Università degli Studi di Torino
**Scuola di Dottorato**

# String Theory and QFT on Time-Dependent Orbifolds

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**Dottorato in Fisica**

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**Andrea Arduino**

**Tutor: Igor Pesando**

*A mia nonna Marisa*

*Happiness is amazing.*
*It's so amazing, it doesn't matter*
*if it's yours or not.*

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Afterlife - Season 1 (Episode 5)

# Abstract

This thesis deals with the study of cosmological Big Bang like singularities in the context of string theory and quantum field theory. Time-dependent orbifolds generated from Minkowski spacetime are used as toy models. Since the computation of tree-level closed and open string scattering amplitudes on these backgrounds exhibits unusual divergences, the aim of the thesis is to understand better their origin and find a way to cure them. We show that the non-existence of a well-defined perturbative expansion into the standard Feynman diagrammatic approach of the underlying effective QFT is at the root of divergent 3-point open string amplitudes involving massive states. Then, besides geometrical regularizations, we propose to introduce a background Kalb-Ramond $B$-field on the orbifolds. The noncommutative field theory which arises as a zero slope $\alpha' \to 0$ decoupling limit after applying the Seiberg-Witten map seems promisingly well-defined.

# Sommario

Questa tesi tratta lo studio di singolarità cosmologiche di tipo Big Bang nel contesto della teoria di stringa e della teoria di campo quantistica. Sono usati come toy model orbifold dipendenti dal tempo e generati dallo spaziotempo di Minkowski. Dal momento che il calcolo di ampiezze di scattering ad albero in stringa chiusa e aperta mostra inusuali divergenze su questi background, lo scopo della tesi è capire meglio la loro origine e trovare un modo per curarle. Viene mostrato che la non esistenza di una ben definita espansione perturbativa nell'approccio diagrammatico standard di Feynman della sottostante teoria di campo effettiva è alla radice delle ampiezze divergenti a 3 punti che includono stati massivi in stringa aperta. Poi, oltre a regolarizzazioni di tipo geometrico, si propone di introdurre sull'orbifold un campo di background di Kalb-Ramond $B$. La teoria di campo noncommutativa che emerge come limite di disaccoppiamento a pendenza nulla $\alpha' \to 0$ dopo aver applicato la mappa di Seiberg-Witten sembra promettentemente ben definita.

# Acknowledgements

I will try to be as brief as possible, but the truth is that there are so many people to thank for having been, in so many different ways, part of the journey of which this thesis is a significant milestone that I don't even know where to start.

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*Andrea Arduino*

# Contents

# Outline

This thesis follows the reasearch work of the author as a PhD student at the University of Torino. The results presented are mainly based on published [1, 2] and ongoing [3] works. The manuscript is divided in five chapters (plus three appendices), where the material is organized in this way:

- In Chapter 1 we explain why the interest for cosmological singularities in the string theory framework is highly motivated. Then time-dependent orbifolds are defined and the computation of string scattering amplitudes on one of this models, the Boost Orbifold, reveals some unusual divergences.

- Chapter 2 deals with the issue of light-cone quantization of a scalar field on time-dependent backgrounds. After defining light-cone evolution, we show that the scalar field we are quantizing is actually a scalar density and that the wave function of the second quantized particle has to be interpreted as a charge density.

- In Chapter 3 we come back to the orbifolds by studying a scalar QED theory on the the Null Boost Orbifold, in order to show that the divergences can be traced back to the ill definition of some interaction vertices. Some meaningful regularization attempts are then proved to fail.

- The first part of Chapter 4 is devoted to the construction of string theory amplitudes on the NBO starting from wave functions defined on the covering spacetime. It comes out that the field theory divergences of the previous chapter are reflected here in pathological 3-point amplitudes involving massive states. Then we propose a geometrical resolution of the NBO. Finally, we show that string theory on the BO suffers similar problems.

- In Chapter 5 we add a background $B$-field on the NBO and study the noncommutative field theory we obtain as a decoupling limit after applying the Seiberg-Witten map. It turns out that the $N$-point scalar amplitude in this framework is now well-defined and this opens up new research paths in this field.

Chapter 1

# An Outstanding String Theory Problem

## 1.1 Introduction and Motivations

Einstein's theory of general relativity predicts the existence of gravitational spacetime singularities at the classical level: the most familiar and fascinating example is the Big Bang from which we suppose our universe began. But what is exactly a spacetime singularity? Roughly speaking, with the term "spacetime singularity"[1] we are referring to a singular point that is related to a singular spacetime, i.e. a geodesically incomplete spacetime.

As is in fact well known, in general relativity free test particles follow geodesics, which are basically the curves representing the shortest path between two points on the spacetime (Riemannian) manifold. Causal geodesics are either null (lightlike) or timelike. The equation for geodesics immediately follows when we first parametrize the path $P$ in terms of an affine parameter $\lambda$ (which is also the eigentime $\tau$ for a massive particle) as

$$P : \lambda \to x^{\mu}(\lambda) \tag{1.1.1}$$

and then we vary the action with respect to the coordinates $x^{\mu}(\lambda)$:

$$\ddot{x}^{\mu} + \Gamma_{\rho\sigma}\dot{x}^{\rho}\dot{x}^{\sigma} = 0. \tag{1.1.2}$$

If the geodesic $P(\lambda)$ cannot be extended for all finite values of the affine parameter $\lambda$, the worldline of a particle that follows this particular geodesic will have a beginning or an end, corresponding to a singularity. Indeed, a singularity results to be "physical" when a test particle can reach it in a finite affine parameter (or finite eigentime). From a more mathematical point of view, what analytically indicates the presence of a gravitational singularity is that some quantities used to measure the gravitational field strength like the scalar invariant curvatures of

[1]Notice that we can just switch names between "spacetime singularity" and "gravitational singularity" because in Einstein's classical theory of gravity the gravitational interaction is encoded in the geometry of the spacetime.

spacetime, which include a measure of the density of matter, become infinite.

If we apply general relativity to derive the evolution of our universe using Friedmann-Lemaître-Robertson-Walker metric, which means to assume homogeneity, isotropy and expansion, we are led to conclude that 13.7 billion years ago the universe began with a Big Bang singularity[2]. Approaching the singularity the energies involved reach the Planck scale which lies at $E_P \sim 1.22 \cdot 10^{19} GeV$ and corresponds to a Planck time of $t_P \sim 10^{-43} s$ after the Big Bang. Before this time the particle energies grow so high that we expect general relativity to break down as a classical theory.

This approach to early universe cosmology may seem old-fashioned in light of the well established paradigm of the inflationary scenario. It cannot be denied that inflation, i.e. a period of accelerated expansion taking place during the early stages of our universe, has proven capable of addressing successfully several cosmological problems like the horizon, flatness, entropy and structure formation ones. Nevertheless, the singularity issue remains an open question, since it doesn't actually disappear in the inflationary paradigm, but it is rather pushed into the past.

There is also a number of conceptual issues surrounding the concept of Big Bang: should we think about it as the beginning of time or did the universe first shrink to a singularity and then expand again? A related question concerns how we define observables: should we impose some initial conditions close to the singularity or should we rather introduce S-matrix type objects using asymptotic regions? And still, how are information related before and after the bounce, if there is a

[2]The metric of a FLRW 4-dimensional spacetime is determined by the scale factor $a(t)$ and the spatial curvature $k$:

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big). \tag{1.1.3}$$

This metric is singular if the scale factor vanishes at a certain time $t^*$, which is what is supposed to happen at $t^* = 0$.

bounce at all?

This is where the more ambitious, even though far from being free of problems, bouncing cosmologies enter the chat. They can be considered either as an alternative to inflation or as part of mixed models in which a bounce reproduces a primordial singularity while a subsequent inflationary phase solves the other puzzles of the standard hot Big Bang. That said, our attention however will be focused on the Big Crunch/Big Bang singularity. Since we believe that the gravitational force, like all the other known ones, has a quantum mechanical nature, the gravitational quantum effects should become important at very high energies around the Planck scale. Therefore the question about the origin of the universe is a question for a theory of quantum gravity.

One of the research lines for a consistent theory of this kind has led to string theory. First of all, string theory is capable of describing the exchange of gravitons which are the quanta of the gravitational force and classical general relativity can be obtained as the low-energy limit of the gravitational interaction. Moreover string theory may unify general relativity and the standard model since it is not only a quantum theory of gravity but it also offers a framework capable of incorporating the other known non-gravitational interactions, i.e. electromagnetic, weak and strong ones. For all these reasons, string theory seems to provide the better theoretical framework for pursuing the study of cosmological singularities.

At the same time, there is also another motivation behind this project. The study of string theory in time-dependent backgrounds throughout the years has proven very challenging and far from completely successful. Also quantum field theories defined on curved time-dependent space-times, to which we will we step back a lot in this thesis, raise non-trivial technical and conceptual issues. From this point of view, cosmological singularities represent a big and hard challenge, but they may provide the way to make important steps in a better understanding of physical theories as a whole.

## 1.2 Time-Dependent Orbifolds

In order to try to address the questions which motivated this work, we will introduce time-dependent orbifolds generated from Minkowski spacetime. They are amongst the simplest toy models backgrounds which can reproduce cosmological singularities and this is the reason why they have been deeply analyzed in the context of string theory. But let us first quickly review the concept of orbifolds, whose relationship with the study of singularities in string theory goes deep back [4, 5].

### 1.2.1 Static Orbifolds

In a purely mathematical context an orbifold is a generalization of a manifold that allows the presence of points whose neighbourhood is diffeomorphic to a quotient of $\mathbb{R}^n$ by a finite group, i.e. $\mathbb{R}^n/G$. In a physical context it usually describes an object that can be globally written as a quotient space $M/G$ where $M$ is a manifold, and $G$ is a group of some of its isometries and/or discrete symmetries. A very trivial example of a quotient space is the periodic segment $\mathbb{R}/T(R)$ where $T(R)$ is a translation that maps $x \in \mathbb{R}$ into $x + 2\pi nR$:

$$x \sim x + 2\pi nR, \quad n \in \mathbb{Z}. \tag{1.2.1}$$

In this simple case the orbifold produces a regular compactified space (the real line is mapped onto a circle of radius $R$) that is invariant under discrete coordinate shifts by $2\pi R$, to which all physical observables should be invariant too. In general, we can say that if the group $G$ has fixed points then the orbifold $M/G$ will have singular points. We will obviously be interested in such orbifolds with singular points because they provide the toy models for a singular spacetime we are looking for.

Let's therefore consider a little more complicated case, the two dimensional space obtained as $\mathbb{R}^2/\mathbb{Z}_2$:

$$(x_1, x_2) \sim (-x_1, -x_2). \tag{1.2.2}$$

The resulting space is a two dimensional cone with a deficit angle $\pi$. Classical general relativity and quantum field theory are singular in this background because of the delta function curvature at the tip of the cone, the fixed point $(0, 0)$. Nevertheless, string theory is solvable and it turns out that the extended nature of strings leads to new degrees of freedom, known as "twisted states", which are localized near the singularity [6]. What is relevant is that quantum physics is completely smooth since string theory sees the spacetime not through the metric $g_{\mu\nu}$ only but by $g_{\mu\nu} + B_{\mu\nu}$[3] in such a way that while $g_{\mu\nu}$ is singular the sum $g_{\mu\nu} + B_{\mu\nu}$ is not. Associated to the non-trivial $B_{\mu\nu}$ there are in fact the twisted states. This is an example of how a timelike static singularity is resolved in string theory by the introduction of degrees of freedom that previously had not been taken into account.

It is obviously possible to consider much more complicated examples of static singularities in time-independent orbifolds which sometimes even need to abandon the perturbative regime in favour of the non-perturbative string approach (as explained in [7]) but we will not discuss these cases since they go beyond the aim of this thesis.

### 1.2.2 $\mathbb{M}^3$ Time-Dependent Orbifolds

We are instead concerned to know how string theory behaves in presence of lightlike (or spacelike) non-static singularities. This kind of singularity is reached when a timelike coordinate approaches a given value or, more physically, we can say that the singularity is present in a certain region of (or, even worse, in the whole) space at a specific value of the time coordinate. Such cosmological singularities are much harder to understand, since the singularity appears in time and then

[3] Here $B_{\mu\nu}$ is an harmonic 2-form whose integral $B = \int_{\mathbb{R}^2/\mathbb{Z}_2} \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ is independent of the scalar curvature $R$.

eventually disappears.

In order to generate a time-dependent orbifold the action of group $G$ has to act non trivially on the time coordinate of target space $M$. We take as covering space $\mathbb{M}^3 \otimes \mathbb{R}^{D-3}$, i.e. the flat three–dimensional Minkowski spacetime with metric $g_{\mu\nu} = (-1, +1, +1)$ plus $D-3$ transverse Euclidean spectator coordinates, and quotient the minkowskian part by the action of subgroups $\Gamma$ of its isometry group $ISO(2,1)$[4]. Then we identify points along the orbits of a generic Killing vector $k$ according to

$$(x^\mu) \sim e^{nk}(x^\mu) = \mathcal{K}^n(x^\mu), \qquad n \in \mathbb{Z}, \tag{1.2.3}$$

where $k$ can be written in its most general form as

$$k = 2\pi i(\alpha^\mu P_\mu + \beta^{\mu\nu} J_{\mu\nu}), \tag{1.2.4}$$

with

$$iJ_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \tag{1.2.5}$$

$$iP_\mu = \partial_\mu \tag{1.2.6}$$

being the usual generators of Poincaré algebra. Notice that we are working in light-cone coordinates $x^\mu = (x^+, x^-, x^2, \vec{x})$.

As shown in [9], the time–dependent orbifolds of three-dimensional Minkowski spacetime can be classified in terms of four different combinations of (1.2.5) and (1.2.6) and just two parameters ($\alpha^2 = R$, $\beta_2 = \Delta$)[5], which describe the inequivalent conjugacy classes of each orbifold. The same conclusion can be reached if we recall the classification of all subgroups, and consequently subalgebras, of $\mathbb{M}^3$ performed in [10]. More specifically, we are restricting our analysis to (symmorphic) subalgebras of dimension one and to those of dimension two which can be written

[4]These models first appeared in the literature in [8].

[5]$\beta^{ab} = \epsilon^{abc}\beta_c$, where $\epsilon^{abc}$ is the Levi-Civita symbol.

as a direct product of a translation generator and a Lorentz generator. For obvious reasons we exclude from our analysis both pure translations and Lorentz generators which do not involve time. Up to $ISO(2,1)$ conjugations, we obtain the following classification:

| Orbifold | Generator |
|---|---|
| Null Boost (NBO) | $\kappa^{NBO} = 2\pi i \Delta J_{+2}$ |
| Boost (BO) | $\kappa^{BO} = 2\pi i \Delta J_{+-}$ |
| O-Plane (OPO) | $\kappa^{OPO} = 2\pi i(\Delta J_{+2} + RP_-)$ |
| Shifted Boost (SBO) | $\kappa^{SBO} = 2\pi i(\Delta J_{+-} + RP_2)$ |

Table 1.1: $\mathbb{M}^3$ time–dependent orbifolds.

All these orbifolds, despite displaying important geometrical differences, appear suitable for our study. They in fact reproduce spacetimes where a circle of an infinite size at $x^- = -\infty$ shrinks to zero size at the singularity $x^- = 0$ and then expands to an infinite size again at $x^- = +\infty$. They provide therefore interesting toy models for understanding a Big Crunch/Big Bang singularity where the coordinate $x^-$ plays a time role. Throughout the thesis we will introduce in more details the BO and the NBO, which are the models we will deal with.

## 1.3 The Boost Orbifold

From now until the end of this chapter we will mostly redo the computations of [11], which led to the discovery on the BO of some unusual string scattering amplitudes divergences.

### 1.3.1 Geometry and Wave Functions

The identification of spacetime points on the BO results in:

$$\begin{pmatrix} x^+ \\ x^- \\ x^2 \end{pmatrix} \sim \begin{pmatrix} e^{2\pi n\Delta}\, x^+ \\ e^{-2\pi n\Delta}\, x^- \\ x^2 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{1.3.1}$$

Since the spatial direction $x^2$ plays no role, from now on it will be considered as a transverse dimension. The Minkowski spacetime (on the left of Figure 1.1) is divided into 4 regions by the action of the orbifold. The dotted lines are the boundaries of the orbifold fundamental region and are identified. The BO geometry (on the right) results to be that of 4 cones. Each quadrant of Minkowski spacetime is mapped in the orbifold to one of the cones, while the origin is a fixed point of the orbifold action. Moreover, points on the light–cone have images arbitrarily close to the origin and, consequently, spacetime is not Hausdorff. We can imagine that this feature may lead to serious issues when trying to cross the singularity[6]. It's also useful to compute the geodesic distance squared between a point and its $n$-th image:

$$\|x_{(n)} - x_{(0)}\|^2 = 8\sinh^2(n\pi\Delta) x^+_{(0)} x^-_{(0)}, \tag{1.3.2}$$

from which it follows immediately that there are CTC's[7] on both left and right quadrants, which are usually called the whiskers. There are lot of studies concerning the role of CTC's in string theory, usually aimed at excising regions containing them. In this case the simplest scenario is the Milne spacetime analyzed in [13], where the authors completely exclude the left and right quadrants from Minkowski spacetime and, as a consequence, the whiskers from the BO.

[6]Notice that the addition of a shift to the BO, which is nothing more than the SBO of Table 1.1, is a sort of regularized space where there are no more fixed points [12].

[7]A Closed Timelike Curve (CTC) is a closed wordline. This means that a particle moving forward in time could theoretically return to its starting point both in space and time.

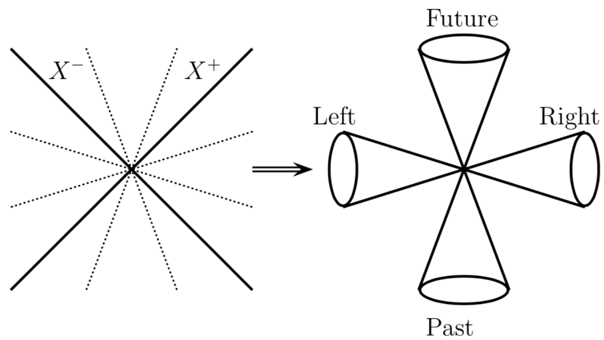


Figure 1.1: The BO geometry [12].

Now we want to construct single particle wave functions invariant under the orbifold action [14, 15]. Let's start by solving the Klein Gordon wave equation

$$\Box\Psi = M^2\Psi \tag{1.3.3}$$

on the covering space $\mathbb{R}^{1,d-1}$. A smooth basis of solutions is obviously given by the usual plane waves:

$$\{e^{i(p^+x^-+p^-x^+)}e^{i\vec{p}\cdot\vec{x}}/(2p^+p^- - \vec{p}^2 = M^2)\}. \tag{1.3.4}$$

But we can also choose an alternative basis of solutions to (1.3.3), which consists of continuous superpositions of (1.3.4) such that the orbifold action $x^\pm \mapsto e^{\pm 2\pi\Delta}x^\pm$ is diagonal. Such a basis is given by

$$\Psi_{l,p^+,p^-,\vec{p}}(x^+,x^-,\vec{x}) = e^{i\vec{p}\cdot\vec{x}}\int_{-\infty}^{+\infty} d\sigma e^{i(p^+x^-e^{\sigma}+p^-x^+e^{-\sigma})}e^{il\sigma},\ l\in\mathbb{R}, \tag{1.3.5}$$

and the orbifold generator acts by multiplication of the phase $e^{2\pi i l\Delta}$. To restrict the wave functions (1.3.5) to the orbifold invariant ones it is therefore sufficient to choose $l \in \frac{1}{\Delta}\mathbb{Z}$. Moreover, we can define the two-dimensional mass $m^2 = 2p^+p^-$ and put the orbifold wave functions in the Klein-Gordon normalized form (where $s = \pm 1$):

$$\Psi_{l,m,s,\vec{p}}([x^+, x^-, \vec{x}]) = \frac{e^{i\vec{p}\cdot\vec{x}}}{2\sqrt{2}\pi i}\int_{-\infty}^{+\infty} d\sigma e^{i(mx^+e^{-\sigma}+smx^-e^{\sigma})}e^{il\sigma},\ l \in \tfrac{1}{\Delta}\mathbb{Z}. \tag{1.3.6}$$

It has been shown that the propagation of free untwisted fluctuations through a Big Crunch/Big Bang singularity of this kind is under control [16, 17]. However, it has turned out to be problematic to include interactions. The situation can be analysed from at least two different points of view. One way is given by the argument, put forward by Horowitz and Polchinski in [18], for the formation of large black holes due to the interaction between image particles on the orbifold. The other possibility, which will be the topic of the next section, is to compute directly scattering amplitudes on the orbifold.

### 1.3.2 $N$-Point Function

Natural objects to compute in string theory are S-matrix elements: send in a number of particles into the past cone and see what comes out in the future cone. Our attitude is to try to apply the usual rules of orbifold field theory (which worked very well for static orbifold singularities) without worrying about apparent pathologies like CTC's. The main motivation is that, at least in the backgrounds like Minkowski spacetime where string theory is well understood, the fundamental observables should always have to do with asymptotic regions. If there are true pathologies, they will presumably show up at some point in the computations of S-matrix elements. From the analytical point of view, the basic tool used to compute tree-level amplitudes in string theory (and quantum field theory) is the "inheritance principle", which states that we can use directly the amplitudes of the parent theory on $\mathbb{M}^3 \times \mathbb{R}^{d-3}$ as long as we restrict our attention to external states which are invariant under the orbifold action [19].

In light of the above, the general expression for the $N$-point amplitude on the BO for $q$ initial states and $N-q$ final states can be written as:

$$A_N = \prod_{i=1}^{q<N} \prod_{j=q+1}^{N} \langle \Psi^*_{l_j,m_j,s_j,\vec{p_j}} \Psi_{l_i,m_i,s_i,\vec{p_i}} \rangle =$$
$$= \int_{-\infty}^{+\infty} d^{d-2}\vec{x} \int_{D_0} d^2x \prod_{i=1}^{q<N} \prod_{j=q+1}^{N} \Psi^*_{l_j,m_j,s_j,\vec{p_j}} \Psi_{l_i,m_i,s_i,\vec{p_i}} \mathcal{A}_{nzm}, \quad (1.3.7)$$

where

$$D_0 = \{(x^+, x^-)/1 \leq \left|\frac{x^-}{x^+}\right| \leq e^{4\pi\Delta}\} \quad (1.3.8)$$

is the fundamental region of Figure 1.1 and $\mathcal{A}_{nzm}$ is the parent amplitude for the non-zero modes. In fact we can split the string coordinates $x^\mu$ as $x^\mu = x_0^\mu + x_{nzm}^\mu$ and factorize the wave functions as $\Psi[x^\mu] = \Psi(x_0^\mu)\Psi[x_{nzm}^\mu]$ so that the zero modes and the non-zero modes can be computed separately.
Using (1.3.6) we can start the full computation of $A_N$ as follows

$$A_N = \prod_{i=1}^{q<N} \prod_{j=q+1}^{N} \tfrac{(-2\sqrt{2}\pi i)^{(q-N)}}{(2\sqrt{2}\pi i)^q} \int_{-\infty}^{+\infty} d^{d-2}\vec{x}\, e^{-i\vec{p_j}\cdot\vec{x}} e^{i\vec{p_i}\cdot\vec{x}} \int_{D_0} d^2x$$
$$\int_{-\infty}^{+\infty} d\sigma_j d\sigma_i e^{-i(m_j x^+ e^{-\sigma_j} + s_j m_j x^- e^{\sigma_j})} e^{i(m_i x^+ e^{-\sigma_i} + s_i m_i x^- e^{\sigma_i})}$$
$$e^{-il_j\sigma_j} e^{il_i\sigma_i} \mathcal{A}_{nzm}. \quad (1.3.9)$$

The integrals over transverse coordinates give just a Dirac delta over the momenta. We introduce for notational simplicity the factor

$$D_{d,q,N} \equiv \tfrac{(-2\sqrt{2}\pi i)^{(q-N)}}{(2\sqrt{2}\pi i)^q} (2\pi)^{(d-2)} \delta\Big(\sum_{i=1}^{q<N} \vec{p_i} - \sum_{j=q+1}^{N} \vec{p_j}\Big), \quad (1.3.10)$$

while it's useful to make the following variable changes:

$$\begin{cases} \sigma_1 = \sigma_1 & (\epsilon_1 = 0) \\ \sigma_i = \epsilon_i + \sigma_1 & \text{for } 2 \leq i \leq q \\ \sigma_j = \epsilon_j + \sigma_1 & \text{for } q+1 \leq j \leq N. \end{cases} \tag{1.3.11}$$

This leads to:

$$\begin{aligned} A_N = D_{d,q,N} \int_{D_0} d^2x \prod_{i=2}^{q<N} \prod_{j=q+1}^{N} \int_{-\infty}^{+\infty} d\sigma_1 d\epsilon_i d\epsilon_j \\ e^{i\sigma_1(\sum_{r=1}^{q} l_r - \sum_{s=q+1}^{N} l_s)} e^{i(\sum_{r=2}^{q} l_r\epsilon_r - \sum_{s=q+1}^{N} l_s\epsilon_s)} \\ e^{ix^+ e^{-\sigma_1}(\sum_{r=1}^{q} m_r e^{-\epsilon_r} - \sum_{s=q+1}^{N} m_s e^{-\epsilon_s})} \\ e^{ix^- e^{\sigma_1}(\sum_{r=1}^{q} s_r m_r e^{\epsilon_r} - \sum_{s=q+1}^{N} s_s m_s e^{\epsilon_s})} \mathcal{A}_{nzm}. \end{aligned} \tag{1.3.12}$$

Now making the substitution $\sigma_1 \mapsto \widehat{\sigma_1} + n2\pi\Delta$ and observing that

- $\int_{-\infty}^{+\infty} d\sigma_1 = \sum_{n=-\infty}^{+\infty} \int_{2\pi n\Delta}^{2\pi(n+1)\Delta} d\sigma_1 = \sum_{n=-\infty}^{+\infty} \int_0^{2\pi\Delta} d\widehat{\sigma_1}$
- $e^{iln2\pi\Delta} = 1 \quad \forall l, \text{ since } l \in \frac{1}{\Delta}\mathbb{Z}$
- $D_n = \{(x^+, x^-)/e^{4n\pi\Delta} \leq \left|\frac{x^-}{x^+}\right| \leq e^{4(n+1)\pi\Delta}\}$
- $\sum_{n=-\infty}^{+\infty} \int_{D_n} d^2x = \int_{\mathbb{M}^2} d^2x,$

the expression for $A_N$ becomes:

$$\begin{aligned} A_N = & D_{d,q,N} \int_0^{2\pi\Delta} d\widehat{\sigma_1} \prod_{i=2}^{q<N} \prod_{j=q+1}^{N} \int_{-\infty}^{+\infty} d\epsilon_i d\epsilon_j \int_{\mathbb{M}^2} d^2x \\ & e^{i\widehat{\sigma_1}(\sum_{r=1}^{q} l_r - \sum_{s=q+1}^{N} l_s)} e^{i(\sum_{r=2}^{q} l_r\epsilon_r - \sum_{s=q+1}^{N} l_s\epsilon_s)} \\ & e^{ix^+ e^{-\widehat{\sigma_1}}(\sum_{r=1}^{q} m_r e^{-\epsilon_r} - \sum_{s=q+1}^{N} m_s e^{-\epsilon_s})} \\ & e^{ix^- e^{\widehat{\sigma_1}}(\sum_{r=1}^{q} s_r m_r e^{\epsilon_r} - \sum_{s=q+1}^{N} s_s m_s e^{\epsilon_s})} \mathcal{A}_{nzm}. \end{aligned} \tag{1.3.13}$$

Performing the integrals over $\widehat{\sigma_1}$ and $d^2x$ we reach the final expression:[8]

$$\begin{aligned} A_N = & C_{d,q,N}\delta_{(\sum_{r=1}^{q} l_r, \sum_{s=q+1}^{N} l_s)} \\ & \prod_{i=2}^{q<N} \prod_{j=q+1}^{N} \int_{-\infty}^{+\infty} d\epsilon_i d\epsilon_j e^{i(\sum_{r=2}^{q} l_r\epsilon_r - \sum_{s=q+1}^{N} l_s\epsilon_s)} \\ & \delta\Big(\sum_{r=1}^{q} m_r e^{-\epsilon_r} - \sum_{s=q+1}^{N} m_s e^{-\epsilon_s}\Big)\delta\Big(\sum_{r=1}^{q} s_r m_r e^{\epsilon_r} - \sum_{s=q+1}^{N} s_s m_s e^{\epsilon_s}\Big)\mathcal{A}_{nzm}. \end{aligned} \tag{1.3.14}$$

#### 1.3.2.1 2- and 3-Point Functions

Let's start by examining the cases of two and three-point functions, which will give finite results. For simplicity here we can assume the parent amplitude to be just a constant, i.e $\mathcal{A}_{nzm} = 1$.
For $N = 2$ and $q = 1$:

$$\begin{aligned} A_2 &= \langle \Psi^*_{l_2,m_2,s_2,\vec{p}_2} \Psi_{l_1,m_1,s_1,\vec{p}_1} \rangle = \\ &= C_{d,1,2}\delta_{l_1,l_2} \int_{-\infty}^{+\infty} d\epsilon_2 e^{-il_2\epsilon_2} \delta(m_1 - m_2 e^{-\epsilon_2})\delta(s_1 m_1 - s_2 m_2 e^{\epsilon_2}). \end{aligned} \tag{1.3.15}$$

Defining

$$e^{\epsilon_2} = v_2, \quad d\epsilon_2 = \frac{dv_2}{v_2} \tag{1.3.16}$$

we find:

$$\begin{aligned} A_2 &= C_{d,1,2}\delta_{l_1,l_2} \int_0^{+\infty} \frac{dv_2}{v_2} v_2^{-il_2} \delta(m_1 - \frac{m_2}{v_2})\delta(s_1 m_1 - s_2 m_2 v_2) \\ &= C_{d,1,2}\delta_{l_1,l_2} (\tfrac{m1}{m2})^{il_2} \delta(s_1 m_1^2 - s_2 m_2^2). \end{aligned} \tag{1.3.17}$$

[8] $C_{d,q,N} \equiv D_{d,q,N}(2\pi\Delta)(2\pi)^2$.

The computation for $N=3$ and $q=2$ is a bit more demanding but gives a finite result as well:

$$A_3 = C_{d,2,3}\delta_{l_1+l_2,l_3}\frac{\left(\frac{m_3^2s_3-m_1^2s_1}{m_1m_2s_2}\right)^{il_2}\left(\frac{m_3}{m_1+\frac{m_1m_2^2s_2}{m_3^2s_3-m_1^2s_1}}\right)^{-il_3}(m_2^2s_2+m_3^2s_3-m_1^2s_1)}{(m_1^2s_1-m_2^2s_2)^2+m_3^2(m_2^2s_2-2m_1^2s_1)s_3+m_3^4}. \tag{1.3.18}$$

#### 1.3.2.2 4-Point Function

The computations made so far can be used to derive results both in closed and open string theory[9], and even in the context of quantum field theory. The orbifold wave functions (1.3.6) can indeed be interpreted as scalar particles in QFT but as tachyons vertex operators as well in (closed and open) string theory. The BO spacetime coordinates $x^\mu$ in the latter case are intended to be functions of the string worldsheet coordinates $\sigma$ and $\tau$, i.e. $x^\mu = x^\mu(\sigma,\tau)$.
For what concerns the 4-point function it will become crucial to make this distinction. It is precisely in a specific kinematic regime of the $2 \to 2$ tree-level scattering amplitude that emerges the pathological divergent behaviour we mentioned earlier and whose interpretation given in the literature [7, 11, 20, 21] we suggest may not be entirely correct. But let's first write the 4-point function on the BO starting from (1.3.14)[10]:

$$\begin{aligned} A_4 &= \langle \Psi^*_{l_4,m_4,\vec{p}_4}\Psi^*_{l_3,m_3,\vec{p}_3}\Psi_{l_2,m_2,\vec{p}_2}\Psi_{l_1,m_1,\vec{p}_1}\rangle = \\ &= C_{d,2,4}\delta_{l_1+l_2,l_3+l_4}\int_{-\infty}^{+\infty} d\epsilon_2 d\epsilon_3 d\epsilon_4 e^{i(l_2\epsilon_2-l_3\epsilon_3-l_4\epsilon_4)} \\ &\quad \delta(m_1+m_2e^{-\epsilon_2}-m_3e^{-\epsilon_3}-m_4e^{-\epsilon_4}) \\ &\quad \delta(m_1+m_2e^{\epsilon_2}-m_3e^{\epsilon_3}-m_4e^{\epsilon_4})\mathcal{A}_{nzm}. \end{aligned} \tag{1.3.19}$$

[9]In the whole thesis we will basically consider only bosonic string theory.
[10]Here for technical simplicity we will choose $s_1=s_2=s_3=s_4=+1$, which means that both the initial and final particles are in the regions without CTC's.

Pursuing three substitutions similar to (1.3.16) we obtain the following expression:

$$\begin{aligned} A_4 =& C_{d,2,4}\delta_{l_1+l_2,l_3+l_4}\int_0^{+\infty}\frac{dv_2}{v_2}\frac{dv_3}{v_3}\frac{dv_4}{v_4}v_2^{il_2}v_3^{-il_3}v_4^{-il_4} \\ & \delta(m_1+\frac{m_2}{v_2}-\frac{m_3}{v_3}-\frac{m_4}{v_4})\delta(m_1+m_2v_2-m_3v_3-m_4v_4)\mathcal{A}_{nzm}. \end{aligned} \tag{1.3.20}$$

The integrals over $v_2$ and $v_3$ can be performed using the two delta functions. We can't find a full analytic solution to (1.3.20) but we are interested to study the behaviour of the remaining integral in the large $v_4$ regime. Looking at the Mandelstam invariants $s$ and $t$ it's in fact easy to recognise that it corresponds to the well-known high-energy Regge limit ($RL$) of large $s$ and small fixed $t$:

$$s = -(p_1+p_2)^2 \overset{v_4\to\infty}{\sim} v_4 \tag{1.3.21}$$

$$t = -(p_1-p_3)^2 \overset{v_4\to\infty}{\sim} \text{const.} \tag{1.3.22}$$

The very relevant result we can obtain from the expression (1.3.20) is that

$$A_4 \overset{v_4\to\infty}{\sim} \int^{\Lambda} dv_4 v_4^{i(l_2-l_4)}\frac{\mathcal{A}_{nzm(RL)}}{v_4^2}. \tag{1.3.23}$$

Now it's clear that the convergence or divergence of (1.3.23) when $\Lambda \to +\infty$ critically depends on the behaviour of $\mathcal{A}_{nzm(RL)}$. All the computations reported in the literature of this $2 \to 2$ tree-level scattering amplitude were intended to be made with closed string. This means that the parent amplitude was basically the Virasoro-Shapiro ($VS$) amplitude for four tachyons whose behavior in the Regge limit

$$\mathcal{A}^{VS}_{nzm(RL)} \sim \frac{s^J}{-t} \tag{1.3.24}$$

$$J = 2 + \tfrac{1}{2}\alpha' t, \tag{1.3.25}$$

results in:

$$A_4 \overset{v_4\to\infty}{\sim} \int^{\Lambda} dv_4 v_4^{i(l_2-l_4)+\frac{1}{2}\alpha' t}. \tag{1.3.26}$$

Therefore the Regge trajectory (1.3.25) for the spin $J$ of the exchanged massless minimally coupled particle makes the amplitude (1.3.23) divergent for $|\vec{t}|$[11] sufficiently small. The physical interpretation given in [7, 11] is that the factor $\frac{1}{t}$ is a pole from graviton exchange and the fast oscillations of wave functions near the singularity give rise to a divergent stress tensor corresponding to an infinite blueshift. This stress energy couples to gravitons and leads to strong gravitational backreaction.
But if we consider the case of open strings, where the Veneziano ($V$) amplitude takes the place of the Virasoro-Shapiro one, the behaviour

$$\mathcal{A}^{V}_{nzm(RL)} \sim \frac{s^J}{-t} \tag{1.3.27}$$

$$J = 1 + \tfrac{1}{2}\alpha' t \tag{1.3.28}$$

results this time in[12]

$$A_4 \overset{v_4\to\infty}{\sim} \int^{\Lambda} dv_4 v_4^{i(l_2-l_4)+\frac{1}{2}\alpha' t-1} \operatorname{tr}\left(\{T_{(1)},T_{(2)}\}\{T_{(3)},T_{(4)}\}\right) \tag{1.3.29}$$

and seems therefore to be responsible of an analogous pathology, at least when $\vec{t} \to 0$. Moreover, the same considerations can be made in quantum field theory where the $\mathcal{A}_{nzm(RL)}$ behaves as (1.3.24) and (1.3.27), with $\alpha' t = 0$ and the only exchanged particle which isn't troublesome from this point of view is a massless scalar with spin $J = 0$. A computation analogous to the one performed in this chapter but which takes the NBO as background spacetime [20] leads to an even worse situation. The corresponding $A_4$ tree-level scattering amplitude in the Regge limit

[11] $\vec{t}$ is the transverse component of (1.3.22).
[12] $T_{(i)}$ are the usual open string Chan-Paton factors.

reads indeed

$$A_4 \overset{v_4 \to \infty}{\sim} \int^{\Lambda} dv_4 \frac{\mathcal{A}_{nzm(RL)}}{v_4}, \tag{1.3.30}$$

which tells us that all the previous worries have here even more motivation. In light of the above, it seems reasonable to believe that the explanation given so far is not satisfactory. In fact, how can this unusual divergence be related only to a gravitational issue if it appears also with open strings, where graviton, at tree-level, is not even present in the mass spectrum? This is actually one of the central questions around which the whole thesis revolves. In order to find some answers, we are going to study (starting from Chapter 3) what happens if we try to construct usual quantum field theories on these backgrounds. At first sight this may seem a bit odd, but we will see that this path leads to a much better understanding of these models while it gives hints on how to solve the problems encountered as well.

