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Generalized Descriptive Set Theory at uncountable cardinals
&
Actions of monoids in combinatorics

Supervisor: Luca Motto Ros

Candidate: Claudio Agostini

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Introduction

This thesis is divided into two parts, the first one focused on generalized descriptive set theory, and the second one on combinatorics and Ramsey theory.

According to Kechris [93], "Descriptive set theory (DST) is the study of "definable sets" in Polish (i.e. separable completely metrizable) spaces". Polish spaces are ubiquitous in mathematics (and not only there), and found application in many different fields. Classical DST has a natural generalization that occurs when countable is replaced by uncountable, called Generalized Descriptive Set Theory (GDST). Until recently, GDST focused mainly on the study of the generalized Baire space ${}^{\kappa}\kappa$ for a cardinal κ satisfying $\kappa^{<\kappa} = \kappa$, obtaining groundbreaking results (see e.g. the wonderful connection with Shelah's stability theory [135, 65, 88, 111]). However, this framework is narrow compared to the one of (classical) DST, focusing mostly on a single space, and heavily relying on cardinal assumptions as the regularity of κ.

The goal of the first part of this work is aimed at filling these gaps. We first develop a solid theoretical framework consisting of a class of spaces that could take the role of Polish spaces in the uncountable regular setting commonly considered in the literature on the subject. Then we extend this theory to all cardinals satisfying $2^{<\kappa} = \kappa$, including in particular singular cardinals. Finally, we provide some examples of substantially new spaces in the theory. This first part is the sum of three distinct works I have done during my PhD together with my supervisor Luca Motto Ros and, partially, with our coauthor Philipp Schlicht [4, 3, 5].

The second part of the project deals with a notion of recent discovery in combinatorics. Hindman's theorem [80], Carlson's theorem [31], Gowers's FIN_κ theorem [74] and Furstenberg-Katznelson's Ramsey theorem [68] are some of the most famous theorems in combinatorics and had a major impact across many fields of mathematics. All these theorems share a common underlying structure: they involve a semigroup with a monoid acting on it. In [142], Solecki isolated these common components into the notions of Ramsey and $\mathbb{Y}$-controllable monoids, and then he provided some necessary and some sufficient conditions for a monoid to satisfy one or the other definition. This result generalized all previous statements at once and showed that a big role in obtaining these theorems is played by the algebraic structure of the monoid acting on the semigroup.

The goal of the second part of this work is to continue the work started by Solecki in [142]. We improve some of the results proved therein, and investigate further this connection between algebra and combinatorics. We also introduce and study other similar classes of monoids defined through combinatorics. The results obtained in this part are taken from two works in collaboration with my PhD colleague and friend Eugenio Colla, [2, 1].

Generalized Descriptive Set Theory

Generalized descriptive set theory (in short, GDST) on regular cardinals is a very active field of research. Basically, the idea is to replace ω with an uncountable regular cardinal κ in the definitions of the Baire space ${}^{\omega}\omega$ and Cantor space ${}^{\omega}2$, as well as in all other topologically-related notions. For example, one considers κ-Borel sets (i.e. sets in the smallest κ^{+}-algebra generated by the topology) instead of Borel sets, κ-Lindelöf spaces (i.e. spaces such that all their open coverings admit a $<\kappa$-sized subcovering) instead of compact spaces, κ-meager sets (i.e. unions of κ-many nowhere dense sets) instead of meager sets, and so on. See [65, 6] for a general introduction and the basics of this subject.

The two spaces lying at the core of the theory are then the generalized or κ-Baire space ${}^{\kappa}\kappa$ and the generalized or κ-Cantor space ${}^{\kappa}2$, with the so-called bounded topology. Since the classical Cantor and Baire spaces are second-countable, it is natural to desire that, accordingly, ${}^{\kappa}\kappa$ and ${}^{\kappa}2$ have weight κ: this amounts to require that $\kappa^{<\kappa} = \kappa$, or, equivalently, κ regular and $2^{<\kappa} = \kappa$.

Despite the achievements already obtained by generalized descriptive set theory, there are still some missing ingredients.

First, the success and strong impact experienced by classical descriptive set theory in other areas of mathematics is partially due to its wide applicability: the theory is developed for arbitrary completely metrizable second-countable (briefly: **Polish**) **spaces** and for **standard Borel spaces**, which are ubiquitous in most mathematical fields. In contrast, generalized descriptive set theory so far concentrated (with a few exceptions) only on ${}^{\kappa}\kappa$ and ${}^{\kappa}2$, and this constitutes a potential limitation.

Our first goal is to fill this gap by considering various generalizations of Polish and standard Borel spaces already proposed in the literature ([7, 42, 71]), adding a few more natural options, and then systematically compare them from various points of view (see Figure 1). Some of these results substantially extend and improve previous work appeared in [42, 71].

Second, GDST so far has been focused almost only on regular cardinals, and this is another potential limitation to the subject. Even in models of ZFC where every regular cardinal κ satisfies $\kappa^{<\kappa} = \kappa$, there are still many cardinals on which there is no GDST. Also, for certain aspects GDST on singular cardinals may reveal to be even more interesting than GDST on regular cardinals. In the forthcoming papers [52, 53], Dimonte and Motto Ros conduct a detailed study of GDST on singular cardinals of countable cofinality, showing that in this context one may recover many theorems of classical descriptive set theory that gets lost in the uncountable regular case.

Our second goal is then to extend the study of GDST to all singular cardinals λ satisfying $2^{<\lambda} = \lambda$, with a particular focus on singular cardinals of uncountable cofinality to complete the theory developed so far. In particular, for the first time one can consistently have GDST on every cardinal (e.g. in models of $\mathsf{ZFC} + \mathsf{GCH}$).

Finally, one of the reasons why DST has been so successful is the abundance of example of Polish spaces with substantially different proprieties. The Cantor and Baire spaces plays certainly a central role in the theory, but we can not avoid the need of other Polish spaces like the real numbers $\mathbb{R}$, the complex field $\mathbb{C}$, the Hilbert cube $[0,1]^{\omega}$, and so on. Our third and last goal is to study the diversity of spaces that belong to the classes we defined before. We provide some new examples that are essentially different from ${}^{\kappa}\kappa$ and ${}^{\kappa}2$, and show that the classes of spaces we defined are as rich as possible, as each contains ${}^{\kappa}2$-many non-homeomorphic spaces.

All together, we believe that our results provide a wide yet well-behaved setup for developing generalized descriptive set theory, opening thus the way to fruitful applications to other areas of mathematics.

This part is divided into three chapters, each corresponding (roughly) to a distinct goal among the ones listed above. There is a partial overlap between the first two chapters, as the result presented in the second chapter are strictly more general and subsume the ones of the first chapter. However, we believed it would be more informative to introduce the concepts gradually. The first chapter has still many points in common with the literature (from the techniques used, to the kind of problems that one has to face), and some proofs are simpler because in the regular case certain subtleties can be avoided. The main focus of the second chapter, instead, is on problems of a different kind, that arise only when the cardinal is singular and have to be faced with new techniques.

GDST for regular cardinals

In Chapter 1, we focus on the study of classes of Polish-like spaces for GDST on a regular uncountable cardinal κ satisfying $\kappa^{<\kappa} = \kappa$. But what do we want from a class of spaces in order to be considered "Polish-like"? There are at least two crucial conditions. First, it should contain the generalized Cantor and Baire spaces. This already compels us to abandon metrizability. Second, we should recover the usual notion of Polish-space when $\kappa = \omega$. Thus, our main focus will be on classes of spaces of weight $\leq \kappa$ that are defined by generalizing complete metrizability or other notions which are equivalent to it in the classical case. When moving to uncountable cardinals, however, part of the theory seems naturally related to other classes of spaces like ultrametrizable and zero-dimensional Polish space: for this reason, we introduce also a few more classes that come from generalizations of other notions and concepts from those areas.

We collected six classes that seem to us the most natural options. The first two come from a direct generalization of the notion of metric. As we said, it is not possible to use classical metrizability in this context: the main reason is that metrizability intrinsically contains a notion of "countable dimension" in itself. More in detail: given a totally ordered Abelian group $\mathbb{G}$, call **degree** of $\mathbb{G}$, denoted by $\mathrm{Deg}(\mathbb{G})$, the smallest length of a sequence of positive elements converging to zero. The problem with metrizability is then that $\mathrm{Deg}(\mathbb{R})$ is countable, and this implies that every metrizable space is first countable. On the other hand the spaces ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ are not first countable if $\mathrm{cof}(\kappa) > \omega$. The idea is then to replace $\mathbb{R}$ with a totally ordered Abelian group $\mathbb{G}$ of degree κ: in this way, we get a notion of metric (called $\mathbb{G}$**-metric**) that is suitable in this context (see Definition 1.1.5 and the preceding paragraph).

When it comes to completeness of $\mathbb{G}$-metrics, there are (at least) two possible ways to proceed. Cauchy completeness can be extended naturally to $\mathbb{G}$-metrics by looking at sequences of length $\mathrm{Deg}(\mathbb{G})$. We call a space $\mathbb{G}$**-Polish** if it has weight $\leq \kappa$ and it is (Cauchy) completely $\mathbb{G}$-metrizable (Definitions 1.1.5). When $\mathrm{Deg}(\mathbb{G})$ is uncountable, the theory of $\mathbb{G}$-metrics is closely related also to the theory of ultrametrics, and for these spaces, we have another natural notion of completeness. We call an ultrametric **spherically complete** if the intersection of every decreasing sequence of balls (it does not matter whether we take them open or closed here, as all balls of an ultrametric are clopen) is nonempty. This notion can be naturally extended to all $\mathbb{G}$-metrics as well. We call a space **spherically complete $\mathbb{G}$-Polish** if furthermore the $\mathbb{G}$-metric can be taken to be spherically complete (Definition 1.1.29).

The idea of using generalized metrics has been rediscovered several times in different fields (see [117, 34, 89, 62], to mention a few instances), and this makes the literature very sparse and difficult to deal with. The first systematic study of generalizations of metrics started around the half of last century ([90, 139]), and later on, a more systematic study of $\mathbb{G}$-metrics has been developed by various authors in general topology (see e.g. [9, 83, 122, 126] and their bibliography). In contrast, the literature on *complete* $\mathbb{G}$-metrics seems scarcer, and it basically reduced to the papers [7, 42, 71]. One of the main results of the theory of generalized metrics is that fixed any totally ordered Abelian group $\mathbb{G}$, if $\deg(\mathbb{G}) = \omega$ then every $\mathbb{G}$-metrizable space is also metrizable (in the classical sense, that is, $\mathbb{R}$-metrizable), while if $\deg(\mathbb{G}) > \omega$, then every $\mathbb{G}$-metrizable space is also $\mathbb{G}'$-metrizable for any other totally ordered Abelian group of the same degree ([126]). In particular, in the countable degree case $\mathbb{G}$-metrizability yields to the usual notion of metrizability, while in the uncountable case we have a theory that is independent of the choice of $\mathbb{G}$. In this work, we prove that similar results hold for *complete* $\mathbb{G}$-metrizability (Corollaries 1.1.22, 1.1.23, and 1.1.33).

The second two classes we consider come from a well-known game-theoretic characterization of Polish spaces: a second countable (regular Hausdorff will always be tacitly assumed for every topological space) space is Polish if and only if it is strong Choquet ([36, 93]). The strong Choquet game has a natural generalization to higher cardinals by simply allowing the two players to play for κ-many rounds. However, when κ is uncountable it may happen that the game stops at a limit ordinal before κ because we already got an empty intersection. In these situations (that are not present in the countable version of the game), we need to choose which of the two players wins. Different choices lead to distinct notions: in the strong κ-Choquet game, we stipulate that player I wins all runs that stop before κ, while in the strong *fair* κ-Choquet game we instead declare that it is player II that wins in those cases. A **strong κ-Choquet space** (or **SC_κ-space**) is then a (regular Hausdorff) space of weight $\leq \kappa$ in which player II has a winning strategy in the strong κ-Choquet game, and we analogously define **strong fair κ-Choquet spaces** (or **$f\mathrm{SC}_\kappa$-spaces**) using the strong fair κ-Choquet game instead of the strong κ-Choquet game (Definitions 1.1.2 and 1.1.3). The notion of SC_κ-spaces and $f\mathrm{SC}_\kappa$-spaces have been introduced quite recently: the notion of strong κ-Choquet space has been studied for the first time in [42], while the notion of strong fair κ-Choquet is introduced in this thesis and in the corresponding paper [4].

Finally, the last two classes we consider in the first chapter are inspired by the following characterization of a zero-dimensionality within the class of Polish spaces: a space is a zero-dimensional Polish space if and only if it is homeomorphic to a closed subset of the Baire space ${}^\omega\omega$. In the classical case, a subset of ${}^\omega\omega$ is closed if and only if it is the set of branches (i.e. the body) of a pruned tree $T \subseteq {}^{<\omega}\omega$, where pruned means that T has no leaves, or, equivalently, no branch of size $< \omega$. Once again, in the uncountable case $\kappa > \omega$ definitions that were equivalent in the classical setup become distinct. Indeed, if we look at bodies of trees $T \subseteq {}^{<\kappa}\kappa$ that have no leaves (called now *weakly* pruned trees), then we get precisely the closed subsets of ${}^{\kappa}\kappa$. However, we can also consider subsets of ${}^{\kappa}\kappa$ which are the body of a tree $T \subseteq {}^{<\kappa}\kappa$ that has no branch of length $< \kappa$: we call them **superclosed** sets. Topological properties of closed subsets of the generalized Baire space have been studied in several papers (see e.g. [6, 65] and their bibliography). Superclosed subsets of the generalized Baire space have been studied, for example, in [42]. However, very little was known about the relationship between these two classes (see e.g. the diagram in [71, p. 25]).

Our first main result instead is to show that this is not the case, and these six classes are nicely divided into two groups: a class of "weakly complete" (Polish-like) spaces, containing $\mathbb{G}$-Polish spaces, closed subsets of the generalized Baire space, and $f\mathrm{SC}_\kappa$-spaces, and a class of "strongly complete" (Polish-like) spaces, containing spherically complete $\mathbb{G}$-Polish spaces, superclosed subsets of the generalized Baire space, and SC_κ-spaces.

Main Theorem 1 (Theorem 1.1.21)**.** *Assume $\kappa^{<\kappa} = \kappa > \omega$. For any (regular Hausdorff) space X the following are equivalent:*

(a) *X is $\mathbb{G}$-Polish;*

(b) *X is a κ-additive $f\mathrm{SC}_\kappa$-space;*

(c) *X is homeomorphic to a closed subset of ${}^{\kappa}\kappa$.*

Main Theorem 2 (Theorem 1.1.32)**.** *Assume $\kappa^{<\kappa} = \kappa > \omega$. For any (regular Hausdorff) space X the following are equivalent:*

(a) *X is a spherically complete $\mathbb{G}$-Polish space;*

(b) *X is a κ-additive SC_κ-space;*

(c) *X is homeomorphic to a superclosed subset of ${}^{\kappa}\kappa$.*

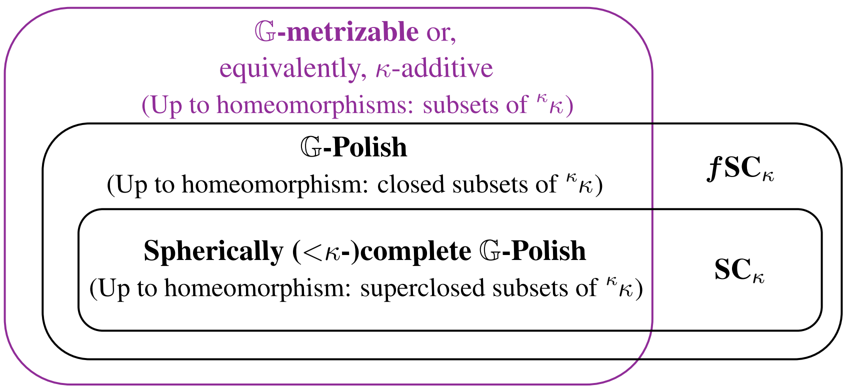

Figure 1: Relationships among different Polish-like classes of regular Hausdorff spaces of weight $\leq \kappa$, for a totally ordered Abelian group $\mathbb{G}$ of degree $\deg(\mathbb{G}) = \kappa$.

When κ is not a weakly compact cardinal, in the above statements we can replace the generalized Baire space ${}^{\kappa}\kappa$ with the generalized Cantor space ${}^{\kappa}2$ (as in this case

${}^{\kappa}\kappa$ and ${}^{\kappa}2$ are homeomorphic). If instead κ is weakly compact, we obtain two analogous results characterizing κ-Lindelöf spaces as closed or superclosed subsets of the generalized Cantor space (see Theorems 1.2.22 and 1.2.23).

The previous results also reveal another important dividing line: κ-additivity, a property which is relevant only if $\kappa > \omega$. More in detail, recall that a topological space is **κ-additive** if every intersection of $< \kappa$-many open sets is still open. This property implies strong forms of zero-dimensionality (see e.g. [10]), thus κ-additivity can be thought of as the uncountable analogue of zero-dimensionality, an important diving line in the context of (classical) Polish spaces. It is immediate to check that (spherically complete) $\mathbb{G}$-Polish spaces are κ-additive (and hence zero-dimensional): this is not too surprising, as $\mathbb{G}$-metrizability coincide with $\mathbb{G}$-ultrametrizability when $\deg(\mathbb{G})$ is uncountable ([121], see also Corollary 2.2.55).

Our analysis of the above-mentioned classes of Polish-like spaces reveals that there is no preferred option among them. Depending on which properties one decides to focus on, certain classes behave better than others, but there is no single class simultaneously sharing all the nice features typically enjoyed by the collection of (classical) Polish spaces. For example, if one is interested in maintaining the usual closure properties of the given class, then the "right" class is arguably the one of "weakly complete" spaces. Indeed, both $f\mathrm{SC}_\kappa$-spaces and $\mathbb{G}$-Polish spaces are closed under continuous open surjections (Theorem 1.4.3), $\leq \kappa$-sized sums and products, and G^κ_δ-subspaces (Theorems 1.4.1 and 1.4.2). In the context of κ-additive spaces, the latter can be turned into a characterization of "weakly complete" subspaces which mimic a well-known classical result.

Main Theorem 3 (Theorems 1.1.28)**.** *Assume $\kappa^{<\kappa} = \kappa > \omega$. A subset Y of a $\mathbb{G}$-Polish space X is $\mathbb{G}$-Polish if and only if Y is G^κ_δ in X.*

In contrast, "strongly complete" spaces miss some of the above closure properties. Indeed, both SC_κ and spherically complete $\mathbb{G}$-Polish spaces are closed under open subspaces, $\leq \kappa$-sized sums and products, and continuous open surjections, but they are not even closed under taking closed subspaces (see [42]). On the other hand, these classes are arguably the "right" ones in other respects. For example, every closed spherically complete $\mathbb{G}$-Polish subspace of a $\mathbb{G}$-Polish space is a retract of it, and thus, every spherically complete $\mathbb{G}$-Polish space is the continuous image of the generalized Baire space ${}^{\kappa}\kappa$ (Corollary 1.1.34). Moreover, we have a Cantor-Bendixson theorem for perfect spaces within such classes (Theorem 1.2.6). All these properties provably fail for the class of $\mathbb{G}$-Polish spaces (Remark 1.2.7 and [105, Theorem 1.5]).

The above different possibilities and behaviors are reconciled at the level of κ-Borel sets: all the proposed classes give rise to the same class of spaces up to κ-Borel isomorphism, and thus they constitute a natural and solid setup to work with if one is interested in developing a (generalized) descriptive set theory from the level of λ-Borel sets onward.

Main Theorem 4 (Theorem 1.1.40)**.** *Assume $\kappa^{<\kappa} = \kappa > \omega$. Up to κ-Borel isomorphism, the following classes of spaces are the same:*

(1) *$f\mathrm{SC}_\kappa$-spaces;*

(2) *SC_κ-spaces;*

(3) *$\mathbb{G}$-Polish spaces;*

(4) *spherically complete $\mathbb{G}$-Polish spaces.*

We also provide a mathematical explanation of the special role played by the κ-Cantor and κ-Baire spaces in the generalized setting. On the one hand, they admit nice characterizations which are analogous to the ones obtained in the classical setup by Brouwer and Alexandrov-Urysohn. These theorems have been extended to higher cardinals many times in different ways (see e.g. [87, Theorem 2.3], [121, Theorems 3.5 and 3.9]). We provide here similar characterisations using the notion of SC_κ-space (Theorems 1.2.10, 1.2.14 and 1.2.15). On the other hand, when restricting to κ-additive spaces all our classes can be described, up to homeomorphism, as collections of simply definable subsets of ${}^\kappa\kappa$ and ${}^\kappa 2$ (Theorems 1.1.21, 1.1.32, 1.2.22, and 1.2.23).

Borel space and standard Borel spaces are other fundamental notions of (classical) descriptive set theory. These notions can be extended to the uncountable case by considering κ^+-algebras instead of σ-algebras ([118, 65, 6]). In the classical case, there are multiple ways to define what a standard Borel space is: they can be defined as the Borel spaces which are Borel isomorphic to some Borel subset of the Baire space ${}^\omega\omega$ (or of any other uncountable Polish space, including ${}^\omega 2$); or, equivalently, they can be defined as the Borel spaces generated by some Polish topology. We show that in the generalized context the two possibilities coincide as well. To fix the terminology, following [118] we introduce the following definition.

Definition. A κ-Borel space $(X, \mathscr{B})$ is **standard** if it is κ-Borel isomorphic to a κ-Borel subset of ${}^\kappa\kappa$.

Having introduced natural classes of generalized Polish spaces, we can now prove the following:

Main Theorem 5 (Theorem 1.3.5)**.** *Assume $\kappa^{<\kappa} = \kappa > \omega$. A κ-Borel space $(X, \mathscr{B})$ is standard if and only if there is a topology τ on X such that $\mathsf{Bor}_\kappa(X, \tau) = \mathscr{B}$ and (X, τ) is an $f\mathrm{SC}_\kappa$-space.*

Moreover, Main Theorem 4 shows that in Main Theorem 5 we may equivalently ask that (X, τ) belongs to any other class of Polish-like spaces considered in this chapter. Main Theorem 5 is obtained via another fundamental technique concerning Borel sets that can be nicely extended to higher cardinals: we can change the topology of a Polish-like space, still maintaining most of its properties and structure, in order to turn κ-Borel sets into clopen sets (Propositions 1.3.1 and 1.3.7).

On the practical side, the usefulness of Main Theorem 5 lies in the following observation. In the literature, there are already a lot of results concerning the κ-Borel subsets of ${}^\kappa\kappa$: thanks to Main Theorem 5, we can now extend them to the κ-Borel structure of an arbitrary Polish-like space. Clearly, this paves the way for a wealth of applications, still to be explored.

We conclude this chapter by collecting some remarks and some open questions on the topic.

GDST for singular cardinals

In Chapter 2, we focus on the study of GDST relatively to an arbitrary cardinal λ of cofinality $\mathrm{cof}(\lambda) = \mu$ satisfying $2^{<\lambda} = \lambda$. The case $\omega < \mu < \lambda$ has never been considered in the literature, and it nicely extends and completes the work from Chapter 1 on regular cardinals and of [52, 53] on singular cardinals of countable cofinality.

