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Isometries between leaf spaces

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Abstract In this paper we prove that an isometry between orbit spaces of two proper isometric actions is smooth if it preserves the codimension of the orbits or if the orbit spaces have no boundary. In other words, we generalize Myers–Steenrod's theorem for orbit spaces. These results are proved in the more general context of singular Riemannian foliations.

Keywords Singular Riemannian foliations · Myers–Steenrod's theorem · Orbit spaces

Mathematics Subject Classification (2010) Primary 53C12 · Secondary 57R30

1 Introduction

Given a Riemannian manifold M on which a compact Lie group G acts by isometries, the quotient M/G is in general not a manifold. Nevertheless, the canonical projection $\pi : M \to M/G$ gives M/G the structure of a Hausdorff metric space. Moreover, following Schwarz [11] one can define a "smooth structure" on M/G to be the $\mathbb{R}$-algebra $C^\infty(M/G)$ consisting of functions $f : M/G \to \mathbb{R}$ whose pullback $\pi^* f$ is a smooth, G-invariant function on M. If M/G is a manifold, the smooth structure defined here corresponds to the more familiar notion of smooth structure. A map $F : M/G \to M'/G'$ is called *smooth* if the pull-back of a smooth function $f \in C^\infty(M'/G')$ is a smooth function on M/G.

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These concepts can actually be formulated in the wider context of singular Riemannian foliations. A singular foliation $\mathcal{F}$ on M is called *singular Riemannian foliation* (SRF for short) if every geodesic perpendicular to one leaf is perpendicular to every leaf it meets, see [9, page 189].

A typical example of a singular Riemannian foliation is the partition of a Riemannian manifold into the connected components of the orbits of an isometric action. Such singular Riemannian foliations are called *Riemannian homogeneous*.

Given $(M, \mathcal{F})$, one can define a quotient $M/\mathcal{F}$, also called *leaf space*. If the leaves of $\mathcal{F}$ are closed, $M/\mathcal{F}$ can again be endowed with a metric structure and a smooth structure, exactly as in the case of group actions.

When dealing with Riemannian manifolds, a theorem of Myers and Steenrod states that the metric structure of a Riemannian manifold uniquely determines its smooth structure. In the same way, one can ask whether the metric structure on a quotient M/G or $M/\mathcal{F}$ uniquely determines its smooth structure in the sense described above. This question can be restated in the following way: given an isometry

$$F : M/\mathcal{F} \to M'/\mathcal{F}'$$

between the quotients of two Riemannian manifolds, is F smooth?

Classic theorems, like the Chevalley Restriction Theorem [3] and the Luna–Richardson Theorem [8] give a positive answer when $\mathcal{F}, \mathcal{F}'$ come from some special group actions. Recently, Alexander Lytchak and the first named author generalized the results above, answering the question in the positive for special foliations $\mathcal{F}, \mathcal{F}'$ (namely *infinitesimally polar* foliations, cf. [1]). Nevertheless, a general answer to this question is not known, even for isometric group actions.

In the present paper we provide a new sufficient condition for an isometry to be smooth.

Theorem 1.1 *Let M_1 and M_2 be complete Riemannian manifolds and $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ be singular Riemannian foliations with closed leaves. Assume that there exists an isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ that preserves the codimension of the leaves. Then φ is a smooth map.*

Remark 1.2 Notice that not every isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$, which preserves the codimension of the leaves, lifts to a foliated diffeomorphism $M_1 \to M_2$. This fact can be illustrated with examples constructed via a procedure called *suspension of homomorphism*, see e.g. [9, Sect. 3.7]. Also notice that in [5] the authors produce arbitrary numbers of pairwise non isometric foliations $(V_i, \mathcal{F}_i)$ on vector spaces of the same dimension, having isometric 2-dimensional leaf spaces and the same codimension of the leaves.

