\mu}C(T)\|v-u\|_2\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\\
&+\frac{C(T)}{(\mu-1)\mu}\|v-u\|_2\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}\leq\frac{(2\mu-1)}{(\mu-1)\mu}C(T)\|v-u\|_2\|\Gamma_t^u\phi_l\|_{L_t^\infty H_x^3}.
\end{aligned}
$$

Thanks to the relation (2.13) and to the Duhamel's formula

$$
\|\Gamma_T^u\phi_l\|_{L_t^\infty H_x^3}\leq\|\phi_l\|_{(3)}+C(T)\|u\|_2\,|||\, B \,|||\,_3\|\Gamma_T^u\phi_l\|_{L_t^\infty H_x^3}
$$

and we obtain

$$
\|\Gamma_T^u\phi_l\|_{L_t^\infty H_x^3}\leq\frac{\|\phi_l\|_{(3)}}{1-C(T)\|u\|_2\,|||\, B \,|||\,_3}\leq\frac{\mu l^3}{\mu-1},
$$

$$
\Longrightarrow\qquad\|H_l(u)-H_l(v)\|_{(3)}\leq\frac{2\mu-1}{(\mu-1)^2}l^3C(T)\|v-u\|_2.
$$

In order to apply Lemma 2.9, we set $M_1=\frac{2\mu-1}{(\mu-1)^2}l^3C(T)$ and we estimate μ such that

$$
M_1\leq\frac{1}{2}M. \tag{2.15}
$$

In other words, we choose $\mu > 1$ such that

$$\frac{2\mu - 1}{(\mu-1)^2} l^3 C(T) \leq \frac{1}{2}\frac{C_l}{\widetilde{C}(T)}$$

and for $a_l = \frac{2C(T)\widetilde{C}(T)l^3}{C_l}$, the inequality is satisfied when

$$\mu \geq a_l + \sqrt{a_l(a_l+1)} + 1. \tag{2.16}$$

Let us establish an upper bound for $C(T)\widetilde{C}(T)$ by studying the constants C_1, C_2 appearing in Ingham's Theorem (Proposition $A.5$, Appendix $A.1$). First, we refer to Remark $A.8$ (Appendix $A.1$) and we set $T = \frac{4\pi}{\mathscr{G}} = \frac{4}{\pi}$ for $\mathscr{G} = \pi^2$. Let I' be such that $|I'| := \frac{\mathscr{G}}{\pi}T = 4$ and

$$\beta = \frac{\pi^2}{4}, \qquad G(0) = \frac{\pi}{2}, \qquad I_0 = [-1,+1], \qquad m = \big(|I'||I_0|^{-1}\big) = 2,$$

$$\alpha = 4R^2, \qquad \widehat{G}(0) = \frac{(R^2-1)\pi}{2}, \qquad R = \frac{|I'|}{2} = 2.$$

By substituting the constants in the proof of Ingham's Theorem [KL05, $pp.$ $62-65$]), we obtain

$$C_2 = \frac{2m\pi G(0)\pi}{\beta\mathscr{G}} = \frac{8}{\pi}, \qquad C_1 = \frac{2\pi\widehat{G}(0)\pi}{\alpha\mathscr{G}} = \frac{3\pi}{16}.$$

The proof of Proposition 2.5 (presented in [BL10]) and the relation ($A.9$) (Remark $A.8$; Appendix $A.1$) imply

$$C\Big(\frac{4}{\pi}\Big) = 3\pi^{-3}\max\left\{\sqrt{2}C_2, \sqrt{\frac{4}{\pi}}\right\} = \frac{24\sqrt{2}}{\pi^4}. \tag{2.17}$$

In addition, we have $\tilde{C}\Big(\frac{4}{\pi}\Big) = 2C_1^{-1}$ and

$$C\Big(\frac{4}{\pi}\Big)\widetilde{C}\Big(\frac{4}{\pi}\Big) \leq \frac{6}{5}.$$

Now, we know that $a_l = \frac{2C(T)\widetilde{C}(T)l^3}{C_l}$ and $a_l \leq \frac{12}{5}\widetilde{a}_l$ for $\widetilde{a}_l := l^3/C_l$ (C_l is defined in Assumptions I). Moreover,

$$C_l \leq |\langle\phi_1, B\phi_l\rangle| \leq \,|||\, B \,|||\, = 1,$$

which ensures that $\widetilde{a}_l > 1$. We need to define μ such that (2.16) is verified and

$$\begin{aligned}a_l + \sqrt{a_l(a_l+1)} + 1 &\le \Big(\frac{12}{5}\widetilde{a}_l + \sqrt{\frac{12}{5}\widetilde{a}_l\Big(\frac{12}{5}\widetilde{a}_l+1\Big)} + 1\Big)\\ &\le \Big(\frac{12}{5}\widetilde{a}_l + \Big(\frac{12}{5}\widetilde{a}_l + 1\Big) + 1\Big) \le \frac{34}{5}\widetilde{a}_l = \frac{34}{5}\frac{l^3}{C_l}.\end{aligned}$$

If we choose $\mu = \frac{34}{5}\frac{l^3}{C_l}$, then $\mu \ge a_l + \sqrt{a_l(a_l+1)} + 1$ as required in (2.16). We recall

$$U = \Big\{ u \in X : \ \|u\|_2 \le (\mu C\Big(\frac{4}{\pi}\Big))^{-1} \Big\}$$

and, thanks to the relation (2.15), Lemma 2.9 is satisfied. In conclusion, the following map is an homeomorphism

$$A_l : U \subseteq L^2((0,4/\pi),\mathbb{R}) \to A(U) \subseteq H^3_{(0)}.$$

3) We show a neighborhood of ϕ_l with respect to the $H^3_{(0)}$-norm included in $A_l(U)$. Let

$$B_X(x,r) := \{\tilde{x} \in X \ \big| \ \|\tilde{x} - x\|_{L^2((0,\frac{4}{\pi}),\mathbb{R})} \le r\}.$$

We notice that $\mu C\Big(\frac{4}{\pi}\Big) < 3\frac{l^3}{C_l}$ and we set

$$\tilde{U} = B_X\left(0, \frac{C_l}{3l^3}\right) \subset U.$$

From the proof of Lemma 2.9, we know that $A_l(U)$ contains a ball of center $A_l(0) = \phi_l\big(\frac{4}{\pi}\big)$ and radius $(M - M_1)\inf(r, r_1/M)$. The parameter $r > 0$ is the radius of a ball contained in U and center $u = 0$, while $r_1 > 0$ is the radius of a ball contained in $F_l(U)$ and center $F_l(0)$. Now, $\tilde{U}$ is a ball contained in U of radius $\frac{C_l}{3l^3}$ and, thanks to (2.12), $F_l(\tilde{U}) \subset F(U)$ contains a ball of radius $M\frac{C_l}{3l^3}$. Hence, $A_l(U)$ contains a ball of radius $(M - M_1)\frac{C_l}{3l^3}$ and center $\phi_l\big(\frac{4}{\pi}\big)$. Thanks to the relation (2.15), we know that

$$M - M_1 \ge \frac{1}{2}M \ge \frac{C_l}{2\widetilde{C}\Big(\frac{4}{\pi}\Big)} \ge \frac{3\pi C_l}{2^5}, \qquad \frac{1}{3}(M - M_1) > 6^{-2}C_l$$

and

$$A_l\left(B_X\left(0, \frac{C_l}{3l^3}\right)\right) \supseteq \widetilde{B}_{H^3_{(0)}}\left(A_l(0), (M - M_1)\frac{C_l}{3l^3}\right) \supseteq \widetilde{B}_{H^3_{(0)}}\left(\phi_l\Big(\frac{4}{\pi}\Big), \frac{C_l^2}{6^2 l^3}\right).$$

In the first part of the proof, we suppose $\| B \|_3 = 1$, but we can generalize the result for $\| B \|_3 \neq 1$ thanks to the identity

$$A + uB = A + u \| B \|_3 \frac{B}{\| B \|_3}.$$

To this purpose, we consider the operator $\frac{B}{\| B \|_3}$ and the control $u \| B \|_3$ and we substitute to C_l with $C_l \| B \|_3^{-1}$ (defined in Assumptions I). We also notice that if

$$\| B \|_3 u \in B_X\left(0, \frac{C_l}{3l^3 \| B \|_3}\right) \qquad \Longrightarrow \qquad u \in B_X\left(0, \frac{C_l}{3l^3 \| B \|_3^2}\right).$$

In conclusion, we obtain

$$\forall \psi \in \widetilde{B}_{H^3_{(0)}}\left(\phi_l\Big(\frac{4}{\pi}\Big), \frac{C_l^2}{6^2 l^3 \| B \|_3^2}\right), \ \exists\, u \in B_X\left(0, \frac{C_l}{3l^3 \| B \|_3^2}\right)$$
$$\text{s.t.} \quad A_l(u) = \psi. \quad \Box$$

2.4 Explicit control function for the global approximate controllability

For $j, k \in \mathbb{N}$, we recall the definition of $B_{j,k} = \langle \phi_j, B\phi_k \rangle$ and we denote

$$T^* = \frac{\pi}{|B_{j,k}|}, \qquad T = \frac{2\pi}{|\lambda_k - \lambda_j|}, \qquad u_n(t) = \frac{\cos\big((k^2 - j^2)\pi^2 t\big)}{n},$$

$$I = \frac{4}{|\lambda_k - \lambda_j|}, \qquad K = \frac{2}{|B_{j,k}|}, \qquad C' = \sup_{(l,m)\in\Lambda'} \left\{ \Big| \sin\Big(\pi \frac{|\lambda_l - \lambda_m|}{|\lambda_k - \lambda_j|}\Big)\Big|^{-1} \right\},$$

$$\Lambda' = \big\{(l,m) \in \mathbb{N}^2 : \ \{l,m\} \cap \{j,k\} \neq \emptyset, \ |\lambda_l - \lambda_m| \leq \frac{3}{2}|\lambda_k - \lambda_j|,$$
$$|\lambda_l - \lambda_m| \neq |\lambda_k - \lambda_j|, \ B_{l,m} \neq 0\big\}.$$

Proposition 2.11. *Let B satisfy Assumptions I. For every $j, k \in \mathbb{N}$, $j \neq k$, and $n \in \mathbb{N}$ such that*

$$n \geq \frac{3(1 + C')|B_{j,k}|^{-1} \| B \|^2}{|k^2 - j^2|}, \tag{2.18}$$

there exist $T_n \in (nT^* - T, nT^* + T)$ and $\theta \in \mathbb{R}$ such that

$$\|\Gamma^{u_n}_{T_n}\phi_j - e^{i\theta}\phi_k\|^2_{\mathscr{H}} \leq \frac{3^2|B_{j,k}|^{-1}(1 + C') \| B \|^2}{n|k^2 - j^2|}.$$

Proof. Thanks to [Cha12, *Proposition* 6], for any $n \in \mathbb{N}$, there exists $T_n \in (nT^* - T, nT^* + T)$ such that

$$\frac{1 - |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle|}{1 + 2K \,|||\, B \,|||} \leq \frac{(1 + C') \,|||\, (\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle) B \,|||\, I}{n}.$$

We point out that the definition of T^* provided in [Cha12, *Proposition* 6] is incorrect and the formulation that we provide can be deduced from [Cha12, *Proposition* 2]. In addition, we have

$$1 - |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle| \leq \frac{(1 + 2K \,|||\, B \,|||\,)(1 + C') \,|||\, B \,|||\, I}{n} =: R_n,$$

$$\begin{aligned}
\implies \quad & \sum_{l \neq k} |\langle \phi_l, \Gamma^{u_n}_{T_n} \phi_j \rangle - \langle \phi_l, \phi_k \rangle|^2 = \sum_{l \neq k} |\langle \phi_l, \Gamma^{u_n}_{T_n} \phi_j \rangle|^2 \\
& = \sum_{l=1}^{\infty} |\langle \phi_l, \Gamma^{u_n}_{T_n} \phi_j \rangle|^2 - |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle|^2 = 1 - |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle|^2 \\
& \leq \big(1 - |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle|\big)\big(1 + |\langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle|\big) \leq 2R_n.
\end{aligned} \tag{2.19}$$

Afterwards, fixed $n \in \mathbb{N}$, there exists $\theta \in \mathbb{R}$ (depending on n) such that

$$|\langle \phi_k, e^{i\theta} \phi_k \rangle - \langle \phi_k, \Gamma^{u_n}_{T_n} \phi_j \rangle|^2 \leq R_n^2. \tag{2.20}$$

From (2.19) and (2.20), we obtain

$$R_n' := \|e^{i\theta} \phi_k - \Gamma^{u_n}_{T_n} \phi_j\|^2 \leq 2R_n + R_n^2. \tag{2.21}$$

As $|B_{j,k}|^{-1} \,|||\, B \,|||\, = \frac{|||\, B \,|||}{|\langle \phi_j, B \phi_k \rangle|} \geq 1$, we have

$$\begin{aligned}
R_n &= \frac{(1 + 2K \,|||\, B \,|||\,)(1 + C') \,|||\, B \,|||\, I}{n} \leq \frac{(1 + C')(|B_{j,k}|^{-1} + 4|B_{j,k}|^{-1}) \,|||\, B \,|||^2 I}{n} \\
&\leq \frac{5(1 + C')|B_{j,k}|^{-1} \,|||\, B \,|||^2 I}{n} \leq \frac{3(1 + C')|B_{j,k}|^{-1} \,|||\, B \,|||^2}{n|k^2 - j^2|}.
\end{aligned}$$

In conclusion, if

$$n \geq \frac{3(1 + C')|B_{j,k}|^{-1} \,|||\, B \,|||^2}{|k^2 - j^2|}, \qquad j \neq k,$$

then $R_n \leq 1$, $R_n^2 \leq R_n$ and

$$\|e^{i\theta} \phi_k - \Gamma^{u_n}_{T_n} \phi_j\|^2 \leq 2R_n + R_n^2 \leq 3R_n \leq \frac{3^2 |B_{j,k}|^{-1}(1 + C') \,|||\, B \,|||^2}{n|k^2 - j^2|}. \quad \square$$

Proposition 2.12. *Let B satisfy Assumptions I. For every $j,k\in\mathbb{N}$, $j\neq k$, and $n\in\mathbb{N}$ satisfying* (2.18) *such that*

$$n\geq 4\,|||\,B\,|||_{(2)}, \tag{2.22}$$

there exists $T_n\in(nT^*-T,nT^*+T)$ and $\theta\in\mathbb{R}$ such that

$$\|\Gamma_{T_n}^{u_n}\phi_j-e^{i\theta}\phi_k\|_{(3)}^8\leq\frac{2^{18}3^{26}\pi^{12}(1+C')e^{\frac{6\,|||\,B\,|||_{(2)}}{|B_{j,k}|}}\,|||\,B\,|||_{(2)}^6\,|||\,B\,|||^2|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|^7n}.$$

Proof. **1) Propagation of regularity from $H_{(0)}^2$ to $H_{(0)}^4$:** In the first part of the proof, we show that the propagator Γ_T^u preserves $H_{(0)}^4$ and $B\in L(H_{(0)}^2)$. Let us introduce the following notation

$$\|f\|_{BV(T)}:=\|f\|_{BV((0,T),\mathbb{R})}=\sup_{\{t_j\}_{j\leq n}\in P}\sum_{j=1}^n|f(t_j)-f(t_{j-1})|,$$

for $f\in BV((0,T),\mathbb{R})$, where P is the set of the partitions of $(0,T)$ such that

$$t_0=0<t_1<...<t_n=T.$$

Let $\epsilon>0$, $\lambda_\epsilon=\,|||\,B\,|||_{(2)}\epsilon^{-1}$, $\tilde\lambda_\epsilon=\lambda_\epsilon+\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)}$ and

$$\widehat{H}_{(0)}^4:=D(A(i\tilde\lambda_\epsilon-A)).$$

We refer to [Kat53] and we prove that the propagator $U_t^{u_n}$ generated by

$$A+u_n(t)B-i\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)}$$

satisfies the condition $\|U_t^{u_n}\psi\|_{(4)}\leq C\|\psi\|_{(4)}$ for every $\psi\in H_{(0)}^4$ and suitable $C>0$. Indeed, if $-i(A+u_n(t)B-i\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)})$ is maximal dissipative, then Hille-Yosida Theorem implies that the semi-group generated by $-i(A+u_n(t)B-i\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)})$ is a semi-group of contraction and the techniques adopted in the proofs of [Kat53, $Theorem$ 2; $Theorem$ 3] are valid. First, $-i(A+u_n(t)B-i\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)})$ is dissipative in $H_{(0)}^2$ as for every $\psi\in H_{(0)}^4$ and $\lambda>0$,

$$\begin{aligned}&\|(\lambda+\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)}+i(A+u_n(t)B))\psi\|_{(2)}\\&\geq\|(\lambda+\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)}+iA)\psi\|_{(2)}-\|u_n(t)B\psi\|_{(2)}\\&\geq(\lambda+\|u_n\|_{L^\infty((0,T),\mathbb{R})}\,|||\,B\,|||_{(2)})\|\psi\|_{(2)}-|u_n(t)|\,|||\,B\,|||_{(2)}\|\psi\|_{(2)}\geq\lambda\|\psi\|_{(2)}.\end{aligned}$$

Second, it is maximal dissipative thanks to Kato-Rellich's Theorem [Dav95, *Theorem* 1.4.2] and, for $n \geq 3\epsilon$, we introduce

$$M := \sup_{t\in[0,T_n]} \|\!| (i\tilde{\lambda}_\epsilon - A - u_n(t)B)^{-1} \|\!|_{L(H^2_{(0)},\widehat{H}^4_{(0)})}$$
$$= \sup_{t\in[0,T_n]} \|\!| (i\tilde{\lambda}_\epsilon - A)(i\tilde{\lambda}_\epsilon - A - u_n(t)B)^{-1} \|\!|_{(2)}$$
$$= \sup_{t\in[0,T_n]} \|\!| (I - u_n(t)B(i\tilde{\lambda}_\epsilon - A)^{-1})^{-1} \|\!|_{(2)}.$$

Now, $\|\!| u_n(t)B(i\tilde{\lambda}_\epsilon - A)^{-1} \|\!|_{(2)} \leq \frac{\|\!| B \|\!|_{(2)} \|\!| (i\tilde{\lambda}_\epsilon - A)^{-1} \|\!|_{(2)}}{n} \leq \frac{\|\!| B \|\!|_{(2)}}{n\lambda_\epsilon} < 1$ and

(2.23)

$$M = \sup_{t\in[0,T_n]} \|\!| \sum_{l=1}^{+\infty} (u_n(t)B(i\tilde{\lambda}_\epsilon - A)^{-1})^l \|\!|_{(2)} \leq \sum_{l=1}^{+\infty} \|\!| n^{-1}B(i\tilde{\lambda}_\epsilon - A)^{-1} \|\!|^l_{(2)}$$
$$\leq \sum_{l=1}^{+\infty} n^{-l} \|\!| B \|\!|^l_{(2)} \|\!| (i\tilde{\lambda}_\epsilon - A)^{-1} \|\!|^l_{(2)} \leq \frac{1}{1 - \|\!| B \|\!|_{(2)} n^{-1}\lambda_\epsilon^{-1}} = \frac{n}{n-\epsilon} < 2.$$

We know that $\|k+f(\cdot)\|_{BV((0,T),\mathbb{R})} = \|f\|_{BV((0,T),\mathbb{R})}$ for every $f \in BV((0,T),\mathbb{R})$ and $k \in \mathbb{R}$. The same idea leads to

$$N := \|\!| i\tilde{\lambda}_\epsilon - A - u_n(\cdot)B \|\!|_{BV\left([0,T_n],L(\widehat{H}^4_{(0)},H^2_{(0)})\right)}$$
$$= \|u_n\|_{BV(T_n)} \|\!| B \|\!|_{L(\widehat{H}^4_{(0)},H^2_{(0)}))}.$$

Thanks to [Kat53, *Section* 3.10], there holds

$$\|(A + u_n(T_n)B - i\tilde{\lambda}_\epsilon)U^{u_n}_{T_n}\phi_j\|_{(2)} \leq Me^{MN}\|(A - i\tilde{\lambda}_\epsilon)\phi_j\|_{(2)}$$
$$\leq Me^{MN}(\lambda_j + \tilde{\lambda}_\epsilon)j^2 \leq Me^{MN}(\pi^2 + \lambda_\epsilon + \|u\|_{L^\infty((0,T),\mathbb{R})} \|\!| B \|\!|_{(2)})j^4.$$

Now, for every $\psi \in \widehat{H}^4_{(0)}$,

$$\|B\psi\|^2_{(2)} \leq \big(\epsilon\|A\psi\|_{(2)} + \|\!| B \|\!|_{(2)}\|\psi\|_{(2)}\big)^2 \leq 2\epsilon^2\big(\|A\psi\|^2_{(2)} + \tilde{\lambda}^2_\epsilon\|\psi\|^2_{(2)}\big).$$

As $\|(A - i\tilde{\lambda}_\epsilon)\psi\|^2_{(2)} = \|A\psi\|^2_{(2)} + \tilde{\lambda}^2_\epsilon\|\psi\|^2_{(2)}$, it follows

$$\|B\psi\|^2_{(2)} \leq 2\epsilon^2(\|(A - i\tilde{\lambda}_\epsilon)\psi\|^2_{(2)}) \leq 2\epsilon^2\|\psi\|^2_{\widehat{H}^4_{(0)}}$$

and $N \leq \epsilon\sqrt{2}\|u_n\|_{BV(T_n)}$. In addition, thanks to the relation (2.23), it is verified that

$$\|\!| A(A + u_n(T_n)B - i\tilde{\lambda}_\epsilon)^{-1} \|\!|_{(2)} \leq M + \|\!| \tilde{\lambda}_\epsilon(A - i\tilde{\lambda}_\epsilon)^{-1} \|\!|_{(2)}M \leq 4.$$

For every $j \in \mathbb{N}$, we know that

$$\|U_t^{u_n}\phi_j\|_{(4)} = e^{-t\|u_n\|_{L^\infty((0,T),\mathbb{R})} \| B \|_{(2)}}\|\Gamma_t^{u_n}\phi_j\|_{(4)} \leq e^{-\frac{t}{n} \| B \|_{(2)}}\|\Gamma_t^{u_n}\phi_j\|_{(4)}$$

and, for n satisfying (2.22),

$$\begin{aligned}
&\|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} = \|A\Gamma_{T_n}^{u_n}\phi_j\|_{(2)} \leq 4e^{\frac{T_n}{n} \| B \|_{(2)}}\|(A+u_n(T_n)B-i\tilde{\lambda}_\epsilon)U_{T_n}^{u_n}\phi_j\|_{(2)} \\
&\leq 8e^{\frac{T_n}{n} \| B \|_{(2)}+2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(\pi^2+\tilde{\lambda}_\epsilon)j^4 \\
&\leq 8e^{\frac{\| B \|_{(2)}}{|B_{j,k}|}+\frac{2\| B \|_{(2)}}{n\pi|k^2-j^2|}+2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(\pi^2+ \| B \|_{(2)}(\epsilon^{-1}+\|u_n\|_{L^\infty((0,T),\mathbb{R})}))j^4 \\
&\leq 8e^{\frac{\| B \|_{(2)}}{|B_{j,k}|}+\frac{1}{2}+2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(\pi^2+ \| B \|_{(2)}(\epsilon^{-1}+n^{-1}))j^4.
\end{aligned}$$

For $\epsilon = (2\sqrt{2}\|u_n\|_{BV(T_n)})^{-1}$, we have

(2.24)

$$\|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} \leq 8e^{\frac{\| B \|_{(2)}}{|B_{j,k}|}+3/2}\big(1+2\sqrt{2}\|u_n\|_{BV(T_n)} \| B \|_{(2)} + \| B \|_{(2)}n^{-1}\big)j^4.$$

The interval $[0, nT^*+T]$ contains less than d quarters of period of the function u_n for $d := 2\pi^2 n|k^2-j^2||B_{j,k}|^{-1}+4$ since

$$u_n(nT^*+T) = \frac{1}{n}\sin\big(\pi^2(k^2-j^2)(nT^*+T)\big) \quad\Rightarrow\quad d = \big(\pi^2(k^2-j^2)(nT^*+T)\big)\frac{2}{\pi}.$$

From (2.22), we know that $n \geq \| B \| (5\pi^{-2}|j^2-k^2|^{-1})$ that implies

$$\pi^2 n|k^2-j^2||B_{j,k}|^{-1} \geq 5$$

and

$$\|u_n\|_{BV(T_n)} \leq \|u_n\|_{BV(nT^*+T)} \leq (d+1)/n \leq 3\pi^2|k^2-j^2||B_{j,k}|^{-1} \tag{2.25}$$

(also the assumption $n \geq 3\epsilon$ is verified). Thanks to $\| B \|_{(2)} \geq |B_{j,k}|$ and $\| B \| \geq |B_{j,k}|$, the relation (2.24) becomes

(2.26)

$$\begin{aligned}
&\|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} \leq 8e^{\frac{\| B \|_{(2)}}{|B_{j,k}|}+3/2}(\pi^2+3\cdot 2\sqrt{2}\pi^2 \| B \|_{(2)}|k^2-j^2||B_{j,k}|^{-1}+ \| B \|_{(2)}n^{-1})j^4 \\
&\leq 8e^{\frac{\| B \|_{(2)}}{|B_{j,k}|}+3/2}(\pi^2+3\cdot 2\sqrt{2}\pi^2 \| B \|_{(2)}|k^2-j^2||B_{j,k}|^{-1} \\
&+5^{-1}\pi^2 \| B \|_{(2)} \| B \|^{-1}|j^2-k^2|)j^4 \leq 2^2 3^4\pi^2 e^{\frac{\| B \|_{(2)}}{|B_{j,k}|}} \| B \|_{(2)}|k^2-j^2||B_{j,k}|^{-1}j^4.
\end{aligned}$$

When $u \in BV(T)$, the propagator Γ_T^u preserves $H^4_{(0)}$ if $B \in L(H^2_{(0)})$.

2) Conclusion: Let $f_n := e^{i\theta}\phi_k - \Gamma_{T_n}^{u_n}\phi_j$. First, we point out that, for every $s > 0$,

$$\|f_n\|_{(s)}^2 \leq (k^s + \|\Gamma_{T_n}^{u_n}\phi_j\|_{(s)})^2.$$

As $\phi_j, \phi_k \in H_{(0)}^s$, for every $s > 0$, the point **1)** ensures that $\Gamma_T^u\phi_j$ and $\Gamma_T^u\phi_j$ belong to $H_{(0)}^4$ for $u \in BV(0,T)$. Thanks to the Cauchy-Schwarz inequality, we have

$$\|A^{\frac{3}{2}} f_n\|^4 = \big(\langle A^{\frac{3}{2}} f_n, A^{\frac{3}{2}} f_n\rangle\big)^2 \leq \big(\langle A^2 f_n, A f_n\rangle\big)^2 \leq \|A^2 f_n\|^2 \|A f_n\|^2,$$

$$\|A f_n\|^2 = \langle A f_n, A f_n\rangle \leq \langle A^2 f_n, f_n\rangle \leq \|A^2 f_n\|\|f_n\|$$

$$\implies \quad \|f_n\|_{(3)}^8 \leq \|f_n\|^2 \|f_n\|_{(4)}^6.$$

For R_n defined in the proof of Proposition 2.11, the relation (2.26) implies

$$\|f_n\|_{(3)}^8 \leq 3R_n(2^3 3^4 \pi^2 e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|}} \|\!| B \|\!|_{(2)} |k^2 - j^2||B_{j,k}|^{-1} \max\{j,k\}^4)^6$$

$$\leq \big(2^{18} 3^{26} \pi^{12} e^{\frac{6\|\!| B \|\!|_{(2)}}{|B_{j,k}|}} \|\!| B \|\!|_{(2)}^6 |k^2 - j^2|^6 |B_{j,k}|^{-6} \max\{j,k\}^{24}\big) \frac{(1+C')|B_{j,k}|^{-1} \|\!| B \|\!|^2}{n|k^2 - j^2|}$$

$$\leq \big(2^{18} 3^{26} \pi^{12} (1+C') e^{\frac{6\|\!| B \|\!|_{(2)}}{|B_{j,k}|}} \|\!| B \|\!|_{(2)}^6 \|\!| B \|\!|^2 |k^2 - j^2|^5 |B_{j,k}|^{-7} \max\{j,k\}^{24}\big) n^{-1}.$$

□

Proposition 2.13. *Let B satisfy Assumptions I. For every $j, k \in \mathbb{N}$, $j \neq k$, and $n \in \mathbb{N}$ satisfying* (2.18), (2.22) *and such that*

$$n \geq e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|} + 3/2} \|\!| B \|\!|_3 \pi^2 \; 6^3 \|\!| B \|\!|_{(2)} |B_{j,k}|^{-1} j^4, \tag{2.27}$$

there exists $\theta \in \mathbb{R}$ such that $\|\Gamma_{nT^*}^{u_n}\phi_j - e^{i\theta}\phi_k\|_{(3)}^8$ is not larger than

$$\frac{6^{26} \pi^{12} (1+C') e^{\frac{6\|\!| B \|\!|_{(2)}}{|B_{j,k}|}} \|\!| B \|\!|_{(2)}^6 \|\!| B \|\!| \max\{\|\!| B \|\!|, \|\!| B \|\!|_3\} |k^2 - j^2|^5 \max\{j,k\}^{24}}{|B_{j,k}|^7 n}.$$

Proof. Fist, we notice that the hypotheses of Proposition 2.12 are verified. Second, we estimate $\sup_{t\in[nT^*-T, nT^*+T]} \|\Gamma_t^{u_n}\phi_j - \Gamma_{T_n}^{u_n}\phi_j\|_{(3)}$ and we consider the arguments adopted in (2.24). The uniformly bounded constant $C(\cdot)$ is increasing and the relation (2.17) implies

$$\sup_{t\in[nT^*-T, nT^*+T]} C(|t - T_n|) \leq C(2T) \leq C(4/\pi) = \frac{24\sqrt{2}}{\pi^4}.$$

Thanks to Proposition 2.5 and Remark 2.6,

$$\sup_{t\in[nT^*-T,nT^*+T]}\|\Gamma_t^{u_n}\phi_j-\Gamma_{T_n}^{u_n}\phi_j\|_{(3)}=\sup\Big\{\sup_{t\in[nT^*-T,T_n]}\|\Gamma_t^{u_n}\phi_j-\Gamma_{T_n-t}^{u_n}\Gamma_t^{u_n}\phi_j\|_{(3)},$$
$$\sup_{t\in[T_n,nT^*+T]}\|\Gamma_{t-T_n}^{u_n}\Gamma_{T_n}^{u_n}\phi_j-\Gamma_{T_n}^{u_n}\phi_j\|_{(3)}\Big\}$$
$$\leq\sup\Big\{\sup_{t\in[nT^*-T,T_n]}C(T_n-t)\,|||\,B\,|||_3\int_t^{T_n}|u_n(s)|ds\|\Gamma_t^{u_n}\phi_j\|_{(3)},$$
$$\sup_{t\in[T_n,nT^*+T]}C(t-T_n)\,|||\,B\,|||_3\int_{T_n}^t|u_n(s)|ds\|\Gamma_{T_n}^{u_n}\phi_j\|_{(3)}\Big\}$$
$$\leq C\Big(\frac{4}{\pi}\Big)\,|||\,B\,|||_3\int_{nT^*-T}^{nT^*+T}|u_n(s)|ds\;\sup\Big\{\|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)},\sup_{t\in[nT^*-T,T_n]}\|\Gamma_t^{u_n}\phi_j\|_{(4)}\Big\}.$$

The techniques adopted in (2.24) lead to

$$\sup_{t\in[nT^*-T,T_n]}\|\Gamma_t^{u_n}\phi_j\|_{(4)}\leq\sup_{t\in[nT^*-T,T_n]}8e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+3/2}\big(1+2\sqrt{2}\|u_n\|_{BV(t)}\,|||\,B\,|||_{(2)}\big)j^4$$
$$\leq 8e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+3/2}\big(1+2\sqrt{2}\|u_n\|_{BV(T_n)}\,|||\,B\,|||_{(2)}\big)j^4.$$

Hence

$$\sup_{t\in(nT^*-T,nT^*+T)}\|\Gamma_t^{u_n}\phi_j-\Gamma_{T_n}^{u_n}\phi_j\|_{(3)}$$
$$\leq C\Big(\frac{4}{\pi}\Big)e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+3/2}\,|||\,B\,|||_3\frac{2T}{n}\;2^23^4\pi^2\,|||\,B\,|||_{(2)}|k^2-j^2||B_{j,k}|^{-1}j^4$$
$$\leq C\Big(\frac{4}{\pi}\Big)e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+3/2}\,|||\,B\,|||_3\frac{2T}{n}\;2^23^4\pi^2\,|||\,B\,|||_{(2)}|k^2-j^2||B_{j,k}|^{-1}j^4$$
$$\leq e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+3/2}\frac{|||\,B\,|||_3}{n}\;6^3\pi^2\,|||\,B\,|||_{(2)}|B_{j,k}|^{-1}j^4.$$

Now, we obtain

(2.28)
$$R_n'':=\|\Gamma_{nT^*}^{u_n}\phi_j-e^{i\theta}\phi_k\|_{(3)}^8$$
$$\leq 2^7\left(\sup_{t\in(nT^*-T,nT^*+T)}\|\Gamma_t^{u_n}\phi_j-\Gamma_{T_n}^{u_n}\phi_j\|_{(3)}^8+\|\Gamma_{T_n}^{u_n}\phi_j-e^{i\theta}\phi_k\|_{(3)}^8\right)$$
$$\leq 2^7\left(\left(e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+3/2}\,|||\,B\,|||_3\;6^3\pi^2\,|||\,B\,|||_{(2)}|B_{j,k}|^{-1}n^{-1}j^4\right)^8+\|f_n\|_{(3)}^8\right).$$

We keep in mind that $\|\!| B \|\!|$ and $\|\!| B \|\!|_{(2)}$ are not smaller than $|B_{j,k}|$ for every $j,k \in \mathbb{N}$. If $n \geq e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|}+3/2} \|\!| B \|\!|_3 \pi^2\, 6^3 \|\!| B \|\!|_{(2)} |B_{j,k}|^{-1} j^4$, then

$$\left(e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|}+3/2} \|\!| B \|\!|_3\, 6^3\pi^2 \|\!| B \|\!|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4\right)^8$$
$$\leq e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|}+3/2} \|\!| B \|\!|_3\, 6^3\pi^2 \|\!| B \|\!|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4$$

and

$$\|\Gamma^{u_n}_{nT^*}\phi_j - e^{i\theta}\phi_k\|^8_{(3)}$$
$$\leq 2^7 \left(e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|}+3/2} \|\!| B \|\!|_3\, 6^3\pi^2 \|\!| B \|\!|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4 + \|f_n\|^8_{(3)}\right)$$
$$\leq 2^7 e^{\frac{\|\!| B \|\!|_{(2)}}{|B_{j,k}|}+3/2} \frac{\pi^2 \|\!| B \|\!|_3 6^3 \|\!| B \|\!|_{(2)} j^4}{n|B_{j,k}|}$$
$$+ \frac{2^{25}3^{26}\pi^{12}(1+C')e^{\frac{6\|\!| B \|\!|_{(2)}}{|B_{j,k}|}} \|\!| B \|\!|^6_{(2)} \|\!| B \|\!|^2 |k^2-j^2|^5 \max\{j,k\}^{24}}{|B_{j,k}|^7 n}$$
$$\leq \frac{6^{26}\pi^{12}(1+C')e^{\frac{6\|\!| B \|\!|_{(2)}}{|B_{j,k}|}} \|\!| B \|\!|^6_{(2)} \|\!| B \|\!| \max\{\|\!| B \|\!|, \|\!| B \|\!|_3\} |k^2-j^2|^5 \max\{j,k\}^{24}}{|B_{j,k}|^7 n}.$$

□

2.5 Proof of Theorem 2.2

The proof follows from the validity of Proposition 2.10 and Proposition 2.13 as the conditions (2.18), (2.22) and (2.27) are satisfied. Let R''_n be introduced in the proof of Proposition 2.13. We know $\lim_{n\to\infty} R''_n = 0$ and there exist n^* and θ (depending on n^*) such that

$$\Gamma^{u_{n^*}}_{n^*T^*}\phi_j \in \widetilde{B}_{H^3_{(0)}}\left(e^{i\theta}\phi_k, C_k^2 (6^2 k^3 \|\!| B \|\!|^2_3)^{-1}\right), \tag{2.29}$$
$$\implies \quad R''_{n^*} \leq \frac{C_k^{16}}{6^2 k^{24} \|\!| B \|\!|^{16}_3}.$$

For $0 \leq s < 3$ and $j,k \in \mathbb{N}$, we know that $\|\!| B \|\!|_{(s)} \geq C_k$ and $\|\!| B \|\!|_{(s)} \geq |B_{j,k}|$. For

$$b := \|\!| B \|\!|^6_{(2)} \|\!| B \|\!| \, \|\!| B \|\!|^{16}_3 \max\left\{\|\!| B \|\!|, \|\!| B \|\!|_3\right\},$$

the relation (2.29) is valid when

$$n^* \geq \frac{6^{42}\pi^{12} e^{\frac{6 \|\| B \|\|_{(2)}}{|B_{j,k}|}} b(1+C')|k^2-j^2|^5 k^{24} \max\{j,k\}^{24}}{C_k^{16}|B_{j,k}|^7}.$$

The local exact controllability is verified in a neighborhood of $\phi_k(4/\pi) = e^{i4k^2\pi}\phi_k = \phi_k$, while our dynamics is pointing $e^{i\theta}\phi_k$. For this reason, we have to pay attention to the phase of the target state. For

$$u_{n^*}(t) = \frac{\cos\big((k^2-j^2)\pi^2 t\big)}{n^*}, \quad n^*T = n^*\frac{\pi}{|B_{j,k}|},$$

thanks to Proposition 2.10 and to the time reversibility of the problem (2.1) (see Paragraph 2.1), there exists $u \in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that

$$\Gamma^u_{\frac{4}{\pi}}\Gamma^{u_{n^*}}_{n^*T}\phi_j = e^{i\theta}\phi_k. \tag{2.30}$$

2.6 Computing the phase

Let $N \in \mathbb{N}$. We define the $N \times N$ matrix M^N such that, for $l, m \in \mathbb{N}$,

$$M^N_{l,m} = \langle \phi_l, M^N\phi_m\rangle = \frac{B_{l,m}}{I}\int_0^I e^{i\pi^2(l^2-m^2)v(x)}dx, \qquad \text{if} \quad \frac{|l^2-m^2|}{|k^2-j^2|} \in \mathbb{N},$$

for $v(t)$ the reciprocal function of $t \mapsto \int_0^t |\cos(\pi^2(k^2-j^2)s)|ds$, otherwise $M^N_{l,m} = 0$. Let $\theta^N \in \mathbb{R}^+$ be the smallest value such that $e^{i\theta^N} = \langle \phi_k, e^{2|B_{k,j}|^{-1}M^N}\phi_j\rangle$ and

$$\widetilde{T}^N = \frac{\theta^N}{(j\pi)^2}.$$

In the following proposition, we provide a similar result of Proposition 2.13 without the presence of the phase ambiguity in the target state.

Proposition 2.14. *Let B satisfy Assumptions I. Let $j, k \in \mathbb{N}$, $j \neq k$, and $n \in \mathbb{N}$ satisfy* (2.18), (2.22) *and* (2.27). *For $N \in \mathbb{N}$ such that*

$$\frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty}|B_{l,k}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty}|B_{l,j}|^2\Big)^{\frac{1}{2}}\Big) \leq \frac{4\|\| B \|\|}{n\pi^2|k^2-j^2|},$$

then

$$\|\Gamma^{u_n}_{nT^*}\Gamma^0_{\widetilde{T}^N}\phi_j - \phi_k\|^8_{(3)} \leq$$

$$\frac{10\ 6^{26}\pi^{12}(1+C')e^{\frac{6\|\| B \|\|_{(2)}}{|B_{j,k}|}} \|\| B \|\|^6_{(2)} \|\| B \|\|^2 \max\{\|\| B \|\|, \|\| B \|\|_3\}|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|^8 n}.$$

Proof. The proof follows from [Cha12] that defines the phase θ introduced in the propositions 2.11, 2.12 and 2.13. By referring to [Cha12, *Section* 3.1], we estimate $N \geq \max\{j,k\}$ such that

$$K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle+\phi_k\langle\phi_k,\cdot\rangle)\| \leq R_n, \tag{2.31}$$

for $\pi_N(\cdot) := \sum_{k=1}^N \phi_k\langle\phi_k,\cdot\rangle$. We have

$$\begin{aligned}
&K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle+\phi_k\langle\phi_k,\cdot\rangle)\| \\
&\leq K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle)\| + K\|(1-\pi_N)B(\phi_k\langle\phi_k,\cdot\rangle)\| \\
&\leq \frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty}|B_{l,k}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty}|B_{l,j}|^2\Big)^{\frac{1}{2}}\Big) \leq R_n.
\end{aligned}$$

As $\frac{4\,|||\,B\,|||}{n\pi^2|k^2-j^2|} \leq R_n$, we impose that $N \geq \max\{j,k\}$ is such that

$$\frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty}|B_{l,k}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty}|B_{l,j}|^2\Big)^{\frac{1}{2}}\Big) \leq \frac{4\,|||\,B\,|||}{n\pi^2|k^2-j^2|}.$$

Let $X^u_{(N)}(t,s)$ be the finite-dimensional propagator defined in the first part of [Cha12, *Section* 2.1]. Thanks to the proof of [Cha12, *Proposition* 2], there exists $T_n \in (nT^*-T, nT^*+T)$ such that

$$|\langle\phi_k, e^{KM^N}\phi_j\rangle - \langle\phi_k, X^{u_n}_{(N)}(T_n,0)\phi_j\rangle| \leq R_n, \qquad \forall n\in\mathbb{N}. \tag{2.32}$$

We point out that $|\langle\phi_k, e^{KM^N}\phi_j\rangle| = 1$ since $M^N = i\tilde{M}$ for $\tilde{M}$ a $N\times N$ matrix with real entries (see also [Cha12, *p.* 5]). Now, $\theta^N \in \mathbb{R}^+$ is the smallest value such that $e^{i\theta^N} = \langle\phi_k, e^{KM^N}\phi_j\rangle$, which follows from [Cha12, *relation* 11]. Indeed, the term $e^{tM^N}z_n(0)$ appearing in the mentioned equation corresponds to the free finite-dimensional propagator after a time reparameterization and the averaging procedure performed in the first part of [Cha12, *Section* 2]. Moreover, from [Cha12, *relation* (14)] and the following one, we can notice that the time reparameterization maps K in T_n. Now, we use [Cha12, *relations* (18), (19)] as in [Cha12, *relation* (20)] and we obtain

$$\begin{aligned}
&|\langle\phi_k, X^{u_n}_{(N)}(T_n,0)\phi_j\rangle - \langle\phi_k, \Gamma^{u_n}_{T_n}\phi_j\rangle| \leq K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle+\phi_k\langle\phi_k,\cdot\rangle)\| \\
&+ 4KR_n\|(1-\pi_N)B\pi_N\| \leq R_n + 8|B_{j,k}|^{-1}\,|||\,B\,|||\,R_n \leq 9|B_{j,k}|^{-1}\,|||\,B\,|||\,R_n.
\end{aligned} \tag{2.33}$$

Hence, from (2.32) and (2.33), it follows

$$
\begin{aligned}
&1-|\langle\phi_k,\Gamma^{u_n}_{T_n}\phi_j\rangle|\le|e^{i\theta^N}-\langle\phi_k,\Gamma^{u_n}_{T_n}\phi_j\rangle|\\
&\le|e^{i\theta^N}-\langle\phi_k,X^{u_n}_{(N)}(T_n,0)\phi_j\rangle|+|\langle\phi_k,X^{u_n}_{(N)}(T_n,0)\phi_j\rangle-\langle\phi_k,\Gamma^{u_n}_{T_n}\phi_j\rangle|\\
&\le R_n+9|B_{j,k}|^{-1}\,|||\,B\,|||\,R_n\le 10|B_{j,k}|^{-1}\,|||\,B\,|||\,R_n=:\widetilde{R}_n.
\end{aligned}
$$

Thus, we substitute R_n with $\widetilde{R}_n$ in the proofs of the propositions 2.11, 2.12 and 2.13 which leads to change the relation (2.28) as follows

$$
\begin{aligned}
&\|\Gamma^{u_n}_{nT^*}\phi_j-e^{i\theta^N}\phi_k\|^8_{(3)}\le 2^7e^{\frac{|||\,B\,|||_{(2)}}{|B_{j,k}|}+\frac{3}{2}}\frac{\pi^2\,|||\,B\,|||_3 6^3\,|||\,B\,|||_{(2)}j^4}{n|B_{j,k}|}\\
&+e^{\frac{6\,|||\,B\,|||_{(2)}}{|B_{j,k}|}}\frac{10\;2^{25}3^{26}\pi^{12}(1+C')\,|||\,B\,|||^6_{(2)}\,|||\,B\,|||^3|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|n}\\
&\le\frac{10\;6^{26}\pi^{12}(1+C')e^{\frac{6\,|||\,B\,|||_{(2)}}{|B_{j,k}|}}\,|||\,B\,|||^6_{(2)}\,|||\,B\,|||^2\max\{\,|||\,B\,|||\,,\,|||\,B\,|||_3\}|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|^8n}.
\end{aligned}
$$

In conclusion, $e^{-i\theta^N}\phi_j=\Gamma^0_{\widetilde{T}^N}\phi_j$ for $\widetilde{T}^N=\lambda_j^{-1}\theta^N$ and then

$$
\begin{aligned}
&\|\Gamma^{u_n}_{nT^*}\Gamma^0_{\widetilde{T}^N}\phi_j-\phi_k\|^8_{(3)}\\
&\le\frac{10\;6^{26}\pi^{12}(1+C')e^{\frac{6\,|||\,B\,|||_{(2)}}{|B_{j,k}|}}\,|||\,B\,|||^6_{(2)}\,|||\,B\,|||^2\max\{\,|||\,B\,|||\,,\,|||\,B\,|||_3\}|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|^8n}.
\end{aligned}
$$

Theorem 2.15. *Let $j,n\in\mathbb{N}$ and $k\in\mathbb{N}$ be such that $k\neq j$ and*

$$(2.34)\qquad m^2-k^2\neq k^2-l^2,\qquad\qquad \forall m,l\in\mathbb{N},\ m,l\neq k.$$

Let $n\ge 6^{42}\;10\;\pi^{12}b\;(1+C')\,|||\,B\,|||\,E(j,k)|B_{j,k}|^{-1}$ and $N\ge\max\{j,k\}$. For $N\in\mathbb{N}$ large enough such that

$$(2.35)\qquad \frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty}|B_{l,k}|^2\Big)^{\frac{1}{2}}+\Big(\sum_{l=N+1}^{\infty}|B_{l,j}|^2\Big)^{\frac{1}{2}}\Big)\le\frac{4\,|||\,B\,|||}{n\pi^2|k^2-j^2|},$$

then

$$\big\|\Gamma^{u_n}_{nT^*}\Gamma^0_{\widetilde{T}^N}\phi_j-\phi_k\big\|_{(3)}\le C_k^2(6^2k^3\,|||\,B\,|||^2_3)^{-1}.$$

Moreover, there exists $u\in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that $\|u\|_{L^2((0,\frac{4}{\pi}),\mathbb{R})}\le\frac{C_k}{3\,|||\,B\,|||^2_3k^3}$ and

$$\Gamma^u_{\frac{4}{\pi}}\Gamma^{u_n}_{nT^*}\Gamma^0_{\widetilde{T}^N}\phi_j=\phi_k.$$

Proof. The proof follows from the validity of Proposition 2.10 and Proposition 2.14 thanks to the arguments of the proof of Theorem 2.2. □

2.7 Example: dipolar moment

In the current paragraph, we retrace the proof of the first point of Theorem 2.2 by fixing B and $j,k\in\mathbb{N}$. For $B:\psi\mapsto x^2\psi$, we define a control function and a time such that the dynamics of (2.1) drives the second eigenstate ϕ_2 in the first ϕ_1.

First, for

$$\langle\phi_j,x^2\phi_k\rangle=2\int_0^1 x^2\sin(\sqrt{\lambda_j}x)\sin(\sqrt{\lambda_k}x)dx=2\int_0^1 x^2\sin(\pi jx)\sin(\pi kx)dx,$$

we notice that Assumptions I are satisfied since $B_{1,1}\in\mathbb{R}$ and

$$\begin{aligned}|\langle\phi_j,x^2\phi_k\rangle|&=\Big|\frac{(-1)^{j-k}}{(j-k)^2\pi^2}-\frac{(-1)^{j+k}}{(j+k)^2\pi^2}\Big|=\frac{4jk}{(j^2-k^2)^2\pi^2}, && j\neq k,\\ |\langle\phi_k,x^2\phi_k\rangle|&=\Big|\frac{1}{3}-\frac{1}{2k^2\pi^2}\Big|, && k\in\mathbb{N}.\end{aligned}$$

Now, for every $\psi\in H^3_{(0)}$, we know that $x^2\psi\in H^3\cap H^1_0$, $\|\partial_x\psi\|\leq\|\partial_x^2\psi\|$ and, thanks to the Poincaré inequality, we have $\|\psi\|\leq\pi^{-1}\|\partial_x\psi\|$ and $\|\partial_x^2\psi\|\leq\pi^{-1}\|\partial_x^3\psi\|$. In addition, we know that $\|x\psi\|\leq\frac{1}{\sqrt{3}}\|\psi\|$, $\|x^2\psi\|\leq\frac{1}{\sqrt{5}}\|\psi\|$ and

$$\begin{aligned}\|\partial_x(x^2\psi)\|&\leq\|2x\psi\|+\|x^2\partial_x\psi\|\leq\frac{2}{\sqrt{3}}\|\psi\|+\frac{1}{\sqrt{5}}\|\partial_x\psi\|\\ &\leq\Big(\frac{2}{\sqrt{3}\pi}+\frac{1}{\sqrt{5}}\Big)\|\partial_x\psi\|\leq\Big(\frac{2\sqrt{5}+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)\|\partial_x^3\psi\|,\\ \|\partial_x^2(x^2\psi)\|&\leq\|2\psi\|+\|4x\partial_x\psi\|+\|x^2\partial_x^2\psi\|\leq\Big(\frac{2\sqrt{15}+4\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)\|\partial_x^3\psi\|,\\ \|\partial_x^3(x^2\psi)\|&\leq\|6\partial_x\psi\|+\|6x\partial_x^2\psi\|+\|x^2\partial_x^3\psi\|\leq\Big(\frac{6\sqrt{15}+6\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi}\Big)\|\partial_x^3\psi\|.\end{aligned}$$

Thus

$$\begin{aligned}
\| B \|_3^2 &= \sup_{\substack{\psi\in H^3_{(0)}\\ \|\psi\|_{(3)}\leq 1}} (\|\partial_x x^2\psi\|^2 + \|\partial_x^2 x^2\psi\|^2 + \|\partial_x^3 x^2\psi\|^2)\\
&\leq \sup_{\substack{\psi\in H^3_{(0)}\\ \|\psi\|_{(3)}\leq 1}} \Big(\frac{2\sqrt{5}+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)^2\|\partial_x^3\psi\|^2 + \Big(\frac{2\sqrt{15}+4\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)^2\|\partial_x^3\psi\|^2\\
&+ \Big(\frac{6\sqrt{15}+6\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi}\Big)^2\|\partial_x^3\psi\|^2 \leq \Big(\frac{2\sqrt{15}+4\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)^2\\
&+ \Big(\frac{2\sqrt{5}+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)^2 + \Big(\frac{6\sqrt{15}+6\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi}\Big)^2
\end{aligned}$$

and $\| B \|_3 \leq 5,93$. Equivalently $\| B \|_{(2)} \leq 3,4$, $\| B \| = 1/\sqrt{5}$, $C' = 0$. Moreover,

$$|B_{1,1}| = C_1 = \frac{2\pi-3}{6\pi^2}, \qquad |B_{1,2}| = C_2 = \frac{8}{9\pi^2}, \qquad I = \frac{4}{3\pi^2}.$$

We retrace the proof of the first point of Theorem 2.2. Let $T = \frac{2}{3\pi}$, $u(t) = \cos(3\pi^2 t)$, $T^* = \frac{9\pi^3}{8}$, $K = \frac{9\pi^2}{4}$. For $u_n := \frac{u}{n}$, there exists $\theta \in \mathbb{R}$ such that

$$\|e^{i\theta}\phi_1 - \Gamma^{u_n}_{T_n}\phi_2\|^2 \leq \frac{3^2|B_{1,2}^{-1}| \| B \|^2}{n|2^2-1^2|} = \frac{27\pi^2}{40n}.$$

Afterwards, for n large enough, thanks to (2.25),

$$\|u_n\|_{BV(0,nT^*+T)} \leq 3\pi^2|k^2-j^2||B_{j,k}|^{-1} \leq 3^4 2^{-3}\pi^4.$$

By following the proof of Theorem 2.2 for $I := [nT^*-T, nT^*+T]$, we have

$$\begin{aligned}
&\|e^{i\theta}\phi_1 - \Gamma^{u_n}_{nT^*}\phi_2\|^8_{(3)} \leq 2^7\Big(\|e^{i\theta}\phi_1 - \Gamma^{u_n}_{T_n}\phi_2\|^6_{(4)}\|e^{i\theta}\phi_1 - \Gamma^{u_n}_{T_n}\phi_2\|^2\Big)\\
&+ \sup_{t\in[nT^*-T,nT^*+T]}\Big(2^7\|\Gamma^{u_n}_{T_n}\phi_2 - \Gamma^{u_n}_t\phi_2\|^8_{(3)}\Big)\\
&\leq 2^7 e^{\frac{9\pi^2}{8}3,4\ 6}\Big(\frac{27\pi^2}{40n}(8e(1+3^4\sqrt{2}\cdot 2^{-2}\cdot 3,4\cdot\pi^4)2^4+1)^6\\
&+ 5,93\cdot 3,4\cdot 6^3\cdot 2\cdot 9\pi^4 n^{-1}\Big) \leq 2,61\cdot 10^{140}n^{-1}.
\end{aligned}$$

In the neighborhood $\widetilde{B}_{H^3_{(0)}}\left(\phi_1,\ 2,4\cdot 10^{-6}\right)$, the local exact controllability is verified and the first point of Theorem 2.2 is satisfied for

$$n = 2,38\cdot 10^{185}.$$

In conclusion, there exists $\theta \in \mathbb{R}$ such that for

$$u(t) = (2,38 \cdot 10^{185})^{-1} \cos(3\pi^2 t), \qquad T = (2,38 \cdot 10^{185}) \frac{9\pi^3}{8},$$

there holds $\left\| e^{i\theta}\phi_1 - \Gamma_T^u \phi_2 \right\|_{(3)} \leq 2,4 \cdot 10^{-6}$. In addition, there exists $\tilde{u} \in L^2((0, \frac{4}{\pi}), \mathbb{R})$ such that

$$\Gamma_T^u \Gamma_{\frac{4}{\pi}}^{\tilde{u}} \phi_2 = e^{i\theta}\phi_1.$$

2.8 Moving forward

The nature of the work opens several questions, first and foremost, if the techniques developed may be adopted in the simultaneous global exact controllability with the approaches of Chapter 3 (see also [MN15]).
Moreover, the results provided in Theorem 2.2 are far from being optimal and one might be interested in optimizing them.

