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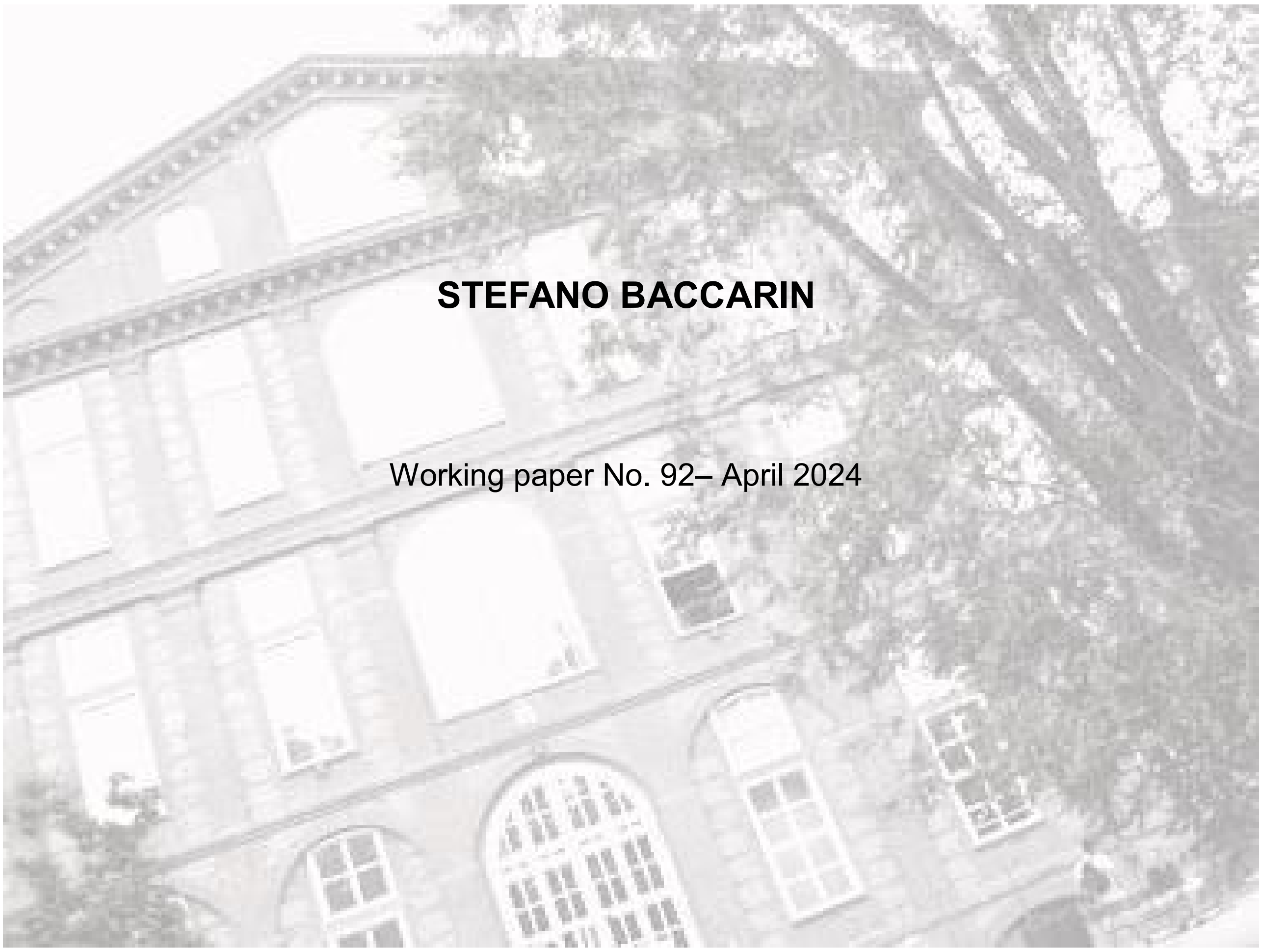
CRRA UTILITY MAXIMIZATION OVER A FINITE HORIZON IN AN EXPONENTIAL LEVY MODEL WITH FINITE ACTIVITY



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CRRA utility maximization over a finite horizon in an exponential Levy model with finite activity

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Abstract

We study a dynamic portfolio optimization problem over a finite horizon with n risky securities and a risk-free asset. The prices of the risky securities are modelled by ordinary exponentials of jump-diffusions. The goal is to maximize the expected discounted utility from both consumption up to the final horizon and terminal wealth. We prove a verification theorem that characterize the value function and the optimal policy by means of a regular solution of a HJB partial integro-differential equation. The verification theorem is used to obtain closed-form expressions for the value function and the optimal policy considering CRRA utility functions $U(x) = \frac{x^\rho}{\rho}$, with $\rho < 1 \wedge \rho \neq 0$, and $U(x) = \ln x$.

Keywords Optimal consumption/investment over a finite horizon, CRRA utility, Dynamic programming, Lévy processes with finite activity, Integro-differential PDE

MSC(2020) 49L20, 91G10, 93E20

1 Introduction

We consider a portfolio problem where the goal is to maximize the expected discounted utility coming from both consumption up to a fixed horizon T and terminal wealth at T. The agent can trade $n+1$ securities, a risk-free asset (bond or bank account) and n risky stocks. Consumption up to T and terminal wealth are competing objectives and the investor has to find, at any time $t \in [0, T]$, the optimal trade-off between immediate consumption and investment. We consider a small investor, whose actions cannot affect the market, and we model the market prices as ordinary exponentials of jump diffusions with finite activity (as in the Merton [9] and Kou [7]).

To compare the value function and the optimal policy with and without the jumps we do not formulate the jump part of the log-prices as martingales. Adding to the diffusion component a compensated Levy process has an effect on the optimal policy similar to an increase in the volatility of the diffusion. As shown in [5] in a two assets model, the result is a decrease in the value function and in the fraction of wealth invested in the risky assets. However if the added process is a compound Poisson process the result on the optimal solution is not determined, we show that it varies with the model parameters. We analyze a model with constant coefficients, also including coefficients which measure the intensity with which the jumps coming from a set of independent compound Poisson processes influence the asset prices. To ensure that the agent's wealth remains positive in the presence of the jumps shortselling and borrowing are not admitted. We prove a verification theorem which characterizes the value function and the optimal policy by means of a sufficiently regular solution of a HJB partial integro-differential equation. Using the verification theorem we solve the model considering power utility $U(x) = \frac{x^\rho}{\rho}$, with $\rho < 1 \wedge \rho \neq 0$, and logarithmic utility $U(x) = \ln x$. Our simple assumptions let us obtain closed-form expressions for the value function, the optimal consumption-investment policy, and the optimal wealth process. The results are first given for a two assets model and the extended to many assets with little difficulties. In [1],[2],[3], [6] the portfolio problem is formulated with infinite horizon where the goal is to maximize only the expected discounted utility of consumption. The prices are modelled by geometric Lévy processes and the value function is characterized as the unique constrained viscosity solution of a HJB integro-differential variational inequality associated to a singular stochastic control problem. Benth et al. [1],[2], incorporate in the utility function the notions of durability and intertemporal substitution and Framstad et al. [6] and De Valliere et al. [3] study the model in the presence of proportional transaction costs and more general conic constraints. Nutz [10] and Egorov and Pergamenchtchikov [4] consider a finite horizon problem where the agent maximizes the expected, but not discounted, utility from both consumption and terminal wealth using a power utility function. In [10] it is shown, under minimal assumptions and any convex set constraint on the portfolio, that the optimal proportions invested in the risky assets are constant over time and the optimal consumption rate is always a deterministic function. In [4] a general verification theorem is proven with a risk parameter $0 < \rho < 1$ and an asymptotic method is developed to compute the solution in the presence of proportional transaction costs.

2 The model

The price P_i of the i-th risky security is modelled by an exponential Levy process $(i = 1, ..., n)$

$$P_i(t) = p_i e^{L_i(t)}$$

where $L_i(t)$ is a jump-diffusion whose jump component is a superposition of indipendent compound Poisson processes

$$dL_i(t) = \alpha_i dt + \sum_{j=1}^{d} \sigma_{i,j} dB_j(t) + \sum_{k=1}^{l} \int_{\mathbb{R}} \gamma_{ik} z_k N_k(dt, dz_k), \ L_i(0) = 0 \ .$$

Here $\{B_j\}$ are standard Brownian motions in $\mathbb{R}$, $\{N_k\}$ are the jump measures coming from l compound Poisson processes η_k with finite Lévy measures v_k and the coefficients α_i, $\sigma_{i,j}$, γ_{ik} are constants $(j = 1, ..., d; k = 1, ..., l)$. All the processes B_j, η_k are defined on a common filtered probability space $(\Omega, F, P, \mathbb{F}_t)$, verifying the usual assumptions, and they are all mutually indipendent and adapted to $\mathbb{F}_t$. The compound Poisson processes represent l independent sources of risk which cause sudden and possibly large jumps in the asset prices: the coefficients γ_{ik} measure the magnitude and direction with which the jumps influence the individual securities. We denote by λ_k the intensity of the compound Poisson process η_k and by l_k its jump size distribution. We have $v_k = \lambda_k l_k$ and $v = \sum_{k=1}^{l} v_k$ is the Lévy measure common to all L_i. Moreover we also assume that the measures v_k verify

$$\int_{|z|>1} e^{\beta z} v_k(dz) < \infty \qquad \forall \beta \in \mathbb{R}, \ k = 1, ..., l. \tag{H}$$