Remark 1.3 The above theorem implies that if $M_i/\mathcal{F}_i$ are isometric orbifolds, then they are diffeomorphic in the sense of Schwarz and hence in the classical sense, see e.g., Strub [12] and Swartz [13, Lemma 1].

In the special case of leaf spaces without boundary (see Definition in Sect. 2), a small modification in the proof of Theorem 1.1 allow us to prove the next result; see Remark 3.2.

Theorem 1.4 *Let $(M_i, \mathcal{F}_i)$, $i = 1, 2$, be singular Riemannian foliations with closed leaves, and $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ be an isometry. If $M_1/\mathcal{F}_1$ has no boundary, then φ is smooth.*

As an immediate corollary of Theorem 1.1, we obtain the following

Corollary 1.5 *Let $(M, \mathcal{F})$ be a singular Riemannian foliation with closed leaves and $\varphi : M/\mathcal{F} \times (-\epsilon, \epsilon) \to M/\mathcal{F}$ a continuous family of isometries $\varphi_t : M/\mathcal{F} \to M/\mathcal{F}$ such that $\varphi_0 = \mathrm{id}_{M/\mathcal{F}}$. Then each φ_t is smooth.*

Remark 1.6 Flows of isometries on the leaf spaces of foliations appear naturally in the study of the dynamical behavior of *non closed* singular Riemannian foliations. Recall that a (locally closed) singular Riemannian foliation $(M, \mathcal{F})$ is locally described by submetries $\pi_\alpha : U_\alpha \to U_\alpha/\mathcal{F}_\alpha$, where $\{U_\alpha\}$ is an open cover of M and $\mathcal{F}_\alpha$ denotes the restriction of $\mathcal{F}$ to U_α. If a leaf L is not closed, one might be interested to understand how it intersects a given neighborhood U_α, and in particular how the closure $\overline{L}$ of L intersects U_α. In the regular case, the local quotient $U_\alpha/\mathcal{F}_\alpha$ is a manifold, and it turns out (cf. [9, Thm 5.2]) that the projection $\pi_\alpha(\overline{L} \cap U_\alpha)$ is a submanifold, which is spanned by flows of isometries φ_α on $U_\alpha/\mathcal{F}_\alpha$. As one tries to generalize this result to singular Riemannian foliations, the main difficulty is that the local quotient $U_\alpha/\mathcal{F}_\alpha$ is no longer a manifold. In particular, when studying the smoothness of the flows of isometries φ_α (which still exist) one cannot rely on classical theorems anymore, hence the need to develop new techniques to deal with these more general situations. Corollary 1.5 is a first result in this direction. Other results on this topic are the center of a forthcoming paper.

We end this introduction sketching the strategy of the proof of Theorem 1.1. This is divided into three steps. First, we linearize the problem and reduce it to the case of foliations $(\mathbb{R}^n, \mathcal{F}_i)$, $i = 1, 2$, with an isometry $\varphi : \mathbb{R}^n/\mathcal{F}_1 \to \mathbb{R}^n/\mathcal{F}_2$ preserving the dimension of the leaves. Secondly, we prove (Proposition 3.1) that in the Euclidean case, the mean curvature vector fields H_i of the regular leaves in $\mathcal{F}_i$ are *basic*, i.e., can be projected to vector fields H_{i*} in $\mathbb{R}^n/\mathcal{F}_i$, and φ takes H_{1*} to H_{2*}. Finally, in the last step we prove (Proposition 3.5) the smoothness of φ. The idea behind the proof of this proposition is to check that for any smooth basic function $f \in C^\infty(\mathbb{R}^n, \mathcal{F}_2)$, the pull-back $\varphi^* f \in C^0(\mathbb{R}^n, \mathcal{F}_1)$ satisfies, in a weak sense, the elliptic equation

$$\triangle \varphi^* f = \varphi^* \triangle f. \tag{1}$$

This is checked using the second step, i.e., the fact that the projections of the mean curvature vector fields are preserved by φ. Then, since φ is C^1 (see Proposition 3.3), we conclude via a bootstrap type argument that $\varphi^* f$ is smooth, which in turns proves the smoothness of φ.