When moving to GDST for cardinals of arbitrary cofinality, various new issues arise. For example, one must now pay attention to the fact that, depending on the situation, ω can be replaced with either λ or its cofinality μ. Indeed, in notions depending on the cardinality it is natural to use λ: for example, we study λ-Borel sets, λ-Lindelöf spaces, λ-meager sets, spaces of weight $\leq \lambda$, and so on. Nevertheless, some other concepts intrinsically depend on the cofinality of λ rather than on λ itself: for example, we look at μ-metrizable spaces (i.e. $\mathbb{G}$-metrizable for some totally ordered Abelian group $\mathbb{G}$ of degree[1] μ), μ-additive spaces, games of length μ, and so on.

Moreover, it turns out that the right generalizations of the Cantor and Baire spaces in this context are ${}^{\lambda}2$ and ${}^{\mu}\lambda$ (rather than ${}^{\lambda}\lambda$). The hypothesis $2^{<\lambda} = \lambda$ then ensures that both spaces have weight λ. Furthermore, when λ is not weakly compact then ${}^{\lambda}2$ and ${}^{\mu}\lambda$ are homeomorphic (see e.g. [121]), and this holds in particular for all singular cardinals.

The (admittedly vague) concept of "right" class of Polish-like spaces changes as well, depending on the cofinality. In fact, as in the regular case we still want that each class we consider contains at least the generalized Cantor and Baire spaces ${}^{\lambda}2$ and ${}^{\mu}\lambda$. However, we now want to develop a theory that is coherent not only with classical DST, but also with the already existing setups of GDST for regular cardinals and for uncountable singular cardinals of countable cofinality. More in detail, when $\mathrm{cof}(\lambda) = \omega$ the spaces ${}^{\lambda}2$ and ${}^{\omega}\lambda$ are completely metrizable, and the (only) right class of Polish-like spaces in this context is arguably the one of **λ-Polish spaces**, i.e. completely metrizable topological spaces of weight $\leq \lambda$ (see [52, 53]): thus we want that when λ has countable cofinality one recovers (classical) complete metrizability. On the other hand, when $\mathrm{cof}(\lambda) = \lambda$, i.e. when λ is regular, we expect to recover (at least) all the classes studied in Chapter 1, although we will also consider other classes of spaces naturally arising in this new context.

All classes of spaces introduced in Chapter 1 can easily be extended to arbitrary cardinals, but only some of them readily give a suitable Polish-like class, while some others require additional conditions (which were implicit in the regular case but become relevant for singular cardinals). Let us start with the ones that do extend in a nice way, without any further effort.

We call **(λ, μ)-Polish** a Cauchy-completely μ-metrizable space of weight $\leq \lambda$, and **spherically complete (λ, μ)-Polish** a (λ, μ)-Polish space where furthermore the $\mathbb{G}$-metric can be taken to be spherically complete. These notions respect all our requirements: when λ has countable cofinality, (λ, ω)-Polish spaces coincide with λ-Polish spaces (Fact 2.3.16), and if we restrict ourselves to ultrametrizable spaces, we get the same result for spherically complete (λ, ω)-Polish spaces (2.2.1). When $\lambda =$

[1] Notice that this is the unique natural choice, as the degree of a totally ordered Abelian group is always a regular cardinal naturally defined in terms of cofinality (or, to be precise, coinitiality).

$\mu > \omega$, instead, we recover exactly the classes of $\mathbb{G}$-Polish spaces and spherically complete $\mathbb{G}$-Polish spaces from Chapter 1 (Fact 2.3.16).

Similarly, we can redefine the classes of spaces homeomorphic to closed or superclosed subsets of ${}^{\mu}\lambda$, without any significant change and in a way that is coherent with the already known setups $\lambda = \mathrm{cof}(\lambda)$ or $\mathrm{cof}(\lambda) = \omega$.

The classes based on Choquet-like games need instead more work. A $\mathrm{SC}^{\lambda}_{\mu}$**-space** (respectively, $f\mathrm{SC}^{\lambda}_{\mu}$**-space**) is a (regular Hausdorff) topological space X of weight $\leq \lambda$ such that player II has a winning strategy in the strong (respectively, fair) μ-Choquet game on X. However, these concepts alone are not enough to grant what we want, as they do not entail any form of (generalized) metrizability if $\lambda > \omega$ is singular. For example, these classes do not coincide with λ-Polish spaces when $\lambda > \mathrm{cof}(\lambda) = \mu = \omega$, as there are examples of non-metrizable $\mathrm{SC}^{\lambda}_{\omega}$-spaces for any λ of countable cofinality.

This happens because Choquet-like games just characterize completeness, but not metrizability. In the classical setting $\lambda = \omega$, metrizability comes for free by second countability, thanks to the well-known Urysohn's metrization theorem. Such theorem has an extension to higher regular cardinals: every μ-additive regular Hausdorff space *of weight* $\leq \mu$ is μ-metrizable (see [139]). This is why in the regular case, where the weight naturally coincides with the additivity, we could ignore metrizability and only care about completeness. However, for λ singular (and more generally for spaces of weight $> \mu$) there is no analogue of the Urysohn's metrization theorem, and thus it becomes crucial to find other suitable generalizations of metrizability. Of course μ-metrizability is a natural option, but when μ is uncountable it implies being μ-additive and Lebesgue zero-dimensional, thus it seems to be too restrictive. Once again, the best way to find alternative notions of (generalized) metrizability is to look at characterizations of (classical) metrizability that hold independently from the weight of the space.

The first notion, leading to $\mathrm{NS}^{\delta}_{\mu}$**-spaces** (for $2 \leq \delta \leq \mu$), is inspired by the Nagata-Smirnov metrization theorem ([119, 141]), one of the most famous characterizations of metrizable spaces holding unconditionally: a topological space is metrizable if and only if it is regular Hausdorff and has a σ-locally finite basis (i.e., in our terminology, a $\mathrm{NS}^{\omega}_{\omega}$-basis). This notion can easily be extended to higher cardinals, and indeed $\mathrm{NS}^{\delta}_{\mu}$-bases (with $\delta = 2$ or $\delta = \omega$) have already been used in the literature to characterize, together with μ-additivity, the class of μ-metrizable spaces ([137, 84]). Motivated by this, we define $\mathrm{NS}^{\delta}_{\mu}$**-spaces** as (regular Hausdorff) spaces having a $\mathrm{NS}^{\delta}_{\mu}$-basis for their topology (Definition 2.2.4). To simplify the notation, we simply write NS_{μ} instead of NS^{μ}_{μ}. Notice that, in contrast to the literature on the subject, we are no longer requiring μ-additivity (and, for the sake of generality, we also allow values of δ in between ω and μ): this allows us to overcome the mentioned restrictions imposed by μ-metrizability itself and naturally include in our classes non-zero-dimensional spaces too. It turns out that having a NS_{μ}-basis is a good substitute for (generalized) metrizability. On the one hand, when $\mu = \omega$ we recover the (classical) metrizable spaces by the mentioned Nagata-Smirnov metrization theorem. On the other hand, every basis of size $\leq \mu$ is trivially a NS^{2}_{μ}-basis, thus if $\lambda = \mathrm{cof}(\lambda) = \mu > \omega$ is regular the condition of being a $\mathrm{NS}^{\delta}_{\mu}$-space (for any $2 \leq \delta \leq \mu$) is automatically satisfied by all spaces considered in Chapter 1, which always have weight $\leq \lambda = \mu$.

Next we consider (μ-)**tree-based spaces**, i.e. (regular Hausdorff) spaces with a basis for the topology that form a tree (of height $\leq \mu$) under the reverse inclusion relation $\supseteq$. Tree-based spaces have been introduced and studied for the first time by Kurepa [99, 100], and later rediscovered in other papers (see e.g. [121]). When $\mu = \omega$, rather than characterizing metrizability this notion characterize ultrametrizability: a space is Hausdorff and ω-tree-based if and only if it is Lebesgue zero-dimensional and metrizable ([44, 121]). Therefore, according to our desiderata, this should exclude this option from the acceptable substitutes for metrizability. However, the class of (μ-)tree-based spaces is quite useful and serves as a bridge between different notions, allowing us to simplify a significant part of the theory. Therefore we nonetheless include it in our analysis.

To the best of our knowledge the last class we introduce has not been considered before, and notably yields a new metrization theorem (even in the classical case $\mu = \omega$). Most of the metrization theorems present in literature depend on the existence of a certain particular basis for the topology. However, such bases might be difficult to find if we are only presented with the topology, without a specific basis already fulfilling the desired requirements. Moreover, in most practical situations what is really used is not the existence of a well-behaved basis of the given metrizable space, but rather the two most fundamental consequences of metrizability: paracompactness and first countability. Even taken together, these two properties are not enough to grant metrizability (consider e.g. to the Sorgenfrey line). We show that by replacing first countability with the following "uniform" (therefore stronger) version of it, we indeed get a characterization of metrizability. Let X be a topological space. The **μ-uniform local basis game** is a game of length μ where at each round $\alpha < \mu$, player I picks a point $x_\alpha \in X$, and player II replies with an open set V_α containing x_α:

I	x_0		x_1		...	x_γ		...
II		V_0		V_1	...		V_γ	...

At the end of the run, player II wins if either $\bigcap_{\alpha<\mu} V_\alpha = \emptyset$, or $\{V_\alpha \mid \alpha < \mu\}$ is a local basis of a point of X; otherwise I wins. A topological space is **μ-uniformly based** if (it is regular Hausdorff and) player II has a winning strategy in the corresponding μ-uniform local basis game. Notice that despite its name, this notion just depends on the topology and not on its bases. Using the above game, we can then give a new characterization of metrizability which has a more descriptive-set-theoretic flavour.

Main Theorem 6 (Theorem 2.2.39)**.** *A topological space is metrizable if and only if it is regular Hausdorff, paracompact and ω-uniformly based.*

As hinted, the advantage to work with this notion rather than with bases is that this game provides a user-friendly concrete tool to verify whether a space is metrizable or not. Consider for example the Sorgenfrey line: it is immediate to see that player I has a winning strategy in the ω-uniform local base game on it (it is enough that (s)he keeps changing point going right in the order), and thus that this space is not metrizable. Conversely, it is less immediate to see that there are no NS_ω-basis for the topology of such space.

Main Theorem 6 has also a natural extension to μ-metrizability (Theorem 2.2.41) once we replace paracompactness with its higher analogue (μ, μ)-paracompactness.[2]

[2]Paracompactness and (μ, μ)-paracompactness coincide when the space is μ-additive, and if furthermore μ is uncountable they coincide also with Lebesgue zero-dimensionality — see [10]).

The last class of spaces we consider is thus the one consisting of (μ, μ)**-paracompact** μ**-uniformly based spaces**. This class satisfies again all the requirements to be "suitable" for developing GDST for singular cardinals. Indeed, when $\mu = \omega$ it coincides with the class of metrizable spaces. Moreover, every regular Hausdorff space of weight $\leq \mu$ is automatically (μ, μ)-paracompact and μ-uniformly based, thus this condition trivializes when $\lambda = \mu$ is a regular cardinal, as in Chapter 1. Finally, this class contains examples of non-zero-dimensional (and non-κ-additive) spaces.

The class of NS^δ_μ-spaces and the class of (μ, μ)-paracompact μ-uniformly based spaces are nicely ordered by inclusion: every NS^δ_μ-space is also a $\mathrm{NS}^{\delta'}_\mu$-space for every $2 \leq \delta \leq \delta' \leq \mu$, and every NS^μ_μ-space is also (μ, μ)-paracompact and μ-uniformly based (Proposition 2.2.36). Without μ-additivity, the class of μ-tree-based does not relate well with NS_μ-spaces: the two classes are incomparable with respect to inclusion because of zero-dimensionality and Proposition 2.5.2. However, every μ-tree-based space is Lebesgue zero-dimensional (thus paracompact) and μ-uniformly based (Proposition 2.2.38), so the former class is included in the latter. Figure 2 sums up the mutual relationships among these classes. Notably, all the distinctions disappear in the realm of μ-additive spaces, where all the notions introduced above coincide (see Theorem 2.2.1 and the following paragraph for the relevant references to the literature). In particular, all these classes coincide up to λ-Borel isomorphism (Theorem 2.4.12).

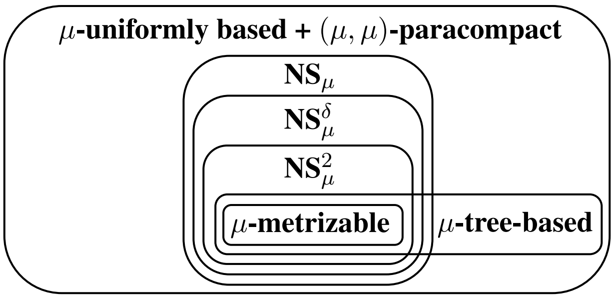

Figure 2: Relationships among different generalizations of metrizability for (regular Hausdorff) topological spaces. All classes coincide assuming μ-additivity.

Having found suitable analogues of metrizability, in Section 2.3 we step back to completeness. The main novelty here is that we introduce two further notions of completeness, closely related to compactness and defined through the existence of a certain basis for the topology (linking thus the work on completeness with the one on metrizability). Every form of compactness brings within itself a form of completeness: for example, it is well-known that every compact metrizable space is completely metrizable, and similar statements hold in the uncountable case as well (Propositions 1.2.18 and 1.2.19). Compactness can be stated (equivalently) as the property that the intersection of any family of closed sets with the finite intersection property is nonempty (a similar statement holds for μ-Lindelöfness and families of closed sets with the $<\mu$-intersection property). It turns out that by limiting this property to (the closure of) sets from a basis for the topology we get precisely a notion of completeness (even for non-compact and non-μ-Lindelöf spaces). More in detail, given a family $\mathcal{B}$ of open sets denote with $\mathrm{cl}[\mathcal{B}]$ the family of the closures of its elements. We say that a space is **compact-based** is it (is regular Hausdorff and) admits a **compact basis**, i.e. a basis $\mathcal{B}$ such that every subfamily of $\mathrm{cl}[\mathcal{B}]$ with the finite intersection property has nonempty intersection. Analogously, we say that a space is **μ-Lindelöf-based** if it (is regular Hausdorff and) admits a **μ-Lindelöf basis**, i.e. a basis $\mathcal{B}$ such that every subfamily of $\mathrm{cl}[\mathcal{B}]$ with the $< \mu$-intersection property has nonempty intersection (Definition 2.3.19. Compare it also with [7]).

Combining these notions with some form of (generalized) metrizability, and especially with the existence of μ-tree-bases, we get a tool that allows us to simplify a significant part of the theory (compare e.g. Proposition 1.1.14 with Proposition 2.3.32) and that better highlights some connections that naturally arise among different concepts. For example, the equivalence between being Polish and being G_δ (and its uncountable analogues, like Main Theorem 3) can be easily explained using the notion of μ-Lindelöf μ-tree-bases (Theorem 2.3.27), and with this notion we get also a (weaker) characterization of spherically complete (λ, μ)-Polish subspaces that we could not get before (Lemma 2.3.30).

We conclude the study of completeness by proving Theorems 2.3.1 and 2.3.2, which show that by adding μ-metrizability we can restore the picture we got in the regular case. In particular, in that case, all the completeness notions get divided into two classes: a first class of "weakly complete" spaces (containing (λ, μ)-Polish spaces, $f\mathrm{SC}^\lambda_\mu$-spaces, μ-Lindel"of-based spaces, and spaces homeomorphic to closed subsets of ${}^\mu\lambda$), and a second class of "strongly complete" spaces (containing spherically complete (λ, μ)-Polish spaces, SC^λ_μ-spaces, compact-based spaces, and spaces homeomorphic to superclosed subsets of ${}^\mu\lambda$). Figure 3 sums up what we obtained so far, and should be compared with Figure 1.

Main Theorem 7 (Theorems 2.3.1 and 2.3.2)**.** *Let X be a (regular Hausdorff) space of weight $\leq \lambda$, and further, assume that X be Lebesgue zero-dimensional if $\mu = \omega$.*

(a) *The following are equivalent:*

  (1) *X is (λ, μ)-Polish;*
  (2) *X is a spherically μ-complete μ-metrizable space;*
  (3) *X is a μ-metrizable $f\mathrm{SC}^\lambda_\mu$-space;*
  (4) *X is a μ-Lindelöf-based μ-metrizable space;*
  (5) *X is homeomorphic to a closed subset of ${}^\mu\lambda$;*
  (6) *X is homeomorphic to a G^μ_δ subset of ${}^\mu\lambda$;*

  If furthermore $\mu = \mathrm{cof}(\lambda)$, then they are also equivalent to the following:

  (7) *X is homeomorphic to a G^μ_δ subset of ${}^\lambda 2$.*

(b) *The following are equivalent:*

  (1) *X is a spherically $<\mu$-complete (λ, μ)-Polish space;*
  (2) *X is a spherically complete μ-metrizable space;*
  (3) *X is a μ-metrizable SC^λ_μ-space;*
  (4) *X is a compact-based μ-metrizable space;*
  (5) *X is homeomorphic to a superclosed subset of ${}^\mu\lambda$.*

In particular, when $\mu = \omega$ then all the above items (1)–(7) from part (a) and (1)–(5) from part (b) (under the appropriate assumptions on λ) are equivalent to each other.

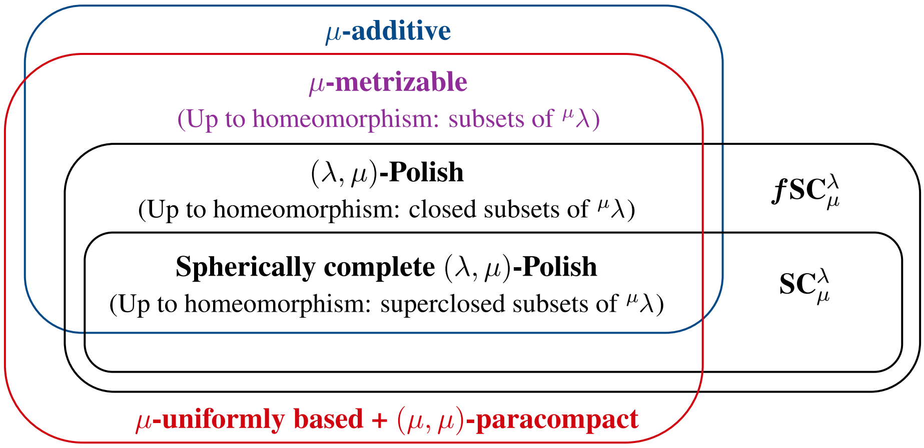

Figure 3: Relationships among different Polish-like classes for regular Hausdorff spaces of weight $\leq \lambda$ and for uncountable cardinals λ and μ.

Finally, in Section 2.4 we study λ-Borel spaces, for an arbitrary cardinal λ with $\mathrm{cof}(\lambda) = \mu$ satisfying $2^{<\lambda} = \lambda$. As discussed, Sections 2.2 and 2.3 introduce a large number of classes of spaces which could claim to be the "right" generalization of Polish spaces: this makes it very challenging to determine which are the best classes to work with. Even if the situation might be chaotic from the topological viewpoint, a better picture can be obtained from the point of view of (generalized) descriptive set theory: all classes of topological spaces considered so far are the same up to λ-Borel isomorphism, and actually, in most cases, the only differences concern the *finite* levels of their λ-Borel hierarchy (even though λ might be very large in the cardinal hierarchy). A first positive result in this direction is Proposition 2.4.3, which shows that most topological properties we considered (being NS_{μ}, μ-tree-based, μ-uniformly based, $f\mathrm{SC}_{\mu}$ or $\mathrm{SC}^{\lambda}_{\mu}$) pass to the smallest μ-additive refinement of the topology; together with Main Theorem 7 and Corollary 2.4.9, this show, for example, that among Lebesgue zero-dimensional NS_{μ}-spaces all classes of complete Polish-like spaces considered so far are the same up to λ-Borel isomorphism. Theorem 2.4.12 improve this result, showing that all the different classes of weakly complete and strongly complete spaces coincide up to λ-Borel isomorphism. Surprisingly, (μ-)metrizability and the other related notions that we defined in Section 2.2 play no role at all from this point of view, as we can always add any of them without altering the λ-Borel hierarchy too much.

Main Theorem 8 (Theorem 2.4.12)**.** *Assume $2^{<\lambda} = \lambda$ and $\mathrm{cof}(\lambda) = \mu$. Up to λ-Borel isomorphism, the following classes of topological spaces coincide:*

(1) $f\mathrm{SC}^{\lambda}_{\mu}$*-spaces;*

(2) $\mathrm{SC}^{\lambda}_{\mu}$*-spaces;*

(3) (λ, μ)*-Polish spaces;*

(4) *(Lebesgue zero-dimensional) spherically complete (λ, μ)-Polish spaces;*

(5) *μ-Lindelöf-based spaces;*

(6) *compact-based spaces.*

As in the regular case, this implies that we have a unique notion of standard λ-Borel space (Theorem 1.3.5), providing thus a solid ground for the development of descriptive set theory from λ-Borel sets onward.

Main Theorem 9 (Theorem 2.4.13)**.** *Assume $2^{<\lambda} = \lambda$ and $\mathrm{cof}(\lambda) = \mu$. Then a λ-Borel space $(X, \mathscr{B})$ is standard if and only if there is a topology τ' on X such that $\mathsf{Bor}_\lambda(X, \tau') = \mathscr{B}$ and τ' belongs to any of the classes from Main Theorem 8.*

Finally, we can classify which subsets of a standard λ-Borel space inherit a standard λ-Borel structure. In a (classical) standard Borel space $(X, \mathscr{B})$, a subset A is again standard Borel (with the inherited structure) if and only if $A \in \mathscr{B}$. This result is a direct consequence of Lusin's separation theorem [110], which is known to fail in the generalized context (see [65]). Nonetheless, we show that a different proof leads to the desired result, unconditionally.

Main Theorem 10 (Theorem 2.4.15)**.** *Assume $2^{<\lambda} = \lambda$ and $\mathrm{cof}(\lambda) = \mu$. Let $(X, \mathscr{B})$ be a standard λ-Borel space, and let $A \subseteq X$. Then $(A, \mathscr{B} \restriction A)$ is a standard λ-Borel space if and only if $A \in \mathscr{B}$.*

Examples of Polish-like spaces

In Chapter 3 we provide numerous examples of spaces inside the classes considered above. This chapter is taken from a preliminary version of a work in progress. The study is here conducted for a regular uncountable cardinal κ satisfying $2^{<\kappa} = \kappa$, with the idea to extend it in the (near) future also to singular cardinals. In Sections 3.1, 3.2 and 3.3, we study some classes of spaces in which one can find examples of non-κ-additive $f\mathrm{SC}_\kappa$ (or SC_κ) spaces. In Section 3.4, we instead go back to κ-additive spaces and show that various classes of $\mathbb{G}$-Polish and spherically complete $\mathbb{G}$-Polish spaces are rich enough and contain 2^κ-many non-homeomorphic spaces. The plan for the future is to extend the work of this third section also to non-κ-additive spaces (possibly using bi-embeddability instead of homeomorphism) by employing the tools developed in the first two sections.

The content of this chapter should serve as a complement to the theoretical study carried out in the previous ones. However, there is also a second motivation behind its first two sections.

Since its very beginning, one natural problem in GDST has been that of finding the right analogue of the real line in that context. Two remarkable works on the topic have been done by Asperó and Tsaprounis, who introduced and studied the so-called field of long reals κ-$\mathbb{R}$ ([12]), and by Galeotti (partially together with Carl and Löwe), who studied the subfield $\mathbb{R}_\kappa$ of the surreals numbers and showed, for example, how one can recover for this space natural analogues of theorems from real analysis ([70, 71, 30]).