# Chapter 2

# Light-Cone Quantization of a Scalar Field

When we have to deal with the orbifold models introduced in the previous chapter it's much easier to work in light-cone coordinates. As a consequence, if we want to construct quantum field theories we are tempted to follow the standard light-cone quantization procedure. However, as we may expect, with time-dependent backgrounds things are not as straightforward as with the usual Minkowski spacetime.

In this chapter, which is based on [2], we first discuss how we can define light-cone evolution and then quantization on a curved time-dependent background, even when it does not admit a null Killing vector. Then we consider the light-cone quantization of a scalar field on a background with a Killing vector and its connection with the second quantization of the particle in the same background. This will lead to some results and considerations which are worthy of attention, even though they are not essential for the study of the orbifold divergences.

## 2.1 Light-Cone Evolution in a Time-Dependent Background

In flat space and with the usual coordinates there are at least four different but equivalent ways to define the light-cone evolution:

(1) The propagation in a null direction, i.e the Hamiltonian generator is associated with a null Killing vector;

(2) The constant time hypersurface is null, i.e. the induced metric is degenerate;

(3) The gauge choice of worldline diffeomorphisms results in a gauge fixed Hamiltonian without square roots;

(4) The Hamiltonian maximizes the number of kinematical generators of the Poincaré algebra.

All of these definitions are indeed satisfied if for the usual light-cone metric[1] $ds^2 = -2dudv + d\vec{x}^2$ we take $u$ as time, $e_u$ as the null evolution vector and consequently $H = p_u = \frac{\vec{p}^2+m^2}{2p_v}$ as the particle Hamiltonian. In the curved time-dependent case we can immediately rule out (4), since the global Poincaré algebra is broken by the explicit time dependence of the metric. As we will see, also (1) is rather sloppy, because what really matters is the orthogonal direction to constant time hypersurfaces, which need to be chosen carefully. Definitions (2) and (3) result instead to be basically equivalent, at least for the cases we are going to take into account, and will provide a "good" and simple way to define light-cone evolution, even in the absence of a Killing vector.

### 2.1.1 A General Procedure

Let's consider a somewhat generic metric with a null Killing vector $k = \partial_v$. All metric components are then $v$ independent and $g_{vv} = 0$, so we can write:[2]

$$\begin{aligned} ds^2 = &-2du\,dv + h(u,x)du^2 + 2l_i(u,x)dudx^i + 2f_i(u,x)dvdx^i \\ &+ g_{ij}(u,x)dx^i dx^j. \end{aligned} \tag{2.1.1}$$

Now we could be tempted to choose $e_v$ as the null evolution vector. In this case the constant time hypersurface $\Sigma_{v_0}$ is described by the induced metric

$$ds^2|_{\Sigma_{v_0}} = h(u,x)du^2 + 2l_i(u,x)dudx^i + g_{ij}(u,x)dx^i dx^j, \tag{2.1.2}$$

which can be either Lorentian or positive definite Riemannian. The causal character of $\Sigma_{v_0}$ may therefore vary from spacelike to timelike along the surface itself and this causes obvious issues in the definition

[1] From now on we will often use the coordinates notation $(v,u)$ in place of $(x^+, x^-)$. The reason will be clearer in the next chapters.

[2] The $x$ dependence of the coefficients $h, f_i, l_i$ and $g_{ij}$ should in general be understood from any $x^i$ component.

of the orthogonal direction.
If instead we choose $e_u$ as evolution vector, the induced metric reads

$$ds^2|_{\Sigma_{u_0}} = 2f_i(u_0, x)dvdx^i + g_{ij}(u_0, x)dx^i dx^j. \tag{2.1.3}$$

The causal character of $\Sigma_{u_0}$ is timelike when $f_i(u,x) \neq 0$ and lightlike if $f_i(u,x) = 0$. We focus on the lightlike case and we are able to define a (null) orthogonal propagation vector $N$ by requiring that $g(N, e_v) = -1$ and $g(N, e_{x^i}) = 0$, which results to be:

$$N = \left(1, \, \frac{h(u,x) - g^{ij}(u,x)l_i(u,x)l_j(u,x)}{2}, \, g^{ij}(u,x)l_j(u,x)\right). \tag{2.1.4}$$

The same problem can be analyzed from the massive particle world-line action point of view. Starting from

$$S = \int d\lambda \, \frac{1}{2}\left[\frac{1}{\zeta}\left(-2\dot{v}\dot{u} + h\dot{u}^2 + 2l_i\dot{u}\dot{x}^i + 2f_i\dot{v}\dot{x}^i + g_{ij}\dot{x}^i\dot{x}^j\right) - \zeta m^2\right], \tag{2.1.5}$$

where $\zeta$ is the one dimensional einbein, we choose the light-cone gauge $u = \tau$, upon which the action becomes:

$$S_{g.f.} = \int d\tau \, \frac{1}{2}\left[\frac{1}{\zeta}\left(-2\dot{v} + h + 2l_i\dot{x}^i + 2f_i\dot{v}\dot{x}^i + g_{ij}\dot{x}^i\dot{x}^j\right) - \zeta m^2\right]. \tag{2.1.6}$$

The only way to obtain a square root free gauge fixed Hamiltonian from this action is to require again that $f_i(u,x) = 0$. Indeed, if in this case we compute the conjugate momenta

$$p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = -\frac{1}{\zeta}, \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{l_i + g_{ij}\dot{x}^j}{\zeta}, \tag{2.1.7}$$

and we move from the lagrangian to the hamiltonian formalism we get:

$$H = \frac{1}{2}\left[p_i\left(l_j g^{ij} + \frac{p_j g^{ij}}{p_v}\right) + \frac{m^2}{p_v} + p_v\left(g^{ij}l_i l_j - h\right)\right], \tag{2.1.8}$$

which does not depend on the einbein $\zeta$. If instead we let $f_i(u,x) \neq 0$ we obtain

$$p_v = \frac{-1 + f_i \dot{x}^i}{\zeta}, \quad p_i = \frac{l_i + f_i \dot{v} + g_{ij}\dot{x}^j}{\zeta}, \tag{2.1.9}$$

and the final result is a nasty Hamiltonian which still depends on $\zeta$. To get rid of it, we need to substitute the e.o.m. for the einbein, which inevitably involves a square root. An analogous situation happens if we choose $v = \tau$, and this time for whichever value of $f_i$. This means that lightlike constant time hypersurfaces are related to square root free Hamiltonian, contrary to timelike and spacelike ones. The connection between the two formalisms can be made more quantitative. Given the induced metric of the $D-1$ dimensional constant time hypersurface $\Sigma_0$

$$ds^2|_{\Sigma_0} = \gamma_{IJ} dx^I dx^J \tag{2.1.10}$$

and the map between velocities and momenta

$$p_I = \frac{1}{\zeta}\left(\gamma_{IJ}\dot{x}^J + \frac{C_I}{2}\right), \tag{2.1.11}$$

it's manifest that both the volume element $Vol(\Sigma_0) = \det\gamma\, d^{D-1}x^I$ and the invertibility of (2.1.11) depend on $\det\gamma$. When this is zero, the hypersurface is lightlike and the map is not invertible, leading to the elimination of the square root, while things work in the opposite way when $\det\gamma \neq 0$.

To be clearer, let's see a basic example. Consider the usual three-dimensional light-cone Minkowski metric $ds^2 = -2dx^+dx^- + dx^2$ and change variables as

$$\begin{cases} x^+ = v + \frac{1}{2}\alpha u^2 \\ x^- = u \end{cases},$$

so that the metric becomes:

$$ds^2 = -2\alpha u du^2 - 2dudv + dx^2. \qquad (2.1.12)$$

If we take $v$ as time and $e_v$ as evolution vector, we see that the induced metric on $\Sigma_{v_0}$ can be either Lorentian or Riemannian, depending on the sign of $-2\alpha u$, and consequently the hypersurface causal character is not uniquely defined, since it varies from timelike to spacelike. In this case the particle Hamiltonian acquires a square root when we solve the einbein e.o.m.:

$$H = -\frac{p_u}{2\alpha u} + \frac{m^2}{\sqrt{-p_u^2 + 2\alpha u(-m^2 + p_x^2)}}. \qquad (2.1.13)$$

Upon the choice of $u$ as time, we get instead an always lightlike hypersurface $\Sigma_{u_0}$. The null orthogonal direction which dictates the evolution is identified by $N = (1, -\alpha u, 0)$ and the particle Hamiltonian reads:

$$H = \frac{1}{2p_v}\left(p_x^2 + m^2\right) + \alpha u p_v. \qquad (2.1.14)$$

Having found an efficient way to define light-cone evolution, let's try to apply this method to a metric which does not admit a Killing vector. We study

$$ds^2 = -2dudr - f(u,r)du^2 + h(u,r)d\theta^2, \qquad (2.1.15)$$

which can be seen as a generalization of the Vaidya metric, the radiating Schwarzschild metric [22]. The choice of $u = \tau$ leads to a lightlike constant time hypersurface described by $ds^2|_{\Sigma_{u_0}} = h(u_0, r)d\theta^2$. From the other side, the action

$$S_{g.f.} = \int d\tau \, \frac{1}{2}\left\{\frac{1}{\zeta}\left[-2\dot{r} - f(\tau, r) + h(\tau, r)\dot{\theta}^2\right] - \zeta m^2\right\} \qquad (2.1.16)$$

results in a gauge fixed Hamiltonian without square root:

$$H_{l.c.} = \frac{1}{2p_r}\left[\frac{p_\theta^2}{h(\tau,r)} + m^2\right] + \frac{1}{2}f(\tau,r)p_r, \tag{2.1.17}$$

where $p_r = \frac{1}{\zeta}$. We can therefore conclude that the equivalence between lightlike constant time hypersurfaces and gauge fixed square root free Hamiltonians holds also in this case.

### 2.1.2 A General Metric

We will consider the following metric:

$$ds^2 = -2du\,dv + h(u,x)du^2 + 2l_i(u,x)dudx^i + g_{ij}(u,x)dx^i dx^j, \tag{2.1.18}$$

which is basically the most general expression for the pp wave metric. It includes the two following special cases which we briefly recall.

- The first one corresponds to Rosen coordinates family of metrics and reads

  $$ds^2 = -2du\,dv + g(u)(dz)^2 + \sum_{i=3}^{D}(dx^i)^2, \tag{2.1.19}$$

  with $i = 3, ...D$. Actually, we can distinguish different subcases:

  $$\begin{cases} g(u) = 1 & \text{Minkowski} \\ g(u) = (\Delta u)^2 & \text{NBO coordinates in Minkowski}^{3} \\ g(u) = u^{2A} & \text{light-cone Kasner-Rosen} \end{cases} \tag{2.1.20}$$

  In particular, $A = 0, 1$ correspond to Minkowski spacetime, while the Kasner-Rosen metric can also be generalized to the case with

[3] We will discuss this case in detail later.

$P$ Kasner exponents $A_I$:

$$ds^2 = -2du\,dv + \sum_{I=1}^{P} u^{2A_I}(dz^I)^2 + \sum_{i=P+1}^{D}(dx^i)^2. \tag{2.1.21}$$

Light-cone quantization can be performed taking $u$ as time, having a degenerate induced metric on $\Sigma_{u_0}$. The evolution vector $\partial_u$ is null and the lightlike trajectories $\gamma(\lambda; v_0, \vec{z}_0, \vec{x}_0) = (u = \lambda, v = v_0, \vec{z} = \vec{z}_0, \vec{x} = \vec{x}_0)$ can be realized with massless physical observers. Notice also for the implication on light-cone quantization that the determinant of the metric is generically $(\sum_I A_I \neq 0)$ light-cone time-dependent and reads:

$$\sqrt{-\det g} = \sqrt{|g|} = |u|^{\sum_I A_I}. \tag{2.1.22}$$

This means that the spacetime at $u = 0$ is singular. However, this does not affect the argument presented in the next sections of this chapter, since we can always consider the evolution far from $u = 0$.

- The second class of family of metrics can be obtained from the previous one by changing to Brinkmann-Fermi coordinates:

$$\begin{cases} v_B = v_R + \frac{1}{2}\sum\limits_{I=1}^{P} A_I \frac{(x^I)^2}{u} \\ z^I_B = z^I_R |u|^{A_I} \end{cases}. \tag{2.1.23}$$

The metric reads:

$$ds^2 = -2du dv + \sum_{I=1}^{P} \frac{A_I(A_I - 1)(z^I)^2}{u^2} du^2 + \sum_{I=1}^{P}(dz^I)^2 + \sum_{i=P+1}^{D}(dx^i)^2, \tag{2.1.24}$$

where for $A_I = 0, 1$ we get the Minkowski spacetime. Notice that for $u_0 \neq 0$ the constant time hypersurfaces are identical, i.e. $\Sigma_{u0(R)} = \Sigma_{u0(B)}$. As before, the light-cone propagation corresponds to the choice of $u$ as time. However, now the trajectories $\gamma(\lambda; v_0, \vec{z}_0, \vec{x}_0) = (u = \lambda, v = v_0, \vec{z} = \vec{z}_0, \vec{x} = \vec{x}_0)$ can be interpreted as physical massive or massless observers. The first case corresponds to the evolution vector $\partial_u$ being timelike, with $\sum_I A_I(A_I - 1) < 0$, while the second one to a lightlike $\partial_u$, i.e. $\sum_I A_I(A_I - 1) = 0$. The observer with $x_0^I = 0$ is always physical and lightlike. Finally we notice that the determinant of the metric is trivial:

$$\sqrt{-\det g} = \sqrt{|g|} = 1. \tag{2.1.25}$$

## 2.2 Quantizing the Complex Scalar Density

We now consider the pp wave metric in order to show a non-trivial point: the field to quantize is not the scalar field but a scalar density. Nevertheless this consideration is independent of the explicit metric taken as example. We will start with a simple particle model and then we will treat the free complex scalar.

### 2.2.1 The Particle Model

In order to mimic the hallmark of light-cone approach, i.e. the appearance of only first order time derivatives, we consider the action

$$S = \int dt \left[ f(t)(y\dot{x} - x\dot{y}) - h(x, y, t) \right], \tag{2.2.1}$$

from which we derive the following e.o.m.:

$$\dot{x} = \frac{1}{2f(t)}\frac{\partial h}{\partial y} - \frac{\dot{f}(t)}{2f(t)}x, \qquad \dot{y} = -\frac{1}{2f(t)}\frac{\partial h}{\partial x} - \frac{\dot{f}(t)}{2f(t)}y. \tag{2.2.2}$$

If we act in a naive way and apply the Dirac procedure for constrained systems we get the classical Dirac bracket

$$\{x, y\}_{DB} = \frac{1}{2f(t)}. \tag{2.2.3}$$

The same result can be obtained if we read from the action the symplectic form $\omega = f(t)(ydx - xdy)$. Unfortunately, these approaches are flawed. The reason is that a symplectic form cannot depend on other coordinates than the symplectic ones, i.e. the explicit time dependence is not allowed. This can indeed be seen directly by checking that there is no Hamiltonian $H(x, y, t)$ which gives the previous e.o.m. using the Dirac bracket. Explicitly, from $\dot{x} = \{x, H\}_{DB}$ we get $H = h - \dot{f}xy$ while from $\dot{y} = \{y, H\}_{DB}$ we get $H = h + \dot{f}xy$. This means that we cannot take $x$ and $y$ as coordinates of the phase space. Therefore if we want to give an Hamiltonian interpretation to these e.o.m. we cannot treat $x$ and $y$ as canonical variables, as the unusual form of (2.2.2) and the previous discussion suggest, and we are forced to perform a change of variables at the Lagrangian level. There are obviously many coordinate redefinitions which eliminate the $t$ factor and we take for example:

$$x = \frac{1}{\sqrt{2f(t)}}\hat{x}, \qquad y = \frac{1}{\sqrt{2f(t)}}\hat{y}. \tag{2.2.4}$$

The action (2.2.1) now reads

$$S = \int dt \left[\frac{1}{2}(\hat{y}\dot{\hat{x}} - \hat{x}\dot{\hat{y}}) - h\left(\frac{1}{\sqrt{2f(t)}}\hat{x}, \frac{1}{\sqrt{2f(t)}}\hat{y}, t\right)\right], \tag{2.2.5}$$

from which follow the usual e.o.m.

$$\dot{\hat{x}} = \frac{\partial h}{\partial \hat{y}}, \qquad \dot{\hat{y}} = -\frac{\partial h}{\partial \hat{x}}, \tag{2.2.6}$$

and therefore the identifications

$$q \equiv \hat{x}, \quad p \equiv \hat{y}, \quad H(\hat{x}, \hat{y}, t) \equiv h\left(\frac{1}{\sqrt{2f(t)}}\hat{x}, \frac{1}{\sqrt{2f(t)}}\hat{y}, t\right). \tag{2.2.7}$$

Given this well-defined phase space, our original variables can be seen as "composite operators" of the true canonical variables and computing the Poisson bracket of the original ones from this point of view we get

$$\{x, y\} = \frac{1}{2f(t)}. \tag{2.2.8}$$

We can also recover the e.o.m. (2.2.2) if we take into account the explicit dependence of $x$ and $y$ on $t$, i.e.:

$$\dot{x} = \{x, H\} + \frac{\partial x}{\partial t}, \qquad \dot{y} = \{y, H\} + \frac{\partial y}{\partial t}. \tag{2.2.9}$$

In other words, this means that $x$ and $y$ have to be seen as time-dependent functions defined on the phase space (2.2.7). The bottom line of this discussion is that variable redefinitions are necessary to get rid of the time dependence (and also of any additional constant) which appears in front of the "kinetic" terms of (2.2.1). We will see in a moment how to rephrase this in field theory.

### 2.2.2 The Complex Scalar Field

We begin with the classical treatment of a complex scalar field and then we move to the quantum one. This background has essentially been considered before in the usual formalism in [23]. The starting point is the kinetic part of the action which reads:

$$
\begin{aligned}
S_2 =& \int du\, dv\, d^{D-2}x \sqrt{|g|} \Big\{ -g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - M^2 \phi^* \phi \Big\} \\
=& \int du\, dv\, d^{D-2}x \sqrt{|g|} \Big\{ \partial_u \phi^* \, \partial_v \phi + \partial_v \phi^* \, \partial_u \phi + \big[ h(u,x) - \bar{l}^2(u,x) \big] \, \partial_v \phi^* \, \partial_v \phi \\
& - \bar{l}^i(u,x) (\partial_i \phi^* \, \partial_v \phi + \partial_v \phi^* \, \partial_i \phi) - g^{ij}(u,x) \partial_i \phi^* \, \partial_j \phi - M^2 \phi^* \phi \Big\} \\
=& \int du\, dv\, d^{D-2}x \sqrt{|g|} \Big\{ \partial_u \phi^* \, \partial_v \phi + \partial_v \phi^* \, \partial_u \phi + h(u,x) \partial_v \phi^* \, \partial_v \phi \\
& - g^{ij}(u,x) \left[ \partial_i \phi^* \, + l_i(u,x) \partial_v \phi^* \right] \left[ \partial_j \phi + l_j(u,x) \partial_v \phi \right] - M^2 \phi^* \phi \Big\},
\end{aligned}
\tag{2.2.10}
$$

where $\bar{l}^i = \bar{g}^{ij} l_j$ and $\bar{l}^2 = \bar{g}^{ij} l_j l_j$, with $\bar{g}^{ij}$ being the inverse of $g_{ij}$. Notice that $g^{ij} = \bar{g}^{ij}$, i.e. the restriction to $i, j$ indices of $g^{\mu\nu}$, matches $\bar{g}^{ij}$ and that $\det(g_{ij}) = \det(\bar{g}_{ij})$.

In the following we consider the complex scalar field coupled to an electromagnetic background. In order to simplify the computations we take $A_\mu$ invariant under the same Killing vector $e_v = \partial_v$. Moreover, we choose the light-cone gauge

$$
A_v = 0, \qquad A_u = A_u(u,x) \qquad A_i = A_i(u,x), \tag{2.2.11}
$$

so that the action reads:

$$
\begin{aligned}
S_2 = \int & du\, dv\, d^{D-2}x \sqrt{|g(u,x)|} \Big\{ \mathcal{D}_u \phi^* \, \partial_v \phi + \partial_v \phi^* \, \mathcal{D}_u \phi + h(u,x) \partial_v \phi^* \, \partial_v \phi \\
& - g^{ij}(u,x) \left[ \mathcal{D}_i \phi^* \, + l_i(u,x) \partial_v \phi^* \right] \left[ \mathcal{D}_j \phi + l_j(u,x) \partial_v \phi \right] - M^2 \phi^* \phi \Big\},
\end{aligned}
\tag{2.2.12}
$$

where $\mathcal{D}_\mu = \partial_\mu - ie\, A_\mu$.

As shown in the particle model in the previous section, we cannot proceed as usual not even with the more formal approach of the Dirac brackets. We can indeed recognize a situation analogous to (2.2.1).

Here the role of $f(t)$ is played by the factor $\sqrt{|g(u,x)|}$ and there is no way to quantize the theory using the fields $\phi$ and $\phi^*$. When moving to the Hamiltonian formulation we would encounter the same issues of the previous section in terms of Dirac brackets and e.o.m.: we hence need to redefine our fields mimicking (2.2.4). The minimal field redefinition for the light-cone quantization is:

$$\phi(u,v,x) = |g(u,x)|^{-\frac{1}{4}}\hat{\phi}(u,v,x), \tag{2.2.13}$$

where the new field is not anymore a scalar but a scalar density[4]. This rescaling is obviously necessary for all the other spins and does not change the physics.
The action becomes:

$$S_2 = \int du\, dv\, d^{D-2}x \Bigg[\mathcal{D}_u\hat{\phi}^*\,\partial_v\hat{\phi} + \partial_v\hat{\phi}^*\,\mathcal{D}_u\hat{\phi} + h\,\partial_v\hat{\phi}^*\,\partial_v\hat{\phi} \\ - g^{ij}\left(|g|^{\frac{1}{4}}\mathcal{D}_i\frac{\hat{\phi}}{|g|^{\frac{1}{4}}} + l_i\partial_v\hat{\phi}\right)^*\left(|g|^{\frac{1}{4}}\mathcal{D}_j\frac{\hat{\phi}}{|g|^{\frac{1}{4}}} + l_j\partial_v\hat{\phi}\right) - M^2\hat{\phi}^*\hat{\phi}\Bigg]. \tag{2.2.14}$$

We perform now a Fourier transform w.r.t. $v$ as:

$$\hat{\phi}(u,v,x) = \int_{-\infty}^{\infty}\frac{dk_v}{(2\pi)^{1/2}}e^{ik_v v}\frac{1}{\sqrt{2|k_v|}}\tilde{\hat{\phi}}(u,k_v,x). \tag{2.2.15}$$

The previous expression can be written for the original scalar field as:

$$\phi(u,v,x) = \frac{1}{|g(u,x)|^{\frac{1}{4}}}\int_{-\infty}^{0}\frac{dk_v}{(2\pi)^{1/2}}e^{ik_v v}\frac{1}{\sqrt{2|k_v|}}\tilde{\hat{\phi}}(u,k_v,x) \\ + \frac{1}{|g(u,x)|^{\frac{1}{4}}}\int_{-\infty}^{0}\frac{dl_v}{(2\pi)^{1/2}}e^{-il_v v}\frac{1}{\sqrt{2|l_v|}}\tilde{\hat{\phi}}(u,-l_v,x), \tag{2.2.16}$$

[4]This scalar density is the object that makes the integrability condition trivial in [24], in particular after the null reduction as in [25].

in a way which is useful to compare with the second quantization of the particle. In particular it can be interpreted as the sum of two particles, the first one with wave function $\tilde{\hat{\phi}}(u, k_v, x)$ and $k_v < 0$, the second one with wave function $\tilde{\hat{\phi}}(u, -k_v, x)$ and $k_v < 0$[5]. This happens because we are quantizing a complex scalar; if we chose instead a real scalar $\phi \to \frac{1}{\sqrt{2}}\phi_{\mathbb{R}}$, we would get $\tilde{\hat{\phi}}_{\mathbb{R}}(u, -k_v, x) = \tilde{\hat{\phi}}_{\mathbb{R}}(u, k_v, x)^*$ and therefore only one particle with wave function $\tilde{\hat{\phi}}_{\mathbb{R}}(u, k_v, x)$.

The original action can then be written in a form which can be interpreted as the sum of two actions for two particles:

$$
\begin{aligned}
S_2 = \int du\, d^{D-2}x \int_{-\infty}^{0} dk_v \Bigg\{ \Bigg[ & i(\tilde{\hat{\phi}}(k_v))^* \,\partial_u \tilde{\hat{\phi}}(k_v) \\
& + \frac{1}{2k_v} g^{ij} \left( |g|^{\frac{1}{4}} \mathcal{D}_i \frac{\tilde{\hat{\phi}}(k_v)}{|g|^{\frac{1}{4}}} + i k_v l_i \,\tilde{\hat{\phi}}(k_v) \right)^* \left( |g|^{\frac{1}{4}} \mathcal{D}_j \frac{\tilde{\hat{\phi}}(k_v)}{|g|^{\frac{1}{4}}} + i k_v l_j \,\tilde{\hat{\phi}}(k_v) \right) \\
& + \left( -eA_u - \frac{1}{2} k_v h + \frac{M^2}{2k_v} \right) (\tilde{\hat{\phi}}(k_v))^* \tilde{\hat{\phi}}(k_v) \Bigg] \\
& + \Bigg[ i(\tilde{\hat{\phi}}^*(k_v))^* \,\partial_u \tilde{\hat{\phi}}^*(k_v) \\
& + \frac{1}{2k_v} g^{ij} \left( |g|^{\frac{1}{4}} \mathcal{D}^*_i \frac{\tilde{\hat{\phi}}^*(k_v)}{|g|^{\frac{1}{4}}} - i k_v l_i \,\tilde{\hat{\phi}}^*(k_v) \right)^* \left( |g|^{\frac{1}{4}} \mathcal{D}^*_j \frac{\tilde{\hat{\phi}}^*(k_v)}{|g|^{\frac{1}{4}}} - i k_v l_j \,\tilde{\hat{\phi}}^*(k_v) \right) \\
& + \left( +eA_u + \frac{1}{2} k_v h + \frac{M^2}{2k_v} \right) (\tilde{\hat{\phi}}^*(k_v))^* \tilde{\hat{\phi}}^*(k_v) \Bigg] \Bigg\},
\end{aligned}
\tag{2.2.17}
$$

where we have integrated by parts in time $u$ in order to get a canonical term $p\dot{q}$ and we have dropped the boundary term $\int dv \partial_v (\dots \hat{\phi}^* \hat{\phi})$ under the assumption that $\hat{\phi} \to 0$ as $v \to \pm\infty$. This asymptotic behavior is

[5]Notice that with our notation $\widetilde{\hat{\phi}^*(v)}(k_v) = \left(\tilde{\hat{\phi}}(-k_v)\right)^*$ so that we can express $\tilde{\hat{\phi}}(-k_v)$ using the field $\widetilde{\hat{\phi}^*(v)}(k_v)$ which has the natural range to be interpreted as a particle.

also important for getting a conserved charge as discussed below. Notice that the interpretation as sum of two independent particles is possible because each contribution in square brackets is real (up to boundary terms). Moreover, only when using the natural fields $\widetilde{\hat{\phi}}(k_v)$ and $\widetilde{\hat{\phi}}^*(k_v)$ ($k_v < 0$) it appears clearly also in the covariant derivative $\mathcal{D}_i^*$ that the antiparticle described by $\widetilde{\hat{\phi}}^*(k_v)$ has the opposite charge $-e$.

In the real case the two contributions are equal so that the action for a real scalar reads:

$$S_{2,\,real} = \int du\, d^{D-2}x \int_{-\infty}^{0} dk_v \Bigg[ i(\widetilde{\hat{\phi}}_{\mathbb{R}}(k_v))^* \,\partial_u \widetilde{\hat{\phi}}_{\mathbb{R}}(k_v) + \frac{1}{2k_v} g^{ij} \left( |g|^{\frac{1}{4}} \partial_i \frac{\widetilde{\hat{\phi}}_{\mathbb{R}}(k_v)}{|g|^{\frac{1}{4}}} + i k_v l_i \,\widetilde{\hat{\phi}}_{\mathbb{R}}(k_v) \right)^* \left( |g|^{\frac{1}{4}} \partial_j \frac{\widetilde{\hat{\phi}}_{\mathbb{R}}(k_v)}{|g|^{\frac{1}{4}}} + i k_v l_j \,\widetilde{\hat{\phi}}_{\mathbb{R}}(k_v) \right) \Bigg]. \tag{2.2.18}$$

The canonical coordinates are

$$\begin{cases} q \sim \widetilde{\hat{\phi}}(k_v, x),\ \left(\widetilde{\hat{\phi}}(-k_v, y)\right)^* \\ p \sim i\left(\widetilde{\hat{\phi}}(k_v, y)\right)^*,\ i\widetilde{\hat{\phi}}(-k_v, y) \end{cases} , \tag{2.2.19}$$

with no time dependence since we are in Hamiltonian formalism, and the canonical commutation relations are (for $k_v < 0$):

$$\begin{aligned} [\widetilde{\hat{\phi}}(k_{1\,v}, x_1),\ \widetilde{\hat{\phi}}(k_{2\,v}, x_2)^*] &= \delta(k_{1\,v} - k_{2\,v})\,\theta(-k_{v1})\,\delta^{D-2}(x_1 - x_2), \\ [\widetilde{\hat{\phi}}^*(k_{1\,v}, x_1),\ \widetilde{\hat{\phi}}^*(k_{2\,v}, x_2)^*] &= \delta(k_{1\,v} - k_{2\,v})\,\theta(-k_{v1})\,\delta^{D-2}(x_1 - x_2). \end{aligned} \tag{2.2.20}$$

As a consequence, if we would consider the Schrödinger formalism the wave functional would depend on the scalar density and not on the field, i.e. $\Psi(\hat{\phi}(v,x), u)$.

### 2.2.3 The Light-Cone Field Expansion and Quantization

We now want to expand the fields in the Heisenberg representation in modes to read the creation and annihilation operators. Looking at (2.2.20) it seems natural to treat $\tilde{\hat{\phi}}(u, k_v, x)$ and $\tilde{\hat{\phi}}^*(u, k_v, x)$ separately and then to join their expansions using (2.2.16). The e.o.m. for $\tilde{\hat{\phi}}(u, k_v, x)$ ($k_v < 0$) is like a Schrödinger equation and reads:

$$\begin{aligned} i\partial_u \tilde{\hat{\phi}}(u, k_v, x) = &\frac{1}{2k_v}\left[-\nabla_i g^{ij}\nabla_j + M^2\right]\tilde{\hat{\phi}}(u, k_v, x) \\ &- \left[eA_u - \frac{1}{2}k_v h\right]\tilde{\hat{\phi}}(u, k_v, x). \end{aligned} \tag{2.2.21}$$

We can then introduce an orthonormal complete basis $\{\tilde{\Psi}_{(n,p_v)}(u, k_v, x; e)\}$[6]. These functions are orthonormal w.r.t. the "spatial coordinates" $k_v, x$, i.e. for all times $u$ we have:

$$\int_{-\infty}^{0} dk_v \int d^{D-2}x\, \tilde{\Psi}^*_{(m,p_v)}(u, k_v, x; e)\, \tilde{\Psi}_{(n,q_v)}(u, k_v, x; e) = \delta_{m,n}\delta(p_v - q_v). \tag{2.2.22}$$

This basis can be obtained from the time evolution of the orthonormal complete basis $\{\tilde{\psi}_n(u_0, x, k_v; e)\}$ as:[7]

$$\tilde{\Psi}_{(n,p_v)}(u, k_v, x; e) = \tilde{\psi}_n(u_0, x; k_v, e)\, \delta(k_v - p_v). \tag{2.2.23}$$

The orthonormal complete basis $\{\tilde{\psi}_n(u_0, x; k_v, e)\}$ is associated with the stationary Schrödinger equation

$$\begin{aligned} &\left[\frac{1}{2k_v}\left(-\nabla_i g^{ij}\nabla_j + M^2\right) - \left(eA_u + \frac{1}{2}k_v h\right)\right]_{u=u_0} \tilde{\psi}_n(u_0, x; k_v, e) \\ &= E_n \tilde{\psi}_n(u_0, x; k_v, e), \end{aligned} \tag{2.2.24}$$

[6] Notice that we have explicitly shown the dependence on the charge $e$.

[7] The $\delta(k_v - p_v)$ factor may at first sight appear strange but it is nothing more than the wave function of the free particle in momentum space associated with the Hamiltonian $H = \frac{p^2}{2m}$.

where $k_v$ is considered a parameter so that:

$$\int d^{D-2}x\, \tilde{\psi}^*_m(u_0, x; k_v, e)\, \tilde{\psi}_n(u_0, x; k_v, e) = \delta_{m,n}. \tag{2.2.25}$$

Then we can expand the field $\tilde{\hat{\phi}}_H(u, k_v, x)$ for $k_v < 0$ in the Heisenberg picture as:

$$\begin{aligned}\tilde{\hat{\phi}}_H(u, k_v, x) &= \sum_n \int_{-\infty}^0 dp_v a_{(n,p_v)H}(u)\, \tilde{\Psi}_{(n,p_v)}(u, k_v, x; e) \\ &= \sum_n a_{(n,k_v)H}(u)\, \tilde{\psi}_n(u, x; k_v, e),\end{aligned} \tag{2.2.26}$$

where the operators $a_{(n,k_v)H}(u)$ are actually constant because of the e.o.m. In a similar way we can expand $\tilde{\hat{\phi}}^*{}_H(u, k_v, x)$ for $k_v < 0$:

$$\tilde{\hat{\phi}}^*{}_H(u, k_v, x) = \sum_n b_{(n,k_v)H}(u)\, \tilde{\psi}^*_n(u, x; k_v, -e), \tag{2.2.27}$$

where the operators $b_{(n,k_v)H}(u)$ are again constant because of the e.o.m. Since we know from (2.2.20) that the basis $\{\tilde{\Psi}_{(n,p_v)}(u, k_v, x; e)\}$ is orthonormal, we get the usual commutation relation:

$$[a_{(m,k_v)H},\, a^\dagger_{(n,p_v)H}] = [b_{(m,k_v)H},\, b^\dagger_{(n,p_v)H}] = \delta_{m,n}\delta(k_v - p_v). \tag{2.2.28}$$

The light-cone vacuum is defined also as

$$a_{(m,k_v)}|\Omega\rangle = b_{(m,k_v)}|\Omega\rangle = 0. \tag{2.2.29}$$

Finally, we can expand the original field in the Heisenberg picture as:

$$\phi_H(u,v,x) = \frac{1}{|g(u,x)|^{\frac{1}{4}}} \int_{-\infty}^{0} \frac{dk_v}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k_v|}}$$
$$\left[ e^{ik_v v} \sum_n a_{(n,k_v)H}\, \tilde{\psi}_n(u,x;k_v,e) + e^{-ik_v v} \sum_n b^{\dagger}_{(n,k_v)H}\, \tilde{\psi}^{*}_n(u,x;k_v,-e) \right]. \tag{2.2.30}$$

Differently from the usual second order evolution, the creation and annihilation operators can be obtained without time derivatives as:

$$\begin{aligned} a_{(m,l_v)} &= \int d^{D-2}x \frac{dv}{\sqrt{2\pi}} e^{-il_v v} |g(u,x)|^{\frac{1}{4}} \tilde{\psi}^{*}_m(u,x;l_v,e) \phi_H(u,v,x), \qquad l_v > 0, \\ b^{\dagger}_{(m,-l_v)} &= \int d^{D-2}x \frac{dv}{\sqrt{2\pi}} e^{-il_v v} |g(u,x)|^{\frac{1}{4}} \tilde{\psi}_n(u,x;-l_v,-e) \phi_H(u,v,x), \;\; l_v < 0. \end{aligned} \tag{2.2.31}$$

As an application for the special cases considered it results that the vacua for the Rosen and Brinkmann coordinates are the same. This happens since if we look at the equal time hypersurfaces we have $v_R = v_B + \dots$, where $\dots$ are terms independent from $v$, and it follows that $k_{v\,R} = k_{v\,B}$. Therefore $a_R$ can be expressed using $a_B$ only.