1. As already mentioned in Remark 2, Theorem 2.2 can be stated for other $\frac{2\pi}{|\lambda_k - \lambda_j|}-$periodic controls by using the theory exposed in [Cha12]. A natural question is when it is possible to retrace the theory of this chapter with different controls and obtain sharper estimates for n.

2. By using the techniques adopted in the proof of Proposition 2.10, one can look for a larger neighborhood of validity of the local exact controllability. A try is to change the time $\frac{4}{\pi}$ and study the variation of the radius as a time-dependent function.

3. The solvability of the moment problem (2.8) can be ensured with "Haraux's Theorem" (Proposition $A.6$, Appendix $A.1$) instead of Ingham's Theorem (Proposition $A.5$, Appendix $A.1$).

   By retracing the steps of the proof of Proposition 2.10, one can establish the new constants and study how the neighborhood changes according to the time.

Chapter 3

Simultaneous global exact controllability in projection

In the present chapter, we consider the Hilbert space $\mathscr{H} = L^2((0,1),\mathbb{C})$. We denote

$$\langle\psi_1,\psi_2\rangle := \langle\psi_1,\psi_2\rangle_{\mathscr{H}} = \int_0^1 \overline{\psi_1}(x)\psi_2(x)dx, \qquad \forall\psi_1,\psi_2\in\mathscr{H}$$

and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. We study the simultaneous global exact controllability of infinitely many (BSE) in $\mathscr{H} = L^2((0,1),\mathbb{C})$, *i.e.* the following infinite Cauchy problems

$$\begin{cases} i\partial_t\psi_j(t) = A\psi_j(t) + u(t)B\psi_j(t), & t\in(0,T), \quad \forall j\in\mathbb{N},\\ \psi_j(0) = \psi_j^0 \end{cases} \tag{3.1}$$

for $T > 0$. The operator $A = -\Delta$ is the Laplacian with Dirichlet type boundary conditions

$$D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C}),$$

u is a $L^2((0,T),\mathbb{R})$ control function and B is a bounded symmetric operator. The state $\psi_j^0(x)$ is the j-th initial state, while the j-th solution of (3.1) is $\psi_j(t) = \Gamma_t^u\psi_j^0$. We call Γ_t^u the unitary propagator of (3.1) when it is defined.

3.1 Framework and main results

We keep the notation introduced in Chapter 2 and we define

$$I^N := \{(j,k)\in\mathbb{N}\times\{1,...,N\} : j\neq k\}, \qquad N\in\mathbb{N}. \tag{3.2}$$

Assumptions (II)**.** The operator B satisfies the following conditions.

1. For any $N\in\mathbb{N}$, there exists $C_N>0$ such that, for every $j,k\in\mathbb{N}$ with $j\leq N$,
$$|\langle\phi_k,B\phi_j\rangle|\geq\frac{C_N}{k^3}.$$
2. $Ran(B|_{H^2_{(0)}})\subseteq H^2_{(0)}$ and $Ran(B|_{H^3_{(0)}})\subseteq H^3\cap H^1_0$.
3. For every $N\in\mathbb{N}$ and $(j,k),(l,m)\in I^N$ such that $(j,k)\neq(l,m)$ and
$$j^2-k^2-l^2+m^2=0,$$
there holds $\langle\phi_j,B\phi_j\rangle-\langle\phi_k,B\phi_k\rangle-\langle\phi_l,B\phi_l\rangle+\langle\phi_m,B\phi_m\rangle\neq0.$

The first two points of Assumptions II compose Assumptions I introduced in the previous chapter. The second condition ensures that B decouples the resonant eigenvalues gaps, *i.e.* $\lambda_j-\lambda_k=\lambda_l-\lambda_m$ for $(j,k),(l,m)\in I$ with $(j,k)\neq(l,m)$.

Example 3.1. *In Paragraph* 2.7*, we prove that Assumptions I are satisfied for $B:\psi\mapsto x^2\psi$. Assumptions II are also verified for this operator since*
$$\begin{cases}|\langle\phi_j,x^2\phi_k\rangle|=\left|\frac{(-1)^{j-k}}{(j-k)^2\pi^2}-\frac{(-1)^{j+k}}{(j+k)^2\pi^2}\right|, & j\neq k,\\ |\langle\phi_k,x^2\phi_k\rangle|=\left|\frac{1}{3}-\frac{1}{2k^2\pi^2}\right|=\frac{1}{3}-\frac{1}{2k^2\pi^2}, & k\in\mathbb{N}.\end{cases}$$
The condition 3) is guaranteed as follows. Let $(j,k),(l,m)\in I^N$ be such that $(j,k)\neq(l,m)$ and
$$j^2-k^2-l^2+m^2=0. \tag{3.3}$$
First, the relation (3.3) *leads to*
$$\begin{aligned}0&=(j^2-k^2)^2-(l^2-m^2)^2=-2j^2k^2+2l^2m^2+j^4+k^4-l^4-m^4\\&=-2j^2k^2+2l^2m^2+(j^2-l^2)(j^2+l^2)+(k^2-m^2)(k^2+m^2)\\&=-2j^2k^2+2l^2m^2+(j^2-l^2)\big((j^2+l^2)+(k^2+m^2)\big).\end{aligned}\tag{3.4}$$
Second, from the relation (3.4)*, we know that $j^2-l^2\neq0$ as $j\neq l$ and*
$$(j^2-l^2)\big((j^2+l^2)+(k^2+m^2)\big)\neq0\qquad\Longrightarrow\qquad j^2k^2\neq l^2m^2.$$
In conclusion, for $j\neq k$, we have
$$\begin{aligned}&j^{-2}-k^{-2}-l^{-2}+m^{-2}=-\frac{j^2-k^2}{j^2k^2}+\frac{l^2-m^2}{l^2m^2}\\&=-(j^2-k^2)\Big(\frac{1}{j^2k^2}-\frac{1}{l^2m^2}\Big)=\frac{j^2-k^2}{j^2k^2l^2m^2}(l^2m^2-j^2k^2)\neq0.\end{aligned}$$

Let $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ and $\mathscr{H}_N(\Psi) := \text{span}\{\psi_j : j \leq N\}$. We define $\pi_N(\Psi)$ the orthogonal projector onto $\mathscr{H}_N(\Psi)$. We call $\{\phi_j\}_{j\in\mathbb{N}}$ a complete orthonormal basis composed by eigenfunctions of A associated to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ such that $\lambda_k = \pi^2 k^2$ and

$$\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j.$$

Definition 3.2. The problems (3.1) are said to be simultaneously globally exactly controllable in projection in $H^3_{(0)}$ if there exist $T > 0$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ such that the following property is verified. For every $\{\psi_j^1\}_{j\in\mathbb{N}}$, $\{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ unitarily equivalent, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi)\psi_j^2 = \pi_N(\Psi)\Gamma_T^u\psi_j^1, \qquad \forall j \in \mathbb{N}.$$

In other words, there holds

$$\langle\psi_k, \psi_j^2\rangle = \langle\psi_k, \Gamma_T^u\psi_j^1\rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N.$$

Definition 3.3. Let us define

$$O_{\epsilon,T} := \Big\{\{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)} \Big|\ \langle\psi_j,\psi_k\rangle = \delta_{j,k};\ \ \sup_{j\in\mathbb{N}} \|\psi_j - \phi_j(T)\|_{(3)} < \epsilon\Big\}.$$

The problems (3.1) are said to be simultaneously locally exactly controllable in projection in $O_{\epsilon,T} \subset H^3_{(0)}$ up to phases if there exist $\epsilon > 0$, $T > 0$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ such that the following property is verified. For every $\{\psi_j^1\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$, there exist $\{\theta_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi)\psi_j^1 = \pi_N(\Psi)e^{i\theta_j}\Gamma_T^u\psi_j, \qquad \forall j \in \mathbb{N}.$$

In other words, there holds

$$\langle\psi_k, \psi_j^1\rangle = e^{i\theta_j}\langle\psi_k, \Gamma_T^u\psi_j\rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N.$$

Let $U(\mathscr{H})$ be the space of the unitary operators on $\mathscr{H}$. We present the simultaneous local exact controllability in projection for any positive times up to phases.

Theorem 3.4. *Let B satisfy Assumptions II. For every $T > 0$, there exist $\epsilon > 0$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ such that the following holds. For any $\{\psi_j^1\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ and $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\psi_j^1\}_{j\in\mathbb{N}} = \{\widehat{\Gamma}\phi_j\}_{j\in\mathbb{N}}$, if*

$$\big\{\widehat{\Gamma}^*\phi_j\big\}_{j\in\mathbb{N}} \subset H^3_{(0)}, \tag{3.5}$$

then there exist $\{\theta_j\}_{j\leq N}\subset\mathbb{R}$ *and* $u\in L^2((0,T),\mathbb{R})$ *such that*

$$\begin{cases}\pi_N(\Psi)\psi_j^1=\pi_N(\Psi)e^{i\theta_j}\Gamma_T^u\psi_j & j\leq N,\\ \pi_N(\Psi)\psi_j^1=\pi_N(\Psi)\Gamma_T^u\psi_j, & j>N.\end{cases}$$

Proof. See Proposition 3.10. □

Now, we present the simultaneous global exact controllability in projection up to phases in the components.

Theorem 3.5. *Let* B *satisfy Assumptions II and* $\Psi^3:=\{\psi_j^3\}_{j\in\mathbb{N}}\subset H^3_{(0)}$ *be an orthonormal system. Let* $\{\psi_j^1\}_{j\in\mathbb{N}},\ \{\psi_j^2\}_{j\in\mathbb{N}},\subset H^3_{(0)}$ *be complete orthonormal systems so that there exists* $\widehat{\Gamma}\in U(\mathscr{H})$ *such that* $\{\widehat{\Gamma}\psi_j^1\}_{j\in\mathbb{N}}=\{\psi_j^2\}_{j\in\mathbb{N}}$*. If*

$$\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}}\subset H^3_{(0)}, \tag{3.6}$$

then for any $N\in\mathbb{N}$*, there exist* $T>0$*,* $u\in L^2((0,T),\mathbb{R})$ *and* $\{\theta_k\}_{k\leq N}\subset\mathbb{R}$ *such that*

$$e^{i\theta_k}\langle\psi_k^3,\psi_j^2\rangle=\langle\psi_k^3,\Gamma_T^u\psi_j^1\rangle,\qquad \forall j,k\in\mathbb{N},\ k\leq N. \tag{3.7}$$

Proof. See Paragraph 3.3. □

In Theorem 3.5, if $\Psi^3=\Psi^2$, then $\widehat{\Gamma}^*\psi_j^3\in H^3_{(0)}$. By considering that

$$e^{i\theta_k}\langle\psi_k^2,\psi_j^2\rangle=e^{i\theta_k}\delta_{k,j}=e^{i\theta_j}\langle\psi_k^2,\psi_j^2\rangle,\qquad \forall j\in\mathbb{N},$$

the relation (3.7) becomes

$$\begin{aligned}\pi_N(\Psi^2)\ e^{i\theta_j}\psi_j^2&=\pi_N(\Psi^2)\Gamma_T^u\psi_j^1, && j\leq N,\\ \pi_N(\Psi^2)\ \psi_j^2&=\pi_N(\Psi^2)\Gamma_T^u\psi_j^1, && j>N.\end{aligned}$$

As Ψ^2 is composed by orthogonal elements, then

$$\pi_N(\Psi^2)\ \psi_j^2=\begin{cases}\psi_j^2, & j\leq N,\\ 0, & j>N\end{cases} \tag{3.8}$$

and the next corollary follows.

Corollary 3.6. *Let B satisfy Assumptions II. Let $\Psi^1 := \{\psi_j^1\}_{j\in\mathbb{N}}$, $\Psi^2 := \{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be complete orthonormal systems. For any $N \in \mathbb{N}$, there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\le N} \subset \mathbb{R}$ such that*

$$\begin{cases} \Gamma_T^u \psi_j^1 = e^{i\theta_j}\psi_j^2, & j \le N, \\ \pi_N(\Psi^2)\ \Gamma_T^u \psi_j^1 = 0, & j > N. \end{cases}$$

Let $P_{\phi_j}^{\perp}$ be the projector onto the orthogonal space of ϕ_j and the operator

$$\widetilde{B}(M,j) = B\big((\lambda_j - A)\big|_{\phi_j^{\perp}}\big)^{-1}\Big(\big((\lambda_j - A)\big|_{\phi_j^{\perp}}\big)^{-1} P_{\phi_j}^{\perp} B\Big)^M P_{\phi_j}^{\perp} B$$

for $M, j \in \mathbb{N}$. When (A,B) satisfies Assumptions II and the following assumptions, the phase ambiguities $\{\theta_j\}_{j\le N} \subset \mathbb{R}$ appearing in Theorem 3.5 can be removed.

Let $\mathbf{0^n} := \{a_j\}_{j\le n} \in \mathbb{Q}^n$ be such that $a_j = 0$ for every $j \le n$ and $n \in \mathbb{N}$.
Assumptions (A)**.** If there exists $\{r_j\}_{j\le n} \in \mathbb{Q}^n \setminus \mathbf{0^n}$ with $n \in \mathbb{N}$ such that

$$r_1 + \sum_{j=2}^{n} r_j\lambda_j = 0,$$

then either we have $\sum_{j=2}^{n} r_j B_{j,j} \neq 0$, or there exists $M \in \mathbb{N}$ such that

$$\sum_{j=2}^{n} r_j \langle \phi_j, \widetilde{B}(M,j)\phi_j\rangle \neq 0.$$

Remark. *When the operator B is such that $\{B_{j,j}\}_{j\in\mathbb{N}}$ are rationally independent, the Assumptions A are verified (also the third point of Assumptions II). In other words, when for any $n \in \mathbb{N}$ and $\{r_j\}_{j\le n}\mathbb{Q}^n \setminus \mathbf{0^n}$, there holds*

$$\sum_{j=1}^{n} r_j B_{j,j} \neq 0.$$

Theorem 3.7. *Let B satisfy Assumptions II and Assumptions A. Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ and $\{\psi_j^1\}_{j\in\mathbb{N}}$, $\{\psi_j^2\}_{j\in\mathbb{N}}, \subset H^3_{(0)}$ such that there exists $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi_j^1\}_{j\in\mathbb{N}} = \{\psi_j^2\}_{j\in\mathbb{N}}$. If*

$$\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}, \tag{3.9}$$

then for any $N \in \mathbb{N}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi^3)\ \psi_j^2 = \pi_N(\Psi^3)\ \Gamma_T^u \psi_j^1, \qquad j \in \mathbb{N}.$$

Proof. See Paragraph 3.3. □

As Corollary 3.6 follows from Theorem 3.5, the next corollary can be deduced from Theorem 3.7.

Corollary 3.8. *Let B satisfy Assumptions II and Assumptions A. Let $\Psi^1 := \{\psi_j^1\}_{j\in\mathbb{N}}$, $\Psi^2 := \{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be unitarily equivalent. For any $N \in \mathbb{N}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\begin{cases} \Gamma_T^u \psi_j^1 = \psi_j^2, & j \leq N, \\ \pi_N(\Psi^2)\ \Gamma_T^u \psi_j^1 = \pi_N(\Psi^2)\ \psi_j^2, & j > N. \end{cases}$$

3.2 Simultaneous locale exact controllability in projection for $T > 0$

3.2.1 Preliminaries

In this paragraph, we discuss the simultaneous local exact controllability in projection. We explain first why we modify the problem.

Let $\Phi = \{\phi_j\}_{j\in\mathbb{N}}$ be an Hilbert basis composed by eigenfunctions of A. We start by studying the local exact controllability in projection in $O_{\epsilon,T}$ with respect to $\pi_N(\Phi)$. We would like to adopt a similar technique of the one adopted in the proof of Theorem 2.8. Let

$$\Gamma_t^u \psi_j = \sum_{k=1}^{\infty} \phi_k(T) \langle \phi_k(T), \Gamma_t^u \phi_j \rangle$$

be the solution of the j-th problem of (3.1). We consider the map $\alpha(u)$ as the infinite matrix with elements

$$\alpha_{k,j}(u) = \langle \phi_k(T), \Gamma_T^u \phi_j \rangle, \qquad k, j \in \mathbb{N}, \quad k \leq N.$$

Our goal is to prove the existence of $\epsilon > 0$ such that for any $\{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Phi)\Gamma_T^u \phi_j = \pi_N(\Phi)\psi_j, \quad \forall j \in \mathbb{N}.$$

This outcome is equivalent to the local surjectivity of the map α for $T > 0$. To this end, we want to use the Generalized Inverse Function Theorem

(Proposition 2.7) and we study the surjectivity of Fréchet derivative of α, $\gamma(v) := (d_u\alpha(0)) \cdot\ v$. The map γ is the infinite matrix with elements

$$\begin{aligned}\gamma_{k,j}(v) :&= \left\langle \phi_k(T), -i\int_0^T e^{-iA(T-s)}v(s)Be^{-iAs}\phi_j ds\right\rangle \\ &= -i\int_0^T v(s)e^{-i(\lambda_j-\lambda_k)s}dsB_{k,j}, \qquad k\leq N,\ j\in\mathbb{N},\end{aligned}$$

for $B_{k,j} = \langle\phi_k, B\phi_j\rangle = \langle B\phi_k, \phi_j\rangle = \overline{B_{j,k}}$. The surjectivity of γ consists in proving the solvability of the moment problem

$$\frac{x_{k,j}}{B_{k,j}} = -i\int_0^T u(s)e^{-i(\lambda_j-\lambda_k)s}ds, \tag{3.10}$$

for each infinite matrix x, with elements $x_{k,j}$, belonging to a suitable space. One would use Haraux Theorem as explained in Remark $A.9$ (Theorem $A.6$, Appendix $A.1$) but the eigenvalues resonances occur: for some $j,k,n,m\in\mathbb{N}$, $(j,k)\neq(n,m)$ and $k,m\leq N$, there holds $\lambda_j-\lambda_k=\lambda_n-\lambda_m$, which implies

$$\begin{aligned}\frac{x_{k,j}}{B_{k,j}} &= -i\int_0^T u(s)e^{-i(\lambda_j-\lambda_k)s}ds \\ &= -i\int_0^T u(s)e^{-i(\lambda_n-\lambda_m)s}ds = \frac{x_{n,m}}{B_{n,m}}.\end{aligned}$$

An example is $\lambda_7-\lambda_1=\lambda_8-\lambda_4$, but they also appear for all the diagonal terms of γ since $\lambda_j-\lambda_k=0$ for $j=k$.

We avoid the problem by adopting the following procedure.

- We decompose

  $$A+u(t)B=(A+u_0B)+u_1(t)B$$

  for $u_0\in\mathbb{R}$ and $u_1\in L^2((0,T),\mathbb{R})$. We consider $A+u_0B$ instead of A and we modify the eigenvalues gaps by using u_0B as a perturbating term in order to remove all the non-diagonal resonances.

- We redefine α in a map $\widehat{\alpha}$ depending on the parameter u_0. We introduce α^{u_0} by acting phase-shifts in order to remove the resonances on the diagonal terms

  $$\widetilde{\psi}_j(t,x)=\frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|}\psi_j(t,x)\Longrightarrow\alpha^{u_0}_{k,j}(u)=\frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|}\widehat{\alpha}_{k,j}(u). \tag{3.11}$$

3.2.2 The modified problem

Let $N \in \mathbb{N}$ and $u(t) = u_0 + u_1(t)$, for u_0 and $u_1(t)$ real. We introduce the following Cauchy problem

(3.12)
$$\begin{cases} i\partial_t \psi_j(t) = (A + u_0 B)\psi_j(t) + u_1(t) B \psi_j(t), & t \in (0,T),\ j \in \mathbb{N}, \\ \psi_j^0 = \psi_j(0). \end{cases}$$

Its solutions are $\psi_j(t) = \Gamma_t^{u_0+u_1}\psi_j^0$, where $\Gamma_t^{u_0+u_1}$ is the unitary propagator of the dynamics, which is equivalent to the one of the problems (3.1).

Remark 3.9. *A bounded perturbation of an operator with compact resolvent is an operator with compact resolvent. Thus, $A + u_0 B$ has pure discrete spectrum.*

Due to Remark 3.9, we call $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ the eigenvalues of $A + u_0 B$ that correspond to an Hilbert basis composed by eigenfunctions $\Phi^{u_0} := \{\phi_j^{u_0}\}_{j\in\mathbb{N}}$. We set

$$\phi_j^{u_0}(T) := e^{-i\lambda_j^{u_0}T}\phi_j^{u_0}.$$

Let us introduce the following space

(3.13)
$$O^{u_0}_{\epsilon_0,T} := \Big\{ \{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)} \Big|\ \langle \psi_j, \psi_k\rangle = \delta_{j,k};\ \sup_{j\in\mathbb{N}} \|\psi_j - \phi_j^{u_0}(T)\|_{(3)} < \epsilon_0 \Big\}.$$

In addition, we choose $|u_0|$ small enough such that $\lambda_k^{u_0} \neq 0$ for every $k \in \mathbb{N}$ (Lemma B.6, Appendix B.1). The introduction of the new Hilbert basis imposes to define

$$\widetilde{H}^3_{(0)} := D(|A + u_0 B|^{\frac{3}{2}}), \qquad \|\cdot\|_{\widetilde{H}^3_{(0)}} = \Big(\sum_{k=1}^{\infty} \big||\lambda_k^{u_0}|^{\frac{3}{2}}\langle \cdot, \phi_k\rangle\big|^2\Big)^{\frac{1}{2}}. \tag{3.14}$$

However, from now on, due to Lemma B.8 (Appendix B.1),

$$\widetilde{H}^3_{(0)} \equiv H^3_{(0)}.$$

We define $\widehat{\alpha}$, the infinite matrices with elements for $k \leq N$ and $j \in \mathbb{N}$ such that $\widehat{\alpha}_{k,j}(u_1) = \langle \phi_k^{u_0}(T), \Gamma_T^{u_0+u_1}\phi_j^{u_0}\rangle$ and the map α^{u_0} with elements

$$\begin{cases} \alpha^{u_0}_{k,j}(u_1) = \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}\widehat{\alpha}_{k,j}(u_1), & j,k \leq N, \\ \alpha^{u_0}_{k,j}(u_1) = \widehat{\alpha}_{k,j}(u_1), & j > N,\ k \leq N. \end{cases} \tag{3.15}$$

Now, for $j \in \mathbb{N}$,

$$\pi_N(\Phi^{u_0})e^{i\theta_j}\Gamma_T^{u_0+u_1}\phi_j^{u_0} = \sum_{k=1}^N \phi_k^{u_0}(T)\alpha_{k,j}^{u_0}(u_1), \qquad e^{i\theta_j} := \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}. \tag{3.16}$$

Thus, the local surjectivity of the map α^{u_0} in a suitable space is equivalent to the simultaneous local exact controllability in projection up to N phases on $O_{\epsilon_0,T}^{u_0}$ for a suitable $\epsilon_0 > 0$.

Let $\gamma^{u_0}(v) = ((d_{u_1}\alpha^{u_0})(0))\cdot v$ be the Fréchet derivative of α^{u_0} and $B_{k,j}^{u_0} = \langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle$ for $k \leq N$ and $j \in \mathbb{N}$. Defined $\widehat{\gamma}_{k,j}(v) = ((d_{u_1}\widehat{\alpha})(0))\cdot v$, we compute $\gamma^{u_0}(v)$ such that

$$\begin{cases} \gamma_{k,j}^{u_0} = \big(\overline{\widehat{\gamma}_{j,j}}\delta_{k,j} + \widehat{\gamma}_{k,j} - \delta_{k,j}\Re(\widehat{\gamma}_{j,j})\big), & j,k \leq N, \\ \gamma_{k,j}^{u_0} = \widehat{\gamma}_{k,j}, & k \leq N,\ j > N. \end{cases}$$

Thus for $k \leq N$ and $j \in \mathbb{N}$,

$$\begin{cases} \gamma_{k,j}^{u_0} = \widehat{\gamma}_{k,j} = -i\int_0^T u_1(s)e^{-i(\lambda_j^{u_0}-\lambda_k^{u_0})s}ds B_{k,j}^{u_0}, & k \neq j, \\ \gamma_{k,k}^{u_0} = \Re(\widehat{\gamma}_{k,k}) = 0, & k = j. \end{cases} \tag{3.17}$$

The relation $\gamma_{k,k}^{u_0} = 0$ comes from the fact that $(i\widehat{\gamma}_{k,k}) \in \mathbb{R}$ since $\widehat{\gamma}_{k,j} = -\overline{\widehat{\gamma}_{j,k}}$ for $j,k \leq N$. Due to (3.11), the diagonal elements of γ^{u_0} are all 0.

Remark. *For every $\{f_k\}_{k\in\mathbb{N}} \in O_{\epsilon_0,T}^{u_0}$ (see (3.13)), we know that*

$$\langle f_k, f_j\rangle = \delta_{k,j}$$

for every $j,k \in \mathbb{N}$. Let $\mathbf{f_t} = \{f_j(t)\}_{j\in\mathbb{N}} : (0,\epsilon) \to O_{\epsilon_0,T}^{u_0}$ be a smooth curve for $\epsilon > 0$ such that, for every $j,k \in \mathbb{N}$,

$$\mathbf{f_0} = \Phi^{u_0} = \{\phi_j^{u_0}\}_{j\in\mathbb{N}}, \qquad \Big(\frac{d}{dt}\mathbf{f_t}\Big)(t=0) = \mathbf{v} = \{v_j\}_{j\in\mathbb{N}}.$$

We notice that

$$0 = \frac{d\langle f_j(t), f_k(t)\rangle}{dt}(0) = \langle v_j, \phi_k^{u_0}\rangle + \langle \phi_j^{u_0}, v_k\rangle,$$

which implies $\langle \phi_k^{u_0}, v_j\rangle = -\overline{\langle \phi_j^{u_0}, v_k\rangle}$. Thus, we can define the tangent space to $O_{\epsilon_0,T}^{u_0}$ at Φ^{u_0} as follows

$$T_{\Phi^{u_0}}O_{\epsilon_0,T}^{u_0} = \big\{\{\psi_j\}_{j\in\mathbb{N}} \subset \ell^\infty(H_{(0)}^3)\big|\ \langle \phi_k^{u_0}, \psi_j\rangle = -\overline{\langle \phi_j^{u_0}, \psi_k\rangle}\big\}.$$

We have $T_{\Phi^{u_0}}O^{u_0}_{\epsilon_0,T}\subset\ell^\infty(H^3_{(0)})$ *since* $\sup_{j\in\mathbb{N}}\|\psi_j-\phi_j^{u_0}\|_{(3)}\leq\epsilon_0$ *for every* $\{\psi_j\}_{j\in\mathbb{N}}\in O^{u_0}_{\epsilon_0,T}$*. Moreover, for every* $k\in\mathbb{N}$*, thanks to Lemma* $B.8$*, there exists* $C>0$ *such that*

$$\sum_{j=1}^{+\infty}j^6|\alpha^{u_0}_{k,j}|^2=\sum_{j=1}^{+\infty}j^6|\langle\widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T\phi_k^{u_0},\phi_j^{u_0}\rangle|^2=\|\widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T\phi_k^{u_0}\|^2_{\widetilde{H}^3_{(0)}}\leq C\|\widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T\phi_k^{u_0}\|^2_{(3)}<\infty$$

as the propagator $\widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T$ *(defined in Paragraph* 2.1*) preserves* $H^3_{(0)}$*. Hence,* $\{\alpha^{u_0}_{k,j}\}_{j\in\mathbb{N}}\in h^3(\mathbb{C})$ *for every* $k\in\mathbb{N}$ *and the maps* α^{u_0} *and* γ^{u_0} *take respectively values in*

$$\begin{aligned}Q^N:=&\Big\{\{x_{k,j}\}_{\substack{k,j\in\mathbb{N}\\k\leq N}}\in(h^3(\mathbb{C}))^N\Big|\ x_{k,k}\in\mathbb{R},\quad k\leq N\Big\},\\G^N:=&\Big\{\{x_{k,j}\}_{\substack{k,j\in\mathbb{N}\\k\leq N}}\in(h^3(\mathbb{C}))^N\Big|\ \ x_{k,j}=-\overline{x_{j,k}},\ x_{k,k}=0\ j,k\leq N\Big\}.\end{aligned}$$

3.2.3 Proof of Theorem 3.4

In the next proposition, we ensure the simultaneous local exact controllability in projection for any $T>0$ up to phases.

Proposition 3.10. *Let* $N\in\mathbb{N}$ *and* B *satisfy Assumptions I. For every* $T>0$*, there exist* $\epsilon>0$ *and* $u_0\in\mathbb{R}$ *such that, for any* $\{\psi_j\}_{j\in\mathbb{N}}\in O_{\epsilon,T}$ *satisfying the relation* (3.5)*, there exist a sequence of real numbers* $\{\theta_j\}_{j\in\mathbb{N}}=\big\{\{\widehat{\theta}_j\}_{j\leq N},0,...\big\}$ *and* $u\in L^2((0,T),\mathbb{R})$ *such that*

$$\pi_N(\Phi^{u_0})\psi_j=\pi_N(\Phi^{u_0})e^{i\theta_j}\Gamma^u_T\phi_j^{u_0},\qquad\forall j\in\mathbb{N}.$$

Proof. **1)** Let u_0 belong to the neighborhoods defined in Appendix $B.1$ by Lemma $B.6$, Lemma $B.7$, Lemma $B.8$ and Remark $B.11$.
First, the relation (3.5) is required for the following reason. Let

$$\{\Gamma^u_T\phi_j^{u_0}\}_{j\in\mathbb{N}}=\{\widehat{\Gamma}\phi_j\}_{j\in\mathbb{N}}$$

for $T>0$, $u\in L^2((0,T),\mathbb{R})$ and $\widehat{\Gamma}\in U(\mathscr{H})$. For $|u_0|$ small enough, thanks to Lemma $B.6$ (Appendix $B.1$), there exists $C_1>0$ such that

$$j^6\leq C_1|\lambda_j^{u_0}|^3.$$

On the one hand, thanks to Lemma $B.8$ (Appendix $B.1$), there exists $C_2>0$ such that, for every $k\in\mathbb{N}$,

$$\sum_{j=1}^{+\infty}j^6|\langle\phi_k,\Gamma^u_T\phi_j^{u_0}\rangle|^2=\sum_{j=1}^{+\infty}j^6|\langle(\Gamma^u_T)^*\phi_k,\phi_j^{u_0}\rangle|^2\leq C_1\|\widetilde{\Gamma}^{\widetilde{u}}_T\phi_k\|^2_{\widetilde{H}^3_{(0)}}\leq C_1C_2\|\widetilde{\Gamma}^{\widetilde{u}}_T\phi_k\|^2_{(3)}<\infty.$$

On the other hand, for every $k \in \mathbb{N}$,

$$\sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j^{u_0}\rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \widehat{\Gamma}\phi_j\rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \widehat{\Gamma}^* \phi_k, \phi_j\rangle|^2 = \|\widehat{\Gamma}^*\phi_k\|_{(3)}^2.$$

Second, thanks to the third point of Remark $B.11$ (Appendix $B.1$), the controllability in $O^{u_0}_{\epsilon_0,T}$ implies the controllability in $O_{\epsilon,T}$ for suitable $\epsilon > 0$. Indeed, if $|u_0|$ is small enough, then $\sup_{j\in\mathbb{N}} \|\phi_j - \phi_j^{u_0}\|_{(3)} \leq \epsilon_0$ (Remark $B.11$). For every $\{\psi_j\}_{j\in\mathbb{N}} \in O^{u_0}_{\epsilon_0,T}$, we have $\{\psi_j\}_{j\in\mathbb{N}} \in O_{2\epsilon_0,T}$ since

$$\sup_{j\in\mathbb{N}} \|\psi_j - \phi_j(T)\|_{(3)} \leq \sup_{j\in\mathbb{N}} \|\phi_j^{u_0} - \phi_j(T)\|_{(3)} + \sup_{j\in\mathbb{N}} \|\psi_j - \phi_j^{u_0}(T)\|_{(3)} \leq 2\epsilon_0.$$

Third, thanks to the discussion about the relation (3.16), the local surjectivity of the map α^{u_0} guarantees the simultaneous local exact controllability in projection up to phases (Definition 3.3) of (3.1) with initial state $\{\phi_j^{u_0}\}_{j\in\mathbb{N}}$ on $O^{u_0}_{\epsilon_0,T}$ for ϵ_0 small enough.
We consider Generalized Inverse function Theorem (see Proposition 2.7) since Q^N and G^N are real Banach spaces. If γ^{u_0} is surjective in G^N, then the local surjectivity of α^{u_0} in Q^N is ensured. The map γ^{u_0} is surjective when the following moment problem is solvable

$$\frac{x^{u_0}_{k,j}}{B^{u_0}_{k,j}} = -i\int_0^T u(s) e^{-i(\lambda_j^{u_0} - \lambda_k^{u_0})s} ds, \qquad j \in \mathbb{N},\ k \leq N,\ k \neq j \tag{3.18}$$

for every $\big\{x^{u_0}_{k,j}\big\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in G^N$. The equations of (3.18) for $k = j$ are redundant as $\gamma^{u_0}_{k,k} = 0$ and $x^{u_0}_{k,k} = 0$ for every $k \leq N$ and $\{x^{u_0}_{k,j}\}_{\substack{k,j\in\mathbb{N}\\ k\leq N}} \in G^N$. Thus, we prove the solvability of the moment problem for $j \neq k$ and $j = k = 1$. Now,

$$\big\{x^{u_0}_{k,j}\big\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (h^3)^N, \qquad \big\{\gamma^{u_0}_{k,j}\big\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (h^3)^N.$$

From Lemma $B.7$ (Appendix $B.1$), it follows

$$\big\{x^{u_0}_{k,j}/B^{u_0}_{k,j}\big\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (\ell^2(\mathbb{C}))^N, \qquad \big\{\gamma^{u_0}_{k,j}/B^{u_0}_{k,j}\big\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (\ell^2(\mathbb{C}))^N.$$

Thanks to Lemma $B.10$ (Appendix $B.1$), for I^N defined in (3.2), there exist

$$\mathscr{G}' := \inf_{\substack{(j,k),(n,m)\in I^N\\ (j,k)\neq(n,m)}} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > 0,$$

$$\mathscr{G} := \sup_{A\subset I^N}\Big(\inf_{\substack{(j,k),(n,m)\in I^N\setminus A\\ (j,k)\neq(n,m)}} |\lambda_j^{u_0}-\lambda_k^{u_0}-\lambda_n^{u_0}+\lambda_m^{u_0}|\Big)\geq \mathscr{G}',$$

where A runs over the finite subsets of I^N. Hence, for $T>\frac{2\pi}{\mathscr{G}}$, Haraux Theorem (Theorem $A.6$, Appendix $A.1$) implies the solvability of the moment problem (3.18) (as explained in Remark $A.9$) by considering the sequence of numbers

$$\{\lambda_j^{u_0}-\lambda_k^{u_0}\}_{\substack{j,k\in\mathbb{N},\ k\leq N\\ j\neq k \text{ or } j=k=1}}.$$

Indeed, $x_{1,1}^{u_0}=0$ and Remark $B.11$ ensures that $\lambda_j^{u_0}-\lambda_k^{u_0}\neq\lambda_l^{u_0}-\lambda_m^{u_0}$ for every $j,k,l,m\in\mathbb{N}$. The proof is achieved since α^{u_0} is locally surjective for $T>0$ large enough.

2) Now, we show that the first point is valid for every $T>0$ by proving that $\mathscr{G}=+\infty$. Let

$$A^M := \{(j,n)\in\mathbb{N}^2|\ j,n\geq M;\ j\neq n\}$$

for $M\in\mathbb{N}$. Thanks to the relation $(B.4)$ in the proof of Lemma $B.6$ (Appendix $B.1$), for $|u_0|$ small enough and for every $K\in\mathbb{R}$, there exists $M_K>0$ large enough such that

$$\inf_{(j,n)\in A^{M_K}}|\lambda_j^{u_0}-\lambda_n^{u_0}|>K.$$

Indeed, the relation $(B.4)$ implies that, for $|u_0|$ small enough,

$$\begin{aligned}|\lambda_j^{u_0}-\lambda_n^{u_0}| &\geq |\lambda_j-\lambda_n|-O(|u_0|)\geq 2\pi^2\min\{\lambda_{j+1}-\lambda_j,\lambda_{n+1}-\lambda_n\}-O(|u_0|)\\ &\geq 2\pi^2\min\{j,n\}-O(|u_0|).\end{aligned}$$

Thus

$$\mathscr{G}\geq\sup_{M\in\mathbb{N}}\big(\inf_{(j,n)\in A^M}|\lambda_j^{u_0}-\lambda_n^{u_0}|-2\lambda_N^{u_0}\big)>0.$$

Now, for $|u_0|$ small enough, Lemma $B.6$ (Appendix $B.1$) implies the existence of $C>0$ such that

$$\begin{aligned}\mathscr{G} &\geq C\big(\lim_{M\to\infty}\inf_{(j,n)\in A^M}|\lambda_j-\lambda_n|-2\lambda_N\big)\\ &\geq C\lim_{M\to\infty}(\lambda_{M+2}-\lambda_{M+1}-2N^2\pi^2)=+\infty.\end{aligned}$$

□

3.3 Simultaneous global exact controllability in projection

The common approach adopted in order to prove the global exact controllability (also simultaneous) consists in gathering the global approximate controllability and the local exact controllability.
However, this strategy can not be used to prove the controllability in projection as the propagator Γ_T^u does not preserve the space $\pi_N(\Psi)H^3_{(0)}$ for any $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$, making impossible to reverse and concatenate dynamics.
We adopt an alternative strategy that we call "transposition argument" (see remark below). In particular, under suitable assumptions, we prove that the controllability in projection onto an N dimensional space is equivalent to the controllability of N problems (without projecting).

Remark 3.11. *The time reversibility (Paragraph* 2.1*) implies that, for every* $j,k\in\mathbb{N}$*,*

$$\begin{aligned}\overline{\langle\phi_k^{u_0}(T),\Gamma_T^u\phi_j^{u_0}\rangle} &= e^{-i\lambda_k^{u_0}T}\langle\Gamma_T^u\phi_j^{u_0},\phi_k^{u_0}\rangle = e^{-i\lambda_k^{u_0}T}\langle\phi_j^{u_0},(\Gamma_T^u)^*\phi_k^{u_0}\rangle \\ &= e^{-i(\lambda_k^{u_0}+\lambda_j^{u_0})T}\langle\phi_j^{u_0}(T),\widetilde{\Gamma}_T^{\widetilde{u}}\phi_k^{u_0}\rangle.\end{aligned}\tag{3.19}$$

Now, $e^{-i(\lambda_k^{u_0}+\lambda_j^{u_0})T}$ *does not depend on* u *and the relation* (3.19) *implies that the surjectivity of the map*

$$\{\langle\phi_k^{u_0}(T),\Gamma_T^u\phi_j^{u_0}\rangle\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} : L^2((0,T),\mathbb{R}) \longrightarrow \{\{x_{k,j}\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} :\ \{x_{k,j}\}_{j\in\mathbb{N}}\in h^3(\mathbb{C}),\ \forall k\leq N\}\tag{3.20}$$

is equivalent to the surjectivity of

$$\{\langle\phi_j^{u_0}(T),\widetilde{\Gamma}_T^{\widetilde{u}}\phi_k^{u_0}\rangle\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} : L^2((0,T),\mathbb{R}) \longrightarrow \{\{x_{j,k}\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} :\ \{x_{j,k}\}_{j\in\mathbb{N}}\in h^3(\mathbb{C}),\ \forall k\leq N\}.\tag{3.21}$$

As explained in Paragraph 3.2.1*, the decomposition*

$$\pi_N(\Phi^{u_0})\Gamma_T^u\phi_j^{u_0} = \sum_{k=1}^N \phi_k^{u_0}(T)\langle\phi_k^{u_0}(T),\Gamma_T^u\phi_j^{u_0}\rangle, \qquad \forall j\in\mathbb{N},$$

ensures that the surjectivity of the map (3.20) *is equivalent to simultaneous global exact controllability in projection of the problems* (3.1)*. From the same idea, the surjectivity of* (3.21) *is equivalent to the surjectivity of the map*

$$\{\widetilde{\Gamma}_T^{(\cdot)}\phi_k^{u_0}\}_{k\leq N} : L^2((0,T),\mathbb{R}) \longrightarrow (H^3_{(0)})^N,$$

which implies the simultaneous global exact controllability of N problems (2.4) *or* (3.1) *(as* (2.4) *represents the reversed dynamics of* (3.1)*). For this reason, the simultaneous global exact controllability in projection onto a suitable N dimensional space is equivalent to the controllability of N problems (without projecting).*

The transposition argument is particularly important as it allows to concatenate and reverse dynamics on $(H^3_{(0)})^N$, which is preserved by the propagator when one wants to prove the controllability in projection.
For the simultaneous local exact controllability result, we can use Proposition 3.10 with the transposition argument, but this is not always the most convenient approach. Indeed, when B satisfies Assumptions A, we consider [MN15, *Theorem* 4.1] that requires stronger assumptions on the operator B but provides the result without phase ambiguities (as in Theorem 3.5).

3.3.1 Approximate simultaneous controllability

In this paragraph, we prove the simultaneous global approximate controllability of the problems (3.1).

Definition 3.12. Let (A, B) be the couples of operator introduced in the problem (3.1). A subset of $\mathbb{N}^2$ is said to couple two levels j and k in $\mathbb{N}$, if there exists a finite sequence $((s_1^1, s_2^1), ..., (s_1^p, s_2^p))$ in S such that

1. $s_1^1 = j$ and $s_2^p = k$;
2. $s_2^l = s_1^{l+1}$ for every $1 \leq l \leq p-1$;
3. $\langle \phi_{s_1^l}, B\phi_{s_2^l} \rangle \neq 0$ for $1 \leq l \leq p$.

S is called a connectedness chain (respectively m-connectedness chain) if S (respectively $S \cap \{1, ..., m^2\}$) couples every pair of levels in $\mathbb{N}$ (respectively $\{1, ..., m\}$).
The couples (A, B) admits a connectedness chain, which is said non-degenerate if, for every (s_1, s_2) in S, such that $B_{s_1,s_2} \neq 0$ and $|\lambda_{s_1} - \lambda_{s_2}| = |\lambda_m - \lambda_l|$ with $m, l \in \mathbb{N}$ implies $\{s_1, s_2\} = \{m, l\}$ or $B_{m,l} = 0$.

Definition 3.13. The problems (3.1) are said to be simultaneously globally approximately controllable in $H^s_{(0)}$ if, for every $N \in \mathbb{N}$, $\psi_1, ..., \psi_N \in H^s_{(0)}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi_1, ..., \widehat{\Gamma}\psi_N \in H^s_{(0)}$ and $\epsilon > 0$, then there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that, for every $1 \leq k \leq N$,

$$\|\widehat{\Gamma}\psi_k - \Gamma^u_T \psi_k\|_{(s)} < \epsilon.$$

Theorem 3.14. *Let B satisfy Assumptions II. The problems* (3.1) *are simultaneously globally approximately controllable in $H^3_{(0)}$.*

Proof. Let $N \in \mathbb{N}$ and u_0 belong to the neighborhoods provided by Remark B.9 and Remark B.11 (Appendix B.1). We define the norms $\||\cdot\||_{(s)} := \||\cdot\||_{L(H^s_{(0)},H^s_{(0)})}$ and

$$\|f\|_{BV(T)} := \|f\|_{BV((0,T),\mathbb{R})} = \sup_{\{t_j\}_{0\leq j\leq n}\in P} \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|,$$

where $f \in BV((0,T),\mathbb{R})$ and P is the set of the partitions of $(0,T)$ such that $t_0 = 0 < t_1 < ... < t_n = T$.

We aim to prove that for every $N \in \mathbb{N}$, $\psi_1, ..., \psi_N \in H^3_{(0)}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi_1, ..., \widehat{\Gamma}\psi_N \in H^3_{(0)}$ and $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that, for every $1 \leq k \leq N$,

$$\|\widehat{\Gamma}\psi_k - \Gamma^u_T\psi_k\|_{(3)} < \epsilon.$$

We consider the techniques adopted in the proof of Proposition 2.12 and developed by Chambrion in [Cha12]. We start by choosing

$$\psi_j = \phi_j, \qquad \forall j \in \{1, ..., N\}.$$

Let π_m be the orthogonal projector

$$\pi_m : \mathscr{H} \to \mathscr{H}_m := \overline{span\{\phi_j \ : \ j \leq m\}}^{L^2}, \qquad \forall m \in \mathbb{N}.$$

The couple $(A + u_0B, B)$ admits a non-degenerate chain of connectedness thanks to Remark B.11 (Appendix B.1). Up to a reordering of $\{\phi_k\}_{k\in\mathbb{N}}$, we can assume that for every $m \in \mathbb{N}$, the couple $(\pi_m(A+u_0B)\pi_m, \pi_mB\pi_m)$ admits a non-degenerate chain of connectedness in $\mathscr{H}_m$.

1) Preliminaries

Claim. For every $\epsilon > 0$, there exist $N_1 \in \mathbb{N}$ and $\widetilde{\Gamma}_{N_1} \in U(\mathscr{H})$ such that $\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} \in SU(\mathscr{H}_{N_1})$ and

$$\|\widetilde{\Gamma}_{N_1}\phi_j - \widehat{\Gamma}\phi_j\|_{(3)} < \epsilon, \qquad \forall j \leq N. \tag{3.22}$$

Let $N'\in\mathbb{N}$ be such that $N'\geq N$. We apply the Gram-Schmidt process to $\{\pi_{N'}\widehat{\Gamma}\phi_j\}_{j\leq N}$. For

$$\varphi_1:=\pi_{N'}\widehat{\Gamma}\phi_1,\qquad \varphi_j:=\pi_{N'}\widehat{\Gamma}\phi_j-\sum_{k=1}^{j-1}\langle\pi_{N'}\widehat{\Gamma}\phi_j,\varphi_k\rangle\varphi_k,\qquad \forall\ 2\leq j\leq N,$$

we denote $\widetilde{\phi}_j:=\frac{\varphi_j}{\|\varphi_j\|}$ for every $j\leq N$. We complete $\{\widetilde{\phi}_j\}_{j\leq N}$ in an orthonormal basis of $\mathscr{H}_{N'}$ that we call $\{\widetilde{\phi}_j\}_{j\leq N'}$. The operator $\widetilde{\Gamma}_{N'}$ is the unitary map such that

$$\widetilde{\Gamma}_{N'}\phi_j=\widetilde{\phi}_j,\qquad\qquad \forall j\leq N'.$$

As $\lim_{N'\to\infty}\|\pi_{N'}\widehat{\Gamma}\phi_k\|=\|\widehat{\Gamma}\phi_k\|=1$ and $\widehat{\Gamma}\phi_k\in H^3_{(0)}$ for every $k\leq N'$, we have

$$\lim_{N'\to\infty}\|\widetilde{\Gamma}_{N'}\phi_1-\widehat{\Gamma}\phi_1\|_{(3)}^2=\lim_{N'\to\infty}\left\|\frac{\pi_{N'}\widehat{\Gamma}\phi_1}{\|\pi_{N'}\widehat{\Gamma}\phi_1\|}-\widehat{\Gamma}\phi_1\right\|_{(3)}^2$$

$$=\lim_{N'\to\infty}\sum_{l=1}^{N'}\left|k^3\left\langle\frac{\widehat{\Gamma}\phi_1}{\|\pi_{N'}\widehat{\Gamma}\phi_1\|}-\widehat{\Gamma}\phi_1,\phi_l\right\rangle\right|^2+\sum_{l=N'+1}^{\infty}\left|k^3\left\langle\widehat{\Gamma}\phi_1,\phi_l\right\rangle\right|^2=0.$$

Equivalently, since $\lim_{N'\to\infty}\langle\pi_{N'}\widehat{\Gamma}\phi_j,\pi_{N'}\widehat{\Gamma}\phi_k\rangle=\delta_{j,k}$ for every $j,k\leq N$, there follows

$$\lim_{N'\to\infty}\|\widetilde{\Gamma}_{N'}\phi_j-\widehat{\Gamma}\phi_j\|_{(3)}^2=0,\qquad \forall j\leq N.$$

Thus, for every $\epsilon>0$, there exists $N'\in\mathbb{N}$ large enough such that

$$\|\widetilde{\Gamma}_{N'}\phi_j-\widehat{\Gamma}\phi_j\|_{(3)}<\epsilon,\qquad \forall j\leq N. \tag{3.23}$$

From now on, we denote N_1 the number $N'\geq N$ such that the relation (3.23) is verified.