The problem with generalizing $\mathbb{R}$ to the uncountable case is that one can not preserve all its different proprieties at the same time. For example, it is well-known that there is no Dedekind-complete totally ordered field of uncountable degree ([43, 71]). So depending on which direction one wants to look at, one should decide which properties (s)he is willing to maintain and which ones (s)he can bear to lose. If one is interested in the algebraic structure of $\mathbb{R}$, then it is natural to ask to preserve the fact that $\mathbb{R}$ is a totally ordered field, at the cost of sacrificing order properties (Dedekind-completeness) and topological proprieties (being connected). For example, this has been the choice in [12] and [71]. In contrast, in topology being connected is certainly a fundamental property of the reals (from basic separation axioms like complete regularity to ideas like path-connectedness and homotopy theory). Also, the theory of $\mathbb{G}$-metrizable spaces shows that the algebraic structure of $\mathbb{G}$ (and thus of $\mathbb{R}$) plays a marginal role in metrization theory (to the point that $\mathbb{G}$ can be taken to be just a semigroup, see [126]), while the order structure of $\mathbb{G}$ plays a fundamental role. For these reasons, one can argue that at least for certain aspects of GDST the "right" generalization of the space of reals $\mathbb{R}$ should be searched within the class of complete linear orders, equipped with their order topology. (Then one might partially reconcile the two different options by looking at order completions of totally ordered fields.)

Hence, we decided to hit two birds with one stone and study linearly ordered spaces as a source of examples of non-κ-additive $f\mathrm{SC}_\kappa$ and SC_κ-spaces. Since our main objective is to find examples of $f\mathrm{SC}_\kappa$ and SC_κ-spaces with different properties, we do not focus directly on proposing alternatives for the real line in the uncountable setting: nevertheless, we believe that our study could contribute to understanding which properties one should expect from such a space, hinting at where to search for it.

In Section 3.1 we study the classes of linearly ordered topological spaces (LOTS) and generalized ordered spaces (GO-spaces). More precisely, a **LOTS** is a linear order equipped with the order-topology generated by its open intervals, while a **GO-space** is a (both order-theoretical and topological) subspace of a LOTS. (Notice that the topology of a GO-space might be strictly finer than the order-topology generated by its open intervals.) This topic has been widely studied, as LOTS and GO-spaces are relatively well-behaved topological spaces and enjoy many interesting properties. We refer to [120] for a good introduction to the topic.

We begin by collecting various results we are interested in, providing also a short proof for those results that are probably folklore but are not explicitly proved in [120]. Then we turn to the relation between order topologies and Choquet games. Our main result on the topic is a theorem providing a sufficient condition for a GO-space (thus including the case of LOTS spaces as well) to be $f\mathrm{SC}_\kappa$ or SC_κ.

Main Theorem 11 (Theorem 3.1.15)**.** *Let $(X, <, \tau)$ be a GO-space of weight $\leq \kappa$. Then X is SC_κ (respectively, $f\mathrm{SC}_\kappa$) if and only if player II has a strategy winning every run of the strong (respectively, fair) κ-Choquet game on (X, τ) in which player I plays only open intervals fully contained in the intersection of the previous moves.*

This (technical) result has several interesting and more concrete consequences. For example, it implies (Corollary 3.1.17) that if a GO-topology τ is SC_κ (or $f\mathrm{SC}_\kappa$), than any other GO-topology $\tau' \supseteq \tau$ of weight $\leq \kappa$ is again SC_κ (or $f\mathrm{SC}_\kappa$). In particular, if the order topology of a linear order $(\mathbb{L}, <)$ is SC_κ (or $f\mathrm{SC}_\kappa$), than any GO-topology of weight $\leq \kappa$ on $(\mathbb{L}, <)$ is again SC_κ (or $f\mathrm{SC}_\kappa$). Main Theorem 11 also shows that, similarly to other compactness notions, completeness properties of

GO-spaces are closely related to the presence or absence of gaps in the order (Proposition 3.1.19 and Corollary 3.1.20).

In Section 3.2 we move to a particular class of LOTS: lexicographic spaces. In particular, we study spaces of the form ${}^{\kappa}\mathbb{L}$, where $\mathbb{L}$ is a given linear order, equipped with the lexicographic order and the corresponding order-topology. Our main goal is actually to study the lexicographic order topology on the κ-Cantor and κ-Baire spaces, but first, we survey the properties of lexicographic topologies in general. We also analyze the relationship between lexicographic spaces and Choquet games. Our main result here is Proposition 3.2.7, where we show that the completeness of the order-topology on the lexicographic space ${}^{\kappa}\mathbb{L}$ depends on the endpoints of $\mathbb{L}$ and on the completeness of the lower-limit and/or upper-limit topology on $\mathbb{L}$.

Main Theorem 12 (Proposition 3.2.7)**.** *Assume $\kappa^{<\kappa} = \kappa$ and let $(\mathbb{L}, <)$ be a linear order of size $\leq \kappa$.*

- *Suppose $\mathbb{L}$ has no maximum nor minimum. Then every GO-topology on $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ is SC_{κ}.*
- *Suppose $\mathbb{L}$ has a maximum but no minimum. Then every GO-topology on $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ is $f\mathrm{SC}_{\kappa}$ (respectively, SC_{κ}) if and only if the lower-limit topology on $(\mathbb{L}, <)$ is $f\mathrm{SC}_{\kappa}$ (respectively, SC_{κ}).*
- *Suppose $\mathbb{L}$ has a minimum but no maximum. Then every GO-topology on $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ is $f\mathrm{SC}_{\kappa}$ (respectively, SC_{κ}) if and only if the upper-limit topology on $(\mathbb{L}, <)$ is $f\mathrm{SC}_{\kappa}$ (respectively, SC_{κ}).*
- *Suppose $\mathbb{L}$ has both maximum and minimum. Then every GO-topology on $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ is $f\mathrm{SC}_{\kappa}$ (respectively, SC_{κ}) if and only if both the upper-limit and lower-limit topologies on $(\mathbb{L}, <)$ are $f\mathrm{SC}_{\kappa}$ (respectively, SC_{κ}).*

For every lexicographic space ${}^{\kappa}\mathbb{L}$, it is possible to define a notion of rationals $\mathbb{Q}({}^{\kappa}\mathbb{L})$ and irrationals $\mathbb{I}({}^{\kappa}\mathbb{L})$ of ${}^{\kappa}\mathbb{L}$. These notions are defined in analogy with the rationals and irrationals of $\mathbb{R}$ in the classical case and have many similarities with them. In this context, we prove that we can extend the usual result on the homeomorphism between the irrationals and the Baire space to the generalized setting.

Main Theorem 13 (Proposition 3.2.6)**.** *Assume $\kappa^{<\kappa} = \kappa$. Suppose $(\mathbb{L}, <)$ is a linear order of size $2 \leq |\mathbb{L}| \leq \kappa$, and that it has an end point. Then the irrationals $\mathbb{I}({}^{\kappa}\mathbb{L})$ (with the lexicographic topology) form a G^{κ}_{δ} subspace of ${}^{\kappa}\mathbb{L}$ homeomorphic to the generalized Baire space ${}^{\kappa}\kappa$ (with its usual bounded topology).*

In Section 3.3 we conclude our study of LOTS and lexicographic spaces by collecting some topological properties of the κ-Cantor set ${}^{\kappa}2$ and the κ-Baire set ${}^{\kappa}\kappa$ with their respective lexicographic topologies, and show that they are both examples of non-κ-additive zero-dimensional SC_{κ}-spaces. We also show that the Dedekind-completion of the κ-Baire set ${}^{\kappa}\kappa$ with lexicographic order is a connected, compact, SC_{κ}-space. Finally, we briefly hint at how to obtain other examples of spaces with different topological properties, like the generalized circle $S^1_1(\mathbb{L})$ generated by a linear order $\mathbb{L}$ with endpoints.

Finally, in Section 3.4 we go back to the realm of κ-additive spaces and show how different classes of $\mathbb{G}$-Polish and spherically complete $\mathbb{G}$-Polish spaces are as rich as possible in cardinality, since they contain 2^{κ}-many non-homeomorphic spaces.

Main Theorem 14 (Theorems 3.4.3, 3.4.6, 3.4.9, 3.4.12, and 3.4.16)**.** *Assume that* $\kappa^{<\kappa} = \kappa$. *The following classes contain* 2^κ*-many pairwise not homeomorphic spaces:*

- *The class of perfect* $\mathbb{G}$*-Polish spaces of size* 2^κ.
- *The class of spherically complete* $\mathbb{G}$*-Polish spaces of size* 2^κ.

If furthermore κ *is weakly compact, the same is true for:*

- *The class of perfect spherically complete* $\mathbb{G}$*-Polish spaces of size* 2^κ.
- *The class of perfect* κ*-Lindelöf* $\mathbb{G}$*-Polish spaces of size* 2^κ.
- *The class of* κ*-Lindelöf spherically complete* $\mathbb{G}$*-Polish spaces of size* 2^κ.

Ramsey theory and combinatorics

Many theorems in combinatorics share a very similar structure: "*Let M be monoid acting by endomorphism on a partial semigroup S. For each finite coloring of S, there is a nice monochromatic subset N (usually closed under the action of M and the operation of S, to a certain degree)*". They differ in the choice of M, S, and N. Once S and the desired properties of N are fixed, each theorem of this form defines a class of monoids that satisfy the above statement. Our aim is to study different classes of monoids defined this way.

It turns out that S plays (almost) no role in the definition of the class of monoids (Main Theorem 16 and Lemma 4.1.10). The condition that really causes different statements to generate different classes of monoids is the structure of the monochromatic subset N.

Following this approach, we define and study four classes of monoids that correspond to different theorems in combinatorics: Ramsey monoids (related to Hindman's theorem or Gowers' theorem or Carlson's theorem), $\mathbb{Y}$-controllable monoids (related to Furstenberg-Katznelson Ramsey theorem), locally Ramsey monoids (related to infinite Carlson's theorem), and locally $\mathbb{Y}$-controllable monoids (related to infinite Furstenberg-Katznelson Ramsey theorem).

The classes of Ramsey and $\mathbb{Y}$-controllable monoids have been introduced and studied by Solecki in [142], where he showed that the possibility to obtain certain combinatorial statements is strictly linked to the algebraic structure of M. He provided some purely algebraic sufficient conditions for a monoid to be Ramsey, which allows to improve several different results in combinatorics (for example, it gives a simultaneous extension of Hindman's theorem, Gowers' theorem, Carlson's theorem, and Furstenberg-Katznelson's Ramsey theorem), and he showed that one of these sufficient conditions is also necessary.

We show here that this link is even stronger, and the possibility to obtain certain combinatorial statements *depends only on the algebraic structure of the monoid*. First, we prove that Ramsey monoids and locally Ramsey monoids can be completely characterized in terms of their algebraic structures (Main Theorem 17 and Main Theorem 20). Furthermore, we provide both necessary and sufficient conditions for a monoid to be $\mathbb{Y}$-controllable and locally $\mathbb{Y}$-controllable (Main Theorems 18, 19, and 21). This, in turn, also gives extensions of some results of Solecki's [142] and of all theorems previously mentioned.

Before presenting our results, let us start with a brief historical introduction. The 20th century has been a time of great development for combinatorics, with a big impulse coming directly from its applications to algebra and number theory. One of the first and most famous results in the field was obtained by Schur [133] while trying to solve a (version of a) very famous problem.

Theorem (Fermat's last theorem $\bmod p$ — Schur, 1916)**.** *For every $k \in \mathbb{N}$ and for every large enough prime p, there exists $x, y, z \in \mathbb{N}$ so that $x^k + y^k \equiv z^k \pmod{p}$.*

This result is based on a wonderful lemma about the combinatorics of natural numbers: "*For every finite partition of $\mathbb{N}$ there exist two numbers $a, b \in \mathbb{N}$ such that a, b and $a + b$ all belongs to the same piece of the partition*".

This lemma soon became a cornerstone of combinatorics, and many different branches of combinatorics originated from it. For example, the set $\{a, a + b\}$ is an arithmetic progression of length 2, and Schur's lemma can be seen as a precursor of further famous theorems about progressions, like Van der Waerden's theorem [156] and its variant by Brauer (see e.g. [24]).

Theorem (Brauer's theorem, 1928)**.** *For every finite coloring of the natural numbers and for every $k \in \mathbb{N}$ there exist two natural numbers $x, d \in \mathbb{N}$ such that the set $\{d\} \cup \{x, x+d, x+2d, \ldots, x+kd\}$ is monochromatic.*

For further results on this research line, see for example [149], [67] or [19].

Schur's lemma can also be seen as one of the first results about partition regularity of families of solutions of linear equations. A family $\mathcal{F} \subseteq \mathscr{P}(X)$ is said to be **weakly partition regular** if whenever we partition X into finitely many pieces, one of these pieces has a subset in $\mathcal{F}$. Research on partition regularity of families of sets defined through equations is currently a very active field of research. Schur's lemma can be seen as the first example of a theorem of this form: "*The family of all possible solutions in $\mathbb{N}$ to the equation $x + y = z$ is weakly partition regular*". For more results in this area, see for example [50], [106], [14].

Or again, one may wonder what happens if we require the monochromatic set to be closed under multiplication instead of sum, or even under both operations. The first result in this direction is easy to achieve and it is an immediate corollary of Schur's lemma: "*For every finite partition of $\mathbb{N}$ there exist two numbers $a, b \in \mathbb{N}$ such that a, b and $a \cdot b$ all belongs to the same piece of the partition*". The second one is instead still wide open.

Open Problem. *Is it true that for every finite coloring of the natural numbers there are $a, b \in \mathbb{N}$ such that the set $\{a, b, a+b, a \cdot b\}$ is monochromatic?*

See also [22], [116] for recent developments on this problem.

The focus of this work is on yet another branch of combinatorics whose results have a different form, so let us restate Schur's lemma one last time. A finite coloring of X is a function $c\colon X \to r$ with finite codomain. There is an obvious natural identification between colorings $c\colon X \to r$ of X and partitions $\{c^{-1}[i] \mid i \in r\}$ of X. Also, the set $\{a, b, a+b\}$ can be seen as a particular example of a more general family of sets. Given a set X with an operation $\cdot$ and a sequence $\bar{s} \in X^{\leq\omega}$, the **span** of $\bar{s}$ is the set

$$\langle \bar{s} \rangle = \{s_{i_0} \cdot \ldots \cdot s_{i_n} \mid i_0 < \cdots < i_n < \mathrm{lh}(s)\}.$$

With this notation and terminology, Schur's result can be formulated in the following way.

Theorem (Schur's lemma,1912)**.** *For every finite coloring $c\colon \mathbb{N} \to r$ of the natural numbers there exists a sequence $\bar{s} = (a, b) \in \mathbb{N}^2$ of length 2 such that $\langle \bar{s} \rangle$ is monochromatic.*

It is natural to wonder whether one can improve this result to obtain sequences of greater length with monochromatic span. The first improvement in this direction appeared in the early 1930s, showing that the statement remains true if we require the sequence to have arbitrary large finite length (this result has been proven independently by different mathematicians almost simultaneously, but it is usually attributed to Rado). In contrast, the question of whether one can obtain an infinite sequence with monochromatic span remained open for a very long time, until Hindman solved in 1974 the conjecture [80].

Theorem (Hindman's finite sum theorem, 1974)**.** *For every finite coloring* $c\colon \mathbb{N} \to r$ *of the natural numbers there exists an infinite sequence* $\bar{s} = (s_i)_{i<\omega} \in \mathbb{N}^\omega$ *such that* $\langle \bar{s} \rangle$ *is monochromatic.*

Around the same time, Galvin and Glazer were working on a simple proof of Hindman's theorem based on a new method combining topological dynamics and combinatorics. The key idea of this method is to find idempotents in the space of ultrafilters over $\mathbb{N}$, i.e. its Stone-Čech compactification $\beta\mathbb{N}$, and then turn these idempotents into theorems in combinatorics. This revolutionary method proved to be successful, to the point that it is now a standard argument in the field (and it is also one of the main arguments used in this paper). See also [81, Section 5.6 (notes)] for a good explanation of the contribution of Galvin and Glazer.

The proof of Galvin and Glazer can easily be adapted to obtain a strengthening of the original Hindman's theorem. In fact, one can further require that the span of the sequence $\bar{s}$ be contained in the span of another sequence $\bar{t}$ chosen in advance. Moreover, it does not use anything specific to the structure of $\mathbb{N}$ except for the associativity of its operation, and thus it works for arbitrary semigroups (see e.g. [81, Corollary 5.15]).

Let $(S, \cdot)$ be a partial semigroup, and consider two sequences $\bar{s} = (s_i)_{i<\alpha}$ and $\bar{t} = (t_i)_{i<\beta} \in S^{\leq\omega}$ of elements of S. We say that $\bar{s}$ is **extracted** from $\bar{t}$, or $\bar{s} \leq \bar{t}$, if there is an increasing sequence $(i_n)_{n\in\gamma}$ of natural numbers $\leq \mathrm{lh}(\bar{t})$ such that $s_n \in \langle t_{i_n}, \ldots, t_{(i_{n+1})-1} \rangle$.

Theorem (Hindman's theorem, stronger version [81])**.** *For every semigroup* S*, every infinite sequence* $\bar{t} = (t_i)_{i<\omega} \in S^\omega$*, and every finite coloring* $c\colon \langle \bar{t} \rangle \to r$ *there exists an infinite sequence* $\bar{s} \leq \bar{t}$ *with monochromatic span* $\langle \bar{s} \rangle$*.*

For those familiar with the subject, the difference between the weak and the strong version (applied to $S = \mathbb{N}$) of Hindman's theorem coincides with the difference between weak partition regularity and (full) partition regularity of IP-sets.

Passing from the natural numbers $\mathbb{N}$ to arbitrary (partial) semigroups has been the key step in extending the range of applicability of combinatorics outside number theory. Two partial semigroups, in particular, immediately gained a central role in the field. Let A be a set, called the alphabet. The **semigroup of words** $W_A = (A^{<\omega}, ^\frown)$ over A is the free semigroup generated by A. In other words, $W_A = (A^{<\omega}, ^\frown)$ is the set of all finite sequences of elements of A with concatenation of sequences as operation. The **partial semigroup of located words** $(\mathrm{FIN}_A, ^\frown)$ on A is the partial subsemigroup of W_Y for $Y = \omega \times A$ consisting of those words $((n_0, a_0), \ldots, (n_i, a_i)) \in W_Y$ such that $n_0 < \cdots < n_i$.

Focusing on these two partial semigroups, many different generalizations of Hindman's theorem (like Carlson's theorem, Gowers' theorem, Furstenberg-Katznelson Ramsey theorem, ...) found soon applications in other branches of mathematics, like functional analysis and ergodic theory. All these theorems extend Hindman's theorem in a similar way, that is, by finding a monochromatic span of a sequence which is furthermore closed under a set of endomorphisms of S. Every set of endomorphisms, when closed under composition and together with the identity function, forms a monoid which naturally acts on the partial semigroup itself. For this reason, we can reduce to work with actions of monoids and introduce the following new class of spans.

Every action of a monoid M on a (partial) semigroup S defines an operation between FIN_M and S^ω: given $\bar{s} = (s_i)_{i<\omega} \in S^\omega$ and $w = (n_i, a_i)_{i<h} \in \mathrm{FIN}_M$, we formally define $w(\bar{s}) = a_0 s_{n_0} \cdots a_h s_{n_h}$. When the semigroup is partial, this expression is not necessarily well-defined: we say that $\bar{s}$ is **basic** if $w(\bar{s})$ is well-defined for every $w \in \mathrm{FIN}_M$.

Definition. Let M be a monoid acting by endomorphisms on a partial semigroup S, and let $\bar{s}$ be a sequence of elements of S. Given a family $C \subseteq \mathrm{FIN}_M$, we define the (combinatorial) C**-span** of $\bar{s}$ as the set

$$\langle \bar{s} \rangle_C = C\bar{s} = \left\{ m_0\, s_{i_0} \cdots m_n\, s_{i_n} \mid (i_h, m_h)_{h \le n} \in C \right\}.$$

We also define the M-span $\langle \bar{s} \rangle_M$ as the set $\mathcal{V}_M \bar{s} = \{w(\bar{s}) \mid w \in \mathcal{V}_M\}$ for $\mathcal{V}_M$ the set of located words of FIN_M containing the identity 1_M (these are called variable located words).

Similarly, we can extend to the new setup the notion of extraction of a sequence.

Definition. Let M be a monoid acting by endomorphism on a partial semigroup $(S, \cdot)$, and let $\bar{s} = (s_i)_{i<\mathrm{lh}(s)}$ and $\bar{t} = (t_i)_{i<\mathrm{lh}(t)}$ be sequences of elements of S. We say that $\bar{s}$ is **extracted** from $\bar{t}$ by M, or $\bar{s} \le_M \bar{t}$, if there is an increasing sequence $(i_n)_{n\in\omega}$ of natural numbers such that $s_n \in \langle t_{i_n}, \ldots, t_{(i_{n+1})-1} \rangle_M$.

These tools allow us to restate many theorems in a very convenient way. For example, the strong version of Hindman's theorem corresponds to the existence of a monochromatic M-span for every action of the trivial monoid $\mathcal{M} = \{1\}$ on an arbitrary semigroup S. We can restate in a similar way other known theorems that we are going to generalize, starting with Carlson's theorem.

Consider a finite alphabet A with a variable x outside A. For every element $a \in A$, we may define a function $f_a : W_{A\cup\{x\}} \to W_A$ that associates to every word $w \in W_{A\cup\{x\}}$ the word $w[a/x] \in W_A$ obtained by replacing each instance of x with a. We call elements of $W_{A\cup\{x\}} \setminus W_A$ **variable words**. It is not difficult to see that each function f_a is an endomorphism that maps $W_{A\cup\{x\}}$ into W_A, and that $f_a(w) \neq w$ if and only if w is a variable word. Furthermore, $f_a \circ f_b = f_b$ for every $a, b \in A$. Thus words over an alphabet with a variable can be seen as a particular case of a more general phenomenon.

Every monoid $(M, *, 1)$ acts on itself by multiplication, and thus it also acts coordinate-wise on the semigroup of words W_M over the alphabet M. If $M = A \cup \{x\}$ and we define the operation on M as $ab = b$ for every $a, b \in A$ and let x be the identity of the monoid, then we get that the coordinate-wise action of M on W_M is exactly the set of endomorphism defined above. We call **Carlson's monoid** a monoid $(M, *, 1)$ such that $ab = b$ for every $a, b \in M \setminus \{1\}$.

Carlson's theorem [31] can then be restated in the following way.

Theorem (Carlson's theorem, 1988)**.** *For every finite Carlson's monoid M, every finite coloring of W_M, and every infinite sequence $\bar{t} \in (W_M)^\omega$ of (variable) words, there exists an infinite sequence $\bar{s} \le_M \bar{t}$ with monochromatic M-span $\langle \bar{s} \rangle_M$.*

Let us now move to Gowers' theorem. It focuses on the partial semigroup of located words over a natural number $k \in \mathbb{N}$, using endomorphisms of the form $f_i \colon \mathrm{FIN}_k \to \mathrm{FIN}_{k-i}$ defined by $f_i((n_j, a_j)_{j<h}) = (n_j, \max(a_j - i, 0))_{j<h}$ for

every located word $(n_j, a_j)_{j<h} \in \mathrm{FIN}_k$ and for every $i \leq k$. The aim is to find an infinite basic sequence $\bar{s} \in (\mathrm{FIN}_k \setminus \mathrm{FIN}_{k-1})^\omega$ such that $\langle \bar{s} \rangle_M$ is monochromatic.