2 Preliminaries

2.1 The leaf space

Let $(M, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. The foliation induces an equivalence relation $\sim$ on M, where $p \sim q$ if and only if p, q lie in the same leaf. The quotient $M/\sim$ is called *leaf space* of $(M, \mathcal{F})$ and is denoted by $M/\mathcal{F}$. The canonical map $\pi : M \to M/\mathcal{F}$ gives $M/\mathcal{F}$ the structure of a Hausdorff metric space, where the distance between two points is given by the distance between the corresponding leaves. Also recall that the image of a *stratum* Σ (the set of leaves with the same dimension) is an orbifold of dimension $\dim \pi(\Sigma) = \dim \Sigma - \dim \mathcal{F}|_\Sigma$. If M_{reg} denotes the *regular stratum* (the set of leaves with maximal dimension), the *quotient codimension* of Σ is

$$qcodim(\Sigma) = \dim \pi(M_{reg}) - \dim \pi(\Sigma) = \dim M - \dim \mathcal{F} - \dim \Sigma + \dim \mathcal{F}|_\Sigma.$$

To say that $M/\mathcal{F}$ *has no boundary* is equivalent to requiring that $qcodim(\Sigma) > 1$ for every singular stratum.

The metric space $M/\mathcal{F}$ has a natural smooth structure. More precisely, one can define the ring $C^\infty(M/\mathcal{F})$ of *smooth functions* on $M/\mathcal{F}$ to be the ring of functions $f : M/\mathcal{F} \to \mathbb{R}$

whose pullback $\pi^* f$ is a smooth function on M. Notice that by contruction $\pi^* f$ is *basic*, i.e., it is constant along the leaves of $\mathcal{F}$.

A map $F : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ is said to be *smooth* if for every smooth function $f \in C^\infty(M_2/\mathcal{F}_2)$ (in the sense defined above) the pullback $F^* f$ is again a smooth function in $C^\infty(M_1/\mathcal{F}_1)$. A smooth map F is a *diffeomorphism* if it is a bijection, and F^{-1} is smooth as well. By definition, the canonical projection $\pi : M \to M/\mathcal{F}$ is smooth and a submetry. Moreover, when restricted to the regular part $M_{reg} \to M_{reg}/\mathcal{F}$ it is a Riemannian submersion.

Given a point $p \in M$ or a vector $x \in T_p M$, we will denote with p_*, x_* the projections $\pi(p), \pi_*(x)$ respectively.

2.2 Non connected foliations

In this paper, we will have to consider Riemannian foliations with non connected leaves. These kind of foliations comes up naturally: consider for example a Riemannian homogeneous foliation (M, G). Even if G itself is connected, some isotropy subgroup might not be, and the orbits of G_p under the slice representation might also be disconnected. Therefore the Riemannian homogeneous foliation $(\nu_p M, G_p)$ would be an example of a disconnected singular Riemannian foliation. In general, a *singular Riemannian foliation with disconnected leaves* is a triple $(M, \mathcal{F}^0, \mathsf{K})$ where $(M, \mathcal{F}^0)$ is a (usual) SRF, K is a discrete group of isometries of $M/\mathcal{F}^0$, and the *non-connected leaves* are just the orbits $\mathsf{K} \cdot L_p$, for $L_p \in \mathcal{F}^0$. By letting $\mathcal{F}$ denote the partition of M into the disconnected leaves, we will sometimes refer to $(M, \mathcal{F})$ instead of $(M, \mathcal{F}^0, \mathsf{K})$.

A leaf L of a disconnected foliation $\mathcal{F}$ is called a *principal leaf* if it satisfies the following conditions:

(1) each connected component of L is a *principal leaf* of $\mathcal{F}^0$, i.e., a *regular leaf* (a leaf with maximal dimension) that has a trivial holonomy; see e.g [9, page 22].
(2) If there exists an isometry $k \in \mathsf{K}$ which fixes any component of L in $M/\mathcal{F}^0$, k is the identity.