## 2.3 Second Quantization of the Particle

In this section we would like to explore how the second quantization of the particle on the light-cone is connected to the light-cone quantization of the scalar field.

### 2.3.1 The Action

The action for the particle in a generic gravitational and electromagnetic background reads:

$$
\begin{aligned}
S_{particle} &= \int d\lambda \left( -m\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} + e_{ph}A_\mu(x)\frac{dx^\mu}{d\lambda}\right) \\
&= \int d\lambda\, \zeta(\lambda)\left[\frac{1}{2}\left(g_{\mu\nu}(x)\frac{dx^\mu}{\zeta(\lambda)d\lambda}\frac{dx^\nu}{\zeta(\lambda)d\lambda} - m^2\right) + e_{ph}A_\mu(x)\frac{dx^\mu}{\zeta(\lambda)d\lambda}\right],
\end{aligned}
\tag{2.3.1}
$$

where $e_{ph}$ is the physical electric charge and $ds^2_{world-line} = -\zeta^2(d\lambda)^2$ is the worldline metric. Notice that in order to reproduce the original action we need

$$\zeta > 0, \tag{2.3.2}$$

and this constraint is important in the following when considering the range of the $v$ momentum. The action has the diffeomorphism invariance

$$d\lambda\,\zeta(\lambda) = d\tau\,\zeta'(\tau), \qquad x^\mu(\lambda) = x'^\mu(\tau). \tag{2.3.3}$$

The e.o.m. read:

$$
\begin{aligned}
-\zeta^2\frac{\delta S}{\delta\zeta} &= \frac{\dot{x}^2}{\zeta^2} + m^2 = 0, \\
\frac{\delta S}{\delta x^\mu} &= -\frac{d}{d\lambda}\left(g_{\mu\nu}(x)\frac{dx^\nu}{\zeta(\lambda)d\lambda}\right) + e_{ph}F_{\mu\nu}(x)\frac{dx^\nu}{d\lambda} = 0.
\end{aligned}
\tag{2.3.4}
$$

Consider now the previous action in the metric (2.1.18) coupled to the electromagnetic background (2.2.11), which as we know are both invariant under the same null Killing vector. We gauge fix the diffeomorphisms as $u = \tau$ so that the action becomes:

$$\begin{aligned}
S_{l.c.} &= \int d\tau \Bigg\{ \frac{1}{\zeta} \left[ -\dot{v} + \frac{1}{2} h(\tau, x) + \frac{1}{2} g_{ij}(\tau, x) \dot{x}^i \dot{x}^j + l_i(\tau, x) \dot{x}^i \right] \\
&\qquad\qquad + e_{ph}\, A_u(\tau, x) + e_{ph}\, A_i(\tau, x) \dot{x}^i - \frac{1}{2} \zeta m^2 \Bigg\} \\
&= \int d\tau \Bigg\{ + p_v \dot{v} + p_i \dot{x}^i \\
&\qquad\qquad - \left[ - \frac{1}{2p_v} \bar{g}^{ij} \left( p_i + l_i p_v - e_{ph}\, A_i \right) \left( p_j + l_j p_v - e_{ph}\, A_j \right) \right. \\
&\qquad\qquad \left. - \frac{m^2}{2p_v} + e_{ph}\, A_u + \frac{1}{2} h p_v \right] \Bigg\},
\end{aligned} \tag{2.3.5}$$

where $\bar{g}^{ij}$ is the inverse of the metric $\bar{g}_{ij} = g_{ij}$. In this formulation $(h, l_i)$ acts as a kind of supplementary gauge field.

### 2.3.2 Light-Cone Hamiltonian Formalism and Quantization

We can read the Poisson brackets

$$\{v, p_v\} = \{x^i, p_i\} = 1, \tag{2.3.6}$$

and the classical light-cone Hamiltonian

$$\begin{aligned}
H_{lc(classical)}(\tau, p_v, x^i, p_i) = &- \frac{1}{2p_v} g^{ij} \left( p_i + l_i p_v - e_{ph}\, A_i \right) \left( p_j + l_j p_v - e_{ph}\, A_j \right) \\
&- \frac{m^2}{2p_v} + e_{ph}\, A_u + \frac{1}{2} h p_v.
\end{aligned} \tag{2.3.7}$$

Now this Hamiltonian suffers from ordering problems. We want it to be Hermitian but it is not uniquely fixed since if we change the measure of integration we get different Hamiltonians: $H = p_i g^{ij} p_j$ is Hermitian w.r.t. $vol = d^{D-2}x$, while $H = \frac{1}{\sqrt{|g|}} p_i \sqrt{|g|} g^{ij} p_j$ is Hermitian w.r.t. $vol = \sqrt{|g|} d^{D-2}x$. Moreover, even when we fix the volume element we do not get a unique result. Indeed, let's consider a light-cone Hamiltonian which is hermitian w.r.t. to $vol = \mu(\tau, x) d^{D-2}x\, dp_v$[8] and reduces to the classical one but differs quantum mechanically; it can be written as:

$$\begin{aligned} H_{lc(1st),\mu,\rho,\sigma}(x, p_v) = &+ \frac{1}{2p_v} \frac{1}{\mu(\tau, x)\rho(\tau, x)} \nabla_i \left( g^{ij}(\tau, x)\sigma(\tau, x) \nabla_j \frac{1}{\rho(\tau, x)} \right) \\ &+ V(\tau, x), \end{aligned} \tag{2.3.8}$$

where $\rho(\tau, x)$ and $\sigma(\tau, x)$ are arbitrary functions and we have introduced the "gauge" covariant derivative

$$\nabla_j = ip_j - i\left(e_{ph}\, A_i - l_j p_v\right). \tag{2.3.9}$$

If we want to reproduce the rescaled complex scalar Hamiltonian we must set

$$\mu = 1, \qquad \sigma = \sqrt{|g|}, \qquad \rho = \sqrt[4]{|g|}, \tag{2.3.10}$$

i.e. we need an Hamiltonian Hermitian w.r.t. $vol = d^{D-2}x$ and we need to know that we are considering a scalar density of weight $\frac{1}{4}$. This information has to be supplied and it does not come out of the formalism automatically. Finally, the first quantized quantum Hermitian Hamiltonian which reproduces the scalar action under a second quantization can be written as:

$$H_{lc(1st)} = + \frac{1}{2p_v} \frac{1}{|g|^{\frac{1}{4}}} \nabla_i \left( \sqrt{|g|} g^{ij} \nabla_j \frac{1}{|\bar{g}|^{\frac{1}{4}}} \right) + e_{ph}\, A_u + \frac{1}{2} h p_v. \tag{2.3.11}$$

This expression follows from the usual one $\frac{1}{\sqrt{|g|}} p_i \sqrt{|g|} g^{ij} p_j$, which is Hermitian w.r.t. $vol = \sqrt{|g|} d^{D-2}x$, by replacing $\nabla_i \to |g|^{\frac{1}{4}} \nabla_i \frac{1}{|g|^{\frac{1}{4}}}$ as suggested by the replacement of a scalar with a scalar density.

[8]The hermiticity w.r.t. $dp_v$ is trivial but in the measure we need it since $p_v$ appears on the same level of $x^i$.

In the case of the pp wave metric in Rosen coordinates, and for all the other metrics whose determinant depends only on light-cone time, the naive connection works since $\rho(\tau)$ filters through the spatial derivatives.

Since we are dealing with a time-dependent Hamiltonian there is no energy conservation and therefore we cannot find a basis of energy eigenfunctions. We can proceed as done in Section 2.2.3. We consider the instantaneous Hamiltonian $H_{lc(1st)}(\tau_0)$, which is Hermitian, and therefore we can find an instantaneous basis $\{\tilde{\Psi}_a(\tau_0, p_v, x) = \tilde{\psi}_n(\tau_0, x, k_v)\,\delta(p_v - k_v)\}$ whose elements are labeled by $a = (n, k_v)$. Using this basis we can expand the second quantized field in the Heisenberg picture as:

$$\tilde{\Psi}_H(\tau, p_v, x) = \sum\!\!\!\!\!\!\!\int_a A_{aH}(\tau;\tau_0)\,\tilde{\Psi}_a(\tau_0, p_v, x) = \sum_n \hat{A}_{(n,p_v)H}(\tau;\tau_0)\,\tilde{\psi}_n(\tau, p_v, x;\tau_0), \tag{2.3.12}$$

where $A_{aH}(\tau;\tau_0)$ are labeled by $\tau_0$ but also by $a$ which includes $k_v$. The $A_{aH}(\tau;\tau_0)$ are annihilators of the second quantized vacuum $|\Omega\rangle$

$$A_{aH}(\tau;\tau_0)|\Omega\rangle = 0 \tag{2.3.13}$$

and satisfy the harmonic oscillator algebra

$$[A_{aH}(\tau;\tau_0), A^\dagger_{bH}(\tau;\tau_0)] = \delta_{a,b} = \delta(n_a - n_b)\,\delta(k_{v\,a} - k_{v\,b}). \tag{2.3.14}$$

While the previous two equations are kinematical statements for $\tau = \tau_0$, for all the other possible $\tau$ they are dynamical and follow from the second quantized action:

$$
\begin{aligned}
S_{2(2nd)} =& \int d\tau\, d^{D-2}x \int_{-\infty}^{0} dp_v\, \tilde{\Psi}_H^\dagger(\tau,p_v,x)\left[i\partial_\tau - H_{lc(1st)}(\tau,x,p_v,p)\right]\tilde{\Psi}_H(\tau,p_v,x)\\
=& \int d\tau\, d^{D-2}x \int_{-\infty}^{0} dp_v\, \tilde{\Psi}_H^\dagger(\tau,p_v,x)\Bigg\{ i\partial_\tau - \Bigg[\frac{1}{2p_v}\frac{1}{|\bar{g}|^{\frac14}}\nabla_i\left(\sqrt{|\bar{g}|}g^{ij}\nabla_j\frac{1}{|\bar{g}|^{\frac14}}\right)\\
&- \frac{m^2}{2p_v} + e_{ph}A_u + \frac{1}{2}hp_v\Bigg]\Bigg\}\tilde{\Psi}_H(\tau,p_v,x) \qquad (2.3.15)\\
=& \sum\!\!\!\!\!\!\int_{a,b} A_{aH}^\dagger(\tau;\tau_0)\left[i\partial_\tau\delta_{ab} - h_{ab}(\tau,\tau_0)\right]A_{bH}(\tau;\tau_0)\\
=& \sum_{n,m}\int_{-\infty}^{0} dp_v\, A_{(n,p_v)H}^\dagger(\tau;\tau_0)\left[i\partial_\tau\delta_{nm} - h_{nm}(\tau,\tau_0,p_v)\right]A_{(m,p_v)H}(\tau;\tau_0),
\end{aligned}
$$

(2.3.16)

where

$$
\begin{aligned}
h_{ab}(\tau,\tau_0) =& \int d^{D-2}x \int_{-\infty}^{0} dp_v\, \tilde{\Psi}_a^*(\tau,p_v,x)H_{lc}(\tau,x,p_v,p)\tilde{\Psi}_b(\tau,p_v,x)\\
&= \delta(k_{v\,a} - k_{v\,b}) \int d^{D-2}x\, \tilde{\psi}_n^*(\tau,p_v,x)H_{lc}(\tau,x,p_v,p)\tilde{\psi}_m(\tau,p_v,x).
\end{aligned}
$$

(2.3.17)

We can therefore write

$$
A_{aH}(\tau;\tau_0) = \sum\!\!\!\!\!\!\int_b A_{bH}(\tau_0;\tau_0)U_{S,ba}(\tau,\tau_0), \qquad (2.3.18)
$$

where $U_{S,ba}(\tau,\tau_0)$ is the Schrödinger evolution operator in the $\{\tilde{\Psi}_a\}$ basis. This means that the time evolution does not mix creation and annihilation operators and the annihilation operators annihilate the vacuum for all times. In other words, the vacuum would in principle depends on $\tau_0$ and $p_v$, but it is actually independent because of the time evolution of $A_H$ which does not involve $A_H^\dagger$.

If we compare the previous quantum action (2.3.16) with the light-cone action of a complex scalar field (2.2.17) we see they match only when we introduce two particles with opposite charges so that we can make the identifications between the quantum fields $\widetilde{\hat{\phi}}_H(u, k_v, x) = \tilde{\Psi}_{1st\, part\ H}(\tau = u, k_v, x)$ and $\widetilde{\hat{\phi}^*}_H(u, k_v, x) = \tilde{\Psi}_{2nd\, part\ H}(\tau = u, k_v, x)$ for $k_v < 0$. For the real scalar we need instead one particle so that we have the match $\widetilde{\hat{\phi}}_{\mathbb{R}}(u, k_v, x) = \tilde{\Psi}_{part\ H}(\tau = u, k_v, x)$ for $k_v < 0$.

Notice that we have coupled one particle with an electromagnetic field and therefore we expect this particle to describe a complex scalar field; nevertheless, we have seen that it may describe also a real scalar, which should not couple to an electromagnetic field. The resolution of this puzzle is that the coupling of a particle with an electromagnetic background describes something like $e^- + \gamma^* \to e^-$ where $\gamma^*$ is a virtual photon. If we want to describe the process $e^+ + \gamma^* \to e^+$ we need another particle with opposite charge and this means exactly the introduction of another second quantized field; then we can match the light-cone action for a complex scalar which is the "double" of the real one.

### 2.3.3 On the Meaning of the Wave Function $\tilde{\psi}(\tau, k_v, x)$

We will now discuss the meaning of the wave function $\tilde{\psi}(\tau, k_v, x)$ associated with the Hamiltonian (2.3.11). The wording would suggest that this is a usual wave function which can be interpreted as probability density. This is not the case since, as we are going to show, its real meaning is a charge density. In order to uncover this, we discuss and compare the conserved currents in the particle and scalar field cases.

#### 2.3.3.1 Conserved Current for the Particle

Let us consider the more general Hamiltonian (2.3.8) to construct and discuss the conserved "probability" current. If we take two solutions of the Schrödinger equation

$$i\frac{\partial}{\partial\tau}\tilde{\psi}(\tau,p_v,x)=H_{lc(1st),\mu,\rho}\tilde{\psi}(\tau,p_v,x), \tag{2.3.19}$$

we can ask whether the quantity

$$Q(\tilde{\psi}_1,\tilde{\psi}_2;\tau_0)=\int d^{D-2}x\,\mu(\tau_0,x)\,\tilde{\psi}_1(\tau_0,x)^*\tilde{\psi}_2(\tau_0,x) \tag{2.3.20}$$

is conserved. In particular, when $\tilde{\psi}_1=\tilde{\psi}_2\equiv\tilde{\psi}$, $Q(\tilde{\psi},\tilde{\psi})$ can be interpreted as probability being always non negative. Using the hermiticity of the Hamiltonian we get:

$$\begin{aligned}Q(\tilde{\psi}_1,\tilde{\psi}_2;\tau_1)-Q(\tilde{\psi}_1,\tilde{\psi}_2;\tau_0)&=\int_{\tau_0}^{\tau_1}d\tau\,\partial_\tau Q(\tilde{\psi}_1,\tilde{\psi}_2;\tau)\\&=\int_{\tau_0}^{\tau_1}d\tau\int d^{D-2}x\,\partial_\tau\mu(\tau,x)\,\tilde{\psi}_1(\tau,x)^*\tilde{\psi}_2(\tau,x).\end{aligned} \tag{2.3.21}$$

Therefore whenever $\partial_\tau\mu(\tau,x)=0$ we get a conserved charge. More explicitly, when $\partial_\tau\mu(\tau,x)=0$ we can introduce the gauge invariant current:

$$\begin{aligned}\tilde{J}_i&=-i\sigma\left(\frac{\tilde{\psi}_1^*}{\rho}\nabla_i\frac{\tilde{\psi}_2}{\rho}-\nabla_i\frac{\tilde{\psi}_1^*}{\rho}\frac{\tilde{\psi}_2}{\rho}\right),\\\tilde{J}_v&=2p_v\tilde{\psi}_1^*\tilde{\psi}_2=(p_v\tilde{\psi}_1)^*\tilde{\psi}_2+\tilde{\psi}_1^*(p_v\tilde{\psi}_2).\end{aligned} \tag{2.3.22}$$

$\tilde{J}_v$ and $\tilde{J}_i$ satisfy the continuity equation

$$-\partial_\tau(\mu\tilde{J}_v)+\mathcal{D}^i\tilde{J}_i=0, \tag{2.3.23}$$

where $\mathcal{D}_i$ is the covariant derivative w.r.t. the "total" gauge field $A_i-\sqrt{2}l_ip_v$ and $\bar{g}_{ij}$, under the assumption that $\tilde{J}_i$ is a 1-form. This construction works for any $\mu$, $\rho$ and $\sigma$, in particular for the special values (2.3.10) required to reproduce the light-cone quantization of the scalar field.

#### 2.3.3.2 Conserved Current and Klein-Gordon Product for the Scalar Field

For a complex scalar field we can define as usual the Klein-Gordon current as:

$$\begin{aligned} iJ_\mu(\phi_1,\phi_2) &= \phi_1^*\partial_\mu\phi_2 - \partial_\mu\phi_1^*\phi_2 \\ = \frac{i}{\sqrt{|g|}}\hat{J}_\mu(\hat{\phi}_1,\hat{\phi}_2) &= \frac{1}{\sqrt{|g|}}\left(\hat{\phi}_1^*\partial_\mu\hat{\phi}_2 - \partial_\mu\hat{\phi}_1^*\hat{\phi}_2\right), \end{aligned} \tag{2.3.24}$$

which is conserved as

$$D^\mu J_\mu = 0, \tag{2.3.25}$$

where $D_\mu$ is the spacetime covariant derivative. For a complex scalar the meaning of $J_\mu(\phi,\phi)$ is that the electrical current associated to the obvious $U(1)$ is conserved. For a real scalar the current $J_\mu(\phi,\phi)$ vanishes identically but $J_\mu(\phi_1,\phi_2)$ can be used to define the conserved Klein-Gordon product.

We can now examine the conditions for the existence of equal $u$

conserved charge by computing:

$$
\begin{aligned}
&\int_{[u_0,u_1]} du \int dv\, d^{D-2}x \sqrt{|\bar{g}|}\, D^\mu J_\mu = \\
&= \int_{[u_0,u_1]} du \int dv\, d^{D-2}x \left\{ -\partial_u \left(\hat{J}_v\right) + \partial_v \left[ -\hat{J}_u + (\bar{l}^2 - h)\hat{J}_v + \bar{l}^i \hat{J}_i \right] \right. \\
&\qquad\qquad\qquad\qquad \left. + \partial_i \left( \bar{l}^i \hat{J}_v + \bar{g}^{ij} \hat{J}_j \right) \right\} = \\
&= -\int dv\, d^{D-2}x\, \hat{J}_v|_{u_1} + \int dv\, d^{D-2}x\, \hat{J}_v|_{u_0} \\
&\quad + \int_{u_0}^{u_1} du\, d^{D-2}x \left[ -\hat{J}_u + (\bar{l}^2 - h)\hat{J}_v + \bar{l}^i \hat{J}_i \right] |_{v=-\infty}^{v=+\infty} \\
&\quad + \sum_{i=2}^{D-2} \int_{u_0}^{u_1} du\, dv\, \frac{d^{D-2}x}{dx^i} \left( \bar{l}^i \hat{J}_v + \bar{g}^{ij} \hat{J}_j \right) |_{x^i=-\infty}^{x^i=+\infty}.
\end{aligned}
\tag{2.3.26}
$$

It follows that the charge

$$
Q(\hat{\phi}_1, \hat{\phi}_2) = \int dv\, d^{D-2}x\, \hat{J}_v(\hat{\phi}_1, \hat{\phi}_2)|_{u_0} \tag{2.3.27}
$$

is conserved if the appropriate boundary conditions are chosen, i.e. when the currents $\hat{J}$ vanish at "space" boundary. The same condition on the $v$ boundary is necessary to write the action (2.2.17) which was obtained by dropping some boundary terms.

#### 2.3.3.3 The Wave Function as a Charge Density

Given the fact the $\tilde{\psi}(\tau, k_v, x)$ follows a Schrödinger equation and that we can find a non-negative conserved density (2.3.20), it would be natural to think of it as a non relativistic wave function. Actually, this is not the right interpretation. The first reason is that the measure used is not the natural and physical one. In fact one would like to take the GR point of view and derive it from the space distance $dl^2$. However, since we are on a null surface, this measure is null. Even forgetting about this and accepting to use $dk_v$ as measure for the partially Fourier transformed wave function $\tilde{\psi}(\tau, k_v, x)$, we have another problem. The spatial distance can be defined and measured using light rays and the volume is $vol = \sqrt{\frac{|g|}{|h|}} d^{D-2}x$. Taking into account that $\tilde{\psi}(\tau, k_v, x)$ is a density, one would like to use $vol = \sqrt{\frac{1}{|h|}} d^{D-2}x$ but this is not the natural measure from the light-cone quantum field theory point of view.

So how can we interpret $\tilde{\psi}(\tau, k_v, x)$? Looking to the way we have arrived to the second quantized theory, it seems natural to interpret $\tilde{\psi}_a(\tau, k_v, x)$ as a mode for the second quantized theory which can be read from the one particle amplitude in light-cone quantum field theory in the Heisenberg picture:

$$\langle\Omega|\tilde{\Psi}_H(\tau, p_v, x)\left(a^{\dagger}_{(n,k_v)}|\Omega\rangle\right) = \tilde{\psi}_n(\tau, k_v, x)\,\delta(p_v - k_v). \qquad (2.3.28)$$

This approach, while technically correct, is not very enlightening. A more physical meaning can be obtained using the conserved current in light-cone quantum field theory. We can use it since if we compare the charge density (2.3.27) and the first quantized charge (2.3.20) we see they essentially match. Let us evaluate the vev of the light-cone quantum field theory charge density in the one particle state. The normal ordered space and time smeared charge in Heisenberg picture reads:

$$
\begin{aligned}
Q_{fH} =& \int du\,dv\,d^{D-2}x\,\sqrt{|g(u,x)|}\,f(u,v,x)\;:J_{vH}(u,v,x):\\
=& \int du\,d^{D-2}x \left[\int_{-\infty}^{0} dk_{1v}\,dk_{2v}\,\frac{\tilde{f}(u,+k_{1v}-k_{2v},x)}{\sqrt{2\pi}}\,\frac{k_{1v}+k_{2v}}{2\sqrt{|k_{1v}k_{2v}|}}\right.\\
&\sum_{n,m} a^{\dagger}_{(n,k_{1v})}\,a_{(m,k_{2v})}\,\tilde{\psi}^{*}_{n}(u,x;k_{1v}u_0,e)\tilde{\psi}_{m}(u,x;k_{2v}u_0,e)\\
&+\int_{-\infty}^{0} dl_{1v}\,dl_{2v}\,\frac{\tilde{f}(u,-l_{1v}+l_{2v},x)}{\sqrt{2\pi}}\,\frac{-l_{1v}-l_{2v}}{2\sqrt{|l_{1v}l_{2v}|}}\\
&\sum_{n,m} :b_{(n,l_{1v})}\,b^{\dagger}_{(m,l_{2v})}:\;\tilde{\psi}_{n}(u,x;l_{1v},u_0,-e)\tilde{\psi}^{*}_{m}(u,x;l_{2v},u_0,-e)\\
&+\int_{-\infty}^{0} dk_{1v}\,dl_{2v}\,\frac{\tilde{f}(u,+k_{1v}-l_{2v},x)}{\sqrt{2\pi}}\,\frac{k_{1v}-l_{2v}}{2\sqrt{|k_{1v}l_{2v}|}}\\
&\sum_{n,m} a^{\dagger}_{(n,k_{1v})}\,b^{\dagger}_{(m,l_{2v})}\,\tilde{\psi}^{*}_{n}(u,x;k_{1v},u_0,e)\tilde{\psi}^{*}_{m}(u,x;k_{2v},u_0,-e)\\
&+\int_{-\infty}^{0} dl_{1v}\,dk_{2v}\,\frac{\tilde{f}(u,-l_{1v}-k_{2v},x)}{\sqrt{2\pi}}\,\frac{-l_{1v}+k_{2v}}{2\sqrt{|l_{1v}k_{2v}|}}\\
&\left.\sum_{n,m} b_{(n,l_{1v})}\,a_{(m,k_{2v})}\,\tilde{\psi}_{n}(u,x;k_{1v},u_0,-e)\tilde{\psi}_{m}(u,x;k_{2v},u_0,e)\right],
\end{aligned}
\tag{2.3.29}
$$

where $f(u,v,x)$ is the smearing function. The previous expression implies:

$$
\begin{aligned}
\left(\langle\Omega|a_{(m,p_v)}\right)Q_{fH}\left(a^{\dagger}_{(n,k_v)}|\Omega\rangle\right) =& \int du\,d^{D-2}x\,\frac{\tilde{f}(u,+p_v-k_v,x)}{\sqrt{2\pi}}\,\frac{p_v+k_v}{2\sqrt{|p_vk_v|}}\\
&\tilde{\psi}^{*}_{m}(u,x;p_v,u_0,e)\tilde{\psi}_{n}(u,x;k_v,u_0,e).
\end{aligned}
\tag{2.3.30}
$$

When specializing the smearing function to the delta as $f_0(u,v,x) =$

$\delta(u-u_0)\,\delta(v-v_0)\,\delta^{D-2}(x-x_0)$ we get the expectation value:

$$\big(\langle\Omega|a_{(n,k_v)}\big) : \left(\sqrt{|g|}J_{vH}\right)(u_0,v_0,x_0) : \left(a^{\dagger}_{(n,k_v)}|\Omega\rangle\right) = -\,|\tilde{\psi}_n(u_0,x_0;k_v,e)|^2, \tag{2.3.31}$$

which clearly shows that $|\tilde{\psi}_n(u,x;k_v,e)|^2$ is a charge density not dependent on the coordinate $v$. The absence of $v_0$ dependence is due to the choice of taking $p_v = k_v$ in the bra and ket states; if we had chosen $p_v \neq k_v$ we would have found a $v_0$ dependence. This is further confirmed by the fact that the antiparticle has opposite sign charge density.

This prompts the question of how it is then possible that the second quantized particle has a conserved current (2.3.22) even if it is neutral. The reason is the absence of interactions. In fact, without interactions events like $e\gamma \to eee^+$ are not possible and the number of positive and negative charged particles is conserved.

# Chapter 3

# The Field Theory Failure

After having shed light on some important features of light-cone quantization on time-dependent backgrounds, we return to the main argument of the thesis. In search for a better understanding of the origin of the divergences, we would like to construct and study a scalar QED theory on the orbifolds introduced in Table 1.1. We will first deal with the NBO[1], where from this point of view things appear more clear. This chapter and the following one are mainly based on [1].

## 3.1 The Null Boost Orbifold Geometry

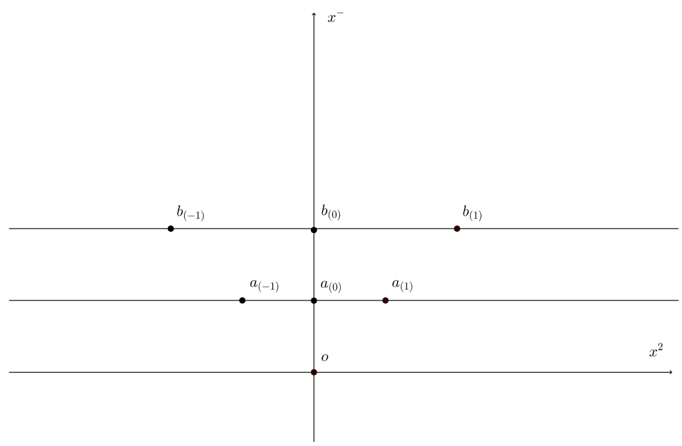


Figure 3.1: The NBO geometry.

Following the procedure described in Section 1.2.2, it comes out that on the NBO points are identified as:

$$\begin{pmatrix} x^- \\ x^+ \\ x^2 \end{pmatrix} \sim \begin{pmatrix} x^- \\ x^+ + 2\pi n \Delta x^2 + 2\pi^2 n^2 \Delta^2 x^- \\ x^2 + 2\pi n \Delta x^- \end{pmatrix} . \tag{3.1.1}$$

In Figure 3.1 we represent the orbits of $\kappa^{NBO}$ on the $(x^-, x^2)$ plane, while image points are displaced in the $x^+$ direction according to (3.1.1).

[1] See [20, 26, 27] for more on this model.

The geodesic distance square between image points is

$$\|x_{(n)} - x_{(0)}\|^2 = (2\pi \Delta n x^-_{(0)})^2, \tag{3.1.2}$$

which vanishes on the surface $x^- = 0$. Therefore there are no CTC's on this orbifold, but there exist CNC's[2] on the $(x^+, x^2)$ plane. Finally, the origin is a fixed point also in this case.

## 3.2 Scalar QED on the NBO

The full action we want to consider is:

$$\mathrm{S}_{sQED} = \int_\Omega \mathrm{d}^D x \sqrt{-\det g} \left[ -(D^\mu \phi)^* D_\mu \phi - M^2 \phi^* \phi - \frac{1}{4} f^{\mu\nu} f_{\mu\nu} - \frac{\lambda_4}{4} |\phi|^4 \right], \tag{3.2.1}$$

with

$$D_\mu \phi = (\partial_\mu - i\, e\, a_\mu)\phi, \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \tag{3.2.2}$$

where $\Omega$ denotes the orbifold. We will construct directly both the scalar and the spin 1 eigenfunctions, which we can use as a starting point for the perturbative computations.

### 3.2.1 Orbifold Coordinates

We perform the following change of coordinates from the usual $(x^\mu) = (x^+, x^-, x^2, \vec{x})$ to $(x^\alpha) = (u, v, z, \vec{x})$:

$$\begin{cases} x^- &= u \\ x^2 &= \Delta u z \\ x^+ &= v + \frac{1}{2}\Delta^2 u z^2 \end{cases} \Leftrightarrow \begin{cases} u &= x^- \\ z &= \frac{x^2}{\Delta\, x^-} \\ v &= x^+ - \frac{1}{2}\frac{(x^2)^2}{x^-} \end{cases}, \tag{3.2.3}$$

[2] Closed Null Curves, which means that only massless particles can "travel back in time".

and consequently the standard light-cone Minkowski metric becomes[3]

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + (\Delta u)^2(\mathrm{d}z)^2 + \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \qquad (3.2.4)$$

We can also compute

$$\det g = -(\Delta u)^2 \qquad (3.2.5)$$

and

$$\Gamma^v_{zz} = \Delta^2 u, \qquad \Gamma^z_{uz} = \frac{1}{u}. \qquad (3.2.6)$$

These are however the only non vanishing geometrical quantities, since all the components of the Riemann and Ricci tensors, and obviously the scalar curvature as well, vanish. This does not come as a surprise: at this stage we only performed a change of coordinates from the Minkowski spacetime and the background is still locally flat. In these coordinates the identifications are much simpler and read:

$$(u, v, z, \vec{x}) \sim (u, v, z + 2\pi n, \vec{x}), \qquad (3.2.7)$$

while $\kappa = 2\pi\partial_z$ is a global Killing vector.

For future use in Section 3.2.6 and inspired by (3.2.5), which is singular at $u = 0$, we can try to regularize the metric (3.2.4) as:

$$ds^2_{\text{reg}} = -2dudv + \Delta^2(u^2 + \epsilon^2)(dz)^2 + \eta_{ij}dx^i dx^j. \qquad (3.2.8)$$

Together with $(\det g_{\text{reg}}) = -[\Delta^2(u^2 + \epsilon^2)]$, we get the following non

[3] We already encountered this metric in (2.1.20).

vanishing geometrical quantities:

$$\text{Christoffel symbols} \qquad \Gamma^v_{zz} = \Delta^2 u, \quad \Gamma^z_{uz} = \frac{u}{u^2+\epsilon^2}; \tag{3.2.9}$$

$$\text{Riemann tensor} \qquad R^z{}_{uzu} = -\frac{\epsilon^2}{(u^2+\epsilon^2)^2}, \quad R^v{}_{zzu} = -\frac{\Delta^2\epsilon^2}{u^2+\epsilon^2}; \tag{3.2.10}$$

$$\text{Ricci tensor} \qquad R_{uu} = -\frac{\epsilon^2}{(u^2+\epsilon^2)^2}. \tag{3.2.11}$$

Notice that $R_{uu} = -\pi^2 \delta^2_{\text{reg}}(u)$, with $\delta_{\text{reg}}(u) = \frac{1}{\pi}\frac{\epsilon}{u^2+\epsilon^2}$, while the scalar curvature $R_{\text{reg}} = 0$.

### 3.2.2 Free Scalar Action

We now want to find the eigenmodes of the Laplacian in order to write in a diagonal way the scalar kinetic term given by:[4]

$$\begin{aligned}
\text{S}_{\text{scalar kin}} &= \int_\Omega \mathrm{d}^D x \sqrt{-\det g}\,\left(-g^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi - M^2\phi^*\phi\right) \\
&= \int \mathrm{d}^{D-3}\vec{x} \int \mathrm{d}u \int \mathrm{d}v \int_0^{2\pi} \mathrm{d}z\, |\Delta u| \Bigg( \partial_u\phi^*\,\partial_v\phi \\
&\quad + \partial_v\phi^*\,\partial_u\phi - \frac{1}{(\Delta u)^2}\partial_z\phi^*\,\partial_z\phi - \partial_i\phi^*\partial_i\phi - M^2\phi^*\phi \Bigg).
\end{aligned} \tag{3.2.12}$$

The solution to the e.o.m. is enough when we want to perform the canonical quantization. Since we want to use the Feynman diagrams, we consider the path integral approach: we take off-shell modes and solve the eigenvalue problem $\Box\phi_r = r\phi_r$. By comparing with the flat case we see that $r$ equals $2k_-k_+ - \vec{k}^2$ when $k$ is the flat coordinates

[4]The factor $-g^{\alpha\beta}$ is due to the choice of the East Coast convention for the metric, i.e.:

$$-g^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi - M^2\phi^*\phi \sim +|\dot\phi|^2 - M^2|\phi|^2 \sim \mathrm{E}^2 - M^2.$$

momentum. We therefore have:

$$-2\partial_u\partial_v\phi_r - \frac{1}{u}\partial_v\phi_r + \frac{1}{(\Delta u)^2}\partial_z^2\phi_r + \partial_i^2\phi_r = r\phi_r. \tag{3.2.13}$$

Using Fourier transforms, it then easily follows that the eigenmodes are

$$\phi_{\{k_+\,l\,\vec{k}\,r\}}(u,v,z,\vec{x}) = e^{ik_+v+ilz+i\vec{k}\cdot\vec{x}}\,\tilde{\phi}_{\{k_+\,l\,\vec{k}\,r\}}(u), \tag{3.2.14}$$

with

$$\tilde{\phi}_{\{k_+\,l\,\vec{k}\,r\}}(u) = \frac{1}{\sqrt{(2\pi)^D\ 2|\Delta k_+|\ |u|}}e^{-i\frac{l^2}{2\Delta^2 k_+}\frac{1}{u}+i\frac{\vec{k}^2+r}{2k_+}u} \tag{3.2.15}$$

and

$$\phi^*_{\{k_+\,l\,\vec{k}\,r\}}(u,v,z,\vec{x}) = \phi_{\{-k_+\,-l\,-\vec{k}\,r\}}(u,v,z,\vec{x}), \tag{3.2.16}$$

where we have chosen the numeric factor in order to get a canonical normalization:

$$\begin{aligned}
&(\phi_{\{k_{(1)+}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}},\,\phi_{\{k_{(2)+}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}})\\
&= \int \mathrm{d}^{D-3}\vec{x}\int \mathrm{d}u\int \mathrm{d}v\int_0^{2\pi}\mathrm{d}z\ |\Delta u|\phi_{\{k_{(1)+}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}}\,\phi_{\{k_{(2)+}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}}\\
&= \delta^{D-3}(\vec{k}_{(1)}+\vec{k}_{(2)})\,\delta(r_{(1)}-r_{(2)})\,\delta(k_{(1)+}+k_{(2)+})\,\delta_{l_{(1)},-l_{(2)}}.
\end{aligned} \tag{3.2.17}$$

We can then perform the off-shell expansion

$$\phi(u,v,z,\vec{x}) = \int \mathrm{d}^{D-3}\vec{k}\int \mathrm{d}r\int \mathrm{d}k_+\sum_{l\in\mathbb{Z}}\mathcal{A}_{\{k_+\,l\,\vec{k}\,r\}}\,\phi_{\{k_+\,l\,\vec{k}\,r\}}(u,v,z,\vec{x}), \tag{3.2.18}$$

so that the scalar kinetic term becomes:

$$\mathrm{S}_{\text{scalar kin}} = \int \mathrm{d}^{D-3}\vec{k} \int \mathrm{d}r \int \mathrm{d}k_+ \sum_{l\in\mathbb{Z}} (r - M^2)\, \mathcal{A}_{\{k_+\, l\, \vec{k}\, r\}}\, \mathcal{A}^*_{\{k_+\, l\, \vec{k}\, r\}}. \tag{3.2.19}$$

We notice from (3.2.15) that, by solving the eigenvalue problem for the scalar field as we just did, we get directly the redefinition factor we discussed in (2.2.13). According to our analysis in Chapter 2, this means that we are dealing with light-cone quantum field theory in the right way.