Once again, the partial semigroup FIN_k with this set of endomorphism can be seen as a particular case of a more general situation. Given a monoid $(M, *, 1)$, then M acts on itself by multiplication, and thus it acts coordinate-wise on the partial semigroup of located words FIN_M over the alphabet M. Notice that $f_i \circ f_j = f_{i \bar{+} j}$ for $i \bar{+} j = \min(i+j, k)$, thus $\mathcal{F} = (\{f_i \mid i \leq k\}, \circ, f_0)$ is isomorphic to the monoid $M = (\{0, \ldots, k\}, \bar{+}, 0)$. Moreover, if f'_i is the coordinate-wise action of $i \in M$ on FIN_M and $\phi\colon \mathrm{FIN}_k \to \mathrm{FIN}_M$ is the isomorphism defined by $\phi((n_j, a_j)_{j<h}) = (n_j, k - a_j)_{j<h}$, we have $\phi(f_i(w)) = f'_i(\phi(w))$. Thus the action of $\mathcal{F}$ on FIN_k is equivalent (up to isomorphism) to the coordinate-wise action of M on FIN_M. Define a **Gower's monoid** as a monoid (isomorphic to one) of the form $G_{k+1} = (\{0, \ldots, k\}, \bar{+}, 0)$ for some $k \in \mathbb{N}$.

Under this notation, Gowers' theorem [74] is the following.

Theorem (Gowers' FIN_k theorem, 1992)**.** *For every (finite) Gowers' monoid M and every finite coloring of FIN_M, there is an infinite basic sequence of variable located words $\bar{s} \in (\mathrm{FIN}_M)^\omega$ with monochromatic M-span $\langle \bar{s} \rangle_M$.*

Solecki proved in [142] a powerful generalization of Gowers' theorem, by extending both the class of partial semigroups and the class of monoids for which the statement of Gowers' theorem hold.

The **partial semigroup of located words** $\langle (X_n)_{n\in\omega} \rangle$ on the family of alphabets $(X_n)_{n\in\omega}$ is the partial subsemigroup of W_Y for $Y = \bigcup_{n\in\omega} X_n$ consisting of all those sequences $x_1 {}^\frown \ldots {}^\frown x_n \in W_Y$ for which there exists $i_1 < \ldots < i_n \in \omega$ such that $x_k \in X_{i_k}$. We say that $(X_n)_{n\in\omega}$ is a uniform sequence of pointed M-sets if M acts on $Y = \bigcup_{n\in\omega} X_n$, and for every $i < \omega$ there is a fixed element $x_i \in X_i$ called a variable such that $Mx_i = X_i$.

Given a monoid M, define $\mathbb{X}(M)$= $\{aM \mid a \in M\}$ to be the set of all principal right ideals of M. We say that M is **almost $\mathcal{R}$-trivial** if for every distinct $a, b \in M$, if $aM = bM$ then $Ma = \{a\}$ (and $Mb = \{b\}$).

Theorem (Solecki's $\mathbb{X}(M)$ theorem [142])**.** *For all finite almost $\mathcal{R}$-trivial monoids M with linear $\mathbb{X}(M)$, for all uniform sequences of pointed M-sets $(X_n)_{n\in\omega}$, and for all finite colorings of the partial semigroup $\langle (X_n)_{n\in\omega} \rangle$, there is a basic sequence of variable words $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ with monochromatic M-span $\langle \bar{s} \rangle_M$.*

Notice that all Gowers' monoids are almost $\mathcal{R}$-trivial and have linear $\mathbb{X}(M)$, and FIN_M can be written as $\langle (X_n)_{n\in\omega} \rangle$ for $X_n = \{n\} \times M$. Thus this theorem extends Gowers' theorems. With some additional work, one can also derive from this theorem the strong version of Hindman's theorem and Carlson's theorem (see Section 4.1 and Main Theorem 16). Nevertheless, we show that it is possible to improve this result so that it directly subsumes Hindman's theorem and Carlson's theorem (and Solecki's $\mathbb{X}(M)$ theorem). Our contribution is twofold: we extend Solecki's theorem by working with the class of *all* partial semigroups; and we extend the class of monoids to the biggest possible class for which a similar statement can be obtained (see Main Theorem 17).

We say that a monoid M is aperiodic if for every $a \in M$ there is $n \in \omega$ such that $a^{n+1} = a^n$. This notion has been widely studied in finite automata theory because of

Schützenberger's theorem [134]. Surprisingly, it appears to be the key notion in this area of combinatorics as well.

Main Theorem 15 (Theorem 5.2.11 and Proposition 4.1.11)**.** *For every finite aperiodic monoid M with linear $\mathbb{X}(M)$, for every partial semigroup S, for every finite coloring c of S, and for every infinite basic sequence $\bar{t} \in S^\omega$, there is an infinite sequence $\bar{s} \leq_M \bar{t}$ with monochromatic M-span $\langle \bar{s} \rangle_M$.*

Notice that every finite almost $\mathcal{R}$-trivial monoid is aperiodic, while there are (even finite) aperiodic monoids that are not almost $\mathcal{R}$-trivial — see Proposition 4.2.6 and the ensuing paragraphs. Morevoer, if $\bar{t} = (x_n)_{n\in\omega}$ are the variables of $(X_n)_{n\in\omega}$, then $\bar{t}$ is basic and every sequence $\bar{s} \leq_M \bar{t}$ consists of variable words (but being extracted from $\bar{t}$ gives a strictly stronger notion: see Remark 4.1.7).

One may wonder whether we can further extend the previous theorem. We already considered the class of *all* partial semigroups, so we can not improve the result in that direction. (Unless we drop associativity, which might have interesting applications e.g. to work with the exponential on the natural numbers like in [51], but it is outside the scope of this work.) But we also show that, as anticipated, we can not extend it to more monoids either.

Before coming to that, however, we need to address one problem. A priori, it might seem possible that each theorem among the ones considered above is true for a different class of monoids. Luckily this is not the case, and all previous statements isolate the same class of monoids. This fact is surprising, since it implies that results on located words (that are often treated as strictly stronger than results on words) can actually be derived from results on words. For example, the Bergelson-Blass-Hindman theorem on located words [18] can be derived directly from Carlson's theorem on variable words [31]. Maybe even more surprisingly, Schur's lemma has the same strength (in terms of the class of monoids isolated by it) as Hindman's theorem, or as any other theorem we listed.

Main Theorem 16 (Proposition 4.1.11 and Theorem 5.2.13)**.** *The following are equivalent for a monoid M:*

(a) For every (partial) semigroup S on which M acts by endomorphisms, every (basic) sequence $\bar{t} \in S^\omega$, and every finite coloring of S there is an infinite sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle_M$ is monochromatic (cf. ***Hindman's theorem*** *and* ***Main Theorem 15****).*

(b) For all sequences of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for all finite colorings of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence of variable words $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ with monochromatic M-span $\langle \bar{s} \rangle_M$ (cf. ***Solecki's*** *$\mathbb{X}(M)$* ***theorem****).*

(c) For every finite coloring of FIN_M there is a basic sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that $\langle \bar{s} \rangle_M$ is monochromatic (cf. ***Gowers' theorem****).*

(d) For all finite colorings of W_M and for all sequences of variable words $\bar{t} \in (W_M)^\omega$ there is an infinite $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s} \rangle_M$ monochromatic (cf. ***Carlson's theorem****).*

(e) *For every finite coloring of W_M there are two variable words $\bar{s} = (s_0, s_1) \in (W_M)^2$ such that $\langle \bar{s} \rangle_M$ monochromatic (cf.* ***Schur's lemma****).*

With this theorem in mind, we can start analyzing the class of monoids defined by these statements. We call a monoid **Ramsey** if any of the conditions above hold. Thus Hindman's theorem (in its strong version for semigroups), Carlson's theorem, and Gowers' theorem can all be interpreted as saying that certain monoids (respectively: the trivial monoid, the Carlson's monoids, and the Gowers' monoids) are Ramsey.

Ramsey monoids have been introduced and studied by Solecki in [142], using point (b) of Main Theorem 16 as their definition. He also provided the first necessary condition for a monoid to be Ramsey: every Ramsey monoid has linear $\mathbb{X}(M)$. Continuing this work, we provide a purely algebraic characterization of Ramsey monoids, showing in particular that all sufficient conditions of Main Theorem 15 are necessary as well.

Main Theorem 17 (Theorems 4.4.7 and 5.2.11)**.** *A monoid is Ramsey if and only if it is finite, aperiodic, and $\mathbb{X}(M)$ is linear.*

The other classes of theorems (and monoids) we consider differ from the above ones because of the monochromatic subsets they aim at finding. Let us first introduce another important generalization of Hindman's theorem: Furstenberg-Katznelson's Ramsey theorem [68]. Given two disjoint sets A, B and a variable x outside $A \cup B$, the (generalized) Furstenberg-Katznelson's Ramsey theorem studies the partial semigroup of located words over the alphabet $X = A \cup B \cup \{x\}$. Given $w \in W_{A\cup B}$, define $\overline{w}$ as the word obtained from w by the reduction $b \to \emptyset$ and $aa \to a$ for every $a \in A$ and $b \in B$. Also, denote with $w[a/x]$ the word obtained by replacing every occurrence of x with a.

Theorem (Generalized Furstenberg-Katznelson's Ramsey theorem, 1989)**.** *For every pair of disjoint sets A, B together with a variable $x \notin A \cup B$, for every finite coloring $c\colon \mathrm{FIN}_{A\cup B} \to r$ of the partial semigroup of located words over $A \cup B$, and for every finite $F \subseteq A^{<\omega}$, there exists an infinite sequence of variable words $\bar{w} \in (\mathrm{FIN}_{B\cup\{x\}} \setminus \mathrm{FIN}_B)^\omega$ such that the set*

$$\big\{w_{i_0}[c_0/x] \cdots w_{i_n}[c_n/x] \mid n \in \omega, i_0 < \cdots < i_n, \overline{c_0 \cdots c_n} = f\big\}$$

is monochromatic for every $f \in F$.

Once again, this theorem can be seen as a particular case of a more general form (thanks to Solecki). Given a monoid M, define $\mathbb{Y}(M) \subseteq \mathscr{P}(\mathbb{X}(M))$ as the family of all non-empty chains (i.e. linear suborders) of $(\mathbb{X}(M), \subseteq)$. Given $x, y \in \mathbb{Y}(M)$, define $x \leq_{\mathbb{Y}} y$ if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x. Then, M acts by endomorphism on $(\mathbb{X}(M), \subseteq)$, and thus also on $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$. Define also $\langle M\mathbf{y}\rangle$, with operation $\vee$, as the semigroup freely generated by the set $M\mathbf{y} = \{m\mathbf{y} \in \mathbb{Y}(M) \mid m \in M\}$ modulo the relations

$$p \vee q = q = q \vee p \text{ for } p \leq_{\mathbb{Y}} q.$$

Given $\mathbf{y} \in \mathbb{Y}(M)$, every element $f \in \langle M\mathbf{y}\rangle$ can be seen as an equivalence class $f = \{(n_i, m_i)_{i\leq h} \in \mathrm{FIN}_M \mid m_0\mathbf{y} \vee \cdots \vee m_h\mathbf{y} = f\}$, and thus if M acts on a partial semigroup S we can define the f-span $\langle \bar{s}\rangle_f$ of a sequence $\bar{s} \in S^\omega$ as the set

$$\langle \bar{s}\rangle_f = f\bar{s} = \big\{m_0\, s_{i_0} \cdots m_n\, s_{i_n} \mid i_0 < \cdots < i_n < \omega, m_0\mathbf{y} \vee \cdots \vee m_h\mathbf{y} = f\big\}.$$

Once the above notation is settled, let us go back to Furstenberg-Katznelson's Ramsey theorem. Given two disjoint sets A, B and a variable x outside $A \cup B$, we can define a monoid operation on $M = A \cup B \cup \{x\}$: we let x be the identity of M, and for every $a, a' \in A$ and $b, b' \in B$ we let $aa' = a$, $ab = ba = b$, and $bb' = b'$. We call these monoids **Furstenberg-Katznelson's monoids**. The action of M on $\mathrm{FIN}_{B\cup\{x\}}$ is by substitution: for every $a \in M$ and $w \in \mathrm{FIN}_{B\cup\{x\}}$, we have $a(w) = w[a/x]$. Also, $M \setminus \{x\}$ acts on FIN_M in the same way as it acts on $\mathrm{FIN}_{B\cup\{x\}}$, because every $a \in A$ behaves like the variable x since $ma = m$ for every $m \in M \setminus \{x\}$. So it is irrelevant whether we obtain a sequence $\bar{s} \in \mathrm{FIN}_{B\cup\{x\}}$ or a sequence $\bar{s}' \in \mathrm{FIN}_M$, since from the latter we can always pass to the former by replacing every $a \in s'_i \cap A$ with x. Finally, notice that $\mathbb{X}(M) = \{B, M\} \cup \bigcup_{a\in A}\{B \cup \{a\}\}$. Let $\mathbf{y} = \{B, M\} \in \mathbb{Y}(M)$: then we have $(c_0, \dots, c_n), (e_0, \dots, e_m) \in W_{A\cup B}$ and $\overline{c_0 \cdots c_n} = \overline{e_0 \cdots e_m}$ if and only if $c_0\mathbf{y} \vee \dots \vee c_n\mathbf{y} = e_0\mathbf{y} \vee \dots \vee e_m\mathbf{y} \in \langle M\mathbf{y} \setminus \{\mathbf{y}\}\rangle$.

Then the Furstenberg-Katznelson's Ramsey theorem can be restated as follows: "*For every finite Furstenberg-Katznelson's monoid M, every finite coloring c of FIN_M, and every finite $F \subseteq \langle M\mathbf{y} \setminus \{\mathbf{y}\}\rangle$, there exists an infinite sequence of variable words $\bar{w} \in (\mathrm{FIN}_M)^\omega$ with monochromatic f-span $\langle \bar{s}\rangle_f$ for every $f \in F$*".

In [142], Solecki proved a much stronger version of this theorem.

Theorem (Solecki's $\mathbb{Y}(M)$ theorem [142])**.** *For all finite almost $\mathcal{R}$-trivial monoids M, maximal $\mathbf{y} \in \mathbb{Y}(M)$ and finite $F \subseteq M\mathbf{y}$, for all uniform sequences of pointed M-sets $(X_n)_{n\in\omega}$, and for all finite colorings of $\langle (X_n)_{n\in\omega}\rangle$, there is a basic sequence of variable words $\bar{s}$ in $(\langle (X_n)_{n\in\omega}\rangle)^\omega$ with monochromatic f-span $\langle \bar{s}\rangle_f$ for all $f \in F$.*

Our second goal is to study the class of monoids that satisfy the above statement. Our first result in this direction improves Solecki's $\mathbb{Y}(M)$ theorem, in that we consider all partial semigroups and a wider class of monoids, including certain infinite monoids. Define $\mathbb{X}_{\mathcal{R}}(M) = \{aM \in \mathbb{X}(M) \mid [a]_{\mathcal{R}} \text{ is non-trivial}\}$. Notice that if M is aperiodic, then $\mathbb{X}_{\mathcal{R}}(M)$ has size one (see Proposition 4.2.6).

Main Theorem 18 (Theorem 5.3.19 and Proposition 4.1.14)**.** *Let M be a (possibly infinite) aperiodic monoid such that each $\mathcal{R}$-class is finite, $\mathbb{X}(M)$ contains no infinite chains, and $\mathbb{X}_{\mathcal{R}}(M)$ is linear. For all maximal $\mathbf{y} \in \mathbb{Y}(M)$ and finite $F \subseteq M\mathbf{y}$, for all partial semigroup S on which M acts by endomorphism, for all finite coloring of S, and for all basic $\bar{t} \in S^\omega$, there is an infinite sequence $\bar{s} \leq_M \bar{t}$ with monochromatic f-span $\langle \bar{s}\rangle_f$ for every $f \in F$.*

In Section 4.6, Theorem 5.3.12, and Section 5.4 we prove that similar results hold for other classes of monoids satisfying different hypotheses, hence the hypothesis above are not optimal. However, some of them are indeed necessary. First of all, as it happens for Ramsey monoids, we show that different statements lead to the same class of monoids.

Proposition (Proposition 4.1.14)**.** *Given a monoid M, a maximal element $\mathbf{y} \in \mathbb{Y}(M)$, and a finite $F \subseteq \langle M\mathbf{y}\rangle$, the following are equivalent:*

(a) For every (partial) semigroup S on which M acts by endomorphisms, every (basic) sequence $\bar{t} \in S^\omega$, and every finite coloring of S, there is a sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s}\rangle_f$ is monochromatic for every $f \in F$.

(b) *For every uniform sequence of pointed M-sets $(X_n)_{n\in\omega}$ and every finite coloring of $\langle (X_n)_{n\in\omega}\rangle$ there is a basic sequence of variable words $\bar{s}$ in $(\langle (X_n)_{n\in\omega}\rangle)^\omega$ with monochromatic f-span $\langle \bar{s}\rangle_f$ for every $f \in F$.*

(c) *For all finite coloring of FIN_M there is a basic sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that $\langle \bar{s}\rangle_f$ is monochromatic for every $f \in F$.*

(d) *There is a rapidly increasing sequence of variable words $\bar{t} \in (W_M)^\omega$ such that for all finite coloring of W_M there is a sequence $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s}\rangle_f$ monochromatic for every $f \in F$.*

We say that a monoid M is $\mathbb{Y}$**-controllable** if for every maximal element $\mathbf{y} \in \mathbb{Y}(M)$ and for every finite $F \subseteq \langle M\mathbf{y}\rangle$, one of the equivalent conditions above holds.

Main Theorem 18 provides sufficient conditions for a monoid to be $\mathbb{Y}$-controllable. We also prove that the following conditions are necessary.

Main Theorem 19 (Propositions 5.2.3 and 5.2.4)**.** *Let M be a $\mathbb{Y}$-controllable monoid. Then M is aperiodic and $(M\mathbf{y}, \leq_{\mathbb{Y}})$ contains no infinite chains for every maximal $\mathbf{y} \in \mathbb{Y}(M)$.*

If moreover we have that $\mathbb{X}(M)$ is linear, then having only finite $\mathcal{R}$-classes becomes necessary as well (see Proposition 5.2.6). As a corollary, we get the following.

Proposition (Corollary 5.2.12)**.** *A monoid is Ramsey if and only if it is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear.*

Finally, we analyze some local (infinite) versions of the theorems stated before. Let M be a (possibly infinite) monoid acting by endomorphisms on a partial semigroup S, and let $\bar{s}$ be a sequence of elements of S. Given a sequence $(M_i)_{i\in\omega}$ of finite subsets of M, we define a local span of $\bar{s}$ by allowing only elements of M_i to act on the i-th coordinate of $\bar{s}$. In other words, using previous notation, this local span is the L-span $\langle \bar{s}\rangle_L$ for $L = \{(n_i, m_i)_{i<h} \in \mathrm{FIN}_M \mid m_i \in M_{n_i}\}$. For ease of notation, given a family $C \subseteq \mathrm{FIN}^{<\omega}$, we define

$$\langle \bar{s}\rangle^C_{(M_i)_{i\in\omega}} = \langle \bar{s}\rangle_C \cap \langle \bar{s}\rangle_L = \big\{ m_0\, s_{i_0} \cdots m_n\, s_{i_n} \mid m_h \in M_{i_h}, ((i_j, m_j))_{j\leq h} \in C \big\}.$$

When $C = \mathcal{V}_M$ is the set of variable located words of FIN_M, we just write $\langle \bar{s}\rangle_{(M_i)_{i\in\omega}}$.

Theorem (Infinite Carlson's theorem [31])**.** *For every (possibly infinite) Carlson's monoid M, every sequence $(M_i)_{i<\omega}$ of finite subsets of M, every finite coloring of W_M, and every infinite sequence $\bar{t} \in (W_M)^\omega$ of (variable) words, there exists an infinite sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s}\rangle_{(M_i)_{i<\omega}}$ is monochromatic.*

Theorem (Infinite Furstenberg-Katznelson's Ramsey theorem [68])**.** *For every (possibly infinite) Furstenberg-Katznelson's monoid M, every sequence $(M_i)_{i<\omega}$ of finite subsets of M, every finite coloring c of FIN_M, and every finite $F \subseteq \langle M\mathbf{y} \setminus \{\mathbf{y}\}\rangle$ there exists an infinite sequence of variable words $\bar{w} \in (\mathrm{FIN}_M)^\omega$ such that $\langle \bar{s}\rangle^f_{(M_i)_{i<\omega}}$ is monochromatic for every $f \in F$.*

These two theorems isolate two distinct classes of monoids extending the ones of Ramsey and $\mathbb{Y}$-controllable monoids.

Definition. A monoid M is called **locally Ramsey** if for every (partial) semigroup S on which M acts by endomorphisms, every (basic) sequence $\bar{t} \in S^\omega$, every finite coloring of S, and every family $(M_i)_{i\in\omega}$ of finite subsets of M, there is a sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle_{(M_i)_{i<\omega}}$ is monochromatic.

Similarly, we define a local version of $\mathbb{Y}$-controllable monoids.

Definition. A monoid M is said **locally $\mathbb{Y}$-controllable** if for every maximal element $\mathbf{y} \in \mathbb{Y}(M)$, every finite $F \subseteq \langle M\mathbf{y} \rangle$, every family $(M_i)_{i\in\omega}$ of finite subsets of M, every (partial) semigroup S on which M acts by endomorphisms, and every (basic) sequence $\bar{t} \in S^\omega$, for every finite coloring of S, there is a sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle^f_{(M_i)_{i<\omega}}$ is monochromatic for every $f \in F$.

As for the non-local case, these definitions could be stated using smaller classes of (partial) semigroups (see Propositions 5.1.1 and 5.1.3). In this fashion, the infinite Carlson's theorem and the infinite Furstenberg-Katznelson's Ramsey theorem are indeed corollaries of the statements that certain (classes of) monoids are locally Ramsey or locally $\mathbb{Y}$-controllable.

Our main contributions in this direction are the following. We provide a full algebraic characterization of locally Ramsey monoids. And we provide some necessary algebraic conditions and some sufficient algebraic conditions for a monoid to be locally $\mathbb{Y}$-controllable. The sufficient conditions for, respectively, locally Ramsey and locally $\mathbb{Y}$-controllable monoids, in turn, extend the infinite Carlson's theorem and the infinite Furstenberg-Katznelson's Ramsey theorem, respectively. It turns out that there is one specific algebraic condition that seems to distinguish the local classes from the non-local ones: the possibility of having infinite $\mathcal{R}$-classes.

Main Theorem 20 (Theorem 5.3.20)**.** *A (possibly infinite) monoid is locally Ramsey if and only if it is aperiodic and $\mathbb{X}(M)$ is finite and linear.*

Main Theorem 21 (Propositions 5.2.3 and 5.2.4 and Theorem 5.3.18)**.** *Suppose that the (possibly infinite) monoid M is aperiodic, $\mathbb{X}(M)$ contains no infinite chains, and $\mathbb{X}_{\mathcal{R}}(M)$ is linear and finite. Then M is locally $\mathbb{Y}$-controllable.*

Conversely, if M is locally $\mathbb{Y}$-controllable, then it is aperiodic and $(M\mathbf{y}, \leq_{\mathbb{Y}})$ contains no infinite chains for every maximal $\mathbf{y} \in \mathbb{Y}(M)$.