2.3 Infinitesimal foliation

Let $(M, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. Given a point $p \in M$, let $V_p^\perp = \nu_p L_p$, and for some $\epsilon > 0$, let $S_p = \exp_p(V_p^\perp) \cap B_\epsilon(p)$ be a *slice* through p, where $B_\epsilon(p)$ is the distance ball of radius ϵ around p. In the definition of S_p, we assume ϵ to be small enough so that S_p does not contain any focal point of L_p. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}|_{S_p}^0$ on S_p by letting the leaves of $\mathcal{F}|_{S_p}^0$ be the connected components of the intersection between S_p and the leaves of $\mathcal{F}$. In general, the foliation $(S_p, \mathcal{F}|_{S_p}^0)$ is not a singular Riemannian foliation with respect to the induced metric on S_p. Nevertheless, the *pull-back* foliation $\exp_p^*(\mathcal{F}^0)$ is a singular Riemannian foliation on $V_p^\perp \cap B_\epsilon(0)$ equipped with the Euclidean metric (cf. [9, Proposition 6.5]), and it is invariant under homotheties fixing the origin (cf. [9, Lemma 6.2]). In particular, it is possible to extend $\exp_p^*(\mathcal{F}^0)$ to all of $V_p^\perp$, giving rise to a singular Riemannian foliation $(V_p^\perp, \mathcal{F}_p^0)$ called the *infinitesimal foliation* of $\mathcal{F}$ at p. The fundamental group $\pi_1(L_p)$ acts on $V_p^\perp/\mathcal{F}_p^0$ by *holonomy maps* in such a way that it induces a disconnected foliation $(V_p^\perp, \mathcal{F}_p) = (M, \mathcal{F}_p^0, \pi_1(L_p))$. Via the exponential map, the leaves of $\mathcal{F}_p$ correspond to the intersections of the leaves of $\mathcal{F}$ with S_p (i.e., we no longer restrict to the connected components), and in particular the exponential map $\exp_p : V_p^\perp \cap B_\epsilon(0) \to S_p \subseteq M$ defines a diffeomorphism $\exp_*$ between $(V_p^\perp \cap B_\epsilon(0))/\mathcal{F}_p$ and a neighborhood of $p_* = \pi(p)$ in $M/\mathcal{F}$.

If (M, G) is Riemannian homogeneous foliation, the infinitesimal foliation $(V_p^\perp, \mathcal{F}_p^0)$ (respectively the disconnected infinitesimal foliation $(V_p^\perp, \mathcal{F}_p)$) is again Riemannian homogeneous foliation, given by the action of the identity component of the isotropy group G_p^0 (respectively the action of the whole isotropy group G_p) on $V_p^\perp$.

2.4 Orbifold part of the leaf space

Let $(M, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. A point $p_* \in M/\mathcal{F}$ is called *orbifold point* of $M/\mathcal{F}$ if there is a neighborhood of p_* isometric to a quotient U/Γ, where U is a Riemannian manifold and Γ is a finite group of isometries. The set of orbifold points of $M/\mathcal{F}$ is denoted by $(M/\mathcal{F})_{orb}$ and called the *orbifold part* of $M/\mathcal{F}$. By [7] the preimage of $(M/\mathcal{F})_{orb}$ consists of those points whose infinitesimal foliation is polar, and the complement of $(M/\mathcal{F})_{orb}$ in $M/\mathcal{F}$ has codimension ≥ 2.

3 Proof of Theorems 1.1 and 1.4

Suppose we have two singular Riemannian foliations $(M_i, \mathcal{F}_i)$, $i = 1, 2$ with closed leaves, and an isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ that preserves the codimension of the leaves. For $p_i \in M_i$, denote p_{i*} its projection under the canonical map $\pi_i \colon M_i \to M_i/\mathcal{F}_i$.