Assumption (H) holds if the l_k are normal distributions. It is obviously verified if the l_k have finite or compact support (this last condition does not seem too restrictive for applications). We will use the following notation (T denotes transposition)

$$\begin{aligned} \alpha &= [\alpha_1, ..., \alpha_n]^T, \ \sigma = [\sigma_{i,j}] \in \mathbb{R}^{n \times d}, \ \sigma_i^T = [\sigma_{i1}, ..., \sigma_{id}], \ \gamma = [\gamma_{i,j}] \in \mathbb{R}^{n \times l} \\ \mu_i &= \alpha_i + \frac{1}{2} |\sigma_i|^2, \ \hat{\mu}_i = \mu_i - r, \ \mu = [\mu_1, ..., \mu_n]^T, \ \hat{\mu} = [\hat{\mu}_1, ..., \hat{\mu}_n]^T \ . \end{aligned}$$

We assume that the matrix $\sigma\sigma^T$ is positive-definite. By Itô formula P_i is the solution of the stochastic differential equation

$$\left| \begin{array}{l} dP_i(t) = P_i(t^-) \left[\mu_i dt + \sigma_i^T dB(t) + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{\gamma_{ik} z_k} - 1) N_k(dt, dz_k) \right] \\ P_i(0) = p_i \end{array} \right. \tag{1}$$

where $B(t)$ is a d-dimensional standard Brownian motion. The price $P_0(t)$ of the risk-free asset grows at a fixed instantaneous rate r

$$\left| \begin{array}{l} dP_0(t) = rP_0(t)dt \\ P_0(0) = 1. \end{array} \right. \tag{2}$$

We denote by $S_i(t)$ the amount of money invested in the stock i and by $R(t)$ the amount of money invested in the risk-free asset. The value of the portfolio, which is equal to the investor's wealth, is $W(t) = R(t) + \sum_{i=1}^{n} S_i(t)$ and we use $\pi_i(t) = \frac{S_i(t)}{W(t)}$ to denote the fraction of the portfolio's value invested in stock i. We will assume that at any time t the investor can control the value of $\pi(t) = [\pi_1(t), ..., \pi_n(t)]^T$ by trading the securities, without transaction costs or other market frictions. Moreover at any time t the agent consumes at a positive wealth consumption rate $c(t)$. We denote by $e^{\gamma z}$ the matrix

$$e^{\gamma z} = [e^{\gamma_{ik} z_k}] \in \mathbb{R}^{n \times l}$$

by $e_k^{\gamma z}$ its k-th column and by $\mathbf{1}$ the n-dimensional all-ones vector. Considering the prices dynamics (1),(2) the evolution of wealth $W_t^{\pi,c}$, controlled by policy $p_t \equiv (\pi_t, c_t) : [0, T] \times \Omega \to \mathbb{R}^{n+1}$, is described by the equation

$$\begin{aligned} dW_t^{\pi,c} &= W_{t^-}^{\pi,c}(r + \hat{\mu}^T \pi_t - c_t)dt + W_{t^-}^{\pi,c} \pi_t^T \sigma dB_t + \\ &\quad + W_{t^-}^{\pi,c} \sum_{k=1}^{l} \int_{\mathbb{R}} \pi_t^T (e_k^{\gamma z} - \mathbf{1}) N_k(dt, dz_k). \end{aligned} \tag{3}$$

Let $\Pi \subset \mathbb{R}^n$ be the set ($\mathbf{0}$ is the all-zeros vector)

$$\Pi = \left\{ \pi \in \mathbb{R}^n : \pi \geq \mathbf{0} \wedge \sum_{i=1}^{n} \pi_i \leq 1 \right\} .$$

A control $p = (\pi, c)$ is called admissible if it verifies the conditions:

1. *π_t, is left continuous with right limits and adapted to $\mathbb{F}_t$*

2. *$\pi_t \in \Pi$, Lebesgue $\times P$ a.e. in $[0,T]\times\Omega$*

3. *c_t is $\mathbb{F}_t -$ adapted*

4. *$c_t > 0$ and $c_t \in C$, with C compact, Lebesgue $\times P$ a.e. in $[0,T]\times\Omega$.*

Condition 2 means that shortselling and borrowing are not admitted and it is sufficient to ensure that wealth remains positive after the jumps. By Itô formula applied to $\ln W_t^{\pi,c}$ the solution of (3) is given by

$$\begin{aligned} W^{\pi,c}(t) &= W^{\pi,c}(0)\exp\left\{\int_0^t (r+\hat{\mu}\pi_s - c_s - \frac{1}{2}\pi_s^T\sigma\sigma^T\pi_s)ds + \right. \\ &\left. + \int_0^t \pi^T\sigma dB_{s-t} + \int_0^t \sum_{k=1}^l \int_{\mathbb{R}} \ln(1+\pi_s^T(e_k^{\gamma z}-\mathbf{1}))N_k(ds,dz_k)\right\} \end{aligned}$$

and $W^{\pi,c}$ stays positive if $W^{\pi,c}(0)>0$. Moreover, always by Itô formula, $y(t)=\frac{1}{W^{\pi,c}(t)}$ verifies the equation

$$\begin{aligned} dy(t) &= -y(t^-)\left[(r+\hat{\mu}^T\pi_t - c_t - \pi_s^T\sigma\sigma^T\pi_s)dt + \pi_t^T\sigma dB_t\right] \\ & -y(t^-)\left[\sum_{k=1}^l \int_{\mathbb{R}} \left(1-\frac{1}{1+\pi_t^T(e_k^{\gamma z}-\mathbf{1})}\right)N_k(dt,dz_k)\right] . \end{aligned} \tag{4}$$

Conditions 1-4 on (π,c), assumption (H) and the linear form of equations (3),(4) are sufficient to ensure that for any $p>0$ we have

$$E_W\left\|W^{\pi,c}(.)\right\|^p < \infty \quad \text{and} \quad E_y\left\|y(.)\right\|^p = E_W\left\|\frac{1}{W^{\pi,c}(.)}\right\|^p < \infty$$

where $\|\|$ is the sup norm on $[0,T]$ (see Menaldi [8], chapter 5). The set of admissible controls in $t=0$ is denoted by $A(0)$. The goal of the investor is to maximize the expected utility from both consumption up to the final date T and terminal wealth at T. She/he has two utility functions, $U_1:(0,\infty)\to\mathbb{R}$ for the consumption flow $C(s)=c(s)W^{\pi,c}(s)$ and $U_2:(0,\infty)\to\mathbb{R}$ for the terminal wealth. We assume that U_1,U_2 are continuous on $(0,\infty)$ and they both verify a growth condition

$$|U_i(x)| \le C(1+|x|^m), \;\; i=1,2$$

for suitable constants $C > 0$, $m \in \mathbb{R}$. We consider the following payoff functional $J^{\pi,c}(W)$ associated to policy $(\pi, c) \in \mathcal{A}(0)$ and initial condition $W = W^{\pi,c}(0)$

$$J^{\pi}(W) = E_W \left[\int_0^T e^{-\delta s} U_1 \left[c(s) W^{\pi,c}(s) \right] ds + e^{-\delta T} U_2(W^{\pi,c}(T)) \right]$$

where δ is a common utility discount rate. The investor's problem is to find M and, if it exists (π^*, c^*), such that

$$M = \sup_{\pi, c \in \mathcal{A}(0)} J^{\pi,c}(W) = J^{\pi^*,c^*}(W).$$

We will study this problem by dynamic programming. Let us define $Q \equiv [0,T) \times (0,\infty)$ and $\bar{Q} \equiv [0,T] \times (0,\infty)$. We consider J as a function of the initial condition $(t, W) \in Q$

$$J^{\pi,c}(t,W) = E_{(t,W)} \left[\int_t^T e^{-\delta(s-t)} U_1 \left[c(s) W^{\pi,c}(s) \right] ds + e^{-\delta(T-t)} U_2(W^{\pi,c}(T)) \right]$$

and we introduce the value function $V : \bar{Q} \to \mathbb{R}$

$$V(t,W) = \sup_{\pi, c \in \mathcal{A}(t)} J^{\pi,c}(t,W) \ . \tag{5}$$

Here, for any $t \in (0,T)$, the reference probability system is given in the interval $[t,T]$ and $\mathcal{A}(t)$ is the set of policies admissible in t, that is the processes π_s, c_s which verify conditions 1-4 for $s \in [t,T]$. The following verification theorem characterizes the value function as a regular solution, if it exists, of a HJB partial integro-differential equation. We use the notation $F_t = \frac{\partial F}{\partial t}$, $F_W = \frac{\partial F}{\partial W}$, $F_{WW} = \frac{\partial^2 F}{\partial W^2}$ for the partial derivatives of a function $F(t,W)$.