As a corollary, we obtain that a monoid M is locally Ramsey if and only if it is locally $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear. Also, M is Ramsey if and only if it is locally Ramsey and every $\mathcal{R}$-class is finite. Finally, we obtain the following:

Proposition (Corollary 5.3.5)**.** *If M is locally $\mathbb{Y}$-controllable and $[a]_{\mathcal{R}}$ is finite for every $a \in M$, then M is $\mathbb{Y}$-controllable.*

In Theorem 5.3.12 and Section 5.4 we also present other results along the same lines, as well as examples of locally $\mathbb{Y}$-controllable monoids that do not satisfy some of the sufficient conditions of Main Theorem 21.

Part I

Generalized Descriptive Set Theory

Chapter 1

Generalized Polish spaces at regular uncountable cardinals

1.1 Polish-like spaces

1.1.1 Spaces, games, and metrics

In this chapter, we study Polish-like classes of spaces in generalized descriptive set theory on an uncountable regular cardinal κ. Throughout the chapter we work in ZFC and assume that κ is an uncountable regular cardinal satisfying $2^{<\kappa} = \kappa$ (equivalently: $\kappa^{<\kappa} = \kappa$). Unless otherwise specified, from now on all topological spaces are assumed to be regular and Hausdorff, and we will refer to them just as "spaces". In this framework, (classical) Polish spaces can equivalently be defined as:

(Pol. 1) completely metrizable second-countable spaces;

(Pol. 2) strong Choquet second-countable spaces, where strong Choquet means that the second player has a winning strategy in a suitable topological game, called strong Choquet game, on the given space (see below for the precise definition).

The two spaces lying at the core of generalized descriptive set theory are:

1. the **generalized Baire space**

   $$^{\kappa}\kappa = \{x \mid x \colon \kappa \to \kappa\}$$

   of all sequences with values in κ and length κ, equipped with the so-called **bounded topology** τ_b, i.e. the topology generated by the sets of the form

   $$\mathbf{N}_s = \{x \in {}^{\kappa}\kappa \mid s \subseteq x\}$$

   with s ranging in the set $^{<\kappa}\kappa$ of sequences with values in κ and length $<\kappa$;

2. the **generalized Cantor space**

   $$^{\kappa}2 = \{x \mid x \colon \kappa \to 2\}$$

   of all binary sequences of length κ, which is a closed subset of $^{\kappa}\kappa$ and is thus equipped with the relative topology.

The assumption $\kappa^{<\kappa} = \kappa$ ensures then that the two spaces ${}^{\kappa}\kappa$ and ${}^{\kappa}2$ have weight κ.

Consider now pairs $(X, \mathscr{B})$ with X a nonempty set and $\mathscr{B}$ a σ-algebra on X. Such pairs are called Borel spaces if $\mathscr{B}$ is countably generated and separates points[1] or, equivalently, if there is a metrizable second-countable topology on X generating $\mathscr{B}$ as its Borel σ-algebra. Standard Borel spaces can then equivalently be defined as:

(St.Bor. 1) Borel spaces $(X, \mathscr{B})$ such that there is a Polish topology on X generating $\mathscr{B}$ as its Borel σ-algebra;

(St.Bor. 2) Borel spaces which are Borel isomorphic to a Borel subset of ${}^{\omega}\omega$ (or any other uncountable Polish space, including ${}^{\omega}2$).

In [118], a notion of standard κ-Borel space was introduced by straightforwardly generalizing the definition given by (St.Bor. 2). Call a pair $(X, \mathscr{B})$ a κ-Borel space if $\mathscr{B}$ is a κ^+-algebra on X which separates points and admits a κ-sized basis. The elements of $\mathscr{B}$ are then called **κ-Borel** sets of X. If $(X, \mathscr{B})$ is a κ-Borel space and $Y \subseteq X$, then setting $\mathscr{B} \restriction Y = \{B \cap Y \mid B \in \mathscr{B}\}$ we get that $(Y, \mathscr{B} \restriction Y)$ is again a κ-Borel space. If $(X, \mathscr{B})$ and $(X', \mathscr{B}')$ are κ-Borel spaces, we say that a function $f \colon X \to X'$ is **κ-Borel (measurable)** if $f^{-1}(B) \in \mathscr{B}$ for all $B \in \mathscr{B}'$. A **κ-Borel isomorphism** between $(X, \mathscr{B})$ and $(X', \mathscr{B}')$ is a bijection f such that both f and f^{-1} are κ-Borel; two κ-Borel spaces are then **κ-Borel isomorphic** if there is a κ-Borel isomorphism between them. Finally, a **κ-Borel embedding** $f \colon X \to X'$ is an injective function which is a κ-Borel isomorphism between $(X, \mathscr{B})$ and $(f(X), \mathscr{B}' \restriction f(X))$. Notice that every T_0 topological space (X, τ) of weight κ can be seen as a κ-Borel space in a canonical way by pairing it with the collection

$$\mathsf{Bor}_{\kappa}(X, \tau)$$

of all its κ-Borel subsets, i.e. with the smallest κ^+-algebra generated by its topology. (We sometimes remove τ from this notation if clear from the context.) If not specified otherwise, we are always tacitly referring to such κ^+-Borel structure when dealing with κ-Borel isomorphisms and κ-Borel embeddings between topological spaces.

We are now ready to generalize (St.Bor. 2).

Definition 1.1.1. A κ-Borel space $(X, \mathscr{B})$ is **standard**[2] if it is κ-Borel isomorphic to a κ-Borel subset of ${}^{\kappa}\kappa$.

Generalizations of (St.Bor. 1) were instead not considered in [118] because at that time no natural generalization of the concept of a Polish space was introduced yet. But clearly, once we are given a notion of a Polish-like space for κ (e.g. the ones we are going to consider below, namely SC_{κ}-spaces, $f\mathrm{SC}_{\kappa}$-spaces, or $\mathbb{G}$-Polish spaces), we can accordingly generalize (St.Bor. 1) by considering those κ-Borel spaces which admit a topology of the desired type generating $\mathscr{B}$ as its κ^+-algebra of κ-Borel sets. This yields to several formally different definitions: we will however show that they all coincide, so that there is no need to notationally and terminologically distinguish them at this point.

[1] A family $\mathscr{B} \subseteq \mathscr{P}(X)$ separates points if for all distinct $x, y \in X$ there is $B \in \mathscr{B}$ with $x \in B$ and $y \notin B$.

[2] Our definition of a standard κ-Borel space is slightly different yet equivalent to the one considered in [118]. Indeed, the difference is that in [118, Definition 3.6] a $\leq \kappa$-weighted topology generating the standard κ-Borel structure is singled out—see also the discussion after Corollary 1.3.11.

We now move to some natural generalizations of Polishness. In [42], the authors considered a natural generalization of (Pol. 2) to uncountable regular κ in order to obtain a notion of ‘ ‘Polish-like” spaces, called therein strong κ-Choquet spaces. Let us recall the relevant definitions. The (classical) Choquet game $\mathrm{G}_\omega(X)$ on a topological space X is the game where two players I and II alternatively pick nonempty open sets U_n and V_n

$$\begin{array}{c|ccccc} \mathrm{I} & U_0 & & U_1 & & \dots \\ \hline \mathrm{II} & & V_0 & & V_1 & & \dots \end{array}$$

so that $U_{n+1} \subseteq V_n \subseteq U_n$; player II wins the run if the set $\bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} V_n$ is nonempty. The strong Choquet game $\mathrm{G}^s_\omega(X)$ is the variant of $\mathrm{G}_\omega(X)$ where I additional plays points $x_n \in U_n$

$$\begin{array}{c|ccccc} \mathrm{I} & (U_0, x_0) & & (U_1, x_1) & & \dots \\ \hline \mathrm{II} & & V_0 & & V_1 & & \dots \end{array}$$

and II ensures that $x_n \in V_n \subseteq U_n$; the winning condition stays the same.

It is (almost) straightforward to generalize such games to uncountable κ's: just let players I and II play for κ-many rounds, and still declare II as the winner of the run if the final intersection $\bigcap_{\alpha<\kappa} U_\alpha = \bigcap_{\alpha<\kappa} V_\alpha$ is nonempty. However, since $\kappa > \omega$ we now have to decide what should happen at limit levels $\gamma < \kappa$. Firstly, since the space X is not necessarily κ-additive we require U_γ, V_γ to be just open *relatively to what has been played so far*, i.e. relatively to $\bigcap_{\alpha<\gamma} U_\alpha = \bigcap_{\alpha<\gamma} V_\alpha$ (this obviously applies to all rounds with index $\gamma \geq \omega$, not only to the limit ones). A more subtle issue is deciding who wins the game if at some limit $\gamma < \kappa$ we already have $\bigcap_{\alpha<\gamma} U_\alpha = \bigcap_{\alpha<\gamma} V_\alpha = \emptyset$, so that the game cannot continue from that round on. Following [42], the (**strong**) **κ-Choquet game** $\mathrm{G}^{(s)}_\kappa(X)$ on X is defined by letting I win in such situations. In other words, II has to ensure that for all limit $\gamma \leq \kappa$ (thus including, in particular, the final stage $\gamma = \kappa$), the intersection $\bigcap_{\alpha<\gamma} U_\alpha = \bigcap_{\alpha<\gamma} V_\alpha$ is nonempty. This leads to the following definition.

Definition 1.1.2. A space X is called **strong κ-Choquet** (or **SC_κ-space**) if it has[3] weight $\leq \kappa$ and player II has a winning strategy in $\mathrm{G}^s_\kappa(X)$.

The other natural option, not yet considered so far in the literature, is to make the game more fair by deciding that I partially shares the burden of having a nonempty intersection and takes care of limit levels $\gamma < \kappa$. In other words: II wins if he can guarantee that $\bigcap_{\alpha<\kappa} U_\alpha = \bigcap_{\alpha<\kappa} V_\alpha \neq \emptyset$, *provided that for all limit $\gamma < \kappa$ the intersection $\bigcap_{\alpha<\gamma} U_\alpha = \bigcap_{\alpha<\gamma} V_\alpha$ is nonempty* (if this fails at some limit stage before κ, then II automatically wins). We call this version of the Choquet game **fair κ-Choquet game** and denote it by $f\mathrm{G}_\kappa(X)$, while its further variant with player I additionally choosing points is called **strong fair κ-Choquet game** and is denoted by $f\mathrm{G}^s_\kappa(X)$, accordingly.

[3]Notice that we are deliberately allowing our spaces to have weight strictly smaller than κ. Although this might sound unnatural at first glance, it allows us to state some of our results in a more elegant form and is perfectly coherent with what is done in the classical case, where one includes among Polish spaces also those of finite weight.

Definition 1.1.3. A space X is called **strong fair κ-Choquet** (or **$f\mathrm{SC}_\kappa$-space**) if it has weight $\leq \kappa$ and player II has a winning strategy in $f\mathrm{G}^s_\kappa(X)$.

Since it is more difficult for player II to win the strong κ-Choquet game than its fair variant, it is clear from the definition that every SC_κ-space is in particular an $f\mathrm{SC}_\kappa$-space. Moreover, both ${}^\kappa\kappa$ and ${}^\kappa 2$ are trivially SC_κ-spaces (any legal strategy where II plays basic open sets is automatically winning in the corresponding strong κ-Choquet games), and thus they are also $f\mathrm{SC}_\kappa$-spaces.

Remark 1.1.4. Although it is not part of the rules in Choquet-like games, in the above definitions one could equivalently require the players to pick only open sets from any given basis of the topological space (possibly intersected with all previous moves, if the space is not κ-additive)—see [42, Lemma 2.5]. This restriction will turn out to be useful in some of the proofs below.

We next move to generalizations of (Pol. 1). This requires to find suitable analogues of metrics over the real line for spaces that are not necessarily first countable. One solution is to consider metrics over a structure other than $\mathbb{R}$. Consider a totally ordered[4] (Abelian) group

$$\mathbb{G} = \langle G, +_\mathbb{G}, 0_\mathbb{G}, \leq_\mathbb{G} \rangle$$

with **degree** $\mathrm{Deg}(\mathbb{G}) = \kappa$, where $\mathrm{Deg}(\mathbb{G})$ denotes the coinitiality of the positive cone $\mathbb{G}^+ = \{\varepsilon \in \mathbb{G} \mid 0_\mathbb{G} <_\mathbb{G} \varepsilon\}$ of $\mathbb{G}$.[5] A **$\mathbb{G}$-metric** on a nonempty space X is then a function $d\colon X^2 \to \mathbb{G}^+ \cup \{0_\mathbb{G}\}$ satisfying the usual rules of a distance function: for all $x, y, z \in X$

- $d(x, y) = 0_\mathbb{G} \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq_\mathbb{G} d(x, y) +_\mathbb{G} d(y, z)$.

Every $\mathbb{G}$-metric space (X, d) is naturally equipped with the (d-)topology generated by its open balls

$$B_d(x, \varepsilon) = \{y \in X \mid d(x, y) <_\mathbb{G} \varepsilon\},$$

where $x \in X$ and $\varepsilon \in \mathbb{G}^+$. If X is already a topological space, we say that the $\mathbb{G}$-metric d is compatible with the topology of X if the latter coincides with the d-topology. A topological space is called **$\mathbb{G}$-metrizable** if it admits a compatible $\mathbb{G}$-metric.

Let (X, d) be a $\mathbb{G}$-metric space. A sequence[6] $(x_i)_{i<\kappa}$ of points from X is called (d-)Cauchy if

$$\forall \varepsilon \in \mathbb{G}^+ \, \exists \alpha < \kappa \, \forall \beta, \gamma \geq \alpha \, (d(x_\beta, x_\gamma) <_\mathbb{G} \varepsilon).$$

The space (X, d) (or the $\mathbb{G}$-metric d) is Cauchy-complete if every Cauchy sequence $(x_i)_{i<\kappa}$ converges to some (necessarily unique) $x \in X$, that is,

$$\forall \varepsilon \in \mathbb{G}^+ \, \exists \alpha < \kappa \, \forall \beta \geq \alpha \, (d(x_\beta, x) <_\mathbb{G} \varepsilon).$$

We are now ready to generalize (Pol. 1).

[4]This means that the order $\leq_\mathbb{G}$ is linear and translation-invariant (on both sides).

[5]This is also called the **base number** of $\mathbb{G}$ in [71] and the **character** of $\mathbb{G}$ in [139].

[6]Notice that when speaking about Cauchy sequences and Cauchy-completeness we always refer to sequences of length $\kappa = \mathrm{Deg}(\mathbb{G})$.

Definition 1.1.5. A space X is $\mathbb{G}$**-Polish** if it is completely $\mathbb{G}$-metrizable and has weight (equivalently, density character) $\leq \kappa$.

Remark 1.1.6. These definitions are not new. Spaces with generalized metrics taking values in a structure different from $\mathbb{R}$ have been introduced in [139] and have been widely studied since then, see for example [137, 126, 122]. To the best of our knowledge, the systematic study of *completely* $\mathbb{G}$-metrizable spaces is instead of more recent interest, and so far it has been developed mainly in [71].

Clearly, $\mathbb{G}$-Polish spaces are closed under closed subspaces. Moreover, the space ${}^{\kappa}\kappa$ (endowed with the bounded topology) is always $\mathbb{G}$-Polish, as witnessed by the $\mathbb{G}$-metric

$$d(x,y) = \begin{cases} 0_{\mathbb{G}} & \text{if } x = y \\ r_\alpha & \text{if } x \restriction \alpha = y \restriction \alpha \text{ and } x(\alpha) \neq y(\alpha) \end{cases} \tag{1.1.1}$$

where $(r_\alpha)_{\alpha<\kappa}$ is a strictly decreasing sequence coinitial in $\mathbb{G}^+$ (the choice of such a sequence is irrelevant). It follows that all closed subspaces of ${}^{\kappa}\kappa$, notably including ${}^{\kappa}2$, are $\mathbb{G}$-Polish for any $\mathbb{G}$ as above. Notice also that commutativity of the group operation is not strictly needed in order to define the metric, but it is usually required to ensure that $\mathbb{G}$ itself form a $\mathbb{G}$-metric space with distance function $d(x,y) = |x -_{\mathbb{G}} y|_{\mathbb{G}}$. Sometimes it is further required that $\mathbb{G}$ is Cauchy-complete with respect to the above metric: in this case $\mathbb{G}$ itself would become $\mathbb{G}$-Polish.

We decided to work with the theory of metrics over a totally ordered Abelian group $\mathbb{G}$ since it is arguably the most common choice in literature. However, other choices are possible. For example, Reichel in [126] studied metrics with values in a totally ordered Abelian semigroup with minimum. Coskey and Schlicht in [42] considered (ultra)metrics with values in a linear order (where the operation $+_{\mathbb{G}}$ is the *minimum* function). Or $\mathbb{G}$ can be non-Abelian as well. All these choices would essentially lead to the same results presented here for Abelian groups: see Remark 1.1.24. The reason why we decided to follow the common practice of sticking to totally ordered Abelian groups is that metrics over groups grant most of the properties of standard metrics. For example, it is easy to show that for every $x \in X$ and every sequence $(r_\alpha)_{\alpha<\kappa}$ coinitial in $\mathbb{G}^+$, the family $\{B_d(x, r_\alpha) \mid \alpha < \kappa\}$ is a local basis of x well-ordered by reverse inclusion $\supseteq$. If one wants to consider metrics taking values in less structured sets, like monoids or semigroups, this condition must be explicitly added to the axioms that define the metric (see e.g. [126]).

We conclude this section by addressing another natural question: is there any advantage in choosing a particular totally ordered Abelian group $\mathbb{G}$ over the others? In the countable case, $\mathbb{R}$ plays a key role among all the possible choices of range for the metrics: for example, every connected (real-valued) metric space does not admit a metric with range contained in $\mathbb{Q}$. In the uncountable case, the situation is the opposite: different choices of $\mathbb{G}$ almost always lead to the same class of spaces, making less relevant the actual choice of the range of the metrics. For example, it is well-known that given an uncountable regular cardinal κ and two totally ordered Abelian groups $\mathbb{G}$ and $\mathbb{G}'$ of degree $\mathrm{Deg}(\mathbb{G}) = \mathrm{Deg}(\mathbb{G}') = \kappa$, a space of weight $\leq \kappa$ is $\mathbb{G}$-metrizable, if and only if it is $\mathbb{G}'$-metrizable if and only if it is κ-additive (see Theorem 1.1.12, which is taken from [139], but see also [137]). In Theorem 1.1.21 and Corollary 1.1.22, we show that a similar statement holds for completely $\mathbb{G}$-metrizable spaces, hence the notion of $\mathbb{G}$-Polish as well is independent from the choice of the actual $\mathbb{G}$.

The fact that there is no preferred structure for the range of our generalized metrics implies that every possible generalization-to-level-κ of the reals yields to an example of $\mathbb{G}$-Polish space (as long as this generalization preserves properties like being Cauchy-complete with respect to its canonical metric over itself). For example, this applies to the long reals introduced by Klaua in [97] and studied by Asperó and Tsaprounis in [12], or to the generalization of $\mathbb{R}$ introduced in [71] using the surreal numbers. See also [43] for other examples of $\mathbb{G}$-Polish spaces, as well as methods to construct Cauchy-complete totally ordered fields.

1.1.2 Relationships

The goal of this subsection is to compare the proposed classes of Polish-like (topological) spaces; in Section 1.3 we will extend our analysis to encompass the various generalizations of standard (κ-)Borel spaces.

Definition 1.1.7. Let X be a space. A set $A \subseteq X$ is G^κ_δ if it can be written as a κ-sized intersection of open sets of X.

It is easy to construct $f\mathrm{SC}_\kappa$-subspaces of, say, the generalized Cantor space ${}^\kappa 2$ which are properly G^κ_δ, e.g.

$$\{x \in {}^\kappa 2 \mid \forall \alpha \exists \beta \geq \alpha\, (x(\beta) = 1)\}. \tag{1.1.2}$$

As in the classical case, this specific example is particularly relevant.

Fact 1.1.8. The generalized Baire space ${}^\kappa\kappa$ is homeomorphic to the G^κ_δ subset of ${}^\kappa 2$ from equation (1.1.2).

The following is a well-known fact, but we reprove it here for the reader's convenience.

Lemma 1.1.9. *Every closed subset C of a space*[7] *X of weight $\leq \kappa$ is G^κ_δ in X.*

Proof. Let $\mathcal{B}$ be a basis for X of size $\leq \kappa$. By regularity of X, for every $x \in X \setminus C$ there is $U \in \mathcal{B}$ such that $x \in U$ and $\mathrm{cl}(U) \subseteq X \setminus C$. Thus

$$C = \bigcap \{X \setminus \mathrm{cl}(U) \mid U \in \mathcal{B} \wedge \mathrm{cl}(U) \cap C = \emptyset\}. \qquad \square$$

Proposition 1.1.10. *If X is an $f\mathrm{SC}_\kappa$-space and $Y \subseteq X$ is G^κ_δ, then Y is an $f\mathrm{SC}_\kappa$-space as well.*

Proof. Let $O_\alpha \subseteq X$ be open sets such that $Y = \bigcap_{\alpha<\kappa} O_\alpha$ and fix a winning strategy τ for II in $f\mathrm{G}^s_\kappa(X)$. We define (by recursion on the round) a strategy for II in $f\mathrm{G}^s_\kappa(Y)$ as follows. Suppose that until a certain round $\alpha < \kappa$, player I has played a sequence $\langle (U_\beta, x_\beta) \mid \beta \leq \alpha \rangle$ following the rules of $f\mathrm{G}^s_\kappa(Y)$. Each set U_β is open in Y relatively to the intersection of all previous moves, hence it can be seen as the intersection of Y (and all previous moves of I) with some open set of X. Proceeding recursively, we can thus associate to each U_β a set $\tilde{U}_\beta \subseteq O_\beta$ such that $U_\beta = \tilde{U}_\beta \cap Y$,

[7] Recall that all spaces are tacitly assumed to be regular Hausdorff.

where $\tilde{U}_\beta$ is open in X relatively to the intersection $\bigcap_{\zeta<\beta} \tilde{U}_\zeta$ of all previous sets (this can be done because each O_β is open in X). Then $\langle(\tilde{U}_\beta, x_\beta) \mid \beta \leq \alpha\rangle$ is a legal sequence of moves for I in $f\mathrm{G}^s_\kappa(X)$. If V_α is what τ requires II to play against $\langle(\tilde{U}_\beta, x_\beta) \mid \beta \leq \alpha\rangle$ in $f\mathrm{G}^s_\kappa(X)$, we get that $V_\alpha \cap Y \neq \emptyset$, as witnessed by x_α itself, and $V_\alpha \subseteq \tilde{U}_\alpha \subseteq O_\alpha$: so we can let II respond to I's move in the game $f\mathrm{G}^s_\kappa(Y)$ on Y with $V_\alpha \cap Y$. By construction, the resulting strategy for II is legal with respect to the rules of $f\mathrm{G}^s_\kappa(Y)$. Moreover, if for all limit $\gamma < \kappa$ the intersection $\bigcap_{\alpha<\gamma}(V_\alpha \cap Y)$ is nonempty, then so is $\bigcap_{\alpha<\gamma} V_\alpha$: since τ is winning in $f\mathrm{G}^s_\kappa(X)$, this means that $\bigcap_{\alpha<\kappa} V_\alpha \neq \emptyset$, whence by $V_\alpha \subseteq O_\alpha$ we also get

$$\bigcap_{\alpha<\kappa}(V_\alpha \cap Y) = \left(\bigcap_{\alpha<\kappa} V_\alpha\right) \cap Y = \bigcap_{\alpha<\kappa} V_\alpha \cap \bigcap_{\alpha<\kappa} O_\alpha = \bigcap_{\alpha<\kappa} V_\alpha \neq \emptyset. \qquad \square$$

Definition 1.1.11. Let ν be an infinite cardinal. A topological space X is **ν-additive** if its topology is closed under intersections of length $< \nu$.