2) Finite dimensional controllability

We denote T_{ad} the set of the admissible transitions, *i.e.* the couples $(j,k)\in\{1,...,N_1\}^2$ such that $B_{j,k}\neq 0$ and $|\lambda_j-\lambda_k|=|\lambda_m-\lambda_l|$ with $m,l\in\mathbb{N}$ implies $\{j,k\}=\{m,l\}$ or $B_{m,l}=0$.
For every $(j,k)\in\{1,...,N_1\}^2$ and $\theta\in[0,2\pi)$, we define $E^{\theta}_{j,k}$ the $N_1\times N_1$ matrix with elements

$$(E^{\theta}_{j,k})_{l,m}=0,\qquad (E^{\theta}_{j,k})_{j,k}=e^{i\theta},\qquad (E^{\theta}_{j,k})_{k,j}=-e^{-i\theta},$$

for $(l,m)\in\{1,...,N_1\}^2\setminus\{(j,k),(k,j)\}$. We call

$$E_{ad}=\big\{E^{\theta}_{j,k}\ :\ (j,k)\in T_{ad},\ \theta\in[0,2\pi)\big\}.$$

Let $Mat_{N_1\times N_1}$ be the space of the $N_1\times N_1$ matrices. For every $M_1,M_2\in Mat_{N_1\times N_1}$, we define $[M_1,M_2]=M_1M_2-M_2M_1$. Let $F,G\subseteq Mat_{N_1\times N_1}$. We denote

$$[F,G]=\big\{M\in Mat_{N_1\times N_1}\mid \exists M_1\in F,\ \exists M_2\in G\ :\ M=[M_1,M_2]\big\}.$$

Let $E_1=E_{ad}$ and $E_j=[E_{ad},E_{j-1}]+E_{j-1}$ for every $j\in\mathbb{N}$ so that $j\geq 2$. As the elements of E_{ad} are $N_1\times N_1$ matrices, we know that there exists $\widetilde{m}\in\mathbb{N}$ such that $\dim(E_{m+1})=\dim(E_m)$ for every $m\geq\widetilde{m}$. We call

$$Lie(E_{ad})=E_{\widetilde{m}}.$$

We introduce the following finite dimensional control system on $SU(\mathscr{H}_{N_1})$

$$\begin{cases}\dot{x}(t)=x(t)v(t), & t\in(0,\tau),\\ x(0)=Id_{SU(\mathscr{H}_{N_1})}\end{cases}\tag{3.24}$$

where the set of admissible controls v is the set of piecewise constant functions taking value in E_{ad} and $\tau>0$.

> **Claim.** The problem (3.24) is controllable, *i.e.* for every $R\in SU(\mathscr{H}_{N_1})$, there exist $p\in\mathbb{N}$, $M_1,...,M_p\in E_{ad}$, $\alpha_1,...,\alpha_p\in\mathbb{R}^+$ such that
>
> $$R=e^{\alpha_1M_1}\circ...\circ e^{\alpha_pM_p}.$$

Thanks to [Sac00, *Theorem* 6.1], the controllability of (3.24) is equivalent to prove that $Lie(E_{ad})\supseteq su(\mathscr{H}_{N_1})$ for $su(\mathscr{H}_{N_1})$ the Lie algebra of $SU(\mathscr{H}_{N_1})$. For every $(j,k)\in\{1,...,N_1\}^2$, we define the following $N_1\times N_1$ matrices:

- $R_{j,k}$ is such that for $(l,m)\in\{1,...,N_1\}^2\setminus\{(j,k),(k,j)\}$,

$$(R_{j,k})_{l,m}=0,\qquad (R_{j,k})_{j,k}=-(R_{j,k})_{k,j}=1;$$

- $C_{j,k}$ is such that for $(l,m)\in\{1,...,N_1\}^2\setminus\{(j,k),(k,j)\}$,

$$(C_{j,k})_{l,m}=0,\qquad (C_{j,k})_{j,k}=(C_{j,k})_{k,j}=i;$$

- D_j is such that for $(l,m) \in \{1,...,N_1\}^2 \setminus \{(1,1),(j,j)\}$,

$$(D_j)_{l,m} = 0, \qquad (D_j)_{1,1} = -(D_j)_{j,j} = i.$$

Now, $\mathbf{e} := \{R_{j,k}\}_{j,k\leq N_1} \cup \{C_{j,k}\}_{j,k\leq N_1} \cup \{D_j\}_{j\leq N_1}$ is a basis of $su(\mathscr{H}_{N_1})$. In order to prove that $Lie(E_{ad}) \supseteq su(\mathscr{H}_{N_1})$, we show that each element of $\mathbf{e}$ belongs to $Lie(E_{ad})$.

- For every $(j,k) \in T_{ad}$, we have $R_{j,k} = E^0_{j,k}$ and $C_{j,k} = E^{\pi/2}_{j,k}$.
- For every $(j,k) \notin T_{ad}$ such that there exists $j_1 \leq N_1$ so that $(j,j_1),(j_1,k) \in T_{ad}$, we have $R_{j,k} = [E^0_{j,j_1}, E^0_{j_1,k}]$ and $C_{j,k} = [E^0_{j,j_1}, E^{\pi/2}_{j_1,k}]$.
- By repeating a finite number of times the previous point, we see that it is possible to generate each element $R_{j,k}$ and $C_{j,k}$ with $(j,k) \in \{1,...,N_1\}^2$. For every $(j,k) \notin T_{ad}$, there exist $m \leq N_1$ and $\{j_l\}_{l\leq m}$ such that
$$(j,j_1),...,(j_m,k) \in T_{ad}.$$
We call $S = \{(j,j_1),...,(j_m,k)\}$. The matrices $R_{j,k}$ and $C_{j,k}$ can be obtained by iterated Lie brackets of $E^\theta_{l,m}$ for $(l,m) \in S$ and $\theta \in [0,2\pi)$.
- If $(1,j) \in T_{ad}$, then $2D_j = [E^0_{1,j}, E^{\frac{\pi}{2}}_{1,j}]$, while if $(1,j) \notin T_{ad}$ and there exists $j_1 \leq N_1$ such that $(1,j_1),(j_1,j) \in T_{ad}$, then
$$-2D_j = \Big[[E^{\frac{\pi}{2}}_{1,j_1}, E^{\frac{\pi}{2}}_{j_1,j}], [E^0_{1,j_1}, E^{\frac{\pi}{2}}_{j_1,j}]\Big].$$
In conclusion, it is possible to obtain the matrices D_j for every $j \leq N_1$ by iterated Lie brackets of elements in E_{ad}.

Then, $Lie(E_{ad}) \supseteq su(\mathscr{H}_{N_1})$ and the controllability of (3.24) follows from [Sac00, $Theorem$ 6.1].

3) Finite dimensional estimates

Thanks to the previous claim and to the fact that $\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}$, $M_1,...,M_p \in E_{ad}$ and $\alpha_1,...,\alpha_p \in \mathbb{R}^+$ such that

$$\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}. \tag{3.25}$$

Claim. For every $l \leq p$ and for each rotation $e^{\alpha_l M_l}$ introduced in (3.25), there exist $\{T_n^l\}_{l\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{u_n^l\}_{n\in\mathbb{N}}$ such that $u_n^l : (0, T_n^l) \to \mathbb{R}$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\|_{(3)} = 0, \qquad \forall k \leq N_1, \tag{3.26}$$

$$\begin{gathered} \sup_{n\in\mathbb{N}} \|u_n^l\|_{BV(T_n)} < \infty, \qquad \sup_{n\in\mathbb{N}} \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})} < \infty, \\ \sup_{n\in\mathbb{N}} \big(T_n \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})}\big) < \infty. \end{gathered} \tag{3.27}$$

As in the proof of Proposition 2.12, we consider the results developed by Chambrion in [Cha12]. Indeed, $e^{\alpha_l M_l}$ is a rotation in a two dimensional space for every $l \in \{1, ..., p\}$. The mentioned work allows to explicit $\{T_n^l\}_{l\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{u_n^l\}_{n\in\mathbb{N}}$ such that $u_n^l : (0, T_n^l) \to \mathbb{R}$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|\pi_{N_1}\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\| = 0, \qquad \forall k \leq N_1,$$

$$\begin{gathered} \sup_{n\in\mathbb{N}} \|u_n^l\|_{BV(T_n)} < \infty, \qquad \sup_{n\in\mathbb{N}} \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})} < \infty, \\ \sup_{n\in\mathbb{N}} \big(T_n \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})}\big) < \infty. \end{gathered}$$

As $e^{\alpha_l M_l} \in SU(\mathscr{H}_{N_1})$ and $\Gamma_{T_n^l}^{u_n^l} \in U(\mathscr{H})$ for every $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \|\pi_{N_1}\Gamma_{T_n^l}^{u_n^l}\phi_k\| = \|e^{\alpha_l M_l}\phi_k\| = 1$ for every $k \leq N_1$. However, $\|\Gamma_{T_n^l}^{u_n^l}\phi_k\| = 1$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - \pi_{N_1}\Gamma_{T_n^l}^{u_n^l}\phi_k\|^2 = \lim_{n\to\infty} \sum_{m=N_1+1}^{\infty} |\langle\phi_m, \Gamma_{T_n^l}^{u_n^l}\phi_k\rangle|^2 = 0,$$

which implies

$$\lim_{n\to\infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\| = 0, \qquad \forall k \leq N_1.$$

In particular, for every $l \leq p$, there exist $(j,k) \in T_{ad}$ and $\theta \in [0, 2\pi)$ such that $M_l = E_{j,k}^\theta$. As in Chapter 2, we can choose

$$u_n^l(t) := \frac{\cos\big((k^2 - j^2)\pi^2 t + \nu\big)}{n},$$

$$T_n^l \in \left(n\frac{\pi}{|B_{k,j}|} - \frac{2\pi}{|\lambda_k - \lambda_j|}, n\frac{\pi}{|B_{k,j}|} + \frac{2\pi}{|\lambda_k - \lambda_j|}\right)$$

with $n \in \mathbb{N}$ and $\nu \in \mathbb{R}$ (ν is required in order to deal with the phase θ). Now, we consider the propagation of regularity adopted in the proof of Proposition 2.12 and developed by Kato in [Kat53]. For every $T > 0$, $u \in BV((0,T),\mathbb{R})$ and $\psi \in H^4_{(0)}$, there exists $C(K) > 0$ depending on

$$K = \big(\|u\|_{BV(T)}, \|u\|_{L^\infty((0,T),\mathbb{R})}, T\|u\|_{L^\infty((0,T),\mathbb{R})}\big)$$

such that $\|\Gamma^u_T\psi\|_{(4)} \leq C(K)\|\psi\|_{(4)}$. Then, thanks to (3.27), there exists a constant $C > 0$ such that

$$|\!|\!|\, \Gamma^{u^l_n}_{T^l_n} \,|\!|\!|_{(4)} \leq C.$$

The interpolation argument adopted in the proof of Proposition 2.12 and the relation (3.26) lead to

$$\lim_{n\to\infty} \|\Gamma^{u^l_n}_{T^l_n}\phi_k - e^{\alpha_l M_l}\phi_k\|_{(3)} = 0, \qquad\qquad \forall k \leq N_1.$$

4) Infinite dimensional estimates

Claim. There exist $K_1, K_2, K_3 > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\|\Gamma^u_T\phi_k - \widehat{\Gamma}\phi_k\|_{(3)} \leq \epsilon, \qquad \forall k \leq N,$$

$$\|u\|_{BV(T)} \leq K_1, \qquad \|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_2, \qquad T\|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_3.$$

Let us assume $p = 2$. However, the following result is valid for any $p \in \mathbb{N}$. Thanks to (3.26) and to the propagation of regularity from [Kat53], for every $\epsilon > 0$ and $N_1 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ large enough such that, for every $k \leq N$,

(3.28)
$$\begin{aligned}
&\|\Gamma^{u^2_n}_{T^2_n}\Gamma^{u^1_n}_{T^1_n}\phi_k - e^{\alpha_2 M_2}e^{\alpha_1 M_1}\phi_k\|_{(3)} \leq \|\Gamma^{u^2_n}_{T^2_n}\big(\Gamma^{u^1_n}_{T^1_n}\phi_k - e^{\alpha_1 M_1}\phi_k\big)\|_{(3)}\\
&+\|\big(\Gamma^{u^2_n}_{T^2_n} - e^{\alpha_2 M_2}\big)e^{\alpha_1 M_1}\phi_k\|_{(3)} \leq |\!|\!|\, \Gamma^{u^2_n}_{T^2_n} \,|\!|\!|_{(3)}\|\Gamma^{u^1_n}_{T^1_n}\phi_k - e^{\alpha_1 M_1}\phi_k\|_{(3)}\\
&+\sum_{l=1}^{N_1}\|\big(\Gamma^{u^2_n}_{T^2_n}\phi_l - e^{\alpha_2 M_2}\phi_l\big)\langle\phi_l, e^{\alpha_1 M_1}\phi_k\rangle\|_{(3)} \leq |\!|\!|\, \Gamma^{u^2_n}_{T^2_n} \,|\!|\!|_{(3)}\|\Gamma^{u^1_n}_{T^1_n}\phi_k - e^{\alpha_1 M_1}\phi_k\|_{(3)}\\
&+\|e^{\alpha_1 M_1}\phi_k\|\Big(\sum_{l=1}^{N_1}\|\big(\Gamma^{u^2_n}_{T^2_n}\phi_l - e^{\alpha_2 M_2}\phi_l\big)\|^2_{(3)}\Big)^{\frac{1}{2}} \leq \epsilon.
\end{aligned}$$

In the previous inequality, we considered that $e^{\alpha_1 M_1}\phi_k \in \mathscr{H}_{N_1}$ and that $\vert\!\vert\!\vert \Gamma_{T_n^2}^{u_n^2} \vert\!\vert\!\vert_{(3)}$ is uniformly bounded in $n \in \mathbb{N}$ thanks to the propagation of regularity from [Kat53] and to (3.27).

The relation (3.28) is valid for every $p \in \mathbb{N}$ and the identity (3.25) leads to the existence of $K_1, K_2, K_3 > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

(3.29)
$$\|u\|_{BV(T)} \leq K_1, \qquad \|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_2, \qquad T\|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_3,$$
$$\|\Gamma_T^u \phi_k - \widetilde{\Gamma}_{N_1}\phi_k\|_{(3)} < \epsilon, \qquad \forall k \leq N.$$

The relation (3.22) and the triangular inequality achieve the claim.

4) Conclusion

For every $\{\psi_j\}_{j\leq N} \subset H^3_{(0)}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi_j\}_{j\leq N} \subset H^3_{(0)}$ and $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for every $l \leq N$,

$$\|\psi_l\|_{(3)} \leq \left\|\sum_{k=1}^{M} \phi_k\langle\phi_k,\psi_l\rangle\right\|_{(3)}^2 + \epsilon, \qquad \|\widehat{\Gamma}\psi_l\|_{(3)} \leq \left\|\sum_{k=1}^{M} \widehat{\Gamma}\phi_k\langle\phi_k,\psi_l\rangle\right\|_{(3)}^2 + \epsilon.$$

The proof is achieved by simultaneously driving $\{\phi_k\}_{k\leq M}$ close enough to $\{\widehat{\Gamma}\phi_k\}_{k\leq M}$ since, for every $l \leq N$, $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ satisfying (3.29),

$$\begin{aligned}
\|\Gamma_T^u\psi_l - \widehat{\Gamma}\psi_l\|_{(3)} &\leq \left\|\sum_{k=1}^{M}(\Gamma_T^u\phi_k - \widehat{\Gamma}\phi_k)\langle\phi_k,\psi_l\rangle\right\|_{(3)} + (\,\vert\!\vert\!\vert \Gamma_T^u \vert\!\vert\!\vert_{(3)} + 1)\epsilon \\
&\leq \sum_{k=1}^{M}\left\|\Gamma_T^u\phi_k - \widehat{\Gamma}\phi_k\right\|_{(3)} |\langle\phi_k,\psi_l\rangle| + (\,\vert\!\vert\!\vert \Gamma_T^u \vert\!\vert\!\vert_{(3)} + 1)\epsilon \\
&\leq \|\psi_l\|\left(\sum_{k=1}^{M}\|\Gamma_T^u\phi_k - \widehat{\Gamma}\phi_k\|_{(3)}^2\right)^{\frac{1}{2}} + (\,\vert\!\vert\!\vert \Gamma_T^u \vert\!\vert\!\vert_{(3)} + 1)\epsilon.
\end{aligned}$$
□

3.3.2 Proofs of Theorem 3.5, Theorem 3.7 and Corollary 3.16

In the current paragraph, we provide the proofs of Theorem 3.5 and Theorem 3.7, which require the following proposition.

Proposition 3.15. *Let $N \in \mathbb{N}$ and B satisfy Assumptions II.*

1. *For any* $\{\psi_k^1\}_{k\leq N}$, $\{\psi_k^2\}_{k\leq N}\subset H^3_{(0)}$ *orthonormal systems, there exist* $T>0$, $u\in L^2((0,T),\mathbb{R})$ *and* $\{\theta_k\}_{k\leq N}\subset\mathbb{R}$ *such that*

$$e^{i\theta_k}\psi_k^2=\widetilde{\Gamma}_T^u\psi_k^1,\quad k\leq N.$$

2. *If* B *satisfies Assumptions A, then for any* $\{\psi_k^1\}_{k\leq N}$, $\{\psi_k^2\}_{k\leq N}\subset H^3_{(0)}$ *orthonormal systems, there exist* $T>0$ *and* $u\in L^2((0,T),\mathbb{R})$ *such that*

$$\psi_k^2=\widetilde{\Gamma}_T^u\psi_k^1,\quad k\leq N.$$

Proof. Let $N\in\mathbb{N}$ and let $u_0\in\mathbb{R}$ belong to the neighborhoods provided by Lemma $B.7$, Lemma $B.8$ and Remark $B.11$ (Appendix $B.1$).

1) Let $\widetilde{\alpha}^{u_0}$ be the map with elements

$$\begin{cases}\frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}\widehat{\alpha}_{k,j}(u_1), & j,k\leq N,\\ \widehat{\alpha}_{k,j}(u_1), & k>N,\ j\leq N.\end{cases}$$

The proof of Proposition 3.10 can be repeated in order to prove the local surjectivity of $\widetilde{\alpha}^{u_0}$ for every $T>0$, instead of α^{u_0} introduced in (3.15). As explained in Remark 3.11, this result corresponds to the simultaneous local exact controllability up to phase of N problems (3.1) in a neighborhood

$$O^N_{\epsilon,T}:=\Big\{\{\psi_j\}_{j\leq N}\subset H^3_{(0)}\Big|\ \langle\psi_j,\psi_k\rangle=\delta_{j,k};\ \ \sum_{j=1}^N\|\psi_j-\phi_j^{u_0}\|_{(3)}<\epsilon\Big\}$$

with $\epsilon>0$ small enough. In other words, for every $\{\psi_k\}_{k\leq N}\in O^N_{\epsilon,T}$, there exist $u\in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N}\subset\mathbb{R}$ such that

$$\Gamma_T^u\phi_j^{u_0}=e^{i\theta_j}\psi_j,\qquad\forall j\leq N.$$

Theorem 3.14 implies the simultaneous global approximate controllability for N problems. For any $\{\psi_j^1\}_{j\leq N}\subset H^3_{(0)}$ composed by orthonormal elements, there exist $T_1>0$ and $u_1\in L^2((0,T_1),\mathbb{R})$ such that

$$\|\Gamma_{T_1}^{u_1}\psi_j^1-\phi_j^{u_0}\|_{(3)}<\frac{\epsilon}{N}$$

for every $j\leq N$ and then

$$\{\Gamma_{T_1}^{u_1}\psi_j^1\}_{j\leq N}\in O^N_{\epsilon,T}.$$

The local controllability is also valid for the reversed dynamics of (2.4), for every $T>0$, there exist $u\in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N}\subset\mathbb{R}$ such that

$$\{\Gamma_{T_1}^{u_1}\psi_j^1\}_{j\leq N}=\{e^{i\theta_j}\widetilde{\Gamma}_T^u\phi_j^{u_0}\}_{j\leq N},$$

which implies

$$\{e^{-i\theta_j}\Gamma_T^{\widetilde{u}}\Gamma_{T_1}^{u_1}\psi_j^1\}_{j\leq N}=\{\phi_j^{u_0}\}_{j\leq N}.$$

Then, there exist $T_2>0$ and $u_2\in L^2((0,T_2),\mathbb{R})$ such that

$$\{e^{-i\theta_j}\Gamma_{T_2}^{u_2}\psi_j^1\}_{j\leq N}=\{\phi_j^{u_0}\}_{j\leq N}.$$

Now, the same property is valid for the reversed dynamics of (2.4) and, for every $\{\psi_j^2\}_{j\leq N}\subset H^3_{(0)}$ composed by orthonormal elements, there exist $T_3>0$, $u_3\in L^2((0,T_3),\mathbb{R})$ and $\{\theta_j'\}_{j\leq N}\subset\mathbb{R}$ such that

$$\{e^{-i\theta_j'}\widetilde{\Gamma}_{T_3}^{u_3}\psi_j^2\}_{j\leq N}=\{\phi_j^{u_0}\}_{j\leq N}.$$

In conclusion, for $\widetilde{u}_3(\cdot)=u_3(T_3-\cdot)$, the proof is achieved as

$$\{e^{-i(\theta_j-\theta_j')}\Gamma_{T_3}^{\widetilde{u}_3}\Gamma_{T_2}^{u_2}\psi_j^1\}_{j\leq N}=\{\psi_j^2\}_{j\leq N}.$$

2) The proof of the second claim follows as the previous one, with the difference that if B satisfies Assumptions A, then Remark $B.12$ provides the validity of a simultaneous local exact controllability without phase ambiguities.

Indeed, keeping in mind our notation, let $H^3_{(V)}$ be the space defined in [MN15]. We know that $H^3_{(V)}$ corresponds to $\widetilde{H}^3_{(0)}$ when $V=u_0B$ and B is a suitable multiplication operator. We consider the assumptions $(C3)$, $(C4)$ and $(C5)$ introduced in [MN15, p. 10]. If we substitute V with u_0B and μ by $-B$, then the statement of [MN15, $Theorem$ 4.1] is valid. The condition $(C3)$ is ensured by Lemma $B.7$ (Appendix $B.1$), while the assumptions $(C4)$ and $(C5)$ respectively follow from the first point of Remark $B.11$ and Remark $B.12$ (Appendix $B.1$).

Lemma $B.7$ (Appendix $B.1$) allows to obtain the result of [MN15, $Theorem$ 4.1], not only in a neighborhood of $\widetilde{H}^3_{(0)}$, but also in $O^N_{\epsilon,T}\subset H^3_{(0)}$ as

$$\|\cdot\|_{(3)}\asymp\|\cdot\|_{\widetilde{H}^3_{(0)}}.$$

Due to [MN15, *Theorem* 4.1], the simultaneous local exact controllability is guaranteed in $O^N_{\epsilon,T}$ for suitable $\epsilon > 0$ and $T > 0$ large enough, *i.e.* for every $\{\psi_k\}_{k\leq N} \in O^N_{\epsilon,T}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\psi_k = \Gamma^u_T \phi^{u_0}_k, \qquad \forall k \leq N.$$

The remaining part of the proof is achieved as in **1)**. □

Proof of Theorem 3.5. Let $N \in \mathbb{N}$ and let $u_0 \in \mathbb{R}$ belong to the neighborhoods provided by Lemma $B.7$, Lemma $B.8$ and Remark $B.11$ (Appendix $B.1$). Let $\Psi^3 := \{\psi^3_j\}_{j\in\mathbb{N}} \in H^3_{(0)}$ be an orthonormal systems. We consider $\{\psi^1_j\}_{j\in\mathbb{N}}$, $\{\psi^2_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ complete orthonormal systems and $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi^1_j = \psi^2_j$ and $\widehat{\Gamma}^*\psi^3_j \in H^3_{(0)}$ for every $j \in \mathbb{N}$. The last relation implies that, for every $k \leq N$,

$$\widetilde{\psi}_k := \sum_{j=1}^{\infty} \psi^1_j \langle \psi^2_j, \psi^3_k \rangle = \sum_{j=1}^{\infty} \psi^1_j \langle \widehat{\Gamma}\psi^1_j, \psi^3_k \rangle = \sum_{j=1}^{\infty} \psi^1_j \langle \psi^1_j, \widehat{\Gamma}^*\psi^3_k \rangle = \widehat{\Gamma}^*\psi^3_k \in H^3_{(0)}.$$

Thanks to the first point of Proposition 3.15, there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_k\}_{k\leq N} \subset \mathbb{R}$ such that

$$e^{i\theta_k}\widetilde{\psi}_k = \widetilde{\Gamma}^u_T \psi^3_k$$

for each $k \leq N$. Hence

$$\langle \psi^1_j, \widetilde{\Gamma}^u_T \psi^3_k \rangle = \langle e^{i\theta_j}\psi^1_j, e^{i\theta_k}\widetilde{\psi}_k \rangle = \langle \psi^2_j, e^{i\theta_k}\psi^3_k \rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N.$$

Thanks to the time reversibility (Paragraph 2.1), we have

$$\langle \Gamma^{\widetilde{u}}_T \psi^1_j, \psi^3_k \rangle = \langle \psi^1_j, \widetilde{\Gamma}^u_T \psi^3_k \rangle = \langle \psi^2_j, e^{i\theta_k}\psi^3_k \rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N. \quad \square$$

Proof of Theorem 3.7. Let $N \in \mathbb{N}$ and let $u_0 \in \mathbb{R}$ belong to the neighborhoods provided by Lemma $B.7$, Lemma $B.8$, Remark $B.11$ and Remark $B.12$ (Appendix $B.1$).

1) Controllability in projection of orthonormal systems: Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \in H^3_{(0)}$ be an orthonormal system. Let us consider $\{\psi_j^1\}_{j\in\mathbb{N}}, \{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be complete orthonormal systems and $\widehat{\Gamma} \in U(\mathscr{H})$ be such that $\widehat{\Gamma}\psi_j^1 = \psi_j^2$ and $\widehat{\Gamma}^*\psi_j^3 \in H^3_{(0)}$ for every $j \in \mathbb{N}$. As in the proof of Theorem 3.5, for every $k \leq N$, we define

$$\widetilde{\psi}_k := \sum_{j=1}^{\infty} \psi_j^1 \langle \psi_j^2, \psi_k^3 \rangle.$$

Thanks to the second point of Proposition 3.15, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\widetilde{\psi}_k = \widetilde{\Gamma}_T^u \psi_k^3$$

for each $k \leq N$. Hence

$$\langle \psi_j^1, \widetilde{\Gamma}_T^u \psi_k^3 \rangle = \langle \psi_j^1, \widetilde{\psi}_k \rangle = \langle \psi_j^2, \psi_k^3 \rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N.$$

Thanks to Paragraph 2.1, we have

$$\langle \Gamma_T^{\widetilde{u}} \psi_j^1, \psi_k^3 \rangle = \langle \psi_j^1, \widetilde{\Gamma}_T^u \psi_k^3 \rangle = \langle \psi_j^2, \psi_k^3 \rangle$$

and then

$$\pi_N(\Psi^3)\psi_j^2 = \pi_N(\Psi^3)\Gamma_T^{\widetilde{u}}\psi_j^1, \qquad \forall j \in \mathbb{N}. \tag{3.30}$$

2) Controllability in projection of unitarily equivalent functions: Let us consider $\{\psi_j^1\}_{j\in\mathbb{N}}, \{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ unitarily equivalent. Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}}$ be an orthonormal system. We suppose the existence of $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi_j^1 = \psi_j^2$ and $\widehat{\Gamma}^*\psi_j^3 \in H^3_{(0)}$ for every $j \in \mathbb{N}$. One knows that, for every $j \in \mathbb{N}$, there exists $\{a_k^j\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ such that

$$\psi_j^1 = \sum_{k\in\mathbb{N}} a_k^j \psi_k^3.$$

However, $\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}}$ is an Hilbert basis of $\mathscr{H}$ and

$$\psi_j^2 = \widehat{\Gamma}\psi_j^1 = \sum_{k\in\mathbb{N}} a_k^j \widehat{\Gamma}\psi_k^3.$$

The point 2) implies that there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi^3)\ \Gamma_T^u \psi_k^3 = \pi_N(\Psi^3)\ \widehat{\Gamma}\psi_k^3$$

for every $k \in \mathbb{N}$, and then for any $j \in \mathbb{N}$,

$$\pi_N(\Psi^3)\ \Gamma_T^u \psi_j^1 = \sum_{k\in\mathbb{N}} a_k^j \big(\pi_N(\Psi^3)\ \Gamma_T^u \psi_k^3\big) = \pi_N(\Psi^3)\ \sum_{k\in\mathbb{N}} a_k^j \widehat{\Gamma}\psi_k^3 = \pi_N(\Psi^3)\ \psi_j^2.$$

3) Controllability in projection with generic projector: Let $\Psi^3 = \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be a sequence of linearly independent elements. For every $N \in \mathbb{N}$, by considering the Gram-Schmidt orthonormalization process, there exists an orthonormal system $\widetilde{\Psi}^3 := \{\{\widetilde{\psi}_j^3\}_{j\leq N}, 0, ...\}$ such that

$$span\{\psi_j^3 :\ j \leq N\} = span\{\widetilde{\psi}_j^3 :\ j \leq N\}.$$

The claim follows since

$$\pi_N(\Psi^3) \equiv \pi_N(\widetilde{\Psi}^3).$$

If $\Psi^3 = \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ is a generic sequence of functions, then one can extract from Ψ^3 a subsequence of linearly independent elements and repeat as above. □

3.4 Global exact controllability in projection of density matrices

Let ψ^1, $\psi^2 \in \mathscr{H}$. We define the rank one operator $|\psi^1\rangle\langle\psi^2|$ such that

$$|\psi^1\rangle\langle\psi^2|\psi = \psi^1\langle\psi^2, \psi\rangle, \qquad \forall\psi \in \mathscr{H}.$$

For any $\widehat{\Gamma} \in U(\mathscr{H})$, we have

$$\widehat{\Gamma}|\psi^1\rangle\langle\psi^2| = |\widehat{\Gamma}\psi^1\rangle\langle\psi^2|$$

and

$$|\psi^1\rangle\langle\psi^2|\widehat{\Gamma}^* = |\psi^1\rangle\langle\widehat{\Gamma}\psi^2|$$

since, for every $\psi \in \mathscr{H}$,

$$|\psi^1\rangle\langle\psi^2|\widehat{\Gamma}^*\psi = \psi^1\langle\psi^2, \widehat{\Gamma}^*\psi\rangle = \psi^1\langle\widehat{\Gamma}\psi^2, \psi\rangle = |\psi^1\rangle\langle\widehat{\Gamma}\psi^2|\psi.$$

In non-relativistic quantum mechanics, any statistical ensemble can be described by a wave function $\psi \in \mathscr{H}$ (pure state) or by a density matrix (mixed state). A density matrix ρ is a positive operator of trace 1 so that there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ such that

$$\rho = \sum_{j\in\mathbb{N}} l_j|\psi_j\rangle\langle\psi_j|, \qquad \sum_{j\in\mathbb{N}} l_j = 1, \quad l_j \geq 0, \ \ \forall j \in \mathbb{N}. \tag{3.31}$$

The sequence $\{\psi_j\}_{j\in\mathbb{N}}$ is a set of eigenvectors of ρ and $\{l_j\}_{j\in\mathbb{N}}$ are the corresponding eigenvalues. If $j_0 \in \mathbb{N}$ is such that $l_{j_0} = 1$, then $l_j = 0$ for each

$j \neq j_0$ and the corresponding density matrix represents a pure state up to a phase. For this reason, the density matrices formalism extends the common formulation of the quantum mechanics in terms of wave function.

Let any couple of unitarily equivalent density matrices $\rho_1,\ \rho_2 \in T(\mathscr{H})$. If there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Gamma_T^u \rho_1 (\Gamma_T^u)^* = \rho_2,$$

then there exist two orthonormal systems $\{\psi_j^1\}_{j\in\mathbb{N}}$, and $\{\psi_j^2\}_{j\in\mathbb{N}}$ composed by eigenfunctions respectively of ρ_1 and ρ_2 such that

$$\sum_{j\in\mathbb{N}} l_j |\psi_j^2\rangle\langle\psi_j^2| = \rho_2 = \Gamma_T^u \rho_1 (\Gamma_T^u)^* = \sum_{j\in\mathbb{N}} l_j |\Gamma_T^u \psi_j^1\rangle\langle\Gamma_T^u \psi_j^1|,$$

for $\{l_j\}_{j\in\mathbb{N}}$ the sequence of eigenvalues of both ρ_1 and ρ_2 (as ρ_1 and ρ_2 are unitarily equivalent, they have the same spectrum). The last spectral decomposition implies that controlling a density matrix is equivalent to the simultaneous controllability of orthonormal systems.

Corollary 3.16. *Let B satisfy Assumptions II and Assumptions A $\rho^1, \rho^2 \in T(\mathscr{H})$ be two density matrices such that $Ran(\rho^1)$, $Ran(\rho^2) \subseteq H^3_{(0)}$. We suppose the existence of $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\rho^2 = \widehat{\Gamma}\rho^1\widehat{\Gamma}^*$. Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be such that*

$$\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)},$$

for every $j \in \mathbb{N}$. For any $N \in \mathbb{N}$, there exist $T > 0$ and a control function $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi^3)\ \Gamma_T^u \rho^1 (\Gamma_T^u)^*\ \pi_N(\Psi^3) = \pi_N(\Psi^3)\ \rho^2\ \pi_N(\Psi^3).$$

Proof. Let $T > 0$ large enough and $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \in H^3_{(0)}$. Let $\rho^1,\ \rho^2 \in T(\mathscr{H})$ be two unitarily equivalent density matrices such that $Ran(\rho^1), Ran(\rho^2) \subseteq H^3_{(0)}$. We suppose that the unitary operator $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\rho^2 = \widehat{\Gamma}\rho^1\widehat{\Gamma}$ satisfies the condition $\widehat{\Gamma}^*\psi_j^3 \in H^3_{(0)}$ for every $j \in \mathbb{N}$. One can ensure the existence of two complete orthonormal systems $\Psi^1 := \{\psi_j^1\}_{j\in\mathbb{N}}$, $\Psi^2 := \{\psi_j^2\}_{j\in\mathbb{N}} \in H^3_{(0)}$ respectively composed by eigenfunctions of ρ^1 and ρ^2 such that

$$\rho^1 = \sum_{j=1}^{\infty} l_j |\psi_j^1\rangle\langle\psi_j^1|, \qquad \rho^2 = \sum_{j=1}^{\infty} l_j |\psi_j^2\rangle\langle\psi_j^2|.$$

The sequence $\{l_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ corresponds to the spectrum of ρ^1 and ρ^2. Now, thanks to Theorem 3.7, there exists a control function $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi^3)\ \Gamma_T^u\psi_j^1 = \pi_N(\Psi^3)\ \psi_j^2.$$

Thus

$$\pi_N(\Psi^3)\ \Gamma_T^u\rho^1(\Gamma_T^u)^*\pi_N(\Psi^3) = \sum_{j=1}^{\infty} l_j|\pi_N(\Psi^3)\ \Gamma_T^u\psi_j^1\rangle\langle\psi_j^1\Gamma_T^u\pi_N(\Psi^3)\ |$$

$$= \sum_{j=1}^{\infty} l_j\pi_N(\Psi^3)\ |\psi_j^2\rangle\langle\psi_j^2|\pi_N(\Psi^3) = \pi_N(\Psi^3)\ \rho^2\pi_N(\Psi^3). \qquad \square$$

Chapter 4

Global exact controllability of the bilinear Schrödinger potential type models on graphs

In this chapter, we study the controllability of the bilinear Schrödinger equation (BSE) on compact graphs (Figure 4.1). We analyze how the boundary conditions and the structure of the graph affect the controllability.

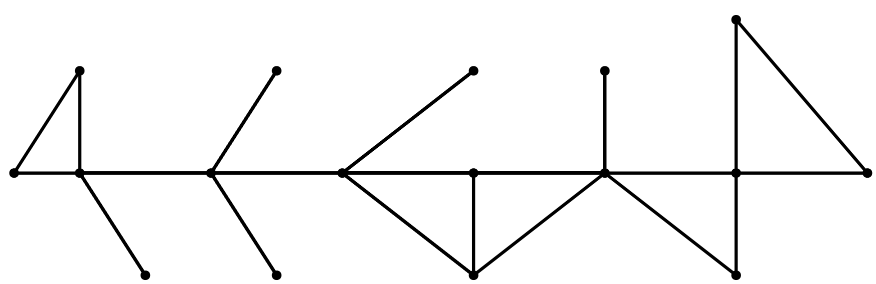


Figure 4.1: Example of compact graph

4.1 Main results

Let $\mathscr{G}$ be a compact graph composed by N edges $\{e_j\}_{j\leq N}$ of lengths $\{L_j\}_{j\leq N}$ connecting M vertices $\{v_j\}_{1\leq j\leq M}$. Let

$$V_e := \{v \in \{v_j\}_{1\leq j\leq M} | \exists! e \in \{e_j\}_{j\leq N} : v \in e\},$$

$$V_i := \{v_j\}_{1\leq j\leq M} \setminus V_e.$$

We respectively call V_e and V_i the external and the internal vertices of $\mathscr{G}$. For each $j \leq M$, we denote

$$N(v_j) := \big\{l \in \{1, ..., N\} \mid v_j \in e_l\big\},$$

$$n(v_j) := |N(v_j)|$$

($n(v_j)$ represents the cardinality of $N(v_j)$). For $f := (f^1, ..., f^N) : \mathscr{G} \to \mathbb{C}$ such that $f^j : e_j \to \mathbb{C}$ for $j \leq N$, we define the Hilbert space

$$\mathscr{H} = L^2(\mathscr{G}, \mathbb{C}) := \prod_{l=1}^{N} L^2((0, L_j), \mathbb{C}),$$

equipped with the scalar product

$$\langle \psi_1, \psi_2 \rangle = \sum_{j=1}^{N} \langle \psi_1^j, \psi_2^j \rangle_{L^2(0,L_j)}.$$

We denote $\|\cdot\| := \|\cdot\|_{\mathscr{H}} = \sqrt{\langle \cdot, \cdot \rangle}$. In $\mathscr{H}$, we consider the following Cauchy problem

$$\begin{cases} i\partial_t \psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0, T), \\ \psi(0) = \psi_0, \end{cases} \tag{4.1}$$

which corresponds to

$$\begin{cases} i\partial_t \psi^j(t, x) = A\psi^j(t, x) + u(t)B\psi^j(t, x), & 1 \leq j \leq N, \quad t \in (0, T), \\ \psi^j(0, x) = \psi_0^j(x), & x \in (0, L_j). \end{cases}$$

The operator B is bounded and symmetric, $\psi_0 = (\psi_0^1, ..., \psi_0^N)$ is the initial state, $u \in L^2((0, T), \mathbb{R})$ and Γ_t^u is the unitary propagator of (4.1). The operator A is a Laplacian with self-adjoint type boundary conditions. Each

$v \in V_i$ is equipped with Neumann-Kirchhoff ($\mathcal{NK}$) boundary conditions when for every $f \in D(A)$,

$$(\mathcal{NK}): \begin{cases} f \text{ is continuous in } v, \\ \sum_{e \in N(v)} \frac{\partial f}{\partial x_e}(v) = 0. \end{cases}$$

The derivatives are assumed to be taken in the directions away from the vertex (outgoing directions). Each $v \in V_e$ is equipped with Dirichlet ($\mathcal{D}$) or Neumann ($\mathcal{N}$) boundary conditions, *i.e.* for every $f \in D(A)$,

$$(\mathcal{D}): \; f(v) = 0, \qquad (\mathcal{N}): \; \frac{\partial f}{\partial x}(v) = 0$$

respectively.

As we consider the self-adjoint operator A on the graph, $\mathscr{G}$ is denoted quantum graph.
We stress the fact that when we introduce a compact quantum graph $\mathscr{G}$, we are implicitly introducing a Laplacian A equipped with self-adjoint type boundary conditions.
From now on, we say that a compact quantum graph is equipped with one of the previous boundary conditions in a vertex $v \in \mathscr{G}$, when each function of $D(A)$ satisfies this boundary condition in v.

We also adopt the following notation.

- We say that a quantum graph $\mathscr{G}$ is equipped with ($\mathcal{D}$)-($\mathcal{NK}$) (or ($\mathcal{N}$)-($\mathcal{NK}$)) when, for every $f \in D(A)$ and $v \in V_e$, the function f satisfies ($\mathcal{D}$) (or ($\mathcal{N}$)) in v and verifies ($\mathcal{NK}$) in every $v \in V_i$.
- We say that a quantum graph $\mathscr{G}$ is equipped with ($\mathcal{D}/\mathcal{N}$)-($\mathcal{NK}$) when, for every $f \in D(A)$ and $v \in V_e$, the function f satisfies ($\mathcal{D}$) or ($\mathcal{N}$) in v and, for every $v \in V_i$, the function f verifies ($\mathcal{NK}$) in v.

For every compact graph $\mathscr{G}$, the operator A admits purely discrete spectrum (see [Kuc04, *Theorem* 18]). We call $\{\lambda_j\}_{j\in\mathbb{N}}$ the non-decreasing sequence of eigenvalues of A and

$$\{\phi_j\}_{j\in\mathbb{N}}$$

a Hilbert basis of $\mathscr{H}$ composed by the corresponding eigenfunctions. Let

$$\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j$$

and $[r]$ be the entire part of a real number $r \in \mathbb{R}$. For $s > 0$, we define the spaces

$$H^s = H^s(\mathscr{G}, \mathbb{C}) := \prod_{j=1}^{N} H^s(e_j, \mathbb{C}),$$

(4.2)

$$H^s_{\mathcal{NK}} := \Big\{ \psi \in H^s \mid \partial_x^{2n}\psi \text{ is continuous in } v, \quad \forall n \in \mathbb{N}\cup\{0\},\ n < \big[(s+1)/2\big]; \\ \sum_{e\in N(v)} \partial_{x_e}^{2n+1} f(v) = 0,\ \forall n \in \mathbb{N}\cup\{0\},\ n < \big[s/2\big],\ \forall v \in V_i \Big\},$$

$$H^s_{\mathscr{G}} = H^s_{\mathscr{G}}(\mathscr{G}, \mathbb{C}) := D(A^{s/2}), \qquad \|\cdot\|_{(s)} := \|\cdot\|_{H^s_{\mathscr{G}}} = \Big(\sum_{k=1}^{\infty} |k^s \langle \cdot, \phi_k\rangle|^2 \Big)^{\frac{1}{2}},$$

$$h^s(\mathbb{C}) := \Big\{ \{a_k\}_{k\in\mathbb{N}} \subset \mathbb{C} \ \Big|\ \sum_{k=1}^{\infty} |k^s a_k|^2 < \infty \Big\}, \qquad \|\cdot\|_{(s)} := \Big(\sum_{k=1}^{\infty} |k^s \cdot|^2 \Big)^{\frac{1}{2}}.$$

Remark 4.1. *If $0 \notin \sigma(A)$ (the spectrum of A), then $\|\cdot\|_{(s)} \asymp \||A|^{\frac{s}{2}} \cdot\|$, i.e. there exist $C_1, C_2 > 0$ such that*

$$C_1 \|\cdot\|^2_{(s)} \le \||A|^{\frac{s}{2}} \cdot\|^2 = \sum_{k=1}^{\infty} |\lambda_k^{\frac{s}{2}} \langle \cdot, \phi_k\rangle|^2 \le C_2 \|\cdot\|^2_{(s)}.$$

Indeed, from [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10], there exist $C_3, C_4 > 0$ such that

$$C_3 k^2 \le \lambda_k \le C_4 k^2$$

for every $k > 2$ when $\lambda_1 = 0$, otherwise for every $k \in \mathbb{N}$ (see Remark 4.17 for further details on this identity). If $0 \in \sigma(A)$, then there exists $c \in \mathbb{R}$ such that $0 \notin \sigma(A + c)$ and

$$\|\cdot\|_{(s)} \asymp \||A + c|^{\frac{s}{2}} \cdot\|.$$

For any compact quantum graph $\mathscr{G}$, the only eigenvalue which can be 0 is λ_1 and there exists $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| > \delta\mathcal{M}. \tag{4.3}$$

Indeed, thanks to [DZ06, *relation* (6.6)], there exist $\mathcal{M} \in \mathbb{N}$ and $\delta' > 0$ such that

$$\inf_{k\in\mathbb{N}} |\sqrt{\lambda}_{k+\mathcal{M}} - \sqrt{\lambda}_k| > \delta'\mathcal{M}$$

and

$$\inf_{k\in\mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \sqrt{\lambda_2}|\sqrt{\lambda}_{n+\mathcal{M}} - \sqrt{\lambda}_n| > \delta'\mathcal{M}.$$

Let $\eta > 0$ and $a \geq 0$. We define the following assumptions on the couple (A, B) for

$$I := \{(j,k) \in \mathbb{N}^2 : j \neq k\}. \tag{4.4}$$

Assumptions (III(η))**.** The operator B satisfies the following conditions.

1. There exists $C > 0$ such that, for every $j \in \mathbb{N}$,

$$|\langle\phi_j, B\phi_1\rangle| \geq \frac{C}{j^{2+\eta}}.$$

2. For every $(j,k),(l,m) \in I$ such that $(j,k) \neq (l,m)$ and

$$\lambda_j - \lambda_k = \lambda_l - \lambda_m,$$

it holds

$$\langle\phi_j, B\phi_j\rangle - \langle\phi_k, B\phi_k\rangle - \langle\phi_l, B\phi_l\rangle + \langle\phi_m, B\phi_m\rangle \neq 0.$$

Assumptions III(η) generalize Assumptions II introduced in the previous chapter. Heuristically speaking, the first condition quantifies how much the operator B mixes the eigenstates of A. The second assumption ensures that B decouples the resonant eigenvalues gaps, *i.e.* $\lambda_j - \lambda_k = \lambda_l - \lambda_m$ for $(j,k),(l,m) \in I$ with $(j,k) \neq (l,m)$.

Assumptions (IV(η, a))**.** Let $Ran(B|_{H^2_{\mathscr{G}}}) \subseteq H^2_{\mathscr{G}}$ and one of the following assumptions be satisfied.

1. When $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ and $a + \eta \in (0, 3/2)$, there exists $d \in [\max\{a + \eta, 1\}, 3/2)$ such that

$$Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^2_{\mathscr{G}}.$$

2. When $\mathscr{G}$ is equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and $a + \eta \in (0, 5/2)$, there exist $d \in [\max\{a + \eta, 2\}, 5/2)$ and $d_1 \in (d, 5/2)$ such that

$$Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^3_{\mathcal{NK}} \cap H^2_{\mathscr{G}}, \qquad Ran(B|_{H^{d_1}}) \subseteq H^{d_1}.$$

3. When $\mathscr{G}$ is equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and $a+\eta\in(0,7/2)$, there exist $d\in[\max\{a+\eta,2\},7/2)$ and $d_1\in(d,7/2)$ such that

$$Ran(B|_{H^{2+d}_{\mathscr{G}}})\subseteq H^{2+d}\cap H^{2+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}},\qquad Ran(B|_{H^{d_1}\cap H^{d_1}_{\mathcal{NK}}})\subseteq H^{d_1}_{\mathcal{NK}}.$$

4. When $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$ and $a+\eta\in(0,5/2)$, there exists $d\in[\max\{a+\eta,1\},5/2)$ such that

$$Ran(B|_{H^{2+d}_{\mathscr{G}}})\subseteq H^{2+d}\cap H^{2+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}.$$

If $a+\eta\geq 2$, then there exists $d_1\in(d,5/2)$ such that

$$Ran(B|_{H^{d_1}})\subseteq H^{d_1}.$$

From now on, we omit the parameters η and a from the notations Assumptions III and Assumptions IV in those contexts where they are not relevant or already defined.

4.1.1 Global exact controllability

Definition 4.2. The problem (4.1) is said to be globally exactly controllable in $H^s_{\mathscr{G}}$ for $s>0$ if, for any $\psi^1,\ \psi^2\in H^s_{\mathscr{G}}$ such that $\|\psi^1\|=\|\psi^2\|$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that

$$\psi^2=\Gamma^u_T\psi^1.$$

Theorem 4.3. *Let $\mathscr{G}$ be a compact quantum graph and let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the spectrum of A.*

1. *Let $\tilde{d}\geq 0$ and $C>0$ be such that it is satisfied*

$$|\lambda_{k+1}-\lambda_k|\geq Ck^{-\frac{\tilde{d}}{\mathcal{M}-1}}\qquad\forall k\in\mathbb{N}.\tag{4.5}$$

If the couple (A,B) satisfies Assumptions III(η) and Assumptions IV($\eta,\tilde{d}$) for some $\eta>0$, then the problem (4.1) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s=2+d$ and d from Assumptions IV.

2. *Let G be an entire function such that $G\in L^\infty(\mathbb{R})$ and there exist $J,I>0$ such that*

$$|G(z)|\leq Je^{I|z|},\qquad\forall z\in\mathbb{C}.$$

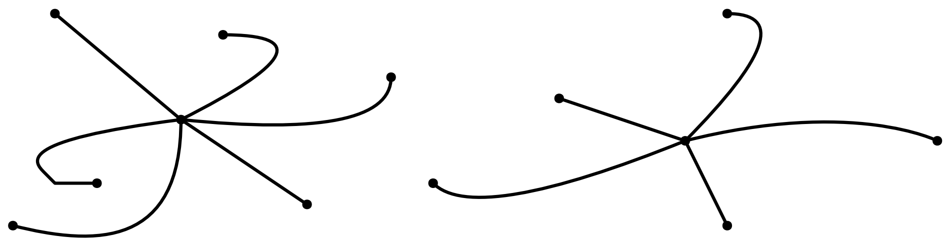


Figure 4.2: Examples of star graphs

The numbers $\{\sqrt{\lambda_j}\}_{j\in\mathbb{N}}$ *and* $\{-\sqrt{\lambda_j}\}_{j\in\mathbb{N}}$ *are simple zeros of* G *and there exist* $\tilde{d}\geq 1$, $C>0$ *such that, for every* $j\in\mathbb{N}$,

$$|G'(\sqrt{\lambda_j})|\geq \frac{C}{j^{\tilde{d}}},\qquad |G'(-\sqrt{\lambda_j})|\geq \frac{C}{j^{\tilde{d}}}.$$

If the couple (A,B) *satisfies Assumptions III*(η) *and Assumptions IV*$(\eta,\tilde{d}-1)$ *for* $\eta>0$ *and* $\epsilon_1>\epsilon$, *then the problem* (4.1) *is globally exactly controllable in* $H^s_{\mathscr{G}}$ *for* $s=2+d$ *and* d *from Assumptions IV.*

Proof. See Paragraph 4.3. □

Definition 4.4. For every $N\in\mathbb{N}$, we define $\mathcal{AL}(N)\subset(\mathbb{R}^+)^N$ as follows. For every $\{L_j\}_{j\leq N}\in\mathcal{AL}(N)$, the numbers $\big\{1,\{L_j\}_{j\leq N}\big\}$ are linearly independent over $\mathbb{Q}$ and all the ratios L_k/L_j are algebraic irrational numbers.

- We denote **tadpole** a compact quantum graph composed by an edge connected to a circle in an internal vertex v.
- We define **two-tails tadpole** a compact quantum graph composed by a circle connected with two edges in an internal vertex v.
- We call **double-rings graph** a compact quantum graph formed by two circles connected in an internal vertex v.
- We denote **star graph** a compact quantum graph composed by $M+1$ edges connected in an internal vertex v (M is the number of vertices). We associate the 0 coordinate with each external vertex. (Figure 4.2)

We show that for these types of graphs (Figure 4.3) the spectral hypothesis of Theorem 4.3 are satisfied.

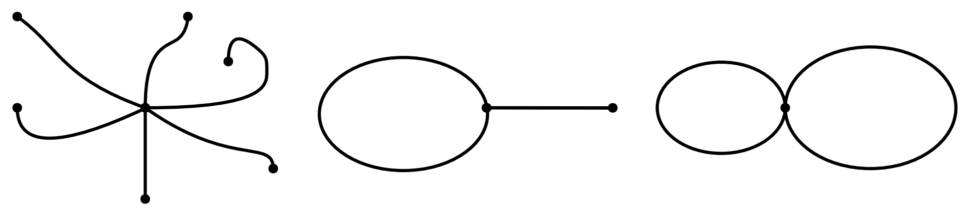


Figure 4.3: Respectively a star graph, a tadpole graph and a double-rings graph

Theorem 4.5. *Let* $\{L_j\}_{j\leq N}\in\mathcal{AL}(N)$.