Theorem 1 *Let $F(t,W) \in C(\bar{Q}) \cap C^{1,2}(Q)$ be a given function. If F verifies:*
a)$|F(t,W)| \leq C(1 + W^m)$ in $\bar{Q}$ for suitable constants $C > 0$ and $m \in \mathbb{R}$ (growth condition for $W \downarrow 0$ or $W \to \infty$)

b)$F(T,W) = U_2(W)$

c) $\underset{\pi\in\Pi, c\in(0,\infty)}{Sup}\left\{-\delta F+F_t+F_W(r+\hat{\mu}^T\pi-c)W+\frac{1}{2}F_{WW}W^2\pi^T\sigma\sigma^T\pi\right.$

$\left.+\sum_{k=1}^{l}\int_{\mathbb{R}}[F(t,W(1+\pi^T(e_k^{\gamma z}-\mathbf{1})))-F(t,W)]v_k(dz_k)+U_1(cW)\right\}=0$ *in* Q

then $F\geq V$.

Moreover suppose that for $(t,W)\in Q$ *there exists an admissible control* $(\pi^*,c^*)\in\mathcal{A}(t)$ *and corresponding wealth process* $W^{\pi^*,c^*}(s)$, *with initial condition* $W^{\pi^*,c^*}(t)=W$, *such that*

$$
\begin{aligned}
&(\pi^*(s),c^*(s))\in\arg\max\left\{-\delta F(s,W^{\pi^*,c^*}(s))+F_t(s,W^{\pi^*,c^*}(s))\right.\\
&+(r+\hat{\mu}^T\pi^*(s)-c^*(s))W^{\pi^*,c^*}(s)F_W(s,W^{\pi^*,c^*}(s))\\
&+\tfrac{1}{2}(W^{\pi^*,c^*}(s))^2\pi^*(s)^T\sigma\sigma^T\pi^*(s)F_{WW}(s,W^{\pi^*,c^*}(s))\\
&+\sum_{k=1}^{l}\int_{\mathbb{R}}[F(s,W^{\pi,c}(s^-))(1+\pi^*(s)^T(e_k^{\gamma z}-\mathbf{1})))-F(s,W^{\pi,c}(s^-))]v_k(dz_k)\\
&\left.+U_1(c^*(s)W^{\pi^*,c^*}(s))\right\}
\end{aligned}
\tag{6}
$$

holds Lebesgue$\times P$ *a.e.in* $[t,T]\times\Omega$. *Then the control* $(\pi^*(s),c^*(s))$ *is optimal for the initial condition* $(t,W)\in Q$, *that is*

$$V(t,W)=F(t,W)=J^{\pi^*,c^*}(t,W).$$

Proof. Let $(t,W)\in Q$ be the initial condition, Q_r be the bounded set $Q_r\equiv[t,T]\times(0,r)$, and θ_r the stopping time

$$\theta_r\equiv\inf\left\{s>0:(s,W^{\pi,c}(s)\notin Q_r\right\}.$$

For any given $(\pi,c)\in\mathcal{A}(t)$ we apply the Itô differential rule to $e^{-\delta(s-t)}F(s,W^{\pi,c}(s))$, integrated from t to $\theta_r\leq T$. Using the notation

$$\Delta_kF(t,W,\pi)=F(t,W(1+\pi^T(e_k^{\gamma z}-\mathbf{1})))-F(t,W)$$

we have (considering also assumption (H))

$$
\begin{aligned}
&e^{-(\theta_r - t)} F(\theta_r, W_t^{\pi,c}(\theta_r)) - F(t,W) = \\
&= \int_t^{\theta_r} e^{-\delta(s-t)} [-\delta F(s, W^{\pi,c}(s)) + F_t(s, W^{\pi,c}(s))] ds + \\
&+ \int_t^{\theta_r} e^{-\delta(s-t)} (r + \hat{\mu}^T \pi(s) - c(s)) W^{\pi,c}(s) F_W(s, W^{\pi,c}(s))] ds + \\
&+ \int_t^{\theta_r} e^{-\delta(s-t)} \tfrac{1}{2} (W^{\pi,c}(s))^2 \pi(s)^T \sigma \sigma^T \pi(s) F_{WW}(s, W^{\pi,c}(s) ds + \\
&+ \int_t^{\theta_r} e^{-\delta(s-t)} \left\{ \sum_{k=1}^{l} \int_{\mathbb{R}} \Delta_k F(s, W^{\pi,c}(s^-), \pi(s)) v_k(dz_k) \right\} ds + \\
&+ \int_t^{\theta_r} e^{-\delta(s-t)} F_W(s, W^{\pi,c}(s)) W^{\pi,c}(s) \pi(s)^T \sigma dB_s + \\
&+ \int_t^{\theta_r} e^{-\delta(s-t)} \left\{ \sum_{k=1}^{l} \int_{\mathbb{R}} \Delta_k F(s, W^{\pi,c}(s^-), \pi(s)) \bar{N}_k(ds, dz_k) \right\}.
\end{aligned}
\tag{7}
$$

where $\bar{N}(dt, dz_k)$ is the compensated jump measure of η_k. Since Q_r is bounded and $F(t,W) \in C(\bar{Q}) \cap C^{1,2}(Q)$ the last two terms in (7) are martingales. Hence adding to both sides $\int_t^{\theta_r} e^{-\delta(s-t)} U_1 [c(s) W^{\pi,c}(s)] \, ds$, taking expectations and using condition c) of the theorem we obtain

$$
E_{(t,W)} \left[\int_t^{\theta_r} e^{-\delta(s-t)} U_1 [c(s) W^{\pi,c}(s)] \, ds + e^{-(\theta_r - t)} F(\theta_r, W_t^{\pi,c}(\theta_r)) \right] \leq F(t,W) \qquad \forall (t,W) \in Q, \, \forall \pi \in \mathcal{A}(t) \; . \tag{8}
$$

As $r \to \infty$, $\theta_r \to T$ almost surely. Since $F \in C(\bar{Q})$ and using condition b) we get $e^{-(\theta_r - t)} F(\theta_r, W^{\pi,c}(\theta_r)) \to e^{-(T-t)} F(T, W^{\pi,c}(T)) = e^{-(T-t)} U_2(W^{\pi,c}(T)$ a.s.. Moreover by condition a)

$$
|F(\theta_r, W^{\pi,c}(\theta_r))| \leq C(1 + \|W^{\pi,c}(.)\|^m)
$$

where $\|\|$ is the sup norm on $[t,T]$. As

$$
E_{(t,W)} \|W^{\pi,c}(.)\|^p < \infty, \quad E_{(t,W)} \left\| \frac{1}{W^{\pi,c}(.)} \right\|^p < \infty \qquad \forall p > 0
$$

if we take $p > |m|$ and $\alpha = \frac{p}{|m|} > 1$, it holds $E_{(t,W)} |F(\theta_r, W^{\pi,c}(\theta_r))|^\alpha < \infty$, whatever the sign of m. Therefore the random variables $e^{-(\theta_r - t)} F(\theta_r, W^{\pi,c}(\theta_r))$

are uniformly integrable and we have

$$\lim_{r\to\infty} E_{(t,W)} e^{-(\theta_r - t)} F(\theta_r, W^{\pi,c}(\theta_r)) = E_{(t,W)} e^{-(T-t)} F(T, W^{\pi,c}(T)) \ .$$

Hence by (8) and the admissibility of $c(s)$ it follows

$$\begin{aligned} F(t,W) &\geq E_{(t,W)}[\int_t^T e^{-\delta(s-t)} U_1 \left[c(s) W^{\pi,c}(s)\right] ds + e^{-(T-t)} F(T, W^{\pi,c}(T))] \\ &= E_{(t,W)}[\int_t^T e^{-\delta(s-t)} U_1 \left[c(s) W^{\pi,c}(s)\right] ds + e^{-(T-t)} U_2(W^{\pi,c}(T))] \end{aligned}$$

for any admissible $(\pi, c) \in \mathcal{A}(t)$ and $\forall (t,W) \in Q$. Consequently we obtain $F(t,W) \geq V(t,W)$. If there is a policy $(\pi^*, c^*) \in \mathcal{A}(t)$ and corresponding wealth process $W^{\pi^*,c^*}(s)$, with $W^{\pi^*,c^*}(t) = W$, which verify (6) a.e. in $[t,T] \times \Omega$ then (8) becomes an equality and since we have $F(t,W) \geq V(t,W)$ it follows (as $r \to \infty$)

$$\begin{aligned} F(t,W) &= E_{(t,W)}[\int_t^T e^{-\delta(s-t)} U_1(c^*(s) W^{\pi^*,c^*}(s)) ds + e^{-(T-t)} U_2(W^{\pi^*,c^*}(T))] \\ &= J^{\pi^*,c^*}(t,W) = V(t,W). \end{aligned}$$

■

In some cases Theorem 1 allows to determine the value function and the optimal policy with closed-form expressions. In the following we will assume $U_1 = U_2 = U$ and we will focus on constant relative risk aversion utility functions.