In particular, every topological space is ω-additive, and the generalized Baire and Cantor spaces ${}^\kappa\kappa$, ${}^\kappa 2$ are both κ-additive when κ is regular. Moreover, if X is regular and ν-additive for some $\nu > \omega$, then X is zero-dimensional (i.e. it has a basis consisting of clopen sets). Indeed, fix a point $x \in X$ and an open neighborhood U of it. Using regularity, recursively construct a sequence $(U_n)_{n\in\omega}$ of open neighborhoods of x such that $U_0 = U$ and $\mathrm{cl}(U_{n+1}) \subseteq U_n$. Then $V = \bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \mathrm{cl}(U_n)$ contains x, it is closed, and it is also open by ν-additivity (here we use $\nu > \omega$). Thus X admits a basis consisting a clopen sets, as required. Notice also that if X has weight κ, then such a clopen basis can be taken of size κ as well.

Recall also the correspondence between closed subsets of ${}^\kappa\kappa$ and trees on κ. Given an ordinal γ and a nonempty set A, we denote by ${}^\gamma A$ the set of all sequences of length γ and values in A. We then set ${}^{<\kappa}\kappa = \bigcup_{\gamma<\kappa} {}^\gamma\kappa$, and for $s \in {}^{<\kappa}\kappa$ we let $\mathrm{lh}(s)$ be the length of s, that is, the unique ordinal $\gamma < \kappa$ such that $s \in {}^\gamma\kappa$. The concatenation between two sequences s, t is denoted by $s^\frown t$, and to simplify the notation we just write $s^\frown i$ and $i^\frown s$ if $t = \langle i\rangle$ is a sequence of length 1. If $\alpha \leq \mathrm{lh}(s)$, we denote by $s \restriction \alpha$ the restriction of s to its first α-many digits. We write $s \subseteq t$ to say that s is an initial segment of t, that is, $\mathrm{lh}(s) \leq \mathrm{lh}(t)$ and $s = t \restriction \mathrm{lh}(s)$. The sequences s and t are **comparable** if $s \subseteq t$ or $t \subseteq s$, and **incomparable** otherwise. A set $T \subseteq {}^{<\kappa}\kappa$ is called **tree** if it is closed under initial segments. For $\alpha < \kappa$ we denote by $\mathrm{Lev}_\alpha(T)$ the α-th level of the tree T, namely,

$$\mathrm{Lev}_\alpha(T) = \{t \in T \mid \mathrm{lh}(t) = \alpha\}.$$

Given $s \in T$, we also define the localization of T at s as

$$T_s = \{t \in T \mid t \text{ is comparable with } s\}.$$

The bounded topology on ${}^\kappa\kappa$ is the unique topology on such a space with the following property: a set $C \subseteq {}^\kappa\kappa$ is closed if and only it there is some tree $T \subseteq {}^{<\kappa}\kappa$ such that $C = [T]$, where the **body** $[T]$ of the tree T is defined by

$$[T] = \{x \in {}^\kappa\kappa \mid \forall\alpha < \kappa\, (x \restriction \alpha \in T)\}.$$

The above tree T can always be required to be **pruned**, that is, for every $s \in T$ there is $x \in [T]$ such that $s \subseteq x$. Indeed, if C is closed, then the tree $T_C = \{x \restriction \alpha \mid x \in$

$C \wedge \alpha < \kappa\}$ is pruned and such that $C = [T_C]$. Sometimes, one needs to consider a further closure property for trees. We say that the tree T is **$<\kappa$-closed** if for all sequences $s \in {}^{\gamma}\kappa$ with $\gamma < \kappa$ limit, if $s \restriction \alpha \in T$ for all $\alpha < \gamma$, then $s \in T$ as well. A tree T is called **superclosed** if it is pruned and $<\kappa$-closed; this in particular implies that if $s \in T$, then $\mathbf{N}_s \cap [T] \neq \emptyset$ or, equivalently, $[T_s] \neq \emptyset$. Not all closed subsets of ${}^{\kappa}\kappa$ are the body of a superclosed tree: consider e.g. the set

$$X_0 = \{x \in {}^{\kappa}2 \mid |\{\alpha < \kappa \mid x(\alpha) = 0\}| < \aleph_0\}. \tag{1.1.3}$$

This justify the following terminology: a closed $C \subseteq {}^{\kappa}\kappa$ is called **superclosed** if $C = [T]$ for some superclosed tree T.

Sikorski proved in [139, Theorem (x)] that every regular κ-additive space of weight $\leq \kappa$ is homeomorphic to a subspace of ${}^{\kappa}2$, and that the latter is $\mathbb{G}$-metrizable. We can sum up his results as follows, where we additionally use Fact 1.1.8 to further add item (d) to the list of equivalent conditions.

Theorem 1.1.12 ([139, Theorem (viii)-(x)])**.** *For any space X of weight $\leq \kappa$ and any totally ordered Abelian group $\mathbb{G}$ with $\mathrm{Deg}(\mathbb{G}) = \kappa$ the following are equivalent:*

(a) *X is κ-additive;*

(b) *X is $\mathbb{G}$-metrizable;*

(c) *X is homeomorphic to a subset of ${}^{\kappa}2$;*

(d) *X is homeomorphic to a subset of ${}^{\kappa}\kappa$.*

Since conditions (a), (c), and (d) do not refer to $\mathbb{G}$ at all, this shows in particular that the choice of the actual group in the definition of the generalized metric is irrelevant. We are now going to prove that analogous results holds also for $f\mathrm{SC}_\kappa$-spaces, SC_κ-spaces, and $\mathbb{G}$-Polish spaces (see Theorems 1.1.21 and 1.1.32).

Proposition 1.1.13. *Let X be a κ-additive $f\mathrm{SC}_\kappa$-space. Then X is homeomorphic to a closed $C \subseteq {}^{\kappa}\kappa$. If furthermore X is an SC_κ-space, then C can be taken to be superclosed.*

Proof. We prove the two statements simultaneously. Let $(B_\alpha)_{\alpha<\kappa}$ be an enumeration of a clopen basis $\mathcal{B}$ of X, possibly with repetitions. Depending on whether X is an SC_κ-space or just an $f\mathrm{SC}_\kappa$-space, let σ be a winning strategy for player II in $\mathrm{G}^s_\kappa(X)$ or $f\mathrm{G}^s_\kappa(X)$. By Remark 1.1.4, without loss of generality we can assume that the range of σ is contained in $\mathcal{B}$. To simplify the notation, given an ordinal β, let $\mathrm{Succ}(\beta)$ be the collection of all successor ordinals $\leq \beta$. Set also

$$^{<\mathrm{Succ}(\kappa)}\kappa = \{s \in {}^{<\kappa}\kappa \mid \mathrm{lh}(s) \in \mathrm{Succ}(\kappa)\}.$$

We will construct a family of the form

$$\mathcal{F} = \left\{x_s, U_s, V_s, \hat{V}_s \mid s \in {}^{<\mathrm{Succ}(\kappa)}\kappa\right\},$$

and set for every $t \in {}^{\leq\kappa}\kappa = {}^{<\kappa}\kappa \cup {}^{\kappa}\kappa$ with $\mathrm{lh}(t) = \gamma \leq \kappa$,

$$V(t) = \bigcap_{\alpha \in \mathrm{Succ}(\gamma)} \hat{V}_{t \restriction \alpha}. \tag{1.1.4}$$

(In particular, when $\gamma = 0$ we get $V(\emptyset) = X$ because $\mathrm{Succ}(0) = \emptyset$.) The family $\mathcal{F}$ will be designed so that for any $\gamma < \kappa$ and $s \in {}^{\gamma+1}\kappa$ the following properties are satisfied:

(i) $x_s \in X$, and U_s, V_s, $\hat{V}_s$ are all clopen in X.

(ii) If $V(s) \neq \emptyset$, then the sequence $\langle (U_{s\restriction\alpha}, x_{s\restriction\alpha}), V_{s\restriction\alpha} \mid \alpha \in \mathrm{Succ}(\gamma+1)\rangle$ is a (partial) run in the strong (fair) κ-Choquet game on X in which II follows σ.

(iii) Either $\hat{V}_s \subseteq B_\gamma$ or $\hat{V}_s \cap B_\gamma = \emptyset$.

(iv) $\hat{V}_s \subseteq V_s \subseteq U_s \subseteq V(s \restriction \gamma)$.

(v) $\{\hat{V}_s \mid s \in {}^{\gamma+1}\kappa\}$ is a partition[8] of X.

Condition (iv) implies that

$$\hat{V}_s \subseteq \hat{V}_{s\restriction\alpha} \tag{1.1.5}$$

for every $s \in {}^{<\mathrm{Succ}(\kappa)}\kappa$ and $\alpha \in \mathrm{Succ}(\mathrm{lh}(s))$. Together with condition (v), this entails that

(v') For any $\gamma < \kappa$, successor or not, $\{V(t) \mid t \in {}^{\gamma}\kappa\}$ is a partition of X.

From condition (v') it easily follows that if $t, t' \in {}^{<\kappa}\kappa$ are such that $V(t) \cap V(t') \neq \emptyset$, then t and t' are comparable. Equation (1.1.5) also implies that if $\mathrm{lh}(t)$ is a successor ordinal, then $V(t) = \hat{V}_t$. If instead $\gamma = \mathrm{lh}(t) \leq \kappa$ is limit, then

$$V(t) = \bigcap_{\alpha \in \mathrm{Succ}(\gamma)} U_{t\restriction\alpha} = \bigcap_{\alpha \in \mathrm{Succ}(\gamma)} V_{t\restriction\alpha} \tag{1.1.6}$$

by condition (iv) again. Notice also that the additional properties discussed in this paragraph have a local (i.e. level-by-level) nature: for example, to have (v') at some level γ, it is enough to have conditions (iv) and (v) at all levels $\gamma' \leq \gamma$.

Given $\mathcal{F}$ as above, one obtains the required homeomorphism of X with a (super)closed set $C \subseteq {}^{\kappa}\kappa$ as follows. Since X is Hausdorff, if $\mathrm{lh}(t) = \kappa$ then $V(t)$ has at most one element by condition (iii). Consider the tree

$$T = \{t \in {}^{<\kappa}\kappa \mid V(t) \neq \emptyset\}.$$

It is pruned by condition (v') and the comment following it. Furthermore, if X is an SC_κ-space (i.e. σ is a winning in the game $\mathrm{G}^s_\kappa(X)$), then T is also $<\kappa$-closed by condition (ii) and equation (1.1.6).

We now prove that the (super)closed set $C = [T]$ is homeomorphic to X. Since σ is a winning strategy in the strong (fair) κ-Choquet game, the set $V(t)$ is nonempty for every $t \in [T]$ by condition (ii) and equation (1.1.6) again, thus it contains exactly one point: let $f \colon [T] \to X$ be the map that associates to every $t \in [T]$ the unique element in $V(t)$. We claim that f is a homeomorphism.

Claim 1.1.13.1. f is bijective.

[8] An indexed family $\{A_i \mid i \in I\}$ of subsets of X is a partition of X if $\bigcup_{i \in I} A_i = X$ and $A_i \cap A_j = \emptyset$ for distinct $i, j \in I$. In particular, some of the A_i's might be empty and for $i \neq j$ we have $A_i = A_j$ if and only if both A_i and A_j are empty.

Proof. To see that f is injective, let $t, t' \in [T]$ be distinct and $\alpha < \kappa$ be such that $t \restriction \alpha \neq t' \restriction \alpha$. By condition (v') we have $V(t \restriction \alpha) \cap V(t' \restriction \alpha) = \emptyset$, and hence $f(t) \neq f(t')$ because $f(t) \in V(t) \subseteq V(t \restriction \alpha)$ and $f(t') \in V(t') \subseteq V(t' \restriction \alpha)$. To see that f is also surjective, fix any $x \in X$. By (v') again (and the comment following it), for each $\alpha < \kappa$ there is a unique t_α of length α with $x \in V(t_\alpha)$, and moreover $t_\alpha \subseteq t_\beta$ for all $\alpha \leq \beta < \kappa$. Let $t = \bigcup_{\alpha<\kappa} t_\alpha$, so that $x \in V(t) = \bigcap_{\alpha<\kappa} V(t_\alpha) = \bigcap_{\alpha<\kappa} V(t \restriction \alpha)$: then x itself witnesses $t \in [T]$, and $f(t) = x$. □

Claim 1.1.13.2. f is a homeomorphism.

Proof. Observe that by definition of f, its surjectivity, and condition (v'),

$$f(\mathbf{N}_s \cap [T]) = V(s) = \hat{V}_s \tag{1.1.7}$$

for all $s \in T$ with $\mathrm{lh}(s) \in \mathrm{Succ}(\kappa)$. Since $\{\mathbf{N}_s \cap [T] \mid s \in T \cap {}^{<\mathrm{Succ}(\kappa)}\kappa\}$ is a basis for the relative topology of $[T]$, while $\{\hat{V}_s \mid s \in T \cap {}^{<\mathrm{Succ}(\kappa)}\kappa\}$ is a basis for X by conditions (i), (iii), and (v), then f and f^{-1} are continuous. □

It remains to construct the required family $\mathcal{F}$ by recursion on $\gamma < \kappa$. We assume that for every $t \in {}^{<\kappa}\kappa$ with $\mathrm{lh}(t) = \gamma$ and all $\alpha \in \mathrm{Succ}(\gamma)$, the elements $x_{t\restriction\alpha}$, $U_{t\restriction\alpha}$, $V_{t\restriction\alpha}$, and $\hat{V}_{t\restriction\alpha}$ have been defined so that conditions (i)–(v) are satisfied up to level γ (when $\gamma > 0$ this is the inductive hypothesis, while if $\gamma = 0$ the assumption is obviously vacuous because $\mathrm{Succ}(0)$ is empty): our goal is to define $x_{t^\frown i}$, $U_{t^\frown i}$, $V_{t^\frown i}$, and $\hat{V}_{t^\frown i}$ for all t as above and $i < \kappa$ in such a way that conditions (i)–(v) are preserved.

Recall the definition of the sets $V(t)$ from equation (1.1.4). If $V(t) = \emptyset$, then we set $U_{t^\frown i} = V_{t^\frown i} = \hat{V}_{t^\frown i} = \emptyset$ for all $i < \kappa$ and let $x_{t^\frown i}$ be an arbitrary point in X. Assume now that $V(t) \neq \emptyset$. Notice that $V(t)$ is clopen: if $\gamma > 0$ this follows from κ-additivity of X and the fact that $\hat{V}_{t\restriction\alpha}$ is clopen for every $\alpha \in \mathrm{Succ}(\gamma)$ by (i), while if $\gamma = 0$ then $V(\emptyset) = X$ by definition. By condition (ii), the sequence $\langle (U_{t\restriction\alpha}, x_{t\restriction\alpha}), V_{t\restriction\alpha} \mid \alpha \in \mathrm{Succ}(\gamma)\rangle$ is a partial run in the corresponding Choquet-like game in which II is following σ. We let such run continue for one more round by letting I play some (U, x) with U clopen and $x \in U \subseteq V(t)$, and II reply with some $V \in \mathcal{B}$ following the winning strategy σ, so that in particular $x \in V \subseteq U$. Let $\{V_j \mid j < \delta\}$ be the collection of all those V's that can be obtained in this way: even if there are possibly more than κ-many moves for I as above, there are at most κ-many replies of II because $|\mathcal{B}| \leq \kappa$, hence $\delta \leq \kappa$. For each $j < \delta$ we then choose one of player I's moves (U_j, x_j) yielding V_j as II's reply. In particular, $x_j \in V_j \subseteq U_j$. Let $(\hat{V}_i)_{i<\nu}$ (where $\nu \leq \kappa$) be an enumeration without repetitions of the nonempty sets in

$$\left\{ \left(V_j \setminus \bigcup\nolimits_{\ell<j} V_\ell\right) \cap B_\gamma \;\middle|\; j < \delta \right\} \cup \left\{ \left(V_j \setminus \bigcup\nolimits_{\ell<j} V_\ell\right) \setminus B_\gamma \;\middle|\; j < \delta \right\},$$

and for each $i < \nu$ let $j(i) < \delta \leq \kappa$ be such that $\hat{V}_i \subseteq V_{j(i)}$. Notice that the $\hat{V}_i$'s are clopen by κ-additivity again. Finally, set

$$x_{t^\frown i} = x_{j(i)} \qquad U_{t^\frown i} = U_{j(i)} \qquad V_{t^\frown i} = V_{j(i)} \qquad \hat{V}_{t^\frown i} = \hat{V}_i$$

if $i < \nu$, and $U_{t^\frown i} = V_{t^\frown i} = \hat{V}_{t^\frown i} = \emptyset$ with $x_{t^\frown i}$ an arbitrary point of X if $\nu \leq i < \kappa$.

It is not hard to see that conditions (i)–(iv) are preserved by construction. As for condition (v), by inductive hypothesis (or $V(\emptyset) = X$ if $\gamma = 0$) we get (v') at level γ, that is, $\{V(t) \mid t \in {}^{\gamma}\kappa\}$ is a partition of X. Thus the desired result straightforwardly follows from the fact that the V_j's cover $V(t)$ because in our construction player I can play any $x \in V(t)$ in her last round (paired with a suitable clopen set U such that $x \in U \subseteq V(t)$, which exists because $V(t)$ is clopen). □

We now consider the problem of simultaneously embedding two κ-additive $f\mathrm{SC}_\kappa$-spaces $X' \subseteq X$ into ${}^{\kappa}\kappa$. Applying Proposition 1.1.13 to X we get a closed C and a homeomorphism $f \colon C \to X$. If X' is a closed in X, it follows that also $C' = f^{-1}(X')$ is closed in C and hence in ${}^{\kappa}\kappa$. However, when X' is an SC_κ-space we would like to have that C' is superclosed. To this aim we need to modify our construction.

Proposition 1.1.14. *Let X be a κ-additive $f\mathrm{SC}_\kappa$-space and $X' \subseteq X$ be a closed SC_κ-subspace. Then there is a closed $C \subseteq {}^{\kappa}\kappa$ and a homeomorphism $f \colon C \to X$ such that $C' = f^{-1}(X')$ is superclosed.*

Proof. The idea is to apply the argument from the previous proof but starting with a strategy σ that is winning for II in $f\mathrm{G}^s_\kappa(X)$ and, when "restricted" to X', in $\mathrm{G}^s_\kappa(X')$ as well. Let $\mathcal{B}$ be a basis for X of size $\leq \kappa$.

Claim 1.1.14.1. There is a winning strategy σ for player II in $f\mathrm{G}^s_\kappa(X)$ with range in $\mathcal{B}$ such that for any (partial) run $\langle (U_\alpha, x_\alpha), V_\alpha \mid \alpha < \gamma\rangle$ in $f\mathrm{G}^s_\kappa(X)$ where player II followed σ, one has $\bigcap_{\alpha<\gamma} V_\alpha \cap X' \neq \emptyset$ if and only if $V_\alpha \cap X' \neq \emptyset$ for every $\alpha < \gamma$.

Proof of the claim. Let σ' be an arbitrary winning strategy for II in $\mathrm{G}^s_\kappa(X')$, and let σ'' be a winning strategy for II in $f\mathrm{G}^s_\kappa(X)$ with range contained in $\mathcal{B}$. Define the strategy σ as follows. Suppose that at stage $\alpha < \kappa$ player I has played the sequence $\langle (U_\beta, x_\beta) \mid \beta \leq \alpha\rangle$ in the game $f\mathrm{G}^s_\kappa(X)$.

(1) As long as all points x_β belongs to X', player II considers the auxiliary partial play $\langle (U_\beta \cap X', x_\beta) \mid \beta \leq \alpha\rangle$ of I in $\mathrm{G}^s_\kappa(X')$ and she uses τ' to get her next move V'_α in the game $\mathrm{G}^s_\kappa(X')$. Since V'_α is open in X', there is W open in X such that $V'_\alpha = W \cap X'$: let II play any $V_\alpha \in \mathcal{B}$ such that $x_\alpha \in V_\alpha \subseteq W \cap \bigcap_{\beta\leq\alpha} U_\beta$ as her next move in the game $f\mathrm{G}^s_\kappa(X)$ (this is possible because $W \cap \bigcap_{\beta\leq\alpha} U_\beta$ is open by κ-additivity).

(2) If α is smallest such that $x_\alpha \notin X'$, from that point on player II uses her strategy σ'' pretending that $(U_\alpha \setminus X', x_\alpha)$ was the first move of I in a new run of $f\mathrm{G}^s_\kappa(X)$.

We claim that σ is as required, so fix any $\gamma \leq \kappa$. Let $\langle (U_\alpha, x_\alpha), V_\alpha \mid \alpha < \gamma\rangle$ be a partial run in which II followed σ and assume that $V_\alpha \cap X' \neq \emptyset$ for every $\alpha < \gamma$. By (2) this implies that $x_\alpha \in X'$ for all $\alpha < \gamma$. If $\gamma = \alpha + 1$ is a successor ordinal, then $\bigcap_{\beta<\gamma} V_\beta \cap X' = V_\alpha \cap X' \neq \emptyset$ by assumption. Assume instead that γ is limit. By $x_\alpha \in X'$ and (1), for all $\alpha < \gamma$ we have

$$U_{\alpha+1} \cap X' \subseteq V_\alpha \cap X' \subseteq V'_\alpha \subseteq U_\alpha \cap X', \tag{1.1.8}$$

where $V'_\alpha \subseteq X'$ is again II's reply to the partial play $\langle (U_\beta \cap X', x_\beta) \mid \beta \leq \alpha\rangle$ of I in $\mathrm{G}^s_\kappa(X')$ according to σ'. It follows that $\langle (U_\alpha \cap X', x_\alpha), V'_\alpha \mid \alpha < \gamma\rangle$ is a (legal) partial run in $\mathrm{G}^s_\kappa(X')$ where II followed σ', and since the latter is winning in such game we get $\bigcap_{\alpha<\gamma} V_\alpha \cap X' = \bigcap_{\alpha<\gamma} V'_\alpha \neq \emptyset$ (the first equality follows from (1.1.8) and the fact that γ is limit). This also implies that σ wins $f\mathrm{G}^s_\kappa(X)$ in all runs where $V_\alpha \cap X' \neq \emptyset$ for all $\alpha < \kappa$; on the other hand, when this is not the case and $\alpha < \kappa$ is smallest such that $V_\alpha \cap X' = \emptyset$, then the tail of the run from level α on is a (legal) run in $f\mathrm{G}^s_\kappa(X)$ in which II followed σ'', thus II won as well. This shows that σ is winning for II in $f\mathrm{G}^s_\kappa(X)$ and concludes the proof. □

Starting from σ as in Claim 1.1.14.1, argue as in the proof of Proposition 1.1.13 to build a family $\mathcal{F} = \left\{x_s, U_s, V_s, \hat{V}_s \mid s \in {}^{<\mathrm{Succ}(\kappa)}\kappa\right\}$ and a homeomorphism $f\colon C \to X$, where $C = [T]$ is the closed subset of ${}^\kappa\kappa$ defined by the tree $T = \{t \in {}^{<\kappa}\kappa \mid V(t) \neq \emptyset\}$, and $f(t)$ is the unique point in $V(t)$ for all $t \in [T]$. Consider now the tree defined by

$$T' = \{t \in {}^{<\kappa}\kappa \mid V(t) \cap X' \neq \emptyset\}.$$

Clearly $T' \subseteq T$. Moreover, for every $t \in T'$ we have $\mathbf{N}_t \cap [T'] \neq \emptyset$: indeed, if $t \in T'$, then there is $x \in V(t) \cap X'$, hence $f^{-1}(x) \supseteq t$ and by construction x witnesses $f^{-1}(x) \restriction \alpha \in T'$ for all $\alpha < \kappa$, so $f^{-1}(x) \in \mathbf{N}_t \cap [T']$. In particular, this implies that T' is pruned. We now prove that T' is also superclosed. Let $t \in {}^\gamma\kappa$ for $\gamma < \kappa$ limit be such that $t \restriction \alpha \in T'$ for all $\alpha < \gamma$. Then $\hat{V}_{t\restriction\alpha} \cap X' \neq \emptyset$ for all $\alpha \in \mathrm{Succ}(\gamma)$, hence also $V_{t\restriction\alpha} \cap X' \neq \emptyset$ by $\hat{V}_{t\restriction\alpha} \subseteq V_{t\restriction\alpha}$. By the choice of σ, it follows that $\bigcap_{\alpha\in\mathrm{Succ}(\gamma)} V_{t\restriction\alpha} \cap X' \neq \emptyset$, hence $t \in T'$ since $V(t) = \bigcap_{\alpha\in\mathrm{Succ}(\gamma)} V_{t\restriction\alpha}$ when t has limit length.