### 3.2.3 Free Photon Action

The photon action can be written as:

$$\mathrm{S}_{\text{spin-1 kin}} = \int_\Omega \mathrm{d}^D x \sqrt{-\det g}\left(-\frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} D_\alpha a_\gamma (D_\beta a_\delta - D_\delta a_\beta)\right). \tag{3.2.20}$$

If we choose the Lorenz gauge[5]

$$D^\alpha a_\alpha = -\frac{1}{u} a_v - \partial_u a_v - \partial_v a_u + \frac{1}{\Delta^2 u^2}\partial_z a_z + \eta^{ij}\partial_i a_j = 0 \tag{3.2.21}$$

[5] Indeed it is exactly the usual Lorenz gauge since locally the spacetime is Minkowski.

and remember that covariant derivatives commute since we are locally flat, the e.o.m. read $(\Box a)_\alpha = 0$. Explicitly, we have:

$$
\begin{aligned}
(\Box a)_u &= \frac{1}{u^2} a_v - \frac{2}{\Delta^2 u^3} \partial_z a_z + \left[ -2\partial_u \partial_v - \frac{1}{u}\partial_v + \frac{1}{\Delta^2 u^2}\partial_z^2 + \eta^{ij}\partial_i \partial_j \right] a_u, \\
(\Box a)_v &= \left[ -2\partial_u \partial_v - \frac{1}{u}\partial_v + \frac{1}{\Delta^2 u^2}\partial_z^2 + \eta^{ij}\partial_i \partial_j \right] a_v, \\
(\Box a)_z &= -\frac{2}{u}\partial_z a_v + \left[ -2\partial_u \partial_v + \frac{1}{u}\partial_v + \frac{1}{\Delta^2 u^2}\partial_z^2 + \eta^{ij}\partial_i \partial_j \right] a_z, \\
(\Box a)_i &= \left[ -2\partial_u \partial_v - \frac{1}{u}\partial_v + \frac{1}{\Delta^2 u^2}\partial_z^2 + \eta^{ij}\partial_i \partial_j \right] a_i.
\end{aligned}
\tag{3.2.22}
$$

As in the scalar case we are actually interested in solving the eigenmodes problem $(\Box a)_\alpha = r\, a_\alpha$. We proceed hierarchically: first we solve for $a_v$ and $a_i$, whose equations are the same as the one for the scalar field, then we insert the solutions as a source in the equation[6] for $a_z$

[6]Notice that inside the square brackets of the differential equation for $a_z$ there is a different sign for the term $\frac{1}{u}\partial_v$ with respect to the equation for the scalar field.

and eventually we solve for $a_u$. We get the solutions:

$$
\begin{aligned}
\| \tilde{a}_{\{k_+\, l\, \vec{k}\, r\}\,\alpha}(u) \| = \begin{pmatrix} \tilde{a}_u \\ \tilde{a}_v \\ \tilde{a}_z \\ \tilde{a}_i \end{pmatrix} &= \sum_{\underline{\alpha}\in\{\underline{u},\underline{v},\underline{z},\underline{i}\}} \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{\alpha}} \, \| \tilde{a}^{\underline{\alpha}}_{\{k_+\, l\, \vec{k}\, r\}\,\alpha}(u) \| \\
&= \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{u}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, l\, \vec{k}\, r\}}(u) \\
&+ \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{v}} \begin{pmatrix} \frac{i}{2k_+ u} + \frac{1}{2}\left(\frac{l}{\Delta k_+}\right)^2 \frac{1}{u^2} \\ 1 \\ \frac{l}{k_+} \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, l\, \vec{k}\, r\}}(u) \\
&+ \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{z}} \begin{pmatrix} \frac{l}{\Delta k_+ |u|} \\ 0 \\ \Delta |u| \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, l\, \vec{k}\, r\}}(u) \\
&+ \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{j}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{i\underline{j}} \end{pmatrix} \tilde{\phi}_{\{k_+\, l\, \vec{k}\, r\}}(u),
\end{aligned}
\tag{3.2.23}
$$

then we can expand the off-shell fields as

$$
a_\alpha(u,v,z,\vec{x}) = \int \mathrm{d}^{D-3}\vec{k} \int \mathrm{d}r \int \mathrm{d}k_+ \sum_{l\in\mathbb{Z}} \sum_{\underline{\alpha}\in\{\underline{u},\underline{v},\underline{z},\underline{i}\}} \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{\alpha}} \, a^{\underline{\alpha}}_{\{k_+\, l\, \vec{k}\, r\}\,\alpha}(u,v,z,\vec{x}),
\tag{3.2.24}
$$

where $a^{\underline{\alpha}}_{\{k_+\, l\, \vec{k}\, r\}\,\alpha}(u,v,z,\vec{x}) = \tilde{a}^{\underline{\alpha}}_{\{k_+\, l\, \vec{k}\, r\}\,\alpha}(u)\, e^{i(k_+ v + l z + \vec{k}\cdot\vec{x})}$.

We can also compute the normalization as:

$$
\begin{aligned}
(a_{(1)}, a_{(2)}) = &\int \mathrm{d}^{D-3}\vec{x} \int \mathrm{d}u \int \mathrm{d}v \int_0^{2\pi} \mathrm{d}z \, |\Delta u| \\
&\times \left( g^{\alpha\beta} a_{\{k_{(1)+}\, l_{(1)}\, \vec{k}_{(1)}\, r_{(1)}\}\,\alpha}\, a_{\{k_{(2)+}\, l_{(2)}\, \vec{k}_{(2)}\, r_{(2)}\}\,\beta} \right) \\
&= \mathcal{E}_{\{k_{(1)+}\, l_{(1)}\, \vec{k}_{(1)}\, r_{(1)}\}} \circ \mathcal{E}_{\{k_{(2)+}\, l_{(2)}\, \vec{k}_{(2)}\, r_{(2)}\}} \\
&\times \delta^{D-3}(\vec{k}_{(1)} + \vec{k}_{(2)})\, \delta(r_{(1)} - r_{(2)})\, \delta(k_{(1)+} + k_{(2)+})\, \delta_{l_{(1)}, -l_{(2)}},
\end{aligned}
\tag{3.2.25}
$$

with[7]

$$
\mathcal{E}_{(1)} \circ \mathcal{E}_{(2)} = -\mathcal{E}_{(1)\,\underline{u}}\ \mathcal{E}_{(2)\,\underline{v}} - \mathcal{E}_{(1)\,\underline{v}}\ \mathcal{E}_{(2)\,\underline{u}} + \mathcal{E}_{(1)\,\underline{z}}\ \mathcal{E}_{(2)\,\underline{z}} + \eta^{\underline{ij}}\, \mathcal{E}_{(1)\,\underline{i}} \mathcal{E}_{(2)\,\underline{j}}.
\tag{3.2.26}
$$

Finally the Lorenz gauge reads

$$
\eta^{\underline{ij}} k_i\, \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{j}} - k_+ \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{u}} - \frac{\vec{k}^2 + r}{2k_+} \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{v}} = 0,
\tag{3.2.27}
$$

which does not impose any constraint on the transverse polarization $\mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}\,\underline{z}}$, while the photon kinetic term becomes:

$$
\mathrm{S}_{\text{spin-1 kin}} = \int \mathrm{d}^{D-3}\vec{k} \int \mathrm{d}r \int \mathrm{d}k_+ \sum_{l \in \mathbb{Z}} \frac{1}{2} r\, \mathcal{E}_{\{k_+\, l\, \vec{k}\, r\}} \circ \mathcal{E}_{\{-k_+\, -l\, -\vec{k}\, r\}}.
\tag{3.2.28}
$$

[7] We use a shortened version of the polarizations $\mathcal{E}$ for the sake of readability. Specifically, we write $\mathcal{E}_{(n)\,\underline{\alpha}} = \mathcal{E}_{\{k_{(n)+}\, l_{(n)}\, \vec{k}_{(n)}\, r_{(n)}\}\,\underline{\alpha}}$ thus hiding the understood dependence of the components of $\mathcal{E}_{(n)}$ on the momenta.

### 3.2.4 Cubic Interaction

With the definition of the d'Alembertian eigenmodes we can now examine the cubic vertex which reads:

$$\mathrm{S}_{\text{cubic}} = \int_{\Omega} \mathrm{d}^D x \, \sqrt{-\det g} \Big( -i \, e \, g^{\alpha\beta} a_\alpha (\phi^* \, \partial_\beta \phi - \partial_\beta \phi^* \, \phi) \Big). \qquad (3.2.29)$$

Its computation involves integrals such as

$$\int \mathrm{d}u \, |\Delta u| \, \left(\frac{l}{u}\right)^2 \prod_{i=1}^{3} \tilde{\phi}_{\{k_{(i)+} \, l_{(i)} \, \vec{k}_{(i)} \, r_{(i)}\}} \sim \int_{u \sim 0} \mathrm{d}u \, \left(\frac{l^2}{|u|^{5/2}}\right) e^{-i \sum_{i=1}^{3} \frac{{l_{(i)}}^2}{2\Delta^2 k_{(i)+}} \frac{1}{u}} \qquad (3.2.30)$$

and

$$\int \mathrm{d}u \, |\Delta u| \, \left(\frac{1}{u}\right) \prod_{i=1}^{3} \tilde{\phi}_{\{k_{(i)+} \, l_{(i)} \, \vec{k}_{(i)} \, r_{(i)}\}} \sim \int_{u \sim 0} \mathrm{d}u \, \left(\frac{1}{u|u|^{1/2}}\right) e^{-i \sum_{i=1}^{3} \frac{{l_{(i)}}^2}{2\Delta^2 k_{(i)+}} \frac{1}{u}}, \qquad (3.2.31)$$

which can be interpreted as hints that the theory may be troublesome. The first integral would diverge if the factor $e^{i\frac{\mathcal{A}}{u}}$ were equal 1. Luckily here it happens only when all $l_{(*)} = 0$ and in this case the integral vanishes (if we set $l_{(*)} = 0$ before its evaluation). This however suggests that when all $l_{(*)} = 0$, i.e. when the eigenfunctions are constant along the compact direction $z$, something is happening near the singularity. On the other side when at least one $l$ is different from zero we have an integral such as

$$\int_{u \sim 0} \mathrm{d}u \, |u|^{-\nu} \, e^{i\frac{\mathcal{A}}{u}} \sim \int_{t \sim \infty} \mathrm{d}t \, t^{\nu-2} \, e^{i\mathcal{A}t}. \qquad (3.2.32)$$

All $l_{(*)}$ are discrete but $k_{(*)+}$ are not: $\mathcal{A}$ can therefore be equal to zero but since it has continuous values it may be given a distributional meaning, similar to a derivative of the $\delta$. In particular, integrals of this kind have been given a rigorous mathematical interpretation by Estrada and

Vindas in [28], under the name of "distributionally integrable functions". This type of distribution will become crucial later on in this thesis.

The second integral has again issues when all $l_{(*)} = 0$ and, since it is not proportional to any $l$, it is divergent unless we take a principal part regularization which may be meaningful.

With all these warnings, we can try anyway to give a meaning to the cubic terms and we get:[8]

$$
\begin{aligned}
\mathrm{S}_{\text{cubic}} =& \prod_{i=1}^{3} \left[ \int \mathrm{d}^{D-3}\vec{k}_{(i)}\, \mathrm{d}r_{(i)}\, \mathrm{d}k_{(i)+} \sum_{l_{(i)}} \right] (2\pi)^{D-1} \delta\left(\sum \vec{k}_{(i)}\right)\, \delta\left(\sum k_{(i)+}\right) \\
& \times \delta_{\left(\sum l_{(i)}\right)} e \left(\mathcal{A}_{\{-k_{(2)+}\, -l_{(2)}\, -\vec{k}_{(2)}\, r_{(2)}\}}\right)^{*} \mathcal{A}_{\{k_{(3)+}\, l_{(3)}\, \vec{k}_{(3)}\, r_{(3)}\}} \\
& \times \Bigg\{ \mathcal{E}_{\{k_{(1)+}\, l_{(1)}\, \vec{k}_{(1)}\, r_{(1)}\}\, \underline{u}}\, k_{(2)+}\, \mathcal{I}^{[0]}_{\{3\}} \\
& + \mathcal{E}_{\{k_{(1)+}\, l_{(1)}\, \vec{k}_{(1)}\, r_{(1)}\}\, \underline{z}}\, \frac{k_{(2)+} l_{(1)} - l_{(2)} k_{(1)+}}{\Delta k_{(1)+}} \mathcal{J}^{[-1]}_{(3)} \\
& + \mathcal{E}_{\{k_{(1)+}\, l_{(1)}\, \vec{k}_{(1)}\, r_{(1)}\}\, \underline{v}} \Bigg[ \frac{\vec{k}^{2}_{(2)} + r_{(2)}}{2k_{(2)+}} \mathcal{I}^{[0]}_{\{3\}} + i \frac{k_{(2)+}}{2k_{(1)+}} \mathcal{I}^{[-1]}_{\{3\}} \\
& + \frac{1}{2} \frac{k_{(2)+}}{\Delta^{2}} \left( \frac{l_{(1)}}{k_{(1)+}} - \frac{l_{(2)}}{k_{(2)+}} \right)^{2} \mathcal{I}^{[-2]}_{\{3\}} \Bigg] \\
& - \eta^{ij}\, \mathcal{E}_{\{k_{(1)+}\, l_{(1)}\, \vec{k}_{(1)}\, r_{(1)}\}\, \underline{i}}\, k_{(2)_j}\, \mathcal{I}^{[0]}_{\{3\}} - \Big( (2) \rightarrow (3) \Big) \Bigg\}.
\end{aligned}
\tag{3.2.33}
$$

[8]The notation $(2) \rightarrow (3)$ means that all previous terms inside the curly brackets appear again in exactly the same structure but with momenta of particle (3) in place of those of particle (2).

We have defined also for future use

$$\begin{aligned}\mathcal{I}^{[\nu]}_{(1)\dots(N)} = \mathcal{I}^{[\nu]}_{\{N\}} &= \int_{-\infty}^{+\infty} \mathrm{d}u\, |\Delta u|\, u^{\nu} \prod_{i=1}^{N} \tilde{\phi}_{\{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}} \\ &= \int_{-\infty}^{+\infty} \mathrm{d}u\, |\Delta u|\, u^{\nu} \prod_{i=1}^{N} \tilde{\phi}_{(i)}, \\ \mathcal{J}^{[\nu]}_{(N)} &= \int_{-\infty}^{+\infty} \mathrm{d}u\, |\Delta| |u|^{\nu+1} \prod_{i=1}^{N} \tilde{\tilde{\phi}}_{\{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}},\end{aligned} \tag{3.2.34}$$

where $\tilde{\phi}_{(i)} = \tilde{\phi}_{\{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}}$ and $\tilde{\tilde{\phi}}_{(i)} = \tilde{\tilde{\phi}}_{\{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}}$ will be used when not causing confusion.

### 3.2.5 Quartic Interactions and Divergences

In the previous section we have seen that the theory may have issues when all $l = 0$, i.e. with eigenfunctions independent of the compact direction $z$, because some integrals were very close to diverge near the singularity. The divergence issue will appear in a clear and unavoidable way when considering the quartic terms:

$$\mathrm{S}_{\text{quartic}} = \int_{\Omega} \mathrm{d}^D x\, \sqrt{-\det g}\, \left( e^2\, g^{\mu\nu}\, a_\mu a_\nu\, |\phi|^2 - \frac{\lambda_4}{4} |\phi|^4 \right), \tag{3.2.35}$$

which can be expressed using the modes as

$$
\begin{aligned}
\mathrm{S}_{\text{quartic}} = & \prod_{i=1}^{4}\left[\int \mathrm{d}^{D-3}\vec{k}_{(i)}\,\mathrm{d}k_{(i)+}\,\mathrm{d}r_{(i)}\sum_{l_{(i)}}\right] \\
& \times (2\pi)^{D-1}\delta\left(\sum \vec{k}_{(i)}\right)\,\delta\left(\sum k_{(i)+}\right)\,\delta_{\sum l_{(i)},0} \\
& \Big\{e^2\,(\mathcal{A}_{\{-k_{(3)+}\,-l_{(3)}\,-\vec{k}_{(3)}\,r_{(3)}\}})^*\mathcal{A}_{\{k_{(4)+}\,l_{(4)}\,\vec{k}_{(4)}\,r_{(4)}\}} \\
& \quad \Big[(\mathcal{E}_{\{k_{(1)+}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}} \circ \mathcal{E}_{\{k_{(2)+}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}})\,\mathcal{I}^{[0]}_{\{4\}} \\
& \quad - i\frac{1}{2}\mathcal{E}_{\{k_{(1)+}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\underline{v}}\mathcal{E}_{\{k_{(2)+}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}\,\underline{v}}\left(\frac{1}{k_{(2)+}}+\frac{1}{k_{(1)+}}\right)\mathcal{I}^{[-1]}_{\{4\}} \\
& \quad + \frac{1}{2}\frac{\mathcal{E}_{\{k_{(1)+}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\underline{v}}\mathcal{E}_{\{k_{(2)+}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}\,\underline{v}}}{\Delta^2}\left(\frac{l_{(1)}}{k_{(1)+}}-\frac{l_{(2)}}{k_{(2)+}}\right)^2\mathcal{I}^{[-2]}_{\{4\}}\Big] \\
& -\frac{\lambda_4}{4}(\mathcal{A}_{\{-k_{(1)+}\,-l_{(1)}\,-\vec{k}_{(1)}\,r_{(1)}\}})^*(\mathcal{A}_{\{-k_{(2)+}\,-l_{(2)}\,-\vec{k}_{(2)}\,r_{(2)}\}})^* \\
& \quad \times \mathcal{A}_{\{k_{(3)+}\,l_{(3)}\,\vec{k}_{(3)}\,r_{(3)}\}}\mathcal{A}_{\{k_{(4)+}\,l_{(4)}\,\vec{k}_{(4)}\,r_{(4)}\}}\,\mathcal{I}^{[0]}_{\{4\}}\Big\}.
\end{aligned}
\tag{3.2.36}
$$

Now when setting $l_{(*)} = 0$ all the surviving terms are divergent, explicitly $\mathcal{I}^{[0]}_{\{4\}} \sim \int du\,|u|^{1-4\times\frac{1}{2}}$ and $\mathcal{I}^{[-1]}_{\{4\}} \sim \int du\,|u|^{1-4\times\frac{1}{2}}\frac{1}{u}$ since $\tilde{\phi}|_{l=0} \sim |u|^{-\frac{1}{2}}$. Obviously higher order terms in the effective field theory will behave even worse. This makes the theory ill-defined.

### 3.2.6 Failure of Obvious Divergences Regularizations

From the discussion of the previous section, the origin of the divergences seems to be related to the sector $l = 0$. Indeed, when $l = 0$ the highest order singularity of the Fourier transformed d'Alembertian equation vanishes. Explicitly we have:

$$
\begin{aligned}
& A\partial_u\tilde{\phi}_{\{k_+\,l\,\vec{k}\,r\}} + B(u)\tilde{\phi}_{\{k_+\,l\,\vec{k}\,r\}} = \\
& Ae^{-\int^u \frac{B(u)}{A}du}\partial_u\left[e^{+\int^u \frac{B(u)}{A}du}\tilde{\phi}_{\{k_+\,l\,\vec{k}\,r\}}\right] = 0,
\end{aligned}
\tag{3.2.37}
$$

with

$$A = (-2i\, k_+), \qquad B(u) = (-\vec{k}^2 - r) + (-ik_+)\frac{1}{u} + \frac{-l^2}{\Delta^2}\frac{1}{u^2}, \qquad (3.2.38)$$

and this in turn implies the absence of the oscillating factor $e^{i\frac{\mathcal{A}}{u}}$ when $l$ goes to zero discretely. It follows that any deformation which makes the coefficient of the highest order singularity continuous can save the situation, letting the integrals being interpretable at least as "distributionally integrable functions".

The first and easiest possibility is to add a Wilson line along $z$, i.e. $a = \theta dz$. This shifts $l \to l - e\theta$ and regularizes the scalar QED. Unfortunately, this does not work for string theory where Wilson lines on $D25$ branes are not felt by the neutral strings starting and ending on the same brane. This happens because not all interactions involve commutators of the Chan-Paton factors as they vanish for neutral strings. Moreover, the interaction among two tachyons and the first massive state involves an anti-commutator, as we discuss later. The anti-commutators are present also in amplitudes of supersymmetric strings with massive states and therefore the issue is not solved by supersymmetry.

A second possibility is to think about higher derivative couplings to curvature, which is also natural in string theory. If we regularize the metric in a minimal way as shown at the end of Section 3.2.1, we see that only $Ric_{uu}$ is non vanishing, therefore it would be natural to consider:

$$S_{\text{higher R}} = \int_\Omega \mathrm{d}^D x \sqrt{-\det g} \Bigg[ \sum_{k\geq 1} \alpha'^{2k-1} \prod_{j=1}^{k} g^{\mu_j \nu_j} \, g^{\rho_j \sigma_j} \, Ric_{\mu_j \rho_j} \times (\sum_{s=0}^{2k} c_{k\,s} \partial^{2k-s} \phi^* \partial^s \phi) \Bigg]$$
$$= \int_\Omega \mathrm{d}^D x \sqrt{-\det g} [\alpha' g^{\mu\nu} \, g^{\rho\sigma} \, Ric_{\mu\rho} \, (c_{12} \phi^* \partial^2_{\nu\sigma} \phi + c_{11} \partial_\nu \phi^* \partial_\sigma \phi + c_{10} \partial^2_{\nu\sigma} \phi^* \phi)], \tag{3.2.39}$$

where $\alpha'$ has been introduced for dimensional reasons and in order to have all $c$'s adimensional. Since only $Ric_{uu}$ is non vanishing and it depends only on $u$, the regularized d'Alembertian eigenmode problem would now read:

$$-2\partial_u \partial_v \phi_r - \frac{u}{u^2+\epsilon^2} \partial_v \phi_r + \frac{1}{\Delta^2 (u^2+\epsilon^2)} \partial_z^2 \phi_r + \sum_{k\geq 1} \alpha'^{2k-1} C_k \, Ric^k_{uu} \, \partial_v^{2k} \phi + \partial_i^2 \phi_r - r\phi_r = 0, \tag{3.2.40}$$

with $C_k = \sum_{s=0}^{2k} (-)^s c_{k\,s}$. We can perform the usual Fourier transform and the function $B(u)$ becomes:

$$B(u) = (-\vec{k}^2 - r) + (-ik_+) \frac{u}{u^2+\epsilon^2} + \frac{-l^2}{\Delta^2} \frac{1}{u^2+\epsilon^2} + \sum_{k\geq 1} \alpha'^{2k-1} C_k \left( \frac{\epsilon^2}{(u^2+\epsilon^2)^2} \right)^k (-ik_+)^{2k}. \tag{3.2.41}$$

Then we examine what happens when $u = 0$:

$$B(0) \sim \frac{-l^2}{\Delta^2} \frac{1}{\epsilon^2} + \sum_{k\geq 1} \alpha'^{2k-1} C_k \, (-ik_+)^{2k} \frac{1}{\epsilon^{2k}}. \tag{3.2.42}$$

Even though it looks as we may have found a way to solve the issue, unfortunately this is not the case. If we consider $\alpha'$ and $\epsilon^2$ uncorrelated we lose predictability, but if we take $\alpha' \sim \epsilon^2$, as it is natural in string theory, we get $B(0) \sim \frac{-l^2}{\Delta^2}\frac{1}{\epsilon^2} + \sum_{k\geq 1} C_k\,(-ik_+)^{2k}\epsilon^{2k-2}$ and the curvature terms are not singular anymore.

We have therefore reached a point where we may have found an important clue as to where the divergences come from, but we seem still far from knowing how to overcome them. The main argument of the next chapter will be to see how this kind of pathological field theory behaviour reflects on string theory amplitudes.

Chapter 4

# String Theory Divergences

## 4.1 NBO Wave Functions from the Covering Spacetime

In this section we will start by recovering the eigenfunctions from the covering Minkowski spacetime in order to elucidate the connection between the polarizations on the NBO and in Minkowski. Moreover, we want to generalize the result to a symmetric two index tensor which is the polarization of the first massive state to compute the two tachyons one massive state string theory amplitude and show that it diverges.

### 4.1.1 Spin 0 Wave Function

We start with the usual plane wave in flat space and we express it in the new coordinates (we do not write the dependence on $\vec{x}$ since it is trivial):

$$\begin{aligned}\psi_{k_+\,k_-\,k_2}(x^+,x^-,x^2) &= e^{i\left(k_+x^+ + k_-x^- + k_2x^2\right)}\\ = \psi_{k_+\,k_-\,k_2}(u,v,z) &= e^{i\left[k_+v+\frac{2k_+k_- - {k_2}^2}{2k_+}u+\frac{1}{2}\Delta^2k_+u\left(z+\frac{k_2}{\Delta k_+}\right)^2\right]}.\end{aligned} \quad (4.1.1)$$

The corresponding wave function on the NBO is obtained by making it periodical in $z$. This can be done in two ways, either in $(x^\mu)$ coordinates or in $(x^\alpha)$ ones. The first way is more useful in deducing how the passage to the orbifold makes the function depend on the equivalence class of momenta. Implementing the projection on periodic $z$ functions we get:

$$\begin{aligned}\Psi_{[k_+\,k_-\,k_2]}([x^+,x^-,x^2]) &= \sum_{n\in\mathbb{Z}} \psi_{k_+\,k_-\,k_2}(\mathcal{K}^n(x^+,x^-,x^2))\\ &= \sum_{n\in\mathbb{Z}} \psi_{\mathcal{K}^{-n}(k_+\,k_-\,k_2)}(x^+,x^-,x^2),\end{aligned} \quad (4.1.2)$$

where we write $[k_+ \, k_- \, k_2]$ because the function depends on the equivalence class of $k_+ \, k_- \, k_2$ only. The equivalence relation is given by

$$k = \begin{pmatrix} k_+ \\ k_- \\ k_2 \end{pmatrix} \sim \mathcal{K}^{-n} k = \begin{pmatrix} k_+ \\ k_- + n(2\pi\Delta)k_2 + \frac{1}{2}n^2(2\pi\Delta)^2 k_+ \\ k_2 + n(2\pi\Delta)k_+ \end{pmatrix} \tag{4.1.3}$$

and allows to choose a representative with

$$\begin{cases} 0 \leq \frac{k_2}{\Delta\,|k_+|} < 2\pi & k_+ \neq 0 \\ 0 \leq \frac{k_-}{\Delta\,|k_2|} < 2\pi & k_+ = 0,\, k_2 \neq 0 \end{cases} . \tag{4.1.4}$$

If we perform the computation in $uvz$ coordinates we get

$$\begin{aligned} \Psi_{[k_+ \, k_- \, k_2]}(u, v, z) &= \sum_{n\in\mathbb{Z}} \psi_{k_+ \, k_- \, k_2}(u, v, z + 2\pi\, n) \\ &= \sum_{n\in\mathbb{Z}} e^{i\left\{ k_+ v + \frac{r}{2k_+} u + \frac{1}{2}(2\pi\Delta)^2 k_+ u \left[ n + \frac{1}{2\pi}\left( z + \frac{k_2}{\Delta k_+} \right) \right]^2 \right\}}, \end{aligned} \tag{4.1.5}$$

with $r = 2k_+k_- - {k_2}^2$ and $Im(k_+u) > 0$, i.e. $k_+u = |k_+u|e^{i\epsilon}$ and $\pi > \epsilon > 0$. Notice that there is no separate dependence on $z$ and on $\frac{k_2}{\Delta k_+}$, therefore one could fix the range $0 \leq z + \frac{k_2}{\Delta k_+} < 2\pi$. However this symmetry is broken when considering the photon eigenfunction.

We can now use the Poisson resummation

$$\sum_n e^{ia(n+b)^2} = \int \mathrm{d}s\, \delta_P(s) e^{ia(s+b)^2} = (2\pi)^2 \frac{e^{-i(\frac{\pi}{4} + \frac{1}{2}arg(a))}}{2\sqrt{\pi|a|}} \sum_m e^{+\frac{\pi^2 m^2}{ia} + i2\pi bm} \tag{4.1.6}$$

to finally get, reintroducing the other variables $\vec{k}, \vec{x}$ and setting therefore

$r = 2k_+k_- - {k_2}^2 - \vec{k}^2$:

$$\begin{aligned}\Psi_{[k_+\,k_-\,k_2\,\vec{k}]}(u,v,z,\vec{x}) &= (2\pi)^2\sqrt{\frac{2}{\pi}}\frac{e^{-i\pi/4}}{(2\pi\Delta)} \\ &\times \sum_l \left[\frac{1}{\sqrt{|k_+u|}} e^{i\left\{k_+v + lz - \frac{l^2}{2\Delta^2 k_+}\frac{1}{u} + \frac{r+\vec{k}^2}{2k_+}u + \vec{k}\cdot\vec{x}\right\}}\right] e^{il\frac{k_2}{\Delta k_+}} \\ &= \mathcal{N}\sum_l \phi_{\{k_+\,l\,\vec{k}\,r\}}(u,v,z,\vec{x}) e^{il\frac{k_2}{\Delta k_+}} \qquad \text{when } k_+ \neq 0,\end{aligned} \tag{4.1.7}$$

with

$$\mathcal{N} = \sqrt{\frac{(2\pi)^D}{\pi\Delta}}\frac{e^{-i\pi/4}}{\pi}. \tag{4.1.8}$$

The fact that $\Psi$ depends only on the equivalence class $[k_+\,k_-\,k_2\,\vec{k}]$ allows to restrict to $0 \le \frac{k_2}{\Delta\,|k_+|} < 2\pi$ so that we can invert the previous expression and get:

$$\phi_{\{k_+\,l\,\vec{k}\,r\}}(u,v,z,\vec{x}) = \frac{1}{\mathcal{N}}\int_0^{2\pi\Delta|k_+|} \frac{\mathrm{d}k_2}{2\pi\Delta|k_+|}\, e^{-il\frac{k_2}{\Delta k_+}}\, \Psi_{[k_+\,k_-\,k_2\,\vec{k}]}(u,v,z,\vec{x}). \tag{4.1.9}$$

### 4.1.2 Spin 1 Wave Function

We can repeat the steps of the previous section in the case of an electromagnetic wave. Again we concentrate on $x^+, x^-$ and $x^2$ coordinates and reinstate $\vec{x}$ at the end. We start with the usual plane wave in flat space $\psi^{[1]}_{k_+\,k_-\,k_2,\epsilon_+\,\epsilon_-\,\epsilon_2}$ and we express it in both Minkowskian and orbifold coordinates. We use the notation $\psi^{[1]}_{k_+\,k_-\,k_2,\epsilon_+\,\epsilon_-\,\epsilon_2}$ to stress that it is the eigenfunction and not the field which is obtained as:

$$A_\mu(x)\,dx^\mu = \int \mathrm{d}^3k\,\sum_\epsilon \psi^{[1]}_{k_+\,k_-\,k_2,\epsilon_+\,\epsilon_-\,\epsilon_2}, \tag{4.1.10}$$

where the sum is performed over $\epsilon$ which are independent and compatible with $k$. The explicit expression for the eigenfunction with $\epsilon$ constant is:[1]

$$\mathcal{N}\psi^{[1]}_{k_+\,k_-\,k_2,\epsilon_+\,\epsilon_-\,\epsilon_2}(x^+,x^-,x^2) = (\epsilon_+ dx^+ + \epsilon_- dx^- + \epsilon_2 dx^2) e^{i\left(k_+x^+ + k_-x^- + k_2x^2\right)}$$
$$= \mathcal{N}\psi^{[1]}_{k_+\,k_-\,k_2,\epsilon_+\,\epsilon_-\,\epsilon_2}(u,v,z) = (\epsilon_u\,\mathrm{d}u + \epsilon_z\,\mathrm{d}z + \epsilon_v\,\mathrm{d}v)$$
$$e^{i\left[k_+v + \frac{2k_+k_- - k_2{}^2}{2k_+}u + \frac{1}{2}\Delta^2 k_+ u\left(z + \frac{k_2}{\Delta k_+}\right)^2\right]}, \tag{4.1.11}$$

with

$$\begin{aligned}\epsilon_v &= \epsilon_+,\\ \epsilon_u(z) &= \epsilon_- + (\Delta\,z)\epsilon_2 + (\frac{1}{2}\Delta^2\,z^2)\epsilon_+,\\ \epsilon_z(u,z) &= (\Delta\,u)(\epsilon_2 + \Delta\,z\,\epsilon_+).\end{aligned} \tag{4.1.12}$$

Notice that we are not imposing any gauge condition. Moreover, if $(\epsilon_+, \epsilon_-, \epsilon_2)$ are constant then $(\epsilon_u, \epsilon_v, \epsilon_z)$ are generic functions. It is worth stressing that $(\epsilon_u, \epsilon_v, \epsilon_z)$ are not the polarizations on the orbifold which are anyhow constant: the fact that they depend on the coordinates is simply the statement that not all eigenfunctions of the vector d'Alembertian are equal.

Building the corresponding function on the orbifold amounts to summing the images:

$$\mathcal{N}\Psi^{[1]}_{[k,\,\epsilon]}([x]) = \sum_n \epsilon\cdot(\mathcal{K}^{-n}\mathrm{d}x)\ \psi_k(\mathcal{K}^{-n}x) = \sum_n \mathcal{K}^n\epsilon\cdot\mathrm{d}x\ \psi_{\mathcal{K}^n k}(x). \tag{4.1.13}$$

This expression makes clear that, under the action of the Killing vector, $\epsilon$ transforms exactly as $k$ since it is induced by $\epsilon\cdot\mathcal{K}^n\mathrm{d}x = \mathcal{K}^{-n}\epsilon\cdot\mathrm{d}x$,

[1]We introduce the normalization factor $\mathcal{N}$ in order to have a less cluttered relation between $\epsilon$ and $\mathcal{E}$.

i.e.:

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_2 \\ \epsilon_- \end{pmatrix} \sim \mathcal{K}^{-n}\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_2 + n(2\pi\Delta)\epsilon_+ \\ \epsilon_- + n(2\pi\Delta)\epsilon_2 + \frac{1}{2}n^2(2\pi\Delta)^2\epsilon_+ \end{pmatrix}. \quad (4.1.14)$$

However the pair $(k,\, \epsilon)$ transforms with the same $n$ since both are "dual" to $x$, i.e. their transformation rules are dictated by the $x$. Therefore there is only one equivalence class $[k,\, \epsilon]$ and not two $[k]$, $[\epsilon]$. In other words, a representative of the combined equivalence class is the one with $0 \le k_2 < 2\pi\Delta|k_+|$ when $k_+ \neq 0$.