3 Two assets and power utility

We have $U(x) = \frac{x^\rho}{\rho}$, with $\rho < 1 \wedge \rho \neq 0$, $\pi \in [0,1]$, $\sigma^T = [\sigma_1, ..., \sigma_d] \in \mathbb{R}^d$, $\alpha \in \mathbb{R}$, $\mu = \alpha + \frac{1}{2} |\sigma|^2$, $\hat{\mu} = \mu - r$, $\gamma^T = [\gamma_1, ..., \gamma_l] \in \mathbb{R}^l$. Let us consider the function $g : [0,1] \to \mathbb{R}$

$$g(\pi) = \hat{\mu} - (1-\rho) |\sigma|^2 \pi + \sum_{k=1}^{l} \int_{\mathbb{R}} (1 + \pi(e^{\gamma_k z_k} - 1))^{-(1-\rho)} (e^{\gamma_k z_k} - 1) v_k(dz_k) \ . \tag{9}$$

We have

$$
\begin{aligned}
g(0) &= \hat{\mu} + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{\gamma_k z_k} - 1) v_k(dz_k) \\
g(1) &= \hat{\mu} - (1-\rho)\,|\sigma|^2 + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{\rho\gamma_k z_k} - e^{-(1-\rho)\gamma_k z_k}) v_k(dz_k) \\
g'(\pi) &= -(1-\rho)[|\sigma|^2 + \sum_{k=1}^{l} \int_{\mathbb{R}} (1+\pi(e^{\gamma_k z_k}-1))^{-(2-\rho)} (e^{\gamma_k z_k}-1)^2 v_k(dz_k)]
\end{aligned}
$$

and $g'(\pi) < 0$ because $\rho < 1$ and $\pi \in [0,1]$. We define

$$
\pi^* = \begin{cases} 0 & \text{if } g(0) \leq 0 \\ \bar{\pi} & \text{if } g(0) > 0 \wedge g(1) < 0 \\ 1 & \text{if } g(1) \geq 0 \end{cases}
$$

where $\bar{\pi} \in (0,1)$ is the only solution of $g(\pi) = 0$ when $g(0) > 0$ and $g(1) < 0$. We also define

$$
\begin{aligned}
A &= \frac{\delta - \rho[r + \hat{\mu}\pi^* - \frac{1}{2}(1-\rho)|\sigma|^2(\pi^*)^2] - \sum_{k=1}^{l} \int_{\mathbb{R}} ((1+\pi^*(e^{\gamma_k z_k}-1))^{\rho} - 1) v_k(dz_k)}{1-\rho} \\
&= \begin{cases} \frac{\delta - \rho r}{1-\rho} & \text{if } g(0) \leq 0 \\ \frac{\delta - \rho[r + \hat{\mu}\bar{\pi} - \frac{1}{2}(1-\rho)|\sigma|^2(\bar{\pi})^2] - \sum_{k=1}^{l} \int_{\mathbb{R}} ((1+\bar{\pi}(e^{\gamma_k z_k}-1))^{\rho} - 1) v_k(dz_k)}{1-\rho} & \text{if } g(0) > 0, g(1) < 0 \\ \frac{\delta - \rho(\mu - \frac{1}{2}(1-\rho)|\sigma|^2) - \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{\rho\gamma_k z_k} - 1) v_k(dz_k)}{1-\rho} & \text{if } g(1) \geq 0\ . \end{cases}
\end{aligned}
$$

Using Theorem 1 we can characterize the value function and the optimal policy. They will depend on the signs of $g(0)$ and $g(1)$.

Theorem 2 *The value function of our control problem considering power utilities $U_1(x) = U_2(x) = \frac{x^\rho}{\rho}$, with $\rho < 1 \wedge \rho \neq 0$, is given by*

$$
V(t,W) = (f(t))^{1-\rho} \frac{W^\rho}{\rho} \qquad (t,W) \in \bar{Q}
$$

where

$$f(t) = \begin{cases} 1+T-t & \text{if } A=0 \\ \\ \frac{1}{A} + e^{-A(T-t)}(1-\frac{1}{A}) & \text{if } A \neq 0 \end{cases} .$$

The optimal policy $p^*(t,W)$ *corresponding to the initial condition* (t,W) *is*

$$p^*(t,W) = \begin{cases} \pi^*(s) = \pi^* \\ & s \in [t,T] \\ c^*(s) = \frac{1}{f(s)} \end{cases}$$

and the optimal wealth $W^{\pi^*,c^*}(s)$ *is given by the process for* $s \in [t,T]$

$$W^{\pi^*,c^*}(s) = \frac{f(s)}{f(t)} W \exp\Big\{(r+\hat{\mu}\pi^* - A)(s-t) - \tfrac{1}{2}(\pi^*)^2 |\sigma|^2 (s-t)$$
$$+\pi^* \sigma^T B_{s-t} + \sum_{k=1}^{l} \int_t^s \int_{\mathbb{R}} \ln(1+\pi^*(e^{\gamma_k z_k}-1)) N(dr,dz_k)\Big\} .$$

Proof. We look for V among functions of the form

$$F(t,W) = (f(t))^{1-\rho} \frac{W^\rho}{\rho} \text{ with } f \in C^1[0,T],\ f(t) > 0 .$$

$F \in C(\bar{Q}) \cap C^{1,2}(Q)$ and it satisfies condition a) of Theorem 1. Setting $f(T) = 1$ it also satisfies condition b). We have

$$\left| \begin{array}{l} \frac{\partial F}{\partial t}(t,W) = (1-\rho)(f(t))^{-\rho} f'(t) \frac{W^\rho}{\rho}, \quad \frac{\partial F}{\partial W}(t,W) = (f(t))^{1-\rho} W^{-(1-\rho)} > 0 \\ \\ \frac{\partial^2 F}{\partial W^2}(t,W) = -(1-\rho)(f(t))^{1-\rho} W^{-(2-\rho)} < 0 . \end{array} \right.$$

For given $(t,W) \in Q$ we consider the function $G(\pi,c) : [0,1] \times [0,\infty) \to \mathbb{R}$

$$\begin{aligned} G(\pi,c) &= -\delta F + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial W}(r+\hat{\mu}\pi - c)W + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} W^2 |\sigma|^2 \pi^2 \\ &\quad + \sum_{k=1}^{l} \int_{\mathbb{R}} [F(t, W(1+\pi(e^{\gamma_k z_k}-1)) - F(t,W)] v_k(dz_k) + \frac{(cW)^\rho}{\rho}. \end{aligned} \tag{10}$$

Substituting the values of F and its derivatives in (10), we obtain

$$\begin{aligned} G(\pi,c) &= -\delta (f(t))^{1-\rho} \frac{W^\rho}{\rho} + (1-\rho)(f(t))^{-\rho} f'(t) \frac{W^\rho}{\rho} \\ &\quad + (f(t))^{1-\rho} W^{-(1-\rho)} (r+\hat{\mu}\pi - c)W - \frac{1}{2}(1-\rho)(f(t))^{1-\rho} W^\rho |\sigma|^2 \pi^2 \\ &\quad + (f(t))^{1-\rho} \frac{W^\rho}{\rho} \sum_{k=1}^{l} \int_{\mathbb{R}} [(1+\pi(e^{\gamma_k z_k}-1))^\rho - 1] v_k(dz_k) + \frac{(cW)^\rho}{\rho} . \end{aligned} \tag{11}$$

Considering the first order necessary conditions to get a maximum of G (and dividing by $(f(t))^{1-\rho}W^{\rho}$) we have

$$\frac{\partial G}{\partial \pi}=0 \Longleftrightarrow g(\pi)=0$$

$$\frac{\partial G}{\partial c}=0 \Longleftrightarrow c=\frac{1}{f(t)}\ .$$

Since $\frac{\partial^2 G}{\partial \pi^2}=W^{\rho}(f(t))^{1-\rho}g'(\pi)<0$, $\frac{\partial^2 G}{\partial c^2}=-W^{\rho}(1-\rho)c^{-(2-\rho)}<0$ and $\frac{\partial^2 G}{\partial \pi \partial c}=0$ we see that the maximum of G in $[0,1]\times[0,\infty)$ is achieved at a single point which is given by

$$\arg\max G(\pi,c)=(\pi^*,c^*)=\begin{cases}(0,\frac{1}{f(t)}) & \text{if } g(0)\leq 0\\ \\ (\bar{\pi},\frac{1}{f(t)}) & \text{if } g(0)>0\wedge g(1)<0\\ \\ (1,\frac{1}{f(t)}) & \text{if } g(1)\geq 0\ .\end{cases}$$