1. *Let* $\mathscr{G}$ *be either a tadpole, a two-tails tadpole, a double-rings graph or a star graph of* $N\leq 4$ *edges and let* $\mathscr{G}$ *be equipped with* $(\mathcal{D}/\mathcal{N})$*-*$(\mathcal{NK})$*. If the couple* (A,B) *satisfies Assumptions III*(η) *and Assumptions IV*(η,ϵ) *for* $\eta,\epsilon>0$*, then the problem* (4.1) *is globally exactly controllable in* $H^s_{\mathscr{G}}$ *for* $s=2+d$ *and* d *from Assumptions IV.*

2. *Let* $\mathscr{G}$ *be a star graph equipped with* $(\mathcal{D})$*-*$(\mathcal{NK})$*. If the couple* (A,B) *satisfies Assumptions III*(η) *and Assumptions IV*(η,ϵ) *for* $\eta,\epsilon>0$*, then the problem* (4.1) *is globally exactly controllable in* $H^s_{\mathscr{G}}$ *for* $s=2+d$ *and* d *from Assumptions IV.*

Proof. See Paragraph 4.4. □

4.1.2 Contemporaneous controllability

Let $\mathscr{G}=\{I_j\}_{j\leq N}$ be a compact quantum graph composed by a set of bounded unconnected intervals. In this context, the problem (4.1) is equivalent to the following Cauchy problems, each one of them in $L^2(I_j,\mathbb{C})$:

$$\begin{cases} i\partial_t\psi^j(t)=A_j\psi^j(t)+u(t)B_j\psi^j(t), & t\in(0,T),\quad 1\leq j\leq N,\\ \psi^j(0)=\psi^j_0, \end{cases} \tag{4.6}$$

for $B_j:=B|_{L^2(I_j)}$. The operator $A_j:=A|_{L^2(I_j)}$ is the Laplacian on I_j equipped with self-adjoint type boundary conditions $(\mathcal{D})$ or $(\mathcal{N})$. Let $\Gamma^{u,j}_t$ be the propagator generated by $A_j+u(t)B_j$ and

$$H^s_{I_j}:=D(A_j^{s/2}),\qquad s>0.$$

Definition 4.6. The problem (4.6) is said to be contemporaneously globally exactly controllable in $\prod_{j=1}^N H^s_{I_j}$ for $s>0$ if, for any $\{\psi^1_j\}_{j\leq N}$, $\{\psi^2_j\}_{j\leq N}$

unitarily equivalent such that $\psi_j^1,\ \psi_j^2 \in H^s_{I_j}$ and $\|\psi_j^1\| = \|\psi_j^2\|$ for every $j \leq N$, then there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Gamma_T^{u,j}\psi_j^1 = \psi_j^2, \qquad \forall j \leq N.$$

Theorem 4.7. *Let $\mathscr{G} = \{I_j\}_{j\leq N}$ be a compact quantum graph composed by a set of bounded unconnected intervals. Let the couple (A,B) satisfy Assumptions III(η) and Assumptions IV(η,ϵ) for some $\eta,\epsilon > 0$. If $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$, then the problem (4.6) is contemporaneously globally exactly controllable in $\prod_{j=1}^N H^s_{I_j}$ for $s = 2 + d$ and d from Assumptions IV.*

Proof. See Paragraph 4.4. □

4.1.3 Energetic controllability

Let $\varphi := \{\varphi_k\}_{k\in\mathbb{N}} \subseteq \Phi$ be an orthonormal system composed by some eigenfunctions of A and corresponding to eigenvalues

$$\{\mu_k\}_{k\in\mathbb{N}} \subseteq \{\lambda_k\}_{k\in\mathbb{N}},$$

i.e. $A\varphi_k = \mu_k\varphi_k$ and $\varphi_k \not\equiv 0$. Let $\eta, a \in (0,4)$ and $\widetilde{\mathscr{H}} := \overline{span\{\varphi_k \mid k \in \mathbb{N}\}}^{L^2}$.

Now, we introduce a set of assumptions for the couple (A,B) which can also be satisfied when the hypotheses of Theorem 4.3 fail.

Assumptions (V(φ,η,a))**.** The couple (A,B) satisfies Assumptions III, Assumptions IV and the hypotheses of one of the two points of Theorem 4.3 in $\widetilde{\mathscr{H}}$.
In other words, the following conditions are satisfied.

1. There exists $C > 0$ such that $|\langle\varphi_j, B\varphi_1\rangle| \geq \frac{C}{j^{2+\eta}}$ for every $j \in \mathbb{N}$.
2. For every $(j,k),(l,m) \in I$ such that $(j,k) \neq (l,m)$ and $\mu_j - \mu_k = \mu_l - \mu_m$, it holds
$$\langle\varphi_j, B\varphi_j\rangle - \langle\varphi_k, B\varphi_k\rangle - \langle\varphi_l, B\varphi_l\rangle + \langle\varphi_m, B\varphi_m\rangle \neq 0.$$
3. $Ran(B|_{H^2_{\mathscr{G}}\cap\widetilde{\mathscr{H}}}) \subseteq H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$.
4. The hypotheses of one of the two points of Theorem 4.3 are satisfied with respect to $\{\mu_k\}_{k\in\mathbb{N}}$.

Let one of the following assumptions be satisfied.

1. When $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ and $a+\eta \in (0,3/2)$, there exists $d \in [\max\{a+\eta, 1\}, 3/2)$ such that
$$Ran(B|_{H^{2+d}_{\mathscr{G}} \cap \widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}.$$

2. When $\mathscr{G}$ is equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and $a+\eta \in (0,5/2)$, there exist $d \in [\max\{a+\eta, 2\}, 5/2)$ and $d_1 \in (d, 5/2)$ such that
$$Ran(B|_{H^{2+d}_{\mathscr{G}} \cap \widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^3_{\mathcal{NK}} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}, \qquad Ran(B|_{H^{d_1}}) \subseteq H^{d_1} \cap \widetilde{\mathscr{H}}.$$

3. When $\mathscr{G}$ is equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and $a+\eta \in (0,7/2)$, there exist $d \in [\max\{a+\eta, 2\}, 7/2)$ and $d_1 \in (d, 7/2)$ such that
$$Ran(B|_{H^{2+d}_{\mathscr{G}} \cap \widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^{2+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}},$$
$$Ran(B|_{H^{d_1} \cap H^{d_1}_{\mathcal{NK}} \cap \widetilde{\mathscr{H}}}) \subseteq H^{d_1}_{\mathcal{NK}} \cap \widetilde{\mathscr{H}}.$$

4. When $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$ and $a+\eta \in (0,5/2)$, there exists $d \in [\max\{a+\eta, 1\}, 5/2)$ such that
$$Ran(B|_{H^{2+d}_{\mathscr{G}} \cap \widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^{2+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}.$$
If $a+\eta \geq 2$, then there exists $d_1 \in (d, 5/2)$ such that
$$Ran(B|_{H^{d_1} \cap \widetilde{\mathscr{H}}}) \subseteq H^{d_1} \cap \widetilde{\mathscr{H}}.$$

From now on, we omit the parameters φ, η and a from the notation Assumptions V in those contexts where they are not relevant or already defined.

Definition 4.8. The problem (4.1) is said to be energetically controllable with respect to $\{\mu_j\}_{j\in\mathbb{N}}$ if, for any $\varphi_m,\ \varphi_n \in \{\varphi_j\}_{j\in\mathbb{N}}$, there exist $T>0$ and a control function $u \in L^2((0,T),\mathbb{R})$ such that
$$\Gamma^u_T \varphi_m = \varphi_n.$$
If $\{\mu_j\}_{j\in\mathbb{N}}$ corresponds to the sequence of eigenvalues of A (not repeated with their multiplicity), then we say that (4.1) is fully energetically controllable in $\{\mu_j\}_{j\in\mathbb{N}}$.

Theorem 4.9. *Let $\mathscr{G}$ be a compact quantum graph and the couple (A, B) satisfy Assumptions $V(\varphi, \eta, \tilde{d})$ for some $\eta, \tilde{d} > 0$. The problem (4.1) is energetically controllable in $\{\mu_j\}_{j\in\mathbb{N}}$ and globally exactly controllable in $H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ for $s = 2 + d$, d from Assumptions V.*

Proof. First, as $B : H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}} \longrightarrow H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$, the propagator Γ^u_t preserves $H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$.
Second, the statement of Theorem 4.3 is valid in $\widetilde{\mathscr{H}}$ implying the global exact controllability in $H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ for some $s > 0$.
In conclusion, the energetic controllability follows from the fact that $\varphi_j \in H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ for every $s > 0$ and $j \in \mathbb{N}$. □

Remark 4.10. *The energetic controllability can be adopted in order to study those complex quantum graphs $\mathscr{G}$ such that the hypotheses of Theorem 4.3 can not be verified. An example is when $\mathscr{G}$ contains a finite number of particular subgraphs $\{\mathscr{G}_j\}_{j\leq\widetilde{N}}$, called* **uniform chains of edges***, each one composed by edges of length L_j such that $\{L_j\}_{j\leq\widetilde{N}} \in \mathcal{AL}(\widetilde{N})$.*

We say that a graph $\widetilde{\mathscr{G}}$ is an **uniform chain of edges** *if $\widetilde{\mathscr{G}}$ is a sequence of edges of equal length connecting $M \in \mathbb{N}$ vertices $\{v_k\}_{k\leq M}$ such that $v_2, ..., v_{M-1} \in V_i$ are equipped with ($\mathcal{NK}$) and one of the following situation is verified.*

- *The vertices $v_1, v_M \in V_e$ are equipped with ($\mathcal{D}$).*
- *The vertices $v_1 = v_M \in V_i$ are equipped with ($\mathcal{NK}$).*

Let us consider the connected graph represented on Figure 4.4 with all the external vertices equipped with ($\mathcal{D}$). Let $\mathscr{G}$ contain a sub-graph $\widetilde{\mathscr{G}}$ composed by two edges e_1 and e_2 of equal length. The edges connect two external vertices to an internal vertex $\tilde{v}$.

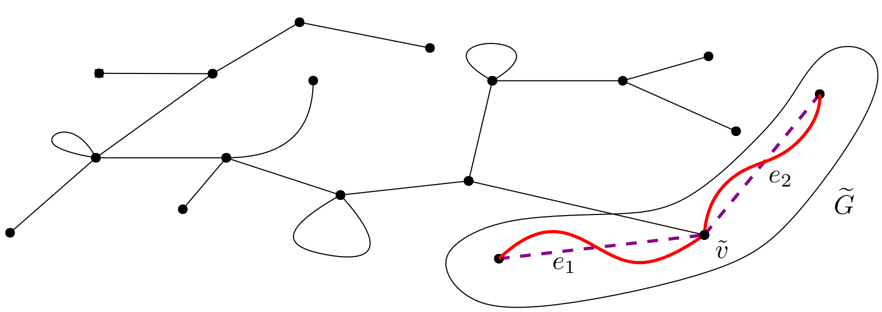


Figure 4.4:

We can exhibit some eigenfunctions of A as follows. For every $j \in \mathbb{N}$, we impose

$$\begin{cases} supp(\varphi_j) = \widetilde{\mathscr{G}}, \\ \varphi_j\big|_{e_1} = -\varphi_j\big|_{e_2}. \end{cases} \tag{4.7}$$

We assume that each function $\varphi\big|_{e_1}$ is the j-th eigenfunction of the operator A_D with domain

$$D(A_D) = H^2((0,L),\mathbb{C}) \cap H^1_0((0,L),\mathbb{C})$$

and such that $A_D f = -\Delta f$ for every $f \in D(A_D)$, where L is the length of each e_1. Now, $\varphi_j(\tilde{v}) = 0$ and the $(\mathcal{NK})$ boundary conditions are satisfied thanks to (4.7). The sequence of functions

$$\{\varphi_l\}_{l\in\mathbb{N}}$$

are eigenfunctions of A on $\mathscr{G}$ corresponding to the eigenvalues

$$\{\mu_j\}_{j\in\mathbb{N}} = \Big\{\frac{j^2\pi^2}{L^2}\Big\}_{j\in\mathbb{N}}.$$

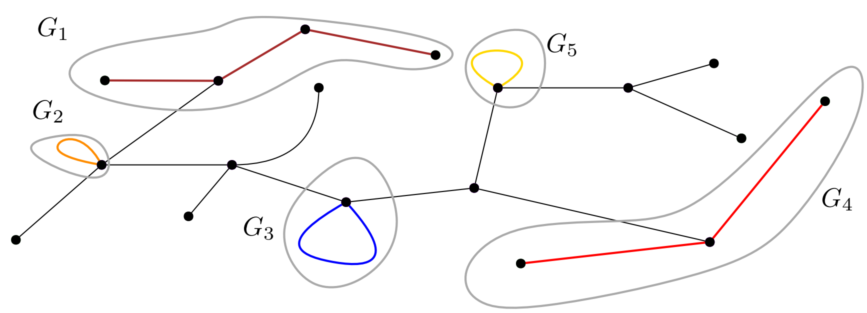


Figure 4.5:

If we repeat this procedure for every uniform chains of edges $\{G_k\}_{k\le 5}$ contained in the graph $\mathscr{G}$ (Figure 4.5), then the previous argument allows to construct 5 sequences of eigenfunctions $\{\varphi_j^k\}_{\substack{j\in\mathbb{N}\\ k\le 5}}$ of A. We define

$$\widetilde{\mathscr{H}} = \overline{span\{\varphi_j^k : \ j\in\mathbb{N}, \ k\le 5\}}.$$

The hypotheses of Theorem 4.3 can verified in $\widetilde{\mathscr{H}}$ as the spectrum and the eigenfunctions of A in $\widetilde{\mathscr{H}}$ are explicit. Then the bilinear Schrödinger equation can be energetically controllable (under suitable assumptions on B) since

$$\{\varphi_j^k\}_{\substack{j,k\in\mathbb{N}\\ k\le 5}} \subset \widetilde{\mathscr{H}} \cap H^s_{\mathscr{G}}, \qquad \forall s>0.$$

4.2 Well-posedness and interpolation properties of the spaces $H^s_{\mathscr{G}}$

Proposition 4.11. *Let the couple (A, B) satisfy Assumptions IV and the hypotheses of one of the two points of Theorem* 4.3.

1. *Let $T > 0$ and $f \in L^2((0,T), H^{2+d} \cap H^2_{\mathscr{G}})$. The map*

$$t \mapsto G(t) = \int_0^t e^{iA\tau} f(\tau) d\tau$$

belongs to $C^0([0,T], H^{2+d}_{\mathscr{G}})$ and there exists $C(T) > 0$ uniformly bounded for T lying on bounded intervals such that

$$\|G\|_{L^\infty((0,T),H^{2+d}_{\mathscr{G}})} \leq C(T)\|f\|_{L^2((0,T),H^{2+d}\cap H^2_{\mathscr{G}})}.$$

2. *Let $T > 0$, $\psi^0 \in H^{2+d}_{\mathscr{G}}$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique mild solution of (*4.1*) in $H^{2+d}_{\mathscr{G}}$ (see Proposition* 2.5*).*

Now, we present some interpolation features of the spaces $H^s_{\mathscr{G}}$ for $s > 0$. The proof of Proposition 4.11 is provided in the end of the paragraph.

Proposition 4.12. *Let $\mathscr{G}$ be a compact graph.*

- *If $\mathscr{G}$ is a graph equipped with ($\mathcal{D}/\mathcal{N}$)-($\mathcal{NK}$), then*

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2}(\mathscr{G},\mathbb{C}) \quad \text{for} \quad s_1 \in \mathbb{N}\cup\{0\},\ s_2 \in [0,1/2).$$

- *If $\mathscr{G}$ is a graph equipped with ($\mathcal{N}$)-($\mathcal{NK}$), then*

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2}_{\mathcal{NK}} \quad \text{for} \quad s_1 \in 2\mathbb{N}\cup\{0\},\ s_2 \in [0,3/2).$$

- *If $\mathscr{G}$ is a graph equipped with ($\mathcal{D}$)-($\mathcal{NK}$), then*

$$H^{s_1+s_2+1}_{\mathscr{G}} = H^{s_1+1}_{\mathscr{G}} \cap H^{s_1+s_2+1}_{\mathcal{NK}} \quad \text{for} \quad s_1 \in 2\mathbb{N}\cup\{0\},\ s_2 \in [0,3/2).$$

Proof. We recall that by defining $\mathscr{G}$ as a quantum graph, we are implicitly introducing a Laplacian A equipped with suitable boundary conditions over the graph $\mathscr{G}$. We also refer to (4.2) for the definitions of the spaces $H^s_{\mathscr{G}}$.

1) (a) Interpolation properties for $\mathscr{G}$ a bounded interval: Let $\mathscr{G}$ be a quantum graph such that $\mathscr{G} = I^{\mathcal{N}}$ for $I^{\mathcal{N}}$ a bounded interval equipped with $(\mathcal{N})$ on the external vertices. Thanks to [Gru16, *p.* 3], for each $s_1 \in 2\mathbb{N}\cup\{0\}$ and $s_2 \in [0, 3/2)$, we have

$$H^{s_1+s_2}_{I^{\mathcal{N}}} = H^{s_1}_{I^{\mathcal{N}}} \cap H^{s_1+s_2}(I^{\mathcal{N}}, \mathbb{C}). \tag{4.8}$$

Indeed, according to [Gru16, *Definition* 2.1], we have $H^{s_2}_{I^{\mathcal{N}}} = H^{s_2}(I^{\mathcal{N}}, \mathbb{C})$ for $s_2 \in [0, 3/2)$, while for $k \in \mathbb{N}\cup\{0\}$ and $1+2k < s_1+s_2-1/2 < 1+2(k+1)$,

$$H^{s_1+s_2}_{I^{\mathcal{N}}} = \{\psi \in H^{s_1+s_2}(I^{\mathcal{N}}, \mathbb{C}) \mid \partial_x A^l\psi|_{\partial I^{\mathcal{N}}} = 0 \quad 0 \le l \le k\}.$$

When the quantum graph $\mathscr{G}$ is an interval $I^{\mathcal{D}}$ equipped with $(\mathcal{D})$, thanks to [Gru16, *p.* 3], there follows that, for each $s_1 \in 2\mathbb{N}\cup\{0\}$, $s_2 \in [0, 3/2)$ and $s_3 \in [0, 1/2)$,

$$H^{s_1+s_2+1}_{I^{\mathcal{D}}} = H^{s_1+1}_{I^{\mathcal{D}}} \cap H^{s_1+s_2+1}(I^{\mathcal{D}}, \mathbb{C}), \qquad H^{s_3}_{I^{\mathcal{D}}} = H^{s_3}(I^{\mathcal{D}}, \mathbb{C}). \tag{4.9}$$

Let the quantum graph $\mathscr{G}$ be an interval $I^{\mathcal{M}}$ equipped with $(\mathcal{D})$ on one external vertex v_1 and $(\mathcal{N})$ on the other v_2. We prove that, for each $s_1 \in \mathbb{N}\cup\{0\}$ and $s_2 \in [0, 1/2)$,

$$H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1}_{I^{\mathcal{M}}} \cap H^{s_1+s_2}(I^{\mathcal{M}}, \mathbb{C}). \tag{4.10}$$

We consider $s_1 = 0$ and $s_2 \in [0, 1/2)$, but the proof is also valid when $s_1 \in \mathbb{N}$.

- We define the quantum graph $\widetilde{I}^{\mathcal{D}} \subseteq I^{\mathcal{M}}$ an interval of length $\frac{3}{4}|I^{\mathcal{M}}|$, containing v_1 and equipped in both the external vertices with $(\mathcal{D})$.
- We define the quantum graph $\widetilde{I}^{\mathcal{N}} \subseteq I^{\mathcal{M}}$ that is an interval of length $\frac{3}{4}|I^{\mathcal{M}}|$, containing v_2 and equipped in both the external vertices with $(\mathcal{N})$.
- We consider $\widetilde{I} \subseteq I^{\mathcal{M}}$ an interval of length $\frac{1}{2}|I^{\mathcal{M}}|$, containing v_1.

Let the partition of unity χ so that $\chi(x) = 1$ in $\widetilde{I}$, $\chi(x) = 0$ in $I^{\mathcal{M}} \setminus I^{\mathcal{D}}$ and $\chi(x) \in (0, 1)$ in $I^{\mathcal{D}} \setminus \widetilde{I}$. There holds that $\chi\psi \in H^2_{I^{\mathcal{D}}}$ and $(1-\chi)\psi \in H^2_{I^{\mathcal{N}}}$ and

$$\psi(x) = \chi(x)\psi(x) + (1 - \chi(x))\psi(x).$$

The same property is valid for functions belonging to $L^2(I^{\mathcal{M}}, \mathbb{C})$ and $H^s(I^{\mathcal{M}}, \mathbb{C})$ for $s \in (0, 2]$. Those decompositions allow to see the functions in these spaces as vectors of functions and

$$H^2_{I^{\mathcal{M}}} = H^2_{\widetilde{I}^{\mathcal{D}}} \times H^2_{\widetilde{I}^{\mathcal{N}}}, \qquad L^2(I^{\mathcal{M}}, \mathbb{C}) = L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}) \times L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}),$$

$$H^s(I^{\mathcal{M}},\mathbb{C}) = H^s(\widetilde{I}^{\mathcal{D}},\mathbb{C}) \ \times \ H^s(\widetilde{I}^{\mathcal{N}},\mathbb{C}).$$

As in [Tri95, *Definition*, *Chapter* 1.9.2], for X and Y suitable spaces, we define

$$\Big[X\,,\,Y\Big]_\theta$$

their complex interpolation for $0<\theta<1$. From [Tri95, *Remark* 1, *Chapter* 1.15.1] as A is a self-adjoint positive operator, [Tri95, *Theorem*, *Chapter* 1.15.3] is valid and

$$\Big[L^2(\widetilde{I}^{\mathcal{N}},\mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}\Big]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}},$$

$$\Big[L^2(\widetilde{I}^{\mathcal{D}},\mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}\Big]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{D}}}.$$

Thanks to [Tri95, *relation* (12), *Chapter* 1.18.1], the interpolation between two products of spaces is the product of the two respective interpolations and

$$H^{s_2}_{I^{\mathcal{M}}} = \Big[L^2(I^{\mathcal{M}},\mathbb{C}), H^2_{I^{\mathcal{M}}}\Big]_{s_2/2} = \Big[L^2(\widetilde{I}^{\mathcal{N}},\mathbb{C}) \ \times \ L^2(\widetilde{I}^{\mathcal{D}},\mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}} \ \times \ H^2_{\widetilde{I}^{\mathcal{D}}}\Big]_{s_2/2}$$
$$= \Big[L^2(\widetilde{I}^{\mathcal{N}},\mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}\Big]_{s_2/2} \ \times \ \Big[L^2(\widetilde{I}^{\mathcal{D}},\mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}\Big]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}} \ \times \ H^{s_2}_{\widetilde{I}^{\mathcal{D}}}.$$

The previous part of the proof leads to

$$H^{s_2}_{I^{\mathcal{M}}} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}} \ \times \ H^{s_2}_{\widetilde{I}^{\mathcal{D}}} = H^{s_2}(\widetilde{I}^{\mathcal{N}},\mathbb{C}) \ \times \ H^{s_2}(\widetilde{I}^{\mathcal{D}},\mathbb{C}) = H^{s_2}(I^{\mathcal{M}},\mathbb{C}).$$

In conclusion, the introduced argument shows that, for each $s_1 \in \mathbb{N}\cup\{0\}$ and $s_2 \in [0,1/2)$,

$$H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1}_{I^{\mathcal{M}}} \cap H^{s_1+s_2}(I^{\mathcal{M}},\mathbb{C}).$$

(b) Interpolation properties for $\mathscr{G}$ a star graph with equal edges: Let $A_{\mathcal{N}}$ be a Laplacian defined on a bounded interval I of length L and equipped with Neumann type boundary conditions. We call $I^{\mathcal{N}}$ the relative quantum graph and

$$\{f_j^1\}_{j\in\mathbb{N}}$$

an Hilbert basis of $L^2(I,\mathbb{C})$ composed by the eigenfunctions of $A_{\mathcal{N}}$.

Let $A_{\mathcal{M}}$ be a Laplacian on I equipped with Dirichlet boundary conditions in one of the external vertices of I and with Neumann boundary conditions in the other. We call $I^{\mathcal{M}}$ the relative quantum graph and

$$\{f_j^2\}_{j\in\mathbb{N}}$$

another Hilbert basis of $L^2(I,\mathbb{C})$ composed by eigenfunctions of $A_{\mathcal{M}}$. Thanks to **1) (a)**, for each $s\in[0,1/2)$ and $\psi\in H^s(I,\mathbb{C})$, we know that

$$\psi\in H^s_{I^{\mathcal{N}}},\qquad \psi\in H^s_{I^{\mathcal{M}}},$$

which imply

$$\sum_{j\in\mathbb{N}}\big|j^s\langle f_j^1,\psi\rangle_{L^2(I,\mathbb{C})}\big|^2<\infty,\qquad \sum_{j\in\mathbb{N}}\big|j^s\langle f_j^2,\psi\rangle_{L^2(I,\mathbb{C})}\big|^2<\infty.$$

Let $\mathscr{G}$ be a star graph equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and composed by N edges of length L. We explicit $\{\phi_k\}_{k\in\mathbb{N}}$ a Hilbert basis of $L^2(\mathscr{G},\mathbb{C})$ composed by eigenfunctions of A. The $(\mathcal{N})$ conditions in the external vertices imply that

$$\phi_k=(a_k^1\cos(\sqrt{\lambda_k}x),...,a_k^N\cos(\sqrt{\lambda_k}x)),\qquad \forall k\in\mathbb{N}$$

and $\{a_k^l\}_{\substack{k\in\mathbb{N}\\ l\leq N}}\subset\mathbb{C}$. For every $l,m\leq N$, the $(\mathcal{NK})$ condition in the internal vertex imply

$$a_k^l\cos(\sqrt{\lambda_k}L)=a_k^m\cos(\sqrt{\lambda_k}L)=c,\qquad \sum_{l=1}^N a_k^l\sin(\sqrt{\lambda_k}L)=0. \tag{4.11}$$

Let $\{\tilde{\lambda}_k\}_{k\in\mathbb{N}}$ be the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ where the eigenvalues are not repeated with their multiplicity. By substituting $c=0$ in the identity (4.11), we obtain the sequence of eigenvalues $\big\{\frac{(1+2n)^2\pi^2}{4L^2}\big\}_{n\in\mathbb{N}\cup\{0\}}$, while for $c\neq0$, we have

$$\left\{\frac{(n-1)^2\pi^2}{L^2}\right\}_{n\in\mathbb{N}}=\{\tilde{\lambda}_j\}_{j\in\mathbb{N}}\setminus\left\{\frac{(1+2n)^2\pi^2}{4L^2}\right\}_{n\in\mathbb{N}\cup\{0\}}$$

of multiplicity (N-1) (see also [BK13, *p.* 15] for further explanations). Now, for every $k\in\mathbb{N}$, there exists $j(k)\in\mathbb{N}$ such that one of the following is verified

$$\begin{aligned}&\text{either}\quad \phi_k^l=c_k^lf_{j(k)}^1\quad\text{for}\quad c_k^l\in\mathbb{C},\ \ |c_k^l|\leq1,\qquad \forall l\in\{1,...,N\},\\ &\text{or}\qquad \phi_k^l=c_k^lf_{j(k)}^2\quad\text{for}\quad c_k^l\in\mathbb{C},\ \ |c_k^l|\leq1,\qquad \forall l\in\{1,...,N\}.\end{aligned} \tag{4.12}$$

We call

$$J^1:=\Big\{k\in\mathbb{N}\,|\,\exists\, j(k)\in\mathbb{N}\ :\ \forall l\in\{1,...,N\},\ \exists\, c_k^l\in\mathbb{C},\ \ |c_k^l|\leq1\ :\ \phi_k^l=c_k^lf_{j(k)}^1\Big\}$$

and $J^2 := \mathbb{N} \setminus J^1$. We have

$$k \leq (N-1) \cdot j(k)$$

since each eigenvalue has multiplicity at most $(N-1)$. We call $\{e_j\}_{j\in\mathbb{N}}$ the edges composing $\mathscr{G}$ and we notice that they correspond to the interval I introduced above as they have the same length L. We consider the quantum graphs $I^{\mathcal{M}}$ and $I^{\mathcal{N}}$ defined in the first part of **1) (b)**. Thanks to **1) (a)**, for $s \in [0, 1/2)$, we have

$$H^s(\mathscr{G}, \mathbb{C}) = \Big(\prod_{j=1}^N H^s(e_j, \mathbb{C})\Big) = \Big(\prod_{j=1}^N H^s_{I^{\mathcal{M}}}\Big),$$
$$H^s(\mathscr{G}, \mathbb{C}) = \Big(\prod_{j=1}^N H^s(e_j, \mathbb{C})\Big) = \Big(\prod_{j=1}^N H^s_{I^{\mathcal{N}}}\Big),$$

which implies that, for every $\psi = (\psi^1, ..., \psi^N) \in H^s(\mathscr{G}, \mathbb{C})$ and $l \leq N$, we have

$$\sum_{k=1}^\infty \big|k^s \langle f_k^1, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\big|^2 < \infty, \qquad \sum_{k=1}^\infty \big|k^s \langle f_k^2, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\big|^2 < \infty.$$

Now, there exists $C_1 > 0$ such that

$$\begin{aligned}
&\|\psi\|_{H^s_{\mathscr{G}}} = \Big(\sum_{k\in\mathbb{N}} \big|k^s \langle \phi_k, \psi\rangle_{L^2(\mathscr{G},\mathbb{C})}\big|^2\Big)^{\frac{1}{2}}\\
&\leq \Big(\sum_{k\in J^1} \Big|k^s \sum_{l=1}^N c_k^l \langle f_{j(k)}^1, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\Big|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k\in J^2} \Big|k^s \sum_{l=1}^N c_k^l \langle f_{j(k)}^2, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\Big|^2\Big)^{\frac{1}{2}}\\
&\leq C_1\Big(\sum_{l=1}^N \sum_{k\in J^1} \Big|k^s \langle f_{j(k)}^1, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\Big|^2\Big)^{\frac{1}{2}} + C_1\Big(\sum_{l=1}^N \sum_{k\in J^2} \Big|k^s \langle f_{j(k)}^2, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\Big|^2\Big)^{\frac{1}{2}}\\
&\leq C_1(N-1)^s \sum_{l=1}^N \Big(\sum_{k\in\mathbb{N}} \Big|j(k)^s \langle f_{j(k)}^1, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\Big|^2\Big)^{\frac{1}{2}}\\
&+ C_1(N-1)^s \sum_{l=1}^N \Big(\sum_{k\in\mathbb{N}} \Big|j(k)^s \langle f_{j(k)}^2, \psi^l\rangle_{L^2(e_l,\mathbb{C})}\Big|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

The function $j(\cdot) : \mathbb{N} \to \mathbb{N}$ is increasing and, for each $n \in \mathbb{N}$, there exist at most $(N-1)$ values $k \in \mathbb{N}$ such that $j(k) = n$ as each eigenvalue of $\mathscr{G}$ has multiplicity at most $(N-1)$. Thus, there exists $C_2 > 0$ such that

$$\|\psi\|_{H^s_{\mathscr{G}}} \leq C_2 \sum_{l=1}^{N} \Big(\sum_{j\in\mathbb{N}} \Big| j^s \langle f_j^1, \psi^l \rangle_{L^2(e_l,\mathbb{C})} \Big|^2 \Big)^{\frac{1}{2}}$$
$$+ C_2 \sum_{l=1}^{N} \Big(\sum_{j\in\mathbb{N}} \Big| j^s \langle f_j^2, \psi^l \rangle_{L^2(e_l,\mathbb{C})} \Big|^2 \Big)^{\frac{1}{2}} < \infty.$$

Thus, $H^s(\mathscr{G},\mathbb{C}) \subseteq H^s_{\mathscr{G}}$, for $s \in [0,1/2)$, which implies $H^s_{\mathscr{G}} = H^s(\mathscr{G},\mathbb{C})$.

The same techniques leads to the claim for every $s_1 \in \mathbb{N}\cup\{0\}$ and $s_2 \in [0,1/2)$ by noticing the two following facts. First, for $s > 0$ and for every

$$\psi = (\psi^1, ..., \psi^N) \in H^s_{\mathscr{G}},$$

we have

$$\psi^l \in H^s(I^{\mathcal{M}},\mathbb{C}), \qquad \psi^l \in H^s(I^{\mathcal{N}},\mathbb{C}), \qquad \forall\, l \leq N$$

thanks to the identities (4.12). Second, as in **1) (a)**, for each $s_1 \in \mathbb{N}\cup\{0\}$ and $s_2 \in [0,1/2)$,

$$H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1}_{I^{\mathcal{M}}} \cap H^{s_1+s_2}(I^{\mathcal{M}},\mathbb{C}),$$

and, for every $s_1 \in 2\mathbb{N}\cup\{0\}$ and $s_2 \in [0,3/2)$,

$$H^{s_1+s_2}_{I^{\mathcal{N}}} = H^{s_1}_{I^{\mathcal{N}}} \cap H^{s_1+s_2}(I^{\mathcal{N}},\mathbb{C}).$$

In conclusion, for $s_1 \in \mathbb{N}\cup\{0\}$ and $s_2 \in [0,1/2)$, it is verified that

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2}(\mathscr{G},\mathbb{C}). \tag{4.13}$$

(c) Interpolation properties for a generic graph $\mathscr{G}$: Let $\mathscr{G}$ be a graph equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ and N be the number of edges of $\mathscr{G}$. Let $\widetilde{L} < \min\{L_k/2 : k \in \{1,...,N\}\}$ and let $v \in V_i \cup V_e$ be a vertex of $\mathscr{G}$. We define $\widetilde{\mathscr{G}}(v)$ as follows.

- If $v \in V_i$, then $\widetilde{\mathscr{G}}(v)$ is a star subgraph of $\mathscr{G}$ equipped with $(\mathcal{N})$-$(\mathcal{NK})$, with $n(v)$ edges of equal lengths $\widetilde{L}$, with internal vertex v.
- If $v \in V_e$, then $\widetilde{\mathscr{G}}(v)$ is an interval of length $\widetilde{L}$ with external vertex v equipped with the same boundary conditions that v has in $\mathscr{G}$. We impose $(\mathcal{N})$ on the other vertex.

Afterwards, we construct a family of intervals $\{I_j\}_{j\leq\hat{N}}$ for $\hat{N}\leq N$ as follows. For each $v,\hat{v}$ such that $\widetilde{\mathscr{G}}(v)$ and $\widetilde{\mathscr{G}}(\hat{v})$ have respectively two external vertices w_1 and w_2 lying on the same edge e and such that $w_1\notin\widetilde{\mathscr{G}}(\hat{v})$, we construct an interval strictly containing w_1 and w_2, strictly contained in e and equipped with $(\mathcal{N})$. Thanks to **1) (a)** and **1) (b)**, for every $v\in V_i\cup V_e$, $j\leq\hat{N}$, $s_1\in\mathbb{N}\cup\{0\}$ and $s_2\in[0,1/2)$,

$$H^{s_1+s_2}_{\widetilde{\mathscr{G}}(v)}=H^{s_1}_{\widetilde{\mathscr{G}}(v)}\cap H^{s_1+s_2}(\widetilde{\mathscr{G}}(v),\mathbb{C}),$$
$$H^{s_1+s_2}_{I_j}=H^{s_1}_{I_j}\cap H^{s_1+s_2}(I_j,\mathbb{C}).$$

We define

$$G=\{G_j\}_{j\leq M+\hat{N}}:=\{\widetilde{\mathscr{G}}(v_j)\}_{j\leq M}\cup\{I_j\}_{j\leq\hat{N}}$$

and we notice that G covers $\mathscr{G}$. As in **1) (a)**, we see each function with domain $\mathscr{G}$ as a vector of functions each one with domain G_j for a suitable $j\leq M+\hat{N}$. Now, [Tri95, *relation* (12), *Chapter* 1.18.1] implies that the interpolation between two products of spaces is the product of the respective interpolations and, for each $s_1\in\mathbb{N}\cup\{0\}$ and $s_2\in[0,1/2)$,

$$H^{s_1+s_2}_{\mathscr{G}}=H^{s_1}_{\mathscr{G}}\cap H^{s_1+s_2}(\mathscr{G},\mathbb{C}).$$

2) Let $\mathscr{G}$ be a graph equipped with $(\mathcal{N})$-$(\mathcal{NK})$. We define $\widetilde{\mathscr{G}}$ from $\mathscr{G}$ as follows. For every external vertex $v\in V_e$, we remove on the edge including v a section of length $\widetilde{L}/2$ containing v. We equip the new external vertex with $(\mathcal{N})$ and we call $N_e\in\mathbb{N}$ the number of external vertices of $\mathscr{G}$. By recalling the definition of the intervals $\widetilde{\mathscr{G}}(v)$ for every $v\in V_e$, introduced in **1) (c)**, we have

$$G'=\{G'_j\}_{j\leq N_e+1}:=\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}\cup\{\widetilde{\mathscr{G}}\}$$

covers $\mathscr{G}$. We see each function with domain $\mathscr{G}$ as a vector of functions, each one with domain G'_j for a suitable $j\leq N_e+1$. Thanks to the argument of **1) (a)**, for every $v\in V_e$, $s_1\in2\mathbb{N}\cup\{0\}$ and $s_2\in[0,3/2)$, we have

$$H^{s_1+s_2}_{\widetilde{\mathscr{G}}(v)}=H^{s_1}_{\widetilde{\mathscr{G}}(v)}\cap H^{s_1+s_2}(\widetilde{\mathscr{G}}(v),\mathbb{C}).$$

The techniques adopted in **1) (a)** and recalled in **1) (c)** lead to the proof.

3) The same arguments mentioned in **2)** ensure the claim by considering $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ as intervals equipped with $(\mathcal{D})$. □

Proof of Proposition 4.11. The proof is inspired by the ones of [BL10, *Lemma* 1] and [BL10, *Proposition* 2]. The first part introduces the techniques that are valid for any type of $\mathscr{G}$, while the second considers $\mathscr{G}$ equipped only with $(\mathcal{N})$ in the external vertices in which stronger results can be achieved.

Generic graphs:
1) Let $f(s) \in H^3 \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$, $t \in (0,T)$ and $f(s) = (f^1(s), ..., f^N(s))$. We prove that $G \in C^0([0,T], H^3_{\mathscr{G}})$. Thanks to the definition of $G(t)$, we have $G(t) = \sum_{k=1}^{\infty} \phi_k \int_0^t e^{i\lambda_k s}\langle \phi_k, f(s)\rangle ds$ and

$$\|G(t)\|_{(3)} = \left(\sum_{k=1}^{\infty}\left|k^3\int_0^t e^{i\lambda_k s}\langle\phi_k, f(s)\rangle ds\right|^2\right)^{\frac{1}{2}}.$$

We estimate the terms $\langle\phi_k, f(s,\cdot)\rangle$ for $k \in \mathbb{N}$ and $s \in (0,t)$. We suppose that $\lambda_1 \neq 0$. Let $\partial_x f(s) = (\partial_x f^1(s), ..., \partial_x f^N(s)$ be the derivative of $f(s)$ and $P(\phi_k) = (P(\phi_k^1), ..., P(\phi_k^N))$ be the primitive of ϕ_k such that

$$P(\phi_k) = -\frac{1}{\lambda_k}\partial_x\phi_k.$$

We call ∂e the set of the two points composing the boundaries of an edge e. For every $v \in V_e$, $\tilde{v} \in V_i$ and $j \in N(\tilde{v})$, there exist $a(v), a^j(\tilde{v}) \in \{-1, +1\}$ such that

(4.14)

$$\begin{aligned}
\langle\phi_k, f(s)\rangle &= \frac{1}{\lambda_k}\langle\phi_k, \partial_x^2 f(s)\rangle = \frac{1}{\lambda_k}\int_{\mathscr{G}}\phi_k(s)\partial_x^2 f(s,y)dy\\
&= \frac{1}{\lambda_k}\sum_{j=1}^{N}\Big[P(\phi_k^j)(x)\ \partial_x^2 f^j(s,x)\Big]_{\partial e_j} - \frac{1}{\lambda_k}\int_{\mathscr{G}} P(\phi_k)(y)\partial_x^3 f(s,y)dy\\
&= \frac{1}{\lambda_k^2}\sum_{v\in V_e} a(v)\partial_x\phi_k(v)\partial_x^2 f(s,v) + \frac{1}{\lambda_k^2}\sum_{v\in V_i}\sum_{j\in N(v)} a^j(v)\partial_x\phi_k^j(v)\partial_x^2 f^j(s,v)\\
&+ \frac{1}{\lambda_k^2}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dy.
\end{aligned}$$

From [BK13, *Theorem* 3.1.8] and [BK13, *Theorem* 3.1.10], there exist $C_1 > 0$ such that $\lambda_k^{-3} \leq C_1 k^{-6}$ for every $k \in \mathbb{N}$ (we provide further explanations on this property in Remark 4.17), then

$$
\begin{aligned}
&\left|k^3\int_0^t e^{i\lambda_k s}\langle\phi_k, f(s)\rangle ds\right| = \left|k^3\int_0^t e^{i\lambda_k s}\frac{1}{\lambda_k}\langle\phi_k, \partial_x^2 f(s)\rangle ds\right| \\
&\leq \frac{C_1}{\lambda_k^{1/2}}\left(\sum_{v\in V_e}\left|\partial_x\phi_k(v)\int_0^t e^{i\lambda_k s}\partial_x^2 f(s,v)ds\right| + \sum_{v\in V_i}\sum_{j\in N(v)}\left|\partial_x\phi_k^j(v)\int_0^t e^{i\lambda_k s}\partial_x^2 f^j(s,v)ds\right|\right. \\
&\left. + \left|\int_0^t e^{i\lambda_k s}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dyds\right|\right).
\end{aligned}
\tag{4.15}
$$

Remark 4.13. *We point out that*

$$A'\lambda_k^{-1/2}\partial_x\phi_k = \lambda_k\lambda_k^{-1/2}\partial_x\phi_k$$

for every $k\in\mathbb{N}$, where $A' = -\Delta$ is a self-adjoint Laplacian with compact resolvent and domain defined as follows. For every vertex $v\in V_e$ where each $g\in D(A)$ satisfies $(\mathcal{D})$, we impose that every $f\in D(A')$ satisfies $(\mathcal{N})$. For every $v\in V_e$ such that each $g\in D(A)$ satisfies $(\mathcal{N})$, we impose that every $f\in D(A')$ satisfies $(\mathcal{D})$. Moreover, for every $v\in V_i$ and $f=(f^1,...,f^N)\in D(A')$, we have

$$\begin{cases}\sum_{j\in\ N(v)} f^j(v) = 0,\\ \frac{\partial f}{\partial_x}\in C^0(\mathscr{G},\mathbb{C}).\end{cases}$$

In addition, $\|\lambda_k^{-1/2}\partial_x\phi_k\|^2 = \langle\lambda_k^{-1/2}\partial_x\phi_k, \lambda_k^{-1/2}\partial_x\phi_k\rangle = \langle\phi_k, \lambda_k^{-1}A\phi_k\rangle = 1$ and then $\{\lambda_k^{-1/2}\partial_x\phi_k\}_{k\in\mathbb{N}}$ is a Hilbert basis of $\mathscr{H}$.

Before studying (4.15), we consider that there exist $\mathbf{a^l}=\{a_k^l\}\subset\mathbb{C}$ and $\mathbf{b^l}=\{b_k^l\}\subset\mathbb{C}$ for every $l\in\{1,...,N\}$ so that

$$\phi_k^l(x) = a_k^l\cos(\sqrt{\lambda_k}x) + b_k^l\sin(\sqrt{\lambda_k}x),$$

$$\lambda_k^{-1/2}\partial_x\phi_k^l(x) = -a_k^l\sin(\sqrt{\lambda_k}x) + b_k^l\cos(\sqrt{\lambda_k}x). \tag{4.16}$$

Now, $2\geq\|\lambda_k^{-1/2}\partial_x\phi_k^l\|^2_{L^2(e^l)} + \|\phi_k^l\|^2_{L^2(e^l)} = (|a_k^l|^2+|b_k^l|^2)|e_l|$ for every $k\in\mathbb{N}$ and $l\in\{1,...,N\}$, which implies $\mathbf{a^l},\mathbf{b^l}\in\ell^\infty(\mathbb{C})$. Thus, from (4.16), there exists $C_2>0$ such that, for every $k\in\mathbb{N}$ and $v\in V_e\cup V_i$,

$$|\lambda_k^{-1/2}\partial_x\phi_k(v)|\leq C_2$$

and

$$\begin{aligned}\left|k^3\int_0^t e^{i\lambda_k s}\langle\phi_k,f(s)\rangle ds\right| \leq C_1\Bigg(C_2\sum_{v\in V_i\cup V_e}\sum_{j\in N(v)}\left|\int_0^t e^{i\lambda_k s}\partial_x^2 f^j(s,v)ds\right| \\ +\frac{1}{\lambda_k^{1/2}}\left|\int_0^t e^{i\lambda_k s}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dyds\right|\Bigg),\end{aligned} \tag{4.17}$$

$$\begin{aligned}\implies \quad \|G(t)\|_{(3)} \leq C_1\Bigg(C_2\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\left\|\int_0^t \partial_x^2 f^j(s,v)e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2} \\ +\left\|\int_0^t\left\langle\lambda_{(\cdot)}^{-1/2}\partial_x\phi_{(\cdot)}(s),\partial_x^3 f(s)\right\rangle e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2}\Bigg).\end{aligned} \tag{4.18}$$

For every $t>0$, thanks to Proposition $A.18$ (Appendix $A.2$), there exists $C_3(t)>0$ uniformly bounded for t lying on bounded intervals, such that for every $v\in V_e\cup V_i$ and $j\in N(v)$,

$$\left\|\int_0^t \partial_x^2 f^j(s,v)e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2} \leq C_3(t)\|\partial_x^2 f^j(\cdot,v)\|_{L^2((0,t),\mathbb{C})}.$$

Thus, there exists $C_4(t)>0$ uniformly bounded for t lying on bounded intervals such that

$$\begin{aligned}\|G\|_{H^3_{\mathscr{G}}} &\leq C_1\Bigg(C_2\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\left\|\int_0^t \partial_x^2 f^j(s,v)e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2} \\ &\quad+\left\|\int_0^t\left\langle\lambda_{(\cdot)}^{-1/2}\partial_x\phi_{(\cdot)}(s),\partial_x^3 f(s)\right\rangle e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2}\Bigg) \\ &\leq C_1\Bigg(C_2\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\left\|\int_0^t \partial_x^2 f^j(s,v)e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2} \\ &\quad+\sqrt{t}\left(\int_0^t\left\|\left\langle\lambda_{(\cdot)}^{-1/2}\partial_x\phi_{(\cdot)}(s),\partial_x^3 f(s)\right\rangle\right\|_{\ell^2}^2 ds\right)^{\frac{1}{2}}\Bigg)\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\|G\|_{H^3_{\mathscr{G}}} &\leq C_1\left(C_2\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\left\|\int_0^t \partial_x^2 f^j(s,v)e^{i\lambda_{(\cdot)}s}ds\right\|_{\ell^2}\right.\\ &\left.+\sqrt{t}\left(\int_0^t\|\partial_x^3 f(s)\|^2_{L^2(\mathscr{G},\mathbb{C})}ds\right)^{\frac{1}{2}}\right)\\ &\leq C_3(t)\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\|\partial_x^2 f^j(\cdot,v)\|_{L^2((0,t),\mathbb{C})}\\ &+\sqrt{t}\|f\|_{L^2((0,t),H^3)}\leq C_4(t)\|f(\cdot,\cdot)\|_{L^2((0,t),H^3\cap H^2_{\mathscr{G}})}.\end{aligned}\tag{4.20}$$

If $\lambda_1=0$, then relations (4.14), (4.15) and (4.17) are still valid for $k>1$ and $\phi_1\equiv 1$. There exists $C_5(t)>0$ uniformly bounded for t lying on bounded intervals such that

$$\left|\int_0^t\langle\phi_1,f(s)\rangle ds\right|\leq C_5(t)\|f(\cdot,\cdot)\|_{L^2((0,t),H^3\cap H^2_{\mathscr{G}})}.$$

Then, we modify (4.18) as follows

$$\begin{aligned}\|G(t)\|_{(3)} &\leq\left(\sum_{k=2}^{\infty}\left|k^3\int_0^t e^{i\lambda_k s}\frac{1}{\lambda_k}\langle\phi_k,\partial_x^2 f\rangle ds\right|^2\right)^{\frac{1}{2}}+\left(\left|\int_0^t\langle\phi_1(\cdot),f(t,\cdot)\rangle|ds\right|^2\right)^{\frac{1}{2}}\\ &\leq C_1\left(\sum_{k=1}^{\infty}\left|k\int_0^t e^{i\lambda_k s}\langle\phi_k,\partial_x^2 f\rangle ds\right|^2\right)^{\frac{1}{2}}+\left(\left|\int_0^t\langle\phi_1(\cdot),f(t,\cdot)\rangle|ds\right|^2\right)^{\frac{1}{2}}.\end{aligned}\tag{4.21}$$

The techniques adopted in the relations (4.19) and (4.20) are still valid and they lead to the existence of $C_6(t)>0$ uniformly bounded for t lying on bounded intervals such that

$$\|G\|_{H^3_{\mathscr{G}}}\leq C_6(t)\|f(\cdot,\cdot)\|_{L^2((0,t),H^3\cap H^2_{\mathscr{G}})}.$$

For every $t\in[0,T]$, the inequality shows that $G(t)\in H^3\cap H^2_{\mathscr{G}}$ and $t\mapsto G(t)$ is continuous since the upper bounds provided are uniformly bounded. The Dominated Convergence Theorem ensures the property.
In conclusion, if $f(s)=(f^1(s),...,f^N(s))\in H^3\cap H^2_{\mathscr{G}}$ for almost every $s\in(0,t)$ and $t\in(0,T)$, then

$$G\in C^0([0,T],H^3_{\mathscr{G}}).$$

Now, if $d\in[1,5/2)$ and $f(s)=(f^1(s),...,f^N(s))\in H^{2+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}$ for almost every $s\in(0,t)$ and $t\in(0,T)$, then the previous steps lead to

$$G\in C^0([0,T],H^3_{\mathscr{G}}).$$

Let $\partial_x A=A'\partial_x$ for A' defined in Remark 4.13. We consider G as a vector of functions such that $G(t)=(G^1(t),...,G^N(t))$ and, for every $l\leq N$,

$$G^l(t)=\int_0^t e^{iAs}f^l(s)ds.$$

If $\sum_{l\in N(v)}\partial_x^{2n+1}f^l(s,v)=0$ for $n\in\mathbb{N}$, $v\in V_e$ and for every $s\in(0,t)$ and $t\in(0,T)$, then

$$\sum_{l\in N(v)}\partial_x^{2n+1}G^l(t,v)=\int_0^t e^{iA's}\left(\sum_{l\in N(v)}\partial_x^{2n+1}f(s,v)\right)ds=0.$$

If $\partial_x^{2n}f(s,v)=0$ for $n\in\mathbb{N}$, $v\in V_i$ and for every $s\in(0,t)$ and $t\in(0,T)$, then

$$\partial_x^{2n}G(t,v)=\int_0^t e^{iAs}\partial_x^{2n}f(s,v)ds=0.$$

Now, we have $G\in C^0([0,T],H^{2+d}_{\mathcal{NK}})$ and, thanks to Proposition 4.12, there follows $H^{2+d}_{\mathcal{NK}}\cap H^3_{\mathscr{G}}=H^{2+d}_{\mathscr{G}}$ for $d\in[1,5/2)$, which implies

$$G\in C^0([0,T],H^{2+d}_{\mathscr{G}}).$$

In conclusion, if $d\in[1,5/2)$ and $f(s)=(f^1(s),...,f^N(s))\in H^{2+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}$ for almost every $s\in(0,t)$ and $t\in(0,T)$, then

$$G\in C^0([0,T],H^{2+d}_{\mathscr{G}}).$$

When $d\in[1,3/2)$ and $f(s)\in H^{2+d}\cap H^2_{\mathscr{G}}$ for almost every $s\in(0,t)$ and $t\in(0,T)$, we have $G\in C^0([0,T],H^3_{\mathscr{G}})$ and $G\in C^0([0,T],H^{2+d})$. Now, $H^{2+d}\cap H^3_{\mathscr{G}}=H^{2+d}_{\mathscr{G}}$ for $d\in[1,3/2)$, thanks to Proposition 4.12. Hence

$$G\in C^0([0,T],H^{2+d}_{\mathscr{G}}).$$

2) Let problem (4.1) verify the first or the fourth point of Assumptions IV. We adopt the techniques of the proof of [BL10, *Proposition* 2]. Thanks to the arguments of Remark 2.1 and to the fact that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^2_{\mathscr{G}}$, we have $B \in L(H^{2+d}_{\mathscr{G}}, H^{2+d} \cap H^2_{\mathscr{G}})$. For every $\psi \in H^{2+d}_{\mathscr{G}}$, we define the map F such that

$$t \mapsto F(\psi)(t) = e^{-iAt} - \int_0^t e^{-iA(t-s)} u(s) B\psi(s) ds \in C^0([0,T], H^{2+d}_{\mathscr{G}}).$$

For every $\psi^1, \psi^2 \in H^{2+d}_{\mathscr{G}}$, thanks to the first point of the proof, there exists $C(t) > 0$ uniformly bounded for t lying on bounded intervals, such that

$$\begin{aligned} \|F(\psi^1)(t) - F(\psi^2)(t)\|_{(2+d)} &\leq \left\| \int_0^t e^{-iA(t-s)} u(s) B(\psi^1(s) - \psi^2(s)) ds \right\|_{(2+d)} \\ &\leq C(t) \|u\|_{L^2((0,t),\mathbb{R})} \, ||| \, B \, ||| _{L(H^{2+d}_{\mathscr{G}}, H^{2+d} \cap H^2_{\mathscr{G}})} \|\psi^1 - \psi^2\|_{L^\infty((0,t), H^{2+d}_{\mathscr{G}})}. \end{aligned}$$

If $\|u\|_{L^2((0,t),\mathbb{R})}$ is small enough, then F is a contraction and Banach Fixed Point Theorem implies that there exists $\psi \in C^0([0,T], H^{2+d}_{\mathscr{G}})$ such that $F(\psi) = \psi$. When $\|u\|_{L^2((0,t),\mathbb{R})}$ is not sufficiently small, one can consider $\{t_j\}_{0 \leq j \leq n}$ a partition of $[0,t]$ for $n \in \mathbb{N}$. We choose a partition such that each $\|u\|_{L^2([t_{j-1},t_j],\mathbb{R})}$ is so small that the map F, defined on the interval $[t_{j-1}, t_j]$, is a contraction. The previous argument leads to the result.