In order to write the eigenfunctions on the orbifold in the orbifold coordinates we notice that $\mathrm{d}u, \mathrm{d}v$ and $\mathrm{d}z$ are invariant. We start from

$$\begin{aligned}\mathcal{N}\Psi^{[1]}_{[k,\,\epsilon]}([x]) &= \sum_n \epsilon \cdot (\mathcal{K}^n \mathrm{d}x)\, \psi_k(\mathcal{K}^n x) \\ &= \mathrm{d}v \left[\epsilon_+ \sum_n \psi_k(\mathcal{K}^n x)\right] \\ &+ \mathrm{d}z\,(\Delta u)\left[\epsilon_2 \sum_n \psi_k(\mathcal{K}^n x) + \epsilon_+\Delta \sum_n (z+2\pi n)\psi_k(\mathcal{K}^n x)\right] \\ &+ \mathrm{d}u \Bigg[\epsilon_- \sum_n \psi_k(\mathcal{K}^n x) + \epsilon_2\Delta \sum_n (z+2\pi n)\psi_k(\mathcal{K}^n x) \\ &\qquad + \frac{1}{2}\epsilon_+\Delta^2 \sum_n (z+2\pi n)^2 \psi_k(\mathcal{K}^n x)\Bigg]. \end{aligned} \quad (4.1.15)$$

From direct computation we get:[2]

$$\begin{aligned}\sum_n (z+2\pi n)\psi_k(\mathcal{K}^n x) &= \left(\frac{1}{i\Delta\, u}\frac{\partial}{\partial k_2} - \frac{k_2}{\Delta k_+}\right)\Psi_{[k]}([x]), \\ \sum_n (z+2\pi n)^2\psi_k(\mathcal{K}^n x) &= \left(\frac{1}{i\Delta\, u}\frac{\partial}{\partial k_2} - \frac{k_2}{\Delta k_+}\right)^2\Psi_{[k]}([x]). \end{aligned} \quad (4.1.16)$$

[2]These expressions may be written using Hermite polynomials.

Then it follows that:

$$\begin{aligned}\mathcal{N}\Psi^{[1]}_{[k,\,\epsilon]}([x]) =& \mathrm{d}v\,\left[\epsilon_+\,\Psi_{[k]}([x])\right]\\ &+\mathrm{d}z\,(\Delta u)\left[\frac{\epsilon_2 k_+ - \epsilon_+ k_2}{k_+}\Psi_{[k]}([x]) + \epsilon_+\,\frac{-i}{u}\frac{\partial}{\partial k_2}\Psi_{[k]}([x])\right]\\ &+\mathrm{d}u\Bigg[\left(\epsilon_- - \epsilon_2\frac{k_2}{k_+} + \frac{1}{2}\epsilon_+\left(\frac{k_2}{k_+}\right)^2\right)\,\Psi_{[k]}([x]) + \frac{i}{2u}\frac{\epsilon_+}{k_+}\,\Psi_{[k]}([x])\\ &\quad + \frac{\epsilon_2 k_+ - \epsilon_+ k_2}{k_+}\frac{-i}{u}\frac{\partial}{\partial k_2}\Psi_{[k]}([x]) + \frac{1}{2}\epsilon_+\frac{-1}{u^2}\frac{\partial^2}{\partial {k_2}^2}\Psi_{[k]}([x])\Bigg].\end{aligned} \tag{4.1.17}$$

We notice that many coefficients of $\Psi$ or its derivatives contain $k_2$. They cannot be expressed using the orbifold quantum numbers $\{k_+\,l\,\vec{k}\,r\}$, but they are invariant on the orbifold and therefore they are new orbifold quantities which we can interpret as orbifold polarizations. Using (4.1.7) we can finally write:

$$\begin{aligned}\Psi^{[1]}_{[k,\,\epsilon]}([x]) = \sum_l \phi_{\{k_+\,l\,\vec{k}\,r\}}(u,v,z,\vec{x})e^{il\frac{k_2}{\Delta k_+}}\Bigg\{&\mathrm{d}v\left[\epsilon_+\right]\\ &+\mathrm{d}z\,(\Delta u)\left[\frac{\epsilon_2 k_+ - \epsilon_+ k_2}{k_+} + \epsilon_+\frac{1}{\Delta u}\frac{l}{k_+}\right]\\ &+\mathrm{d}u\Bigg[\left(\epsilon_- - \epsilon_2\frac{k_2}{k_+} + \frac{1}{2}\epsilon_+\left(\frac{k_2}{k_+}\right)^2\right) + \frac{i}{2u}\frac{\epsilon_+}{k_+}\\ &\quad + \frac{\epsilon_2 k_+ - \epsilon_+ k_2}{k_+}\frac{1}{u}\frac{l}{\Delta k_+} + \epsilon_+\frac{1}{2u^2}\left(\frac{l}{\Delta k_+}\right)^2\Bigg]\Bigg\}.\end{aligned} \tag{4.1.18}$$

If we compare the last expression with (3.2.23) we find:

$$\begin{aligned}
\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{v}} &= \epsilon_+ \\
\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{z}} &= \mathrm{sgn}(u)\frac{\epsilon_2 k_+ - \epsilon_+ k_2}{k_+} \\
\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{u}} &= \epsilon_- - \epsilon_2\frac{k_2}{k_+} + \frac{1}{2}\epsilon_+\left(\frac{k_2}{k_+}\right)^2,
\end{aligned} \tag{4.1.19}$$

which implies that the true polarizations $(\epsilon_+, \epsilon_-, \epsilon_2)$ and $\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{*}}$ are constant as it turns out from direct computation. A different way of reading the previous result is that the polarizations on the orbifold are the coefficients of the highest power of $u$.

We can also invert the previous relations to get:

$$\begin{aligned}
\epsilon_+ &= \mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{v}} \\
\epsilon_2 &= \mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{z}}\,\mathrm{sgn}(u) + \frac{k_2}{k_+}\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{v}} \\
\epsilon_- &= \mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{u}} + \frac{k_2}{k_+}\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{z}}\,\mathrm{sgn}(u) + \frac{1}{2}\left(\frac{k_2}{k_+}\right)^2\mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{v}},
\end{aligned} \tag{4.1.20}$$

and use them in the Lorenz gauge $k \cdot \epsilon = 0$ in order to obtain its expression with orbifold polarizations. If their definition is right, the result cannot depend on $k_2$ since it is not a quantum number of orbifold eigenfunctions. Taking into account $k_- = \frac{\vec{k}^2 + {k_2}^2 + r}{2k_+}$ in $k \cdot \epsilon = 0$ we get exactly the expression for the Lorenz gauge for orbifold polarizations (3.2.21).

### 4.1.3 Spin 2 Wave Function

We can use the analysis of the previous section in the case of a second order symmetric tensor wave function. Again we suppress the dependence on $\vec{x}$ and $\vec{k}$ with a caveat: the Minkowskian polarizations $S_{+\,i}$, $S_{-\,i}$ and $S_{2\,i}$ do transform non trivially, therefore we give the full expressions in Appendix A even if these components contribute in a somewhat trivial way since they behave effectively as a vector of the orbifold.

We start with the usual wave in flat space and we express it either in the Minkowskian coordinates

$$\begin{aligned}\mathcal{N}\psi^{[2]}_{k\,S}(x^+,x^-,x^2) &= S_{\mu\nu}\,\psi_k(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= (S_{+\,+}\,\mathrm{d}x^+\,\mathrm{d}x^+ \quad +2S_{+\,2}\,\mathrm{d}x^+\,\mathrm{d}x^2 + 2S_{+\,-}\,\mathrm{d}x^+\,\mathrm{d}x^- \\ &\qquad +2S_{2\,2}\,\mathrm{d}x^2\,\mathrm{d}x^2 + 2S_{2\,-}\,\mathrm{d}x^2\,\mathrm{d}x^- \\ &\qquad + 2S_{-\,-}\,\mathrm{d}x^-\,\mathrm{d}x^-) \\ &\quad\times e^{i\left(k_+x^+ + k_-x^- + k_2x^2\right)},\end{aligned} \tag{4.1.21}$$

or in the orbifold ones

$$\begin{aligned}\mathcal{N}\psi^{[2]}_{k\,S}(x) =& S_{\alpha\beta}\,\psi_k(x)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \\ =& \Big\{(\mathrm{d}v)^2\,[S_{+\,+}] \\ &+ \mathrm{d}v\,\mathrm{d}z\,\Delta u[2S_{+\,2} + S_{+\,+}\Delta z] \\ &+ \mathrm{d}v\,\mathrm{d}u\,[2S_{+\,-} + 2S_{+\,2}\Delta z + S_{+\,+}\Delta^2 z^2] \\ &+ \mathrm{d}z^2\,\Delta^2 u^2\,[S_{2\,2} + 2S_{+\,2}\Delta z + S_{+\,+}\Delta^2 z^2] \\ &+ \mathrm{d}z\,\mathrm{d}v\,\Delta u\,[2S_{-\,2} + 2(S_{2\,2} + S_{+\,-})\Delta z + 3S_{+\,2}\Delta^2 z^2 + S_{+\,+}\Delta^3 z^3] \\ &+ \mathrm{d}u^2\,[S_{-\,-} + 2S_{-\,2}\Delta z + (S_{2\,2} + S_{+\,-})\Delta^2 z^2 + S_{+\,2}\Delta^3 z^3 \\ &\qquad + \frac{1}{4}S_{+\,+}\Delta^4 z^4]\Big\} \\ &\times e^{i\left[k_+v + \frac{2k_+k_- - k_2{}^2}{2k_+}u + \frac{1}{2}\Delta^2 k_+ u\left(z + \frac{k_2}{\Delta k_+}\right)^2\right]}.\end{aligned} \tag{4.1.22}$$

Now we define the tensor on the orbifold as a sum over all images as:

$$\begin{aligned}\mathcal{N}\Psi^{[2]}_{[k\,S]}([x]) &= \sum_n (\mathcal{K}^n dx)\cdot S\cdot(\mathcal{K}^n\mathrm{d}x)\;\psi_k(\mathcal{K}^n x) \\ &= \sum_n \mathrm{d}x\cdot(\mathcal{K}^{-n}S)\cdot\mathrm{d}x\;\psi_{\mathcal{K}^{-n}k}(x).\end{aligned} \tag{4.1.23}$$

In the last line we have defined the induced action of the Killing vector on $(k, S)$, which can be explicitely written as:

$$\mathcal{K}^{-n}\begin{pmatrix} S_{++} \\ S_{+2} \\ S_{+-} \\ S_{22} \\ S_{2-} \\ S_{--} \end{pmatrix} = \begin{pmatrix} S_{++} \\ S_{+2} + n\Delta S_{++} \\ S_{+-} + n\Delta S_{+2} + \frac{1}{2}n^2\Delta^2 S_{++} \\ S_{22} + 2n\Delta S_{+2} + n^2\Delta^2 S_{++} \\ S_{2-} + n\Delta(S_{22} + S_{+-}) + \frac{3}{2}n^2\Delta^2 S_{+2} + \frac{1}{2}n^3\Delta^3 S_{++} \\ S_{--} + 2n\Delta S_{-2} + n^2\Delta^2(S_{22} + S_{+-}) + n^3\Delta^3 S_{+2} + \frac{1}{4}n^4\Delta^4 S_{++} \end{pmatrix}. \tag{4.1.24}$$

Computing the tensor on the orbifold in its own coordinates is equivalent to summing over all the shifts $z \to (z + 2\pi n)$ using the generalization of (4.1.16), i.e. to substitute $(\Delta\, z)^j \psi_k \to \left(\frac{1}{iu}\frac{\partial}{\partial k_2} - \frac{k_2}{\Delta k_+}\right)^j \Psi_{[k]}([x])$. When expressing all in the $\phi$ basis this last step is equivalent to $(\Delta\, z)^j \psi_k \to \left(\frac{l}{\Delta\, u\, k_+}\right)^j + \dots$. We identify the basic polarizations on the orbifold by considering the highest power in $u$:

$$\begin{aligned}
\mathcal{S}_{u\,u} &= \frac{1}{4}K^4\, S_{+\,+} + K^2\, S_{+\,-} - K^3\, S_{+\,2} + S_{-\,-} - 2\,K\,S_{-\,2} + S_{2\,2}\,K^2 \\
\mathcal{S}_{u\,v} &= \frac{1}{2}K^2\, S_{+\,+} + S_{+\,-} - K\,S_{+\,2} \\
\mathcal{S}_{u\,z} &= -\frac{1}{2}K^3\, S_{+\,+} - K\,S_{+\,-} + \frac{3}{2}K^2\, S_{+\,2} + S_{-\,2} - K\,S_{2\,2} \\
\mathcal{S}_{v\,v} &= S_{+\,+} \\
\mathcal{S}_{v\,z} &= S_{+\,2} - K\,S_{+\,+} \\
\mathcal{S}_{z\,z} &= K^2\, S_{+\,+} - 2\,K\,S_{+\,2} + S_{2\,2},
\end{aligned} \tag{4.1.25}$$

where $K = \frac{k_2}{k_+}$. The previous equations can be inverted to get:

$$
\begin{aligned}
S_{--} &= K^2\ (\mathcal{S}_{z\,z} + \mathcal{S}_{u\,v}) + K^3\,\mathcal{S}_{v\,z} + \frac{1}{4}K^4\,\mathcal{S}_{v\,v} + 2\,K\,\mathcal{S}_{u\,z} + \mathcal{S}_{u\,u} \\
S_{+-} &= K\,\mathcal{S}_{v\,z} + \frac{1}{2}K^2\,\mathcal{S}_{v\,v} + \mathcal{S}_{u\,v} \\
S_{-\,2} &= K\ (\mathcal{S}_{z\,z} + \mathcal{S}_{u\,v}) + \frac{3}{2}\,K^2\,\mathcal{S}_{v\,z} + \frac{1}{2}K^3\,\mathcal{S}_{v\,v} + \mathcal{S}_{u\,z} \\
S_{++} &= \mathcal{S}_{v\,v} \\
S_{+\,2} &= \mathcal{S}_{v\,z} + K\,\mathcal{S}_{v\,v} \\
S_{2\,2} &= \mathcal{S}_{z\,z} + 2\,K\,\mathcal{S}_{v\,z} + K^2\,\mathcal{S}_{v\,v}.
\end{aligned}
\tag{4.1.26}
$$

Since we plan to use the previous expressions in the case of the first massive string state we compute some relevant quantities. In particulare we have the trace

$$
\mathrm{tr}(S) = \mathcal{S}_{z\,z} - 2\,\mathcal{S}_{u\,v} \tag{4.1.27}
$$

and the transversality conditions

$$
\begin{aligned}
\text{trans } \mathcal{S}_v =& (k\cdot S)_+ = -\frac{(r+\vec{k}^2)}{2\,k_+}\,\mathcal{S}_{v\,v} - k_+\,\mathcal{S}_{u\,v}, \\
\text{trans } \mathcal{S}_z =& (k\cdot S)_2 - K(k\cdot S)_+ = -\frac{(r+\vec{k}^2)}{2\,k_+}\,\mathcal{S}_{v\,z} - k_+\,\mathcal{S}_{u\,z}, \\
\text{trans } \mathcal{S}_u =& (k\cdot S)_- - K(k\cdot S)_2 + \frac{1}{2}K^2(k\cdot S)_+ = -\frac{(r+\vec{k}^2)}{2\,k_+}\,\mathcal{S}_{u\,v} - k_+\,\mathcal{S}_{u\,u},
\end{aligned}
\tag{4.1.28}
$$

where we used $k_- = \frac{(r+\vec{k}^2+k_2^2)}{(2k_+)}$. These conditions do no depend on $K$ since $k_2$ is not an orbifold quantum number.

The final expression for the orbifold symmetric tensor is:

$$
\begin{aligned}
\Psi^{[2]}_{[k,S]}([x]) = \sum_l \phi_{\{k_+\, l\, \vec{k}\, r\}}(u,v,z,\vec{x}) e^{il\frac{k_2}{\Delta k_+}} \\
&\Big\{ (\mathrm{d}v)^2 \left[\mathcal{S}_{vv}\right] \\
&+ 2\Delta\, u\, \mathrm{d}v\, \mathrm{d}z \left[\mathcal{S}_{v\,z} + \left(\frac{L\mathcal{S}_{v\,v}}{\Delta}\right)\frac{1}{u}\right] \\
&+ 2\mathrm{d}v\,\mathrm{d}u \left[\mathcal{S}_{u\,v} + \left(\frac{L\,\mathcal{S}_{v\,z}}{\Delta} + \frac{i\,\mathcal{S}_{v\,v}}{2\,k_+}\right)\frac{1}{u} + \left(\frac{L^2\,\mathcal{S}_{v\,v}}{2\,\Delta^2}\right)\frac{1}{u^2}\right] \\
&+ (\Delta\, u)^2 \mathrm{d}z^2 \left[\mathcal{S}_{z\,z} + \left(\frac{2\,L\,\mathcal{S}_{v\,z}}{\Delta} + \frac{i\,\mathcal{S}_{v\,v}}{k_+}\right)\frac{1}{u} + \left(\frac{L^2\,\mathcal{S}_{v\,v}}{\Delta^2}\right)\frac{1}{u^2}\right] \\
&+ 2\Delta u\,\mathrm{d}z\,\mathrm{d}u \left[\mathcal{S}_{u\,z} + \left(\frac{L\,\mathcal{S}_{z\,z}}{\Delta} + \frac{3\,i\,\mathcal{S}_{v\,z}}{2\,k_+} + \frac{L\,\mathcal{S}_{u\,v}}{\Delta}\right)\frac{1}{u}\right. \\
&\qquad \left. + \left(\frac{3\,L^2\,\mathcal{S}_{v\,z}}{2\,\Delta^2} + \frac{3\,i\,L\,\mathcal{S}_{v\,v}}{2\,\Delta\,k_+}\right)\frac{1}{u^2} + \left(\frac{L^3\,\mathcal{S}_{v\,v}}{2\,\Delta^3}\right)\frac{1}{u^3}\right] \\
&+ \mathrm{d}u^2 \left[\mathcal{S}_{u\,u} + \left(\frac{i\,\mathcal{S}_{z\,z}}{k_+} + \frac{2\,L\,\mathcal{S}_{u\,z}}{\Delta} + \frac{i\,\mathcal{S}_{u\,v}}{k_+}\right)\frac{1}{u}\right. \\
&\quad + \left(\frac{L^2\,\mathcal{S}_{z\,z}}{\Delta^2} + \frac{3\,i\,L\,\mathcal{S}_{v\,z}}{\Delta\,k_+} - \frac{3\,\mathcal{S}_{v\,v}}{4\,k_+^2} + \frac{L^2\,\mathcal{S}_{u\,v}}{\Delta^2}\right)\frac{1}{u^2} \\
&\quad \left. + \left(\frac{L^3\,\mathcal{S}_{v\,z}}{\Delta^3} + \frac{3\,i\,L^2\,\mathcal{S}_{v\,v}}{2\,\Delta^2\,k_+}\right)\frac{1}{u^3} + \left(\frac{L^4\mathcal{S}_{v\,v}}{4\Delta^4}\right)\frac{1}{u^4}\right]\Big\},
\end{aligned}
\tag{4.1.29}
$$

where $L = \frac{l}{k_+}$.

## 4.2 Wave Functions Overlaps

In this section we compute overlaps of wave functions. We give their expressions using both integrals over the eigenfunctions and sum of products of $\delta$. The latter is the expression which is naturally obtained by computing tree-level string amplitudes on the orbifold when one starts with Minkowski amplitudes and adds the images. This is equivalent to computing emission vertices on the orbifold and then their correlation functions since it amounts to transfer the sum over the spacetime images to the sum over the polarizations images. Finally, we consider also when and if they diverge.

### 4.2.1 Overlaps without Derivatives

Let us start with the simplest case of the overlap of $N$ scalar wave functions. We compute the overlap of orbifold wave functions and then we re-express it as sum of images of the corresponding Minkowski overlap, thus establishing a dictionary between Minkowski and orbifold overlaps. Explicitely we consider the following overlap where all the polarizations $\mathcal{A}_{(i)}$ have been set to one:

$$
\begin{aligned}
I^{(N)} &= \int_\Omega \mathrm{d}^3x \sqrt{-\det g} \prod_{i=1}^N \Psi_{[k_{(i)+}\, k_{(i)-}\, k_{(i)2}]}([x^+, x^-, x^2]) \\
&= \int_{\mathbb{R}^{1,2}} \mathrm{d}^3x \sqrt{-\det g}\ \psi_{k_{(1)+}\, k_{(1)-}\, k_{(1)2}}(x^+, x^-, x^2) \\
&\times \prod_{i=2}^N \sum_{m_{(i)}\in\mathbb{Z}} \psi_{k_{(i)+}\, k_{(i)-}\, k_{(i)2}}(\mathcal{K}^{m_{(i)}}(x^+, x^-, x^2)) \\
&= \int_{\mathbb{R}^{1,2}} \mathrm{d}^3x \sqrt{-\det g}\ \psi_{k_{(1)+}\, k_{(1)-}\, k_{(1)2}}(x^+, x^-, x^2) \\
&\times \prod_{i=2}^N \sum_{m_{(i)}\in\mathbb{Z}} \psi_{\mathcal{K}^{m_{(i)}}(k_{(i)+}\, k_{(i)-}\, k_{(i)2})}(x^+, x^-, x^2) \\
&= (2\pi)^3 \delta(\sum_i k_{(i)+}) \prod_{i=2}^N \sum_{m_{(i)}\in\mathbb{Z}} \delta\left(\sum_i \mathcal{K}^{m_{(i)}} k_{(i)2}\right) \delta\left(\sum_i \mathcal{K}^{m_{(i)}} k_{(i)-}\right) \Bigg|_{m_{(1)}=0},
\end{aligned}
\tag{4.2.1}
$$

where $\Omega = \mathbb{R}^{1,2}/\Gamma$ is the orbifold fundamental region. We used the unfolding trick to rewrite the integral over $R^{1,2}$ by dropping the sum over the images of particle (1). Then we moved the action of the Killing vector from $x$ to $k$ and finally we used the usual $\delta$ definition. $I^{(N)}$ can be expressed explicitely as

$$
\begin{aligned}
I^{(N)} &= \mathcal{N}^N \sum_{\{l_{(i)}\}\in\mathbb{Z}^N} e^{i\sum_{i=1}^N l_{(i)}\frac{k_{(i)2}}{\Delta k_{(i)+}}} \int_\Omega \mathrm{d}^3x\,\sqrt{-\det g}\,\prod_{i=1}^N \phi_{\{k_{(i)+}\,k_{(i)-}\,l_{(i)}\,r_{(i)}\}}([x])) \\
&= \mathcal{N}^N \sum_{\{l_{(i)}\}\in\mathbb{Z}^N} e^{i\sum_{i=1}^N l_{(i)}\frac{k_{(i)2}}{\Delta k_{(i)+}}} (2\pi)^2\delta\left(\sum k_{(i)+}\right)\delta_{\sum l_{(i)}}\,\mathcal{I}^{[0]}_{\{N\}},
\end{aligned}
\tag{4.2.2}
$$

from which we can rewrite the overlap of the wave functions as:

$$
\begin{aligned}
\int_\Omega \mathrm{d}^3x \prod_{i=1}^N \phi_{\{k_{(i)+}\,k_{(i)-}\,l_{(i)}\,r_{(i)}\}}\,([x])) &= \frac{1}{\mathcal{N}^N}\prod_{i=1}^N \int_0^{2\pi\Delta|k_{(i)+}|} \frac{\mathrm{d}k_{(i)2}}{2\pi\Delta|k_{(i)+}|}\, e^{-il_{(i)}\frac{k_{(i)2}}{\Delta k_{+i}}}\, I^{(N)} \\
= \quad (2\pi)^3\delta\left(\sum_i k_{(i)+}\right) &\frac{1}{\mathcal{N}^N}\prod_{i=1}^N \int_0^{2\pi\Delta|k_{(i)+}|} \frac{\mathrm{d}k_{(i)2}}{2\pi\Delta|k_{(i)+}|}\, e^{-il_{(i)}\frac{k_{(i)2}}{\Delta k_{(i)+}}} \\
\times \prod_{j=2}^N \sum_{m_{(j)}\in\mathbb{Z}} &\delta\left(\sum_j \mathcal{K}^{m_{(j)}} k_{(j)2}\right)\delta\left(\sum_j \mathcal{K}^{m_{(j)}} k_{(j)-}\right).
\end{aligned}
\tag{4.2.3}
$$

In particular it follows from the explicit expression of $\mathcal{I}^{[0]}_{\{N\}}$ that all overlaps $I^{(N)}$ for $N \geq 4$ are infinite.

Is there any intuitive reason for the divergence of the overlapping? It's true that we are summing over infinite distributions with accumulation points of their support. Nevertheless, the existence of the accumulation point is not sufficient since the three scalars overlap, i.e. the three tachyons amplitude, converges (see also (1.3.18) on the BO): the coefficients of the deltas matter too and the convergence issue must be analyzed in more detail.

### 4.2.2 An Overlap with One Derivative

Since we will also compute the amplitude involving two tachyons and one photon, as a preliminary step we consider the overlap in Minkowski space:

$$J_{Mink} = i\,(\epsilon_{(1)} \cdot k_{(2)2})\,(2\pi)^3 \delta\left(\sum_i k_{(i)+}\right)\,\delta\left(\sum_i k_{(i)2}\right)\,\delta\left(\sum_i k_{(i)-}\right). \tag{4.2.4}$$

Applying the recipe of summing over momenta and polarization images of all but one particle, we obtain:

$$J([k_{(1)}, \epsilon_{(1)}],\, [k_{(2)}],\, [k_{(3)}]) = i\,(2\pi)^3 \delta\left(\sum_i k_{(i)+}\right)$$
$$\times \sum_{\{m_{(i)}\}\in\mathbb{Z}^3} \delta_{m_{(1)},1}\,(\mathcal{K}^{m_{(1)}}\epsilon_{(1)} \cdot \mathcal{K}^{m_{(2)}} k_{(2)2})\,\delta\left(\sum_i \mathcal{K}^{m_{(i)}} k_{(i)2}\right)\,\delta\left(\sum_i \mathcal{K}^{m_{(i)}} k_{(i)-}\right). \tag{4.2.5}$$

Notice that, under $(k_{(1)}, \epsilon_{(1)}) \to \mathcal{K}^s(k_{(1)}, \epsilon_{(1)})$, we can use $\mathcal{K}^s a \cdot b = a \cdot \mathcal{K}^{-s} b$ and the invariance of deltas $\delta^3(\mathcal{K}^s a) = \delta^3(a)$ to prove that the latest expression depends only on equivalence classes. Now it is not difficult to show that it can also be written as:

$$J = \int_\Omega \mathrm{d}^3x\, \eta^{\mu\nu}\, \Psi^{[1]}_{[k_{(1)},\epsilon_{(1)}]\,\mu}([x])\,\partial_\nu \Psi_{[k_{(2)}]}([x])\,\Psi_{[k_{(3)}]}([x]), \tag{4.2.6}$$

where we performed the unfolding using $a_{[k_{(1)},\epsilon_{(1)}]\,\mu}([x])$. Obviously we can choose whichever other field to do the unfolding trick and this amount to keep the corresponding $m_{(i)}$ fixed in place of $m_{(1)}$. Notice also that the previous expression is invariant despite the fact that the derivatives $\partial_\mu$ are not well defined on the orbifold, since this is compensated by $\Psi^{[1]}_\mu$.

We can then evaluate (4.2.6) with Minkowskian polarizations using (4.1.18), which is nothing else but a rearrangement of terms of (4.2.5), to write:

$$
\begin{aligned}
J =& i\,\mathcal{N}^2 \sum_{\{l_{(i)}\}\in\mathbb{Z}^3} e^{i\sum_{i=1}^{3} l_{(i)} \frac{k_{(i)2}}{\Delta k_{(i)+}}} (2\pi)^2 \delta\left(\sum k_{(i)+}\right) \delta_{\sum l_{(i)}} \\
& \times \int_\Omega \mathrm{d}^3 x \prod_{i=1}^{3} \phi_{\{k_{(i)+}\, k_{(i)-}\, l_{(i)}\, r_{(i)}\}}([x])) \\
& \Bigg\{ \epsilon_{(1)+} \left[ + \frac{i}{2u} + \frac{{l_{(2)}}^2}{k_{(2)+}} \frac{1}{2\Delta^2\, u^2} + \frac{r_{(2)}}{2k_{(2)+}} \right] \\
& + \frac{1}{\Delta\, u} \left[ \epsilon_{(1)2} + \frac{1}{\Delta u} \epsilon_{(1)+} \frac{l_{(1)}}{k_{(1)+}} \right] l_{(2)} \\
& + \left[ \epsilon_{(1)-} + \epsilon_{(1)2} \frac{1}{\Delta u} \frac{l_{(1)}}{k_{(1)+}} + \epsilon_{(1)+} \frac{1}{2(\Delta u)^2} \frac{{l_{(1)}}^2}{{k_{(1)+}}^2} \right] k_{(2)+} \Bigg\}.
\end{aligned} \tag{4.2.7}
$$

Divergences occur when $l = 0$ because of the absence of the factor $e^{i\frac{A}{u}}$. However, all explicit factors $\frac{1}{u}$ come always with $l$, therefore when $l = 0$ they do not give any contribution. The divergence in this case comes actually only from the contribution of the first line $\partial_u \phi|_{l=0} = -\frac{1}{2u}\phi|_{l=0}$, but it cancels in scalar QED or with abelian tachyons since we have to subtract the contribution obtained exchanging (2) and (3). It does instead not cancel when considering the non abelian case because of color factors, unless one uses a kind of principal part prescription, since replacing $\int_{-|a|}^{|b|} \mathrm{d}u\, \frac{\mathrm{sgn}(u)}{|u|^{3/2}}$ with $\lim_{\delta\to 0} \left[\int_{-|a|}^{-|\delta|} + \int_{-|\delta|}^{|b|}\right] \mathrm{d}u\, \frac{\mathrm{sgn}(u)}{|u|^{3/2}}$ gives a finite result.

### 4.2.3 An Overlap with Two Derivatives

We can generalize the previous expressions to other cases.[3] Having in mind the amplitudes with two tachyons and one massive state, we

[3] Since we use the results from Section 4.1, we miss some non-trivial contributions from polarizations like $\mathcal{S}_{vi}$. This does not alter the discussion. However, we give for completeness the lengthy full expressions in Appendix B.

can consider an expression like

$$K = \int_\Omega \mathrm{d}^3x \sqrt{-\det g}\, \eta^{\mu\nu}\, \eta^{\rho\sigma}\, \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\mu\rho}([x])\, \partial^2_{\nu\sigma} \Psi_{[k_{(2)}]}([x])\, \Psi_{[k_{(1)}]}([x]) \tag{4.2.8}$$

in Minkowskian coordinates or

$$K = \int_\Omega \mathrm{d}^3x \sqrt{-\det g}\; g^{\alpha\beta}\, g^{\gamma\delta}\, \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\alpha\gamma}([x])\, D_\beta \partial_\delta \Psi_{[k_{(2)}]}([x])\, \Psi_{[k_{(1)}]}([x]) \tag{4.2.9}$$

in orbifold coordinates, where we need to use covariant derivatives. Using the unfolding trick over wave function (3) we get:

$$\begin{aligned} K =& (2\pi)^3 \delta\left(\sum_i k_{(i)+}\right) \prod_{i=2}^N \sum_{m_{(i)}\in\mathbb{Z}} S_{(3)\mu\rho}\, (\mathcal{K}^{m_{(2)}} k_{(2)2})^\mu (\mathcal{K}^{m_{(2)}} k_{(2)2})^\rho \\ & \times \delta\left(\sum_i \mathcal{K}^{m_{(i)}} k_{(i)2}\right)\, \delta\left(\sum_i \mathcal{K}^{m_{(i)}} k_{(i)-}\right). \end{aligned} \tag{4.2.10}$$

Explicitly, in orbifold coordinates we can write:

$$\begin{aligned} K = \int_\Omega \mathrm{d}^3x \sqrt{-\det g} \Big[ & + \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,uu}\, \partial_v^2 \Psi_{[k_{(2)}]} \\ & - 2\frac{1}{(\Delta u)^2} \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,uz}\, \partial_v \partial_z \Psi_{[k_{(2)}]} \\ & + 2\Psi^{[2]}_{[k_{(3)},S_{(3)}]\,uv}\, \partial_v \partial_u \Psi_{[k_{(2)}]} \\ & + \frac{1}{(\Delta u)^4} \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,zz}\, (\partial_z^2 \Psi_{[k_{(2)}]} - \Delta^2 u \partial_v \Psi_{[k_{(2)}]}) \\ & - 2\frac{1}{(\Delta u)^2} \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,zv}\, (\partial_z \partial_u \Psi_{[k_{(2)}]} - \frac{1}{u} \partial_z \Psi_{[k_{(2)}]}) \\ & + \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,vv}\, \partial_u^2 \Psi_{[k_{(2)}]} \Big] \Psi_{[k_{(1)}]}. \end{aligned} \tag{4.2.11}$$

Keeping the terms which do not vanish when all $l = 0$ and considering only the leading order in $\frac{1}{u}$ we get:

$$K \sim \int \mathrm{d}u\, |u|\, \frac{3}{4} \frac{(k_{(2)+} + k_{(3)+})^2}{{k_{(3)+}}^2}\, \mathcal{S}_{vv(3)}\, \frac{1}{u^2} \prod_{i=1}^{3} \phi_{(i)} \Big|_{l_{(*)}=0}, \qquad (4.2.12)$$

which is divergent as $\frac{1}{|u|^{5/2}}$.

## 4.3 3-Point String Theory Amplitudes

We consider now string theory amplitudes which involve string massive states. They are obtained using the inheritance principle and therefore they are connected to the integrals and relations derived in Section 4.2.3. In particular, we want to use the inheritance principle on the momenta and polarizations, i.e. we start from amplitudes in Minkowski expressed with momenta and polarizations and then we implement on them the projection to the orbifold. It is worth stressing that, since there is one Killing vector acting on the spacetime coordinates, there is only one common Killing vector action on all the momenta and polarizations of each field as discussed in the spin 1 and spin 2 cases. Moreover, this approach gives the complete answer only for tree-level amplitudes since inside the loops twisted states may be created in pairs.