It is important to note that π^* never depends on t or W. Considering the maximizing (π^*,c^*) condition c) of Theorem 1 becomes

$$G(\pi^*,c^*)=0\ \forall(t,W)\in Q. \tag{12}$$

Substituting (π^*,c^*) in (11) and dividing (12) by $\frac{W^{\rho}}{\rho}(f(t))^{-\rho}$ we obtain

$$-\delta f(t)+(1-\rho)f'(t)+\rho f(t)[r+\hat{\mu}\pi^*-\tfrac{1}{f(t)}]-\tfrac{1}{2}\rho(1-\rho)f(t)\left|\sigma\right|^2(\pi^*)^2$$

$$+f(t)\sum_{k=1}^{l}\int_{\mathbb{R}}((1+\pi^*(e^{\gamma_k z_k}-1))^{\rho}-1)v_k(dz_k)+1=0 \qquad \forall t\in[0,T]$$

that is the differential equation

$$f'(t)=Af(t)-1 \qquad\qquad t\in[0,T]. \tag{13}$$

If $f(t)$ is the solution of (13) with final condition $f(T)=1$ then $F(t,W)=(f(t))^{1-\rho}\frac{W^{\rho}}{\rho}$ verifies condition a)-c) of Theorem 1 and we have $(f(t))^{1-\rho}\frac{W^{\rho}}{\rho}\geq V(t,W)$. The solution of (13) with final condition $f(T)=1$ is

$$f(t)=\begin{cases}1+T-t & \text{if } A=0\\ \\ \frac{1}{A}+e^{-A(T-t)}(1-\frac{1}{A}) & \text{if } A\neq 0\ .\end{cases}$$

Note that $f(t) > 0$ in $[0,T]$ since if $A \neq 0$ we have $f(0) = \frac{1}{A} + e^{-AT}(1 - \frac{1}{A})$, $f(T) = 1$ and $f'(t) = e^{-A(T-t)}(A-1)$. Moreover the policy

$$p^*(t,W) = \begin{cases} \pi^*(s) = \pi^* \\ & s \in [t,T] \\ c^*(s) = \frac{1}{f(s)} \end{cases}$$

is admissible and the corresponding wealth process $W^{\pi^*,c^*}(s)$ in $s \in [t,T]$ is given by

$$\begin{aligned} W^{\pi^*,c^*}(s) = W \exp \Big\{ &(r + \hat{\mu}\pi^*)(s-t) - \textstyle\int_t^s \frac{1}{f(r)} dr - \frac{1}{2}(\pi^*)^2 \left|\sigma\right|^2 (s-t) \\ &+ \pi^* \sigma^T B_{s-t} + \int_t^s \sum_{k=1}^l \int_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) N(dr, dz_k) \Big\} . \end{aligned}$$

By construction $p^*(t,W)$ verifies (6) of Theorem 1, that is it is optimal for the initial condition (t,W). It follows

$$V(t,W) = (f(t))^{1-\rho} \frac{W^\rho}{\rho} = J^{p^*}(t,W) \ .$$

Since

$$\int_t^s \frac{1}{f(r)} dr = \begin{cases} \ln \frac{1+T-t}{1+T-s} & \text{if } A = 0 \\ A(s-t) + \ln \frac{1+(A-1)e^{-A(T-t)}}{1+(A-1)e^{-A(T-s)}} & \text{if } A \neq 0 \end{cases}$$

the optimal wealth process is equal to

$$\begin{aligned} W^{\pi^*,c^*}(s) = \tfrac{f(s)}{f(t)} W \exp \Big\{ &(r + \hat{\mu}\pi^* - A)(s-t) - \tfrac{1}{2}(\pi^*)^2 \left|\sigma\right|^2 (s-t) \\ &+ \pi^* \sigma^T B_{s-t} + \int_t^s \sum_{k=1}^l \int_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) N(dr, dz_k) \Big\} . \end{aligned}$$

■

As in a Merton's model without price jumps the optimal proportion of wealth invested in the risky asset is constant. The optimal consumption rate $c^*(s) = \frac{1}{f(s)}$ is equal to one at T, it is decreasing in $[t,T]$ if $A > 1$, increasing if $A < 1$. If $g(0) \leq 0$, that is if $\hat{\mu} \leq \sum_{k=1}^{l} \int_{\mathbb{R}} (1 - e^{\gamma_k z_k}) v_k(dz_k)$, it follows $\pi^* = 0$. The risk sources coming from the l jump processes are so high that the investor buys only the risk-free asset. We have $A = \frac{\delta - \rho r}{1-\rho}$ and $A = 0$ if $\delta = \rho r$. We obtain $(t \leq s \leq T)$

$$W^{\pi^*,c^*}(s) = \frac{f(s)}{f(t)} W e^{(r-A)(s-t)} \ .$$

If $g(1) \geq 0$, that is if $\hat{\mu} \geq (1-\rho)\left|\sigma\right|^2 + \sum\limits_{k=1}^{l} \int\limits_{\mathbb{R}} (e^{-(1-\rho)\gamma_k z_k} - e^{\rho\gamma_k z_k}) v_k(dz_k)$ it follows $\pi^* = 1$. The risk premium $\hat{\mu}$ is sufficiently high with respect to the price volatility and the risk coming from the jumps that the investor buys only the risky security. We have
$A = \frac{1}{1-\rho}[\delta - \rho(\mu - \frac{1}{2}(1-\rho)\left|\sigma\right|^2) - \sum\limits_{k=1}^{l} \int\limits_{\mathbb{R}} (e^{\rho\gamma_k z_k} - 1) v_k(dz_k)]$
and $A = 0$ if $\delta = \rho(\mu - \frac{1}{2}(1-\rho)\left|\sigma\right|^2) + \sum\limits_{k=1}^{l} \int\limits_{\mathbb{R}} (e^{\rho\gamma_k z_k} - 1) v_k(dz_k)$. We obtain $(t \leq s \leq T)$

$$W^{\pi^*,c^*}(s) = \frac{f(s)}{f(t)} W \exp\left\{ (\mu - A)(s-t) + \sigma^T B_{s-t} + \sum_{k=1}^{l} \sum_{r=1}^{N_k(s-t)} l_{kr} \right\} .$$

If $g(0) > 0$ and $g(1) < 0$, that is if

$$\sum_{k=1}^{l} \int\limits_{\mathbb{R}} (1 - e^{\gamma_k z_k}) v_k(dz_k) < \hat{\mu} < (1-\rho)\left|\sigma\right|^2 + \sum_{k=1}^{l} \int\limits_{\mathbb{R}} (e^{-(1-\rho)\gamma_k z_k} - e^{\rho\gamma_k z_k}) v_k(dz_k)$$

it follows $0 < \pi^* = \bar{\pi} < 1$. This is the most interesting case when the investor buys both securities. To compute the value of $\bar{\pi}$ it is necessary to solve the equation $g(\pi) = 0$, possibly by numerical methods. We have

$$A = \frac{\delta - \rho[r + \hat{\mu}\bar{\pi} - \frac{1}{2}(1-\rho)\left|\sigma\right|^2 (\bar{\pi})^2] - \sum\limits_{k=1}^{l} \int\limits_{\mathbb{R}} ((1 + \bar{\pi}(e^{\gamma_k z_k} - 1))^\rho - 1) v_k(dz_k)}{1-\rho}$$

and $A = 0$ if

$$\delta = \rho[r + \hat{\mu}\bar{\pi} - \frac{1}{2}(1-\rho)\left|\sigma\right|^2 (\bar{\pi})^2] + \sum_{k=1}^{l} \int\limits_{\mathbb{R}} ((1 + \bar{\pi}(e^{\gamma_k z_k} - 1))^\rho - 1) v_k(dz_k) .$$

The optimal wealth process is $(t \leq s \leq T)$

$$W^{\pi^*,c^*}(s) = \tfrac{f(s)}{f(t)} W \exp\{(r + \hat{\mu}\bar{\pi} - A)(s-t)$$
$$\left. -\tfrac{1}{2}(\bar{\pi})^2 \left|\sigma\right|^2 (s-t) + \bar{\pi}\sigma^T B_{s-t} + \int\limits_t^s \sum\limits_{k=1}^{l} \int\limits_{\mathbb{R}} \ln(1 + \bar{\pi}(e^{\gamma_k z_k} - 1)) N(dr, dz_k) \right\} .$$

Apart from the same form for the value function and the optimal policy the values assumed by π^*, A and f will depend on the size distributions l_k, the intensities λ_k of the jumps and the values assumed by the other model parameters r, α, σ, ρ.

An interesting example is when the jumps are normally distributed, that is the Levy measure v_k has density

$$v_k(dz_k) = \lambda_k \frac{1}{\theta_k\sqrt{2\pi}} \exp(-\frac{(z_k - m_k)^2}{2\theta_k^2})dz_k.$$

It holds ($\beta \in \mathbb{R}$)

$$\int_{\mathbb{R}} e^{\beta z_k} v_k(dz_k) = \lambda_k \exp(\frac{\beta^2\theta_k^2}{2} + \beta m_k)$$

and assumption (H) is verified. We have $g(0) > 0$ if

$$\hat{\mu} > \sum_{k=1}^{l} \lambda_k (1 - \exp(\frac{\gamma_k^2\theta_k^2}{2} + \gamma_k m_k)) \tag{14}$$

and $g(1) < 0$ if

$$\begin{aligned} \hat{\mu} \quad &< \quad (1-\rho)\,|\sigma|^2 \\ &+ \sum_{k=1}^{l} \lambda_k [\exp(\frac{(1-\rho)^2\gamma_k^2\theta_k^2}{2} - (1-\rho)\gamma_k m_k) - \exp(\frac{\rho^2\gamma_k^2\theta_k^2}{2} + \rho\gamma_k m_k)]. \end{aligned} \tag{15}$$

If the processes η_k are identically distributed with the jump size a standard normal distribution, that is

$$v_k(dz_k) = \lambda \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \qquad \forall k = 1, ..., l \tag{16}$$