In order to avoid cumbersome notations, we will denote each basic function $f \colon M_i \to \mathbb{R}$ and the induced function on $M_i/\mathcal{F}_i$ by the same letter f.

We now prove Theorem 1.1, closely following the steps presented at the end of the introduction. We first observe that the main problem can be reduced to a problem in Euclidean space, following standard arguments from the theory of SRF's; see [2,7,9]. Fixing p_1, p_2 in such a way that $\varphi(p_{1*}) = p_{2*}$, φ restricts to an isometry $\varphi : (S_{p_1}, g_1)/\mathcal{F}_1 \to (S_{p_2}, g_2)/\mathcal{F}_2$. Recall that the flat metrics g_{p_i} are the limit of metrics $g_\lambda^i = \frac{1}{\lambda^2} h_\lambda^* g_i$ as $\lambda \to 0$, where h_λ denotes the homothetic transformation around p_i. In particular, since the isometry φ induces an isometry $\varphi : (S_{p_1}, g_\lambda^1)/\mathcal{F}_1 \to (S_{p_2}, g_\lambda^2)/\mathcal{F}_2$ for any $\lambda \in (0, 1)$, by taking the limit as $\lambda \to 0$ we obtain an isometry

$$\varphi_* : (V_{p_1}^\perp, g_{p_1})/\mathcal{F}_{p_1} \to (V_{p_2}^\perp, g_{p_2})/\mathcal{F}_{p_2}.$$

This is an isometry between leaf spaces of foliations in Euclidean space. Moreover, around p_1, φ can be written as $(\exp_2)_* \circ \varphi_* \circ (\exp_1)_*^{-1}$, where $(\exp_i)_*$ are diffeomorphisms, and therefore φ is smooth around p_1 if and only if φ_* is smooth. Thus in order to prove the theorem, it is enough to check it on Euclidean spaces.

Proposition 3.1 *Let $(\mathbb{R}^{n_1}, \mathcal{F}_1)$, $(\mathbb{R}^{n_2}, \mathcal{F}_2)$ be two (possibly non-connected) SRF's with closed leaves, and let $\varphi : \mathbb{R}^{n_1}/\mathcal{F}_1 \to \mathbb{R}^{n_2}/\mathcal{F}_2$ be an isometry that preserves the codimension of the leaves. Then the mean curvature vector fields of the corresponding principal leaves are basic and φ preserves the projections of those vector fields.*

Proof This result was proved in Gromoll and Walschap [6, Theorem 4.1.1] in the case of regular Riemannian foliations. In what follows we will explain how that proof can be adapted in the case of SRF's.

For $i = 1, 2$ let $p_i \in M_i = \mathbb{R}^{n_i}$ be a principal point of $\mathcal{F}_i$ such that $\varphi(p_{1*}) = p_{2*}$. Moreover, let $x_i \in V_{p_i}^\perp$, $i = 1, 2$ be horizontal vectors such that $\varphi_*(x_{1*}) = x_{2*}$. Finally, define $\gamma_i(t) = p_i + t x_i$.

In order to prove the proposition, it is enough to show that $\mathrm{tr}(\mathsf{S}_{x_1}) = \mathrm{tr}(\mathsf{S}_{x_2})$, where S_{x_i} is the *shape operator* of the leaf L_{p_i} through p_i. We will actually show something stronger, namely that every nonzero eigenvalue of S_{x_1} is an eigenvalue of S_{x_2} of the same multiplicity, for almost every x_1.

Since the complement of the orbifold part $(M_1/\mathcal{F}_1)_{orb}$ has codimension ≥ 2, almost every projected horizontal geodesic stays in $(M_1/\mathcal{F}_1)_{orb}$ for all time, and in what follows we will assume that our fixed geodesic γ_1 has this property.