Finally, we want to show that $f^{-1}(X') = [T']$. Given $x \in X'$, then x itself witnesses $f^{-1}(x) \in [T']$. Conversely, if $t \in [T']$ then $V_{t\restriction\alpha} \cap X' \supseteq V(t \restriction \alpha) \cap X' \neq \emptyset$ for all $\alpha \in \mathrm{Succ}(\kappa)$, hence by the choice of σ again we have that $\bigcap_{\alpha\in\mathrm{Succ}(\kappa)} V_{t\restriction\alpha} \cap X' \neq \emptyset$. Since $\bigcap_{\alpha\in\mathrm{Succ}(\kappa)} V_{t\restriction\alpha} = V(t) = \{f(x)\}$, it follows that $f(x) \in X'$ as desired. □

Proposition 1.1.14 allows us to considerably extend [105, Proposition 1.3] from superclosed subsets of ${}^\kappa\kappa$ to arbitrary closed SC_κ-subspaces of a κ-additive $f\mathrm{SC}_\kappa$-space.

Corollary 1.1.15. *Let X be a κ-additive $f\mathrm{SC}_\kappa$-space. Then every closed SC_κ-subspace Y of X is a retract of it.*

Proof. By Proposition 1.1.14, without loss of generality we may assume that X is a closed subspace of ${}^\kappa\kappa$ and $Y \subseteq X$ a superclosed set. By [105, Proposition 1.3] there is a retraction r from ${}^\kappa\kappa$ onto Y. Then $r \restriction X$ is a retraction of X onto Y. □

None of the conditions on Y can be dropped in the above result: every retract of a Hausdorff space is necessarily closed in it, and by [105, Proposition 1.4] the space X_0 from equation (1.1.3) is a closed $f\mathrm{SC}_\kappa$-subspace of the SC_κ-space ${}^\kappa 2$ which is not a retract of it. Notice also that there are even clopen (hence strong κ-Choquet) subspaces of ${}^\kappa\kappa$ which are not superclosed, for example $\{x \in {}^\kappa\kappa \mid \exists n < \omega\, (x(n) \neq 0)\}$. This shows that even in the special case $X = {}^\kappa\kappa$, our Corollary 1.1.15 properly extends [105, Proposition 1.3].

Lemma 1.1.9, Proposition 1.1.10 and Proposition 1.1.13 together lead to the following characterization of κ-additive $f\mathrm{SC}_\kappa$-spaces.

Theorem 1.1.16. *For any space*[9] X *the following are equivalent:*

(a) X *is a* κ*-additive* $f\mathrm{SC}_\kappa$*-space;*

(b) X *is homeomorphic to a* G^κ_δ *subset of* ${}^\kappa\kappa$*;*

(c) X *is homeomorphic to a closed subset of* ${}^\kappa\kappa$*.*

In particular, ${}^\kappa\kappa$ *is universal for* κ*-additive* $f\mathrm{SC}_\kappa$*-spaces, and hence also for* κ*-additive* SC_κ*-spaces.*

Proof. The implication from (a) to (c) is Proposition 1.1.13, while (c) trivially implies (b) by Lemma 1.1.9. Finally, (b) implies (a) because ${}^\kappa\kappa$ is trivially a κ-additive $f\mathrm{SC}_\kappa$-space and such spaces are closed under G^κ_δ subspaces by Proposition 1.1.10. □

From Proposition 1.1.13 we also get a characterization of κ-additive SC_κ-spaces. (The fact that every superclosed subset of ${}^\kappa\kappa$ is an SC_κ-space is trivial.)

Theorem 1.1.17. *For any space* X *the following are equivalent:*

(a) X *is a* κ*-additive* SC_κ*-space;*

(b) X *is homeomorphic to a superclosed subset of* ${}^\kappa\kappa$*.*

Remark 1.1.18. Since ${}^\kappa\kappa$ is κ-additive and the latter is a hereditary property, Theorems 1.1.16 and 1.1.17 can obviously be turned into a characterization of κ-additivity inside the classes of $f\mathrm{SC}_\kappa$-spaces and SC_κ-spaces.

Recall that an uncountable cardinal κ is (**strongly**) **inaccessible** if it is regular and strong limit, that is, $2^\lambda < \kappa$ for all $\lambda < \kappa$. An uncountable cardinal κ is **weakly compact** if and only if it is inaccessible and has the tree property: $[T] \neq \emptyset$ for every tree $T \subseteq {}^{<\kappa}\kappa$ satisfying $1 \leq |\mathrm{Lev}_\alpha(T)| < \kappa$ for all $\alpha < \kappa$. A topological space X is κ**-Lindelöf** if all its open coverings admit a subcovering of size $< \kappa$. (Thus ω-Lindelöfness is ordinary compactness.) It turns out that the space ${}^\kappa 2$ is κ-Lindelöf if and only if κ is weakly compact [118, Theorem 5.6], in which case ${}^\kappa 2$ and ${}^\kappa\kappa$ are obviously not homeomorphic; if instead κ is not weakly compact, then ${}^\kappa 2$ is homeomorphic to ${}^\kappa\kappa$ by [86, Theorem 1]. This implies that if κ is not weakly compact, then we can replace ${}^\kappa\kappa$ with ${}^\kappa 2$ in both Proposition 1.1.13 and Theorem 1.1.16. Moreover, since one can easily show that if κ is not weakly compact then there are homeomorphisms between ${}^\kappa\kappa$ and ${}^\kappa 2$ preserving superclosed sets, for such κ's we can replace ${}^\kappa\kappa$ with ${}^\kappa 2$ in Theorem 1.1.17 as well. As for weakly compact cardinals κ, the equivalence between (a) and (b) in Theorem 1.1.16 still holds replacing ${}^\kappa\kappa$ with ${}^\kappa 2$ by Fact 1.1.8, but the same does not apply to part (c) and Theorem 1.1.17 because for such a κ all (super)closed subsets of ${}^\kappa 2$ are κ-Lindelöf—see Theorems 1.2.22 and 1.2.23.

We now move to $\mathbb{G}$-Polish spaces. Our goal is to show that such spaces coincide with the κ-additive $f\mathrm{SC}_\kappa$-spaces, and thus that the definition is in particular independent of the chosen $\mathbb{G}$. Along the way, we also generalize some results independently

[9] Recall that all spaces are tacitly assumed to be regular Hausdorff.

obtained in [71, Section 2.3] and close some open problems and conjectures contained therein, obtaining a fairly complete picture of the relationships among all the proposed generalizations of Polish spaces.

In the subsequent results, $\mathbb{G}$ is a totally ordered Abelian group with $\mathrm{Deg}(\mathbb{G}) = \kappa$. The next lemma was essentially proved in [139, Theorem (viii)] and it corresponds to (b) $\Rightarrow$ (a) in Theorem 1.1.12. We reprove it here for the reader's convenience.

Lemma 1.1.19. *Every $\mathbb{G}$-metric space X is κ-additive, hence also zero-dimensional.*

Proof. Let $\gamma < \kappa$ and $(U_\alpha)_{\alpha<\gamma}$ be a sequence of nonempty open sets. If $\bigcap_{\alpha<\gamma} U_\alpha \neq \emptyset$, consider an arbitrary $x \in \bigcap_{\alpha<\gamma} U_\alpha$. The family $\{B_d(x, \varepsilon) \mid \varepsilon \in \mathbb{G}^+\}$ is a local basis of x, so for every $\alpha < \gamma$ we may find $\varepsilon_\alpha \in \mathbb{G}^+$ such that $B_d(x, \varepsilon_\alpha) \subseteq U_\alpha$. Since $\mathrm{Deg}(\mathbb{G}) = \kappa > \gamma$, there is $\varepsilon \in \mathbb{G}^+$ such that $\varepsilon \leq_{\mathbb{G}} \varepsilon_\alpha$ for all $\alpha < \gamma$: thus $x \in B_d(x, \varepsilon) \subseteq \bigcap_{\alpha<\gamma} B_d(x, \varepsilon_\alpha) \subseteq \bigcap_{\alpha<\gamma} U_\alpha$. □

Lemma 1.1.20. *Every $\mathbb{G}$-Polish space X is strong fair κ-Choquet.*

Proof. Fix a compatible Cauchy-complete metric d on X and a strictly decreasing sequence $(r_\alpha)_{\alpha<\kappa}$ coinitial in $\mathbb{G}^+$. Consider the strategy τ of II in $f\mathrm{G}^s_\kappa(X)$ in which he replies to player I's move (U_α, x_α) by picking a ball $V_\alpha = B_d(x_\alpha, \varepsilon_\alpha)$ with $\varepsilon_\alpha \in \mathbb{G}^+$ small enough so that $\varepsilon_\alpha \leq_{\mathbb{G}} r_\alpha$ and $\mathrm{cl}(V_\alpha) \subseteq U_\alpha$. In particular, we will thus have $\mathrm{cl}(V_{\alpha+1}) \subseteq V_\alpha$. Suppose that $\langle (U_\alpha, x_\alpha), V_\alpha \mid \alpha < \kappa \rangle$ is a run in $f\mathrm{G}^s_\kappa(X)$ in which $\bigcap_{\alpha<\gamma} V_\alpha \neq \emptyset$ for every limit $\gamma < \kappa$. Then the choice of the ε_α's ensures that $(x_\alpha)_{\alpha<\kappa}$ is a Cauchy sequence, and thus it converges to some $x \in X$ by Cauchy-completeness of d. It follows that $x \in \bigcap_{\alpha<\kappa} \mathrm{cl}(V_\alpha) = \bigcap_{\alpha<\kappa} V_\alpha \neq \emptyset$, and thus τ is a winning strategy for player II. □

Theorem 1.1.21. *For any space X the following are equivalent:*

(a) *X is $\mathbb{G}$-Polish;*

(b) *X is a κ-additive $f\mathrm{SC}_\kappa$-space;*

(c) *X is homeomorphic to a G^κ_δ subset of ${}^\kappa\kappa$;*

(d) *X is homeomorphic to a closed subset of ${}^\kappa\kappa$.*

Proof. The equivalence of (b), (c), and (d) is Theorem 1.1.16, and (d) easily implies (a). The remaining implication, (a) implies (b), follows from Lemma 1.1.19 and Lemma 1.1.20. □

As usual, when κ is not weakly compact we can replace ${}^\kappa\kappa$ with its homeomorphic copy ${}^\kappa 2$ in conditions (c) and (d) above. When κ is instead weakly compact, by Fact 1.1.8 we can still replace ${}^\kappa\kappa$ with ${}^\kappa 2$ in condition (c), but the same does not apply to condition (d) because of κ-Lindelöfness—see Theorem 1.2.22. In view of this observation, the implication (a) $\Rightarrow$ (c) in Theorem 1.1.21 is just a reformulation of [71, Corollary 2.36], which is thus nicely complemented by our result.

Theorem 1.1.21 shows in particular that the notion of $\mathbb{G}$-Polish space does not depend on the particular choice of the group $\mathbb{G}$.

Corollary 1.1.22. *Let $\mathbb{G}, \mathbb{G}'$ be two totally ordered (Abelian) groups, both of degree κ, and X be a space. Then X is $\mathbb{G}$-Polish if and only if it is $\mathbb{G}'$-Polish.*

Notice that thanks to Choquet Theorem ([36], see also [93]), Lemma 1.1.20 implies that when $\mathbb{G}$ has countable degree we recover the usual notion of Polish space.

Corollary 1.1.23. *Let $\mathbb{G}$ be a totally ordered (Abelian) group of countable degree. Then every $\mathbb{G}$-Polish space is Polish.*

For this reason, from now on will systematically avoid to specify which kind of $\mathbb{G}$ we are considering and freely use the term "$\mathbb{G}$-Polish" as a shortcut for "$\mathbb{G}$-Polish with respect to a(ny) totally ordered (Abelian) group of degree κ".

Remark 1.1.24. In the proofs of Lemma 1.1.19 and Lemma 1.1.20, the only property required of the metric d is that

$$\text{For all } x \in X, \text{ the family } \{B_d(x,\varepsilon) \mid \varepsilon \in \mathbb{G}^+\} \text{ is a local basis of } x. \tag{1.1.9}$$

Hence, Theorem 1.1.12 and Theorem 1.1.21 (and Corollary 1.1.22) can be extended to metrics taking values in any other kind of structure, as long as equation (1.1.9) is still satisfied. (In particular, commutativity of $\mathbb{G}$ is not really needed.) This includes the case of completely S-quasimetrizable spaces for a totally ordered semigroup S considered in [126], or spaces admitting a complete κ-ultrametric as defined in [42]. In particular, the concepts of (complete) metric space and (complete) ultrametric space lead to the same class of spaces in generalized descriptive set theory. This is in strong contrast to what happens in the classical setting, where Polish ultrametric spaces form a proper subclass of arbitrary Polish spaces because admitting a compatible ultrametric implies zero-dimensionality.

Another easy corollary of Theorem 1.1.21 is that a G^κ_δ subset of a $\mathbb{G}$-Polish space is necessarily $\mathbb{G}$-Polish as well. We complement this in Corollary 1.1.27, using an extension result for continuous functions (Proposition 1.1.26). These results are the natural generalization of the classical arguments in [93, Theorems 3.8 and 3.11], and already appeared in [71, Theorems 2.34 and 2.35] where, as customary in the subject, the fact that $\mathbb{G}$ is Abelian is assumed and used. However, we fully reprove both results for the sake of completeness and to confirm that also in this case commutativity of $\mathbb{G}$ is not required.

Lemma 1.1.25. *Let $\mathbb{G}$ be a totally ordered (non-necessarily Abelian) group with arbitrarily small positive elements. Then for every $\varepsilon \in \mathbb{G}^+$ and every $n \in \omega$ there is $\delta \in \mathbb{G}^+$ with*[10] *$n\delta \leq_{\mathbb{G}} \varepsilon$.*

Proof. It is clearly enough to prove the result for $n = 2$. Let $\varepsilon' \in \mathbb{G}^+$ be such that $0_{\mathbb{G}} <_{\mathbb{G}} \varepsilon' <_{\mathbb{G}} \varepsilon$ and set $\delta = \min\{\varepsilon', -\varepsilon' +_{\mathbb{G}} \varepsilon\}$. Since $\leq_{\mathbb{G}}$ is translation-invariant on both sides we get

$$\delta +_{\mathbb{G}} \delta \leq_{\mathbb{G}} \varepsilon' +_{\mathbb{G}} (-\varepsilon' +_{\mathbb{G}} \varepsilon) = \varepsilon.$$

□

Proposition 1.1.26. *Let X be a $\mathbb{G}$-metrizable space, and (Y, d) be a Cauchy-complete $\mathbb{G}$-metric space. Let $A \subseteq X$ be any set and $f \colon A \to Y$ be continuous. Then there is a G^κ_δ set $B \subseteq X$ and a continuous function $g \colon B \to Y$ such that $A \subseteq B \subseteq \mathrm{cl}(A)$ and g extends f, i.e. $g \restriction A = f$.*

[10] As customary, we denote by $n\delta$ the finite sum $\underbrace{\delta +_{\mathbb{G}} \ldots +_{\mathbb{G}} \delta}_{n \text{ times}}$.

Proof. Given any $\varepsilon \in \mathbb{G}^+$, let O_ε be the collection of those $x \in X$ admitting an open neighborhood U such that $d(f(y), f(z)) <_\mathbb{G} \varepsilon$ for all $y, z \in U \cap A$. By definition, each O_ε is open in X, and since $f \colon A \to Y$ is continuous then $A \subseteq O_\varepsilon$ for all $\varepsilon \in \mathbb{G}^+$ (here we are implicitly using Lemma 1.1.25). Fix a strictly decreasing sequence $(r_\alpha)_{\alpha<\kappa}$ coinitial in $\mathbb{G}^+$, and set

$$B = \mathrm{cl}(A) \cap \bigcap_{\alpha<\kappa} O_{r_\alpha},$$

so that $A \subseteq B \subseteq \mathrm{cl}(A)$ and B is G^κ_δ by Lemma 1.1.9. Fix $x \in B$, and for every $\alpha < \kappa$ fix an open neighborhood U^x_α of x witnessing $x \in O_{r_\alpha}$. Without loss of generality we may assume that $U^x_\beta \subseteq U^x_\alpha$ if $\alpha \leq \beta < \kappa$ (if not, then $\tilde{U}^x_\beta = \bigcap_{\zeta \leq \beta} U^x_\zeta$ is as desired by κ-additivity of X). Since $x \in B \subseteq \mathrm{cl}(A)$, for each $\alpha < \kappa$ we can pick some $y_\alpha \in U^x_\alpha \cap A$. The sequence $(f(y_\alpha))_{\alpha<\kappa}$ is d-Cauchy by construction, thus it converges to some $y \in Y$ by Cauchy-completeness of d: set $g(x) = y$. By uniqueness of limits, it is easy to check that the map g is well-defined (i.e. the value $g(x)$ is independent of the choice of the U^x_α's and y_α's), and that $g(x) = f(x)$ for all $x \in A$. It remains to show that g is also continuous at every $x \in B$. Given any $\varepsilon \in \mathbb{G}^+$, we want to find an open neighborhood U of x such that $g(U \cap B) \subseteq B_d(g(x), \varepsilon)$. Let U^x_α and y_α be as in the definition of $g(x)$. Using Lemma 1.1.25, find $\delta \in \mathbb{G}^+$ such that $3\delta \leq_\mathbb{G} \varepsilon$. Let α be large enough so that $d(f(y_\alpha), g(x)) < \delta$ and $r_\alpha < \delta$, so that $f(U^x_\alpha \cap A) \subseteq B_d(g(x), 2\delta)$. We claim that $U = U^x_\alpha$ is as required. Indeed, if $z \in U \cap B$, then when defining $g(z)$ we may without loss of generality pick U^z_α so that $U^z_\alpha \subseteq U^x_\alpha$: it then follows that

$$g(z) \in \mathrm{cl}(f(U^z_\alpha \cap A)) \subseteq \mathrm{cl}(f(U^x_\alpha \cap A)) \subseteq \mathrm{cl}\left(B_d(g(x), 2\delta)\right) \subseteq B_d(g(x), \varepsilon),$$

as required. □

Corollary 1.1.27. *Let X be a $\mathbb{G}$-metrizable space, and let $Y \subseteq X$ be a completely $\mathbb{G}$-metrizable subspace of X. Then Y is a G^κ_δ subset of X.*

Proof. Apply Proposition 1.1.26 with $A = Y$ and f the identity map from Y to itself. The resulting $g \colon B \to Y$ is then the identity map on B, hence $Y = B$ and thus Y is G^κ_δ. □

In [71] it is asked whether the reverse implication holds, i.e. whether G^κ_δ subsets of $\mathbb{G}$-Polish spaces need to be $\mathbb{G}$-Polish as well (see the discussion in the paragraph after [71, Theorem 2.10]): our Theorem 1.1.21 already yields a positive answer, and thus it allows us to characterize which subspaces of a $\mathbb{G}$-Polish space are still $\mathbb{G}$-Polish.

Theorem 1.1.28. *Let X be $\mathbb{G}$-Polish and $Y \subseteq X$. Then Y is $\mathbb{G}$-Polish if and only if Y is G^κ_δ in X.*

Proof. One direction follows from Corollary 1.1.27. For the other direction, since X is homeomorphic to a closed subset of ${}^\kappa\kappa$ by Theorem 1.1.21, every G^κ_δ subspace $Y \subseteq X$ is homeomorphic to a G^κ_δ subset of ${}^\kappa\kappa$. Using again Theorem 1.1.21, it follows that Y is $\mathbb{G}$-Polish as well. □

By Theorem 1.1.21, Theorem 1.1.28 admits a natural counterpart characterizing $f\mathrm{SC}_\kappa$-subspaces of κ-additive $f\mathrm{SC}_\kappa$-spaces.

To complete the description of how our classes of spaces relate one to the other, we just need to characterize those spaces which are in all of them and thus have the richest structure (this includes e.g. the generalized Cantor and Baire spaces). To this aim, we need to introduce one last notion inspired by [42, Definition 6.1] and [71].

Definition 1.1.29. A $\mathbb{G}$-metric d on a space X is called **spherically complete**[11] if the intersection of every *decreasing* (with respect to inclusion) sequence of open balls is nonempty. If in the definition we consider only sequences of order type κ (respectively, $<\kappa$ or $\leq\kappa$) we say that the metric is **spherically κ-complete** (respectively, **spherically $<\kappa$-complete** or **spherically $\leq\kappa$-complete**).

Remark 1.1.30. Let (X, d) be a $\mathbb{G}$-metric space.

(i) If the space X has weight κ, then the metric d is spherically complete if and only if it is spherically $\leq\kappa$-complete. For the non trivial direction, fix in $\mathbb{G}^+$ a decreasing sequence $(\varepsilon_i)_{i<\kappa}$ converging to $0_\mathbb{G}$ and consider an arbitrary decreasing chain of balls $B_\alpha = B_d(x_\alpha, r_\alpha)$ for $\alpha < \lambda$ with λ a regular cardinal greater than κ. If for all $i < \kappa$ there is $\alpha_i < \lambda$ such that $r_{\alpha_i} < \varepsilon_i$, then by spherically κ-completeness we have $\bigcap_{i<\kappa} B_{\alpha_i} = \{x\}$ for some x. It follows that $B_\alpha = \{x\}$ for all $\alpha \geq \sup_{i<\kappa} \alpha_i$, whence $\bigcap_{\alpha<\lambda} B_\alpha = \{x\} \neq \emptyset$. The remaining case is when there is $\delta \in \mathbb{G}^+$ such that $r_\alpha \geq \delta$ for all $\alpha < \lambda$. If $\bigcap_{\alpha<\lambda} B_\alpha = \emptyset$, then we could recursively construct an increasing sequence $(\alpha_\beta)_{\beta<\lambda}$ of ordinals $< \lambda$ such that $x_{\alpha_\beta} \notin B_{\alpha_{\beta'}}$ for all $\beta < \beta' < \lambda$. By case assumption, we thus have $d(x_{\alpha_\beta}, x_{\alpha_{\beta'}}) \geq \delta$ for all distinct $\beta, \beta' < \lambda$, against the fact that X has weight $\kappa < \lambda$.