(14) and (15) are simplified to

$$\lambda \sum_{k=1}^{l} (1 - e^{\frac{\gamma_k^2}{2}}) < \hat{\mu} < (1-\rho)\,|\sigma|^2 + \lambda \sum_{k=1}^{l} (e^{\frac{(1-\rho)^2\gamma_k^2}{2}} - e^{\frac{\rho^2\gamma_k^2}{2}}).$$

In the absence of the jumps (or if $\gamma_k = 0$, $\forall k$) the agent buys both securities if $0 < \hat{\mu} < (1-\rho)\,|\sigma|^2$ and the optimal fraction of wealth invested in the stock is given by $\pi_M = \frac{\hat{\mu}}{(1-\rho)|\sigma|^2}$, the Merton proportion. We cannot say a priori if in the presence of the jumps the optimal proportion $\bar{\pi}$ will be greater or less than π_M, it depends on the model parameters. For instance if $\gamma_k = \gamma \neq 0$, $\forall k$ and (16) holds true we have

$$\begin{aligned} g(0) &= \hat{\mu} + l\lambda(e^{\frac{\gamma^2}{2}} - 1) \\ g(1) &= \hat{\mu} - (1-\rho)\,|\sigma|^2 + l\lambda[e^{\frac{\rho^2\gamma^2}{2}}(1 - e^{\frac{\gamma^2(1-2\rho)}{2}})] \ . \end{aligned}$$

Denoting by $g_a(\pi) = \hat{\mu} - (1-\rho)\,|\sigma|^2$ the function g without the jumps it follows $g'(\pi) < g'_a(\pi)$ and for $\frac{1}{2} < \rho < 1$ sufficiently near to $\frac{1}{2}$ we have

$$\begin{aligned} g(0) &> g_a(0) > 0 \\ g_a(1) &< g(1) < 0 \ . \end{aligned}$$

Therefore we obtain $\bar{\pi} > \pi_M$, that is in this case if the agent is sufficiently risk tolerant he/she invests more in the stock with the jumps than without them, for all values of $\gamma \neq 0$. On the contrary increasing the risk aversion for $\rho \to -\infty$ we have

$$\begin{aligned} \lim_{\rho\to-\infty} g(\pi_M) &= \lim_{\rho\to-\infty} l \int_{\mathbb{R}} \frac{e^{\gamma z} - 1}{(1 + \frac{\hat{\mu}(e^{\gamma z}-1)}{|\sigma|^2}\frac{1}{1-\rho})^{(1-\rho)}} v(dz) \qquad (17) \\ &= l \int_{\mathbb{R}} (\frac{e^{\gamma z} - 1}{\exp(\frac{\hat{\mu}}{|\sigma|^2}(e^{\gamma z} - 1))} v(dz) \\ &= \frac{l\lambda}{\sqrt{2\pi}} \int_0^{+\infty} (\frac{e^{\gamma z} - 1}{\exp(\frac{\hat{\mu}}{|\sigma|^2}(e^{\gamma z} - 1))} - \frac{1 - e^{-\gamma z}}{\exp(\frac{\hat{\mu}}{|\sigma|^2}(e^{-\gamma z} - 1))}) e^{-\frac{z^2}{2}} dz \ . \end{aligned}$$

If for instance $\frac{\hat{\mu}}{|\sigma|^2} \geq \frac{1}{2}$ it follows

$$\frac{e^{\gamma z} - 1}{\exp(\frac{\hat{\mu}}{|\sigma|^2}(e^{\gamma z} - 1))} - \frac{1 - e^{-\gamma z}}{\exp(\frac{\hat{\mu}}{|\sigma|^2}(e^{-\gamma z} - 1))} < 0 \qquad \forall z \neq 0$$

and the integral in (17) is certainly negative. When ρ has a sufficiently large negative value it follows $\bar{\pi} < \pi_M$ because $g'(\pi) < 0$. Therefore in this case if $\frac{\hat{\mu}}{|\sigma|^2} \geq \frac{1}{2}$ and the agent is sufficiently risk averse he/she invests less in the stock in the presence of the jumps than without them, for all values of $\gamma \neq 0$.

4 Two assets and logarithmic utility

Now we consider

$$U_1(x) = U_2(x) = \ln x$$

and by the same reasoning of the previous section we obtain a theorem equivalent to Theorem 2. We define $g:[0,1]\to\mathbb{R}$

$$g(\pi)=\hat{\mu}-|\sigma|^2\pi+\sum_{k=1}^{l}\int_{\mathbb{R}}\frac{e^{\gamma_k z_k}-1}{1+\pi(e^{\gamma_k z_k}-1)}v_k(dz_k)$$

which corresponds to set $\rho=0$ in (9). We have

$$\begin{aligned}
g(0) &= \hat{\mu}+\sum_{k=1}^{l}\int_{\mathbb{R}}(e^{\gamma_k z_k}-1)v_k(dz_k)\\
g(1) &= \hat{\mu}-|\sigma|^2+\sum_{k=1}^{l}\int_{\mathbb{R}}(1-e^{-\gamma_k z_k})v_k(dz_k)\\
g'(\pi) &= -|\sigma|^2-\sum_{k=1}^{l}\int_{\mathbb{R}}\left(\frac{e^{\gamma_k z_k}-1}{1+\pi(e^{\gamma_k z_k}-1)}\right)^2 v_k(dz_k)<0
\end{aligned}$$

and we set

$$\pi^*=\begin{cases}0 & \text{if } g(0)\le 0\\ \bar{\pi} & \text{if } g(0)>0\wedge g(1)<0\\ 1 & \text{if } g(1)\ge 0\end{cases}$$

where $\bar{\pi}\in(0,1)$ is the only solution of $g(\pi)=0$ when $g(0)>0$ and $g(1)<0$. We also define

$$A=\delta-r-\hat{\mu}\pi^*+\frac{1}{2}|\sigma|^2(\pi^*)^2-\sum_{k=1}^{l}\int_{\mathbb{R}}\ln(1+\pi^*(e^{\gamma_k z_k}-1))v_k(dz_k).$$

Theorem 3 *The value function assuming $U_1(x)=U_1(x)=\ln x$ is*

$$V(t,W)=p(t)\ln\frac{W}{p(t)}+q(t)\quad (t,W)\in\bar{Q}$$

where

$$\begin{aligned}
p(t) &= \frac{1}{\delta}(1-e^{-\delta(T-t)})+e^{-\delta(T-t)}\\
q(t) &= -A\left[\frac{1}{\delta^2}(1-e^{-\delta(T-t)})+(T-t)e^{-\delta(T-t)}(1-\frac{1}{\delta})\right].
\end{aligned}$$

The optimal policy $p^*(t,W)$ corresponding to the initial condition (t,W) is

$$p^*(t,W) = \begin{cases} \pi^*(s) = \pi^* & \\ & s \in [t,T] \\ c^*(s) = \frac{1}{p(s)} & \end{cases}$$

and the optimal wealth $W^{\pi^*,c^*}(s)$ is given by the process, for $s \in [t,T]$

$$W^{\pi^*,c^*}(s) = \tfrac{p(s)}{p(t)} W \exp\Big\{ (r + \hat{\mu}\pi^* - \delta)(s-t) - \tfrac{1}{2}(\pi^*)^2 |\sigma|^2 (s-t) \\ + \pi^* \sigma^T B_{s-t} + \sum_{k=1}^{l} \int_t^s \int_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) N(dr, dz_k) \Big\} .$$

Proof. We assume V of the form

$$\begin{aligned} F(t,W) &= p(t) \ln \frac{W}{p(t)} + q(t) \quad \text{with} \\ p(t), q(t) &\in C^1[0,T],\ p(t) > 0,\ p(T) = 1,\ q(T) = 0 \ . \end{aligned}$$

$F \in C(\bar{Q}) \cap C^{1,2}(Q)$ and it satisfies condition b) of Theorem 1. It holds $|F(t,W)| \leq C(1 + \frac{1}{W} + W)$ in $\bar{Q}$ for suitable constant C and condition a) of Theorem 1 can be replaced by this condition. We have

$$\begin{aligned} \frac{\partial F}{\partial t} &= p'(t) \left(\ln \frac{W}{p(t)} - 1 \right) + q'(t) \\ \frac{\partial F}{\partial W} &= \frac{p(t)}{W} > 0, \quad \frac{\partial^2 F}{\partial W^2} = -\frac{p(t)}{W^2} < 0. \end{aligned}$$

Inserting the values of F and its derivatives in

$$\begin{aligned} G(\pi,c) \equiv\ & -\delta F + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial W}(r + \hat{\mu}\pi - c)W + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} W^2 |\sigma|^2 \pi^2 \\ & + \sum_{k=1}^{l} \int_{\mathbb{R}} [F(t, W(1 + \pi(e^{\gamma_k z_k} - 1)) - F(t,W)] v_k(dz_k) + \ln(cW) \end{aligned}$$

we obtain

$$\begin{aligned} G(\pi,c) &= -\delta p(t) \ln \frac{W}{p(t)} - \delta q(t) + p'(t) \ln \frac{W}{p(t)} - p'(t) \\ & + q'(t) + p(t)(r + \hat{\mu}\pi - c) - \frac{1}{2} p(t) |\sigma|^2 \pi^2 \\ & + p(t) \sum_{k=1}^{l} \int_{\mathbb{R}} \ln(1 + \pi(e^{\gamma_k z_k} - 1)) v_k(dz_k) + \ln(cW) \ . \end{aligned}$$