Because $\varphi((M_1/\mathcal{F}_1)_{orb}) = (M_2/\mathcal{F}_2)_{orb}$, and φ takes projected horizontal geodesics in $(M_1/\mathcal{F}_1)_{orb}$ to projected horizontal geodesics in $(M_2/\mathcal{F}_2)_{orb}$, we conclude that $\varphi(\pi_1 \circ \gamma_1) = \pi_2 \circ \gamma_2$; see [13].

On the one hand, since $\pi_i \circ \gamma_i$ are contained in $(M_i/\mathcal{F}_i)_{orb}$, we know that the φ preserves conjugate points along $\pi_i \circ \gamma_i$, as well as their multiplicity. We also know, by hypothesis, that φ preserves codimension of the singular points contained in γ_1.

On the other hand, by [7, Lemma 5.2] the focal index, i.e., the number of focal points of L_{p_i} along γ_i counted with multiplicity, is a sum of two indices, namely:

- the horizontal index, which counts conjugate points of $\pi_i(p_i)$ with their multiplicity along $\pi_i \circ \gamma_i$. The notion of conjugare point along $\pi_i \circ \gamma_i$ makes sense, since $\pi_i \circ \gamma_i$ is contained in the orbifold part of $M_i/\mathcal{F}_i$.
- The vertical index, which counts the singular points of $\mathcal{F}_i$ contained in γ_i, their multiplicity being the jump in codimension $\mathrm{codim}\, L_{\gamma_i(t)} - \mathrm{codim}\, \mathcal{F}$ at those points; see also the discussion in [7, Sect. 5.2].

These facts combined, imply that φ preserves the focal points of L_{p_i} along γ_i and their multiplicities.

Finally recall that, since M_i are Euclidean spaces, the focal points of L_{p_1} along γ_1 are at distance $1/\lambda_1, \ldots 1/\lambda_r$, where $\{\lambda_1, \ldots, \lambda_r\}$ are the eigenvalues of S_{x_1} counted with the same multiplicity, see [10, Proposition 4.1.8]. Since φ preserves focal points and their multiplicities, we infer that the shape operator S_{x_2} of the leaf L_{p_2} has the same eigenvalues as those of S_{x_1}, counted with the same multiplicity. In particular, $\mathrm{tr}(\mathsf{S}_{x_1}) = \mathrm{tr}(\mathsf{S}_{x_2})$ whenever the projection of $\gamma_1(t) = p_1 + tx_1$ is entirely contained in $(M_1/\mathcal{F}_1)_{orb}$. Because this condition is open and dense, the fact that φ preserves mean curvature vector field follows from the continuity of the mean curvature form.

Remark 3.2 1. The above proposition implies that, given a SRF $\mathcal{F}$ on $\mathbb{R}^n$, then each principal leaf L of $\mathcal{F}$ is a *generalized isoparametric* submanifold, i.e., the principal curvatures along a basic vector field of L are constant.

2. If $M_1/\mathcal{F}_1$, has no boundary, then almost every horizontal geodesic γ_1 stays in the regular stratum of M_1. By the proof of Proposition 3.1 all the focal points of L_{p_1} along γ_1 correspond to conjugate points of $\pi_1(p_1)$ along $\pi_1 \circ \gamma_1$. In particular, if $M_1/\mathcal{F}_1$ has no boundary, φ preserves the mean curvature even without the assumption of preserving the codimension of the leaves.

By the discussion above, Theorems 1.1 and 1.4 will both be proved once we show that any isometry between leaf spaces preserving the (basic) mean curvature vector fields is smooth. In order to do this, we show:

Proposition 3.3 *Let M_1 and M_2 be complete Riemannian manifolds and $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be SRF's with closed leaves. Then an isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ is of class C^1, i.e., for each smooth basic function f on M_2, the basic function $\varphi^* f$ on M_1 is of class C^1.*

Proof For $i = 1, 2$ let p_i be a point in M_i, let P_i be a small tubular neighborhood of L_{p_i} in the stratum containing p_i, and let U_i be a small tubular neighborhood of P_i of radius ϵ, with closest-point projection $\mathsf{p}_i : U_i \to P_i$. We can make these choices so that $\varphi(\pi_1(p_1)) = \pi_2(p_2)$ and $\varphi(\pi_1(P_1)) = \pi_2(P_2)$.