The final result is that the open string amplitude with two tachyons and the first massive (level 2) state diverges and there is no obvious way of curing it since the divergence is also present in the Abelian sector. The open string expansion we use is:

$$X(u, \bar{u}) = x_0 - i\, 2\alpha' p\, \ln(|u|) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n}{n} \left( u^{-n} + \bar{u}^{-n} \right). \qquad (4.3.1)$$

### 4.3.1 Level 2 Massive State

Before computing the amplitude we would like to review the possible polarizations of the first massive state in open string. The first massive vertex is

$$V_M(x;k,S,\xi) = :\left(\frac{i}{\sqrt{2\alpha'}}\xi\cdot\partial_x^2 X(x,x) + \left(\frac{i}{\sqrt{2\alpha'}}\right)^2 S_{\mu\nu}\partial_x X^\mu(x,x)\partial_x X^\nu(x,x)\right) e^{ik\cdot X(x,x)}: , \tag{4.3.2}$$

and the corresponding state is

$$\lim_{x\to 0} V_M(x;k,S,\xi)|0\rangle = |k,S,\xi\rangle = (\xi\cdot\alpha_{-2} + \alpha_{-1}\cdot S\cdot\alpha_{-1})\,|k\rangle. \tag{4.3.3}$$

For the state to be physical we require:

$$\begin{aligned} (L_0-1)|k,S,\xi\rangle = 0 \Rightarrow &\qquad \alpha' k^2 = -1 \\ L_1|k,S,\xi\rangle = 0 \Rightarrow &\qquad S\cdot k + \xi = 0 \\ L_2|k,S,\xi\rangle = 0 \Rightarrow &\qquad k\cdot\xi + \mathrm{tr}(S) = 0. \end{aligned} \tag{4.3.4}$$

String gauge invariance allows to add

$$L_{-1}(\chi\cdot\alpha_{-1}|k\rangle) = (\chi\cdot\alpha_{-2} + \chi\cdot\alpha_{-1}\,k\cdot\alpha_{-1})|k\rangle, \tag{4.3.5}$$

subject to the physical constraints, i.e.:

$$\alpha' k^2 = -1, \qquad \chi\cdot k = 0. \tag{4.3.6}$$

Actually, in critical string theory there is another gauge invariance generated by $L_{-2} + \frac{3}{2}L_{-1}^2$: in this case we can add a multiple of

$$(L_{-2} + \frac{3}{2}L_{-1}^2)|k\rangle = (\frac{5}{2}k\cdot\alpha_{-2} + \frac{3}{2}(k\cdot\alpha_{-1})^2 + \frac{1}{2}\alpha_{-1}^2)|k\rangle \tag{4.3.7}$$

to set $a = 0$. Therefore the only non-trivial d.o.f. refer to $S^{TT}$, i.e.:

$$\mathrm{tr}(S^{TT}) = k \cdot S^{TT} = \xi = 0. \tag{4.3.8}$$

In view of the computation for the orbifold, we can check that given $k = (k_+, k_-, k_2, \vec{k})$ such that $-2k_+k_- + {k_2}^2 + \vec{k}^2 = -1$ we can find a non-trivial $S^{TT}$ with non vanishing components in the directions $+, -$ and 2 only. In fact, we find a two parameters family of solutions. The parameters may be taken to be $S_{++}$ and $S_{+2}$. Explicitly we have:

$$\begin{pmatrix} S_{++} \\ S_{+-} \\ S_{+2} \\ S_{--} \\ S_{-2} \\ S_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{k_-}{k_+} \\ 0 \\ \frac{k_-(k_-k_+ - 2{k_2}^2)}{{k_+}^3} \\ -2\frac{k_-k_2}{{k_+}^2} \\ -2\frac{k_-}{k_+} \end{pmatrix} S_{++} + \begin{pmatrix} 0 \\ \frac{k_2}{k_+} \\ 1 \\ \frac{2k_2(-k_-k_+ + {k_2}^2)}{{k_+}^3} \\ \frac{k_-k_+ - 2{k_2}^2}{{k_+}^2} \\ 2\frac{k_2}{k_+} \end{pmatrix} S_{+2} \tag{4.3.9}$$

There is even a non-trivial solution for the more special case $k = (k_+, k_- = 1/k_+, k_2 = 0, \vec{k} = \vec{0})$.

Similarly, using the expressions for $S^{TT}$ in orbifold coordinates we check that there are two possible independent polarizations $\mathcal{S}_{vv}$ and $\mathcal{S}_{vz}$ which correspond to the ones used above. Then the non-trivial solution reads:

$$\begin{pmatrix} \mathcal{S}_{vv} \\ \mathcal{S}_{uv} \\ \mathcal{S}_{vz} \\ \mathcal{S}_{uu} \\ \mathcal{S}_{uz} \\ \mathcal{S}_{zz} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{r+\vec{k}^2}{2{k_+}^2} \\ 0 \\ \left(\frac{r+\vec{k}^2}{2{k_+}^2}\right)^2 \\ 0 \\ -2\frac{r+\vec{k}^2}{2{k_+}^2} \end{pmatrix} \mathcal{S}_{vv} + \begin{pmatrix} 0 \\ -\frac{r+\vec{k}^2}{2{k_+}^2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathcal{S}_{vz}. \tag{4.3.10}$$

### 4.3.2 Two Tachyons First Massive State Amplitude

This Minkowskian full amplitude is given by the sum of two color ordered ones as

$$\mathcal{A}_{TTM} = A_{T_{(1)}T_{(2)}M_{(3)}}\,\mathrm{tr}(T_{(1)}T_{(2)}T_{(3)}) + A_{T_{(2)}T_{(1)}M_{(3)}}\,\mathrm{tr}(T_{(2)}T_{(1)}T_{(3)}), \tag{4.3.11}$$

where an easy computation gives:

$$\begin{aligned}
A_{T_{(1)}T_{(2)}M_{(3)}} &= \langle\langle k_{(1)}|\, V_T(1;k_{(2)})\,(\alpha_{-1}\cdot S^{TT}_{(3)}\cdot\alpha_{-1}|k_{(3)}\rangle)\\
&= \langle\langle k_{(1)}|\, e^{i\,k_{(2)}\cdot x_0} e^{-\sqrt{2\alpha'}k_{(2)}\cdot\alpha_1}(\alpha_{-1}\cdot S^{TT}_{(3)}\cdot\alpha_{-1}|k_{(3)}\rangle)\\
&= (2\pi)^D\delta^D\left(\sum k_{(i)}\right)\quad(\sqrt{2\alpha'})^2\,k_{(2)}\cdot S^{TT}_{(3)}\cdot k_{(2)}.
\end{aligned} \tag{4.3.12}$$

Because of transversality of $S^{TT}_{(3)}$, $A_{T_{(2)}T_{(1)}M_{(3)}}$ gives the same result of $A_{T_{(1)}T_{(2)}M_{(3)}}$, hence the final Minkowskian amplitude is:

$$\mathcal{A}_{TTM} = (2\pi)^D\delta^D\left(\sum k_{(i)}\right)\;2(\sqrt{2\alpha'})^2\,k_{(2)}\cdot S^{TT}_{(3)}\cdot k_{(2)}\,\mathrm{tr}\left(\{T_{(1)},T_{(2)}\}T_{(3)}\right). \tag{4.3.13}$$

Then we can compute the orbifold amplitude as

$$\begin{aligned}
\mathcal{A}_{TTM} &= (2\pi)^{D-2}\delta^{D-3}\left(\sum\vec{k}_{(i)}\right)\delta\left(\sum k_{(i)+}\right)\\
&\quad 2(\sqrt{2\alpha'})^2\sum_{\{m_{(1)},m_{(2)},m_{(3)}\}\in\mathbb{Z}^3}\delta_{m_{(3)},1}\,(\mathcal{K}^{m_{(2)}}k_{(2)})\cdot S^{TT}_{(3)}\cdot(\mathcal{K}^{m_{(2)}}k_{(2)})\\
&\qquad \delta\left(\sum(\mathcal{K}^{m_{(i)}}k_{(i)2}\right)\delta\left(\sum(\mathcal{K}^{m_{(i)}}k_{(i)-}\right)\;\mathrm{tr}\left(\{T_{(1)},T_{(2)}\}T_{(3)}\right).
\end{aligned}$$

Finally, we can rewrite the previous expression using an overlap as:

$$\begin{aligned}\mathcal{A}_{TTM} &= 2(-i\sqrt{2\alpha'})^2 \int_\Omega \mathrm{d}^3x\, g^{\mu\nu}\, g^{\rho\sigma}\, \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\mu\rho}([x])\, \partial^2_{\nu\sigma}\Psi_{[k_{(2)}]}([x])\, \Psi_{[k_{(1)}]}([x]) \\ &\qquad \mathrm{tr}\left(\{T_{(1)}, T_{(2)}\}T_{(3)}\right), \\ &= 2(-i\sqrt{2\alpha'})^2 \int_\Omega \mathrm{d}^3x\, g^{\alpha\beta}\, g^{\gamma\delta}\, \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\alpha\gamma}([x])\, D_\beta\partial_\delta\Psi_{[k_{(2)}]}([x])\, \Psi_{[k_{(1)}]}([x]) \\ &\qquad \mathrm{tr}\left(\{T_{(1)}, T_{(2)}\}T_{(3)}\right). \end{aligned} \tag{4.3.14}$$

As discussed in the Section 4.2.3 the last integral is divergent when $S_{++} = \mathcal{S}_{vv} \neq 0$, in the specific sector where all $l_{(*)} = 0$. The divergence cannot be avoided even introducing a Wilson line around $z$ since the amplitude involves an anticommutator which does not vanish in the Abelian sector.

Therefore this is probably the main result of the thesis so far: we have found in a string theory amplitude the same pathological behaviour which occurs in quantum field theory. This seems to suggest that string theory, at least at this level, is unable to get past the absence of a well-defined underlying effective QFT near the singularity.

## 4.4 A Geometrical Regularization

We introduce here the Generalized Null Boost Orbifold (GNBO) by inserting one additional non-compact direction w.r.t. the NBO and show that divergences no longer occur. As for the NBO, we first present the geometry of the GNBO and study scalar and spin 1 eigenfunctions to build the scalar QED on the orbifold. We then show how the presence of a non-compact direction can cure the theory when considering amplitudes and overlaps.

We would like to stress out that this is more a geometrical trick than a real “physical” regularization. Nevertheless, it’s useful to see how things work in this case.

### 4.4.1 The GNBO Geometry

We consider Minkowski spacetime and the change of coordinates from the light-cone set $(x^\mu) = (x^+, x^-, x^2, x^3, \vec{x})$ to $(x^\alpha) = (u, v, w, z, \vec{x})$:

$$\begin{cases} x^- &= u \\ x^+ &= v + \frac{\Delta_2^2}{2}u(z+w)^2 + \frac{\Delta_3^2}{2}u(z-w)^2 \\ x^2 &= \Delta_2 u(z+w) \\ x^3 &= \Delta_3 u(z-w) \end{cases} \Leftrightarrow \begin{cases} u &= x^- \\ v &= x^+ - \frac{1}{2x^-}\left((x^2)^2 + (x^3)^2\right) \\ w &= \frac{1}{2x^-}\left(\frac{x^2}{\Delta_2} - \frac{x^3}{\Delta_3}\right) \\ z &= \frac{1}{2x^-}\left(\frac{x^2}{\Delta_2} + \frac{x^3}{\Delta_3}\right) \end{cases}, \tag{4.4.1}$$

where we do not perform any change on the transverse coordinates $\vec{x}$. The metric in these coordinates is non diagonal and reads:

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + (\Delta_2^2+\Delta_3^2)u^2(\mathrm{d}w^2+\mathrm{d}z^2) + 2(\Delta_2^2-\Delta_3^2)u^2\mathrm{d}w\mathrm{d}z + \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.4.2}$$

while its determinant is

$$\det g = -4\Delta_2^2\Delta_3^2u^4. \tag{4.4.3}$$

From the previous expressions we can also derive the non vanishing Christoffel symbols:

$$\begin{aligned} \Gamma^v_{ww} = \Gamma^v_{zz} &= (\Delta_2^2 + \Delta_3^2)u, \\ \Gamma^v_{wz} &= (\Delta_2^2 - \Delta_3^2)u, \\ \Gamma^w_{uw} = \Gamma^z_{uz} &= \frac{1}{u}, \end{aligned} \tag{4.4.4}$$

which however produce, as for the NBO coordinates (3.2.3), a vanishing Riemann tensor, Ricci tensor and curvature scalar, since we are considering Minkowski spacetime anyway.

We now introduce the GNBO by identifying points in spacetime along the orbits of the Killing vector[4]:

$$\begin{aligned}\kappa^{\mathrm{GNBO}} &= -2\pi i(\Delta_2 J_{+2} + \Delta_3 J_{+3}) \\ &= 2\pi(\Delta_2 x^2 + \Delta_3 x^3)\partial_+ + 2\pi\Delta_2 x^- \partial_2 + 2\pi\Delta_3 x^- \partial_3 \\ &= 2\pi\partial_z .\end{aligned} \tag{4.4.5}$$

The identification

$$(x^\mu) \sim e^{n\kappa^{\mathrm{GNBO}}}(x^\mu), \qquad n \in \mathbb{Z} \tag{4.4.6}$$

leads to

$$\begin{pmatrix} x^- \\ x^2 \\ x^3 \\ x^+ \\ \vec{x} \end{pmatrix} \sim \begin{pmatrix} x^- \\ x^2 + 2\pi n\Delta_2 x^- \\ x^3 + 2\pi n\Delta_3 x^- \\ x^+ + 2\pi n\Delta_2 x^2 + 2\pi n\Delta_3 x^3 + (2\pi n)^2 \frac{\Delta_2^2+\Delta_3^2}{2} x^- \\ \vec{x} \end{pmatrix}, \tag{4.4.7}$$

or to the simpler

$$(u, v, w, z) \sim (u, v, w, z + 2\pi n) \tag{4.4.8}$$

using the map to the orbifold coordinates (4.4.1), where the Killing vector $\kappa = 2\pi\partial_z$ does not depend on the local spacetime configuration. As in the previous case, the difference between Minkowski spacetime and the GNBO is therefore global.

The geodesic distance between the n-th copy and the base point on

[4]Notice that, unlike those introduced in Section 1.2.2, the model we are considering now is an orbifold of $\mathbb{M}^4$.

the orbifold can be computed in any set of coordinates and is:

$$\|x_{(n)} - x_{(0)}\|^2 = (\Delta_2^2 + \Delta_3^2)(2\pi n x^-_{(0)})^2. \tag{4.4.9}$$

From this point of view the situation is analogous to the NBO: CTC's are avoided, but there are CNC's on the surface $x^- = u = 0$, where the Killing vector $\kappa^{\text{GNBO}}$ vanishes, while the origin is still a fixed point.

### 4.4.2 Free Scalar Action

In order to build a quantum theory on the GNBO using Feynman's approach to quantization, we proceed as we did for the NBO: first we solve the eigenvalue equations for the fields and then we derive their off-shell expansion. We start from a complex scalar field and then consider the free photon before moving to the sQED interactions on the GNBO.

Consider the action for a complex scalar field:

$$\begin{aligned}
\text{S}_{\text{scalar kin}} &= \int_\Omega \mathrm{d}^D x \sqrt{-\det g}\left(-g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - M^2\phi^*\phi\right) \\
&= \int \mathrm{d}^{D-4}\vec{x} \int \mathrm{d}u \int \mathrm{d}v \int \mathrm{d}w \int_0^{2\pi} \mathrm{d}z \;\; 2\left|\Delta_2\Delta_3\right| u^2 \\
&\times \Bigg[\partial_u\phi^*\partial_v\phi + \partial_v\phi^*\partial_u\phi - \frac{1}{4u^2}\Bigg(\left(\frac{1}{\Delta_2^2} + \frac{1}{\Delta_3^2}\right)\Big(\partial_w\phi^*\partial_w\phi + \partial_z\phi^*\partial_z\phi\Big) \\
&+ \left(\frac{1}{\Delta_2^2} - \frac{1}{\Delta_3^2}\right)\Big(\partial_w\phi^*\partial_z\phi + \partial_z\phi^*\partial_w\phi\Big)\Bigg) - \eta^{ij}\partial_i\phi^*\partial_j\phi - M^2\phi^*\phi\Bigg].
\end{aligned} \tag{4.4.10}$$

As in the case of the NBO, the solutions to the e.o.m. are necessary to provide the modes of the quantum fields. We study the eigenvalue equation $\Box\phi_r = r\phi_r$, where $r$ is $2k_+k_- - \vec{k}$ by comparison with the flat case ($k$ is the momentum associated to the flat coordinates). We therefore need to solve:

$$\begin{aligned}\Bigg\{ &-2\partial_u\partial_v - \frac{2}{u}\partial_v + \frac{1}{4u^2}\Big[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)\left(\partial_w^2+\partial_z^2\right) \\ &+2\left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)\partial_w\partial_z\Big] + \eta^{ij}\partial_i\partial_j - r\Bigg\}\phi_r = 0.\end{aligned} \tag{4.4.11}$$

To this purpose, we introduce a Fourier transformation over $v, w, z, \vec{x}$:

$$\phi_r(u,v,w,z,\vec{x}) = \sum_{l\in\mathbb{Z}}\int \mathrm{d}^{D-4}\vec{k}\int \mathrm{d}k_+\int \mathrm{d}p\ e^{i\left(k_+v+pw+lz+\vec{k}\cdot\vec{x}\right)}\tilde{\phi}_{\{k_+\,p\,l\,\vec{k}\,r\}}(u), \tag{4.4.12}$$

where we defined $k_+, p, l, \vec{k}$ as associated momenta to $v, w, z, \vec{x}$ respectively, and we find

$$\phi_{\{k_+\,p\,l\,\vec{k}\,r\}}(u,v,w,z,\vec{x}) = e^{i\left(k_+v+pw+lz+\vec{k}\cdot\vec{x}\right)}\tilde{\phi}_{\{k_+\,p\,l\,\vec{k}\,r\}}(u), \tag{4.4.13}$$

with

$$\tilde{\phi}_{\{k_+\,p\,l\,\vec{k}\,r\}}(u) = \frac{1}{2\sqrt{(2\pi)^D|\Delta_2\Delta_3k_+|}}\ \frac{1}{|u|}e^{-i\left(\frac{1}{8k_+u}\left[\frac{(l+p)^2}{\Delta_2^2}+\frac{(l-p)^2}{\Delta_3^2}\right]-\frac{\vec{k}^2+r}{2k_+}u\right)}. \tag{4.4.14}$$

These solutions present the right normalization, as we can verify through the product:

$$\begin{aligned}&\left(\phi_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}},\ \phi_{\{k_{(2)+}\,p_{(2)}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}}\right) = 2\,|\Delta_2\Delta_3| \\ &\times\int \mathrm{d}^{D-4}\vec{x}\int \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w\ \int_0^{2\pi}\mathrm{d}z\ u^2\ \phi_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}}\ \phi_{\{k_{(2)+}\,p_{(2)}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}} \\ &= \delta^{D-4}(\vec{k}_{(1)}+\vec{k}_{(2)})\ \delta(k_{(1)+}+k_{(2)+})\ \delta(p_{(1)}+p_{(2)})\ \delta(r_{(1)}+r_{(2)})\ \delta_{l_{(1)},l_{(2)}}.\end{aligned} \tag{4.4.15}$$

Then we have the off-shell expansion:

$$\phi_r(u,v,w,z,\vec{x}) = \frac{1}{2\sqrt{(2\pi)^D\,|\Delta_2\Delta_3 k_+|}} \sum_{l\in\mathbb{Z}} \int \mathrm{d}^{D-4}\vec{k} \int \mathrm{d}k_+ \int \mathrm{d}p \int \mathrm{d}r \\ \times \frac{\mathcal{A}_{\{k_+\,p\,l\,\vec{k}\,r\}}}{|u|} e^{i\left(k_+ v + pw + lz + \vec{k}\cdot\vec{x} - \frac{1}{8k_+ u}\left[\frac{(l+p)^2}{\Delta_2^2} + \frac{(l-p)^2}{\Delta_3^2}\right] + \frac{\vec{k}^2 + r}{2k_+}u\right)}. \tag{4.4.16}$$

### 4.4.3 Free Photon Action

We study the action of the free photon field using the Lorenz gauge, which in the orbifold coordinates reads:

$$D^\alpha a_\alpha = -\frac{2}{u}a_v - \partial_v a_u - \partial_u a_v \\ + \frac{1}{4u^2}\left[\left(\frac{1}{\Delta_2^2} + \frac{1}{\Delta_3^2}\right)\Big(\partial_w a_w + \partial_z a_z\Big) + \left(\frac{1}{\Delta_2^2} - \frac{1}{\Delta_3^2}\right)\Big(\partial_w a_z + \partial_z a_w\Big)\right] \\ + \eta^{ij}\partial_i a_j = 0. \tag{4.4.17}$$

We then solve the eigenvalue equations $(\Box a_r)_\nu = r\ a_{r\,\nu}$, which in components read:

$$
\begin{aligned}
&(\Box a_r)_u = \\
&\frac{2}{u^2} a_{r\,v} - \frac{1}{2u^3}\left[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)(\partial_w a_{r\,w}+\partial_z a_{r\,z}) + \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)(\partial_w a_{r\,z}+\partial_z a_{r\,w})\right] \\
&+\left\{-2\partial_u\partial_v - \frac{2}{u}\partial_v + \frac{1}{4u^2}\left[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)(\partial_w^2+\partial_z^2) + \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)2\partial_w\partial_z\right] + \nabla_T^2\right\} a_{r\,u}, \\
&(\Box a_r)_v = \\
&\left\{-2\partial_u\partial_v - \frac{2}{u}\partial_v + \frac{1}{4u^2}\left[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)(\partial_w^2+\partial_z^2) + \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)2\partial_w\partial_z\right] + \nabla_T^2\right\} a_{r\,v}, \\
&(\Box a_r)_w = -\frac{2}{u}\partial_w a_{r\,v} \\
&+\left\{-2\partial_u\partial_v + \frac{1}{4u^2}\left[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)(\partial_w^2+\partial_z^2) + \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)2\partial_w\partial_z\right] + \nabla_T^2\right\} a_{r\,w}, \\
&(\Box a)_z = -\frac{2}{u}\partial_z a_{r\,v} \\
&+\left\{-2\partial_u\partial_v + \frac{1}{4u^2}\left[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)(\partial_w^2+\partial_z^2) + \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)2\partial_w\partial_z\right] + \nabla_T^2\right\} a_{r\,z}, \\
&(\Box a)_i = \\
&\left\{-2\partial_u\partial_v - \frac{2}{u}\partial_v + \frac{1}{4u^2}\left[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)(\partial_w^2+\partial_z^2) + \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)2\partial_w\partial_z\right] + \nabla_T^2\right\} a_{r\,i},
\end{aligned}
\tag{4.4.18}
$$

where $\nabla_T^2 = \eta^{ij}\partial_i\partial_j$ is the Laplace operator in the transverse coordinates $\vec{x}$. These equations can be solved using standard techniques through a Fourier transform:

$$
a_{r\,\alpha}(u,v,w,z,\vec{x}) = \sum_{l\in\mathbb{Z}} \int \mathrm{d}^{D-4}\vec{k} \int \mathrm{d}k_+ \int \mathrm{d}p\ e^{i\left(k_+ v + pw + lz + \vec{k}\cdot\vec{x}\right)} \tilde{a}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\alpha}(u).
\tag{4.4.19}
$$

We first solve the equations for $\tilde{a}_{\{k_+\,p\,l\,\vec{k}\,r\}\,v}$ and $\tilde{a}_{\{k_+\,p\,l\,\vec{k}\,r\}\,i}$ since they are identical to the scalar equation (4.4.11). Then we insert their solutions as sources for the equations for $\tilde{a}_{\{k_+\,p\,l\,\vec{k}\,r\}\,u}$, $\tilde{a}_{\{k_+\,p\,l\,\vec{k}\,r\}\,w}$ and $\tilde{a}_{\{k_+\,p\,l\,\vec{k}\,r\}\,z}$.

We can write the results as:

$$
\begin{aligned}
\parallel \tilde{a}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\alpha}(u) \parallel = \begin{pmatrix} \tilde{a}_u \\ \tilde{a}_v \\ \tilde{a}_w \\ \tilde{a}_z \\ \tilde{a}_i \end{pmatrix} &= \sum_{\underline{\alpha}\in\{\underline{u},\underline{v},\underline{w},\underline{z},\underline{i}\}} \mathcal{E}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\underline{\alpha}} \parallel \tilde{a}^{\underline{\alpha}}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\alpha}(u) \parallel \\
&= \mathcal{E}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\underline{u}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, p\, l\, \vec{k}\, r\}} \\
&+ \mathcal{E}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\underline{v}} \begin{pmatrix} \frac{i}{2k_+u} + \frac{1}{8{k_+}^2u^2}\left(\frac{(l+p)^2}{\Delta_2^2} + \frac{(l-p)^2}{\Delta_3^2}\right) \\ 1 \\ \frac{p}{k_+} \\ \frac{l}{k_+} \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, p\, l\, \vec{k}\, r\}} \\
&+ \mathcal{E}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\underline{w}} \begin{pmatrix} \frac{1}{4k_+|u|}\left(\frac{l+p}{\Delta_2^2} - \frac{l-p}{\Delta_3^2}\right) \\ 0 \\ |u| \\ 0 \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, p\, l\, \vec{k}\, r\}} \\
&+ \mathcal{E}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\underline{z}} \begin{pmatrix} \frac{1}{4k_+|u|}\left(\frac{l+p}{\Delta_2^2} + \frac{l-p}{\Delta_3^2}\right) \\ 0 \\ 0 \\ |u| \\ 0 \end{pmatrix} \tilde{\phi}_{\{k_+\, p\, l\, \vec{k}\, r\}} \\
&+ \mathcal{E}_{\{k_+\, p\, l\, \vec{k}\, r\}\,\underline{j}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \delta_{i\underline{j}} \end{pmatrix} \tilde{\phi}_{\{k_+\, p\, l\, \vec{k}\, r\}}.
\end{aligned}
\tag{4.4.20}
$$

We consider the Fourier transformed functions

$$a^{\underline{\alpha}}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\alpha}(u,v,w,z,\vec{x}) = e^{i(k_+v+pw+lz+\vec{k}\cdot\vec{x})}\tilde{a}^{\underline{\alpha}}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\alpha}(u), \tag{4.4.21}$$

then we can expand the off-shell fields as:

$$a_\alpha(x) = \sum_{l\in\mathbb{Z}} \int \mathrm{d}^{D-4}\vec{k} \int \mathrm{d}k_+ \int \mathrm{d}p \int \mathrm{d}r \sum_{\underline{\alpha}\in\{\underline{u},\underline{v},\underline{w},\underline{z},\underline{i}\}} \mathcal{E}_{\{k_+\,l\,\vec{k}\,r\}\,\underline{\alpha}}\, a^{\underline{\alpha}}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\alpha}(x). \tag{4.4.22}$$

We can also compute the normalization as:

$$\begin{aligned}
\left(a_{(1)},\, a_{(2)}\right) &= \int_{\mathbb{R}^{D-4}} \mathrm{d}^{D-4}\vec{x} \int_{\mathbb{R}^3} \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w \int_0^{2\pi} \mathrm{d}z\ 2|\Delta_2\Delta_3| u^2 \\
&\quad\times \left(g^{\alpha\beta}\ a_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\alpha}\ a_{\{k_{(2)+}\,p_{(2)}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}\,\beta}\right) \\
&= \delta^{D-4}(\vec{k}_{(1)}+\vec{k}_{(2)})\ \delta(p_{(1)}+p_{(2)})\ \delta(k_{(1)+}+k_{(2)+})\ \delta_{l_{(1)}+l_{(2)},0} \\
&\quad\times \delta(r_1-r_2)\mathcal{E}_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}} \circ \mathcal{E}_{\{k_{(2)+}\,p_{(2)}\,l_{(2)}\,\vec{k}_{(2)}\,r_{(2)}\}},
\end{aligned} \tag{4.4.23}$$

where

$$\begin{aligned}
\mathcal{E}_{(1)} \circ \mathcal{E}_{(2)} &= -\mathcal{E}_{(1)\,\underline{u}}\ \mathcal{E}_{(2)\,\underline{v}} - \mathcal{E}_{(1)\,\underline{v}}\ \mathcal{E}_{(2)\,\underline{u}} \\
&\quad+ \frac{1}{4}\Bigg[\left(\frac{1}{\Delta_2^2}+\frac{1}{\Delta_3^2}\right)\left(\mathcal{E}_{(1)\,\underline{w}}\ \mathcal{E}_{(2)\,\underline{w}} + \mathcal{E}_{(1)\,\underline{z}}\ \mathcal{E}_{(2)\,\underline{z}}\right) \\
&\quad+ \left(\frac{1}{\Delta_2^2}-\frac{1}{\Delta_3^2}\right)\left(\mathcal{E}_{(1)\,\underline{w}}\ \mathcal{E}_{(2)\,\underline{z}} + \mathcal{E}_{(1)\,\underline{z}}\ \mathcal{E}_{(2)\,\underline{w}}\right)\Bigg]
\end{aligned} \tag{4.4.24}$$

is independent of the coordinates. The Lorenz gauge now reads:

$$\eta^{\underline{ij}}k_i\,\mathcal{E}_{\{k_+\,p\,l\,\vec{k}\,r\}\underline{j}} - k_+\mathcal{E}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\underline{u}} - \frac{\vec{k}^2+r}{2k_+}\mathcal{E}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\underline{v}} = 0. \tag{4.4.25}$$

As in the NBO case, it does not pose any constraint on the transverse polarizations $\mathcal{E}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\underline{w}}$ and $\mathcal{E}_{\{k_+\,p\,l\,\vec{k}\,r\}\,\underline{z}}$.

### 4.4.4 Cubic Interaction

As previously studied on the NBO, we can now analyze the sQED 3-points vertex computation using the eigenmodes. The presence of a continuous momentum in the non-compact direction plays a major role in saving the convergence of the integrals. In the case of the GNBO we find:

$$
\begin{aligned}
\mathrm{S_{cubic}} &= \int_\Omega \mathrm{d}^D x \sqrt{-\det g}\,\left(-ieg^{\mu\nu} a_\mu \left(\phi^* \partial_\nu \phi - \partial_\nu \phi^* \phi\right)\right) \\
&= \prod_{i=1}^{3} \sum_{l_{(i)}\in\mathbb{Z}} \int \mathrm{d}^{D-4}\vec{k}_{(i)} \int \mathrm{d}k_{(i)+} \int \mathrm{d}p_{(i)} \int \mathrm{d}r_{(i)} \\
&\times (2\pi)^{D-1}\ \delta^{D-4}(\sum_{i=1}^{3}\vec{k}_{(i)})\ \delta(\sum_{i=1}^{3} p_{(i)})\ \delta(\sum_{i=1}^{3} k_{(i)+})\ \delta_{\sum_{i=1}^{3} l_{(i)},\,0} \\
&\times e\mathcal{A}^*_{\{-k_{(2)+}\,-kwN2\,-l_{(2)}\,-\vec{k}_{(2)}\,r_{(2)}\}} \mathcal{A}_{\{k_{(3)+}\,p_{(3)}\,l_{(3)}\,\vec{k}_{(3)}\,r_{(3)}\}} \\
&\times \Bigg\{ \mathcal{E}_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\underline{u}}\ k_{(2)+}\ \mathcal{I}^{[0]}_{\{3\}} \\
&+ \mathcal{E}_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\underline{v}} \left[ \left( \frac{\vec{k}^2_{(2)} + r_{(2)}}{2k_{(2)+}} \right) \mathcal{I}^{[0]}_{\{3\}} + i\frac{k_{(2)+}}{k_{(1)+}}\ \mathcal{I}^{[-1]}_{\{3\}} \right. \\
&+ \frac{k_{(2)+}}{8}\Bigg[ \frac{1}{\Delta_2^2}\left( \frac{l_{(1)}+p_{(1)}}{k_{(1)+}} + \frac{l_{(2)}+p_{(2)}}{k_{(2)+}} \right)^2 \\
&\qquad\quad \left. + \frac{1}{\Delta_3^2}\left( \frac{l_{(1)}-p_{(1)}}{k_{(1)+}} + \frac{l_{(2)}-p_{(2)}}{k_{(2)+}} \right)^2 \Bigg]\ \mathcal{I}^{[-2]}_{\{3\}} \right] \\
&+ \left( \mathcal{E}_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\underline{w}} - \mathcal{E}_{\{k_{(1)+}\,p_{(1)}\,l_{(1)}\,\vec{k}_{(1)}\,r_{(1)}\}\,\underline{z}} \right) \\
&\times \left[ \frac{1}{\Delta_2^2} \left( \frac{k_{(1)+}(l_{(2)}+p_{(2)}) + k_{(2)+}(l_{(1)}+p_{(1)})}{k_{(1)+}} \right) \right. \\
&\left. - \frac{1}{\Delta_3^2} \left( \frac{k_{(1)+}(l_{(2)}-p_{(2)}) + k_{(2)+}(l_{(1)}-p_{(1)})}{k_{(1)+}} \right) \right]\ \mathcal{J}^{[-1]}_{(3)} \\
&+ \Big( (2) \leftrightarrow (3) \Big) \Bigg\},
\end{aligned}
\tag{4.4.26}
$$

where we defined:

$$
\begin{aligned}
\mathcal{I}^{[\nu]}_{\{N\}} &= \int_{\mathbb{R}} \mathrm{d}u \; 2|\Delta_2\Delta_3|u^2 \; u^\nu \; \prod_{i=1}^{N} \tilde{\phi}_{\{k_{(i)+}\, p_{(i)}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}}, \\
\mathcal{J}^{[\nu]}_{(N)} &= \int_{\mathbb{R}} \mathrm{d}u \; 2|\Delta_2\Delta_3|u^2 \; |u|^\nu \; \prod_{i=1}^{N} \tilde{\phi}_{\{k_{(i)+}\, p_{(i)}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}}.
\end{aligned}
\tag{4.4.27}
$$

While on the NBO case we needed to regularize the integrals at least taking their principal part when all $l_{(*)} = 0$ in (3.2.31), the GNBO does not require any specific manipulation. In fact the form of $\tilde{\phi}_{\{k_{(i)+}\, p_{(i)}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}}$ in (4.4.14) prevents the formation of isolated zeros in the phase factor proportional to $u^{-1}$: the presence of the continuous momentum $p$, contrary to the NBO where all momenta are discrete, gives the integrals a distributional interpretation à la Estrada-Vindas.

### 4.4.5 Quartic Interactions

As for the NBO, we consider the quartic interaction for the sQED action:

$$
\begin{aligned}
\mathrm{S}_{\text{quartic}} &= \int_{\Omega} \mathrm{d}^D x \sqrt{-\det g} \left( e^2\ g^{\mu\nu}\ a_\mu a_\nu |\phi|^2 - \frac{\lambda_4}{4} |\phi|^4 \right) \\
&= \prod_{i=1}^{3} \left( \frac{1}{4\pi \sqrt{((2\pi)^D |\Delta_2 \Delta_3 k_{(i)+}|}} \right) \\
&\times \sum_{l_{(i)} \in \mathbb{Z}} \int \mathrm{d}^{D-4} \vec{k}_{(i)} \int \mathrm{d}k_{(i)+} \int \mathrm{d}p_{(i)} \int \mathrm{d}r_{(i)} \\
&\times (2\pi)^{D-1}\ \delta^{D-4}(\sum_{i=1}^{3} \vec{k}_{(i)})\ \delta(\sum_{i=1}^{3} p_{(i)})\ \delta(\sum_{i=1}^{3} k_{(i)+})\ \delta_{\sum_{i=1}^{3} l_{(i)}, 0} \\
&\times \Bigg\{ e^2 \mathcal{A}^*_{\{-k_{(3)+}\ -kwN3\ -l_{(3)}\ -\vec{k}_{(3)}\ r_{(3)}\}} \mathcal{A}_{\{k_{(4)+}\ p_{(4)}\ l_{(4)}\ \vec{k}_{(4)}\ r_{(4)}\}} \\
&\times \Bigg[ \mathcal{E}_{\{k_{(1)+}\ p_{(1)}\ l_{(1)}\ \vec{k}_{(1)}\ r_{(1)}\}} \circ \mathcal{E}_{\{k_{(2)+}\ p_{(2)}\ l_{(2)}\ \vec{k}_{(2)}\ r_{(2)}\}}\ \mathcal{I}^{[0]}_{\{4\}} \\
&- i \mathcal{E}_{\{k_{(1)+}\ p_{(1)}\ l_{(1)}\ \vec{k}_{(1)}\ r_{(1)}\}\ \underline{v}}\ \mathcal{E}_{\{k_{(2)+}\ p_{(2)}\ l_{(2)}\ \vec{k}_{(2)}\ r_{(2)}\}\ \underline{v}} \left( \left( \frac{1}{k_{(1)+}} + \frac{1}{k_{(2)+}} \right)\ \mathcal{I}^{[-1]}_{\{4\}} \right. \\
&\left. - i \left( \frac{\mathcal{G}_{+\,(1,2)}}{\Delta_2^2} + \frac{\mathcal{G}_{-\,(1,2)}}{\Delta_3^2} \right)\ \mathcal{I}^{[-2]}_{\{4\}} \right) \\
&+ \frac{1}{4} \left( \tilde{\mathcal{E}}_{+\,(1,2)}\ \frac{\mathcal{G}_{+\,(1,2)}}{\Delta_2^2} - \tilde{\mathcal{E}}_{-\,(1,2)}\ \frac{\mathcal{G}_{-\,(1,2)}}{\Delta_2^2} \right)\ \mathcal{J}^{[-1]}_{(4)} \Bigg] \\
&- \frac{\lambda_4}{4} \mathcal{A}^*_{\{-k_{(1)+}\ -kwN1\ -l_{(1)}\ -\vec{k}_{(1)}\ r_{(1)}\}} \mathcal{A}^*_{\{-k_{(2)+}\ -kwN2\ -l_{(2)}\ -\vec{k}_{(2)}\ r_{(2)}\}} \\
&\times \mathcal{A}_{\{k_{(3)+}\ p_{(3)}\ l_{(3)}\ \vec{k}_{(3)}\ r_{(3)}\}} \mathcal{A}_{\{k_{(4)+}\ p_{(4)}\ l_{(4)}\ \vec{k}_{(4)}\ r_{(4)}\}} \mathcal{I}^{[0]}_{\{4\}} \Bigg\},
\end{aligned}
\tag{4.4.28}
$$

where we defined for simplicity:

$$\mathcal{G}_{\pm\,(a,b)} = \frac{l_{(a)} \pm p_{(a)}}{k_{(a)+}} - \frac{l_{(b)} \pm p_{(b)}}{k_{(b)+}},$$

$$\tilde{\mathcal{E}}_{\pm\,(a,b)} = \mathcal{E}_{\{k_{(a)+}\, p_{(a)}\, l_{(a)}\, \vec{k}_{(a)}\, r_{(a)}\}\, \underline{v}} \left( \mathcal{E}_{\{k_{(b)+}\, p_{(b)}\, l_{(b)}\, \vec{k}_{(b)}\, r_{(b)}\}\, \underline{w}} \pm \mathcal{E}_{\{k_{(b)+}\, p_{(b)}\, l_{(b)}\, \vec{k}_{(b)}\, r_{(b)}\}\, \underline{z}} \right) - \mathcal{E}_{\{k_{(b)+}\, p_{(b)}\, l_{(b)}\, \vec{k}_{(b)}\, r_{(b)}\}\, \underline{v}} \left( \mathcal{E}_{\{k_{(a)+}\, p_{(a)}\, l_{(a)}\, \vec{k}_{(a)}\, r_{(a)}\}\, \underline{w}} \pm \mathcal{E}_{\{k_{(a)+}\, p_{(a)}\, l_{(a)}\, \vec{k}_{(a)}\, r_{(a)}\}\, \underline{z}} \right). \tag{4.4.29}$$

As the 4-point function in the NBO case shows with clear evidence the presence of divergences when all $l_{(*)} = 0$, the GNBO allows a distributional interpretation of the integrals $\mathcal{I}^{[\nu]}_{\{N\}}$ and $\mathcal{J}^{[\nu]}_{(N)}$ in the previous expression. In fact the regularization occurs in the same way as in the 3-point function: the phase factor proportional to $u^{-1}$ has a continuous value due to the continuous momentum $p$ and it does not present isolated zeros.