Considering that

$$\frac{\partial G}{\partial \pi} = 0 \Longleftrightarrow g(\pi) = 0 \qquad \frac{\partial G}{\partial c} = 0 \Longleftrightarrow c = \frac{1}{p(t)}$$

$$\frac{\partial^2 G}{\partial \pi^2} = p(t)g'(\pi) < 0, \ \frac{\partial^2 G}{\partial c^2} = -\frac{1}{W^2} < 0, \ \frac{\partial^2 G}{\partial \pi \partial c} = 0$$

we see that the maximum of G in $[0,1] \times [0,\infty)$ is achieved at the single point

$$\arg\max G(\pi, c) = (\pi^*, c^*) = \begin{cases} (0, \frac{1}{p(t)}) & \text{if } g(0) \leq 0 \\ (\bar{\pi}, \frac{1}{p(t)}) & \text{if } g(0) > 0 \wedge g(1) < 0 \\ (1, \frac{1}{p(t)}) & \text{if } g(1) \geq 0 \ . \end{cases} .$$

Injecting (π^*, c^*) into $G(\pi, c)$ we obtain

$$\begin{aligned} G(\pi^*, c^*) &= [-\delta p(t) + p'(t) + 1] \ln \frac{W}{p(t)} - \delta q(t) - p'(t) \\ &\quad + q'(t) + p(t)(r + \hat{\mu}\pi^*) - 1 - \frac{1}{2} p(t) \left|\sigma\right|^2 (\pi^*)^2 \\ &\quad + p(t) \sum_{k=1}^{l} \int_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) v_k(dz_k) \ . \end{aligned}$$

The optimality condition $G(\pi^*, c^*) = 0$, $\forall (t, W) \in Q$, implies

$$\begin{cases} -\delta p(t) + p'(t) + 1 = 0 \\ -\delta q(t) - p'(t) + q'(t) + p(t)(r + \hat{\mu}\pi^*) - 1 - \frac{1}{2} p(t) \left|\sigma\right|^2 (\pi^*)^2 \\ + p(t) \sum\limits_{k=1}^{l} \int\limits_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) v_k(dz_k) = 0 \ . \end{cases} \tag{18}$$

The solution in $[0, T]$ of $-\delta p(t) + p'(t) + 1 = 0$ with $p(T) = 1$ is

$$p(t) = \frac{1}{\delta}(1 - e^{-\delta(T-t)}) + e^{-\delta(T-t)} > 0 \ .$$

Setting

$$A = \delta - r - \hat{\mu}\pi^* + \frac{1}{2} \left|\sigma\right|^2 (\pi^*)^2 - \sum_{k=1}^{l} \int_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) v_k(dz_k)$$

the second equation in (18) can be written

$$q'(t) = \delta q(t) + Ap(t) \; . \tag{19}$$

Since

$$\int e^{-\delta t} p(t) dt = -\frac{1}{\delta^2} e^{-\delta t} + t e^{-\delta T}(1 - \frac{1}{\delta}) + C$$

the solution of (19) with final condition $q(T) = 0$ is

$$q(t) = -A \left[\frac{1}{\delta^2}(1 - e^{-\delta(T-t)}) + (T-t)e^{-\delta(T-t)}(1 - \frac{1}{\delta}) \right] .$$

Moreover the policy

$$p^*(t, W) = \begin{cases} \pi^*(s) = \pi^* \\ \\ c^*(s) = \frac{1}{p(s)} \end{cases} \quad s \in [t, T]$$

is admissible and since by construction it verifies (6) of Theorem 1, it is optimal. Therefore

$$V(t, W) = p(t) \ln \frac{W}{p(t)} + q(t) = J^{p^*}(t, W) \; .$$

Given that

$$\int_t^s \frac{1}{p(r)} dr = \delta(s-t) + \ln \frac{p(t)}{p(s)}$$

the optimal wealth process is equal to

$$\begin{aligned} W^{\pi^*, c^*}(s) = \tfrac{p(s)}{p(t)} W \exp \Big\{ & (r + \hat{\mu}\pi^* - \delta)(s-t) - \tfrac{1}{2}(\pi^*)^2 \left|\sigma\right|^2 (s-t) \\ & + \pi^* \sigma^T B_{s-t} + \int\limits_t^s \sum_{k=1}^{l} \int\limits_{\mathbb{R}} \ln(1 + \pi^*(e^{\gamma_k z_k} - 1)) N(dr, dz_k) \Big\} . \end{aligned}$$

■

With logarithmic utility the consumption rate $c(t) = \frac{1}{p(t)}$ depends only on δ and it is increasing for realistic values $0 < \delta < 1$. Considered in $[0,1] \times \mathbb{R}$ the function

$$g(\pi, \rho) = \hat{\mu} - (1-\rho) \left|\sigma\right|^2 \pi + \sum_{k=1}^{l} \int\limits_{\mathbb{R}} (1 + \pi(e^{\gamma_k z_k} - 1))^{-(1-\rho)} (e^{\gamma_k z_k} - 1) v_k(dz_k)$$

verifies $\frac{\partial g}{\partial \pi} < 0$, $\frac{\partial g}{\partial \rho} > 0$ and thus the optimal proportion $\pi^*(\rho)$, defined by $g(\pi, \rho) = 0$, is a decreasing function of the risk aversion parameter $1 - \rho$.

Therefore the logarithmic utility optimal proportion $\pi^*(0)$ is greater than the power utility proportion $\pi^*(\rho)$ if $\rho < 0$, smaller if $0 < \rho < 1$. The optimal policy and wealth with logarithmic utility depend on the signs of $g(0)$ and $g(1)$ with the same interpretation, in terms of jump risk and model parameters, to that given for power utility.

5 CRRA utility and many assets

From a theoretical standpoint the extension of the model to n risky assets is not difficult. We still look for a value function of the form $F(t,W) = (f(t))^{1-\rho}\frac{W^\rho}{\rho}$ in case of power utility and of the form $F(t,W) = p(t)\ln\frac{W}{p(t)} + q(t)$ in case of logarithmic utility. If we consider $F(t,W) = (f(t))^{1-\rho}\frac{W^\rho}{\rho}$ the function $G(\pi,c) : \Pi \times (0,\infty)$, becomes

$$\begin{aligned} G(\pi,c) &= -\delta(f(t))^{1-\rho}\frac{W^\rho}{\rho} + (1-\rho)(f(t))^{-\rho}f'(t)\frac{W^\rho}{\rho} \\ &\quad + (f(t))^{1-\rho}W^{-(1-\rho)}(r + \hat{\mu}\pi - c)W \\ &\quad - \frac{1}{2}(1-\rho)(f(t))^{1-\rho}W^\rho\pi^T\sigma\sigma^T\pi \\ &\quad + (f(t))^{1-\rho}\frac{W^\rho}{\rho}\sum_{k=1}^{l}\int_{\mathbb{R}}[(1+\pi^T(e_k^{\gamma z} - \mathbf{1}))^\rho - 1]v_k(dz_k) + \frac{(cW)^\rho}{\rho} . \end{aligned} \tag{20}$$

We have $(i,j = 1,...,n)$

$$\left|\begin{aligned} &\frac{\partial G}{\partial \pi_i}(\pi,c) = f(t)^{1-\rho}W^\rho \times \Big\{\hat{\mu}_i - (1-\rho)\sigma_i^T\sigma^T\pi \\ &\qquad\qquad + \sum_{k=1}^{l}\int_{\mathbb{R}}(1+\pi^T(e_k^{\gamma z} - \mathbf{1}))^{-(1-\rho)}(e^{\gamma_{ik}z_k} - 1)v_k(dz_k)\Big\} \\ &\frac{\partial^2 G}{\partial \pi_i \partial \pi_j}(\pi,c) = -f(t)^{1-\rho}W^\rho(1-\rho) \times \{(\sigma_i,\sigma_j) \\ &\qquad + \sum_{k=1}^{l}\int_{\mathbb{R}}(1+\pi^T(e_k^{\gamma z} - \mathbf{1}))^{-(2-\rho)}(e^{\gamma_{ik}z_k} - 1)(e^{\gamma_{jk}z_k} - 1)v_k(dz_k)\Big\} \\ &\frac{\partial G}{\partial c}(\pi,c) = -W^\rho f(t)^{1-\rho} + W^\rho c^{-(1-\rho)} \\ &\frac{\partial^2 G}{\partial c^2}(\pi,c) = -W^\rho(1-\rho)c^{-(2-\rho)}, \qquad \frac{\partial^2 G}{\partial \pi_i \partial c}(\pi,c) = 0 \end{aligned}\right.$$

We can prove that $G(\pi,c)$ is strictly concave in its domain and therefore that there exists a single maximum (π^*,c^*) of G in $\Pi \times (0,\infty)$. Since we have