If f is a smooth basic function on M_2, let f_0 be the smooth basic function on U_2 defined as $f_0 = \mathsf{p}_2^*(f|_{P_2})$. Since the gradient of f at p_2 is tangent to the stratum, $\nabla f_0 = \nabla f$ at p_2. Therefore, if we rewrite f as $f = f_0 + R$ (locally this is the Taylor formula), we conclude that $\nabla R = 0$ at p_2.

The pullback of f_0 under φ is

$$\varphi^* f_0 = \varphi^*(\mathsf{p}_2^*(f|_{P_2})) = \mathsf{p}_1^*\big((\varphi^* f)|_{P_1}\big).$$

It is easy to prove that $\varphi^* f$ is smooth on each stratum of M_1, in particular $(\varphi^* f)|_{P_1}$ is smooth and thus $\varphi^* f_0$ is smooth on U_1. If we write

$$\varphi^* f = \varphi^* f_0 + \varphi^* R,$$

it now follows that $\varphi^* R$ is smooth on each stratum, and it makes sense to define the gradient $\nabla\varphi^* R$ on each stratum. Moreover, since $\varphi^* R$ is basic, $\nabla\varphi^* R$ is always horizontal and we can compute $\lim_{p\to p_1} \|\nabla\varphi^* R\|$ from the quotient:

$$\lim_{p\to p_1} \|\nabla\varphi^* R\|(p) = \lim_{p''\to\varphi(\pi_1(p))} \|\nabla R\|(p'') = 0, \tag{2}$$

where we used the fact that φ is an isometry.

Equation (2) implies that $\varphi^* R$ is of class C^1 ar p_1 and $\nabla\varphi^* R(p_1) = 0$. In particular $\varphi^* f = \varphi^* f_0 + \varphi^* R$ is C^1 at p_1, and this proves the proposition.

Remark 3.4 1. In Proposition 3.3, the fact that φ is an isometry is used only in Eq. (2). Here, all we have really used is the fact that the derivative of φ (restricted to each stratum) is locally bounded.
2. Observe that Proposition 3.3 does not use the assumption that φ preserves the codimension of the leaves. In particular, every isometry between leaf spaces is of class C^1.

The next proposition concludes the proof of Theorem 1.1 and Theorem 1.4.

Proposition 3.5 *Let M_1 and M_2 be complete Riemannian manifolds and $(M_i, \mathcal{F}_i)$ SRF's with closed leaves such that the mean curvature vector fields H_i of the corresponding principal leaves are basic. Assume that there exists an isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ that preserves the mean curvature vector fields restricted to the principal stratum. Then φ is a smooth map.*

Proof Let g_i denote the metric on M_i. Recall that we are using the notation H_{i*} to denote the projection $\pi_{i*}H_i$ of the mean curvature vector field on the regular part of M_i. For $i = 1, 2$, let p_i be a regular point in M_i, and let U_i be a neighborhood of p_i that admits a local quotient $\mathsf{q}_i : U_i \to B_i$, where the manifold B_i is the local model of the orbifold $\pi_i(U_i) \subseteq M_i/\mathcal{F}_i$. We can make these choices so that $\varphi(\pi_1(p_1)) = \pi_2(p_2)$ and $\varphi(\pi_1(U_1)) = \pi_2(U_2)$. Since

$$\varphi|_{\pi_1(U_1)} : \pi_1(U_1) \to \pi_2(U_2)$$

is an isometry, by [13] it lifts to an isometry $\overline{\varphi} : B_1 \to B_2$.