Graphs equipped with $(\mathcal{N})$ on the external vertices:

1) Let $f(s) \in H^4 \cap H^2_{\mathscr{G}} \cap H^3_{\mathcal{NK}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$. We notice the spectrum of A is simple and $\lambda_1 = 0$ under the chosen hypotheses. We proceed as in (4.21) and we notice that the first two terms of the last line of (4.14) are equal to zero. First, $\partial_x^2 f(s) \in C^0$ as $f(s) \in H^3_{\mathcal{NK}}$ and, for $v \in V_e$, we have $\partial_x \phi_k(v) = 0$ thanks to $(\mathcal{N})$ boundary conditions. Second, for every $v \in V_i$, thanks to the $(\mathcal{NK})$ in $v \in V_i$, we have

$$\sum_{j \in N(v)} a^j(v) \partial_x \phi_k^j(v) = 0.$$

Indeed, the terms $a^j(v)$ assume different signs according to the orientation of the edges connected in v. If their orientations are assumed in the directions away from the vertex, then we have $a^j(v) = a^l(v)$ for every $l, j \in N(v)$. Thus, the first two terms of the relation (4.14) become, for $s \in (0,t)$ and

$t \in (0,T)$,

$$\frac{1}{\lambda_k^2}\sum_{v\in V_e} a(v)\partial_x\phi_k(v)\partial_x^2 f(s,v) + \frac{1}{\lambda_k^2}\sum_{v\in V_i}\sum_{j\in N(v)} a^j(v)\partial_x\phi_k^j(v)\partial_x^2 f^j(s,v)$$
$$= \frac{1}{\lambda_k^2}\sum_{v\in V_i}\partial_x^2 f(s,v)\sum_{j\in N(v)} a^j(v)\partial_x\phi_k^j(v) = 0$$

In the relation (4.14), integration by parts leads to gain one order of regularity since

(4.22)
$$\langle\phi_k, f(s)\rangle = -\frac{1}{\lambda_k^2}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dy = -\frac{1}{\lambda_k^2}\sum_{v\in V_e} a(v)\phi_k(v)\partial_x^3 f(s,v)$$
$$-\frac{1}{\lambda_k^2}\sum_{v\in V_i}\sum_{j\in N(v)} a^j(v)\phi_k^j(v)\partial_x^3 f^j(s,v) + \frac{1}{\lambda_k^2}\int_{\mathscr{G}}\phi_k(y)\partial_x^4 f(s,y)dy.$$

Since $\{\phi_k\}_{k\in\mathbb{N}}$ is a Hilbert basis, we can proceed as before by using Proposition $A.18$ (Appendix $A.2$) as done for generic graphs. Hence, there exists $C_1(t) > 0$ uniformly bounded for t lying in bounded intervals such that

$$\|G\|_{H^4_{\mathscr{G}}} \leq C_1(t)\|f(\cdot,\cdot)\|_{L^2((0,t),H^4\cap H^3_{\mathscr{G}})}.$$

When $d \in [2, 5/2)$ and $f(s) \in H^{2+d}\cap H^2_{\mathscr{G}}\cap H^3_{\mathcal{NK}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then we have

$$G \in C^0([0,T], H^{2+d}_{\mathscr{G}}),$$

thanks to $H^{2+d} \cap H^4_{\mathscr{G}} = H^{2+d}_{\mathscr{G}}$ for $d \in [2, 5/2)$ due to Proposition 4.12.

If $d \in [2, 7/2)$ and $f(s) \in H^{2+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then

$$G \in C^0([0,T], H^{2+d}_{\mathscr{G}}),$$

thanks to $H^{2+d}_{\mathcal{NK}} \cap H^4_{\mathscr{G}} = H^{2+d}_{\mathscr{G}}$ for $d \in [2, 7/2)$ due to Proposition 4.12.

2) Let (4.1) verify the second or the third point of Assumptions IV. The second claim follows as in the proof dedicated to the generic graphs. □

4.3 Proof of Theorem 4.3

The proof follows the ideas of the ones of Theorem 2.2 and Theorem 3.5.

4.3.1 Local exact controllability in $H^s_{\mathscr{G}}$

By referring to the proofs of Theorem 2.8 or Proposition 3.10 (see also [Bea05], [BL10], [Mor14] and [MN15]), it is possible to see the following fact. Let Assumptions III be verified and let

$$O^s_{\epsilon,T} := \Big\{\psi \in H^s_{\mathscr{G}} \Big| \ \|\psi\| = 1, \ \|\psi - \phi_1(T)\|_{(s)} < \epsilon\Big\}$$

Let us consider the decomposition

$$\Gamma^u_t \phi_1 = \sum_{k=1}^{\infty} \phi_k(t)\langle\phi_k(t), \Gamma^u_t\phi_1\rangle.$$

As in the proof of Theorem 2.8, we define the map α, the sequence with elements

$$\alpha_k(u) = \langle\phi_k(T), \Gamma^u_T\phi_1\rangle, \qquad \forall k \in \mathbb{N}$$

so that

$$\alpha : L^2((0,T),\mathbb{R}) \longrightarrow Q := \{\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in h^s(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

Ensuring the local existence of the control function for a time $T > 0$ is equivalent to prove the local surjectivity of α. To this end, we use the Generalized Inverse Function Theorem (Proposition 2.7). As in the proof of Theorem 2.8, for $\alpha(0) = \delta = \{\delta_{k,1}\}_{k\in\mathbb{N}}$, we study the surjectivity of $\gamma(v) := (d_u\alpha(0)) \cdot \ v$ the Fréchet derivative of α such that

$$\gamma : L^2((0,T),\mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}$$

(see the proof of Theorem 2.8 for further details on $T_\delta Q$). The map γ is the sequence with elements

$$\begin{aligned}\gamma_k(v) : &= \Big\langle \phi_k(T), -i\int_0^T e^{-iA(T-\tau)}v(s)Be^{-iA\tau}\phi_1 d\tau\Big\rangle \\ &= -i\int_0^T v(\tau)e^{i(\lambda_k-\lambda_1)s}d\tau\langle\phi_k, B\phi_1\rangle, \qquad k\in\mathbb{N}.\end{aligned}$$

Thanks to Proposition 4.11, the well-posedness of (4.1) is guaranteed in $H^s_{\mathscr{G}}$ and both α and γ take value in h^s. Thus, the local surjectivity of α can be proved by ensuring the solvability of the moment problem

$$\frac{x_k}{\langle\phi_k, B\phi_1\rangle} = -i\int_0^T u(\tau)e^{i(\lambda_k-\lambda_1)\tau}d\tau. \tag{4.23}$$

1) In the hypotheses of the first point of Theorem 4.3, Proposition $A.14$ (Appendix $A.2$) leads to the solvability of (4.23) in $h^{\tilde{d}}$. Indeed, if we consider the sequence of numbers

$$\{\lambda_k - \lambda_1\}_{k\in\mathbb{N}}$$

and $\{x_k\}_{k\in\mathbb{N}} \in h^{\tilde{d}+2+\eta}$, then the hypotheses of Proposition $A.14$ are satisfied since $B_{1,1} \in \mathbb{R}$ as B is symmetric, the element $ix_1/B_{1,1} \in \mathbb{R}$ and $\{x_k B_{k,l}^{-1}\}_{k\in\mathbb{N}} \in h^{\tilde{d}}$ thanks to the first point of Assumptions III.
In conclusion, $\{\gamma_k(u)\}_{k\in\mathbb{N}} \in h^s$ for every $u \in L^2((0,T),\mathbb{R})$ and the moment problem (4.23) is solvable for $\{x_k\}_{k\in\mathbb{N}} \in h^s \subseteq h^{2+\tilde{d}+\eta}$ with $s = d+2$ since $d \geq \tilde{d} + \eta$.

2) In the hypotheses of the second point of Theorem 4.3, the solvability of (4.23) is guaranteed by Proposition $A.17$ (Appendix $A.2$) in $h^{\tilde{d}-1}$ thanks to the relation (4.3).
Indeed, from [BK13, $Theorem$ 3.1.8] and [BK13, $Theorem$ 3.1.10], there exist $C_1, C_2 > 0$ such that $C_1 k \leq \sqrt{\lambda_k} \leq C_2 k$, for every $k \in \mathbb{N}$ (in Remark 4.17 we provide further explanations on this property).
By considering the first point of Assumptions III, $\{x_k\}_{k\in\mathbb{N}}$ has to be in $h^{\tilde{d}+1+\eta}$, which is true since $\{\gamma_k(u)\}_{k\in\mathbb{N}} \in h^s$ for every $u \in L^2((0,T),\mathbb{R})$ and then $\{x_k\}_{k\in\mathbb{N}} \in h^s \subseteq h^{1+\tilde{d}+\eta}$ for $s = d+2$.

4.3.2 Global approximate controllability in $H^s_{\mathscr{G}}$:

Let $s = d+2$ for d introduced in Assumptions IV. The approximate controllability of the problem (4.1) in $H^s_{\mathscr{G}}$ follows from the proof of Theorem 3.14. In other words, for every $\psi \in H^s_{\mathscr{G}}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi \in H^s_{\mathscr{G}}$ and $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\|\widehat{\Gamma}\psi - \Gamma^u_T\psi\|_{(s)} < \epsilon.$$

The only difference with the mentioned proof is that the propagation of regularity from Kato [Kat53] has to be applied by considering different spaces. Let $B : H^{s_1}_{\mathscr{G}} \to H^{s_1}_{\mathscr{G}}$ for $s_1 \geq 0$. As in the proof of Proposition 2.12, for every $T > 0$, $u \in BV((0,T),\mathbb{R})$ and $\psi \in H^{s_1+2}_{\mathscr{G}}$, there exists $C(K) > 0$ depending on $K = \big(\|u\|_{BV((0,T),\mathbb{R})}, \|u\|_{L^\infty((0,T),\mathbb{R})}, T\|u\|_{L^\infty((0,T),\mathbb{R})}\big)$ such that

$$\|\Gamma^u_T\psi\|_{(s_1+2)} \leq C(K)\|\psi\|_{(s_1+2)}.$$

This last result and the proof of Theorem 3.14 lead to the global approximate controllability in $H^s_{\mathscr{G}}$ with $s \in [s_1, s_1+2)$ when $B : H^{s_1}_{\mathscr{G}} \to H^{s_1}_{\mathscr{G}}$.

Let d be introduced in Assumptions IV and the hypotheses of Theorem 4.3 be satisfied.

- If $d < 2$, then $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$ and the global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d + 2 < 4$.

- If $d \in [2, 5/2)$ and the second or the fourth point of Assumptions IV is verified, then $B : H^{d_1} \to H^{d_1}$ for $d_1 \in (d, 5/2)$ from Assumptions IV. Now, $H^{d_1}_{\mathscr{G}} = H^{d_1} \cap H^2_{\mathscr{G}}$, thanks to Proposition 4.12, and $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$ implies $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d + 2 < d_1 + 2$.

- If $d \in [5/2, 7/2)$, then $B : H^{d_1}_{\mathcal{NK}} \to H^{d_1}_{\mathcal{NK}}$ for $d_1 \in (d, 7/2)$ and $H^{d_1}_{\mathscr{G}} = H^{d_1}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ from Proposition 4.12. Now, $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$ that implies $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d + 2 < d_1 + 2$.

4.3.3 Global exact controllability in $H^s_{\mathscr{G}}$

The global exact controllability follows by gathering the local exact controllability and the global approximate controllability as in the proof of Proposition 3.15, thanks to the time reversibility (see Paragraph 2.1).

We recall that $s = 2 + d$ for d defined in Assumptions IV. The global approximate controllability and the local exact controllability are valid for the problem (4.1) and for the reversed dynamics (2.4). For any $\psi^1, \psi^2 \subset H^s_{\mathscr{G}}$ so that $\|\psi^1\| = \|\psi^2\| = p$, there exist $T_1, T_2 > 0$ and $u_1 \in L^2((0, T_1), \mathbb{R})$, $u_2 \in L^2((0, T_2), \mathbb{R})$ such that

$$p^{-1}\Gamma^{u_1}_{T_1}\psi^1 \in O^s_{\epsilon,T}, \qquad p^{-1}\widetilde{\Gamma}^{u_2}_{T_2}\psi^2 \in O^s_{\epsilon,T}.$$

Thanks to the local exact controllability, there exist $T > 0$ and $u_3, u_4 \in L^2((0, T), \mathbb{R})$ such that

$$p^{-1}\Gamma^{u_3}_{T}\Gamma^{u_1}_{T_1}\psi_1 = \phi_1 = p^{-1}\widetilde{\Gamma}^{u_4}_{T}\widetilde{\Gamma}^{u_2}_{T_2}\psi_2.$$

In conclusion, the time reversibility (Paragraph 2.1) leads to

$$\Gamma^{\widetilde{u}_2}_{T_2}\Gamma^{\widetilde{u}_4}_{T}\Gamma^{u_3}_{T}\Gamma^{u_1}_{T_1}\psi_1 = \psi_2.$$

4.4 Proofs of Theorem 4.5 and Theorem 4.7

We rephrase in the following proposition the so-called Roth's theorem, which represents an important tool for the proofs of Theorem 4.5 and Theorem 4.7.

Proposition 4.14. *(Roth's Theorem; [Rot56]) If z is an algebraic irrational number, then for every $\epsilon > 0$ the inequality*

$$\Big|z - \frac{n}{m}\Big| \leq \frac{1}{m^{2+\epsilon}}$$

is satisfied for at most a finite number of $n, m \in \mathbb{Z}$.

Lemma 4.15. *Let $\{\lambda_k^1\}_{k\in\mathbb{N}}$ and $\{\lambda_k^2\}_{k\in\mathbb{N}}$ be sequences of numbers respectively obtained by reordering*

$$\Big\{\frac{k^2\pi^2}{L_l^2}\Big\}_{\substack{k,l\in\mathbb{N},\\ l\leq N_1}} \qquad \Big\{\frac{k^2\pi^2}{\tilde{L}_i^2}\Big\}_{\substack{k,i\in\mathbb{N},\\ i\leq N_2}}$$

for $N_1, N_2 \in \mathbb{N}$ and $\{L_l\}_{l\leq N_1}, \{\tilde{L}_i\}_{i\leq N_2} \subset \mathbb{R}$. If all the ratios $\tilde{L}_i/L_l$ are algebraic irrational numbers, then for every $\epsilon > 0$, there exists a constant $C > 0$ such that

$$|\lambda_{k+1}^1 - \lambda_k^2| \geq \frac{C}{k^\epsilon}, \qquad \forall k \in \mathbb{N}.$$

Proof. Let z be an algebraic irrational number. Roth's Theorem introduced in Proposition 4.14 implies that, for every $\epsilon > 0$,

$$\Big|z - \frac{n}{m}\Big| \geq \frac{1}{m^{2+\epsilon}},$$

for every $m, n \in \mathbb{N}$, up to a finite number of $n, m \in \mathbb{N}$. Moreover, $z \neq \frac{m}{n}$ for every $n, m \in \mathbb{N}$. Then, for every $\epsilon > 0$ and for $C > 0$ small enough, there holds

$$\Big|z - \frac{n}{m}\Big| \geq \frac{C}{m^{2+\epsilon}}, \qquad \forall m, n \in \mathbb{N}.$$

Now, for every $k \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ and $i, l \leq N$ such that $\lambda_{k+1}^1 = \frac{m^2\pi^2}{L_l^2}$, $\lambda_k^2 = \frac{n^2\pi^2}{\tilde{L}_i^2}$, $\lambda_{k+1}^1 \neq \lambda_k^2$. We suppose $L_l < \tilde{L}_i$. If $m < n$, then for each $\epsilon > 0$, there exists $C_1 > 0$ small enough

$$\begin{aligned}\Big|\frac{m^2\pi^2}{L_l^2} - \frac{n^2\pi^2}{\tilde{L}_i^2}\Big| &= \Big|\Big(\frac{m\pi}{L_l} + \frac{n\pi}{\tilde{L}_i}\Big)\Big(\frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i}\Big)\Big| \geq \frac{2m\pi}{\tilde{L}_i}\Big|\frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i}\Big| \\ &\geq \frac{2m^2\pi^2}{\tilde{L}_i^2}\Big|\frac{\tilde{L}_i}{L_l} - \frac{n}{m}\Big| \geq \frac{2C_1 m^2\pi^2}{m^{2+\epsilon}\tilde{L}_i^2} \geq \frac{2C_1\pi^2}{m^\epsilon\tilde{L}_i^2}.\end{aligned}$$

When $m \geq n$, it follows

$$\Big|\frac{m^2\pi^2}{L_l^2} - \frac{n^2\pi^2}{\tilde{L}_i^2}\Big| \geq n^2\pi^2\Big(\frac{1}{L_l^2} - \frac{1}{\tilde{L}_i^2}\Big).$$

In conclusion, there exists $C_2 > 0$ such that

$$|\lambda_{k+1}^1 - \lambda_k^2| \geq \frac{C_2}{(k+1)^\epsilon} \geq \frac{C_2}{2^\epsilon k^\epsilon}$$

for $k \in \mathbb{N}$ and the proof is achieved. □

The following proposition rephrases the results of [BK13, *Theorem* 3.1.8] and [BK13, *Theorem* 3.1.10]. Let $\{\lambda_k^{\widehat{\mathscr{G}}}\}_{k\in\mathbb{N}}$ be the spectrum of A on a generic compact quantum graph $\widehat{\mathscr{G}}$.

Proposition 4.16.

1. *Let w, v be two vertices of $\mathscr{G}$ equipped with $(\mathcal{NK})$ or $(\mathcal{N})$ boundary conditions. If $\mathscr{G}'$ is the graph obtained by merging in $\mathscr{G}$ the vertices w and v in one unique vertex equipped with $(\mathcal{NK})$, then*

$$\lambda_k^{\mathscr{G}} \leq \lambda_k^{\mathscr{G}'} \leq \lambda_{k+1}^{\mathscr{G}}, \qquad \forall k \in \mathbb{N}.$$

2. *Let w be a vertex of $\mathscr{G}$. If $\mathscr{G}^D$ is the graph obtained by imposing $(\mathcal{D})$ boundary condition on w, then*

$$\lambda_k^{\mathscr{G}} \leq \lambda_k^{\mathscr{G}^D} \leq \lambda_{k+1}^{\mathscr{G}}, \qquad \forall k \in \mathbb{N}.$$

Remark 4.17. *Let $\mathscr{G}$ be any compact quantum graphs composed by edges of lengths $\{L_l\}_{l\leq N}$. Thanks to Proposition* 4.16, *there exist $C_1, C_2 > 0$ such that for $k \geq 2$,*

$$C_1 k^2 \leq \lambda_k^{\mathscr{G}} \leq C_2 k^2. \tag{4.24}$$

Indeed, we define the quantum graph $\mathscr{G}^{\mathcal{D}}$ from $\mathscr{G}$ by imposing $(\mathcal{D})$ boundary conditions in each vertex. We also denote $\mathscr{G}^{\mathcal{N}}$ the quantum graph obtained from $\mathscr{G}$ by disconnecting each edge and by imposing $(\mathcal{N})$ boundary conditions in each vertex. The graphs $\mathscr{G}^{\mathcal{D}}$ and $\mathscr{G}^{\mathcal{N}}$ are respectively obtained in at most M and $2N$ steps from $\mathscr{G}$ (M and N are respectively the numbers of vertices and edges). Thanks to Proposition 4.16, *for $k > 2N$, we have*

$$\lambda_{k-2N}^{\mathscr{G}^{\mathcal{N}}} \leq \lambda_k^{\mathscr{G}} \leq \lambda_{k+M}^{\mathscr{G}^{\mathcal{D}}}.$$

The sequences $\lambda_k^{\mathscr{G}\mathcal{N}}$ *and* $\lambda_k^{\mathscr{G}\mathcal{D}}$ *are respectively obtained by reordering*

$$\left\{\frac{k^2\pi^2}{L_l^2}\right\}_{\substack{k\in\mathbb{N}\\ l\leq N}}, \qquad \left\{\frac{(k-1)^2\pi^2}{L_i^2}\right\}_{\substack{k\in\mathbb{N}\\ i\leq N}}.$$

Thus, for each $l > 2N+1$,

$$\lambda_{l-2N}^{\mathscr{G}\mathcal{N}} \geq \frac{(l-2N-1)^2\pi^2}{N^2\max\{L_j^2 : j\leq N\}} \geq \frac{l^2\pi^2}{2^{2(2N+1)}N^2\max\{L_j^2 : j\leq N\}},$$

and

$$\lambda_{l+M}^{\mathscr{G}\mathcal{D}} \leq \frac{(l+M)^2\pi^2}{\min\{|L_j|^2 : j\leq N\}} \leq \frac{l^2 2^{2M}\pi^2}{\min\{L_j^2 : j\leq N\}}.$$

The claim is valid for every $k\geq 2$ *as* $\lambda_k \neq 0$. *In conclusion, if* $\lambda_1\neq 0$, *then there exists* $C_3, C_4 > 0$ *such that*

$$C_3k^2 \leq \lambda_k^{\mathscr{G}} \leq C_4k^2, \qquad \forall k\in\mathbb{N}.$$

Proof of Theorem 4.5. Let $\{\lambda_k^{\widetilde{\mathscr{G}}}\}_{k\in\mathbb{N}}$ be the eigenvalues A for a graph $\widetilde{\mathscr{G}}$. Let $\mathscr{G}$ be a tadpole graph equipped with $(\mathcal{D})$-$(\mathcal{NK})$. Let $\mathscr{G}^D$ be the graph obtained from $\mathscr{G}$ by imposing $(\mathcal{D})$ on $v\in V_i$. Let e_1 be the edge connecting v to itself. We define $\mathscr{G}^N$ the graph obtained by disconnecting e_1 in one side and by imposing $(\mathcal{N})$ on the new external vertex of e_1. Thanks to Proposition 4.16, for $k\in\mathbb{N}$, it holds

$$\text{(4.25)}\qquad ...\leq\lambda_k^{\mathscr{G}}\leq\lambda_k^{\mathscr{G}^D}\leq\lambda_{k+1}^{\mathscr{G}}\leq ..., \qquad ...\leq\lambda_k^{\mathscr{G}}\leq\lambda_{k+1}^{\mathscr{G}^N}\leq\lambda_{k+1}^{\mathscr{G}}\leq ...$$

Now, $\{\lambda_k^{\mathscr{G}^D}\}_{k\in\mathbb{N}}$ and $\{\lambda_k^{\mathscr{G}^N}\}_{k\in\mathbb{N}}$ are respectively obtained by reordering

$$\left\{\frac{k^2\pi^2}{L_j^2}\right\}_{\substack{k\in\mathbb{N}\\ j\in\{1,2\}}}, \qquad \left\{\frac{(2k-1)^2\pi^2}{4(L_1+L_2)^2}\right\}_{k\in\mathbb{N}}.$$

If $\{L_1, L_2\}\in\mathcal{AL}$, then $\{L_1, L_2, L_1+L_2\}\in\mathcal{AL}$. Thanks to the techniques leading to Proposition 4.15, there exists a constant $C>0$ such that, for every $\epsilon>0$, there holds $|\lambda_{k+1}^{\mathscr{G}}-\lambda_k^{\mathscr{G}}|\geq|\lambda_{k+1}^{\mathscr{G}^N}-\lambda_k^{\mathscr{G}^D}|\geq Ck^{-\epsilon}$, for each $k\in\mathbb{N}$. Hence, the relation (4.5) is verified and the claim is guaranteed by the first point of Theorem 4.3.

- The same techniques can be used when $\mathscr{G}$ is a tadpole equipped with $(\mathcal{N})$-$(\mathcal{NK})$.

- When $\mathscr{G}$ is a two-tails tadpole equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$, we define $\mathscr{G}^{\mathcal{N}}$ by disconnecting the ring v_1 from $\mathscr{G}$. The graph $\mathscr{G}^{\mathcal{N}}$ is composed by a ring v_1 and an edge $v_2 + v_3$. By defining $\mathscr{G}^{\mathcal{D}}$ as before, the introduced argument leads to the claim.

- When $\mathscr{G}$ is a double-rings graph, it is possible to define $\mathscr{G}^{D}$ as before and $\mathscr{G}^{N}$ by dividing $\mathscr{G}$ in two rings. The spectra of A on $\mathscr{G}^{D}$ and $\mathscr{G}^{\mathcal{N}}$ are explicit, which implies to the result.

- In conclusion, the same procedure in valid when $\mathscr{G}$ is a generic star graph with $N \leq 4$ edges. Indeed, if we define $\mathscr{G}^{\mathcal{N}}$ by disconnecting v_1 and v_2 from $\mathscr{G}$ and by connecting them together in a new internal vertex equipped with $(\mathcal{NK})$, the previous techniques lead to the result.

2) Let $\mathscr{G}$ be a star graph equipped with $(\mathcal{D})$-$(\mathcal{NK})$. The conditions $(\mathcal{D})$ on the external vertices of $\mathscr{G}$ imply that

$$\phi_k = (a_k^1 \sin(\sqrt{\lambda_k}x), ..., a_k^n \sin(\sqrt{\lambda_k}x))$$

for $\{a_k^l\}_{l\leq N} \subset \mathbb{C}$. By imposing $(\mathcal{NK})$ in the internal vertex v_0, we obtain

$$a_k^1 \sin(\sqrt{\lambda_k}L_1) = ... = a_k^N \sin(\sqrt{\lambda_k}L_N),$$
$$\sum_{l=1}^{N} a_k^l \cos(\sqrt{\lambda_k}L_l) = 0.$$

Then, $\{\sqrt{\lambda_k}\}_{k\in\mathbb{N}}$ are the zeros of the function $\sum_{l=1}^{N}\cot(xL_l)$, *i.e.*

$$\sum_{l=1}^{N}\cot(\sqrt{\lambda_k}L_l) = 0, \qquad \forall k \in \mathbb{N}.$$

Let us define the maps

$$\begin{aligned} G(x) :=& \prod_{l=1}^{N}\sin(xL_l)\sum_{l=1}^{N}\cot(xL_l) = \sum_{l=1}^{N}\cos(xL_l)\prod_{m\neq l}\sin(xL_m),\\ \tilde{G}(x) :=& \prod_{l=1}^{N}\sin(xL_l)\sum_{l=1}^{N}\frac{L_l}{\sin^2(xL_l)}. \end{aligned} \tag{4.26}$$

First, we notice that G is an entire function such that $G \in L^\infty(\mathbb{R})$ and, for every $z \in \mathbb{C}$,

$$|G(z)| \leq 2^{(N-1)}Ne^{|z|\sum_{l=1}^{N}L_l}$$

since

$$|\cos(zL_l)| \leq 2e^{L_l|z|}, \qquad |\sin(zL_l)| \leq 2e^{L_l|z|}, \qquad \forall l \leq N.$$

We prove that G satisfies the hypotheses of the second point of Theorem 4.3. For $L^* := \min\{L_l : 1 \leq l \leq N\}$ and for every $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
|\tilde{G}(x)| &= \frac{\prod_{l=1}^N |\sin(xL_l)| \sum_{l=1}^N L_l \prod_{k\neq l} \sin^2(xL_k)}{\prod_{l=1}^N \sin^2(xL_l)} \\
&= \frac{\sum_{l=1}^N L_l \prod_{k\neq l} \sin^2(xL_k)}{\prod_{l=1}^N |\sin(xL_l)|} = \sum_{l=1}^N L_l \frac{\prod_{k\neq l} |\sin(xL_k)|}{|\sin(xL_l)|} \\
&\geq L^* \sum_{l=1}^N \prod_{k\neq l} |\sin(xL_k)|
\end{aligned} \tag{4.27}$$

and

$$|\tilde{G}(\sqrt{\lambda_n})| \geq L^* \sum_{l=1}^N \prod_{k\neq l} |\sin(\sqrt{\lambda_n} L_k)|.$$

Now, $G'(\sqrt{\lambda_n}) = -\tilde{G}(\sqrt{\lambda_n})$ for each $n \geq 2$ since

$$G'(x) = -\tilde{G}(x) + H(x), \qquad H(x) := \frac{d}{dx}\big(\prod_{l=1}^N \cos(xL_l)\big) \sum_{l=1}^N \cot(xL_l)$$

and $H(\sqrt{\lambda_n}) = 0$. We refer to [DZ06, *Corollary A*.10; (2)] which contains a misprint as the relation is valid for every

$$\lambda > \frac{1}{2} \max\{L_k/L_j \ : \ j,k \leq N\}.$$

Then, for every $n \geq \hat{n}$ with $\hat{n} \in \mathbb{N}$ such that

$$\lambda_{\hat{n}} > \frac{1}{2} \max\{L_k/L_j \ : \ j,k \leq N\}$$

and for every $\epsilon > 0$, there exists $C_1 > 0$ such that

$$|G'(\sqrt{\lambda_n})| \geq L^* \sum_{l=1}^N \prod_{k\neq l} |\sin(\sqrt{\lambda_n} L_k)| \geq \frac{C_1}{(\sqrt{\lambda_n})^{1+\epsilon}}.$$

Remark 4.18. *For every $n \in \mathbb{N}$ and $k \leq N$, we have*

$$|\phi_n^k(L_k)| \neq 0,$$

otherwise the $(\mathcal{NK})$ conditions would imply the existence of at least two indices $l, m \leq N$ so that

$$\phi_n^l(L_l) = a_n^l \sin(L_l\sqrt{\lambda_n}) = a_n^m \sin(L_m\sqrt{\lambda_n}) = \phi_n^m(L_m) = 0,$$

$$\phi_n^l \not\equiv 0, \qquad \phi_n^m \not\equiv 0,$$

which is absurd since

$$a_n^l, a_n^m \neq 0, \qquad \{L_l\} \in \mathcal{AL}.$$

Thus, $\sin(L_k\sqrt{\lambda_n}) \neq 0$ *for each $k \leq N$ and $n \in \mathbb{N}$ (see also* [*BK13*, p. 15] *where it is explained that the spectrum of A is simple).*

In conclusion, thanks to Remark 4.18, we have $|G'(\sqrt{\lambda_n}) \neq 0$ for every $n < \hat{n}$ and, for $\epsilon > 0$, there exists $C_2 > 0$ such that

$$|G'(\sqrt{\lambda_n})| \geq \frac{C_3}{n^{1+\epsilon}}, \qquad \forall n \in \mathbb{N},$$

thanks to Remark 4.17. The function $G(x)$ satisfies the hypotheses of the second point of Theorem 4.3 for $\tilde{d} = 1 + \epsilon$. Indeed, the numbers $\{\sqrt{\lambda_j}\}_{j\in\mathbb{N}}$ and $\{-\sqrt{\lambda_j}\}_{j\in\mathbb{N}}$ are simple zeros of G and there exist $\tilde{d} \geq 1$, $C > 0$ such that, for every $j \in \mathbb{N}$,

$$|G'(\sqrt{\lambda_j})| \geq \frac{C}{j^{\tilde{d}}}, \qquad |G'(-\sqrt{\lambda_j})| \geq \frac{C}{j^{\tilde{d}}}. \quad \square$$

Proof of Theorem 4.7. The claim follows from [Rot56] since

$$\{\lambda_j\}_{j\in\mathbb{N}} \subset \left\{\frac{(k-1)^2\pi^2}{4L_j^2}\right\}_{\substack{k,j\in\mathbb{N}\\ j\leq N}}.$$

In fact, thanks to the arguments adopted in the proof of Proposition 4.15, for every $\epsilon > 0$, there exists $C_1 > 0$ such that

$$|\lambda_{k+1} - \lambda_k| \geq \frac{C_1}{k^\epsilon}, \qquad \forall k \in \mathbb{N}.$$

The proof is achieved thanks to Theorem 4.3. $\square$

4.5 Examples

Let B be a bounded operator on $\mathscr{H}$. We define $B|_{L^2(e_k)}$ the action of B on the k-th component of any function $\psi = (\psi^1, ..., \psi^N) \in \mathscr{H}$ such that

$$B\psi = (B|_{L^2(e_1)}\psi^1, ..., B|_{L^2(e_N)}\psi^N).$$

Example 4.19. *Let $\mathscr{G}$ be a star graph equipped with $(\mathcal{D})$-$(\mathcal{NK})$ and B be such that*

$$B|_{L^2(e_1)} = (x - L_1)^4, \qquad B|_{L^2(e_k)} = 0$$

for $k \in \{2, ..., N\}$. There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable such that, for every $\{L_j\}_{j \le N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, the problem (4.1) is globally exactly controllable in

$$H^{4+\epsilon}_{\mathscr{G}} \qquad \epsilon \in (0, 1/2).$$

Proof. First, we prove that the couple (A, B) satisfies Assumptions III($2+\epsilon$) for $\epsilon > 0$. The conditions $(\mathcal{D})$ on the external vertices imply that each eigenfunction

$$\phi_j = (\phi_j^1, ..., \phi_j^N), \qquad \forall j \in \mathbb{N},$$

satisfies $\phi_j^l(0) = 0$ for every $l \le N$. Then

$$\phi_j = (a_j^1 \sin(x\sqrt{\lambda_j}), ..., a_j^N \sin(x\sqrt{\lambda_j}))$$

for $\{a_j^l\}_{l \le N} \subset \mathbb{C}$ such that $\{\phi_j\}_{j \in \mathbb{N}}$ forms a Hilbert basis of $\mathscr{H}$, *i.e.*

$$\begin{aligned} 1 = \|\phi_j\|^2 &= \sum_{l=1}^{N} \int_0^{L_l} |a_j^l|^2 \sin^2(x\sqrt{\lambda_j})) \\ &= \sum_{l=1}^{N} |a_j^l|^2 \left(\frac{L_l}{2} + \frac{\cos(L_l\sqrt{\lambda_j}) \sin(L_l\sqrt{\lambda_j})}{2\sqrt{\lambda_j}} \right). \end{aligned}$$

Thanks to the condition $(\mathcal{NK})$ on the internal vertex, for every $j \in \mathbb{N}$, there hold

$$a_j^1 \sin(\sqrt{\lambda_j}L_1) = ... = a_j^N \sin(\sqrt{\lambda_j}L_N), \qquad \sum_{l=1}^{N} a_j^l \cos(\sqrt{\lambda_j}L_l) = 0.$$

Hence, for every $j \in \mathbb{N}$,

$$\sum_{l=1}^{N} \cot(\sqrt{\lambda_j}L_l) = 0, \qquad \sum_{l=1}^{N} |a_j^l|^2 \sin(L_l\sqrt{\lambda_j}) \cos(L_l\sqrt{\lambda_j}) = 0.$$

Now, $1 = \sum_{l=1}^N |a_j^l|^2 L_l/2$ and the continuity implies

$$a_j^l = a_j^1 \frac{\sin(\sqrt{\lambda_j} L_1)}{\sin(\sqrt{\lambda_j} L_l)}$$

for every $l \neq 1$ and $j \in \mathbb{N}$. For every $j \in \mathbb{N}$, we have

$$|a_j^1|^2 \left(L_1 + \sum_{l=2}^N L_l \frac{\sin^2(\sqrt{\lambda_j} L_1)}{\sin^2(\sqrt{\lambda_j} L_l)} \right) = 2.$$

Thus,

$$|a_j^1|^2 = \frac{2\prod_{m\neq 1} \sin^2(\sqrt{\lambda_j} L_m)}{\sum_{k=1}^N L_k \prod_{m\neq k} \sin^2(\sqrt{\lambda_j} L_m)}, \qquad \forall j \in \mathbb{N}. \tag{4.28}$$

Each a_j^l can be computed from a_j^1 which is defined, up to phase, from the orthonormality of $\{\phi_j\}_{j\in\mathbb{N}}$. Thanks to [DZ06, *Proposition A*.11] and Remark 4.17, for every $\epsilon > 0$, there exist $C_1, C_2 > 0$ such that, for every $j \in \mathbb{N}$,

$$\begin{aligned} |a_j^1| &= \sqrt{\frac{2\prod_{l=1}^N \sin^2(\sqrt{\lambda_j} L_l)}{\sum_{l=1}^N L_l \prod_{m\neq l} \sin^2(\sqrt{\lambda_j} L_m)}} = \sqrt{\frac{2}{\sum_{l=1}^N L_l \sin^{-2}(\sqrt{\lambda_j} L_l)}} \\ &\geq \sqrt{\frac{2}{\sum_{l=1}^N L_l C_1^{-2} \lambda_j^{1+\epsilon}}} \geq \frac{C_2}{j^{1+\epsilon}}. \end{aligned} \tag{4.29}$$

We notice $\langle \phi_k^l, B\phi_j^l \rangle = 0$ for every $2 \leq l \leq N$ and $k, j \in \mathbb{N}$. Moreover, by calculation, we have

$$B_{1,1} = |a_k^1|^2 \frac{-30\sqrt{\lambda_1} L_1 + 20\sqrt{\lambda_1}^3 L_1^3 + 4\sqrt{\lambda_1}^5 L_1^5 + 15\sin(2\sqrt{\lambda_1} L_1)}{40\sqrt{\lambda_1}^5}.$$

When $j \in \mathbb{N} \setminus \{1\}$, the scalar product $B_{1,j} = \langle \phi_1^1, B\phi_j^1 \rangle$ corresponds to

$$\begin{aligned} &2\overline{a_1^1} a_j^1 \frac{-6(\sqrt{\lambda_1} - \sqrt{\lambda_j}) L_1 + (\sqrt{\lambda_1} - \sqrt{\lambda_j})^3 L_1^3 + 6\sin((\sqrt{\lambda_1} - \sqrt{\lambda_j}) L_1)}{(\sqrt{\lambda_1} - \sqrt{\lambda_j})^5} \\ &- 2\overline{a_1^1} a_j^1 \frac{-6(\sqrt{\lambda_1} + \sqrt{\lambda_j}) L_1 + (\sqrt{\lambda_1} + \sqrt{\lambda_j})^3 L_1^3 + 6\sin((\sqrt{\lambda_1} + \sqrt{\lambda_j}) L_1)}{(\sqrt{\lambda_1} + \sqrt{\lambda_j})^5}, \end{aligned}$$

$$\implies \quad |B_{1,j}| = \frac{8\sqrt{\lambda_1} |a_j^l a_1^l| L_1^3}{\sqrt{\lambda_j}^3} + O((\sqrt{\lambda_j})^{-5}), \qquad \forall j \geq 2. \tag{4.30}$$

For $j \geq 2$, we define the non-constant analytic functions

$$B_1(x) := \frac{-30\sqrt{\lambda_1}x + 20\sqrt{\lambda_1}^3 x^3 + 4\sqrt{\lambda_1}^5 x^5 + 15\sin(2\sqrt{\lambda_1}x)}{40\sqrt{\lambda_1}^5},$$

$$B_j(x) := 2\frac{-6(\sqrt{\lambda_1} - \sqrt{\lambda_j})x + (\sqrt{\lambda_1} - \sqrt{\lambda_j})^3 x^3 + 6\sin((\sqrt{\lambda_1} - \sqrt{\lambda_j})x)}{(\sqrt{\lambda_1} - \sqrt{\lambda_j})^5}$$
$$- 2\frac{-6(\sqrt{\lambda_1} + \sqrt{\lambda_j})x + (\sqrt{\lambda_1} + \sqrt{\lambda_j})^3 x^3 + 6\sin((\sqrt{\lambda_1} + \sqrt{\lambda_j})x)}{(\sqrt{\lambda_1} + \sqrt{\lambda_j})^5},$$

such that $B_{1,j} = \overline{a_1^1} a_j^1 B_j(L_1)$ for every $j \in \mathbb{N}$. The set of positive zeros of each function B_j, that we denote $\tilde{V}_j$, is a discrete subset of $\mathbb{R}^+$ and $\tilde{V} = \bigcup_{j\in\mathbb{N}} \tilde{V}_j$ is countable. For every $\{L_l\}_{l\leq N} \in \mathcal{AL}(N)$ such that $L_1 \notin \tilde{V}$, we have $|B_{1,j}| \neq 0$ for every $j \in \mathbb{N}$. Thanks to Remark 4.17, we use the inequality (4.29) in (4.30) and the first point of Assumptions III$(2+\epsilon)$ is verified since, for each $\epsilon > 0$, there exists $C_3 > 0$ such that

$$|B_{1,j}| \geq \frac{C_3}{j^{4+\epsilon}}, \qquad \forall j \in \mathbb{N}.$$

Let $(k,j), (m,n) \in I$, $(k,j) \neq (m,n)$ for I defined in (3.2). By calculation, we have

$$B_{j,j} = |a_j^1|^2 \frac{-30\sqrt{\lambda_j}L_1 + 20\sqrt{\lambda_j}^3 L_1^3 + 4\sqrt{\lambda_j}^5 L_1^5 + 15\sin(2\sqrt{\lambda_j}L_1)}{40\sqrt{\lambda_j}^5}.$$

For every $j \in \mathbb{N}$, we define the map

(4.31)
$$a_j(x) = \frac{2\prod_{m\neq 1}\sin^2(\sqrt{\lambda_j}L_m)}{\sum_{k=2}^N L_k \sin^2(\sqrt{\lambda_j}x)\prod_{\substack{m\neq k\\ m\neq 1}}\sin^2(\sqrt{\lambda_j}L_m) + x\prod_{m\neq 1}\sin^2(\sqrt{\lambda_j}L_m)}$$

such that $a_j(L_1) = |a_j^1|^2$. Thanks to Remark 4.18, for every $j \in \mathbb{N}$, the map $a_j(x)$ is analytic for $x > 0$. We define the map

$$F_j(x) := a_j(x)\frac{-30\sqrt{\lambda_k}x + 20\sqrt{\lambda_k}^3 x^3 + 4\sqrt{\lambda_k}^5 x^5 + 15\sin(2\sqrt{\lambda_k}x)}{40\sqrt{\lambda_k}^5}.$$

We notice that $F_j(L_1) = B_{j,j}$ and we denote

$$F_{j,k,l,m}(x) = F_j(x) - F_k(x) - F_l(x) + F_m(x).$$

Now, $F_{j,k,l,m}(L_1) = B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m}$ and $F_{j,k,l,m}(x)$ is a non-constant analytic function for $x > 0$. Furthermore $V_{j,k,l,m}$, the set of the positive zeros of $F_{j,k,l,m}(x)$, is discrete and

$$V := \bigcup_{\substack{j,k,l,m \in \mathbb{N} \\ j \neq k \neq l \neq m}} V_{j,k,l,m}$$

is a countable subset of $\mathbb{R}^+$. For each $\{L_l\}_{l \leq N} \in \mathcal{AL}(N)$ such that $L_1 \notin V \cup \tilde{V}$, Assumptions III$(2+\epsilon)$ are verified.

The fourth point of Assumptions IV$(2+\epsilon_1, \epsilon_2)$ is valid for each $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 \in (0, 1/2)$ since B stabilizes $H^2_{\mathscr{G}}$, H^m and $H^m_{\mathcal{NK}}$ for $m \in (0, 9/2)$. Indeed, let $v \in V_i$.

- For every $\psi \in H^1_{\mathcal{NK}}$, we have $B\psi(v) = 0$, $B\psi \in C^0(\mathscr{G}, \mathbb{C})$ and $B\psi \in H^1_{\mathcal{NK}}$.
- For every $\psi \in H^2_{\mathcal{NK}}$, we have $\partial_x(B\psi)(v) = 0$, which implies $B\psi \in H^2_{\mathcal{NK}}$.
- For every $\psi \in H^3_{\mathcal{NK}}$, there hold $\partial_x^2(B\psi)(v) = 0$, $\partial_x^2(B\psi) \in C^0(\mathscr{G}, \mathbb{C})$ and $B\psi \in H^3_{\mathcal{NK}}$.
- For every $\psi \in H^4_{\mathcal{NK}}$, there hold $\partial_x^3(B\psi)(v) = 0$ and $B\psi \in H^4_{\mathcal{NK}}$.

In conclusion, from the second point of Theorem 4.5, the controllability is achieved in $H^{4+\epsilon}_{\mathscr{G}}$ for every $\epsilon \in (0, 1/2)$. $\square$

Example 4.20. *Let $\mathscr{G} = \{I_j\}_{j \leq N}$ be a compact quantum graph composed by a set of bounded unconnected intervals equipped with $(\mathcal{D})$. Let B be such that*

$$B : \psi = (\psi^1, ...\psi^N) \mapsto \left(\sum_{j=1}^N \frac{L_j^{\frac{1}{2}} x^2}{L_1^{\frac{1}{2}}} \psi^j \Big(\frac{L_j}{L_1} x \Big), ..., \sum_{j=1}^N \frac{L_j^{\frac{1}{2}} x^2}{L_N^{\frac{1}{2}}} \psi^j \Big(\frac{L_j}{L_N} x \Big) \right).$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable such that, for every $\{L_j\}_{j \leq N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, the problem (4.1) *is contemporaneously globally exactly controllable in*

$$\prod_{j=1}^N H^{3+\epsilon}_{I_j}, \qquad \forall \epsilon \in (0, 3/2).$$

Proof. First, the conditions $(\mathcal{D})$ imply that each eigenfunction

$$\phi_k = (\phi_k^1, ..., \phi_k^N)$$

is such that

$$\phi_k^l(0) = 0, \qquad \phi_k^l(L_l) = 0, \qquad \forall l \leq N.$$

The fact that $\{L_l\}_{l\leq N} \in \mathcal{AL}(N)$ implies that, for each $k \in \mathbb{N}$, there exist $m(k) \in \mathbb{N}$ and $l(k) \leq N$ such that

$$\lambda_k = \frac{m(k)^2\pi^2}{L_{l(k)}^2}, \qquad \phi_k^{l(k)}(x) = \sqrt{\frac{2}{L_{l(k)}}} \sin(\sqrt{\lambda_k} x),$$

$$\phi_k^n \equiv 0, \qquad n \neq l(k).$$

Hence, $\{\lambda_k\}_{k\in\mathbb{N}}$ is the sequence obtained by reordering $\left\{\frac{m^2\pi^2}{L_l^2}\right\}_{\substack{m,l\in\mathbb{N}\\ l\leq N}}$. Now

$$\begin{aligned}|B_{1,j}| &= \left|\left\langle \phi_1^{l(1)}(x), \frac{L_{l(j)}^{\frac{1}{2}} x^2}{L_{l(1)}^{\frac{1}{2}}} \phi_j^{l(j)}\Big(\frac{L_{l(j)}}{L_{l(1)}} x\Big)\right\rangle_{L^2(I_{l(1)},\mathbb{C})}\right| \\ &= 2L_{l(1)} \Big| \int_0^{L_{l(1)}} \frac{1}{L_{l(1)}^2} x^2 \sin\Big(\frac{m(j)\pi}{L_{l(j)}} \frac{L_{l(j)}}{L_{l(1)}} x\Big) \sin\Big(\frac{m(1)\pi}{L_{l(1)}} x\Big) dx\Big| \\ &\geq 2\min\{L_l^2 :\ l \leq N\} \Big| \int_0^1 x^2 \sin(m(j)\pi x) \sin(m(1)\pi x) dx\Big|.\end{aligned}$$

It is the same integral that we treat in Paragraph 2.7 and in Example 3.1. Then, for every $j \in \mathbb{N}$, there exists $C_1 > 0$ such that

$$|B_{j,1}| \geq \frac{C_1}{m(j)^3} \geq \frac{C_1}{j^3}, \qquad \forall j \in \mathbb{N}$$

since $m(j) \leq j$. Now

$$B_{j,j} = 2L_{m(j)}^2 \int_0^1 x^2 \sin^2(m(j)\pi x) dx = \frac{L_{m(j)}^2}{3} - \frac{L_{m(j)}^2}{2m(j)^2\pi^2}. \tag{4.32}$$

As done in the proof of Example 4.19, we define the maps

$$F_j(x_1, ..., x_n) := \frac{x_{m(j)}^2}{3} - \frac{x_{m(j)}^2}{2m(j)^2\pi^2}, \qquad \forall j \in \mathbb{N}.$$

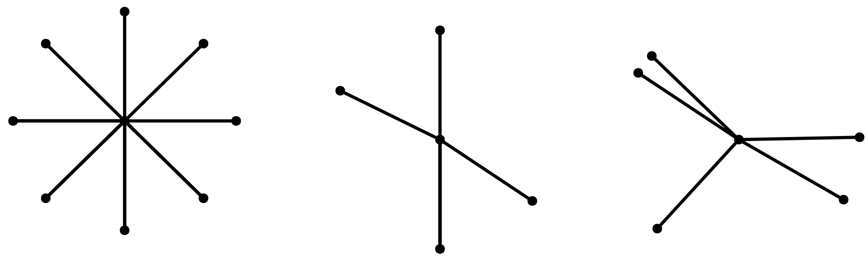


Figure 4.6: Examples of star graphs with equal length edges

We notice that $F_j(L_1, ..., L_N) = B_{j,j}$ and we denote, for $j, k, l, m \in \mathbb{N}$,

$$F_{j,k,l,m}(x_1, ..., x_N) = F_j(x_1, ..., x_N) - F_k(x_1, ..., x_N) - F_l(x_1, ..., x_N) + F_m(x_1, ..., x_N).$$

Now, $F_{j,k,l,m}(L_1, ..., L_N) = B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m}$ and each $F_{j,k,l,m}(x_1, ..., x_n)$ is a non-constant analytic function for $(x_1, ..., x_N) \in (\mathbb{R}^+)^N$. Furthermore $V_{j,k,l,m}$, the set of the positive zeros of $F_{j,k,l,m}$, is discrete and

$$V := \bigcup_{\substack{j,k,l,m \in \mathbb{N} \\ j \neq k \neq l \neq m}} V_{j,k,l,m}$$

is countable. For each $\{L_l\}_{l \leq N} \in \mathcal{AL}(N) \backslash V$, Assumptions III(1) are verified.