Looking back at the metric (4.4.2) and at the identifications (4.4.8) where we compactified only the coordinate $z$ through the Killing vector $2\pi\partial_z$, it seems reasonable to wonder what would happen if we acted in the same way over $w$, since $2\pi\partial_w$ is a Killing vector as well and it commutes with $2\pi\partial_z$. However, the lesson we learnt from our whole study on NBO and GNBO is that in the absence of at least one continuous transverse direction it is not possible to avoid the divergences associated with discrete zero energy modes.

Notice moreover that instead of the GNBO we could have also chosen the OPO of Table (1.1), or the null brane of [21], as in those models the radius $R$ may play the role of the continuous momentum. However, we chose the path of this section since we found it more straightforward.

## 4.5 Analysis of the BO

In this section we would like to perform on the BO (which we introduced in detail in Section 1.3.1) a similar analysis to the one we did for the NBO. We will see that the results are not very different apart from the fact that divergences are milder. In fact, it is possible to construct the full sQED but from the effective field theory side higher derivative terms are ill-defined: this results again in some divergent string theory 3-point amplitudes with a massive state.

### 4.5.1 Orbifold Coordinates

We consider the change of coordinates:

$$\begin{cases} x^+ = t e^{+\Delta\varphi} \\ x^- = \sigma_-\, t e^{-\Delta\varphi} \end{cases} \quad \Leftrightarrow \quad \begin{cases} t = \operatorname{sgn}(x^+)\,\sqrt{|x^+ x^-|} \\ \varphi = \frac{1}{2\Delta}\log\left|\frac{x^+}{x^-}\right| \\ \sigma_- = \operatorname{sgn}(x^+ x^-) \end{cases} , \tag{4.5.1}$$

where $\sigma_- = \pm 1$ and $t,\ \varphi \in \mathbb{R}$. The metric reads

$$\begin{aligned} \mathrm{d}s^2 &= -2\mathrm{d}x^+\,\mathrm{d}x^+ \\ &= -2\sigma_-(\mathrm{d}t^2 - (\Delta t)^2\,\mathrm{d}\varphi^2), \end{aligned} \tag{4.5.2}$$

its determinant is

$$\det g = -4\Delta^2 t^2, \tag{4.5.3}$$

while the non vanishing Christoffel symbols are

$$\Gamma^{t}_{\varphi\varphi} = \Delta^2 t, \qquad \Gamma^{\varphi}_{t\varphi} = \frac{1}{t}. \tag{4.5.4}$$

Using the orbifold coordinates $(t, \varphi)$, the BO is obtained by requiring the identification $\varphi \sim \varphi + 2\pi$ along the orbit of the global Killing vector $\kappa_\varphi = 2\pi\partial_\varphi$. We will therefore use the recurrent parameter $\Lambda = e^{2\pi\Delta}$ in what follows.

### 4.5.2 Free Scalar Action

The action for a complex scalar $\phi$ is given by:

$$S_{\text{scalar kin}} = \int d^D x \sqrt{-\det g}\Big(-g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - M^2\phi^*\phi\Big)$$
$$= \sum_{\sigma_-\in\{\pm1\}} \int d^{D-2}\vec{x} \int dt \int_0^{2\pi} d\phi\, \Delta|t|$$
$$\left(\frac{1}{2}\sigma_-\,\partial_t\phi^*\,\partial_t\phi\, + \frac{1}{2}\sigma_-\frac{1}{(\Delta t)^2}\partial_\varphi\phi^*\,\partial_\varphi\phi\, - \partial_i\phi^*\partial_i\phi - M^2\phi^*\phi\right). \quad (4.5.5)$$

As before we solve the associated eigenfunction problem for the d'Alembertian operator:

$$-\frac{1}{2}\sigma_-\partial_t^2\phi_r - \frac{1}{2}\sigma_-\frac{1}{t}\partial_t\phi_r + \frac{1}{2}\sigma_-\frac{1}{(\Delta\, t)^2}\partial_\varphi^2\phi_r + \partial_i^2\phi_r = r\phi_r, \qquad (4.5.6)$$

with

$$r = 2k_+k_- - \vec{k}^2 = 2\varsigma_- m^2 - \vec{k}^2. \qquad (4.5.7)$$

For later convenience (see the transformation of $k$ under the induced action of the Killing vector (4.5.17)), we parameterize the momenta as the coordinates:

$$\begin{cases} k_+ = m\, e^{+\Delta\beta} \\ k_- = \varsigma_- m\, e^{-\Delta\beta} \end{cases} \quad \Leftrightarrow \quad \begin{cases} m = \text{sgn}(k_+)\sqrt{|k_+k_-|} \\ \beta = \frac{1}{2\Delta}\log\left|\frac{k_+}{k_-}\right| \\ \varsigma_- = \text{sgn}(k_+k_-) \end{cases}, \qquad (4.5.8)$$

where $\varsigma_- = \pm1$ and $m,\ \beta \in \mathbb{R}$. To solve the problem we use the usual techniques and perform the Fourier transform w.r.t. $\varphi$ and $\vec{x}$ as:

$$\phi(t,\varphi,\vec{x}) = \int d^{D-2}\vec{k} \sum_{l\in\mathbb{Z}} e^{i\vec{k}\cdot\vec{x}} e^{il\varphi} H_{l\,\vec{k}\,r\,\sigma_-}(t), \qquad (4.5.9)$$

so that the new function $H_{l\,\vec{k}\,r\,\sigma_-}$ satisfies

$$\partial_t^2 H_{l\,\vec{k}\,r\,\sigma_-} + \frac{1}{t}\partial_t H_{l\,\vec{k}\,r\,\sigma_-} + \left[\frac{l^2}{(\Delta\,t)^2} + 2\sigma_-(r+\vec{k}^2)\right] H_{l\,\vec{k}\,r\,\sigma_-} = 0. \tag{4.5.10}$$

Upon the introduction of the natural quantities (see also (4.5.19) for an explanation of the naturalness of $\lambda$)

$$\tau = mt, \qquad \lambda = e^{\Delta(\varphi+\beta)}, \qquad \hat{\sigma}_- = \sigma_-\varsigma_-, \tag{4.5.11}$$

the actual dependence on parameters is

$$H_{l\,\vec{k}\,r\,\sigma_-}(t) = \tilde{\phi}_{l\,\hat{\sigma}_-}(\tau), \tag{4.5.12}$$

so that:

$$\partial_\tau^2 \tilde{\phi}_{l\,\hat{\sigma}_-} + \frac{1}{\tau}\partial_\tau \tilde{\phi}_{l\,\hat{\sigma}_-} + \left[\frac{l^2}{(\Delta\,\tau)^2} + 4\hat{\sigma}_-\right] \tilde{\phi}_{l\,\hat{\sigma}_-} = 0. \tag{4.5.13}$$

The solutions have asymptotics

$$\tilde{\phi}_{l\,\hat{\sigma}_-} \sim \begin{cases} A_+|\tau|^{i\frac{l}{\Delta}} + A_-|\tau|^{-i\frac{l}{\Delta}} & l \neq 0 \\ A_+\log(|\tau|) + A_- & l = 0 \end{cases}, \tag{4.5.14}$$

and we will be more concerned on the $l = 0$ case as before.

### 4.5.3 BO Wave Functions from the Covering Spacetime

We now repeat the essential part of the analysis performed in the NBO case.

#### 4.5.3.1 Spin 0 Wave Function

We start as usual with the Minkowskian wave function and we write only the dependence on $x^+$ and $x^-$ since all the other coordinates are spectators:

$$
\begin{aligned}
\psi_{k_+k_-}(x^+,x^-) &= e^{i\left(k_+x^+ + k_-x^-\right)} \\
= \psi_{k_+k_-}(t,\varphi,\sigma_-) &= e^{imt\left[e^{+\Delta(\varphi+\beta)}+\hat{\sigma}_- t e^{-\Delta(\varphi+\beta)}\right]}.
\end{aligned} \tag{4.5.15}
$$

We can compute the wave function on the orbifold by summing over all images:

$$
\begin{aligned}
\Psi_{[k_+k_-]}([x^+,x^-]) &= \sum_{n\in\mathbb{Z}} \psi_{k_+k_-}(\mathcal{K}^n(x^+,x^-)) \\
&= \sum_{n\in\mathbb{Z}} \psi_{k_+k_-}(x^+e^{2\pi\Delta\, n}, x^-e^{-2\pi\Delta\, n}) \\
&= \sum_{n\in\mathbb{Z}} e^{i\left\{[k_+e^{2\pi\Delta\, n}]x^+ + [k_-e^{-2\pi\Delta\, n}]x^-\right\}} \\
&= \sum_{n\in\mathbb{Z}} \psi_{\mathcal{K}^{-n}(k_+k_-)}(x^+,x^-),
\end{aligned} \tag{4.5.16}
$$

where we write $[k_+k_-]$ because the function depends on the equivalence class of $k_+k_-$ only. The equivalence relation is given by:

$$
k = \begin{pmatrix} k_+ \\ k_- \end{pmatrix} \equiv \mathcal{K}^{-n}k = \begin{pmatrix} k_+e^{2\pi\Delta\, n} \\ k_-e^{-2\pi\Delta\, n} \end{pmatrix}. \tag{4.5.17}
$$

The previous equation explains the reason behind the parametrization (4.5.8): we can always choose a representative

$$
0 \le \beta < 2\pi, \quad m \neq 0, \tag{4.5.18}
$$

or, in other words, $\beta \sim \beta + 2\pi$ and therefore we can use the dual quantum number $l$ via a Fourier transform. Through the well adapted set of coordinates we can write the spin 0 wave function in a way which shows the natural variables as:

$$\Psi_{[k_+k_-]}([x^+,x^-]) = \sum_n e^{i\,\tau\left[\lambda e^{+2\pi\Delta\, n}+\hat{\sigma}_-\lambda^{-1}e^{-2\pi\Delta\, n}\right]} = \hat{\Psi}(\tau,\lambda,\hat{\sigma}_-). \tag{4.5.19}$$

Again, the scalar eigenfunction has a unique equivalence class which mixes coordinates and momenta.

We can also use the basic trick of the Poisson resummation to write:

$$\begin{aligned}
\Psi_{[k_+k_-]}([x^+,x^-]) &= \int_{-\infty}^{\infty} ds\ \delta_P(s)\, e^{i\left\{k_+x^+\Lambda^s+k_-x^-\Lambda^{-s}\right\}} \\
&= \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} \left|\frac{k_+x^+}{k_-x^-}\right|^{-i\frac{l}{2\Delta}} \int_{-\infty}^{\infty} ds\ e^{i\,2\pi ls} e^{i\,sgn(k_+x^+)\sqrt{|k_+k_-x^+x^-|}\left\{\Lambda^s+\sigma_-\varsigma_-\Lambda^{-s}\right\}} \\
&= \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} \left(e^{\Delta(\varphi+\beta)}\right)^{-i\frac{l}{\Delta}} \int_{-\infty}^{\infty} ds\ e^{i\,2\pi ls} e^{i\,mt\,\left\{\Lambda^s+\sigma_-\varsigma_-\Lambda^{-s}\right\}} \\
&= \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} e^{il\beta}\left[e^{il\varphi}\int_{-\infty}^{\infty} ds\ e^{-i\,2\pi ls} e^{i\,mt\,\left\{\Lambda^s+\sigma_-\varsigma_-\Lambda^{-s}\right\}}\right].
\end{aligned} \tag{4.5.20}$$

The last line of the previous expression represents the change of quantum number from $m\beta$ to $ml$ and allows us to identify:

$$\mathcal{N}_{BO}\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\ e^{-i\,2\pi ls} e^{i\,\tau\,\left\{\Lambda^s+\hat{\sigma}_-\Lambda^{-s}\right\}}, \tag{4.5.21}$$

where $\mathcal{N}_{BO}$ is a constant which depends on the normalization chosen for $\tilde{\phi}_{l\,\hat{\sigma}_-}$. We have obtained an integral representation of the o.d.e. solutions.

#### 4.5.3.2 Spin 2 Wave Function

We start with the Minkowskian tensorial wave function where we suppress all directions but $x^+, x^-$ and $x^2$, since all the other coordinates behave as $x^2$. In this case, differently from spin 0, we need to keep the dependence on $x^2$ since it is needed for non-trivial physical polarizations as it enters in the transversality conditions. Explicitly:

$$\begin{aligned}\mathcal{N}_{BO}\psi^{[2]}_{k\,S}(x^+,x^-,x^2) &= S_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu\,\psi_k(x)\\ &= \Big[S_{++}\,(\mathrm{d}x^+)^2 + 2S_{+-}\,\mathrm{d}x^+\mathrm{d}x^- + +2S_{+2}\,\mathrm{d}x^+\mathrm{d}x^2\\ &\quad + S_{--}\,(\mathrm{d}x^-)^2 + 2S_{-2}\,\mathrm{d}x^2\mathrm{d}x^2 +\\ &\quad + S_{22}\,(\mathrm{d}x^2)^2\Big]\,e^{i\left(k_+x^+ + k_-x^- + k_2x^2\right)},\end{aligned} \tag{4.5.22}$$

which we rewrite in orbifold coordinates as

$$\begin{aligned}\mathcal{N}_{BO}\psi^{[2]}_{k\,S}(t,\varphi,x^2,\sigma_-) &= S_{\alpha\beta}\,\mathrm{d}x^\alpha\mathrm{d}x^\beta\,\psi_k(x)\\ &\Big[\mathrm{d}t^2\,\left(2\,S_{+-}\,\sigma_- + S_{++}\,e^{2\,\Delta\,\varphi} + S_{--}\,e^{-2\,\Delta\,\varphi}\right)\\ &+2\,\Delta\,t\,\mathrm{d}t\,\mathrm{d}\varphi\,\left(S_{++}\,e^{2\,\Delta\,\varphi} - S_{--}\,e^{-2\,\Delta\,\varphi}\right)\\ &+\Delta^2\,t^2 d\varphi^2\,\left(-2\,S_{+-}\,\sigma_- + S_{++}\,e^{2\,\Delta\,\varphi} + S_{--}\,e^{-2\,\Delta\,\varphi}\right)\\ &+2\mathrm{d}t\,\mathrm{d}x^2\,\left(S_{-2}\,e^{-\Delta\,\varphi}\,\sigma_- + S_{+2}\,e^{\Delta\,\varphi}\right)\\ &+2\Delta\,t\,\mathrm{d}x^2\,d\varphi\,\left(S_{+2}\,e^{\Delta\,\varphi} - S_{-2}\,e^{-\Delta\,\varphi}\,\sigma_-\right)\\ &+(\mathrm{d}x^2)^2\,S_{2\,2}\Big]\,e^{imt\left[\;e^{+\Delta(\varphi+\beta)} + \hat{\sigma}_- e^{-\Delta(\varphi+\beta)}\right] + ik_2x^2}.\end{aligned} \tag{4.5.23}$$

Now we take the tensor wave on the orbifold as a sum over all images:

$$\begin{aligned}\mathcal{N}_{BO}\Psi^{[2]}_{[k\,S]}([x]) &= \sum_n (\mathcal{K}^n dx)\cdot S\cdot(\mathcal{K}^n dx)\,\psi_k(\mathcal{K}^n x)\\ &= \sum_n dx\cdot(\mathcal{K}^{-n}S)\cdot dx\,\psi_{\mathcal{K}^{-n}k}(x).\end{aligned} \tag{4.5.24}$$

In the last line we have defined the induced action of the Killing vector on $(k, S)$ which can be explicitly written as:

$$\mathcal{K}^{-n}\begin{pmatrix} S_{++} \\ S_{+-} \\ S_{--} \\ S_{+2} \\ S_{-2} \\ S_{22} \end{pmatrix} = \begin{pmatrix} e^{2n\Delta\varphi}S_{++} \\ S_{+-} \\ e^{-2n\Delta\varphi}S_{--} \\ e^{n\Delta\varphi}\Delta S_{+2} \\ e^{-n\Delta\varphi}S_{-2} \\ S_{22} \end{pmatrix}, \tag{4.5.25}$$

and it amounts to a trivial scaling.

In orbifold coordinates, computing the tensor wave on the orbifold simply results in summing over all the shifts $\varphi \to \varphi + 2\pi n$. Then we have to give a close expression for the sum involving powers $e^{2\pi\Delta n}$. Explicitly we find

$$\begin{aligned} &\sum_n \left(e^{2\pi\Delta n}\right)^N e^{i\,\tau\left[\lambda e^{+2\pi\Delta\, n}+\hat{\sigma}_- \frac{1}{\lambda}e^{-2\pi\Delta\, n}\right]} \\ &= \begin{cases} \left[\frac{1}{2}\left(\frac{1}{\lambda}\partial_\tau + \frac{1}{\tau}\partial_\lambda\right)\right]^N \hat{\Psi}(\tau,\lambda,\hat{\sigma}_-) & N>0 \\ \left[\frac{1}{2}\left(\lambda\partial_\tau - \frac{\lambda^2}{\tau}\partial_\lambda\right)\right]^N \hat{\Psi}(\tau,\lambda,\hat{\sigma}_-) & N<0 \end{cases}, \end{aligned} \tag{4.5.26}$$

where $\tau$ derivatives of $\tilde{\phi}_{l\,\hat{\sigma}_-}$ higher than 2 can be reduced with the help of the differential equation (4.5.13).

We now have to identify the basic polarizations on the orbifold. However the quantum number $\beta$ is no longer a good quantum number on the orbifold and it is replaced by $l$. The relations among orbifold polarizations and Minkowski polarizations may depend on $\beta$ as long as the traceless and transversality conditions on the orbifold are independent of it.[5] Finally, it seems reasonable to use the natural variable

[5]These conditions may be a linear combinations of the Minkowski ones.

$\lambda = e^{\Delta(\varphi+\beta)}$. Therefore we have:

$$\begin{aligned}
\mathcal{S}_{t\,t} &= e^{-2\,\Delta\,\beta}\, S_{+\,+} \\
\mathcal{S}_{t\,\varphi} &= S_{+\,-} \\
\mathcal{S}_{t\,2} &= e^{-\Delta\,\beta}\, S_{+\,2} \\
\mathcal{S}_{\varphi\,\varphi} &= e^{2\,\Delta\,\beta}\, S_{-\,-} \\
\mathcal{S}_{\varphi\,2} &= e^{\Delta\,\beta}\, S_{-\,2} \\
\mathcal{S}_{2\,2} &= S_{2\,2},
\end{aligned} \tag{4.5.27}$$

which can be trivially inverted as:

$$\begin{aligned}
S_{+\,+} &= e^{2\,\Delta\,\beta}\, \mathcal{S}_{t\,t} \\
S_{+\,-} &= \mathcal{S}_{t\,\varphi} \\
S_{+\,2} &= e^{\Delta\,\beta}\, \mathcal{S}_{t\,2} \\
S_{-\,-} &= e^{-2\,\Delta\,\beta}\, \mathcal{S}_{\varphi\,\varphi} \\
S_{-\,2} &= e^{-\Delta\,\beta}\, \mathcal{S}_{\varphi\,2} \\
S_{2\,2} &= \mathcal{S}_{2\,2}.
\end{aligned} \tag{4.5.28}$$

When they are inserted into the trace condition we obtain

$$\mathrm{tr}(S) = -2\mathcal{S}_{t\,\varphi} + \mathcal{S}_{2\,2}, \tag{4.5.29}$$

while the transversality conditions become

$$\begin{aligned}
(k\cdot S)_{+} &= -\, e^{\Delta\,\beta}\, \left(m\,\hat{\sigma}_{-}\,\sigma_{-}\,\mathcal{S}_{t\,t} + m\,\mathcal{S}_{t\,\varphi} - k_2\,\mathcal{S}_{t\,2}\right) \\
(k\cdot S)_{-} &= -\, e^{-\Delta\,\beta}\, \left(m\,\hat{\sigma}_{-}\,\sigma_{-}\,\mathcal{S}_{t\,\varphi} + m\,\mathcal{S}_{\varphi\,\varphi} - k_2\,\mathcal{S}_{\varphi\,2}\right) \\
(k\cdot S)_{2} &= -\left(m\,\hat{\sigma}_{-}\,\sigma_{-}\,\mathcal{S}_{t\,2} + m\,\mathcal{S}_{\varphi\,2} - k_2\,\mathcal{S}_{2\,2}\right),
\end{aligned} \tag{4.5.30}$$

which are independent of $\beta$ when set to zero.

The final expression for the wave function for the symmetric tensor on the orbifold reads:

$$\Psi^{[2]}_{[k\,S]}([x]) = \sum_{l\in\mathbb{Z}} e^{il\beta}\Bigg[S_{ml,tt}\,(\mathrm{d}t)^2 + 2S_{ml,t\varphi}\,\mathrm{d}t\mathrm{d}\varphi + +2S_{ml,t2}\,\mathrm{d}t\mathrm{d}x^2$$
$$+ S_{ml,\varphi\varphi}\,(\mathrm{d}\varphi)^2 + 2S_{ml,\varphi 2}\,\mathrm{d}\varphi\mathrm{d}x^2 +$$
$$+ S_{ml,22}\,(\mathrm{d}x^2)^2\Bigg], \tag{4.5.31}$$

where the explicit expressions for the components are

$$\begin{aligned} S_{ml,tt} = &+\left[-\frac{\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,l\,\lambda^{\frac{i\,l}{\Delta}}\,\left(l\,\mathcal{S}_{t\,t} + i\,\Delta\,\mathcal{S}_{t\,t} + l\,\mathcal{S}_{\varphi\,\varphi} - i\,\Delta\,\mathcal{S}_{\varphi\,\varphi}\right)}{2\,\Delta^2}\right]\frac{1}{\tau^2} \\ &+\left[\frac{1}{2\,\Delta}\frac{d}{d\,\tau}\,\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,\lambda^{\frac{i\,l}{\Delta}}\,\left(i\,l\,\mathcal{S}_{t\,t} - i\,l\,\mathcal{S}_{\varphi\,\varphi} - \Delta\,\mathcal{S}_{t\,t} - \Delta\,\mathcal{S}_{\varphi\,\varphi}\right)\right]\frac{1}{\tau} \\ &+\left[\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,\lambda^{\frac{i\,l}{\Delta}}\,\left(\hat{\sigma}_-\,\mathcal{S}_{t\,t} + 2\,\sigma_-\,\mathcal{S}_{t\,\varphi} + \hat{\sigma}_-\,\mathcal{S}_{\varphi\,\varphi}\right)\right], \end{aligned} \tag{4.5.32}$$

$$\begin{aligned} S_{ml,t\varphi} = &+\left[-\frac{\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,l\,\lambda^{\frac{i\,l}{\Delta}}\,\left(l\,\mathcal{S}_{t\,t} + i\,\Delta\,\mathcal{S}_{t\,t} - l\,\mathcal{S}_{\varphi\,\varphi} + i\,\Delta\,\mathcal{S}_{\varphi\,\varphi}\right)}{2\,\Delta\,m}\right]\frac{1}{\tau} \\ &+\left[\frac{\frac{d}{d\,\tau}\,\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,\lambda^{\frac{i\,l}{\Delta}}\,\left(i\,l\,\mathcal{S}_{t\,t} - \Delta\,\mathcal{S}_{t\,t} + i\,l\,\mathcal{S}_{\varphi\,\varphi} + \Delta\,\mathcal{S}_{\varphi\,\varphi}\right)}{2\,m}\right] \\ &+\left[\frac{\Delta\,\hat{\sigma}_-\,\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,\lambda^{\frac{i\,l}{\Delta}}\,\left(\mathcal{S}_{t\,t} - \mathcal{S}_{\varphi\,\varphi}\right)}{m}\right]\tau, \end{aligned} \tag{4.5.33}$$

$$\begin{aligned} S_{ml,\varphi\varphi} = &+\left[-\frac{1}{2\,m^2}\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,l\,\lambda^{\frac{i\,l}{\Delta}}\,\left(l\,(\mathcal{S}_{t\,t} + \mathcal{S}_{\varphi\,\varphi}) + i\,\Delta\,(\mathcal{S}_{t\,t} - \mathcal{S}_{\varphi\,\varphi})\right)\right] \\ &+\left[\frac{1}{2\,m^2}\Delta\,\left(\frac{d}{d\,\tau}\,\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\right)\,\lambda^{\frac{i\,l}{\Delta}}\,\left(i\,l\,\mathcal{S}_{t\,t} - i\,l\,\mathcal{S}_{\varphi\,\varphi} - \Delta\,\mathcal{S}_{t\,t} - \Delta\,\mathcal{S}_{\varphi\,\varphi}\right)\right]\tau \\ &+\left[\frac{1}{m^2}\Delta^2\,\tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\,\lambda^{\frac{i\,l}{\Delta}}\,\left(\hat{\sigma}_-\,\mathcal{S}_{t\,t} + \hat{\sigma}_-\,\mathcal{S}_{\varphi\,\varphi} - 2\,\sigma_-\,\mathcal{S}_{t\,\varphi}\right)\right]\tau^2, \end{aligned} \tag{4.5.34}$$

together with the effectively vector components in the orbifold directions

$$S_{ml,t2} = + \left[ \frac{i}{2\,\Delta} \tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\, l\, \lambda^{\frac{i\,l}{\Delta}}\; (\mathcal{S}_{t\,2} - \mathcal{S}_{\varphi\,2}\,\sigma_-) \right] \frac{1}{\tau} + \left[ \frac{1}{2} \frac{d}{d\,\tau}\, \tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\, \lambda^{\frac{i\,l}{\Delta}}\; (\mathcal{S}_{t\,2} + \mathcal{S}_{\varphi\,2}\,\sigma_-) \right], \tag{4.5.35}$$

$$S_{ml,\varphi 2} = + \left[ \frac{i}{2\,m} \tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\, l\, \lambda^{\frac{i\,l}{\Delta}}\; (\mathcal{S}_{t\,2} + \mathcal{S}_{\varphi\,2}\,\sigma_-) \right] + \left[ \frac{1}{2\,m} \Delta \left( \frac{d}{d\,\tau}\, \tilde{\phi}_{l\,\hat{\sigma}_-}(\tau) \right) \lambda^{\frac{i\,l}{\Delta}}\; (\mathcal{S}_{t\,2} - \mathcal{S}_{\varphi\,2}\,\sigma_-) \right] \tau, \tag{4.5.36}$$

and the effectively scalar component

$$S_{ml,22} = \mathcal{S}_{2\,2}\, \tilde{\phi}_{l\,\hat{\sigma}_-}(\tau)\, \lambda^{\frac{i\,l}{\Delta}}. \tag{4.5.37}$$

### 4.5.4 Wave Functions Overlaps and a 3-Point String Theory Amplitude

Now we consider some overlaps as done for the NBO. The connection between the overlaps on the orbifold and the sums of images remains the same when we change the Killing vector $k$, hence we can limit ourselves to discuss the integrals on the orbifold space.

#### 4.5.4.1 Overlaps without Derivatives

Let us start with the simplest case of the overlap of $N$ scalar wave functions:

$$I^{(N)} = \int_\Omega \mathrm{d}^3 x\, \sqrt{-\det g} \prod_{i=1}^{N} \Psi_{[k_{+(i)} k_{-(i)}]}([x^+, x^-, x^2]) = \mathcal{N}_{BO}^N \sum_{\{l_{(i)}\} \in \mathbb{Z}^N} e^{i \sum_{i=1}^N l_{(i)} \beta_{(i)}} \int_\Omega \mathrm{d}^3 x\, \sqrt{-\det g} \prod_{i=1}^{N} \phi_{l_{(i)}\,\hat{\sigma}_{-(i)}}. \tag{4.5.38}$$

This is always a distribution since the problematic $l_{(*)} = 0$ sector gives a divergence like $(\log(|t|))^N$ around zero. All the other sectors have no issues because of the asymptotic behaviours (4.5.14).

#### 4.5.4.2 An Overlap with Two Derivatives

We consider in orbifold coordinates the overlap needed for the amplitude involving two tachyons and one massive state, i.e.:

$$K = \int_\Omega \mathrm{d}^3x \sqrt{-\det g}\, g^{\alpha\beta}\, g^{\gamma\delta}\, \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\alpha\gamma}([x])\, D_\beta \partial_\delta \Psi_{[k_{(2)}]}([x])\, \Psi_{[k_{(1)}]}([x]). \tag{4.5.39}$$

Since we want to use the traceless condition we need to keep all momenta and polarizations and not only the ones along the orbifold, then we can write:

$$\begin{aligned} K = \int_\Omega \mathrm{d}^3x \sqrt{-\det g}\Big[ &+\Psi^{[2]}_{[k_{(3)},S_{(3)}]\,t\,t}\, \partial_t^2 \Psi_{[k_{(2)}]} \\ &- 2\left(\frac{1}{\Delta t}\right)^2 \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,t\,\varphi}\, \left(\partial_t \partial_\varphi \Psi_{[k_{(2)}]} - \frac{1}{t}\partial_\varphi \Psi_{[k_{(2)}]}\right) \\ &+ \left(\frac{1}{\Delta t}\right)^4 \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\varphi\,\varphi}\, \left(\partial_\varphi^2 \Psi_{[k_{(2)}]} - \Delta^2 t \partial_t \Psi_{[k_{(2)}]}\right) \\ &- 2\Psi^{[2]}_{[k_{(3)},S_{(3)}]\,t\,2}\, \partial_t \partial_2 \Psi_{[k_{(2)}]} \\ &+ 2\left(\frac{1}{\Delta t}\right)^2 \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,\varphi\,2}\, \partial_\varphi \partial_2 \Psi_{[k_{(2)}]} \\ &+ \Psi^{[2]}_{[k_{(3)},S_{(3)}]\,2\,2}\, \partial_2^2 \Psi_{[k_{(2)}]} \Big] \Psi_{[k_{(1)}]}. \end{aligned} \tag{4.5.40}$$

Now we consider the behavior for $l_{(*)} = 0$ for small $t$. All the $\partial_\varphi$ can be dropped since they lower a $l_{(2)}$. The leading contributions from spin 2 components are $S_{ml\,tt} \sim \frac{1}{t^2}$, $S_{ml\,\varphi\varphi}, S_{ml\,2\,2} \sim 1$ and $S_{ml\,t2} \sim \frac{1}{t}$, therefore the leading $\frac{1}{t^4}$ reads:

$$K \sim \int_{t\sim 0} \mathrm{d}t\,|t| \Bigg[ -\frac{1}{2}\frac{d}{d\,\tau}\,\tilde{\phi}_{l\,\hat{\sigma}_-}\,(\mathcal{S}_{t\,t}+\mathcal{S}_{\varphi\,\varphi})\frac{1}{\tau} \times \partial_t^2 \Psi_{[k_{(2)}]} + \left(\frac{1}{\Delta t}\right)^4 \times \frac{-\Delta^2}{2\,m^2}\frac{d}{d\,\tau}\,\tilde{\phi}_{l\,\hat{\sigma}_-}\,(\mathcal{S}_{t\,t}+\mathcal{S}_{\varphi\,\varphi})\,\tau \times \left(-\Delta^2 t \partial_t \Psi_{[k_{(2)}]}\right)\Bigg] \Psi_{[k_{(3)}]}. \tag{4.5.41}$$

In the limit of our interest $\Psi_{[k]}|_{l=0} \sim \tilde{\phi}_{l\,\hat{\sigma}_-}|_{l=0} \sim \log(|t|)$, then the two terms add together because of sign of the covariant derivative to give

$$K \sim \int_{t\sim 0} \mathrm{d}t\,|t|\,\left[\left(\frac{1}{2}+\frac{1}{2}\right)\frac{\mathcal{S}_{t\,t}+\mathcal{S}_{\varphi\,\varphi}}{m^4}\frac{\log(|t|)}{t^4} + O\left(\frac{\log^2(|t|)}{t}\right)\right], \tag{4.5.42}$$

which is divergent for the physical polarization $\mathcal{S}_{t\,t} = \mathcal{S}_{\varphi\,\varphi} = -\hat{\sigma}_-\sigma_-\mathcal{S}_{t\,\varphi} = -\frac{1}{2}\hat{\sigma}_-\sigma_-\mathcal{S}_{22}$.

## 4.6 Final Considerations

For what concerns the models we are interested in, i.e. the NBO and the BO, the bottom line of the discussion so far is that the non-existence of a well-defined underlying effective QFT, due to the pathological behaviour of the interaction contact terms we studied in details, reflects in unexpected and dramatic divergences of open string theory amplitudes which involve massive states, already at tree-level. This does not mean however that the gravitational backreaction, which was referred in the literature as the main cause of the issue, is not going to play any role, as in the open string case it may reappear at the one loop level.

Actually, in [29] it was also suggested a deeper meaning to the breakdown we encounter. In particular, the root of these divergences seems to lie ultimately in the failure of the perturbative expansion through the singularity of the usual time evolution operator in the interaction picture. This results in the collapse of the particle interpretation of interactions and, as a consequence, of the standard Feynman diagrammatic approach, as we have shown. Nevertheless, it was also pointed out that a perturbative formulation should still be possible, as at the quantum mechanical level the full Hamiltonian theory is well-defined, at least in the minisuperspace approximation. Therefore, this means that either we move on to a non perturbative treatment or we find a way to recover a meaningful perturbation theory on this background. In the next and final chapter we are going to explore the second option invoking the aid of noncommutativity.

# Chapter 5

# The Role of Noncommutativity on the NBO

Throughout the thesis we have made it clear that the convergence issue of the amplitudes on the orbifolds is intimately related to the presence of a continuous coefficient in the phase factors which become heavily oscillating near the singularity. So far we have already explored different plausible regularizations: some, like the attempts of Section 3.2.6, failed to do the job, while others, like the GNBO of Section 4.4, seemed to work.

In this chapter, which is based on some ongoing works [3] and whose results are still to be considered as preliminary, we will propose a new direction which seems very promising, at least for the NBO. The idea has its roots in the resolution we mentioned in Section 1.2.1 of some static singularities and was already suggested also for this models [7, 30], but never completely explored. In a nutshell, the hope is that the introduction of a Kalb-Ramond $B$-field could help in the construction of a well-defined perturbative string theory. More specifically, since closed strings twisted around the orbifold become massless near the singularity, they should somehow be included in a low-energy description. In particular, they should generate a background potential $B_{\mu\nu}$ which is equivalent to an electromagnetic background from the open string perspective and which could play a decisive role in the resolution of the singularity.