Let f be a smooth basic function of $(M_2, \mathcal{F}_2)$. We want to prove that $\varphi^* f$ is a smooth basic function of $(M_1, \mathcal{F}_1)$.

Clearly f stays basic with respect to $\mathcal{F}_2|_{U_2}$ and thus it defines a function on B_2, which we still denote f.

We recall that (see e.g., [6, page 53])

$$\triangle_{U_i} f = \triangle_{B_i} f - g_i(\nabla f, H_{i*}). \tag{3}$$

Set $u := \triangle_{M_2} f$. Equation (3) implies that u is a smooth basic function on $(U_2, \mathcal{F}_2)$.

Since $\overline{\varphi} : B_1 \to B_2$ is an isometry and $\varphi_*(H_{1*}) = H_{2*}$ by assumption, it easily follows from Eq. (3) that

$$\triangle_{M_1} (\varphi^* f) = \varphi^* u \qquad \text{on } U_1. \tag{4}$$

Since p_1, p_2 were chosen arbitrarily, it follows that $\triangle(\varphi^* f) = \varphi^* u$ in the regular part $(M_1)_{reg}$. Since the complement of $(M_1)_{reg}$ in M_1 is a locally finite union of submanifolds of codimension ≥ 2, Eq. (4) holds weakly on the whole M_1 by the following Lemma.

Lemma 3.6 *Let f, u be C^1 functions on a manifold M, and let M' be a submanifold of M such that:*

- *$M \setminus M'$ is a locally finite union of submanifolds of codimension ≥ 2.*
- *f, u are smooth on M', and $\triangle f = u$ on M'.*

Then $\triangle f = u$ holds weakly on M, i.e.,

$$\int_M f \cdot \triangle h = \int_M u \cdot h$$

for every smooth function h with compact support on M.

Proof Let W be a neighborhood of $M \setminus M'$ with smooth boundary that ∂W. Let h be a smooth function with compact support on M. By Green's second identity

$$\int_{M-W} \triangle f \cdot h - \int_{M-W} f \cdot \triangle h = \int_{\partial W} h \cdot g(\nabla f, \eta) - f \cdot g(\nabla h, \eta), \tag{5}$$

where η is the normal vector field of ∂W.

Since $\triangle f = u$ on $M - W \subseteq M'$, Eq. (5) becomes

$$\int_{M-W} u \cdot h - \int_{M-W} f \cdot \triangle h = \int_{\partial W} h \cdot g(\nabla f, \eta) - f \cdot g(\nabla h, \eta). \tag{6}$$

Given $\epsilon > 0$, it is possible to choose a small neighborhood W so that

$$\left| \int_M u \cdot h - \int_{M-W} u \cdot h \right| < \frac{\epsilon}{3}, \tag{7}$$

$$\left| \int_M f \cdot \triangle h - \int_{M-W} f \cdot \triangle h \right| < \frac{\epsilon}{3}. \tag{8}$$

Since $M \setminus M'$ has codimension ≥ 2, we can choose W with boundary of arbitrarily small volume. In particular we can assume

$$\left| \int_{\partial W} h \cdot g(\nabla f, \eta) - f \cdot g(\nabla h, \eta) \right| \leq |\partial W| \cdot \sup_M |h \cdot g(\nabla f, \eta) - f \cdot g(\nabla h, \eta)| < \frac{\epsilon}{3}. \tag{9}$$

Equations (6) through (9) now prove the Lemma.

By Lemma 3.6 above, the equation $\triangle \varphi^* f = \varphi^* u$ holds weakly on the whole M_1. Since $\varphi^* u$ is a function of class C^1 (recall Proposition 3.3) we can apply the regularity theory of solutions of linear elliptic equations (see e.g., the proof of Theorem 3, Sect. 6.3.1 of Evans [4]), and this proves the smoothness of f. Therefore φ is smooth as well, and Proposition 3.5 follows.

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