The fourth point of Assumptions IV$(1, \epsilon)$ is valid for each $\epsilon \in (0, 3/2)$ since B stabilizes $H^2_{\mathscr{G}}$ and H^m for $m > 0$ ($H^m \equiv H^m_{\mathcal{NK}}$ as there are not internal vertices in $\mathscr{G}$). Moreover, B maps

$$H^{2+d}_{\mathscr{G}} \to H^{2+d}, \qquad H^{2+d}_{\mathscr{G}} \subset H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}.$$

Thus, for every $d > 0$, B maps $H^{2+d}_{\mathscr{G}}$ in $H^{2+d} \cap H^2_{\mathscr{G}} \cap H^{2+d}_{\mathcal{NK}}$. In conclusion, Theorem 4.7 achieves the controllability for every $\epsilon \in (0, 3/2)$ in

$$H^{3+\epsilon}_{\mathscr{G}} = \prod_{j=1}^{N} H^{3+\epsilon}_{I_j}. \qquad \square$$

Example 4.21. *Let $\mathscr{G}$ be a star graph with edges of equal length L (Figure* 4.6*) and equipped with $(\mathcal{D})$-$(\mathcal{NK})$. Let the operator B be such that*

$$B|_{L^2(e_1)} = (x - L)^2,$$

$$B|_{L^2(e_k)} = 0, \qquad 2 \leq k \leq N.$$

There exists $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \{\phi_j\}_{j\in\mathbb{N}}$ *such that* (4.1) *is globally exactly controllable in*

$$H^{3+\epsilon}_{\mathscr{G}} \cap \widetilde{\mathscr{H}}, \qquad \forall \epsilon \in [0, 1/2)$$

and fully energetically controllable in

$$\Big\{\frac{k^2\pi^2}{4L^2}\Big\}_{k\in\mathbb{N}}.$$

Proof. For $N = 3$, the $(\mathcal{D})$ conditions lead to

$$\exists a_k^1, a_k^2, a_k^3 : \ \phi_k = (a_k^1 \sin(\sqrt{\lambda_k}x), a_k^2 \sin(\sqrt{\lambda_k}x), a_k^3 \sin(\sqrt{\lambda_k}x)).$$

By imposing $(\mathcal{NK})$ in the internal vertex v_0, we obtain

$$a_k^1 \sin(\sqrt{\lambda_k}L) = a_k^2 \sin(\sqrt{\lambda_k}L) = a_k^3 \sin(\sqrt{\lambda_k}L) = c$$
$$a_k^1 \cos(\sqrt{\lambda_k}L) + a_k^2 \cos(\sqrt{\lambda_k}L) + a_k^3 \cos(\sqrt{\lambda_k}L) = 0.$$

For $c \neq 0$, we can compute the sequence of eigenvalues

$$\Big\{\frac{(1+2n)^2\pi^2}{4L^2}\Big\}_{n\in\mathbb{N}\cup\{0\}}$$

corresponding to the eigenfunctions $\{g_n\}_{n\in\mathbb{N}}$ such that each g_n is equal to

$$\bigg(\sqrt{\frac{2}{3L}}\sin\Big(\frac{(1+2n)\pi}{2L}x\Big), \sqrt{\frac{2}{3L}}\sin\Big(\frac{(1+2n)\pi}{2L}x\Big), \sqrt{\frac{2}{3L}}\sin\Big(\frac{(1+2n)\pi}{2L}x\Big)\bigg).$$

For $c = 0$, we obtain

$$\Big\{\frac{n^2\pi^2}{L^2}\Big\}_{n\in\mathbb{N}} \subset \{\lambda_j\}_{j\in\mathbb{N}}$$

of multiplicity two that we associate to couples of eigenfunctions f_n^1 and f_n^2 such that

$$f_n^1 := \Big(-\sqrt{\frac{4}{3L}}\sin\Big(\frac{n\pi}{L}x\Big), \sqrt{\frac{1}{3L}}\sin\Big(\frac{n\pi}{L}x\Big), \sqrt{\frac{1}{3L}}\sin\Big(\frac{n\pi}{L}x\Big)\Big),$$

$$f_n^2 := \Big(0, -\sqrt{\frac{1}{L}}\sin\Big(\frac{n\pi}{L}x\Big), \sqrt{\frac{1}{L}}\sin\Big(\frac{n\pi}{L}x\Big)\Big).$$

Moreover,

$$\{f_n^1\}_{n\in\mathbb{N}} \cup \{f_n^2\}_{n\in\mathbb{N}} \cup \{g_n\}_{n\in\mathbb{N}}$$

is an Hilbert basis of $\mathscr{H}$. Without considering the multiplicity, the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ is obtained by reordering

$$\left\{\frac{n^2\pi^2}{L^2}\right\}_{n\in\mathbb{N}} \cup \left\{\frac{(1+2n)^2\pi^2}{4L^2}\right\}_{n\in\mathbb{N}\cup\{0\}}$$

(see also [BK13, p. 15] for further explanations). The operator $A+uB$ maps $\overline{span\{f_n^2 :\ n\in\mathbb{N}\}}^{L^2}$ in itself as the propagator Γ_t^u. We call $\varphi = \{\varphi_j\}_{j\in\mathbb{N}}$ the sequence obtained by reordering

$$\{f_n^1\}_{n\in\mathbb{N}} \cup \{g_n\}_{n\in\mathbb{N}}$$

and Γ_t^u stabilizes

$$\widetilde{\mathscr{H}} := \overline{span\{\varphi_n\}_{n\in\mathbb{N}}}^{L^2}.$$

The second point of Assumptions III(1) follows since there exist $C_3, C_4 > 0$ such that, for every $j\in\mathbb{N}$, we have

$$\tilde{B}_{j,j} := \langle\varphi_j, B\varphi_j\rangle = C_3 + \frac{C_4}{j^2} \in \mathbb{R}^+$$

and $\mu_j = \frac{\pi^2 j^2}{4L}$. Thus, thanks to Example 3.1, there holds

$$\mu_j - \mu_k - \mu_l + \mu_m = 0 \qquad \Longrightarrow \qquad \tilde{B}_{j,j} - \tilde{B}_{k,k} - \tilde{B}_{l,l} + \tilde{B}_{m,m} \neq 0.$$

The first point of Assumptions III(1) is verified in $\widetilde{\mathscr{H}}$ since $B_{1,1}\in\mathbb{R}^+$ and there exists $C_1, C_2 > 0$ such that

$$|\langle\varphi_1, B\varphi_k\rangle| = |\langle\varphi_1^1, B\varphi_k^1\rangle| \geq \frac{C_1\sqrt{\lambda_k}\sqrt{\lambda_1}}{(\lambda_k-\lambda_1)^2} \geq \frac{C_2}{k^3}, \qquad \forall k\in\mathbb{N}.$$

The first point of Assumptions IV(1, 0) in $\widetilde{\mathscr{H}}$ is achieved since B stabilizes

$$\widetilde{\mathscr{H}} \cap H^2_{\mathscr{G}}.$$

Now, $\{\mu_j\}_{j\in\mathbb{N}}$, the eigenvalues corresponding to $\{\varphi_k\}_{k\in\mathbb{N}}$, is the sequence of eigenvalues of A (not repeated with their multiplicity). Thanks to

$$\inf_{j,k\in\mathbb{N}} |\mu_k - \mu_j| = \frac{\pi^2}{4L^2},$$

the hypotheses of Theorem 4.3 are satisfied in $\widetilde{\mathscr{H}}$. Then, the couple (A, B) satisfies Assumptions V$(1, 0)$ and the problem is globally exactly controllable in

$$H_{\mathscr{G}}^{3+\epsilon} \cap \widetilde{\mathscr{H}}$$

for each $\epsilon \in [0, 1/2)$ and fully energetically controllable.

When $N > 3$, the spectrum contains simple eigenvalues and eigenvalues of multiplicity $N-1$ (see [BK13, *p.* 15]). To each $(N-1)$-tuple, we construct $N-1$ eigenfunctions such that $N-2$ of them have null component in e_1. We call $\mathscr{H}'$ the closure with respect to the L^2-norm of the span of all those $(N-2)$-tuples. The propagator Γ_t^u stabilizes $\mathscr{H}'$ and its orthogonal complement $\widetilde{\mathscr{H}}$. The spectrum of A in $\widetilde{\mathscr{H}}$ corresponds to $\{\mu_j\}_{j\in\mathbb{N}}$, which allows to achieve the proof as before. □

Example 4.22. *Let $\mathscr{G}$ be a star graph containing two edges e_1 and e_2 of equal length L connecting the internal vertex, equipped with ($\mathcal{NK}$), with two external vertices both equipped with ($\mathcal{D}$). Let B be such that*

$$B|_{L^2(e_1)}\psi^1 = -B|_{L^2(e_2)}\psi^2 = x^2(\psi^1(x) - \psi^2(x)),$$

$$B|_{L^2(e_k)}\psi^k = 0, \qquad \forall k \in \{3, ..., N\}.$$

for every $\psi = (\psi^1, ..., \psi^N) \in \mathscr{H}$. There exists $\{\varphi_k\}_{k\in\mathbb{N}} \subset \{\phi_j\}_{j\in\mathbb{N}}$ such that (4.1) *is globally exactly controllable in*

$$H_{\mathscr{G}}^{3+\epsilon} \cap \widetilde{\mathscr{H}}, \qquad \forall \epsilon \in [0, 1/2).$$

and energetically controllable in

$$\Big\{\frac{k^2\pi^2}{L^2}\Big\}_{k\in\mathbb{N}}.$$

Proof. The proof follows the techniques of the proof of Example 4.21. One can compute a sequence of eigenfunction $\boldsymbol{\varphi} = \{\varphi_k\}_{k\in\mathbb{N}}$, corresponding to the eigenvalues $\big\{\frac{k^2\pi^2}{L^2}\big\}_{k\in\mathbb{N}} \subset \{\lambda_k\}_{k\in\mathbb{N}}$, so that

$$\varphi_k^1 = -\varphi_k^2 = \sqrt{\frac{1}{L}}\sin\Big(\frac{k\pi}{L}x\Big), \qquad \varphi_k^l = 0, \qquad 3 \leq l \leq N.$$

In addition, $\langle\varphi_k, B\varphi_j\rangle = 4\langle\varphi_k^1, x^2\varphi_j^1\rangle$ and Assumptions III(1) follow thanks to Example 3.1. Now, we set

$$\widetilde{\mathscr{H}} = \overline{span\{\varphi_n : \ n \in \mathbb{N}\}}^{L^2}$$

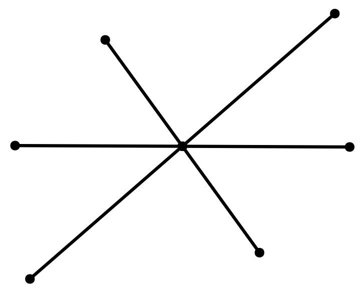


Figure 4.7: Example of graph described in Example 4.23

and we notice that, for every $f = (f^1, ..., f^n) \in \widetilde{\mathscr{H}} \cap D(A)$ and $v \in V_i$, there hold

$$\begin{cases} f(v) = 0, \\ \frac{\partial f^1}{\partial x}(v) + \frac{\partial f^2}{\partial x}(v) = 0. \end{cases}$$

Then, for every $\psi \in \widetilde{\mathscr{H}} \cap D(A)$, we have

$$(B\psi)^1(v) = L^2(\psi^1(L) - \psi^2(L)) = L^2(0-0) = 0 = (B\psi)^2(v) = ... = (B\psi)^N(v),$$

$$\sum_{j\in N(v)} \frac{(\partial B\psi)^j}{\partial x}(v) = 2L(\psi^1(L) - \psi^2(L)) - 2L(\psi^1(L) - \psi^2(L)) + L^2\Big(\frac{\partial \psi^1}{\partial x} + \frac{\partial \psi^2}{\partial x}\Big) - L^2\Big(\frac{\partial \psi^1}{\partial x} + \frac{\partial \psi^2}{\partial x}\Big) = 0.$$

Now, $B\psi \in \widetilde{\mathscr{H}} \cap D(A)$ and, as in Example 4.21, the propagator Γ_t^u stabilizes $\widetilde{\mathscr{H}} \cap D(A)$. The result is achieved equivalently to Example 4.21. □

Example 4.23. *Let $\mathscr{G}$ be a star graph containing an even number of external vertices equipped with ($\mathcal{D}$) and an internal vertex equipped with ($\mathcal{NK}$). Let $\mathscr{G}$ be composed by $N/2$ couples of edges $\{e_{2k-1}, e_{2k}\}_{k\leq N/2}$ of lengths $\{L_k\}_{k\leq N/2} \in \mathcal{AL}(N/2)$ (Figure 4.7). Let B be such that*

$$B|_{L^2(e_{2k})}\psi^{2k} = -B|_{L^2(e_{2k-1})}\psi^{2k-1} = \sum_{j=1}^{N/2} \frac{L_j^{\frac{1}{2}}}{L_k^{\frac{1}{2}}} x^2 \Big(\psi^{2j}\Big(\frac{L_j}{L_k}x\Big) - \psi^{2j-1}\Big(\frac{L_j}{L_k}x\Big)\Big),$$

for every $\psi = (\psi^1, ..., \psi^N) \in \mathscr{H}$ and $k \leq N/2$. There exists $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \{\phi_j\}_{j\in\mathbb{N}}$ such that (4.1) *is globally exactly controllable in*

$$H_{\mathscr{G}}^{3+\epsilon} \cap \widetilde{\mathscr{H}}, \qquad \forall \epsilon \in (0, 1/2)$$

and energetically controllable in

$$\Big\{\frac{k^2\pi^2}{L_j^2}\Big\}_{\substack{k,j\in\mathbb{N}\\ j\leq N/2}}.$$

Proof. The example follows the idea of Example 4.22 by using the arguments of Example 4.20. For every couple of edges of the same length L_j, one can compute a sequence of eigenfunction $\{\varphi_k^j\}_{k\in\mathbb{N}}$, corresponding to the eigenvalues

$$\left\{\frac{k^2\pi^2}{L_j^2}\right\}_{k\in\mathbb{N}}\subset\{\lambda_k\}_{k\in\mathbb{N}},$$

so that, for $l\leq N$, $l\neq 2j-1$ and $l\neq 2j$, there holds

$$\varphi_k^{2j-1}=-\varphi_k^{2j}=\sqrt{\frac{1}{L_j}}\sin\Big(\frac{k\pi}{L_j}x\Big),\qquad \varphi_k^l=0.$$

Let $\{\mu_k\}_{k\in\mathbb{N}}$ be obtained by reordering

$$\left\{\frac{k^2\pi^2}{L_j^2}\right\}_{\substack{k\in\mathbb{N}\\ j\leq N/2}}.$$

For each $k\in\mathbb{N}$, there exist $m(k)\in\mathbb{N}$ and $l(k)\leq N/2$ such that

$$\mu_k=\frac{m(k)^2\pi^2}{L_{l(k)}^2},\qquad \varphi_k^{2l(k)-1}(x)=-\varphi_k^{2l(k)}(x)=\sqrt{\frac{1}{L_{l(k)}}}\sin\big(\sqrt{\lambda_k}x\big),$$

$$\varphi_k^n\equiv 0,\qquad n\neq 2l(k),\quad n\neq 2l(k)-1.$$

Now, for $[\cdot]$ the entire part of a number, the number $|B_{k,1}|$ corresponds to

$$\left|\sum_{l=1}^{N}\left\langle\phi_k^l(x),\sum_{n=1}^{N/2}\frac{L_n^{\frac12}x^2}{L_{[(l+1)/2]}^{\frac12}}\Big(\phi_1^{2n-1}\Big(\frac{L_n}{L_{[(l+1)/2]}}x\Big)-\phi_1^{2n}\Big(\frac{L_n}{L_{[(l+1)/2]}}x\Big)\Big)\right\rangle_{L^2(e_l,\mathbb{C})}\right|$$

$$=2\left|\left\langle\phi_k^{2l(k)}(x),\frac{2L_{l(1)}^{\frac12}x^2}{L_{l(k)}^{\frac12}}\phi_1^{2l(1)}\Big(\frac{L_{l(1)}}{L_{l(k)}}x\Big)\right\rangle_{L^2(e_{l(k)},\mathbb{C})}\right|$$

$$=4L_{l(k)}\Big|\int_0^{L_{l(k)}}\frac{1}{L_{l(k)}^2}x^2\sin\Big(\frac{m(1)\pi}{L_{l(1)}}\frac{L_{l(1)}}{L_{l(k)}}x\Big)\sin\Big(\frac{m(k)\pi}{L_{l(k)}}x\Big)dx\Big|$$

$$\geq 4\min\{L_l^2:\ l\leq N\}\Big|\int_0^1 x^2\sin(m(1)\pi x)\sin(m(k)\pi x)dx\Big|.$$

Thus, Assumptions III(1) are verified in $\widetilde{\mathscr{H}}$, as in Example 4.20 and Example 4.22. Thanks the techniques leading to Lemma 4.15, already adopted in the proofs of Theorem 4.5 and Theorem 4.7, we have the validity of the condition (4.5). Indeed, if we call $\{\mu_k\}_{k\in\mathbb{N}}$ the sequence reordered of

$$\left\{\frac{k^2\pi^2}{L_j^2}\right\}_{\substack{k\in\mathbb{N}\\ j\leq N/2}},$$

then we know that, for every $\epsilon>0$, there exists $C_1>0$ such that, for every $k\in\mathbb{N}$,

$$|\mu_{k+1}-\mu_k|\geq\frac{C_1}{k^\epsilon}.$$

In conclusion, the techniques adopted in Example 4.22 imply the validity of the first point of Assumptions IV$(1,\epsilon)$ for $\epsilon\in(0,1/2)$ in $\widetilde{\mathscr{H}}$ and Theorem 4.9 ensures the claim. □

Appendix A

Moment problem

Let H be a Hilbert space over a field $\mathcal{K}$ for $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$ and $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence of elements of H. In the current appendix, we study the so-called "moment problem", which consists in finding $v \in H$ such that, for a given $\{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathcal{K})$, there holds

$$x_n = \langle f_n, v\rangle_H, \qquad n \in \mathbb{Z}. \tag{A.1}$$

A possible way that we can follow is to look for $\{v_k\}_{k\in\mathbb{Z}} \in H$ such that

$$\delta_{j,k} = \langle f_j, v_k\rangle_H, \qquad \forall j, k \in \mathbb{Z}.$$

The sequence $\{v_k\}_{k\in\mathbb{Z}} \in H$ is said to be biorthogonal to $\{f_k\}_{n\in\mathbb{Z}}$ and it can be used in order to solve the moment problem. Indeed, under additional summability conditions on $\{x_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathcal{K})$ as

$$\sum_{k\in\mathbb{Z}} |x_k| \|v_k\|_H < \infty,$$

the function

$$v(t) = \sum_{k\in\mathbb{Z}} x_k v_k(t)$$

satisfies the relation $(A.1)$. This approach leads to the solvability of $(A.1)$, but it is not the only one as we show in the current appendix. This type of problems appears in a natural way in the study of control problems. In this work, it is crucial to prove the local exact controllability of the bilinear Schrödinger equation as in Theorem 2.8. In our framework, we assume $H = L^2((0,T),\mathbb{R})$ and $\{f_n\}_{n\in\mathbb{N}} = \{e^{i\lambda_n(\cdot)}\}_{n\in\mathbb{N}}$ that lead to the moment problem

$$x_n = \int_0^T e^{i\lambda_n s} u(s) ds, \qquad \{x_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{C}). \tag{A.2}$$

In the next paragraph of this appendix, we present another approach leading to the solvability of $(A.2)$.

Definition A.1. A family of functions $\{f_n\}_{n\in\mathbb{Z}}$ belonging to an Hilbert space $\mathscr{H}$ is a Riesz basis of

$$\overline{span\{f_k:\ k\in\mathbb{Z}\}}^{\mathscr{H}}$$

if it is the image of some orthonormal family by an isomorphism of $\mathscr{H}$.

Proposition A.2. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a family of functions belonging to an Hilbert space $\mathscr{H}$ over $\mathbb{C}$. If $\{f_k\}_{k\in\mathbb{Z}}$ is a Riesz basis of*

$$\overline{span\{f_k:\ k\in\mathbb{Z}\}}^{\mathscr{H}},$$

then there exist $C_1, C_2>0$ such that

$$C_1\|\mathbf{x}\|_{\ell^2}\le\|u\|_{\mathscr{H}}\le C_2\|\mathbf{x}\|_{\ell^2} \tag{A.3}$$

for every $\mathbf{x}=\{x_k\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$ and $u(t)=\sum_{k\in\mathbb{Z}}f_kx_k$.

Proof. First, there is not ambiguity in the definition of the series $u(t)=\sum_{k\in\mathbb{Z}}f_kx_k$. Indeed, $\{f_k\}_{k\in\mathbb{Z}}$ is the image of an orthonormal family $\{e_k\}_{k\in\mathbb{Z}}\subset\mathscr{H}$ by an isomorphism $V:\mathscr{H}\to\mathscr{H}$. For every $\{x_k\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$, the element $\sum_{k\in\mathbb{Z}}e_kx_k\in\mathscr{H}$ and $V\left(\sum_{k\in\mathbb{Z}}e_kx_k\right)\in X$ thanks to the definition of V. Then

$$V\left(\sum_{k\in\mathbb{Z}}e_kx_k\right)=\sum_{k\in\mathbb{Z}}V(e_k)x_k=\sum_{k\in\mathbb{Z}}f_kx_k.$$

Second, for every $\mathbf{x}=\{x_k\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$, thanks to Parseval's identity, we know that for $C_2=|||\,V\,|||_{L(\mathscr{H})}$,

$$\begin{aligned}\|\sum_{k\in\mathbb{Z}}f_kx_k\|_{\mathscr{H}}&\le\|V\Big(\sum_{k\in\mathbb{Z}}e_kx_k\Big)\|_{\mathscr{H}}\le|||\,V\,|||_{L(\mathscr{H})}\|\sum_{k\in\mathbb{Z}}e_kx_k\|_{\mathscr{H}}\\&\le|||\,V\,|||_{L(\mathscr{H})}\|\mathbf{x}\|_{\ell^2}\le C_2\|\mathbf{x}\|_{\ell^2}.\end{aligned}$$

The opposite inequality is verified for $C_1=|||\,V^{-1}\,|||^{-1}_{L(\mathscr{H})}$ as

$$\|\mathbf{x}\|_{\ell^2}\le\|\sum_{k\in\mathbb{Z}}e_kx_k\|_{\mathscr{H}}=\|V^{-1}\Big(\sum_{k\in\mathbb{Z}}f_kx_k\Big)\|_{\mathscr{H}}\le|||\,V^{-1}\,|||_{L(\mathscr{H})}\|\sum_{k\in\mathbb{Z}}f_kx_k\|_{\mathscr{H}}.$$

□

Remark A.3. *Sometimes the inequality* $(A.3)$ *is used as definition for a Riesz basis since it is possible to prove that a family of functions* $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz basis if and only if* $(A.3)$ *is verified.*

Remark A.4. *When Proposition A.2 is satisfied,* $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz Basis in*

$$X = \overline{span\{f_k : \ k \in \mathbb{Z}\}}^{\mathscr{H}} \subseteq \mathscr{H}.$$

For $\{v_k\}_{k\in\mathbb{Z}}$ *the unique biorthogonal family to* $\{f_k\}_{k\in\mathbb{Z}}$ *(*$[BL10, Remark\ 7]$*),* $\{v_k\}_{k\in\mathbb{Z}}$ *is also a Riesz Basis of* X *(*$[BL10, Remark\ 9]$*). If* $\{f_k\}_{k\in\mathbb{Z}}$ *is the image of an orthonormal family* $\{e_k\}_{k\in\mathbb{Z}} \subset \mathscr{H}$ *by an isomorphism* $V : \mathscr{H} \to \mathscr{H}$*, then* $\{v_k\}_{k\in\mathbb{Z}}$ *is the image of* $\{e_k\}_{k\in\mathbb{Z}} \subset \mathscr{H}$ *by the isomorphism* $(V^*)^{-1} : \mathscr{H} \to \mathscr{H}$*. Indeed, for every* $k, n \in \mathbb{Z}$*, we have*

$$\delta_{k,j} = \langle v_k, f_j\rangle_{\mathscr{H}} = \langle v_k, V(e_j)\rangle_{\mathscr{H}} = \langle V^*(v_k), e_j\rangle_{\mathscr{H}}$$

that implies $(V^*)^{-1}(e_k) = v_k$ *for every* $k \in \mathbb{Z}$*. Thus, the arguments of the proof of Proposition A.2 and the relations*

$$(V^*)^{-1} = (V^{-1})^*, \quad |||\, V^* \,|||_{L(\mathscr{H})} = |||\, V \,|||_{L(\mathscr{H})}, \quad |||\, (V^{-1})^* \,|||_{L(\mathscr{H})} = |||\, V^{-1} \,|||_{L(\mathscr{H})}$$

lead to a similar inequality to $(A.3)$ *as*

$$C_2^{-2} \sum_{k\in\mathbb{Z}} |x_k|^2 \le \|u\|_{\mathscr{H}}^2 \le C_1^{-2} \sum_{k\in\mathbb{Z}} |x_k|^2,$$

for every $u(t) = \sum_{k\in\mathbb{Z}} x_k v_k(t)$ *with square-summable complex coefficients* x_k*. The constants* $C_1, C_2 > 0$ *are the same of the relation* $(A.3)$*. Moreover, for every* $u \in X$*, we know that*

$$u = \sum_{k\in\mathbb{Z}} v_k \langle f_k, u\rangle_{\mathscr{H}}$$

since $\{f_k\}_{k\in\mathbb{Z}}$ *and* $\{v_k\}_{k\in\mathbb{Z}}$ *are reciprocally biorthonoromal (see* $[BL10, Remark\ 9]$*) and*

$$C_2^{-1} \left(\sum_{k\in\mathbb{Z}} |\langle f_k, u\rangle_{\mathscr{H}}|^2 \right)^{\frac{1}{2}} \le \|u\|_{\mathscr{H}} \le C_1^{-1} \left(\sum_{k\in\mathbb{Z}} |\langle f_k, u\rangle_{\mathscr{H}}|^2 \right)^{\frac{1}{2}}. \tag{A.4}$$

A.1 Uniformly separated sequences of real numbers

Now, we present Ingham's Theorem and Haraux's Theorem that are two important results implying the solvability of $(A.2)$.

Proposition A.5. [*KL05, Theorem* 4.3] *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a family of real numbers satisfying the uniform gap condition*

$$G := \inf_{k\neq j} |\lambda_k - \lambda_j| > 0.$$

If I *is a bounded interval of length* $|I| > \frac{2\pi}{G}$, *then there exist* $C_1, C_2 > 0$ *such that*

$$C_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \leq \int_I |u(t)|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} |x_k|^2, \tag{A.5}$$

for every $u(t) = \sum_{k\in\mathbb{Z}} x_k e^{i\lambda_k t}$ *with square-summable complex coefficients* x_k.

Proposition A.6. [*KL05, Theorem* 4.6] *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a family of real numbers satisfying the uniform gap condition*

$$G := \inf_{k\neq j} |\lambda_k - \lambda_j| > 0$$

and such that

$$G' := \sup_{K\subset\mathbb{Z}} \inf_{\substack{k\neq j \\ k,j\in\mathbb{Z}\setminus K}} |\lambda_k - \lambda_j| > 0$$

where K *runs over the finite subsets of* $\mathbb{Z}$. *For every bounded interval* $|I| > \frac{2\pi}{G'}$, *there exist* $C_1, C_2 > 0$ *such that*

$$C_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \leq \int_I |u(t)|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} |x_k|^2, \tag{A.6}$$

for every $u(t) = \sum_{k\in\mathbb{Z}} x_k e^{i\lambda_k t}$ *with square-summable complex coefficients* x_k.

In Proposition $A.5$ and Proposition $A.6$, there is no ambiguity on the interpretation of the convergence of the sums. As in the case of orthogonal series, the series that we provide have only countable non-zero terms and they converge in norm unconditionally. The relations ($A.5$) and ($A.6$) lead to the fact that the family of functions $\{e^{i\lambda_k}t\}_{k\in\mathbb{Z}}$ is a Riesz Basis. The same argument is valid for the infinite sums that we treat in the following part of this appendix.

Remark A.7. *For* $T > 0$ *large enough, the relations* ($A.5$) *or* ($A.6$) *guarantee that* $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ *is a Riesz Basis in*

$$X = \overline{span\{e^{i\lambda_k(\cdot)}: \ k \in \mathbb{Z}\}}^{L^2} \subseteq L^2((0,T),\mathbb{C})$$

(see Remark A.3). Thanks to Remark A.4 (relation (A.4)), for $\{v_k\}_{k\in\mathbb{Z}}$ *the unique biorthogonal family to* $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ *and* $\mathscr{H} = L^2((0,T),\mathbb{C})$*, the following inequality is satisfied*

(A.7)
$$C_2^{-1}\left(\sum_{k\in\mathbb{Z}}|\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}|^2\right)^{\frac{1}{2}} \le \|u\|_{\mathscr{H}} \le C_1^{-1}\left(\sum_{k\in\mathbb{Z}}|\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}|^2\right)^{\frac{1}{2}}.$$

Then, the map

$$F : u \in X \longmapsto \left\{\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}\right\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$$

is invertible. For every sequence $\{x_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$*, there exists a function* $u \in \mathscr{H}$ *such that*

$$x_k = \int_0^T u(s)e^{-i\lambda_k s}ds, \qquad \forall k \in \mathbb{Z}.$$

Remark A.8. *We refer to the proof of Theorem* 2.8 *and we consider* $\{\lambda_k\}_{k\in\mathbb{N}} = \{\pi^2(k^2-l^2)\}_{k\in\mathbb{N}}$ *for* $l \in \mathbb{N}$ *such that*

$$\text{(A.8)} \qquad \lambda_k - \lambda_l = \pi^2(k^2-l^2) \neq \pi^2(l^2-j^2) = \lambda_l - \lambda_j, \qquad \forall k,j \in \mathbb{N}.$$

For $k > 0$*, we call* $\omega_k = -\lambda_k$*, while we impose* $\omega_k = \lambda_{-k}$ *for* $k < 0$ *and* $k \neq -l$*. We call* $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$*. The sequence* $\{\omega_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}}$ *satisfies the hypotheses of Proposition A.5 thanks to the relation (A.8), which implies*

$$G := \inf_{k\neq j}|\omega_k - \omega_j| \ge \pi^2.$$

Given $\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$*, we introduce* $\{\tilde{x}_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}} \in \ell^2(\mathbb{C})$ *such that* $\tilde{x}_k = x_k$ *for* $k > 0$*, while* $\tilde{x}_k = \overline{x}_{-k}$ *for* $k < 0$ *and* $k \neq -l$*. Thanks to Remark A.7, for* $T > 2\pi/G$*, there exists* $u \in L^2((0,T),\mathbb{C})$ *such that*

$$\tilde{x}_k = \int_0^T u(s)e^{-i\omega_k s}ds, \qquad \forall k \in \mathbb{Z}^*\setminus\{-l\},$$

$$\Longrightarrow \quad \begin{cases} x_k = \int_0^T u(s)e^{i\lambda_k s}ds, & k \in \mathbb{N}\setminus\{l\},\\ \overline{x}_k = \int_0^T u(s)e^{-i\lambda_k s}ds & k \in \mathbb{N}\setminus\{l\},\\ x_k = \int_0^T u(s)ds, & k = l, \end{cases}$$

$$\Longrightarrow \quad \begin{cases} x_k = \int_0^T u(s)e^{i\lambda_k s}ds, & k \in \mathbb{N}\setminus\{l\},\\ x_k = \int_0^T \overline{u}(s)e^{i\lambda_k s}ds & k \in \mathbb{N}\setminus\{l\},\\ x_k = \int_0^T u(s)ds, & k = l, \end{cases}$$

which implies that, if $x_l \in \mathbb{R}$, then u is real. For $\{v_k\}_{k\in\mathbb{N}}$ the biorthogonal family to $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{N}}$, we have $v_l \in \mathbb{R}$ and $\{\overline{v}_k\}_{k\in\mathbb{N}}$ is the biorthogonal family to $\{e^{-i\lambda_k(\cdot)}\}_{k\in\mathbb{N}}$. Thus

$$u(t) = \sum_{k\in\mathbb{N}} \widetilde{x}_k v_k(t) + \sum_{k\in\mathbb{N}\setminus\{l\}} \widetilde{x}_{-k}\overline{v_k}(t) = x_l v_l(t) + 2\sum_{k\in\mathbb{N}\setminus l} \Re(x_k v_k(t))$$

The relation (A.7) leads to

$$\text{(A.9)}\qquad \begin{aligned} &C_2^{-1}\left(\sum_{k\in\mathbb{Z}^*\setminus\{-l\}} |x_k|^2\right)^{\frac{1}{2}} \leq \|u\|_{L^2((0,T),\mathbb{R})} \leq C_1^{-1}\left(\sum_{k\in\mathbb{Z}^*\setminus\{-l\}} |x_k|^2\right)^{\frac{1}{2}}, \\ &C_2^{-1}\left(\sum_{k\in\mathbb{N}} |x_k|^2\right)^{\frac{1}{2}} \leq \|u\|_{L^2((0,T),\mathbb{R})} \leq 2C_1^{-1}\left(\sum_{k\in\mathbb{N}} |x_k|^2\right)^{\frac{1}{2}}. \end{aligned}$$

For $\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}}$ belonging to

$$\ell_l^2(\mathbb{C}) := \{\{x_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}} : \{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C});\ x_{-k} = \overline{x_k},\ -k \in \mathbb{N}\setminus\{l\};\ x_l \in \mathbb{R}\},$$

we define $u_{\mathbf{x}}(t) = x_l v_l + 2\sum_{k\in\mathbb{N}\setminus\{l\}} \Re(x_k v_k)$ and

$$X := \{u_{\mathbf{x}} :\ \mathbf{x} \in \ell_l^2(\mathbb{C})\}.$$

From (A.9), the map $J : u \in X \longmapsto \{\langle u, e^{i\omega_k(\cdot)}\rangle\}_{k\in\mathbb{Z}^*\setminus\{-l\}} \in \ell_l^2(\mathbb{C})$ is an homeomorphism (for $\{\omega_k\}_{k\in\mathbb{N}}$ defined above), which implies that

$$\tilde{J} : u \in X \longmapsto \{\langle u, e^{i\lambda_k(\cdot)}\rangle\}_{k\in\mathbb{N}} \in \{\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C}) :\ x_l \in \mathbb{R}\}$$

is also an homeomorphism.

Remark A.9. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be an ordered sequence of real numbers such that $\lambda_k \neq -\lambda_l$ for every $k, l \in \mathbb{N}$. If*

$$G := \inf_{k\neq j} |\lambda_k - \lambda_j| > 0,$$

$$G' := \sup_{K\subset\mathbb{N}} \inf_{\substack{k\neq j \\ k,j\in\mathbb{N}\setminus K}} |\lambda_k - \lambda_j|,$$

where K runs over the finite subsets of $\mathbb{Z}$, then a similar result of Remark A.8 is valid. Indeed, as in the mentioned remark, for every $\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and for $T > 2\pi/G'$ there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\tilde{x}_k = \int_0^T u(s) e^{i\lambda_k s} ds, \qquad \forall k \in \mathbb{N}.$$

A.2 Sequences of pairwise distinct real numbers

Let $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. We treat the solvability of the moment problem $(A.2)$ when the numbers $\mathbf{\Lambda} = \{\lambda_k\}_{k\in\mathbb{Z}^*}$ are not uniformly separated but pairwise distinct. We also assume that there exist $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$\inf_{k\in\mathbb{Z}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}. \tag{A.10}$$

From $(A.10)$, we notice that there does not exist $\mathcal{M}$ consecutive $k \in \mathbb{Z}^*$ such that

$$|\lambda_{k+1} - \lambda_k| < \delta.$$

This fact leads to a partition of $\mathbb{Z}^*$ in subsets that we call E_m with $m \in \mathbb{Z}^*$. By definition, for every $m \in \mathbb{Z}^*$, if $k, n \in E_m$, then

$$|\lambda_k - \lambda_n| < \delta(\mathcal{M} - 1),$$

while if $k \in E_m$ and $n \notin E_m$, then

$$|\lambda_k - \lambda_n| \geq \delta.$$

Moreover, the partition defines an equivalence relation in $\mathbb{Z}^*$ such that $k, n \in \mathbb{Z}^*$ are equivalent if and only if there exists $m \in \mathbb{Z}^*$ such that $k, n \in E_m$. The sets $\{E_m\}_{m\in\mathbb{Z}}$ are the corresponding equivalence classes and $i(m) := |E_m| \leq \mathcal{M} - 1$. For every sequence $\mathbf{x} := \{x_l\}_{l\in\mathbb{Z}^*}$, we define the vectors

$$\mathbf{x}^m := \{x_l\}_{l\in E_m}, \qquad m \in \mathbb{Z}^*.$$

Let $\widehat{\mathbf{h}} = \{h_j\}_{j\leq i(m)} \in \mathbb{C}^{i(m)}$. We denote $F_m(\widehat{\mathbf{h}}) : \mathbb{C}^{i(m)} \to \mathbb{C}^{i(m)}$ the matrix with elements, for every $j, k \leq i(m)$,

$$F_{m;j,k}(\widehat{\mathbf{h}}) := \begin{cases} \prod_{\substack{l\neq j \\ 1\leq l\leq k}} (h_j - h_l)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k. \end{cases}$$

Let us introduce the following linear operator on the Hilbert space $\ell^2(\mathbb{C})$

$$F(\mathbf{\Lambda}) : D(F(\mathbf{\Lambda})) \to \ell^2(\mathbb{C}).$$

For each $k \in \mathbb{Z}^*$, we know that there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$ and we introduce

$$(F(\mathbf{\Lambda})\mathbf{x})_k = \Big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)}\Big)_k, \qquad \forall \mathbf{x} = \{x_l\}_{l\in\mathbb{Z}^*} \in D(F(\mathbf{\Lambda})),$$

$$H(\mathbf{\Lambda}) := D(F(\mathbf{\Lambda})) = \left\{\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C}) \ : \ F(\mathbf{\Lambda})\mathbf{x} \in \ell^2(\mathbb{C})\right\}.$$

For $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$ and for $\mathcal{A} = \mathbb{N}$ or $\mathbb{Z}^*$, we define

$$h^s(\mathcal{K}) = \left\{\mathbf{x} = \{x_k\}_{k\in\mathcal{A}} \in \ell^2(\mathcal{K}) \ : \sum_{k\in\mathcal{A}} |k^s x_k|^2 < \infty\right\}, \qquad s > 0.$$

Proposition A.10. *Let $\mathbf{\Lambda} := \{\lambda_k\}_{k\in\mathbb{Z}^*}$ be an ordered sequence of real numbers satisfying (A.10). If there exist $\tilde{d} \geq 0$ and $C > 0$ such that*

$$|\lambda_{k+1} - \lambda_k| \geq C|k|^{-\frac{\tilde{d}}{\mathcal{M}-1}} \qquad \forall k \in \mathbb{Z}^*, \tag{A.11}$$

then $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}(\mathbb{C})$.

Proof. Thanks to $(A.11)$, for every $m \in \mathbb{Z}^*$ and $j,k \in E_m$, we have

$$|\lambda_j - \lambda_k| \geq C \min\{|l|^{-1} \in E_m\}^{\frac{\tilde{d}}{\mathcal{M}-1}}.$$

There exists $C_1 > 0$ such that, for every $1 < j,k \leq i(m)$,

$$\begin{aligned}
|F_{m;j,k}(\mathbf{\Lambda}^m)| &\leq C_1\big(\max\{|l| \in E_m\}^{\frac{\tilde{d}}{\mathcal{M}-1}}\big)^{k-1} \\
&\leq C_1\big(\max\{|l| \in E_m\}^{\frac{\tilde{d}}{\mathcal{M}-1}}\big)^{\mathcal{M}-1} \leq C_1 2^{|E_m|\tilde{d}} \min\{|l| \in E_m\}^{\tilde{d}} \\
&\leq C_1 2^{(\mathcal{M}-1)\tilde{d}} \min\{|l| \in E_m\}^{\tilde{d}}
\end{aligned}$$

and $|F_{m;1,1}(\mathbf{\Lambda}^m)| = 1$. The last relation implies that there exists $C_2 > 0$ such that, for every $j \leq i(m)$,

$$\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big)_{j,j} \leq C_2 \min\{|l| \in E_m\}^{2\tilde{d}},$$

for $F_m(\mathbf{\Lambda}^m)^*$ the transposed matrix of $F_m(\mathbf{\Lambda}^m)$. Thus, there exists $C_3 > 0$ such that

$$Tr\Big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\Big) \leq C_3 \min\{|l| \in E_m\}^{2\tilde{d}}.$$

By calling $\rho(\cdot)$ the spectral radius of a matrix, we denote $|||\, M \,||| = \sqrt{\rho(M^*M)}$ the euclidean norm of a matrix M. As $\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big)$ is positive-definite, for each $m \in \mathbb{Z}^*$, we have

$$|||\, F_m(\mathbf{\Lambda}^m) \,|||^2 = \rho\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big) \leq C_3 \min\{|l| \in E_m\}^{2\tilde{d}}.$$

In conclusion, the proof is achieved since, for every $\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$, it follows

$$\begin{aligned}\|F(\mathbf{\Lambda})\mathbf{x}\|_{\ell^2}^2 &\leq \sum_{l\in\mathbb{Z}^*} |(F(\mathbf{\Lambda})\mathbf{x})_l|^2 \leq \sum_{m\in\mathbb{Z}^*} \interleave F_m(\mathbf{\Lambda}^m) \interleave^2 \sum_{l\in E_m} |x_l|^2 \\ &\leq C_3 \sum_{m\in\mathbb{Z}^*} \min\{|l| \in E_m\}^{2\tilde{d}} \sum_{l\in E_m} |x_l|^2 \leq C_3 \sum_{l\in\mathbb{Z}^*} |l|^{2\tilde{d}}|x_l|^2 \\ &= C_3\|\mathbf{x}\|_{h^{\tilde{d}}}^2.\end{aligned}$$

□

Corollary A.11. *If $\mathbf{\Lambda} := \{\lambda_k\}_{k\in\mathbb{Z}^*}$ is an ordered sequence of pairwise distinct real numbers satisfying* (*A*.10)*, then $F(\mathbf{\Lambda})$ is an invertible map from $H(\mathbf{\Lambda})$ to $Ran(F(\mathbf{\Lambda}))$.*

Proof. By referring to [DZ06, *p.* 48], if the elements of $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ are pairwise distinct, then we can define $F_m(\mathbf{\Lambda}^m)^{-1}$ as the inverse matrix of $F_m(\mathbf{\Lambda}^m)$ for every $m \in \mathbb{Z}^*$. We call $F(\mathbf{\Lambda})^{-1}$ the operator such that, for every $\mathbf{x} \in Ran(F(\mathbf{\Lambda}))$ and $k \in \mathbb{Z}^*$, there holds

$$(F(\mathbf{\Lambda})^{-1}\mathbf{x})_k = \Big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^{-1}\mathbf{x}^{m(k)}\Big)_k,$$

which implies $F(\mathbf{\Lambda})^{-1}F(\mathbf{\Lambda}) = Id_{H(\mathbf{\Lambda})}$, $F(\mathbf{\Lambda})F(\mathbf{\Lambda})^{-1} = Id_{Ran(F(\mathbf{\Lambda}))}$. Hence, $F(\mathbf{\Lambda})^{-1}$ is the inverse operator of $F(\mathbf{\Lambda})$. □

For every $k \in \mathbb{Z}^*$, we know the existence of $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. We define $F(\mathbf{\Lambda})^*$ the infinite matrix such that, for every sequence $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}^*}$ and $k \in \mathbb{Z}^*$,

$$(F(\mathbf{\Lambda})^*\mathbf{x})_k = \Big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*\mathbf{x}^{m(k)}\Big)_k$$

where $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$ is the transposed matrix of $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$. For $T > 0$, we introduce

$$\mathbf{e} := \{e^{i\lambda_j t}\}_{j\in\mathbb{Z}^*} \subset L^2((0,T),\mathbb{C}).$$

Let $t \in (0,T)$ with $T > 0$. We call

$$\xi_k(t) = \big(F(\mathbf{\Lambda})^*\{e^{i\lambda_j t}\}_{j\in\mathbb{Z}^*}\big)_k$$

for every $k \in \mathbb{Z}^*$. By considering each $\xi_k(t)$ as time-dependent function, we denote

$$\mathbf{\Xi} := \{\xi_k(t)\}_{k\in\mathbb{Z}^*} = F(\mathbf{\Lambda})^*\mathbf{e} \subset L^2((0,T),\mathbb{C}).$$

Remark A.12. *Thanks to Proposition* A.10*, when* $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ *satisfies* (A.10)*, the space* $H(\mathbf{\Lambda})$ *is dense in* $\ell^2(\mathbb{C})$ *as* $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}$*. Indeed, for every* $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}^*} \in \ell^2$ *and* $\epsilon > 0$*, there exists* $M \in \mathbb{N}$ *such that*

$$\left(\sum_{l=M+1}^{\infty} |x_k|^2 + \sum_{l=-M-1}^{-\infty} |x_k|^2\right)^{1/2} \leq \epsilon.$$

Now, $\mathbf{x}' = \{x_k\}_{-M\leq k\leq M} \in h^{\tilde{d}}$ *and* $\|\mathbf{x}-\mathbf{x}'\|_{\ell^2} \leq \epsilon$*, which implies that* $h^{\tilde{d}}$ *is dense in* ℓ^2 *with respect to the* ℓ^2*-norm. As* $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}$*,* $H(\mathbf{\Lambda})$ *is dense in* ℓ^2 *with respect to the* ℓ^2*-norm. In this case, we can consider the infinite matrix*

$$F(\mathbf{\Lambda})^*$$

as the unique adjoint operator of $F(\mathbf{\Lambda})$ *with domain* $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*) \subseteq \ell^2(\mathbb{C})$*. By transposing each* $F_m(\mathbf{\Lambda}^m)$ *for* $m \in \mathbb{Z}^*$*, we obtain the following properties of the operator* $F(\mathbf{\Lambda})^*$*.*

- *The arguments of the proof of Corollary* A.11 *lead to the invertibility of the map* $F(\mathbf{\Lambda})^* : H(\mathbf{\Lambda})^* \to Ran(F(\mathbf{\Lambda})^*)$ *and* $(F(\mathbf{\Lambda})^*)^{-1} = (F(\mathbf{\Lambda})^{-1})^*$*.*
- *Thanks to the techniques of the proof of Proposition* A.10*, we know that* $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}$*.*

In the following theorem, we rephrase a result of Avdonin and Moran [AM01], which is also proved by Baiocchi, Komornik and Loreti in [BKL02].

Theorem A.13 (Theorem 3.29; [DZ06])**.** *Let* $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ *be an ordered sequence of pairwise distinct real numbers satisfying* (A.10)*. If* $T > 2\pi/\delta$*, then* $\{\xi_k\}_{k\in\mathbb{Z}^*}$ *forms a Riesz Basis in the space*

$$X := \overline{span\{\xi_k|\ k \in \mathbb{Z}^*\}}^{L^2} \subseteq L^2((0,T),\mathbb{C}).$$

Proposition A.14. *Let* $\{\omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \cup \{0\}$ *be an ordered sequence of real numbers with* $\omega_1 = 0$ *such that there exist* $\tilde{d}, \delta, C > 0$ *and* $\mathcal{M} \in \mathbb{N}$ *with*

$$\begin{cases} \inf_{k\in\mathbb{N}} |\omega_{k+\mathcal{M}} - \omega_k| \geq \delta\mathcal{M}, \\ |\omega_{k+1} - \omega_k| \geq Ck^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{N}. \end{cases}$$

Then, for $T > 2\pi/\delta$ *and* $\{x_k\}_{k\in\mathbb{N}} \in h^{\tilde{d}}(\mathbb{C})$ *with* $x_1 \in \mathbb{R}$*, there exists* $u \in L^2((0,T),\mathbb{R})$ *such that*

$$x_k = \int_0^T u(\tau) e^{i\omega_k\tau} d\tau, \qquad \forall k \in \mathbb{N}. \tag{A.12}$$

Proof. Let $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ be an ordered sequence of real numbers satisfying $(A.10)$ and $(A.11)$. Thanks to Theorem $A.13$, Proposition $A.2$ is valid. The arguments of Remark $A.7$ imply that

$$M : g \in X \mapsto \{\langle \xi_k, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C})$$

is invertible and, for every $k \in \mathbb{Z}^*$, we have

$$\langle \xi_k, g\rangle_{L^2((0,T),\mathbb{C})} = (F(\mathbf{\Lambda})^*\langle \mathbf{e}, g\rangle_{L^2((0,T),\mathbb{C})})_k.$$

Thanks to Remark $A.12$, $(F(\mathbf{\Lambda})^*)^{-1} : Ran(F(\mathbf{\Lambda})^*) \to H(\mathbf{\Lambda})^*$ is invertible and $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}(\mathbb{C})$. Thus, for $\tilde{X} := M^{-1} \circ F(\mathbf{\Lambda})^*(h^{\tilde{d}}(\mathbb{C}))$, the following map is invertible

$$(F(\mathbf{\Lambda})^*)^{-1} \circ M : g \in \tilde{X} \mapsto \{\langle \mathbf{e}, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C}).$$

Now, we define the complex conjugation map $I : \mathbf{x} \in \ell^2(\mathbb{C}) \mapsto \overline{\mathbf{x}} \in \ell^2(\mathbb{C})$ and

$$I \circ (F(\mathbf{\Lambda})^*)^{-1} \circ M : g \in \tilde{X} \mapsto \{\langle g, \mathbf{e}\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$$

is invertible. For every $\{x_k\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$, there exists $g \in \tilde{X}$ such that

$$x_k = \int_0^T \overline{g}(\tau)e^{i\lambda_k\tau}d\tau, \qquad \forall k \in \mathbb{Z}^*.$$

For $u = \overline{g} \in L^2((0,T),\mathbb{C})$, we have

$$x_k = \int_0^T u(\tau)e^{i\lambda_k\tau}d\tau, \qquad \forall k \in \mathbb{Z}^*.$$

When $k > 0$, we call $\lambda_k = \omega_k$, while we impose $\lambda_k = -\omega_{-k}$ for $k < 0$ such that $k \neq -1$. The sequence $\{\lambda_k\}_{k\in\mathbb{Z}^*\setminus\{-1\}}$ is such that there exists $C_1 > 0$ satisfying

$$\begin{cases} \inf_{k\in\mathbb{N}} |\lambda_{k+2\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}, \\ |\lambda_{k+1} - \lambda_k| \geq C_1|k|^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{Z}^* \setminus \{-1\}. \end{cases}$$

As in Remark $A.8$, the solvability of $(A.12)$ is guaranteed for u real when $x_1 \in \mathbb{R}$. □

Lemma A.15. *Let* $\nu := \{\nu_k\}_{k\in\mathbb{Z}^*}$ *be an ordered sequence of pairwise distinct real numbers satisfying* $(A.10)$. *Let* G *be an entire function such that* $G \in L^\infty(\mathbb{R})$ *and there exist* $J, I > 0$ *such that*

$$|G(z)| \leq Je^{I|z|}, \qquad \forall z \in \mathbb{C}.$$

If $\{\nu_j\}_{j\in\mathbb{Z}^*}$ *are simple zeros of* G *such that there exist* $\tilde{d} \geq 0$, $C > 0$ *such that*

$$\text{(A.13)} \qquad |G'(\nu_j)| \geq \frac{C}{|j|^{\tilde{d}}}, \qquad \forall j \in \mathbb{Z}^*,\ \nu_j \neq 0,$$

then there exists $C > 0$ *such that*

$$Tr\Big(F_m(\nu^m)^* F_m(\nu^m)\Big) \leq C \min\{|l| \in E_m\}^{2\tilde{d}}, \qquad \forall m \in \mathbb{Z}^*.$$

Proof. **Construction of a biorthogonal sequence to** $\{e^{i\nu_k(\cdot)}\}_{k\in\mathbb{Z}^*}$**:** The sequence $\{\nu_k\}_{k\in\mathbb{Z}^*}$ satisfies $(A.10)$ and there exist $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$\inf_{k\in\mathbb{Z}^*} |\nu_{k+\mathcal{M}} - \nu_k| \geq \delta\mathcal{M}.$$

If $2I \leq 2\pi/\delta$, then, for every $I_1 \geq I$,

$$|G(z)| \leq Je^{I|z|} \leq Je^{I_1|z|}.$$

We set $T > 2\pi/\delta$ and, for every $k \in \mathbb{Z}^*$, we define the function

$$G_k(z) := \frac{G(z)}{(z - \nu_k)}.$$

Thanks to the Paley-Wiener's Theorem [DZ06, $Theorem$ 3.19], for every $k \in \mathbb{Z}^*$, there exists $w_k \in L^2$ with support in $[0, T]$ such that

$$G_k(z) = \int_{\mathbb{R}} e^{izt} e^{-iz\frac{T}{2}} w_k(t)dt = \int_0^T e^{izt} e^{-iz\frac{T}{2}} w_k(t)dt.$$

For $j, k \in \mathbb{Z}^*$ and $c_k := G'(\nu_k)$, we call $v_k(t) := e^{i\nu_k\frac{T}{2}}\overline{w_k}(t)$ and

$$\langle v_k, e^{i\nu_j(\cdot)}\rangle_{L^2((0,T),\mathbb{C})} = \delta_{k,j} G_k(\nu_k) = \delta_{k,j} G'(\nu_k) = \delta_{k,j} c_k.$$

The sequence $\{v_k\}_{k\in\mathbb{Z}^*}$ is biorthogonal to $\{e^{i\nu_k(\cdot)}/c_k\}_{k\in\mathbb{Z}^*}$ and $\{v_k/c_k\}_{k\in\mathbb{Z}^*}$ is biorthogonal to $\{e^{i\nu_k(\cdot)}\}_{k\in\mathbb{Z}^*}$.