$\frac{\partial^2 G}{\partial c^2}(\pi, c) < 0$ and $\frac{\partial^2 G}{\partial \pi_i \partial c}(\pi, c) = 0$ it is sufficient to show that the Hessian matrix of G, considering only the variables π_i, $H_{G(\pi)}$, is positive-definite. We have

$$H_{G(\pi)} = -f(t)^{1-\rho} W^\rho (1-\rho) \times \left\{ \sigma\sigma^T + \sum_{k=1}^{l} \int_{\mathbb{R}} M_k \right\}$$

where the matrix M_k is given by

$$M_k(i,j) = (1 + \pi^T (e_k^{\gamma z} - \mathbf{1}))^{-(2-\rho)} (e^{\gamma_{ik} z_k} - 1)(e^{\gamma_{jk} z_k} - 1) \quad i, j = 1, ..., n.$$

By the linearity property of integrals and since $f(t)^{1-\rho} W^\rho (1-\rho) > 0$ and $\sigma\sigma^T$ is positive-definite it is sufficient to prove that every M_k is positive semi-definite $(k = 1, ..., l)$. This holds true because

$$x^T M_k x = (1 + \pi^T (e_k^{\gamma z} - \mathbf{1}))^{-(2-\rho)} \left(\sum_{i=1}^{n} x_i (e^{\gamma_{ik} z_k} - 1) \right)^2 \geq 0, \qquad \forall x \in \mathbb{R}^n.$$

The optimal c is always $c^* = \frac{1}{f(t)}$ where $\frac{\partial G}{\partial c}(\pi, c^*) = 0$. Inserting (π^*, c^*) in (20) we can repeat the proof of Theorem 2 Now we have

$$A = \frac{\delta - \rho[r + \hat{\mu}^T \pi^* - \frac{1}{2}(1-\rho)\pi^{*T} \sigma\sigma^T \pi^*] - \sum_{k=1}^{l} \int_{\mathbb{R}} ((1 + \pi^{*T} (e_k^{\gamma z} - \mathbf{1}))^\rho - 1) v_k(dz_k)}{1-\rho}$$

and the statement of Theorem 2 remains the same with many assets if we consider $\hat{\mu}, \pi^* \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n \times d}$ and we replace $(\pi^*)^2 |\sigma|^2$ with $\pi^{*T} \sigma\sigma^T \pi^*$ and $1 + \pi^*(e^{\gamma_k z_k} - 1)$ with $1 + \pi^{*T}(e_k^{\gamma z} - \mathbf{1})$. To compute the optimal proportions π^* it is possible to use the Kuhn-Tucker necessary conditions which become sufficient by the strict concavity of $G(\pi, c)$. The Lagrangian function is

$$L(\lambda, \pi, c) = G(\pi, c) + \lambda \left(1 - \sum_{i=1}^{n} \pi_i \right)$$

and the Kuhn-Tucker conditions are $(i = 1, ..., n)$

$$\left| \begin{array}{l} \frac{\partial G}{\partial \pi_i}(\pi, c) \leq \lambda, \quad \left(\frac{\partial G}{\partial \pi_i}(\pi, c) - \lambda \right) \pi_i = 0, \; c \frac{\partial G}{\partial c}(\pi, c) = 0 \\ \pi_i \geq 0, \quad \left(1 - \sum_{i=1}^{n} \pi_i \right) \geq 0, \quad c \geq 0 \\ \lambda \geq 0, \quad \lambda \left(1 - \sum_{i=1}^{n} \pi_i \right) = 0. \end{array} \right.$$

The agent invests only in the risk-free asset if and only if ($\mathbf{0} \in \mathbb{R}^n$ the all-zeros vector)

$$\frac{\partial G}{\partial \pi_i}(\mathbf{0}, c^*) \leq 0 \qquad \forall i = 1, ..., n \ .$$

In fact the vector $[\lambda, \pi^T, c] = [0, \mathbf{0}^T, c^*]$ verifies the Kuhn-Tucker conditions. Given that

$$\frac{\partial G}{\partial \pi_i}(\mathbf{0}, c^*) = f(t)^{1-\rho} W^\rho \times (\hat{\mu}_i + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{\gamma_{ik} z_k} - 1) v_k(dz_k)$$

it must hold

$$\hat{\mu}_i \leq \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{\gamma_{ik} z_k} - 1) v_k(dz_k) \quad \forall i = 1, ..., n \ .$$

A sufficient condition for investing only in the risky securities is that there exists an asset k such that

$$\frac{\partial G}{\partial \pi_k}(\pi, c) \geq 0 \qquad \forall \pi : \sum_{i=1}^{n} \pi_i = 1 \ .$$

Indeed suppose $\bar{\pi}$ is optimal with $\sum_{i=1}^{n} \bar{\pi}_i < 1$, then $\bar{\pi}_k < 1$. Setting

$$\hat{\pi}^T = [\bar{\pi}_1, ..., \bar{\pi}_{k-1}, \ \bar{\pi}_k + 1 - \sum_{i=1}^{n} \bar{\pi}_i \ , \bar{\pi}_{k+1}, ..., \bar{\pi}_n]$$

we have $\frac{\partial G}{\partial \pi_k}(\hat{\boldsymbol{\pi}}, c) \geq 0$. Since $\frac{\partial^2 G}{\partial \pi_k^2}(\pi, c) < 0$, $\forall \pi \in \Pi$ it follows that $\frac{\partial G}{\partial \pi_k}(\bar{\pi}, c) > 0$ and $\bar{\pi}$ cannot be optimal. By the Kuhn-Tucker conditions a portfolio π^* made only of risky securities, $\sum_{i=1}^{n} \pi_i^* = 1$, with $0 < \pi_i < 1$, $\forall i = 1, ..., n$, is optimal if and only if (C a constant)

$$\frac{\partial G}{\partial \pi_i}(\pi^*, c) = C \geq 0 \quad \forall i = 1, ..., n \ .$$

If there exists k such that $\pi_k = 1$ the necessary and sufficient condition becomes

$$\frac{\partial G}{\partial \pi_k}(\pi^*, c) \geq 0 \text{ and } \frac{\partial G}{\partial \pi_k}(\pi^*, c) \geq \frac{\partial G}{\partial \pi_i}(\pi^*, c) \ \ \forall i \neq k \ .$$

If $U(x) = \ln x$ and we consider $F(t, W) = p(t) \ln \frac{W}{p(t)} + q(t)$ the function $G(\pi, c) : \Pi \times (0, \infty)$ is

$$\begin{aligned} G(\pi, c) &= -\delta p(t) \ln \frac{W}{p(t)} - \delta q(t) + p'(t) \ln \frac{W}{p(t)} - p'(t) + q'(t) \\ &\quad + p(t)(r + \hat{\mu}^T \pi - c) - \frac{1}{2} p(t) \pi^T \sigma \sigma^T \pi \\ &\quad + p(t) \sum_{k=1}^{l} \int_{\mathbb{R}} \ln(1 + \pi^T (e_k^{\gamma z} - \mathbf{1})) v_k(dz_k) + \ln(cW) \end{aligned}$$

with

$$\left| \begin{array}{l} \frac{\partial G}{\partial \pi_i}(\pi, c) = p(t) \times \left\{ \hat{\mu}_i - \sigma_i^T \sigma^T \pi + \sum_{k=1}^{l} \int_{\mathbb{R}} \frac{e^{\gamma_{ik} z_k} - 1}{1 + \pi^T (e_k^{\gamma z} - \mathbf{1})} v_k(dz_k) \right\} \\ \\ \frac{\partial^2 G}{\partial \pi_i \partial \pi_j}(\pi, c) = -p(t) \times \left\{ (\sigma_i, \sigma_j) + \sum_{k=1}^{l} \int_{\mathbb{R}} \frac{(e^{\gamma_{ik} z_k} - 1)(e^{\gamma_{jk} z_k} - 1)}{(1 + \pi^T (e_k^{\gamma z} - \mathbf{1}))^2} v_k(dz_k) \right\} \\ \\ \frac{\partial G}{\partial c}(\pi, c) = -p(t) + \frac{1}{c}, \quad \frac{\partial^2 G}{\partial c^2}(\pi, c) = -\frac{1}{c^2}, \quad \frac{\partial^2 G}{\partial \pi_i \partial c}(\pi, c) = 0 \ . \end{array} \right.$$

As before we can show that the function $G(\pi, c)$ is strictly concave; there is a single optimal (π^*, c^*) with $c^* = \frac{1}{p(t)}$. Setting

$$A = \delta - r - \hat{\mu} \pi^* + \frac{1}{2} \pi^{*T} \sigma \sigma^T \pi^* - \sum_{k=1}^{l} \int_{\mathbb{R}} \ln(1 + \pi^{*T} (e_k^{\gamma z} - \mathbf{1})) v_k(dz_k)$$

the statement of Theorem 3 remains the same with many assets replacing $(\pi^*)^2 |\sigma|^2$ with $\pi^{*T} \sigma \sigma^T \pi^*$ and $1 + \pi^*(e^{\gamma_k z_k} - 1)$ with $1 + \pi^{*T}(e_k^{\gamma z} - \mathbf{1})$.

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