String theory computations in a framework of this kind are far from trivial and, at the same time, we have already seen that important results can be obtained also stepping back to simpler QFT models. In a certain sense, we are going to do that again.

## 5.1 Seiberg-Witten Map and Field Theory Limit

Let's consider the nonlinear sigma model of the bosonic open string, whose worldsheet action with Lorentzian signature on $\Sigma$ is:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, \left( g_{\mu\nu} \partial_a x^\mu \partial^a x^\nu - 2\pi\alpha' \epsilon^{ab} B_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \right). \tag{5.1.1}$$

Since the NBO is locally flat, and therefore $H = dB = 0$, we introduce a constant[1] $B$-field on the covering Minkowski spacetime.[2] We also want it to be globally defined on the orbifold: looking back at the identifications (3.1.1), it's obvious that the only option is to take

$$B = B_{-2} \, dx^- \wedge dx^2. \tag{5.1.2}$$

The Seiberg-Witten map

$$\frac{1}{(g + 2\pi\alpha' B)_{\mu\nu}} = (G + \frac{\theta}{2\pi\alpha'})^{\mu\nu} \tag{5.1.3}$$

enables to go from the description in terms of the "closed string metric" $g_{\mu\nu}$ and the Kalb-Ramond $B_{\mu\nu}$ field to the study of a theory which depends on the "open string metric" $G^{\mu\nu}$ and the anti-symmetric non-commutative matrix $\theta^{\mu\nu}$ [32]. Applying (5.1.3) we get:

$$G^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -(2\pi\alpha' b)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (2\pi\alpha')^2 b \\ 0 & -(2\pi\alpha')^2 b & 0 \end{pmatrix}, \tag{5.1.4}$$

with $b = B_{-2}$. The same result can be obtained more formally using the original method of the symplectic form for quantized open strings: the full computation can be found in Appendix C.

We would like to consider now the decoupling zero slope $\alpha' \to 0$

[1]Notice however that the relation which follows holds not only for a constant $B$, but also when the Kalb-Ramond field varies with $x$ [31].

[2]Here we will ignore the $D-3$ Euclidean spectators coordinates.

limit defined in [32]. This is particularly interesting when we take into account the boundary propagator

$$\langle x^\mu(\tau)x^\nu(\tau')\rangle = -\alpha' G^{\mu\nu}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau')\,, \tag{5.1.5}$$

since, if $G^{\mu\nu}$ remains finite in the $\alpha' \to 0$ limit, we are left with a noncommutative quantum field theory living on the D2-brane wrapping $x^-, x^+$ and $x^2$. The configuration (5.1.4) is rather peculiar and was discussed in detail in [33]. The presence of a $\theta^{+2} \neq 0$ term raises indeed genuine questions about the meaning of a theory which at first sight appears nonlocal in the time coordinate. However, Gomis et al. showed[3] that as long as we perform a light-cone quantization in which $x^-$ is our time coordinate and $\theta^{-i} = 0$ the theory is unitary[4][5]. As explained in [36], (5.1.4) can also be seen as the infinitely-boosted limit of a unitary theory.

In [33] the authors discuss the $\alpha' \to 0$ field theory limit where $\theta^{+2}$ stays finite, which means $b \sim (\alpha')^{-2}$. This results in a strange open string metric with an infinite $G^{++}$ component. However, as explained by Seiberg (see again [33]), this can be easily fixed by the following coordinates change, which turns $G^{\mu\nu}$ into $\eta^{\mu\nu}$:

$$\begin{cases} y^- &= x^- \\ y^+ &= x^+ - \frac{1}{2}(2\pi\alpha' b)^2 x^- \\ y^2 &= x^2 \end{cases}. \tag{5.1.6}$$

[3] Here the role of $x^+$ and $x^-$ is reversed w.r.t. [33].

[4] For the sake of completeness, notice that doubts regarding the preservation of (micro)causality in this situation have been raised [34, 35].

[5] For this reason, the argument presented in this chapter doesn't seem to hold for the BO model. Indeed, we could in principle think to add $B = B_{+-}\, dx^+ \wedge dx^-$, but this would result in a $\theta^{+-}$ term.

## 5.2 Noncommutative QFT

We can now come back to the study of a simple QFT model on the NBO in the renewed framework of noncommutative spacetime. As we have seen in Chapter 3, in the perturbative expansions contact terms involving only scalars already display the pathological divergences. Here therefore we will consider for simplicity a basic scalar theory with a $N$-point interaction. The arguments which follow can be easily generalized to amplitudes involving higher spin particles. The action we look at is therefore:

$$S = \int_\Omega \mathrm{d}^D x \sqrt{-\det g} \left[\frac{1}{2}\partial_\mu \Psi \partial^\mu \Psi - \frac{M^2}{2}\Psi^2 - \frac{\lambda_N}{N!}\Psi^N\right]. \tag{5.2.1}$$

Let's then just suppose to replace our usual coordinates $x^\mu$ with the Hermitian operators $\hat{x}^\mu$ which obey the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{5.2.2}$$

where $\theta^{\mu\nu}$ are the constant components of an antisymmetric matrix $\theta$. Now when we compute the $N$-point scalar amplitude we need to take into account an additional momentum dependent phase factor. Indeed, as a straightforward application of the Baker–Campbell–Hausdorff formula, the composition of two plane waves reads

$$e^{ik_{(1)\mu}\hat{x}^\mu} e^{ik_{(2)\mu}\hat{x}^\mu} = e^{i(k_{(1)}+k_{(2)})_\mu \hat{x}^\mu} e^{-\frac{i}{2}k_{(1)\mu}\theta^{\mu\nu}k_{(2)\nu}}, \tag{5.2.3}$$

and the rule can be easily generalized [37] for the $N$ interaction vertex to:

$$\prod_{s=1}^{N} e^{ik_{(s)\mu}\hat{x}^\mu} = e^{i\left(\sum_{s=1}^N k_{(s)}\right)_\mu \hat{x}^\mu} e^{-\frac{i}{2}\sum_{r<s} k_{(r)\mu}\theta^{\mu\nu}k_{(s)\nu}}. \tag{5.2.4}$$

### 5.2.1 $N$-Point Scalar Amplitude

We now briefly review and manipulate a bit the expression for the $N$-point scalar function on the NBO. We start from the wave function defined on the orbifold as:

$$\Psi_{[k]}^{NBO}([x]) = \sum_{n\in\mathbb{Z}} \Psi_k(\mathcal{K}^n x). \tag{5.2.5}$$

The action of the orbifold group can be easily transported from coordinates to momenta:

$$\Psi_{[k]}^{NBO}([x]) = \sum_{n\in\mathbb{Z}} \Psi_k(\mathcal{K}^n x) = \sum_{n\in\mathbb{Z}} \Psi_{\mathcal{K}^n k}(x). \tag{5.2.6}$$

We can also Fourier transform $\Psi_{[k]}^{NBO}([x])$ with respect to $k_2$ defining $\tilde{\Psi}_{[\tilde{k}]}^{NBO}([x])$, where $[\tilde{k}]$ includes the discrete integer valued momentum $l$ in place of $k_2$:

$$\Psi_{[k]}^{NBO}([x]) = \sum_n e^{i\{k_+x^+ + k_-[x^- + (2\pi\Delta)nx^2 + \frac{1}{2}(2\pi\Delta)^2 n^2 x^+] + k_2[x^2 + (2\pi\Delta)nx^+] + \vec{k}\cdot\vec{x}\}},$$

$$\tilde{\Psi}_{[\tilde{k}]}^{NBO}([x]) = \frac{1}{2\pi\Delta k_+}\int_0^{2\pi\Delta k_+} dk_2\, e^{-il\frac{k_2}{\Delta k_+}}\, \Psi_{[k]}^{NBO}([x])\,. \tag{5.2.7}$$

The orbifold $N$-point scalar tree-level vertex reads[6]:

$$\begin{aligned} I_N^{NBO}(\{[\tilde{k}_{(s)}]\}) &= -i\lambda_N \int_\Omega d^D[x] \prod_{s=1}^N \tilde{\Psi}_{[\tilde{k}_{(s)}]}^{NBO}([x]) \\ &= -i\lambda_N \prod_{s=1}^N \frac{1}{2\pi\Delta k_{(s)+}} \int_{\omega_{k_{(s)2}}} dk_{(s)2}\, e^{-i\frac{l_{(s)}k_{(s)2}}{\Delta k_{(s)+}}} \\ &\quad \int_\Omega d^D[x] \sum_{\{n_{(s)}\}\in\mathbb{Z}^N} \Psi_{k_{(s)}}(\mathcal{K}^{n_{(s)}}x). \end{aligned} \tag{5.2.8}$$

[6] $\omega_{k_{(s)}}$ is a shortened notation for the orbifold fundamental domain in the momenta space.

We now change variables as $y = \mathcal{K}^{n_{(N)}} x$ and then we define:

$$\begin{cases} m_{(r)} &= n_{(r)} - n_{(N)} \qquad \text{for} \quad r = 1, ... N-1 \\ m_{(N)} &= 0 \\ n &= n_{(N)} \end{cases} \tag{5.2.9}$$

in order to apply the unfolding trick which extends the integration domain to the whole $\mathbb{M}^3 \otimes \mathbb{R}^{D-3}$ spacetime. We obtain:

$$I_N^{NBO}(\{[\tilde{k}_{(s)}]\}) = -\, i\lambda_N \prod_{s=1}^{N} \frac{1}{2\pi \Delta k_{(s)+}} \int_{\omega_{k_{(s)2}}} dk_{(s)2}\, e^{-i \frac{l_{(s)} k_{(s)2}}{\Delta k_{(s)+}}} \int_{\mathbb{M}^3 \otimes \mathbb{R}^{D-3}} d^D y \sum_{\{m_{(s)}\} \in \mathbb{Z}^{N-1} \otimes \{0\}} \Psi_{k_{(s)}}(\mathcal{K}^{m_{(s)}} y). \tag{5.2.10}$$

We make use of (5.2.6) to transfer the orbifold action on the momenta and then we define $\{q_{(s)}\} = \{\mathcal{K}^{m_{(s)}} k_{(s)}\}$, so that $q_{(N)} = k_{(N)}$. This allows us to extend to all the real values the integrals over $q_{(r)2}$, for $1 \le r \le N-1$:

$$I_N^{NBO}(\{[\tilde{q}_{(s)}]\}) = \frac{-i\lambda_N}{2\pi \Delta q_{(N)+}} \int_{\omega_{q_{(N)2}}} dq_{(N)2} \prod_{r=1}^{N-1} \frac{1}{2\pi \Delta q_{(r)+}} \int_{\mathbb{R}} dq_{(r)2}\, e^{-i \sum_{s=1}^{N} \frac{l_{(s)} q_{(s)2}}{\Delta q_{(N)+}}} \prod_{s=1}^{N} \int_{\mathbb{M}^3 \otimes \mathbb{R}^{D-3}} d^D y\ \Psi_{q_{(s)}}(y). \tag{5.2.11}$$

Then we can rewrite $I_N^{NBO}(\{[\tilde{q}_{(s)}]\})$ in terms of the vertex $I_N(\{\tilde{q}_{(s)}\})$ defined on the covering Minkowski spacetime:

$$I_N^{NBO}(\{[\tilde{q}_{(s)}]\}) = \frac{-i\lambda_N}{2\pi \Delta q_{(N)+}} \int_{\omega_{q_{(N)2}}} dq_{(N)2} \prod_{r=1}^{N-1} \frac{1}{2\pi \Delta q_{(r)+}} \int_{\mathbb{R}} dq_{(r)2}\, e^{-i \sum_{s=1}^{N} \frac{l_{(s)} q_{(s)2}}{\Delta q_{(N)+}}} I_N(\{q_{(s)}\}). \tag{5.2.12}$$

Since we are working at tree-level with plane waves, the expression can also be reduced to:

$$I_N^{NBO}(\{[\tilde{q}_{(s)}]\}) = \frac{-i\lambda_N}{2\pi\Delta q_{(N)+}} \int_0^{2\pi\Delta q_{(N)+}} dq_{(N)2} \prod_{r=1}^{N-1} \frac{1}{2\pi\Delta q_{(r)+}} \int_{\mathbb{R}} dq_{(r)2}$$
$$e^{-i\sum_{s=1}^{N} l_{(s)} \frac{q_{(s)2}}{\Delta q_{(s)+}}} \, \delta(\sum_{s=1}^{N} q_{(s)+}) \, \delta(\sum_{s=1}^{N} q_{(s)2}) \, \delta(\sum_{s=1}^{N} \vec{q}_{(s)})$$
$$\delta(\sum_{s=1}^{N} \frac{(q_{(s)2})^2 + (\vec{q}_{(s)})^2 + (M_{(s)})^2}{2q^+_{(s)}}), \tag{5.2.13}$$

where we used the mass-shell condition in the last $\delta$ function.

Now we will make explicit the divergence of $I_N^{NBO}$ in a slightly different way w.r.t. the previous chapters. For the sake of simplicity, let us forget the terms which are irrelevant from this point of view and concentrate only on the behaviour of the vertex. First we rewrite the conservation of $q_2$ and $q_-$ using a Fourier representation of the $\delta$ function as:

$$I_N^{NBO}(\{[\tilde{q}_{(s)}]\}) \propto \int_0^{2\pi\Delta q_{(N)+}} dq_{(N)2} \int_{-\infty}^{+\infty} d\alpha d\beta \prod_{r=1}^{N-1} \int_{\mathbb{R}} dq_{(r)2}$$
$$e^{i\sum_{s=1}^{N} q_{(s)2}\left(\alpha - l_{(s)} \frac{1}{\Delta q_{(s)+}}\right)}$$
$$e^{i\beta \sum_{s=1}^{N} \frac{(q_{(s)2})^2 + (\vec{q}_{(s)})^2 + M^2_{(s)}}{q_{(s)+}}}. \tag{5.2.14}$$

Then we integrate over $q_{(r)2}$ for $1 \leq r \leq N-1$ and over $\alpha$ to get:

$$I_N^{NBO}(\{[\tilde{q}_{(s)}]\}) \propto \int_0^{2\pi\Delta q_{(N)+}} dq_{(N)2} \int_{-\infty}^{+\infty} d\beta \frac{1}{\beta^{\frac{N}{2}-1}}$$
$$e^{-\frac{i}{2\beta\Delta^2} \sum_{s=1}^{N} \frac{l^2_{(s)}}{q_{(s)+}}}. \tag{5.2.15}$$

As expected, this expression is clearly divergent for $\beta \sim 0$ when $N \geq 4$

and all $l_{(s)} = 0$.

However this is the result on the usual commutative spacetime, but it is not the end of the story in the noncommutative framework. We need indeed to add a phase factor of the form (5.2.4), which on the NBO, where the only non-vanishing component of $\theta$ is $\theta^{+2}$, reads:

$$\begin{aligned} & e^{-\frac{i}{2}\theta^{+2}\sum_{r=1}^{N} k_{(r)+}\left[-\sum_{s=1}^{r-1}\mathcal{K}_{(s)2}k_{(s)2}+\sum_{s=r+1}^{N}\mathcal{K}_{(s)2}k_{(s)2}\right]} \\ =& e^{-\frac{i}{2}\theta^{+2}\sum_{r=1}^{N} k_{(r)+}\sum_{s=1}^{N}\mathrm{sgn}(s-r)\mathcal{K}_{(s)2}k_{(s)2}} \\ =& e^{-\frac{i}{2}\theta^{+2}\sum_{r=1}^{N} q_{(r)+}\sum_{s=1}^{N}\mathrm{sgn}(s-r)q_{(s)2}}. \end{aligned} \tag{5.2.16}$$

If we carry on a computation analogous to the one we just did but with the additional phase from the beginning, we end up this time with the result:

$$\begin{aligned} I_N^{NBO}(\{[\tilde{q}_{(s)}]\}) \propto \int_0^{2\pi\Delta q_{(N)+}} dq_{(N)2}\int_{-\infty}^{+\infty} d\beta \frac{1}{\beta^{\frac{N}{2}-1}} \\ e^{-\frac{i}{2\beta}\sum_{s=1}^{N} q_{(s)+}\left(\frac{l_{(s)}}{\Delta q_{(s)+}}+\theta^{+2}\sum_{r=1}^{N}\mathrm{sgn}(s-r)q_{(r)+}\right)^2}. \end{aligned} \tag{5.2.17}$$

This means that we have obtained a regularization which makes plausible the interpretation of the amplitude as a distribution, since now isolated zeros are avoided thanks to $\theta$. But the real novelty and the most important result is that for the first time we have a regularization with a truly stringy origin.

# Chapter 6

# Conclusions and Outlooks

Our journey through the study of the divergences on time-dependent orbifolds has gone a long way. First of all, we reviewed thoroughly the original computations where they were discovered. We came to the same result of the literature for the closed string amplitudes, but their unexpected appearance in the open string sector suggested us to investigate deeper. Our efforts haven't been in vain, since a careful analysis of simple quantum field theories defined on these backgrounds has revealed even more serious issues. Indeed, we came across a dramatic failure of the standard Feynman diagrammatic approach. This pathological behaviour is reflected in tree-level 3-point open string amplitudes involving massive states. This therefore seems to be the real root at the basis of the origin of the divergences, whereas the gravitational backreaction, which was originally accused of being the main motivation behind the breakdown, may eventually cause additional troubles at a later stage where one and higher loops diagrams are taken into account.

The consequent work of the thesis revolves around finding a way to cure the divergences, in order to recover a meaningful perturbative treatment. The first real attempt relies on the geometrical regularization achieved with the GNBO. Despite its success and the useful hints it gives, it leaves a rather nasty taste the idea of having to drastically change the construction and the geometry of the background. Conversely, the results we have obtained after the introduction of a Kalb-Ramond $B$-field, through the well-defined noncommutative QFT which arises in the decoupling limit, appear undoubtedly promising. This line of work is still at an early stage, therefore Chapter 5 has to be seen more as a starting point then as a closing of the circle, since a lot of directions can now be explored.

Reaching a better understanding of the noncommutative QFT is above all necessary. On a more general level, a next step should be the computation of string amplitudes involving massive states in presence of the Kalb-Ramond field even going beyond tree-level, to see if we can really recover a full perturbation theory. Showing analytically that the

$B$-field is the one generated by twisted closed string, as we suggested earlier, would also be a great result. Then it would be nice to see if there is a way to cure the BO, for which the noncommutative field theory limit doesn't apply well for unitarity reason.[1]

Finally, we would also like to mention that a new theory for a background similar to the NBO was introduced in [38] (see also [39]), where the term $\alpha' G^{\mu\nu}$ in (5.1.5) remains finite in the $\alpha' \to 0$ limit. In this case the modes of closed strings decouple but the massive modes from the open strings do not, resulting in a noncommutative open string theory (NCOS) [40, 41]. If we consider the following coordinates rescaling on the NBO

$$\begin{cases} x^+ \to \beta x^+ \\ x^- \to \gamma x^- \\ x^2 \to \sigma x^2 \end{cases},$$

apply the Seiberg-Witten map and take

$$\beta \sim (\alpha')^{\frac{3}{2}} \qquad \gamma \sim (\alpha')^{-\frac{1}{2}} \qquad \sigma \sim (\alpha')^{\frac{1}{2}} \qquad b \sim (\alpha')^{-2}, \tag{6.0.1}$$

we have exactly that $G \sim \alpha'$, while $\theta$ stays finite. It could therefore be of interest to explore the relationship between the field theory and this NCOS limit, similarly to what was done in [36]. Another non-trivial point would be to understand how this background behaves under S-duality. The original NCOS theories were S-dual to noncommutative gauge theories on D3-branes with space/space noncommutativity [40]. Is there a supergravity S-dual to the NBO background, and what type of theory does this NCOS theory map into under S-duality? Finally and to speculate a bit, this may also be a way to understand if and how this theory fits into the OM-theory [42].

[1] Nevertheless, notice that the SBO of Table 1.1 could serve at least as a geometrical regularization of the BO, with the radius $R$ playing the role of $\theta$.

# Appendix A

# Complete Tensor Wave Function on the NBO

For the sake of completeness we report the expression of the full NBO tensor wave function. In what follows $L = \frac{l}{k_+}$. We have:

$$
\begin{pmatrix} S_{uu} \\ S_{uv} \\ S_{uz} \\ S_{ui} \\ S_{vv} \\ S_{vz} \\ S_{vi} \\ S_{zz} \\ S_{zi} \\ S_{ii} \end{pmatrix}
= \mathcal{S}_{uu} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_{\{k_+\, l\, \vec{k}\, r\}}
+ \mathcal{S}_{uv} \begin{pmatrix} \frac{i}{k_+\, u} + \frac{L^2}{\Delta^2\, u^2} \\ 1 \\ L \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_{\{k_+\, l\, \vec{k}\, r\}}
+ \mathcal{S}_{uz} \begin{pmatrix} \frac{2\, L}{\Delta\, u} \\ 0 \\ \Delta\, u \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_{\{k_+\, l\, \vec{k}\, r\}}
$$

$$
+ \mathcal{S}_{ui} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_{\{k_+\, l\, \vec{k}\, r\}}
+ \mathcal{S}_{vv} \begin{pmatrix} -\frac{3}{4\, k_+^2\, u^2} + \frac{3\, i\, L^2}{2\, \Delta^2\, k_+\, u^3} + \frac{L^4}{4\, \Delta^4\, u^4} \\ \frac{i}{2\, k_+\, u} + \frac{L^2}{2\, \Delta^2\, u^2} \\ \frac{3\, i\, L}{2\, k_+\, u} + \frac{L^3}{2\, \Delta^2\, u^2} \\ 0 \\ 1 \\ L \\ 0 \\ \frac{i\, \Delta^2\, u}{k_+} + L^2 \\ 0 \\ 0 \end{pmatrix} \phi_{\{k_+\, l\, \vec{k}\, r\}}
$$

$$
+\mathcal{S}_{vz}\begin{pmatrix} \frac{3\,i\,L}{\Delta\,k_+\,u^2}+\frac{L^3}{\Delta^3\,u^3} \\ \frac{L}{\Delta\,u} \\ \frac{3\,L^2}{2\,\Delta\,u}+\frac{3\,i\,\Delta}{2\,k_+} \\ 0 \\ 0 \\ \Delta\,u \\ 0 \\ 2\,\Delta\,L\,u \\ 0 \\ 0 \end{pmatrix}\phi_{\{k_+\,l\,\vec{k}\,r\}}+\mathcal{S}_{vi}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{i}{2\,k_+\,u}+\frac{L^2}{2\,\Delta^2\,u^2} \\ 0 \\ 0 \\ 1 \\ 0 \\ L \\ 0 \end{pmatrix}\phi_{\{k_+\,l\,\vec{k}\,r\}}
$$

$$
+\,\mathcal{S}_{zz}\begin{pmatrix} \frac{i}{k_+\,u}+\frac{L^2}{\Delta^2\,u^2} \\ 0 \\ L \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta^2\,u^2 \\ 0 \\ 0 \end{pmatrix}\phi_{\{k_+\,l\,\vec{k}\,r\}}+\mathcal{S}_{zi}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{L}{\Delta\,u} \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta\,u \\ 0 \end{pmatrix}\phi_{\{k_+\,l\,\vec{k}\,r\}}+\mathcal{S}_{ij}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \delta_{ij} \end{pmatrix}\phi_{\{k_+\,l\,\vec{k}\,r\}}.
$$

# Appendix B

# Complete Overlap With Two Derivatives on the NBO

We report the full expression of the overlap with two derivatives considered in the main text which corresponds to the colour ordered amplitude of two tachyons and one level 2 massive state

$$
\begin{aligned}
K = \mathcal{N}^2 \int \mathrm{d}^D x \sqrt{-\det g} \Bigg[ & \mathfrak{s}^{(-3)} \left( \{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}_{i=1,2,3},\ \{\mathcal{S}\} \right) u^{-3} \\
& + \mathfrak{s}^{(-2)} \left( \{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}_{i=1,2,3},\ \{\mathcal{S}\} \right) u^{-2} \\
& + \mathfrak{s}^{(-1)} \left( \{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}_{i=1,2,3},\ \{\mathcal{S}\} \right) u^{-1} \\
& + \mathfrak{s}^{(0)} \left( \{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}_{i=1,2,3},\ \{\mathcal{S}\} \right) \\
& + \mathfrak{s}^{(1)} \left( \{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}_{i=1,2,3},\ \{\mathcal{S}\} \right) u \Bigg] \\
& \times \prod_{i=1}^{3} \phi_{\{k_{(i)+}\, l_{(i)}\, \vec{k}_{(i)}\, r_{(i)}\}},
\end{aligned}
$$

where:

$$\begin{aligned}
\mathfrak{s}^{(-3)} \quad &= \left( - \frac{k_{(2)+}^{4} {l_{(3)}}^{4} - 4\, k_{(2)+}^{3} k_{(3)+} l_{(2)} {l_{(3)}}^{3}}{4\, k_{(2)+}^{2} k_{(3)+}^{4} \Delta^{3}} \right. \\
&\quad \left. - \frac{6\, k_{(2)+}^{2} k_{(3)+}^{2} {l_{(2)}}^{2} {l_{(3)}}^{2} + k_{(3)+}^{4} {l_{(2)}}^{4}}{4\, k_{(2)+}^{2} k_{(3)+}^{4} \Delta^{3}} \right) \mathcal{S}_{v\,v},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{s}^{(-2)} \quad &= \left( - \frac{i\, \left( 3\, k_{(2)+}^{2} k_{(3)+} {l_{(3)}}^{2} + 3\, k_{(2)+}^{3} {l_{(3)}}^{2} - 4\, k_{(2)+} k_{(3)+}^{2} l_{(2)} l_{(3)} - 6\, k_{(2)+}^{2} k_{(3)+} l_{(2)} l_{(3)} \right)}{2\, k_{(2)+} k_{(3)+}^{3} \Delta} \right. \\
&\quad \left. + \frac{-i\, \left( +3\, k_{(3)+}^{3} {l_{(2)}}^{2} + 3\, k_{(2)+} k_{(3)+}^{2} {l_{(2)}}^{2} \right)}{2\, k_{(2)+} k_{(3)+}^{3} \Delta} \right) \mathcal{S}_{v\,v} \\
&\quad + \left( - \frac{l_{(3)}\, \left( k_{(2)+}^{2} {l_{(3)}}^{2} - 3\, k_{(2)+} k_{(3)+} l_{(2)} l_{(3)} + 3\, k_{(3)+}^{2} {l_{(2)}}^{2} \right)}{k_{(3)+}^{3} \Delta^{2}} \right) \mathcal{S}_{v\,z},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{s}^{(-1)} \quad &= \left( - \frac{\left( k_{(2)+} l_{(3)} - k_{(3)+} l_{(2)} \right)^{2}}{k_{(3)+}^{2} \Delta} \right) \mathcal{S}_{u\,v} \\
&\quad + \left( - \frac{2\, k_{(2)+}^{2} {l_{(3)}}^{2} \left( r_{(2)} + \vec{k}_{(2)}^{2} \right) + 2\, k_{(3)+}^{2} {l_{(2)}}^{2} \left( r_{(2)} + \vec{k}_{(2)}^{2} \right) - 8\, k_{(2)+}^{3} k_{(3)+} l_{(2)} l_{(3)}}{4\, k_{(2)+}^{2} k_{(3)+}^{2} \Delta} \right. \\
&\quad \left. - \frac{-3\, k_{(2)+}^{2} k_{(3)+}^{2} \Delta^{2} - 6\, k_{(2)+}^{3} k_{(3)+} \Delta^{2} - 3\, k_{(2)+}^{4} \Delta^{2}}{4\, k_{(2)+}^{2} k_{(3)+}^{2} \Delta} \right) \mathcal{S}_{v\,v} \\
&\quad + \left( - \frac{i\, \left( 3\, k_{(2)+} k_{(3)+} l_{(3)} + 3\, k_{(2)+}^{2} l_{(3)} - 2\, k_{(3)+}^{2} l_{(2)} - 3\, k_{(2)+} k_{(3)+} l_{(2)} \right)}{k_{(3)+}^{2}} \right) \mathcal{S}_{v\,z} \\
&\quad + \left( \frac{k_{(2)\,i}\, l_{(3)}\, \left( k_{(2)+} l_{(3)} - 2\, k_{(3)+} l_{(2)} \right)}{k_{(3)+}^{2} \Delta} \right) \mathcal{S}_{v\,i} \\
&\quad + \left( - \frac{\left( k_{(2)+} l_{(3)} - k_{(3)+} l_{(2)} \right)^{2}}{k_{(3)+}^{2} \Delta} \right) \mathcal{S}_{z\,z},
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{s}^{(0)} \quad &= \left(-\frac{i\,k_{(2)\,+}\,\left(k_{(3)\,+}+k_{(2)\,+}\right)\,\Delta}{k_{(3)\,+}}\right)\,\mathcal{S}_{u\,v} \\
&+ \left(-\frac{2\,k_{(2)\,+}\,\left(k_{(2)\,+}\,l_{(3)}-k_{(3)\,+}\,l_{(2)}\right)}{k_{(3)\,+}}\right)\,\mathcal{S}_{u\,z} \\
&+ \left(-\frac{i\,\left(k_{(3)\,+}+k_{(2)\,+}\right)\,\Delta\,(r_{(2)}+\vec{k}^2_{(2)})}{2\,k_{(2)\,+}\,k_{(3)\,+}}\right)\,\mathcal{S}_{v\,v} \\
&+ \left(-\frac{l_{(3)}\,(r_{(2)}+\vec{k}^2_{(2)})\,-2\,k_{(2)\,+}\,k_{(3)\,+}\,l_{(2)}}{k_{(3)\,+}}\right)\,\mathcal{S}_{v\,z} \\
&+ \left(\frac{i\,k_{(2)\,i}\,k_{(2)\,+}\,\Delta}{k_{(3)\,+}}\right)\,\mathcal{S}_{v\,i} \\
&+ \left(-\frac{i\,k_{(2)\,+}\,\left(k_{(3)\,+}+k_{(2)\,+}\right)\,\Delta}{k_{(3)\,+}}\right)\,\mathcal{S}_{z\,z} \\
&+ \left(\frac{2\,k_{(2)\,i}\,\left(k_{(2)\,+}\,l_{(3)}-k_{(3)\,+}\,l_{(2)}\right)}{k_{(3)\,+}}\right)\,\mathcal{S}_{z\,i},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{s}^{(1)} \quad &= \left(-k^2_{(2)\,+}\,\Delta\right)\,\mathcal{S}_{u\,u} \\
&+ \left(-\Delta\,(r_{(2)}+\vec{k}^2_{(2)})\right)\,\mathcal{S}_{u\,v} \\
&+ \left(2\,k_{(2)\,i}\,k_{(2)\,+}\,\Delta\right)\,\mathcal{S}_{u\,i} \\
&+ \left(-\frac{\Delta\,(r_{(2)}+\vec{k}^2_{(2)})^{\,2}}{4\,k^2_{(2)\,+}}\right)\,\mathcal{S}_{v\,v} \\
&+ \left(2\,k_{(2)\,i}\,k_{(2)\,+}\,\Delta\right)\,\mathcal{S}_{v\,i} \\
&+ \left(-k_{(2)\,i}k_{(2)\,j}\,\Delta\right)\,\mathcal{S}_{i\,j}.
\end{aligned}
$$

# Appendix C

# Stringy Derivation of Noncommutativity

We would like to derive the noncommutative parameter (5.1.4) of the field theory limit on the NBO directly from the quantized open string in presence of the $B$-field (5.1.2). Starting from the action

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \, \left(g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - 2\pi\alpha' \epsilon^{ab} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right)$$

we find the usual e.o.m.

$$g_{\mu\nu}\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\nu = 0$$

together with the Neumann boundary conditions

$$\left[g_{\mu\nu}\partial_\sigma X^\nu + (2\pi\alpha')B_{\mu\nu}\partial_\tau X^\nu\right]\Big|_{\sigma=0}^{\sigma=\pi} = 0.$$

These explicitly read

$$\begin{aligned} \left[\partial_\sigma X^-\right]\Big|_{\sigma=0}^{\sigma=\pi} &= 0 \\ \left[\partial_\sigma X^+ + (2\pi\alpha')B_{-2}\,\partial_\tau X^2\right]\Big|_{\sigma=0}^{\sigma=\pi} &= 0 \\ \left[\partial_\sigma X^2 + (2\pi\alpha')B_{-2}\,\partial_\tau X^-\right]\Big|_{\sigma=0}^{\sigma=\pi} &= 0, \end{aligned}$$

from which we derive:

$$X^- = x_0^- + 2\alpha' p^- \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \alpha_n^- \cos(n\sigma)$$

$$X^+ = x_0^+ + 2\alpha' p^+ \tau - 2\alpha' p^2 (2\pi\alpha') B_{-2}\sigma + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \alpha_n^+ \cos(n\sigma)$$

$$- \sqrt{2\alpha'} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \alpha_n^2 (2\pi\alpha') B_{-2} \sin(n\sigma)$$

$$X^2 = x_0^2 + 2\alpha' p^2 \tau - 2\alpha' p^- (2\pi\alpha') B_{-2}\sigma + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \alpha_n^2 \cos(n\sigma)$$

$$- \sqrt{2\alpha'} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \alpha_n^- (2\pi\alpha') B_{-2} \sin(n\sigma).$$

Then we compute the canonical momenta $P^\mu$, whose expressions are:

$$P^- = -\frac{2}{\pi} p^+ + 8\pi\alpha'^2 B_{-2}^2 p^- - \frac{\sqrt{2}}{\sqrt{\alpha'}\pi} \sum_{n\neq 0} e^{-in\tau} \alpha_n^+ \cos(n\sigma)$$

$$+ 4\sqrt{2}\pi\alpha'^{\frac{3}{2}} B_{-2}^2 \sum_{n\neq 0} e^{-in\tau} \alpha_n^- \cos(n\sigma)$$

$$P^+ = -\frac{2}{\pi} p^- - \frac{\sqrt{2}}{\sqrt{\alpha'}\pi} \sum_{n\neq 0} e^{-in\tau} \alpha_n^- \cos(n\sigma)$$

$$P^2 = \frac{1}{\pi} p^2 + \frac{1}{\sqrt{2\alpha'}\pi} \sum_{n\neq 0} e^{-in\tau} \alpha_n^2 \cos(n\sigma).$$

Now, following a procedure well-known in the literature [43, 44, 45, 46, 47], we can evaluate the symplectic form in terms of the string mode expansion:

$$\Omega = \int_0^\pi d\sigma\, dP_\mu dX^\mu = \, dx_0^- dp^+ + \, dx_0^+ dp^- + \, dx_0^2 dp^2$$
$$- (2\pi\alpha')^2 B_{-2}^2\, dx_0^- dp^- + (2\pi\alpha')^2 B_{-2}\, dp^- dp^2$$
$$+ \sum_{n>0} \frac{i}{n}\, d\alpha_n^- d\alpha_{-n}^+ + \sum_{n>0} \frac{i}{n}\, d\alpha_n^+ d\alpha_{-n}^- + \sum_{n>0} \frac{-i}{n}\, d\alpha_n^2 d\alpha_{-n}^2$$
$$+ \sum_{n>0} \frac{-i}{n} (2\pi\alpha')^2 B_{-2}^2\, d\alpha_n^- d\alpha_{-n}^-.$$

From this expression it's straightforward to derive the commutation relations

$$[x_0^-, p^+] = -i \quad [x_0^+, p^-] = -i \quad [x_0^2, p^2] = +i$$
$$[x_0^+, p^+] = -i(2\pi\alpha')^2 B_{-2}^2 \quad [x_0^+, x_0^2] = +i(2\pi\alpha')^2 B_{-2}$$
$$[\alpha_m^-, \alpha_n^+] = -m\,\delta_{m+n} \quad [\alpha_m^2, \alpha_n^2] = +m\delta_{m+n} \quad [\alpha_m^-, \alpha_n^-] = +m(2\pi\alpha')^2 B_{-2}^2 \delta_{m+n}$$

which are in accordance with

$$[x_0^\mu, p^\nu] = iG^{\mu\nu} \quad [x_0^\mu, x_0^\nu] = i\theta^{\mu\nu} \quad [\alpha_m^\mu, \alpha_n^\nu] = mG^{\mu\nu}\delta_{m+n}$$

and with the results (5.1.4) obtained by applying the Seiberg-Witten map.

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