Thanks to the Plancherel's identity, $\|v_k\|_{L^2((0,T),\mathbb{C})} = \|G_k\|_{L^2(\mathbb{R},\mathbb{R})}$. We show that from the Phragmén-Lindelöf Theorem ($e.g.$ [You80, $p.$ 82; $Theorem$ 11]), there exists $C_1 > 0$ such that

$$\|v_k\|_{L^2((0,T),\mathbb{C})} = \|G_k\|_{L^2(\mathbb{R},\mathbb{R})} \leq C_1, \qquad \forall k \in \mathbb{Z}^*. \tag{A.14}$$

Indeed, G is an entire function such that there exist I and J such that $|G(z)| \leq Je^{I|z|}$ for every $z \in \mathbb{C}$. Moreover, there exists $M > 0$ such that $|G(x)| \leq M$ for every $x \in \mathbb{R}$. From [You80, $p.$ 82; $Theorem$ 11], we have

$$|G(x+iy)| \leq Me^{I|y|}, \qquad \forall x, y \in \mathbb{R}.$$

For every $k \in \mathbb{Z}^*$, we consider $\|G_k\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overline{G_k(x)} G_k(x)\, dx = \int_{\mathbb{R}} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx$ and there exists $c_1 > 0$ not depending on k such that

$$\begin{aligned}
\|G_k\|^2_{L^2(\mathbb{R})} &= \int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx + \int_{|x-\nu_k|\geq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx \\
&= \int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx + \int_{\mathbb{R}\setminus(-1,+1)} \frac{\overline{G(x+\nu_k)}G(x+\nu_k)}{x^2}\, dx \\
&\leq \int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx + M^2 \int_{\mathbb{R}\setminus(-1,+1)} \frac{1}{x^2}\, dx \\
&\leq \int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx + M^2 c_1.
\end{aligned}$$

We analyze the term $\int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx$ and we notice that $z \mapsto \frac{\overline{G(\overline{z})}G(z)}{(z-\nu_k)^2}$ is an entire function. Hence, by Cauchy Integral Theorem

$$\int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx + \int_{|z-\nu_k|=1, \Im z>0} \frac{\overline{G(\overline{z})}G(z)}{(z-\nu_k)^2}\, dz = 0$$

$$\implies \qquad \int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx = -\int_0^{\pi} \overline{G(\overline{\nu_k + e^{i\theta}})} G(\nu_k + e^{i\theta}) i e^{-i\theta}\, d\theta.$$

Now, there exists $c_2 > 0$ not depending on k such that

$$\int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2} dx \leq \int_0^{\pi} |\overline{G(\overline{\nu_k + e^{i\theta}})}||G(\nu_k + e^{i\theta})| d\theta \leq M^2 \int_0^{\pi} e^{2I\sin(\theta)}\, d\theta = M^2 c_2.$$

In conclusion, the relation $(A.14)$ is valid as $M^2 c_1$ and $M^2 c_2$ do not depend on k, then $\|G_k\|^2_{L^2(\mathbb{R})} \leq M^2(c_1 + c_2)$ for every $k \in \mathbb{Z}^*$.

Construction of a Riesz basis and conclusion: Let $\boldsymbol{\nu} := \{\nu_k\}_{k\in\mathbb{Z}^*}$ and $\mathbf{e} := \{e^{i\nu_k(\cdot)}\}_{k\in\mathbb{Z}^*} \subset L^2((0,T),\mathbb{C})$. For $\boldsymbol{\Xi} = \{\xi_k\}_{k\in\mathbb{Z}^*} := \{(F(\boldsymbol{\nu})^*\mathbf{e})_k\}_{k\in\mathbb{Z}^*}$, thanks to Proposition $A.13$, the sequence of functions $\{\xi_k\}_{k\in\mathbb{Z}^*}$ forms a Riesz basis in

$$X := \overline{span\{\xi_k \ : \ k \in \mathbb{Z}^*\}}^{L^2} \subseteq L^2((0,T),\mathbb{C}).$$

We call $\widetilde{v} := \{\widetilde{v}_k\}_{k\in\mathbb{Z}^*}$ the corresponding biorthogonal sequence which is also a Riesz basis of X. Thanks to Remark $A.12$, the map $F(\boldsymbol{\nu})$ is invertible from $H(\boldsymbol{\nu})^*$ to $Ran(F(\boldsymbol{\nu})^*)$ and

$$(F(\boldsymbol{\nu})^*)^{-1} = (F(\boldsymbol{\nu})^{-1})^*.$$

As $\mathbf{v}/\mathbf{c} = \{v_k/c_k\}_{k\in\mathbb{Z}^*}$ is biorthogonal to $\{e^{i\nu_k(\cdot)}\}_{k\in\mathbb{Z}^*}$, we have $\{v_k/c_k\}_{k\in\mathbb{Z}^*} = F(\boldsymbol{\nu})\widetilde{v}$. Indeed, for every $j,k \in \mathbb{N}$, it holds

$$\begin{aligned}\delta_{k,j} &= \langle v_k/c_k, e^{i\lambda_j(\cdot)}\rangle_{L^2((0,T),\mathbb{C})} = \langle v_k/c_k, ((F(\boldsymbol{\Lambda})^*)^{-1}\boldsymbol{\Xi})_j\rangle_{L^2((0,T),\mathbb{C})}\\ &= \langle (F(\boldsymbol{\Lambda})^{-1}\mathbf{v}/\mathbf{c})_k, \xi_j\rangle_{L^2((0,T),\mathbb{C})},\end{aligned}$$

which implies that $(F(\boldsymbol{\Lambda})^{-1}\mathbf{v}/\mathbf{c})_k = \widetilde{v}_k$. The uniqueness of the biorthogonal family to $\boldsymbol{\Xi}$ implies the uniqueness of the biorthogonal family to $\mathbf{e}$. From Theorem $A.2$, there exist $C_2, C_3 > 0$ such that

$$C_2\|\mathbf{x}\|_{\ell^2}^2 \leq \int_0^T |u(s)|^2 ds \leq C_3\|\mathbf{x}\|_{\ell^2}^2,$$

for $u(t) = \sum_{k\in\mathbb{Z}^*} \xi_k x_k$ and $\mathbf{x} \in \ell^2(\mathbb{C})$. Thanks to the biorthogonality, we have

$$x_k = \langle \tilde{v}_k, u\rangle_{L^2((0,T),\mathbb{C})} = \int_0^T \overline{\tilde{v}_k(\tau)}u(\tau)d\tau, \qquad \forall k \in \mathbb{Z}^*.$$

For every $k \in \mathbb{Z}^*$, we call $m(k) \in \mathbb{Z}^*$ the number such that $k \in E_{m(k)}$. Thanks to $(A.13)$ and $(A.14)$, there exist $C_4, C_5 > 0$ such that, for every $k \in \mathbb{Z}^*$, we have

$$\begin{aligned}|(F(\boldsymbol{\nu})\mathbf{x})_k| &= |\langle (F(\boldsymbol{\nu})\{\langle \tilde{v}_l, u\rangle_{L^2((0,T),\mathbb{C})}\}_{l\in\mathbb{Z}})_k|\\ &= |\langle (F(\boldsymbol{\nu})\tilde{v})_k, u\rangle_{L^2((0,T),\mathbb{C})}| = |\langle v_k/c_k, u\rangle_{L^2((0,T),\mathbb{C})}|\\ &\leq \frac{\|v_k\|_{L^2((0,T),\mathbb{C})}\|u\|_{L^2((0,T),\mathbb{C})}}{|c_k|} \leq C_3^{\frac{1}{2}}\frac{\|G_k\|_{L^2(\mathbb{R},\mathbb{R})}\|\mathbf{x}\|_{\ell^2}}{|G'(\nu_k)|}\\ &\leq C_4|k|^{\tilde{d}}\|\mathbf{x}\|_{\ell^2} \leq C_5 \min\{|l| \in E_{m(k)}\}^{\tilde{d}}\|\mathbf{x}\|_{\ell^2}.\end{aligned}$$

Then, for every $j,k \leq i(m)$, we obtain

$$|(F_{m;j,k}(\boldsymbol{\nu}^m))| \leq C_6 \min\{|l| \in E_m\}^{\tilde{d}}.$$

The arguments of the proof of Proposition $A.10$ lead to the existence of $C > 0$ such that

$$Tr\Big(F_m(\boldsymbol{\nu}^m)^* F_m(\boldsymbol{\nu}^m)\Big) \leq C \min\{|l| \in E_m\}^{2\tilde{d}}. \quad \square$$

Proposition A.16. *Let $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ be an ordered sequence of distinct real numbers and*

$$\{\nu_k\}_{k\in\mathbb{Z}^*} = \Big\{sgn(\lambda_k)\sqrt{|\lambda_k|}\Big\}_{k\in\mathbb{Z}^*}$$

satisfy (A.10). *We assume that there exist $C_1, C_2 > 0$ such that, for every $k \in \mathbb{Z}^*$ with $\nu_k \neq 0$, the following inequality is verified*

$$C_1|k| \leq |\nu_k| \leq C_2|k|. \tag{A.15}$$

Let G be an entire function such that $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ such that

$$|G(z)| \leq Je^{I|z|}, \qquad \forall z \in \mathbb{C}.$$

If $\{\nu_j\}_{j\in\mathbb{Z}^*}$ are simple zeros of G and there exist $\tilde{d} \geq 1$, $C > 0$ such that

$$|G'(\nu_j)| \geq \frac{C}{|j|^{\tilde{d}}}, \qquad \forall j \in \mathbb{Z}^*,\ \nu_j \neq 0,$$

then the space $H(\boldsymbol{\Lambda})$ contains $h^{\tilde{d}-1}$.

Proof. If $\{\nu_k\}_{k\in\mathbb{Z}^*} = \Big\{sgn(\lambda_k)\sqrt{|\lambda_k|}\Big\}_{k\in\mathbb{Z}^*}$, then $\lambda_k = sgn(\nu_k)\nu_k^2$ for every $k \in \mathbb{Z}^*$. There exist $\delta > 0$ and $\mathcal{M} \in \mathbb{N}$ such that

$$\inf_{k\in\mathbb{Z}^*} |\nu_{k+\mathcal{M}} - \nu_k| \geq \delta\mathcal{M} \geq \delta \inf_{\substack{j\in\mathbb{Z}^*\\ \nu_j\neq 0}} \{|\nu_j|, 1\}\mathcal{M}.$$

For every $k \in \mathbb{N}$ such that $\lambda_{k+\mathcal{M}}$ and λ_k have the same sign, we have

$$|\lambda_{k+\mathcal{M}} - \lambda_k| = |\nu_{k+\mathcal{M}} - \nu_k||\nu_{k+\mathcal{M}} + \nu_k| \geq \delta \inf_{\substack{j\in\mathbb{Z}^*\\ \nu_j\neq 0}} \{|\nu_j|, 1\}\mathcal{M}.$$

For every $a, b > 0$, we know that $|a^2 + b^2| \geq \min\{a, b\}|a + b|$ and, for every $k \in \mathbb{Z}^*$ such that $\lambda_{k+\mathcal{M}}$ and λ_k have opposite signs, we obtain

$$|\lambda_{k+\mathcal{M}} - \lambda_k| = |\nu_{k+\mathcal{M}}^2 + \nu_k^2| \geq \inf_{\substack{j\in\mathbb{Z}^*\\ \nu_j\neq 0}} \{|\nu_j|, 1\}|\nu_{k+\mathcal{M}} - \nu_k| \geq \inf_{\substack{j\in\mathbb{Z}^*\\ \nu_j\neq 0}} \{|\nu_j|, 1\}\delta\mathcal{M}.$$

Both sequences $\boldsymbol{\Lambda} := \{\lambda_k\}_{k\in\mathbb{Z}^*}$ and $\boldsymbol{\nu} := \{\nu_k\}_{k\in\mathbb{Z}^*}$ satisfy $(A.10)$ with respect to $\delta' := \inf_{\substack{j\in\mathbb{Z}^*\\ \nu_j\neq 0}} \{|\nu_j|, 1\}\delta$ and $\mathcal{M}$, which leads to the same the equivalence classes E_m in $\mathbb{Z}^*$. Then, the theory exposed in the current appendix is valid for both the sequences $\mathbf{\Lambda}$ and $\boldsymbol{\nu}$. Now, we notice that $\{\nu_k\}_{k\in\mathbb{Z}^*}$ verifies the hypotheses of Lemma $A.15$ with respect to δ' and $\mathcal{M}$ and

$$Tr\Big(F_m(\boldsymbol{\nu}^m)^*F_m(\boldsymbol{\nu}^m)\Big) \leq C\min\{|l|\in E_m\}^{2\tilde{d}}, \qquad \forall m\in\mathbb{Z}^*.$$

As above, we notice

$$|\lambda_{k+1}-\lambda_k| = |sgn(\nu_{k+1})\nu_{k+1}^2 - sgn(\nu_k)\nu_k^2| \geq \min\{|\nu_k|,|\nu_{k+1}|\}|\nu_{k+1}-\nu_k|.$$

For every $m\in\mathbb{Z}^*$ and $I\subseteq E_m$ such that $|I|\geq 2$, we have $|I|\leq|E_m|\leq\mathcal{M}-1$. For $C_1 = \min_{\substack{l\in Z^*\\ |\nu_l|\neq 0}}\{|\nu_l|^{\mathcal{M}-3},|\nu_l|\}$, there holds

$$\begin{aligned}\prod_{j,k\in I}|\lambda_k-\lambda_j| &\geq (\min\{|\nu_l|:\ l\in I,|\nu_l|\neq 0\})^{|I|-1}\prod_{j,k\in I}|\nu_k-\nu_j|\\ &\geq \min_{\substack{l\in Z^*\\ |\nu_l|\neq 0}}\{|\nu_l|^{\mathcal{M}-3},|\nu_l|\}\min\{|\nu_l|:\ l\in I,|\nu_l|\neq 0\}\prod_{j,k\in I}|\nu_k-\nu_j|\\ &\geq C_1\min\{|\nu_l|:\ l\in I,|\nu_l|\neq 0\}\prod_{j,k\in I}|\nu_k-\nu_j|.\end{aligned}$$

For every $m\in\mathbb{Z}^*$ and $j,k\in E_m$, the following inequality is valid

$$|F_{m;j,k}(\mathbf{\Lambda}^m)| \leq C_1\frac{|F_{m;j,k}(\boldsymbol{\nu}^m)|}{\min\{|\nu_l|:\ l\in E_m,\nu_l\neq 0\}}.$$

Thanks to the arguments adopted in the proof of Proposition $A.10$, from Proposition $A.15$, there exists $C_2>0$ such that

$$\begin{aligned}Tr\Big(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)\Big) &\leq C_1^2\frac{Tr\Big(F_m(\boldsymbol{\nu}^m)^*F_m(\boldsymbol{\nu}^m)\Big)}{\min\{\nu_l^2:\ l\in E_m,\nu_l\neq 0\}}\\ &\leq C_1^2\frac{(\mathcal{M}-1)^2C_2^2\min\{|l|\in E_m\}^{2\tilde{d}}}{\min\{\nu_l^2:\ l\in E_m,\nu_l\neq 0\}}.\end{aligned}$$

As in the proof of Proposition $A.10$, thanks to the relation $(A.15)$, for every $m\in\mathbb{Z}^*$, there exists $C_3>0$ such that

$$|||\, F_m(\mathbf{\Lambda}^m)\, |||^{\,2} \leq C_3\min\{|l|\in E_m\}^{2(\tilde{d}-1)},$$

which leads to $h^{\tilde{d}-1}\subset H(\mathbf{\Lambda})$. □

Proposition A.17. *Let $\{\sqrt{\omega}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \cup \{0\}$ be an ordered sequence of pairwise distinct numbers such that $\omega_1 = 0$ and there exist $\delta, C > 0$ and $\mathcal{M} \in \mathbb{N}$ such that*

$$\inf_{k\in\mathbb{N}} |\sqrt{\omega}_{k+\mathcal{M}} - \sqrt{\omega}_k| \geq \delta\mathcal{M}.$$

We assume that there exist $C_1, C_2 > 0$ such that, for every $k \in \mathbb{N}$ with $\omega_k \neq 0$, the following inequality is verified

$$C_1 k \leq \omega_k \leq C_2 k.$$

Let G be an entire function such that $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ such that

$$|G(z)| \leq J e^{I|z|}, \qquad \forall z \in \mathbb{C}.$$

If $\{\sqrt{\omega_k}\}_{k\in\mathbb{N}}$ and $\{-\sqrt{\omega_k}\}_{k\in\mathbb{N}}$ are simple zeros of G and there exist $\tilde{d} \geq 1$, $C > 0$ such that

$$|G'(\sqrt{\omega_k})| \geq \frac{C}{k^{\tilde{d}}}, \qquad |G'(-\sqrt{\omega_k})| \geq \frac{C}{k^{\tilde{d}}}, \qquad \forall j \in \mathbb{N},$$

then, for $T > 2\pi/\delta$ and for every $\{x_k\}_{k\in\mathbb{N}} \in h^{\tilde{d}-1}(\mathbb{C})$ with $x_1 \in \mathbb{R}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$x_k = \int_0^T u(\tau) e^{i\omega_k \tau} d\tau, \qquad \forall k \in \mathbb{N}.$$

Proof. First, we construct a sequence $\boldsymbol{\nu} := \{\nu_k\}_{k\in\mathbb{Z}^*}$ such that $\nu_k = \sqrt{\omega_k}$ for $k > 0$ and $\nu_k = -\sqrt{\omega_{-k}}$ for $k < 0$. Second, we call $\boldsymbol{\Lambda} := \{\lambda_k\}_{k\in\mathbb{Z}^*}$ such that $\lambda_k = \omega_k$ for $k > 0$ and $\lambda_k = -\omega_{-k}$ for $k < 0$ with $k \neq -1$. Now, the hypotheses of Proposition $A.16$ are satisfied with respect to $\boldsymbol{\nu}$ and $\boldsymbol{\Lambda}$ that imply

$$H(\boldsymbol{\Lambda}) \supseteq h^{\tilde{d}-1}.$$

The validity of Remark $A.12$ is guaranteed and $H(\boldsymbol{\Lambda})^* \supseteq h^{\tilde{d}-1}$. In conclusion, as in Proposition Remark $A.8$ and $A.14$, for every $\{x_k\}_{k\in\mathbb{N}} \in h^{\tilde{d}-1}(\mathbb{C})$ with $x_1 \in \mathbb{R}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\tilde{x}_k = \int_0^T u(s) e^{i\lambda_k s} ds, \qquad \forall k \in \mathbb{N}.$$

□

Proposition A.18. *Let the hypotheses of one of the two points of Theorem 4.3 be satisfied. For every $T > 0$, there exists $C(T) > 0$ uniformly bounded* *for T lying on bounded intervals such that, for every $g \in L^2((0,T),\mathbb{C})$, we have*

$$\left\|\int_0^T e^{i\lambda_{(\cdot)}s}g(s)ds\right\|_{\ell^2} \leq C(T)\|g\|_{L^2((0,T),\mathbb{C})}.$$

Proof. **1) Uniformly separated eigenvalues:** Let $\{\omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that

$$\gamma := \inf_{k\neq j}|\omega_k - \omega_j| > 0$$

Thanks to Proposition $A.5$ and Remark $A.7$, for $T > \frac{2\pi}{\gamma}$, the family of functions $\{e^{i\omega_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz Basis in

$$X = \overline{span\{e^{i\omega_k(\cdot)}:\ k \in \mathbb{N}\}}^{L^2} \subseteq L^2((0,T),\mathbb{C})$$

(Remark $A.3$). Moreover, as in the relation $(A.7)$, there exists $C_1(T) > 0$ such that

$$\left(\sum_{k\in\mathbb{N}}|\langle e^{i\omega_k(\cdot)}, u\rangle_{L^2((0,T),\mathbb{C})}|^2\right)^{\frac{1}{2}} \leq C_1(T)\|u\|_{L^2((0,T),\mathbb{C})}$$

for every $u \in X$. We denote with P the orthogonal projector mapping $L^2((0,T),\mathbb{C})$ in X and, for every $g \in L^2((0,T),\mathbb{C})$,

$$\left\|\{\langle e^{i\omega_k(\cdot)}, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{N}}\right\|_{\ell^2} = \left\|\{\langle e^{i\omega_k(\cdot)}, Pg\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{N}}\right\|_{\ell^2}$$

$$\leq C_1(T)\|Pg\|_{L^2((0,T),\mathbb{C})} \leq C_1(T)\|g\|_{L^2((0,T),\mathbb{C})}.$$

2) Pairwise distinct eigenvalues: Let the hypotheses of one of the two points of Theorem 4.3 be satisfied. For any graph, there exists $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}}|\lambda_{k+\mathcal{M}} - \lambda_k| > \delta\mathcal{M}.$$

We can define $\{\lambda_k^j\}_{\substack{k,j\in\mathbb{N}\\ j\leq\mathcal{M}}}$ such that $\{\lambda_k\}_{k\in\mathbb{N}}$ is obtained by reordering $\{\lambda_k^j\}_{\substack{k,j\in\mathbb{N}\\ j\leq\mathcal{M}}}$ and

$$\inf_{k\neq l}|\lambda_k^j - \lambda_l^j| > \delta\mathcal{M}, \qquad \forall j \leq \mathcal{M}.$$

Now, for every $j \leq \mathcal{M}$, we apply the result of the point **1)** by considering

$$\{\omega_k\}_{k\in\mathbb{N}} = \{\lambda_k^j\}_{k\in\mathbb{N}}.$$

For every $T > 2\pi/\delta\mathcal{M}$ and $g \in L^2((0,T),\mathbb{C})$, there exist $\{C_j(T)\}_{j\leq\mathcal{M}} \subset \mathbb{R}^+$ and $C(T) > 0$ uniformly bounded for T lying on bounded intervals such that

$$\left\|\{\langle e^{i\lambda_k(\cdot)}, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{N}}\right\|_{\ell^2} \leq \sum_{j=1}^{\mathcal{M}} \left\|\{\langle e^{i\lambda_k^j(\cdot)}, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{N}}\right\|_{\ell^2}$$

$$\leq \sum_{j=1}^{\mathcal{M}} C_j(T)\|g\|_{L^2((0,T),\mathbb{C})} \leq \mathcal{M}C(T)\|g\|_{L^2((0,T),\mathbb{C})},$$

which implies that

$$\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}\overline{g}(\tau)dt\right\|_{\ell^2} = \left\|\int_0^T e^{-i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2}$$

$$\leq \mathcal{M}C(T)\|g\|_{L^2((0,T),\mathbb{C})} = \mathcal{M}C(T)\|\overline{g}\|_{L^2((0,T),\mathbb{C})}.$$

Then, for every $g \in L^2((0,T),\mathbb{C})$,

$$\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2} \leq \mathcal{M}C(T)\|g\|_{L^2((0,T),\mathbb{C})}.$$

In conclusion, for $T > 2\pi/\delta\mathcal{M}$, we choose the smallest value possible for $C(T)$. When $T \leq 2\pi/\delta\mathcal{M}$, for $g \in L^2((0,T),\mathbb{C})$, we define

$$\tilde{g} \in L^2((0, 2\pi/\delta\mathcal{M}+1),\mathbb{C})$$

such that $\tilde{g} = g$ on $(0,T)$ and $\tilde{g} = 0$ in $(T, 2\pi/\delta\mathcal{M}+1)$. Then

$$\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2} = \left\|\int_0^{2\pi/\delta\mathcal{M}+1} e^{i\lambda_{(\cdot)}\tau}\tilde{g}(\tau)dt\right\|_{\ell^2} \leq \mathcal{M}C(2\pi/\delta\mathcal{M}+1)\|g\|_{L^2((0,T),\mathbb{C})}.$$

Let $0 < T_1 < T_2 < +\infty$, $g \in L^2(0,T_1)$ and $\tilde{g} \in L^2(0,T_2)$ be defined as $\tilde{g} = g$ on $(0,T_1)$ and $\tilde{g} = 0$ on (T_1,T_2). By applying the inequality on $\tilde{g}$, we obtain $C(T_1) \leq C(T_2)$. □

Appendix B

Analytic Perturbation

B.1 Bilinear Schrödinger equation on a bounded interval

Let us consider the problem (3.12) and the eigenvalues $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ of the operator $A + u_0 B$. Let B be a bounded symmetric operator satisfying Assumptions II. We introduce some results from Kato [Kat95].

Definition B.1. Let D be a domain of the complex plan. A family $T(z)$ for $z \in D$ of closed operators from a Banach space X to a Banach space Y is said to be a holomorphic family of type (A) when $D(T(z))$ is independent of z and $T(z)u$ is holomorphic for $z \in D$ and for every $u \in D(T(z))$.

Theorem B.2. [*Kat95*, *Theorem VII*.3.9] *Let $T(z)$ be a self-adjoint holomorphic family of type (A) defined for z in a neighborhood of an interval $I \subset \mathbb{R}$. Furthermore, let $T(z)$ have a compact resolvent. Then all eigenvalues of $T(z)$ can be represented by functions that are holomorphic in I. More precisely, there is a sequence of scalar-valued functions $z \mapsto \{\lambda_n(z)\}_{n\in\mathbb{N}}$ and operator-valued functions $z \mapsto \{\phi_n(z)\}_{n\in\mathbb{N}}$, all holomorphic on I, such that for $z \in I$, the sequence $\{\lambda_n(z)\}_{n\in\mathbb{N}}$ represents all the repeated eigenvalues of $T(z)$ and $\{\phi_n(z)\}_{n\in\mathbb{N}}$ forms a complete orthonormal family of the associated eigenvectors of $T(z)$.*

When B is a bounded symmetric operator satisfying Assumptions II and $A = -\Delta$ is the Laplacian with Dirichlet type boundary conditions

$$D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C}),$$

thanks to [Kat95, *Theorem VII*.2.6], there exists a neighborhood $D \subset \mathbb{R}$ containing 0 such that the self-adjoint family of operators $A + u_0 B$ is holomorphic of type (A) (Definition $B.1$) for $u_0 \in D$. Then, the following proposition follows from Theorem $B.2$.

Proposition B.3. *Let B be a bounded symmetric operator satisfying Assumptions II. There exists a neighborhood D of $u = 0$ in $\mathbb{R}$ small enough where the maps $u \mapsto \lambda_j^u$ are analytic for every $j \in \mathbb{N}$.*

The next lemma proves the existence of perturbations, which do not shrink the eigenvalues gaps. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a complete orthonormal system of $L^2((0,1),\mathbb{C})$ composed by eigenfunctions of A and associated to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ ($\lambda_k = \pi^2 k^2$).

Lemma B.4. *Let B be a bounded symmetric operator satisfying Assumptions II. There exists a neighborhood $U(0)$ in $\mathbb{R}$ of $u = 0$ such that, there exists $r > 0$ such that, for every $u_0 \in U(0)$ and $j \in \mathbb{N}$,*

$$\mu_j := \frac{\lambda_j + \lambda_{j+1}}{2} \in \rho(A + u_0 B), \qquad \vert\vert\vert\, (A + u_0 B - \mu_j)^{-1} \,\vert\vert\vert \leq r.$$

Proof. Let D be the neighborhood provided by Proposition $B.3$. First, we prove that, for $u_0 \in D$ small enough, the operator $(A+u_0B-\mu_j)$ is invertible with bounded inverse for every $j \in \mathbb{N}$. We know that $(A - \mu_j)$ is invertible in a bounded operator and $\mu_j \in \rho(A)$ (resolvent set of A). Let

$$\delta := \min_{j\in\mathbb{N}}\{|\lambda_{j+1} - \lambda_j|\}.$$

We know that

$$\vert\vert\vert\, (A - \mu_j)^{-1} \,\vert\vert\vert \leq \sup_{k\in\mathbb{N}} \frac{1}{|\mu_j - \lambda_k|} = \frac{2}{|\lambda_{j+1} - \lambda_j|} \leq \frac{2}{\delta}.$$

Thus

$$\vert\vert\vert\, (A - \mu_j)^{-1} u_0 B \,\vert\vert\vert \leq |u_0| \,\vert\vert\vert\, (A - \mu_j)^{-1} \,\vert\vert\vert\, \vert\vert\vert\, B \,\vert\vert\vert \leq \frac{2}{\delta} |u_0| \,\vert\vert\vert\, B \,\vert\vert\vert$$

and if

$$|u_0| \leq \frac{\delta(1-\epsilon)}{2 \,\vert\vert\vert\, B \,\vert\vert\vert} \quad \text{for} \quad \epsilon \in (0,1),$$

then we have

$$\vert\vert\vert\, (A - \mu_j)^{-1} u_0 B \,\vert\vert\vert \leq 1 - \epsilon.$$

The operator $(A + u_0 B - \mu_j)$ is invertible and, for every $\psi \in D(A)$,

$$\|(A + u_0 B - \mu_j)\psi\| \geq \|(A - \mu_j)\psi\| - \|u_0 B\psi\| \geq \Big(\frac{\delta}{2} - \frac{\delta(1-\epsilon)}{2}\Big)\|\psi\| = \frac{\delta\epsilon}{2}\|\psi\|.$$

In conclusion, $\vert\vert\vert\, (A + u_0 B - \mu_j)^{-1} \,\vert\vert\vert \leq \frac{2}{\delta\epsilon}$. □

Lemma B.5. *Let B be a bounded symmetric operator satisfying Assumptions II. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that, for every $u_0 \in U(0)$,*

$$(A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})$$

is invertible with bounded inverse from $D(A) \cap \phi_k^{\perp}$ to $\phi_k^{\perp}$, for every $k \in \mathbb{N}$ and $P_{\phi_k}^{\perp}$ is the projector onto the orthogonal space of ϕ_k.

Proof. Let D be the neighborhood provided by Lemma $B.4$. For any $u_0 \in D$, one can consider the decomposition

$$(A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0}) = (A - \lambda_k^{u_0}) + u_0 P_{\phi_k}^{\perp} B.$$

The operator $A - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and we estimate

$$|||\, ((A - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1} u_0 P_{\phi_k}^{\perp} B \,||| \,.$$

However, for every $\psi \in D(A) \cap Ran(P_{\phi_k}^{\perp})$ such that $\|\psi\| = 1$, we have

$$\|(A - \lambda_k^{u_0})\psi\| \geq \min\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\}\|\psi\|.$$

Let

$$\delta_k := \min\left\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\right\}.$$

Thanks to Lemma $B.4$, for $|u_0|$ small enough, $\lambda_k^{u_0} \in \left(\frac{\lambda_{k-1}+\lambda_k}{2}, \frac{\lambda_k+\lambda_{k+1}}{2}\right)$ and then

$$\begin{aligned}\delta_k &\geq \min\left\{\left|\lambda_{k+1} - \frac{\lambda_k + \lambda_{k+1}}{2}\right|, \left|\frac{\lambda_{k-1} + \lambda_k}{2} - \lambda_{k-1}\right|\right\} \\ &\geq \frac{(2k-1)\pi^2}{2} > k.\end{aligned}$$

Afterwards,

$$|||\, ((A - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1} u_0 P_{\phi_k}^{\perp} B \,||| \,\leq \frac{1}{\delta_k}|u_0| \,|||\, B \,|||$$

and, if $|u_0| \leq (1 - r)\frac{\delta_k}{|||\, B\, |||}$ for $r \in (0, 1)$, then it follows

$$|||\, ((A - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1} u_0 P_{\phi_k}^{\perp} B \,||| \,\leq (1 - r) < 1.$$

The operator $A_k := (A - \lambda_k^{u_0} + u_0 P_{\phi_k}^{\perp} B)$ is invertible when it acts on the orthogonal space of ϕ_k and

$$\begin{aligned}\|| A_k \|| \geq \|| A - \lambda_k^{u_0} \|| - \|| u_0 P_{\phi_k}^{\perp} B \|| &\geq \delta_k - \|| u_0 P_{\phi_k}^{\perp} B \|| \\ g \geq \delta_k - |u_0| \|| B \|| \geq \delta_k - (1-r)\delta_k &= r\delta_k.\end{aligned}$$

In conclusion, as

$$\|| ((A - \lambda_k^{u_0} + u_0 P_{\phi_k}^{\perp} B)\big|_{\phi_k^{\perp}})^{-1} \|| \leq \frac{1}{r\delta_k} < \frac{1}{rk}, \tag{B.1}$$

the proof is achieved. □

Lemma B.6. *Let B be a bounded symmetric operator satisfying Assumptions II. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0 \in U(0)$, we have $\lambda_j^{u_0} \neq 0$ and $\lambda_j^{u_0} \asymp \lambda_j$ for every $j \in \mathbb{N}$. In other words, there exist two constants $C_1, C_2 > 0$ such that, for each $j \in \mathbb{N}$,*

$$C_1 \lambda_j \leq \lambda_j^{u_0} \leq C_2 \lambda_j. \tag{B.2}$$

Proof. Let $u_0 \in D$ for D the neighborhood provided by Lemma $B.5$. We decompose the eigenfunction $\phi_j^{u_0} = a_j \phi_j + \eta_j$, where a_j is an orthonormalizing constant and η_j is orthogonal to ϕ_j. Hence

$$\lambda_k^{u_0} \phi_k^{u_0} = (A + u_0 B)(a_k \phi_k + \eta_k)$$

and

$$\lambda_k^{u_0} a_k \phi_k + \lambda_k^{u_0} \eta_k = A a_k \phi_k + A\eta_k + u_0 B a_k \phi_k + u_0 B \eta_k.$$

By projecting onto the orthogonal space of ϕ_k,

$$\begin{aligned}\lambda_k^{u_0} \eta_k &= A\eta_k + u_0 P_{\phi_k}^{\perp} B a_k \phi_k + u_0 P_{\phi_k}^{\perp} B \eta_k \\ (A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})\eta_k &= -u_0 P_{\phi_k}^{\perp} B a_k \phi_k.\end{aligned}$$

However, Lemma $B.5$ ensures that $A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and then

$$\eta_k = -a_k((A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1} u_0 P_{\phi_k}^{\perp} B \phi_k. \tag{B.3}$$

Afterwards,

$$\begin{aligned}\lambda_j^{u_0} &= \langle a_j \phi_j + \eta_j, (A + u_0 B)(a_j \phi_j + \eta_j)\rangle \\ &= |a_j|^2 \lambda_j + u_0 \langle a_j \phi_j, B a_j \phi_j\rangle + \langle a_j \phi_j, (A + u_0 B)\eta_j\rangle \\ &\quad + \langle \eta_j, (A + u_0 B) a_j \phi_j\rangle + \langle \eta_j, (A + u_0 B)\eta_j\rangle.\end{aligned}$$

By using the relation $(B.3)$,

$$\begin{aligned}\langle\eta_j,(A+u_0B)\eta_j\rangle=&\langle\eta_j,(A+u_0P^{\perp}_{\phi_k}B-\lambda_j^{u_0})\eta_j\rangle+\lambda_j^{u_0}\|\eta_j\|^2\\ =&\lambda_j^{u_0}\|\eta_j\|^2+\Big\langle\eta_j,-a_j(A+u_0P^{\perp}_{\phi_j}B-\lambda_j^{u_0})\\ &\cdot((A+u_0P^{\perp}_{\phi_j}B-\lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1}u_0P^{\perp}_{\phi_j}B\phi_j\Big\rangle.\end{aligned}$$

However, $(A+u_0P^{\perp}_{\phi_j}B-\lambda_j^{u_0})((A+u_0P^{\perp}_{\phi_j})B-\lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1}=Id$ and then

$$\langle\eta_j,(A+u_0B)\eta_j\rangle=\lambda_j^{u_0}\|\eta_j\|^2-u_0a_j\langle\eta_j,P^{\perp}_{\phi_j}B\phi_j\rangle.$$

Moreover, we have

$$\langle\phi_j,(A+u_0B)\eta_j\rangle=u_0\langle\phi_j,B\eta_j\rangle=u_0\langle P^{\perp}_{\phi_j}B\phi_j,\eta_j\rangle$$

and equivalently $\langle\eta_j,(A+u_0B)\phi_j\rangle=u_0\langle\eta_j,P^{\perp}_{\phi_j}B\phi_j\rangle$. Thus

$$\lambda_j^{u_0}=|a_j|^2\lambda_j+u_0|a_j|^2B_{j,j}+\lambda_j^{u_0}\|\eta_j\|^2+u_0\overline{a_j}\langle P^{\perp}_{\phi_j}B\phi_j,\eta_j\rangle. \tag{B.4}$$

One can notice that $|a_j|\in[0,1]$ and $\|\eta_j\|$ are uniformly bounded in j. We show that the first accumulates at 1 and the second at 0. Indeed, from $(B.1)$ and $(B.3)$, we know that there exists a constant $C_1>0$ such that

$$\begin{aligned}\|\eta_j\|^2\leq&|u_0|^2\,|||\,((A+u_0P^{\perp}_{\phi_j}B-\lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1}\,|||\,^2|a_j|^2\|B\phi_j\|^2\\ \leq&\frac{|u_0|^2\|B\phi_j\|^2}{r^2j^2}\leq\frac{C_1}{j^2}\end{aligned} \tag{B.5}$$

for $r\in(0,1)$, which implies that $\lim_{j\to\infty}\|\eta_j\|=0$. Afterwards, by contradiction, if $|a_j|$ does not converge to 1, then there exists $\{a_{j_k}\}_{k\in\mathbb{N}}$ a subsequence of $\{a_j\}_{j\in\mathbb{N}}$ such that $|a_{j_\infty}|:=\lim_{k\to\infty}|a_{j_k}|\in[0,1)$. Now, we have

$$1=\lim_{k\to\infty}\|\phi_{j_k}^{u_0}\|\leq\lim_{k\to\infty}|a_{j_k}|\|\phi_{j_k}\|+\|\eta_{j_k}\|=\lim_{k\to\infty}|a_{j_k}|+\|\eta_{j_k}\|=|a_{j_\infty}|<1$$

that is absurd. Then, $\lim_{j\to\infty}|a_j|=1$. From $(B.4)$, it follows $\lambda_j^{u_0}\asymp\lambda_j$ for $|u_0|$ small enough. The relation also implies that $\lambda_j^{u_0}\neq 0$ for every $j\in\mathbb{N}$ and $|u_0|$ small enough. □

Lemma B.7. *Let B be a bounded symmetric operator satisfying Assumptions II. For every $N\in\mathbb{N}$, there exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that there exists $\widetilde{C}_N>0$ such that, for any $u_0\in U(0)$, $k\in\mathbb{N}$ and $j\leq N$,*

$$|\langle\phi_k^{u_0},B\phi_j^{u_0}\rangle|\geq\frac{\widetilde{C}_N}{k^3}. \tag{B.6}$$

Proof. We prove $(B.6)$ for fixed $j \leq N$, then the generalization follows by using the minimum of all the constants defined for every $j \leq N$.

We start by choosing $k \in \mathbb{N}$ such that $k \neq j$ and $u_0 \in D$ for D the neighborhood provided by Lemma $B.6$. Thanks to Assumptions II, we have

$$\begin{aligned} &|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle| = |\langle a_k\phi_k + \eta_k, B(a_j\phi_j + \eta_j)\rangle| \\ &\geq C_N \frac{\overline{a_k} a_j}{k^3} - \big|\overline{a_k}\langle \phi_k, B\eta_j\rangle + a_j\langle \eta_k, B\phi_j\rangle + \langle \eta_k, B\eta_j\rangle\big|. \end{aligned} \tag{B.7}$$

1) Expansion of $\langle \eta_k, B\phi_j\rangle$, $\langle \phi_k, B\eta_j\rangle$, $\langle \eta_k, B\eta_j\rangle$:
Thanks to $(B.3)$, for every $k \in \mathbb{N}$ and $j \leq N$,

$$\begin{aligned} &\langle \eta_k, B\phi_j\rangle = \langle \eta_k, P_{\phi_k}^{\perp} B\phi_j\rangle = \\ &\langle -a_k((A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1} u_0 P_{\phi_k}^{\perp} B\phi_k, P_{\phi_k}^{\perp} B\phi_j\rangle. \end{aligned} \tag{B.8}$$

For $|u_0|$ small enough,

$$\begin{aligned} &\big((A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})\big|_{\phi_k^{\perp}}\big)^{-1} = \\ &((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} \sum_{n=0}^{\infty} \big(u_0((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp} B P_{\phi_k}^{\perp}\big)^n \end{aligned}$$

and by defining

$$M_k := \sum_{n=0}^{\infty} \big(u_0((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp} B\big)^n P_{\phi_k}^{\perp},$$

the relation $(B.8)$ becomes

$$\langle \eta_k, B\phi_j\rangle = -u_0\langle a_k M_k B\phi_k, ((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp} B\phi_j\rangle. \tag{B.9}$$

Thanks to $B : D(A) \to D(A)$, for every $k \in \mathbb{N}$ and $j \leq N$,

$$\begin{aligned} &((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp} B\phi_j = P_{\phi_k}^{\perp} B((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1}\phi_j \\ &- \big[P_{\phi_k}^{\perp} B, ((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp}\big]\phi_j = P_{\phi_k}^{\perp} B((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1}\phi_j \\ &- ((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp}[B, A]((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1}\phi_j \end{aligned}$$

and by calling

$$\widetilde{B}_k := ((A - \lambda_k^{u_0}) P_{\phi_k}^{\perp})^{-1} P_{\phi_k}^{\perp}[B, A],$$

we have

$$\text{(B.10)}\qquad \begin{aligned}((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}B\phi_j &= P_{\phi_k}^{\perp}(B+\widetilde{B}_k)((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}\phi_j\\ &= P_{\phi_k}^{\perp}(B+\widetilde{B}_k)(\lambda_j-\lambda_k^{u_0})^{-1}\phi_j.\end{aligned}$$

Let us consider $(B.9)$. From $(B.10)$, for every $k\in\mathbb{N}$ and $j\leq N$, we have

$$\text{(B.11)}\qquad \langle\eta_k, B\phi_j\rangle = -\frac{u_0}{\lambda_j-\lambda_k^{u_0}}\langle a_kM_kB\phi_k,(B+\widetilde{B}_k)\phi_j\rangle.$$

Now, one can use the same techniques. For every $k\in\mathbb{N}$ and $j\leq N$, we obtain

$$\text{(B.12)}\qquad \begin{aligned}&|\langle\eta_k, B\eta_j\rangle| = |\langle B\eta_k,\eta_j\rangle| = |\langle u_0a_kB((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}M_kB\phi_k,\\ &u_0a_j((A-\lambda_j^{u_0})P_{\phi_j}^{\perp})^{-1}M_jB\phi_j\rangle\Big| = \Big|\frac{a_j\overline{a_k}u_0^2}{\lambda_k-\lambda_j^{u_0}}\big\langle\phi_k, L_{k,j}\phi_j\big\rangle\Big|\end{aligned}$$

with

$$L_{k,j} := (A-\lambda_j^{u_0})BM_k((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}B((A-\lambda_j^{u_0})P_{\phi_j}^{\perp})^{-1}M_jB.$$

Now, there exists $\epsilon>0$ such that $|a_l|\in(\epsilon,1)$ for every $l\in\mathbb{N}$. Thanks to $(B.11)$, $(B.12)$ and $(B.7)$, there exists $\widehat{C}_N$ such that

$$\text{(B.13)}\qquad \begin{aligned}|\langle\phi_k^{u_0}, B\phi_j^{u_0}\rangle| &\geq \widehat{C}_N\frac{1}{k^3} - \Big|\frac{u_0}{\lambda_j-\lambda_k^{u_0}}\langle M_kB\phi_k,(B+\widetilde{B}_k)\phi_j\rangle\Big|\\ &- \Big|\frac{u_0}{\lambda_k-\lambda_j^{u_0}}\langle(B+\widetilde{B}_j)\phi_k, M_jB\phi_j\rangle\Big| - \Big|\frac{u_0^2}{\lambda_k-\lambda_j^{u_0}}\big\langle\phi_k, L_{k,j}\phi_j\big\rangle\Big|.\end{aligned}$$

2) Features of the operators M_k, $\widetilde{B}_k$, $L_{k,j}$:
Let $k\in\mathbb{N}$. First, the operators M_k are uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$ when u_0 is small enough such that

$$|||\, u_0((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}BP_{\phi_k}^{\perp}\,|||_{L(H^2_{(0)})} < 1.$$

Second, the relation $(B.10)$ implies that

$$\widetilde{B}_kP_{\phi_k}^{\perp} = ((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}B(A-\lambda_k^{u_0})P_{\phi_k}^{\perp} - P_{\phi_k}^{\perp}BP_{\phi_k}^{\perp}.$$

Hence, the operators $\widetilde{B}_k$ are uniformly bounded in k in

$$L\big(H^2_{(0)}\cap Ran(P_{\phi_k}^{\perp}), H^2_{(0)}\cap Ran(P_{\phi_k}^{\perp})\big).$$

Third, one can notice that

$$B((A-\lambda_j^{u_0})P_{\phi_j}^{\perp})^{-1}M_jB \in L(H^2_{(0)}, H^2_{(0)}), \qquad \forall j \in \mathbb{N}.$$

Then, for every $k \in \mathbb{N}$ and $j \leq N$

$$\begin{aligned}
&(A-\lambda_j^{u_0})BM_k((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}\\
&=(A-\lambda_j^{u_0})B((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}\sum_{n=0}^{\infty}\big(u_0P_{\phi_k}^{\perp}B((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}\big)^nP_{\phi_k}^{\perp}\\
&=(A-\lambda_j^{u_0})((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}(\widetilde{B}_k+B)\widetilde{M}_k
\end{aligned}$$

for

$$\widetilde{M}_k := \sum_{n=0}^{\infty}\big(u_0P_{\phi_k}^{\perp}B((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}\big)^nP_{\phi_k}^{\perp}.$$

Now, the operators $\widetilde{M}_k$ are uniformly bounded in $L(H^2_{(0)}, H^2_{(0)})$ as M_k. Hence $L_{k,j}$ are uniformly bounded in $L(H^2_{(0)}, H^2_{(0)})$.

Let $\{F_l\}_{l\in\mathbb{N}}$ be an infinite uniformly bounded family of operators in $L(H^2_{(0)}, H^2_{(0)})$. We know that, for every $l, j \in \mathbb{N}$, there exists $c_{l,j} > 0$ such that

$$\sum_{k=1}^{\infty}|k^2\langle\phi_k, F_l\phi_j\rangle|^2 < \infty, \implies |\langle\phi_k, F_l\phi_j\rangle| \leq \frac{c_{l,j}}{k^2}, \qquad \forall k \in \mathbb{N}.$$

Now, the constant $c_{l,j}$ can be assumed uniformly bounded in l since, for every $k, j \in \mathbb{N}$,

$$\sup_{l\in\mathbb{N}}|k^2\langle\phi_k, F_l\phi_j\rangle|^2 \leq \sup_{l\in\mathbb{N}}\sum_{m\in\mathbb{N}}|m^2\langle\phi_m, F_l\phi_j\rangle|^2 \leq \sup_{l\in\mathbb{N}}\|F_l\phi_j\|^2_{(2)} < \infty.$$

Thus, for every infinite uniformly bounded family of operators $\{F_l\}_{l\in\mathbb{N}}$ in $L(H^2_{(0)}, H^2_{(0)})$ and for every $j \in \mathbb{N}$, there exists a constant c_j such that

$$|\langle\phi_k, F_l\phi_j\rangle| \leq \frac{c_j}{k^2}, \qquad \forall k, l \in \mathbb{N}. \tag{B.14}$$

3) Conclusion:
We know that $|\lambda_j - \lambda_k^{u_0}|^{-1}$ and $|\lambda_k - \lambda_j^{u_0}|^{-1}$ asymptotically behave as k^{-2} thanks to Lemma $B.6$. From the previous point, the families of operators $\{BM_k(B+\widetilde{B}_k)\}_{k\in\mathbb{N}}$, $\{L_{k,j}\}_{k\in\mathbb{N}}$ are uniformly bounded in $L(H^2_{(0)}, H^2_{(0)})$ and $BM_j(B+\widetilde{B}_j) \in L(H^2_{(0)}, H^2_{(0)})$ for every $1 \leq j \leq N$. Hence, we use the relation $(B.14)$ in $(B.13)$ and there exist $C_1, C_2, C_3, C_4 > 0$ depending on $j \in \mathbb{N}$ such that, for $|u_0|$ small enough and $k \in \mathbb{N}$ large enough,

$$\begin{aligned}|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle| = |\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle| &\geq \widehat{C}_N \frac{1}{k^3} - \frac{C_1|u_0|}{|\lambda_j - \lambda_k^{u_0}|k^2} \\ &- \frac{C_2|u_0|}{|\lambda_k - \lambda_j^{u_0}|k^2} - \frac{C_3|u_0|^2}{|\lambda_k - \lambda_j^{u_0}|k^2} \geq C_4 \frac{1}{k^3}.\end{aligned} \tag{B.15}$$

Let $K \in \mathbb{N}$ be so that

$$|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle| \geq C_4 \frac{1}{k^3}, \qquad \forall k > K.$$

For $j \in \mathbb{N}$ fixed, the zeros of the analytic map $u_0 \mapsto \{|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle|\}_{k\leq K} \in \mathbb{R}^K$ are discrete. Then, for $|u_0|$ small enough,

$$|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle| \neq 0, \qquad \forall k \leq K.$$

Thus, for every $j \in \mathbb{N}$ and $|u_0|$ small enough, there exists $C_j > 0$ such that

$$|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle| \geq \frac{C_j}{k^3}, \qquad \forall k \in \mathbb{N}.$$

In conclusion, the identity $(B.6)$ is valid for every $k \in \mathbb{N}$ and $j \leq N$ by considering $\widetilde{C}_N = \min\{C_j : \ j \leq N\}$. □

Lemma B.8. *Let B be a bounded symmetric operator satisfying Assumptions II. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ contained in the one introduced in Lemma $B.6$ such that, for any $u_0 \in U(0)$,*

$$\Big(\sum_{j=1}^{\infty} \big||\lambda_j^{u_0}|^{\frac{3}{2}} \langle \phi_j^{u_0}, \cdot\rangle\big|^2\Big)^{\frac{1}{2}} \asymp \Big(\sum_{j=1}^{\infty} |j^3 \langle \phi_j, \cdot\rangle|^2\Big)^{\frac{1}{2}}.$$

Proof. Let D be the neighborhood provided by Lemma $B.6$. For $|u_0|$ small enough, we prove that there exist $C_1 > 0$ such that

$$\||A + u_0 B|^{\frac{s}{2}}\psi\| \leq C_1 \||A|^{\frac{s}{2}}\psi\|$$

for $s=3$. We start by assuming $s=4$ and we recall that $B \in L(H^2_{(0)})$ thanks to Remark 2.1. For any $\psi \in H^4_{(0)}$, there exist $C_2, C_3 > 0$ such that

$$\begin{aligned}\||A+u_0B|^2\psi\| = \|(A+u_0B)^2\psi\| &\leq \|A^2\psi\| + |u_0|^2\|B^2\psi\| \\ &+ |u_0|\|AB\psi\| + |u_0|\|BA\psi\| \leq \|A^2\psi\| \\ &+ |u_0|^2\|B^2\psi\| + |u_0| \, \|\!| B |\!\|_{L(H^2_{(0)})}\|A\psi\| \\ &+ |u_0| \, \|\!| B |\!\| \, \|A\psi\| \leq C_2\|A^2\psi\| + C_3\|\psi\| \\ &\leq (C_2+C_3)\||A|^2\psi\|.\end{aligned}$$

Thus, there exists $C(2)>0$ such that, for every $\tilde{\psi} \in H^4_{(0)}$,

$$\sum_{n=1}^{+\infty} |\lambda_n^{u_0}|^4 |\langle \phi_n^{u_0}, \tilde{\psi}\rangle|^2 \leq C(2) \sum_{n=1}^{+\infty} |\lambda_n|^4 |\langle \phi_n, \tilde{\psi}\rangle|^2$$

and

$$\sum_{n=1}^{+\infty} |\lambda_n^{u_0}|^4 \left| \sum_{l=1}^{+\infty} \langle \phi_n^{u_0}, \phi_l\rangle\langle\phi_l, \tilde{\psi}\rangle \right|^2 \leq C(2) \sum_{n=1}^{+\infty} |\lambda_n|^4 |\langle \phi_n, \tilde{\psi}\rangle|^2$$

The operators $|A|$ and $|A+u_0B|$ are positive and invertible for $|u_0| < \pi^2/ \|\!| B |\!\|$. For every $\tilde{\psi} \in D(A)$, we consider $\psi \in \mathscr{H}$ such that $\tilde{\psi} = |A|^{-1}\psi = \sum_{l=1}^{\infty} \lambda_n^{-1}\phi_n\langle\phi_n, \psi\rangle$ and

$$\sum_{n=1}^{+\infty} |\lambda_n^{u_0}|^4 \overline{\sum_{l=1}^{+\infty} \lambda_l^{-1} \langle \phi_n^{u_0}, \phi_l\rangle\langle\phi_l, \psi\rangle} \sum_{k=1}^{+\infty} \lambda_k^{-1}\langle \phi_n^{u_0}, \phi_k\rangle\langle\phi_k, \psi\rangle \leq C(2)\|\psi\|^2.$$

Let $\psi \in \mathscr{H}$ and

$$f_\psi : z = s+iy \mapsto \sum_{n=1}^{+\infty} (\lambda_n^{u_0})^{2(s+iy)} \langle |A|^{-s+iy}\psi, \phi_n^{u_0}\rangle\langle\phi_n^{u_0}, |A|^{-s-iy}\psi\rangle$$

where, for every $z \in \mathbb{C}$, $|A|^z\psi = \sum_{j=1}^{+\infty} \lambda_j^z \phi_j\langle\phi_j, \psi\rangle$. Then, by [BBR10] for $s=0$ and $s=2$, there exists $C(s)>0$ such that

$$|f_\psi(s+iy)| \leq C(s)\||A|^{-s+iy}\psi\|_{(s)}\||A|^{-s-iy}\psi\|_{(s)} \leq C(s)\|\psi\|^2.$$

If ψ is finite linear combination of the vectors $\{\phi_j\}_{j\in\mathbb{N}}$, then the function f_ψ is analytic on the strip $\{z \in \mathbb{C} : \ 0 < \Re(s) < z\}$ and continuous on its closure as uniform limits of a partial sum in n. Since it is bounded on the boundary, by Hadamar Three-Lines Theorem [RS80, *Appendix IX*.4], it is bounded on the strip and $\log(\sup_{\Re(z)=s}|f_\psi(z)|)$ is a convex function of $s\in[0,2]$. For $s\in(0,2)$, we obtain

$$\sum_{n=1}^{+\infty}|\lambda_n^{u_0}|^{2s}\langle\phi_n^{u_0},\psi\rangle|^2\leq C(2)^{\frac{s}{2}}\sum_{n=1}^{+\infty}|\lambda_n|^{2s}|\langle\phi_n,\psi\rangle|^{2s}.$$

Then, there exists $C>0$ such that, for every $\psi\in H^3_{(0)}$,

$$\|\psi\|_{\widetilde{H}^3_{(0)}}=\||A+u_0B|^{\frac{3}{2}}\psi\|\leq C\||A|^{\frac{3}{2}}\psi\|.$$

Now, $H^2_{(0)}=D(|A|)=D(|A+u_0B|)=\widetilde{H}^2_{(0)}$ and B preserves $\widetilde{H}^2_{(0)}$ since $B:H^2_{(0)}\longrightarrow H^2_{(0)}$. The arguments of Remark 2.1 imply that $B\in L(\widetilde{H}^2_{(0)})$ and the opposite inequality follows as above thanks to the identity $A=(A+u_0B)-u_0B$. □

Remark B.9. *Let B be a bounded symmetric operator satisfying Assumptions II. The techniques of the proof of Lemma B.8 also allow to prove that, for $s\in(0,3)$, there exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0\in U(0)$,*

$$\|\cdot\|_{\widetilde{H}^s_{(0)}}=\Big(\sum_{j=1}^{\infty}\big|(\lambda_j^{u_0})^{\frac{s}{2}}\langle\phi_j^{u_0},\cdot\rangle\big|^2\Big)^{\frac{1}{2}}\asymp\Big(\sum_{j=1}^{\infty}|j^s\langle\phi_j,\cdot\rangle|^2\Big)^{\frac{1}{2}}=\|\cdot\|_{(s)}.$$

Lemma B.10. *Let B be a bounded symmetric operator satisfying Assumptions II and $N\in\mathbb{N}$. Let $\epsilon>0$ small enough and I^N be the set defined in (3.2). There exists $U_\epsilon\subset\mathbb{R}\setminus\{0\}$ of positive measure such that, for each $u_0\in U_\epsilon$,*

$$\inf_{\substack{(j,k),(n,m)\in I^N\\(j,k)\neq(n,m)}}|\lambda_j^{u_0}-\lambda_k^{u_0}-\lambda_n^{u_0}+\lambda_m^{u_0}|>\epsilon.$$

Moreover, for every $\delta>0$ small there exists $\epsilon>0$ such that $dist(U_\epsilon,0)<\delta$.

Proof. Let us consider the neighborhood D provided by Lemma $B.5$. The maps $\lambda_j^u-\lambda_k^u-\lambda_n^u+\lambda_m^u$ are analytic for each $j,k,n,m\in\mathbb{N}$ and $u\in D$. One can notice that the number of elements such that

$$\lambda_j-\lambda_k-\lambda_n+\lambda_m=0,\qquad j,n\in\mathbb{N},\quad k,m\leq N \tag{B.16}$$

is finite. Indeed $\lambda_k=k^2\pi^2$ and $(B.16)$ corresponds to $j^2-k^2=n^2-m^2$. We have

$$|j^2-n^2|=|k^2-m^2|\leq N^2-1,$$

which is satisfied for a finite number of elements. Thus, for I^N (defined in (3.2), the following set is finite

$$R := \{((j,k),(n,m)) \in (I^N)^2 \ : \ (j,k) \neq (n,m);\ \lambda_j - \lambda_k - \lambda_n + \lambda_m = 0\}.$$

1) Let $((j,k),(n,m)) \in R$, the set

$$V_{(j,k,n,m)} = \{u \in D \big|\ \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$$

is a discrete subset of D or equal to D. Thanks to the relation $(B.4)$,

$$\begin{aligned}
&\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = \\
&|a_j|^2\lambda_j + u|a_j|^2 B_{j,j} + \lambda_j^u\|\eta_j\|^2 + u\overline{a_j}\langle P_{\phi_j}^{\perp} B\phi_j, \eta_j\rangle \\
&- |a_k|^2\lambda_k - u|a_k|^2 B_{k,k} - \lambda_k^u\|\eta_k\|^2 - u\overline{a_k}\langle P_{\phi_k}^{\perp} B\phi_k, \eta_k\rangle \\
&- |a_n|^2\lambda_n - u|a_n|^2 B_{n,n} - \lambda_n^u\|\eta_n\|^2 - u\overline{a_n}\langle P_{\phi_n}^{\perp} B\phi_n, \eta_n\rangle \\
&+ |a_m|^2\lambda_m + u|a_m|^2 B_{m,m} + \lambda_m^u\|\eta_m\|^2 + u\overline{a_m}\langle P_{\phi_m}^{\perp} B\phi_m, \eta_m\rangle
\end{aligned}$$

(B.17)

$$\begin{aligned}
\Longrightarrow \qquad \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u &= |a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m \\
&+ (|a_j|^2 B_{j,j} - |a_k|^2 B_{k,k} - |a_n|^2 B_{n,n} + |a_m|^2 B_{m,m})u + o(u).
\end{aligned}$$

For $|u|$ small enough, thanks to $lim_{|u|\to 0}|a_j|^2 = 1$ and to the third point of Assumptions I, $\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$ can not be constantly equal to 0. Then, $V_{(j,k,n,m)}$ is discrete and

$$V = \{u \in D \big|\ \exists (j,k,n,m) \in R : \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$$

is a discrete subset of D. As R is a finite set

$$\widetilde{U}_\epsilon := \{u \in D : \forall (j,k,n,m) \in R \big|\ |\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \epsilon\}$$

has positive measure for $\epsilon > 0$ small enough. Moreover, for any $\delta > 0$ small, there exists $\epsilon_0 > 0$ such that $dist(0, \widetilde{U}_{\epsilon_0}) < \delta$.

2) Let $((j,k),(n,m)) \in (I^N)^2 \setminus R$ be different numbers. We know that

$$|\lambda_j^0 - \lambda_k^0 - \lambda_n^0 + \lambda_m^0| = \pi^2|j^2 - k^2 - n^2 + m^2| > \pi^2.$$

First, thanks to $(B.4)$,

$$\lambda_j^u \leq |a_j|^2\lambda_j + |u|C_1, \qquad \lambda_j^u \geq |a_j|^2\lambda_j - |u|C_2$$

for suitable constants $C_1, C_2 > 0$ non depending on the index j. Thus

$$|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq ||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \\ - |u|(2C_1 + 2C_2).$$

Now, thanks to the relation (2.21), $\lim_{k\to\infty} |a_k|^2 = 1$. For any u in D and ϵ small enough, there exists $M_\epsilon \in \mathbb{N}$ such that, for $R^C := (I^N)^2 \setminus R$,

$$||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \geq \pi^2 - \epsilon, \\ \forall((j,k),(n,m)) \in R^C, \quad j,k,n,m \geq M_\epsilon.$$

However $\lim_{|u|\to 0} |a_k|^2 = 1$ uniformly in k thanks to $(B.5)$ and then there exists a neighborhood $W_\epsilon \subseteq D$ such that, for each $u \in W_\epsilon$,

$$||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \geq \pi^2 - \epsilon, \\ \forall((j,k),(n,m)) \in R^C, \quad 1 \leq j,k,n,m < M_\epsilon.$$

Thus, for each $u \in W_\epsilon$ and $((j,k),(n,m)) \in R^C$ such that $(j,k) \neq (n,m)$,

$$|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \pi^2 - \epsilon.$$

3) The proof is achieved since, for $\epsilon_1 > 0$ small enough, $\widetilde{U}_{\epsilon_1} \cap W_\epsilon$ is a non-zero measure subset of D. For any $u \in \widetilde{U}_{\epsilon_1} \cap W_\epsilon$ and for any $((j,k),(n,m)) \in (I^N)^2$ such that $(j,k) \neq (n,m)$, we have

$$|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \min\{\pi^2 - \epsilon, \epsilon_1\}. \qquad \square$$

Remark B.11. *Let B be a bounded symmetric operator satisfying Assumptions II. By using the techniques of the proofs of Lemma $B.7$ and Lemma $B.10$, one can ensure the existence of a neighborhood U_1 of u_0 in $\mathbb{R}$ and U_2, a countable subset of $\mathbb{R}$ such that, for any $u_0 \in U(0) := (U_1 \setminus U_2) \setminus \{0\}$, we have:*

1. *For every $N \in \mathbb{N}$, $(j,k),(n,m) \in I^N$ (see (3.2)) such that $(j,k) \neq (n,m)$, there holds*
$$\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0} \neq 0.$$
2. *$B_{j,k}^{u_0} = \langle \psi_j^{u_0}(T), B\phi_k^{u_0}(T)\rangle \neq 0$ for every $j,k \in \mathbb{N}$.*
3. *For $\epsilon > 0$, if $|u_0|$ is small enough, then*
$$\sup_{j\in\mathbb{N}} \|\phi_j - \phi_j^{u_0}\|_{(3)} \leq \epsilon.$$

Let $\mathbf{0^n} := \{a_j\}_{j\leq n} \in \mathbb{Q}^n$ be such that $a_j = 0$ for every $j \leq n$ and $n \in \mathbb{N}$.

Remark B.12. *Let B be a bounded symmetric operator satisfying Assumptions II and Assumptions A. As Remark B.11, there exists a neighborhood U_1 of u_0 in $\mathbb{R}$ and U_2, a countable subset of $\mathbb{R}$ containing $u = 0$ such that, for any $u_0 \in U(0) := (U_1 \setminus U_2) \setminus \{0\}$, the numbers*

$$\{1\} \cup \big\{\lambda_j^{u_0}\big\}_{j\in\mathbb{N}}$$

are rationally independent, i.e. for any $n \in \mathbb{N}$ and $\{r_j\}_{j\leq n} \in \mathbb{Q}^n \setminus \mathbf{0^n}$, it holds

$$r_1 + \sum_{j=2}^{n} r_j \lambda_j^{u_0} \neq 0.$$

Indeed, we notice that $(1 - \|\eta_j\|^2) = |\alpha_j|^2$ for every $j \in \mathbb{N}$ and we denote

$$x_{j,M}^{u_0} := B\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1}\Big(\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1} P_{\phi_j}^\perp B\Big)^M P_{\phi_j}^\perp B, \qquad \forall j, M \in \mathbb{N}.$$

By using (B.3) in the relation (B.4), for $|u_0|$ small enough, we obtain

(B.18)

$$\begin{aligned}
\lambda_j^{u_0} &= \frac{|\alpha_j|^2}{1 - \|\eta_j\|^2}\lambda_j + u_0 \frac{|\alpha_j|^2}{1 - \|\eta_j\|^2} B_{j,j} \\
&\quad - u_0 \frac{|\alpha_j|^2}{1 - \|\eta_j\|^2} \langle P_{\phi_j}^\perp B\phi_j, ((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})\big|_{\phi_j^\perp})^{-1} u_0 P_{\phi_j}^\perp B\phi_j\rangle \\
&= \lambda_j + u_0 B_{j,j} - u_0 \langle P_{\phi_j}^\perp B\phi_j, ((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})\big|_{\phi_j^\perp})^{-1} u_0 P_{\phi_j}^\perp B\phi_j\rangle \\
&= \lambda_j + u_0 B_{j,j} \\
&\quad + u_0^2 \Big\langle \phi_j, B\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1}\Big(\big(I - u_0\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1} P_{\phi_j}^\perp B\big)\big|_{\phi_j^\perp}\Big)^{-1} P_{\phi_j}^\perp B\phi_j\Big\rangle \\
&= \lambda_j + u_0 B_{j,j} \\
&\quad + u_0^2 \Big\langle \phi_j, B\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1} \sum_{M=0}^{+\infty} \Big(u_0\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1} P_{\phi_j}^\perp B\Big)^M P_{\phi_j}^\perp B\phi_j\Big\rangle \\
&= \lambda_j + u_0 B_{j,j} + u_0^2 \Big\langle \phi_j, \sum_{M=0}^{+\infty} \big(u_0^M x_{j,M}^{u_0}\big)\, \phi_j\Big\rangle.
\end{aligned}$$

Now, for every $j, M \in \mathbb{N}$, we have

$$\lim_{|u_0|\to 0} x_{j,M}^{u_0} = x_{j,M} := \Big\langle \phi_j, B\big((\lambda_j - A)\big|_{\phi_j^\perp}\big)^{-1}\Big(\big((\lambda_j - A)\big|_{\phi_j^\perp}\big)^{-1} P_{\phi_j}^\perp B\Big)^M P_{\phi_j}^\perp B\phi_j\Big\rangle.$$

We underline that, for every $j, M \in \mathbb{N}$,

$$x_{j,M} = \langle \phi_j, \widetilde{B}(M,j)\phi_j \rangle$$

with $\widetilde{B}(M,j)$ *introduced in Assumptions A. Let* $n \in \mathbb{N}$ *and* $\mathbf{r} := \{r_j\}_{j\leq n} \in \mathbb{Q}^n \setminus \mathbf{0^n}$. *Thanks to Assumptions A, the map* $u \mapsto r_1 + \sum_{j=2}^n r_j \lambda_j^u$ *is non-constant and analytic. The set* $V_{\mathbf{r}}$ *of its positive zeros is discrete. The property is valid for* $U_2 := \cup_{n\in\mathbb{N}} \cup_{\mathbf{r}\in\mathbb{Q}^n} V_{\mathbf{r}}$ *that is countable.*

B.2 Bilinear Schrödinger equation on compact graphs

The aim of this paragraph is to adapt the perturbation theory techniques provided in Appendix $B.1$ where we consider the bilinear Schrödinger equation (4.1) in $L^2(\mathscr{G},\mathbb{C})$ for $\mathscr{G} = (0,1)$ and A the Dirichlet Laplacian. In the mentioned framework, we have

$$\inf_{k,l\in\mathbb{N}} |\lambda_k - \lambda_l| > 0,$$

which is not guaranteed if $\mathscr{G}$ is a generic compact graph. Even though we know that there exist $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \inf_{k\in\mathbb{N}} \sqrt{\lambda_2} |\sqrt{\lambda_{k+\mathcal{M}}} - \sqrt{\lambda_k}| \geq \delta\mathcal{M}$$

thanks to [DZ06, $Proposition$ 6.2; 3)]. First, we modify (4.1) as in Appendix $B.1$. Let $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ be the spectrum of $A + u_0 B$ corresponding to some eigenfunctions $\{\phi_j^{u_0}\}_{j\in\mathbb{N}}$. We refer to the definition of the equivalence classes E_m with $m \in \mathbb{Z}^*$ provided in the first part of Appendix $A.2$. We denote

- $n : \mathbb{N} \to \mathbb{N}$ maps $j \in \mathbb{N}$ in the value $n(j)$ such that $j \in E_{n(j)}$;
- $s : \mathbb{N} \to \mathbb{N}$ is such that $\lambda_{s(j)} = \inf\{\lambda_k > \lambda_j \mid k \notin E_{n(j)}\}$;
- $m : \mathbb{N} \to \mathbb{N}$ is such that $\lambda_{m(j)} = \sup\{\lambda_k < \lambda_j \mid k \notin E_{n(j)}\}$;
- $p : \mathbb{N} \to \mathbb{N}$ is such that $\lambda_{p(j)} = \sup\{k \in E_{n(j)}\}$.

The proofs of Lemma $B.4$ and Lemma $B.5$, imply the following lemma.

Lemma B.13. *Let the hypotheses of Theorem* 4.3 *be satisfied. Let* $j \in \mathbb{N}$ *and* $P_j^\perp$ *be the projector onto*

$$\overline{span\{\phi_m \ : \ m \notin E_{n(j)}\}}^{\,L^2}.$$

There exists a neighborhood $U(0)$ *small enough of* $u = 0$ *in* $\mathbb{R}$ *such that*

1. *there exists* $c > 0$ *such that, for every* $u_0 \in U(0)$ *and* $k \in \mathbb{N}$
$$|||\,(A + u_0 B - \nu_k)^{-1}\,||| \leq c,$$
with
$$\nu_k := \frac{\lambda_{s(k)} - \lambda_{p(k)}}{2};$$
2. *for every* $u_0 \in U(0)$*, the operator* $(A + u_0 P_k^{\perp} B - \lambda_k^{u_0})$ *is invertible with bounded inverse from* $D(A) \cap Ran(P_k^{\perp})$ *to* $Ran(P_k^{\perp})$ *for every* $k \in \mathbb{N}$*.*

Lemma B.14. *Let the hypotheses of Theorem* 4.3 *be satisfied. For each neighborhood small enough* $U(0)$ *of* $u = 0$ *in* $\mathbb{R}$ *up to a countable subset* Q *we have*
$$\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} \neq 0, \qquad \langle \phi_k^{u_0}, B\phi_j^{u_0} \rangle \neq 0, \qquad u_0 \in U(0) \setminus Q$$
for every $(k, j), (m, n) \in I,\ (k, j) \neq (m, n)$ *(see* (4.4)*).*

Proof. For $k \in \mathbb{N}$, we decompose the perturbed eigenfunction as follows
$$\phi_k^{u_0} = a_k \phi_k + \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k \phi_j + \eta_k, \tag{B.19}$$
where $a_k \in \mathbb{C}$, $\{\beta_j^k\} \subset \mathbb{C}$ and η_k is orthogonal to ϕ_l for every $l \in E_{n(k)}$. Moreover, $\lim_{|u_0| \to 0} |a_k| = 1$ and $\lim_{|u_0| \to 0} |\beta_j^k| = 0$ for every $j, k \in \mathbb{N}$. By following the techniques of the proof of Lemma $B.6$,
$$\lambda_k^{u_0} \phi_k^{u_0} = (A + u_0 B)(a_k \phi_k + \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k \phi_j + \eta_k) = A a_k \phi_k$$
$$+ \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k A\phi_j + A\eta_k + u_0 B a_k \phi_k + u_0 \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k B\phi_j + u_0 B\eta_k.$$

Now, Lemma $B.13$ leads to the existence of $C_1 > 0$ such that, for every $k \in \mathbb{N}$,

(B.20)
$$\eta_k = -\left((A + u_0 P_k^{\perp} B - \lambda_k^{u_0}) P_k^{\perp}\right)^{-1} u_0 \Big(a_k P_k^{\perp} B\phi_k + \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k P_k^{\perp} B\phi_j\Big),$$

$$\Longrightarrow \quad \|\eta_k\| \leq C_1 |u_0|. \tag{B.21}$$

We compute $\lambda_k^{u_0} = \langle \phi_k^{u_0}, (A+u_0B)\phi_k^{u_0}\rangle$ and for $B_{k,j} := \langle \phi_k, B\phi_j\rangle$,

$$\begin{aligned}\lambda_k^{u_0} &= |a_k|^2\lambda_k + \langle \eta_k, (A+u_0B)\eta_k\rangle + \sum_{j\in E_{n(k)}\setminus\{k\}} \lambda_j|\beta_j^k|^2\\ &+ u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2 B_{k,k} + u_0 \sum_{j,l\in E_{n(k)}\setminus\{k\}\ j\neq l} \overline{\beta_j^k}\beta_l^k B_{j,l}\\ &+ 2u_0\Re\Big(\sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k\langle \eta_k, B\phi_j\rangle + \overline{a_k}\sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k B_{k,j} + \overline{a_k}\langle\phi_k, B\eta_k\rangle\Big)\\ &+ u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2(B_{j,j} - B_{k,k}) + u_0|a_k|^2 B_{k,k}.\end{aligned}$$

Thanks to $(B.20)$, it follows $\langle \eta_k, (A+u_0B)\eta_k\rangle = \lambda_k^{u_0}\|\eta_k\|^2 + O(u_0^2)$ and there exist f_k, f_k' such that $\lim_{|u_0|\to 0} f_k = 0$, $\lim_{|u_0|\to 0} f_k' = 0$ uniformly in k and

$$\begin{aligned}\lambda_k^{u_0} &= (1-\|\eta_k\|^2)^{-1}\Big(|a_k|^2\lambda_k + u_0|a_k|^2B_{k,k}\\ &+ \sum_{j\in E_{n(k)}\setminus\{k\}} (\lambda_j - \lambda_k)|\beta_j^k|^2 + u_0 f_k + \lambda_k \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2\\ &+ u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2(B_{j,j} - B_{k,k}) + u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2 B_{k,k}\Big) + O(u_0^2)\\ &= (1-\|\eta_k\|^2)^{-1}\Big(|a_k|^2 + \sum_{j\in E_{n(k)}\setminus\{k\}} \lambda_j/\lambda_k|\beta_j^k|^2\Big)\lambda_k\\ &+ u_0(1-\|\eta_k\|^2)^{-1}\Big(|a_k|^2 + \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2\Big)B_{k,k}\\ &+ u_0 f_k' + O(u_0^2).\end{aligned}$$

For $\widehat{a}_k := (1-\|\eta_k\|^2)^{-1}(|a_k|^2 + \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2)$ thanks to $(B.21)$, it follows

$$\lim_{|u_0|\to 0} |\widehat{a}_k| = 1$$

uniformly in k. From [DZ06, *Proposition* 6.2; 5)], we have

$$\lim_{n\to+\infty} \frac{\lambda_n}{n^2} = \frac{\pi^2}{\big(\sum_{l=1}^N L_l\big)^2}$$

and, thanks to $\sup_{j\in\mathbb{N}} |E_j| < N$, we obtain

$$\lim_{k\to+\infty} \inf_{j\in E_{n(k)}\setminus\{k\}} \lambda_j{\lambda_k}^{-1} = \lim_{k\to+\infty} \sup_{j\in E_{n(k)}\setminus\{k\}} \lambda_j{\lambda_k}^{-1} = 1.$$

For
$$\widetilde{a}_k := (1 - \|\eta_k\|^2)^{-1}(|a_k|^2 + \sum_{j\in E_{n(k)}\setminus\{k\}} \lambda_j/\lambda_k|\beta_j^k|^2),$$
$\lim_{|u_0|\to 0} |\widetilde{a}_k| = 1$ uniformly in k and

$$\lambda_k^{u_0} = \widetilde{a}_k\lambda_k + u_0\widehat{a}_k B_{k,k} + u_0 f_k' + O(u_0^2). \tag{B.22}$$

When $\lambda_k = 0$, the result is still valid. For each $(k,j),(m,n) \in I$ such that $(k,j) \neq (m,n)$, there exists $f_{k,j,m,n}$ such that $\lim_{|u_0|\to 0} f_{k,j,m,n} = 0$ uniformly in k,j,m,n and

$$\begin{aligned}\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} &= \widetilde{a}_k\lambda_k - \widetilde{a}_j\lambda_j - \widetilde{a}_m\lambda_m + \widetilde{a}_n\lambda_n + u_0 f_{k,j,m,n}\\ &+ u_0(\widehat{a}_k B_{k,k} - \widehat{a}_j B_{j,j} - \widehat{a}_m B_{m,m} + \widehat{a}_n B_{n,n}) = \widetilde{a}_k\lambda_k - \widetilde{a}_j\lambda_j\\ &- \widetilde{a}_m\lambda_m + \widetilde{a}_n\lambda_n + u_0(\widehat{a}_k B_{k,k} - \widehat{a}_j B_{j,j} - \widehat{a}_m B_{m,m} + \widehat{a}_n B_{n,n}) + O(u_0^2).\end{aligned}$$

Thanks to the third point of Assumptions III, there exists $U(0)$ a neighborhood of $u = 0$ in $\mathbb{R}$ small enough such that, for each $u \in U(0)$, we have that every function $\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0}$ is not constant.

Now
$$V_{(k,j,m,n)} = \{u \in D \big|\ \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$$
is a discrete subset of D and

$$V = \{u \in D \big|\ \exists((k,j),(m,n)) \in I^2 : \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$$

is a countable subset of D.

The second relation is achieved with the same technique by considering that, for every $j,k \in \mathbb{N}$, the analytic functions $u_0 \to \langle\phi_j^{u_0}, B\phi_k^{u_0}\rangle$ can not be constantly zero since $\langle\phi_j, B\phi_k\rangle \neq 0$. In fact, one can prove that

$$W = \{u \in D \big|\ \exists(k,j) \in I : \langle\phi_j^{u_0}, B\phi_k^{u_0}\rangle = 0\}$$

is a countable subset of D. □

Lemma B.15. *Let the hypotheses of Theorem* 4.3 *be satisfied. Let* $T > 0$ *and* $s = d + 2$ *for* d *introduced in Assumptions IV. Let* $c \in \mathbb{R}$ *such that* $0 \notin \sigma(A + u_0B + c)$ *(the spectrum of* $A + u_0B + c$*) and such that* $A + u_0B + c$ *is a positive operator. There exists a neighborhood* $U(0)$ *of* 0 *in* $\mathbb{R}$ *such that, for any* $u_0 \in U(0)$,

$$\Big\||A + u_0B + c|^{\frac{s}{2}}\cdot\Big\| \asymp \|\cdot\|_{(s)}.$$

Proof. Let D be the neighborhood provided by Lemma $B.14$. We define a neighborhood $U(0) \subseteq D$ such that the claim is achieved. The proof follows the one of Lemma $B.8$. We suppose that $0 \notin \sigma(A+u_0B)$ and $A+u_0B$ is positive such that we can assume $c=0$. If $c \neq 0$, then the proof follows from the same arguments. We prove the existence of a neighborhood $U(0) \subset D$ such that, for any $u_0 \in U(0)$,

$$\Big(\sum_{j=1}^{\infty} \big||\lambda_j^{u_0}|^{\frac{s}{2}} \langle \phi_j^{u_0}, \cdot\rangle\big|^2\Big)^{\frac{1}{2}} \asymp \Big(\sum_{j=1}^{\infty} |j^s \langle \phi_j, \cdot\rangle|^2\Big)^{\frac{1}{2}}.$$

Thanks to Remark 4.1, we have $\|\cdot\|_{(s)} \asymp \||A|^{\frac{s}{2}} \cdot\|$. We prove the existence of $C_1, C_2, C_3 > 0$ such that, for every $\psi \in D(|A+u_0B|^{\frac{s}{2}}) = D(|A|^{\frac{s}{2}})$,

$$\begin{aligned} \||A+u_0B|^{\frac{s}{2}}\psi\| = \|(A+u_0B)^{\frac{s}{2}}\psi\| \leq C_1\|A^{\frac{s}{2}}\psi\| \\ + C_2\|\psi\| \leq C_3\|A^{\frac{s}{2}}\psi\|. \end{aligned} \tag{B.23}$$

Let $s/2 = k \in \mathbb{N}$. The relation $(B.23)$ is proved by iterative argument. First, it is true for $k=1$ and $k=2$ since if $B \in L(D(A^{k_1}))$ for $1 \leq k_1 \leq 2$, then there exists $C>0$ such that $\|AB\psi\| \leq C \, |||\, B \,|||_{D(A^{k_1})} \|A^{k_1}\psi\|$ for every $\psi \in D(A)$. As $B \in L(\mathscr{H})$, there exist $C_4, C_5, C_6, C_7 > 0$ such that, for every $\psi \in D(A^2)$,

$$\begin{aligned} \|(A+u_0B)^2\psi\| &\leq \|A^2\psi\| + |u_0|^2\|B^2\psi\| + |u_0|\|AB\psi\| + |u_0|\|BA\psi\| \\ &\leq \|A^2\psi\| + |u_0|^2 \,|||\, B^2 \,|||\, \|\psi\| \\ &+ C_4|u_0| \,|||\, B \,|||_{L(D(A^{k_1}))} \|\psi\|_{(k_1)} + |u_0| \,|||\, B \,|||\, \|\psi\|_{(2)} \\ &\leq C_5\|A^2\psi\| + C_6\|\psi\| \leq C_7\|A^2\psi\|. \end{aligned}$$

Second, we assume the validity of $(B.23)$ for $k \in \mathbb{N}$ when $B \in L(D(A^{k_j}))$ for $k-j-1 \leq k_j \leq k-j$ and for every $j \in \{0, ..., k-1\}$. We prove the relation $(B.23)$ for $k+1$ when $B \in L(D(A^{k_j}))$ for $k-j \leq k_j \leq k-j+1$ and for every $j \in \{0, ..., k\}$. There exists $C > 0$ such that $\|A^kB\psi\| \leq C \, |||\, B \,|||_{D(A^{k_0})} \|A^{k_0}\psi\|$ for every $\psi \in D(A^{k+1})$. Thus, there exist $C_8, C_9, C_{10}, C_{11} > 0$ such that, for every $\psi \in D(A^{k+1})$,

$$\begin{aligned} &\|(A+u_0B)^{k+1}\psi\| = \|(A+u_0B)^k(A+u_0B)\psi\| \\ &\leq C_8\|A^k(A+u_0B)\psi\| + C_9\|(A+u_0B)\psi\| \\ &\leq C_8\|A^{k+1}\psi\| + C_8|u_0|\|A^kB\psi\| + C_9\|A\psi\| \\ &+ |u_0|C_{10}\|B\psi\| \leq C_8\|A^{k+1}\psi\| + C_{10}|u_0| \,|||\, B \,|||_{L(D(A^{k_0}))} \|\psi\| \\ &+ C_9\|\psi\|_{(2)} + |u_0|C_9 \,|||\, B \,|||\, \|\psi\| \leq C_{11}\|A^{k+1}\psi\|. \end{aligned}$$

As in the proof of Lemma $B.8$, the relation $(B.23)$ is valid for any $s \leq k$ when $B \in L(D(A^{k_0}))$ for $k-1 \leq k_0 \leq s$ and $B \in L(D(A^{k_j}))$ for $k-j-1 \leq k_j \leq k-j$ and for every $j \in \{1, ..., k-1\}$. The opposite inequality follows by decomposing $A = A + u_0 B - u_0 B$.

In our framework, Assumptions IV ensure that $s = 2 + d$.

- If the third point of Assumptions IV is verified for $s \in [4, 11/2)$, then B preserves $H^{d_1}_{\mathcal{NK}}$ and $H^2_{\mathscr{G}}$ for d_1 introduced in Assumptions IV. Proposition 4.12 claims that $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$ and the argument of Remark 2.1 implies $B \in L(H^{d_1}_{\mathscr{G}})$.
- If the second or the fourth point of Assumptions IV is verified for $s \in [4, 9/2)$, then $B \in L(\mathscr{H})$, $B \in L(H^2_{\mathscr{G}})$ and $B \in L(H^{d_1}_{\mathscr{G}})$ for $d_1 \in [d, 9, 2)$ since B stabilizes H^{d_1} and $H^2_{\mathscr{G}}$ for d_1 introduced in Assumptions IV. Thanks to Proposition 4.12, $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$ and the argument of Remark 2.1 implies $B \in L(H^{d_1}_{\mathscr{G}})$.
- If $s < 4$ instead, then the conditions $B \in L(\mathscr{H})$ and $B \in L(H^2_{\mathscr{G}})$ are sufficient (see Remark 2.1). □

Remark B.16. *The techniques of Lemma* $B.15$ *allow to prove the following claim. Let the hypotheses of Theorem* 4.3 *be satisfied and* $0 < s_1 < d+2$ *for* d *introduced in Assumptions IV. Let* $c \in \mathbb{R}$ *such that* $0 \notin \sigma(A + u_0 B + c)$ *and such that* $A + u_0 B + c$ *is a positive operator. There exists a neighborhood* $U(0)$ *of* 0 *in* $\mathbb{R}$ *such that, for any* $u_0 \in U(0)$,

$$\Big(\sum_{j=1}^{\infty} \big|\langle |A + u_0 B + c|^{\frac{s_1}{2}} \phi_j^{u_0}, \cdot\rangle\big|^2\Big)^{\frac{1}{2}} \asymp \Big(\sum_{j=1}^{\infty} |j^{s_1} \langle \phi_j, \cdot\rangle|^2\Big)^{\frac{1}{2}}.$$

Notation

We set some notation that we widely use in the work.

- Let $\mathscr{H} = L^2(\Omega, \mathbb{C})$ for Ω the bounded interval $(0,1)$ or a generic compact graph. We denote

$$\langle \psi_1, \psi_2 \rangle := \langle \psi_1, \psi_2 \rangle_{\mathscr{H}} = \int_\Omega \overline{\psi_1}(x)\psi_2(x)dx, \qquad \forall \psi_1, \psi_2 \in \mathscr{H},$$

$$\|\psi\| := \|\psi\|_{\mathscr{H}} = \sqrt{\int_\Omega |\psi(x)|^2 dx}, \qquad \forall \psi \in \mathscr{H}.$$

  We call $H^s := H^s(\Omega, \mathbb{C})$, $H^s_0 := H^s_0(\Omega, \mathbb{C})$ and $U(\mathscr{H})$ the space of the unitary operators on $\mathscr{H}$.

- When Ω is a compact graph composed by $N \in \mathbb{N}$ edges $\{e_j\}_{j \le N}$, any function $f \in \mathscr{H} = L^2(\mathscr{G}, \mathbb{C})$ can be denoted as vector of functions such that

$$f = (f^1, ..., f^N), \qquad f^j \in L^2(e_j, \mathbb{C}), \quad \forall j \le N$$

  and

$$\langle \psi_1, \psi_2 \rangle = \int_\Omega \overline{\psi_1}(x)\psi_2(x)dx = \sum_{j=1}^N \int_{e_j} \overline{\psi_1^j}(x)\psi_2^j(x)dx, \qquad \forall \psi_1, \psi_2 \in \mathscr{H},$$

$$\|\psi\| = \sqrt{\int_\Omega |\psi(x)|^2 dx} = \sqrt{\sum_{j=1}^N \int_{e_j} |\psi^j(x)|^2 dx}, \qquad \forall \psi \in \mathscr{H}.$$

- In the current work, A is the Laplacian equipped with self-adjoint type boundary conditions. The sequence $\{\phi_k\}_{k \in \mathbb{N}}$ is an Hilbert basis of $\mathscr{H}$ composed by eigenfunctions of A associated to the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ and

$$\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j.$$

- The operator B is bounded and symmetric and

$$B_{j,k} = \langle \phi_j, B\phi_k \rangle, \qquad \forall j,k \in \mathbb{N}.$$

For $u \in L^2((0,T),\mathbb{R})$, $T>0$ and $t \in (0,T)$, we denote Γ_t^u the unitary propagator generated by

$$A + u(t)B$$

and $\widetilde{\Gamma}_t^{\widetilde{u}}$ the unitary propagator generated by

$$-A - \widetilde{u}(t)B$$

for $\widetilde{u}(\cdot) = u(T - \cdot)$.

- When $\Omega = (0,1)$ and A is such that

$$D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})),$$

$$A\psi = -\Delta\psi, \qquad \forall \psi \in D(A),$$

we introduce

$$H^s_{(0)} = H^s_{(0)}((0,1),\mathbb{C}) := D(A^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} = \left(\sum_{k=1}^{\infty} |k^s \langle \phi_k, \cdot\rangle|^2\right)^{\frac{1}{2}}.$$

- When $\Omega = \mathscr{G}$ is a compact graph, we call

$$H^s_{\mathscr{G}} = H^{\mathscr{G}}_{(0)}(\mathscr{G},\mathbb{C}) := D(A^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} = \left(\sum_{k=1}^{\infty} |k^s \langle \phi_k, \cdot\rangle|^2\right)^{\frac{1}{2}}.$$

- We use the following notation for $s > 0$

$$|||\cdot||| := |||\cdot|||_{L(\mathscr{H},\mathscr{H})}, \qquad |||\cdot|||_{(s)} := |||\cdot|||_{L(D(A^{\frac{s}{2}}),D(A^{\frac{s}{2}}))},$$

$$|||\cdot|||_3 := |||\cdot|||_{L(D(A^{\frac{3}{2}}),H^3\cap D(A^{\frac{1}{2}}))}.$$

- We denote the following norm

$$\|f\|_{BV(T)} := \|f\|_{BV((0,T),\mathbb{R})} = \sup_{\{t_j\}_{j\le n}\in P} \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|,$$

for $f \in BV((0,T),\mathbb{R})$, where P is the set of the partitions of $(0,T)$ such that

$$t_0 = 0 < t_1 < ... < t_n = T.$$

- For $\mathbf{x^1} = \{x_k^1\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and $\mathbf{x^2} = \{x_k^2\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$, we define

$$\langle \mathbf{x^1}, \mathbf{x^2} \rangle_{\ell^2} = \sum_{k=1}^{+\infty} \overline{x_k^1} x_k^2, \qquad \|\mathbf{x^1}\|_{\ell^2} = \sqrt{\sum_{k=1}^{+\infty} |x_k^1|^2}.$$

  Moreover, for $s > 0$,

$$h^s(\mathbb{C}) = \Big\{ \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{C} \Big| \sum_{j=1}^{\infty} |j^s x_j|^2 < \infty \Big\}, \qquad \|\cdot\|_{h^s} = \left(\sum_{k=1}^{\infty} |k^s \cdot|^2 \right)^{\frac{1}{2}}.$$

- For any Hilbert space X, we call $\langle \cdot, \cdot \rangle_X$ its scalar product.

- For any Banach manifold X equipped with the norm $\|\cdot\|_X$, we denote

$$B_X(\psi, r) = \{\tilde{\psi} \in X : \ \|\tilde{\psi} - \psi\|_X \le r\}, \qquad \psi \in X, \quad r > 0.$$

  For every $x \in X$, we define $T_x X$ the tangent space to X at the point x.

  Let $F : X \to Y$ be a differentiable map between two Banach manifolds X and Y. For $x_0 \in X$, we call $d_{x_0} F : T_{x_0} X \to T_{f(x_0)} Y$ its linear differential map.

  We denote with $L(X, Y)$ the space of the bounded operators from X to Y, while with $L(X, X)$ or $L(X)$ the space of the bounded operators from X to X.

- For any real number r, we call

$$[r] := r - \min_{\substack{n\in\mathbb{N} \\ n \le r}} |r - n|,$$

  its entire part. If $r \in \mathbb{C}$, then we call $\Re(r)$ its real part and $Imm(r)$ its imaginary part.

- For every interval I, we denote its length as $|I|$.

Notation Chapter 2

- We define

$$|||\cdot|||_{L(L^2((0,T),\mathbb{R}),H^3_{(0)})} = |||\cdot|||_{(L^2_t, H^3_x)}, \quad |||\cdot|||_{L(H^3_{(0)}, L^2((0,T),\mathbb{R}))} = |||\cdot|||_{(H^3_x, L^2_t)},$$

$$\|\cdot\|_{L^\infty((0,T),H^3_{(0)})} = \|\cdot\|_{L^\infty_t H^3_x}, \quad \|\cdot\|_{L^2((0,T),\mathbb{R})} = \|\cdot\|_2,$$

- We call

$$u_n(t) := \frac{\cos\left((k^2-j^2)\pi^2 t\right)}{n}, \qquad C' := \sup_{(l,m)\in\Lambda'}\left\{\left|\sin\left(\pi\frac{|l^2-m^2|}{|k^2-j^2|}\right)\right|^{-1}\right\},$$

$$\Lambda' := \Big\{(l,m)\in\mathbb{N}^2 :\ \{l,m\}\cap\{j,k\}\neq\emptyset,\ |l^2-m^2|\leq\frac{3}{2}|k^2-j^2|,$$
$$|l^2-m^2|\neq|k^2-j^2|,\ \langle\phi_l, B\phi_m\rangle\neq 0\Big\},$$

$$T^* := \frac{\pi}{|B_{k,j}|}, \qquad T=\frac{2}{\pi|k^2-j^2|}, \qquad I=\frac{4}{\pi^2|k^2-j^2|}.$$

- Let $N\in\mathbb{N}$. We define the $N\times N$ matrix M^N such that, for $l,m\in\mathbb{N}$,

$$M^N_{l,m}=\langle\phi_l, M^N\phi_m\rangle=\frac{B_{l,m}}{I}\int_0^I e^{i\pi^2(l^2-m^2)v(x)}dx, \quad \text{if} \quad \frac{|l^2-m^2|}{|k^2-j^2|}\in\mathbb{N},$$

  for $v(t)$ the reciprocal function of $t\mapsto\int_0^t|\cos(\pi^2(k^2-j^2)s)|ds$, otherwise $M^N_{l,m}=0$.

- Let $\theta^N\in\mathbb{R}^+$ be the smallest value such that $e^{i\theta^N}=\langle\phi_k, e^{2|B_{k,j}|^{-1}M^N}\phi_j\rangle$ and

$$\widetilde{T}^N=\frac{\theta^N}{(j\pi)^2}.$$

Notation Chapter 3

- Let $\Psi:=\{\psi_j\}_{j\in\mathbb{N}}\subset\mathscr{H}$ and $\mathscr{H}_N(\Psi):=\text{span}\{\psi_j : j\leq N\}$. We define $\pi_N(\Psi)$ the orthogonal projector onto $\mathscr{H}_N(\Psi)$.

- We denote

$$O_{\epsilon,T} := \Big\{\{\psi_j\}_{j\in\mathbb{N}}\subset H^3_{(0)}\Big|\ \langle\psi_j,\psi_k\rangle=\delta_{j,k};\ \sup_{j\in\mathbb{N}}\|\psi_j-\phi_j(T)\|_{(3)}<\epsilon\Big\}.$$

- Let $u(t)=u_0+u_1(t)$, for u_0 and $u_1(t)$ real. A bounded perturbation of an operator with compact resolvent is an operator with compact resolvent. Thus, $A+u_0B$ has pure discrete spectrum. We call $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ the eigenvalues of $A+u_0B$ that correspond to an Hilbert basis of $\mathscr{H}$ composed by eigenfunctions $\Phi^{u_0}:=\{\phi_j^{u_0}(x)\}_{j\in\mathbb{N}}$. We set

$$\phi_j^{u_0}(T) := e^{-i\lambda_j^{u_0}T}\phi_j^{u_0}.$$

- We denote

$$O^{u_0}_{\epsilon_0,T} := \Big\{\{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}\Big|\ \langle\psi_j,\psi_k\rangle = \delta_{j,k};\ \sup_{j\in\mathbb{N}} \|\psi_j - \phi^{u_0}_j(T)\|_{(3)} < \epsilon_0\Big\}.$$

- We define

$$\widetilde{H}^3_{(0)} := D(|A+u_0B|^{\frac{3}{2}}), \qquad \|\cdot\|_{\widetilde{H}^3_{(0)}} = \Big(\sum_{k=1}^{\infty}\big||\lambda^{u_0}_k|^{\frac{3}{2}}\langle\cdot,\phi_k\rangle\big|^2\Big)^{\frac{1}{2}}.$$

- Let ψ^1, $\psi^2 \in \mathscr{H}$. We call $|\psi^1\rangle\langle\psi^2|$ the rank one operator such that

$$|\psi^1\rangle\langle\psi^2|\psi = \psi^1\langle\psi^2,\psi\rangle, \qquad \forall\psi\in\mathscr{H}.$$

- A density matrix ρ is a positive operator of trace 1 such that there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ such that

$$\rho = \sum_{j\in\mathbb{N}} l_j|\psi_j\rangle\langle\psi_j|, \qquad \sum_{j\in\mathbb{N}} l_j = 1, \qquad l_j \geq 0\ \ \forall j\in\mathbb{N}.$$

Notation Chapter 4

- Let $\mathscr{G}$ be a compact graph composed by N edges $\{e_j\}_{j\leq N}$ of lengths $\{L_j\}_{j\in\mathbb{N}}$ connecting M vertices $\{v_j\}_{1\leq j\leq M}$. Let

$$V_e := \{v\in\{v_j\}_{1\leq j\leq M} | \exists! e\in\{e_j\}_{j\leq N} : v\in e\}, \qquad V_i := \{v_j\}_{1\leq j\leq M}\setminus V_e.$$

  We respectively call V_e and V_i the external and the internal vertices of $\mathscr{G}$. For each $j\leq M$,

$$N(v_j) := \big\{l\in\{1,...,N\}\ |\ v_j\in e_l\big\}, \qquad n(v_j) := |N(v_j)|,$$

  where $|N(v_j)|$ represents the cardinality of the set $N(v_j)$.

- Each $v\in V_i$ is equipped with $(\mathcal{NK})$ (Neumann-Kirchhoff boundary conditions) when for every $f\in\ D(A)$,

$$(\mathcal{NK}) :\ \begin{cases} f \text{ is continuous in } v, \\ \sum_{e\in N(v)} \frac{\partial f}{\partial x_e}(v) = 0. \end{cases}$$

  The derivatives are assumed to be taken in the directions away from the vertex (outgoing directions).

  Each $v\in V_e$ is equipped $(\mathcal{D})$ when is equipped with Dirichlet boundary conditions. In addition, v is equipped with $(\mathcal{N})$ when it is equipped with Neumann boundary conditions.

- We say that a graph $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$ (or $(\mathcal{N})$-$(\mathcal{NK})$) when every $v \in V_e$ is equipped with $(\mathcal{D})$ (or $(\mathcal{N})$) and every $v \in V_i$ is equipped with $(\mathcal{NK})$.

  We say that a graph $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ when, for every $v \in V_e$, v is equipped with $(\mathcal{D})$ or $(\mathcal{N})$ and every $v \in V_i$ is equipped with $(\mathcal{NK})$.

- We introduce the following space for $s > 0$ ($[\cdot]$ denote the entire part of a number)

$$H^s_{\mathcal{NK}} := \Big\{\psi \in H^s \mid \partial_x^{2n}\psi \in C^0(\mathscr{G},\mathbb{C})\ \forall n \in \mathbb{N}\cup\{0\},\ n < \big[(s+1)/2\big];$$
$$\sum_{e\in N(v)} \partial_{x_e}^{2n+1} f(v) = 0,\ \forall v \in V_i,\ \forall n \in \mathbb{N}\cup\{0\},\ n < \big[s/2\big]\Big\}.$$

- For every $N \in \mathbb{N}$, we define $\mathcal{AL}(N) \subset (\mathbb{R}^+)^N$ as follows. For every $\{L_j\}_{j\leq N} \in \mathcal{AL}(n)$, the numbers $\big\{1, \{L_j\}_{j\leq N}\big\}$ are linearly independent over $\mathbb{Q}$ and all the ratios L_k/L_j are algebraic irrational numbers.

- We usually denote $\varphi := \{\varphi_k\}_{k\in\mathbb{N}} \subseteq \{\phi_k\}_{k\in\mathbb{N}}$ an orthonormal system of eigenfunctions of A corresponding to the eigenvalues $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \{\lambda_k\}_{k\in\mathbb{N}}$, *i.e.* $A\varphi_k = \mu_k\varphi_k$ and $\varphi_k \not\equiv 0$.

- As in [Tri95, $Definition$, $Chapter$ 1.9.2], we define $[\cdot,\cdot]_\theta$ the complex interpolation for $0 < \theta < 1$.

Notation Appendix A

- Let $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$ and $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ be pairwise distinct. We assume that there exists $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ such that

$$\inf_{k\in\mathbb{Z}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}. \tag{24}$$

  From $(A.10)$, we notice that there does not exist $\mathcal{M}$ consecutive $k \in \mathbb{Z}^*$ such that

$$|\lambda_{k+1} - \lambda_k| < \delta.$$

  This fact leads to a partition of $\mathbb{Z}^*$ in subsets that we call E_m with $m \in \mathbb{Z}^*$. The partition defines an equivalence relation in $\mathbb{Z}^*$ such that $k, n \in \mathbb{Z}^*$ are equivalent if and only if there exists $m \in \mathbb{Z}^*$ such that $k, n \in E_m$. The sets $\{E_m\}_{m\in\mathbb{Z}}$ are the corresponding equivalence classes and $i(m) := |E_m| \leq \mathcal{M} - 1$. For $\mathbf{\Lambda} := \{\lambda_l\}_{l\in\mathbb{Z}^*}$, we define the vectors

$$\mathbf{\Lambda}^m := \{\lambda_l\}_{l\in E_m}, \qquad m \in \mathbb{Z}^*.$$

- Let $\widehat{\mathbf{h}} = \{h_j\}_{j\leq i(m)} \in \mathbb{C}^{i(m)}$. We denote $F_m(\widehat{\mathbf{h}}) : \mathbb{C}^{i(m)} \to \mathbb{C}^{i(m)}$ the matrix with elements, for every $j, k \leq i(m)$,

$$F_{m;j,k}(\widehat{\mathbf{h}}) := \begin{cases} \prod_{\substack{l\neq j \\ 1\leq l\leq k}} (h_j - h_l)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k. \end{cases}$$

- On the Hilbert space $\ell^2(\mathbb{C})$, we introduce the linear operator $F(\mathbf{\Lambda}) : D(F(\mathbf{\Lambda})) \to \ell^2(\mathbb{C})$ as follows. For every $k \in \mathbb{Z}^*$, we know that there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$ and, for every $\mathbf{x} = \{x_l\}_{l\in\mathbb{Z}^*} \in D(F(\mathbf{\Lambda}))$, we define

$$(F(\mathbf{\Lambda})\mathbf{x})_k = \Big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)}\Big)_k,$$

$$H(\mathbf{\Lambda}) := D(F(\mathbf{\Lambda})) = \big\{\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C}) \ : \ F(\mathbf{\Lambda})\mathbf{x} \in \ell^2(\mathbb{C})\big\}.$$

- When $H(\mathbf{\Lambda})$ is dense in $\ell^2(\mathbb{C})$, we can define

$$F(\mathbf{\Lambda})^*$$

the unique adjoint operator of $F(\mathbf{\Lambda})$ of domain $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*)$. We know that, for $k \in \mathbb{Z}^*$, there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. The operator $F(\mathbf{\Lambda})^*$ is the infinite matrix such that, for every sequence $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}^*} \in H(\mathbf{\Lambda})^*$ and $k \in \mathbb{Z}^*$,

$$(F(\mathbf{\Lambda})^*\mathbf{x})_k = \Big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*\mathbf{x}^{m(k)}\Big)_k$$

where $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$ is the transposed matrix of $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$.

- For $T > 0$, we introduce

$$\mathbf{e} := \{e^{i\lambda_j t}\}_{j\in\mathbb{Z}^*} \subset L^2((0,T),\mathbb{C}), \qquad \mathbf{\Xi} := \{\xi_k\}_{k\in\mathbb{Z}^*} = F(\mathbf{\Lambda})^*\mathbf{e}.$$

- Let $\mathscr{H}$ be an Hilbert space. The families of functions $\{f_k\}_{k\in\mathbb{Z}^*}$, $\{g_k\}_{k\in\mathbb{Z}^*} \subset \mathscr{H}$ are biorthogonal if

$$\langle f_k, g_l\rangle_{\mathscr{H}} = \delta_{k,l}, \qquad \forall k, l \in \mathbb{Z}^*.$$

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