(ii) If d is spherically κ-complete, then it is also Cauchy-complete (independently of the weight of the space). Thus if d is spherically complete, then it is both spherically $<\kappa$-complete and Cauchy-complete.

(iii) The converse does not hold: there are examples of $\mathbb{G}$-metric spaces (X, d) of weight κ such that d is both Cauchy-complete and spherically $<\kappa$-complete, yet it is not spherically κ-complete. For example, consider the subspace $X = \{x_\alpha \in {}^\kappa 2 \mid \alpha < \kappa\}$ of ${}^\kappa 2$, where $x_\alpha(\alpha) = 1$ and $x_\alpha(\beta) = 0$ for all $\beta \neq \alpha$. Fix a decreasing sequence $(r_\alpha)_{\alpha<\kappa}$ converging to 0 in the distance group $\mathbb{G}$ and a strictly positive element $s \in \mathbb{G}$. The ultrametric $d(x_\alpha, x_\beta) = s + \max\{r_\alpha, r_\beta\}$ on X is discrete and hence trivially Cauchy-complete. Moreover, it is $<\kappa$-spherically complete. But the decreasing sequence $(B_d(x_\alpha, s + r_\alpha))_{\alpha<\kappa}$ has empty intersection. Thus for a given $\mathbb{G}$-metric d being Cauchy-complete and spherically $<\kappa$-complete is strictly weaker than being spherically ($\leq\kappa$-)complete.

[11] This notion is defined in multiple ways in different parts of the literature: for example, sometimes a metric is called spherically complete if the intersection of every decreasing sequence of *closed balls* is nonempty, or sometimes it is requirement that the balls are open, but the closure of each ball is contained in all previous ones. These alternative definitions closer resemble the original characterization of metrizability given by Cantor's intersection theorem. While there is difference between these definitions at the level of ($\mathbb{G}$-)metrics, the difference disappears at the level of the induced topologies (at least for $\mathbb{G}$-metrics for $\mathbb{G}$ of uncountable degree) and the two definitions lead to the same class of topological spaces. We opted for the current definition to be consistent with [42].

Definition 1.1.31. A $\mathbb{G}$-Polish space is **spherically** ($<\kappa$-)**complete** if it admits a compatible Cauchy-complete metric which is also spherically ($<\kappa$-)complete.

In [71], spherically $<\kappa$-complete $\mathbb{G}$-Polish spaces are also called strongly κ-Polish spaces. Although in view of Remark 1.1.30(iii) this seems to be the weakest among the two possibilities considered in Definition 1.1.31, it will follow from Theorem 1.1.32 that they are indeed equivalent: if a space of weight $\leq \kappa$ admits a compatible Cauchy-complete spherically $<\kappa$-complete $\mathbb{G}$-metric, then it also admits a (possibly different) compatible Cauchy-complete $\mathbb{G}$-metric which is (fully) spherically complete. We point out that the implication (c) $\Rightarrow$ (a) already appeared in [71, Theorem 2.45], although with a different terminology.

Theorem 1.1.32. *For any space X the following are equivalent:*

(a) *X is a κ-additive SC_κ-space;*

(b) *X is both an SC_κ-space and $\mathbb{G}$-Polish;*

(c) *X is a spherically $<\kappa$-complete $\mathbb{G}$-Polish space;*

(d) *X is a spherically complete $\mathbb{G}$-Polish space;*

(e) *X is homeomorphic to a superclosed subset of ${}^\kappa\kappa$.*

Proof. Item (b) implies (a) because all $\mathbb{G}$-Polish spaces are κ-additive (Lemma 1.1.19), while (a) implies (e) by Theorem 1.1.17. Moreover, any superclosed subset of ${}^\kappa\kappa$ is trivially spherically complete with respect to the $\mathbb{G}$-metric on ${}^\kappa\kappa$ defined in equation (1.1.1), thus (e) implies (d), and (d) obviously implies (c). Finally, to prove that (c) implies (b), recall that every $\mathbb{G}$-Polish space X is an $f\mathrm{SC}_\kappa$-space by Theorem 1.1.21. Fix a compatible spherically $<\kappa$-complete $\mathbb{G}$-metric on X and a winning strategy τ for II in $f\mathrm{G}^s_\kappa(X)$, and observe that by Remark 1.1.4 we can assume that τ requires II to play only open d-balls V_α because the latter form a basis for the topology of X. Then τ is also winning in $\mathrm{G}^s_\kappa(X)$ because spherically $<\kappa$-completeness implies that $\bigcap_{\alpha<\gamma} V_\alpha \neq \emptyset$ for every limit $\gamma < \kappa$. □

As any superclosed subset of ${}^\kappa\kappa$ is spherically complete $\mathbb{G}$-metrizable over any totally ordered (Abelian) groups of degree κ, Theorem 1.1.32 shows also that the notion of spherically complete $\mathbb{G}$-Polish space does not depend on the choice of $\mathbb{G}$.

Corollary 1.1.33. *Let $\mathbb{G}, \mathbb{G}'$ be two totally ordered (Abelian) groups, both of degree κ, and X be a space. Then X is spherically complete $\mathbb{G}$-Polish if and only if it is spherically complete $\mathbb{G}'$-Polish.*

Theorems 1.1.21 and 1.1.32 allow us to reformulate our Corollary 1.1.15 on retractions in terms of $\mathbb{G}$-Polish spaces. (Again, we have that none of the conditions on Y can be dropped, see the comment after Corollary 1.1.15.)

Corollary 1.1.34. *If X is $\mathbb{G}$-Polish, then all its closed subspaces Y which are also spherically complete $\mathbb{G}$-Polish (possibly with respect to a different $\mathbb{G}$-metric) are retracts of X.*

Moreover, using the results obtained so far, one can easily observe that the classes of SC_κ-spaces and $\mathbb{G}$-Polish spaces do not coincide. On the one hand, there are $\mathbb{G}$-Polish spaces which are not SC_κ-spaces: in [71, Theorem 2.41] it is observed that Sikorski's κ-$\mathbb{R}$ is such an example, but it is also enough to consider any closed subset of ${}^\kappa\kappa$ which is not strong κ-Choquet, such as the one defined in equation (1.1.3). Conversely, there are SC_κ-spaces which are not $\mathbb{G}$-Polish (to the best of our knowledge, examples of this kind were not yet provided in the literature): just take any non-κ-additive SC_κ-space, such as ${}^\kappa\kappa$ equipped with the order topology induced by the lexicographical ordering.

In a different direction, Theorem 1.1.32 allows us to characterize inside one given class those spaces which happen to also belong to a different one in a very natural way. For example, among SC_κ-spaces we can distinguish those that are also $\mathbb{G}$-Polish by checking κ-additivity. Conversely, working in the class of $\mathbb{G}$-Polish spaces we can isolate those spaces X in which player II wins the strong κ-Choquet game $\mathrm{G}^s_\kappa(X)$ by checking spherical completeness.

Figure 1 sums up the relationship among the various classes of (regular Hausdorff) spaces of weight $\leq \kappa$ considered so far. At the end of Section 1.2 we will further enrich this picture by distinguishing the class of κ-Lindelöf spaces—see Theorems 1.2.22 and 1.2.23.

Despite the fact that the classes we are considering are all different from each other, we now show that one can still pass from one to the other by changing (and sometimes even refining) the underlying topology yet maintaining the same notion of κ-Borelness.

Proposition 1.1.35. *Let (X,τ) be an $f\mathrm{SC}_\kappa$-space (respectively, SC_κ-space). Then there is $\tau' \supseteq \tau$ such that $\mathsf{Bor}_\kappa(X,\tau') = \mathsf{Bor}_\kappa(X,\tau)$ and (X,τ') is a κ-additive $f\mathrm{SC}_\kappa$-space (respectively, SC_κ-space).*

Proof. It is enough to let τ' be the topology generated by the $<\kappa$-sized intersections of τ-open sets. Arguing as in [42, Proposition 4.3 and Lemma 4.4], player II still has a winning strategy in the relevant Choquet-like game on (X,τ'). Moreover the weight of (X,τ') is still $\leq \kappa$ because we assumed $\kappa^{<\kappa} = \kappa$. Finally, κ-Borel sets do not change because by definition $\tau \subseteq \tau' \subseteq \mathsf{Bor}_\kappa(X,\tau)$. □

This allows us to strengthen [42, Theorem 3.3] and extend it to $f\mathrm{SC}_\kappa$-spaces.

Corollary 1.1.36. *If X is an $f\mathrm{SC}_\kappa$-space, then there is a pruned tree $T \subseteq {}^{<\kappa}\kappa$ and a continuous bijection $f\colon [T] \to X$. Moreover, if X is an SC_κ-space then T can be taken to be superclosed.*

Proof. Refine the topology τ of X to a topology $\tau' \supseteq \tau$ as in Proposition 1.1.35. Then use Theorem 1.1.13 to find a pruned (superclosed, if X was SC_κ) tree $T \subseteq {}^{<\kappa}\kappa$ and a homeomorphism $f\colon [T] \to (X,\tau')$. Since f remains a continuous bijection when stepping back to τ, we get that T and f are as required. □

By Proposition 1.1.35 (together with Theorem 1.1.21), every $f\mathrm{SC}_\kappa$-space, and thus every SC_κ-space, can be turned into a $\mathbb{G}$-Polish space sharing the same κ-Borel structure by suitably refining its topology. In contrast, it is not always possible to *refine* the topology τ of an $f\mathrm{SC}_\kappa$-space X to turn it into an SC_κ-space, even if we start with a κ-additive (hence $\mathbb{G}$-Polish) one and we further allow to change its κ-Borel structure. Indeed, as shown in the next example, there are κ-additive strongly fair κ-Choquet (i.e. $\mathbb{G}$-Polish) spaces (X, τ) such that for every topology $\tau' \supseteq \tau$, the space (X, τ') is not an SC_κ-space.

Example 1.1.37. Consider a closed (hence $\mathbb{G}$-Polish) subspace $C \subseteq {}^\kappa\kappa$ which is not a continuous image of ${}^\kappa\kappa$. Such a set exists by [105, Theorem 1.5]: we can e.g. let C be the set of well-orders on κ (coded as elements of ${}^\kappa 2 \subseteq {}^\kappa\kappa$ via the usual Gödel pairing function). If one could find a refinement τ' of the bounded topology on C such that (C, τ') is an SC_κ-space (recall that any SC_κ-space has weight $\leq \kappa$ by definition), then (C, τ') would be a continuous image of ${}^\kappa\kappa$ by [42, Theorem 3.5] and thus so would be (C, τ), contradicting the choice of C.

Nevertheless, if we drop the requirement that τ' refines the original topology τ of X, then we can get a result along the lines above. This is due to the next technical lemma, which will be further extended in Section 1.3 (see Corollary 1.3.3).

Lemma 1.1.38. *Every closed $C \subseteq {}^\kappa\kappa$ is κ-Borel isomorphic to a superclosed set $C' \subseteq {}^\kappa\kappa$.*

Proof. If C has $\leq\kappa$-many points, then any bijection between C and $C' = \{\alpha ^\frown 0^{(\kappa)} \mid \alpha < |C|\}$, where $0^{(\kappa)}$ is the constant sequence with length κ and value 0, is a κ-Borel isomorphism between C and the superclosed set C', hence we may assume without loss of generality that $|C| > \kappa$. Let $T \subseteq {}^{<\kappa}\kappa$ be a pruned tree such that $C = [T]$. Let $L(T)$ be the set of sequences $s \in {}^{<\kappa}\kappa$ of limit length such that $s \notin T$ but $s \restriction \alpha \in T$ for all $\alpha < \mathrm{lh}(s)$. (Clearly, the set $L(T)$ is empty if and only if C is already superclosed). Set $C' = [T']$ with

$$T' = T \cup \{s ^\frown 0^{(\alpha)} \mid s \in L(T) \wedge \alpha < \kappa\},$$

where $0^{(\alpha)}$ denotes the sequence of length α constantly equal to 0. The tree T' is clearly pruned and $<\kappa$-closed, hence C' is superclosed. Notice also that $C' \setminus C = \{s ^\frown 0^{(\kappa)} \mid s \in L(T)\}$ has size $\leq \kappa$. Pick a set $A \subseteq C$ of size κ and fix any bijection $g \colon A \to A \cup (C' \setminus C)$. Since both C and C' are Hausdorff, it is easy to check that the map

$$f \colon C \to C', \qquad x \mapsto \begin{cases} g(x) & \text{if } x \in A \\ x & \text{otherwise} \end{cases}$$

is a κ-Borel isomorphism. ∎

Combining this lemma with Proposition 1.1.35 and Theorem 1.1.16 we thus get

Proposition 1.1.39. *Let (X, τ) be an $f\mathrm{SC}_\kappa$-space. Then there is a topology τ' on X such that $\mathsf{Bor}_\kappa(X, \tau') = \mathsf{Bor}_\kappa(X, \tau)$ and (X, τ') is a κ-additive SC_κ-space (equivalently, a spherically complete $\mathbb{G}$-Polish space).*

As a corollary, we finally obtain:

Theorem 1.1.40. *Up to κ-Borel isomorphism, the following classes of spaces are the same:*

(1) *$f\mathrm{SC}_\kappa$-spaces;*

(2) SC_κ*-spaces;*

(3) $\mathbb{G}$*-Polish spaces;*

(4) κ*-additive* SC_κ*-spaces or, equivalently, spherically complete* $\mathbb{G}$*-Polish spaces.*

Theorem 1.1.40 shows that, as we already claimed after Definition 1.1.1, we can considered any class of Polish-like spaces to generalize (St.Bor. 1): they all yield the same notion, and it is thus not necessary to formally specify one of them. Furthermore, in Section 1.3 we will prove that the class of κ-Borel spaces obtained in this way coincide with the class of all standard κ-Borel spaces as defined in Definition 1.1.1, so we do not even need to introduce a different terminology.

The sweeping results obtained so far allow us to improve some results from the literature and close some open problems contained therein, so let us conclude this section with a brief discussion on this matter. In [71, Theorem 2.51] it is proved that, in our terminology, if X is a spherically $<\kappa$-complete $\mathbb{G}$-Polish space and κ is weakly compact, then every SC_κ-subspace $Y \subseteq X$ is G^κ_δ in X. By Theorem 1.1.21 and Corollary 1.1.27, we actually have that every SC_κ-subspace Y of a $\mathbb{G}$-metrizable space X is G^κ_δ in X: hence the further hypotheses on κ and X required in [71, Theorem 2.51] are not necessary. Furthermore, in [71, Lemma 2.47] the converse is shown to hold assuming that X is a $\mathbb{G}$-metric SC_κ-space (which through κ-additivity implies that X is $\mathbb{G}$-Polish by Theorem 1.1.32 again) and Y is spherically $<\kappa$-complete. Theorems 1.1.28 and 1.1.32 show that we can again weaken the hypotheses on X by dropping the requirement that X be a SC_κ-space: if X is $\mathbb{G}$-Polish and $Y \subseteq X$ is spherically $<\kappa$-complete and G^κ_δ, then Y is a SC_κ-space. Finally, Theorem 1.1.32 shows that [71, Theorem 2.53] and [42, Proposition 3.1] deal with the same phenomenon: if X is a κ-perfect SC_κ-space, there is a continuous injection f from the generalized Cantor space into X, and if furthermore X is κ-additive, then f can be taken to be an homeomorphism on the image. This will be slightly improved in Theorem 1.2.6, where we show that in the latter case the range of f can be taken to be superclosed.

Summing up the results above, one can now complete and improve the diagram in [71, p. 25], which corresponds to Arrows 1–7 of Figure 1.1 (although [71] sometimes requires additional assumption on the space Y or on the cardinal κ, see the discussion below).

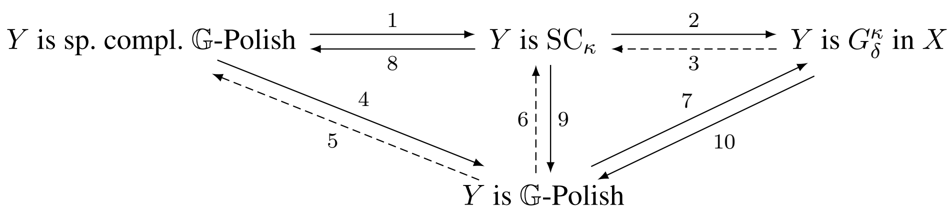

Figure 1.1: Properties of subspaces $Y \subseteq X$ for X a $\mathbb{G}$-Polish space. A line means implication without further assumptions, while a dotted line means that the implication holds under the further assumption that Y is spherically complete or, equivalently, an SC_κ-space.

Here is a list of our improvements:

- First of all, the ambient space X can be any $\mathbb{G}$-Polish space, and need not to be spherically complete as assumed in [71].
- The new Arrows 8 and 9 hold because by $\mathbb{G}$-metrizability Y is κ-additive, and hence a spherically $<\kappa$-complete $\mathbb{G}$-Polish space.
- Arrow 10 holds as well by Theorem 1.1.28.
- The implication Arrow 2 holds unconditionally (κ needs not to be weakly compact, as originally required in [71]).
- The requirement that, in our terminology, Y be spherically $<\kappa$-complete cannot instead be dropped in the implication Arrow 3: indeed, there are even closed subsets of $X = {}^{\kappa}\kappa$ which are not homeomorphic to a superclosed subset of ${}^{\kappa}\kappa$, and hence they are not strong κ-Choquet. Thus in this case the hypothesis in [71] were already optimal.
- We now obtained that Arrows 5 and 6, which were forbidden in [71], holds when additionally requiring that Y be spherically $< \kappa$-complete (same hypothesis as in Arrow 3): taking into account Galeotti's counterexamples, such hypothesis cannot be dropped.

1.2 Characterizations of ${}^{\kappa}\kappa$ and ${}^{\kappa}2$

The (classical) Cantor and Baire spaces play a central role in classical descriptive set theory. It is remarkable that they admit a purely topological characterization (see [93, Theorems 7.4 and 7.7]).

Theorem 1.2.1.

(1) *(Brouwer) Up to homeomorphism, the Cantor space ${}^{\omega}2$ is the unique nonempty perfect compact metrizable zero-dimensional space.*

(2) *(Alexandrov-Urysohn) Up to homeomorphism, the Baire space ${}^{\omega}\omega$ is the unique nonempty Polish zero-dimensional space such that all its compact subsets have empty interior.*

Our next goal is to find analogous characterizations of the generalized Baire and Cantor spaces. To this aim, we first have to generalize the above mentioned topological notions to our setup.

First of all, we notice that a special feature of ${}^{\kappa}\kappa$ and ${}^{\kappa}2$ which is not shared by some of the other SC_{κ}-spaces is κ-additivity: since this condition already implies that the space be zero-dimensional, the latter will always be absorbed by κ-additivity and will not explicitly appear in our statements. As for compactness, it is natural to replace it with the property of being κ-Lindelöf. Notice that this condition may play a role in the characterization of ${}^{\kappa}2$ only when κ is weakly compact, as otherwise ${}^{\kappa}2$ is not κ-Lindelöf. However, this is not a true limitation, because if κ is not weakly compact, then the spaces ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ are homeomorphic, and thus the characterization of ${}^{\kappa}\kappa$ takes care of both. In view of the Hurewicz dichotomy [93, Theorem 7.10], which in [104] has been analyzed in detail in the context of generalized descriptive set theory, we will also consider K_κ**-sets**, i.e. sets in a topological space which can be written as unions of κ-many κ-Lindelöf sets.

We now come to perfectness. The notion of an isolated point may be transferred to the generalized context in (at least) two natural ways:

- keeping the original definition: a point x is **isolated** in X if there is an open set $U \subseteq X$ such that $U = \{x\}$;
- allowing short intersections of open sets (see e.g. [42, Section 3]): a point x is κ**-isolated** in X if there are $<\kappa$-many open sets whose intersection is $\{x\}$.

A topological space is then called (κ**-**)**perfect** if it has no (κ-)isolated points.

If we restrict the attention to κ-additive spaces, as we do in this section, the two notions coincide. However, the notion of κ-perfectness is in a sense preferable when the space X is not κ-additive because it implies that X has weight at least κ and that all its nonempty open sets have size $\geq \kappa$ (use the regularity of κ and the fact that all our spaces are Hausdorff). If we further require X to be strong κ-Choquet, we get the following strengthening of the last property.

Lemma 1.2.2. *Let X be an SC_κ-space. If X is κ-perfect, then every open set $U \subseteq X$ has size 2^κ.*

Proof. If X is κ-perfect, then so is every open $U \subseteq X$. Since U is strong κ-Choquet as well, there is a continuous injection from ${}^\kappa 2$ into U by [42, Proposition 3.1], hence $|U| = 2^\kappa$. □

In the statement of Lemma 1.2.2 one could further replace the open set U with a $<\kappa$-sized intersection of open sets. The lemma is instead not true for arbitrary $f\mathrm{SC}_\kappa$-spaces, even when requiring κ-additivity (and thus it does not work for arbitrary $\mathbb{G}$-Polish spaces as well). For a counterexample, consider the closed subspace X_0 of ${}^\kappa 2$ defined in equation (1.1.3): by Theorem 1.1.21, X_0 is a κ-additive $f\mathrm{SC}_\kappa$-space (equivalently, a $\mathbb{G}$-Polish space), it is clearly κ-perfect, yet it has size κ.

In the next lemma we crucially use the fact that κ is such that $\kappa^{<\kappa} = \kappa$.

Lemma 1.2.3. *If Y is a T_0-space of size $> \kappa$, then Y has weight $\geq \kappa$.*

Proof. Let $\mathcal{B}$ be any basis of Y. Then the map sending each point of Y into the set of its basic open neighborhoods is an injection into $\mathscr{P}(\mathcal{B})$. Thus if there is such a $\mathcal{B}$ of size $\nu < \kappa$ then $|Y| \leq 2^\nu \leq \kappa^{<\kappa} = \kappa$. □

A tree $T \subseteq {}^{<\kappa}\kappa$ is **splitting** if for every $s \in T$ there are incomparable $t, t' \in T$ extending s (without loss of generality we can further require that $\mathrm{lh}(t) = \mathrm{lh}(t')$). We now show that the splitting condition captures the topological notion of perfectness for κ-additive SC_κ-spaces. (Notice that the equivalence between items (a) and (e) in Lemma 1.2.4 may be seen as the analogue of Theorem 1.1.17 for (κ-)perfect κ-additive SC_κ-spaces.)

Lemma 1.2.4. *Let X be a κ-additive SC_κ-space. The following are equivalent:*

(a) *X is (κ-)perfect;*

(b) *every nonempty open subset of X has size $> \kappa$;*

(c) *every nonempty open subspace of X has weight κ;*

(d) *every superclosed $T \subseteq {}^{<\kappa}\kappa$ such that X is homeomorphic to $[T]$ is splitting;*

(e) *there is a splitting superclosed*[12] *tree $T \subseteq {}^{<\kappa}\kappa$ with $[T]$ homeomorphic to X.*

Proof. The implication (a) $\Rightarrow$ (b) is Lemma 1.2.2, while the implication (b) $\Rightarrow$ (c) follows from Lemma 1.2.3. In order to prove (c) $\Rightarrow$ (d), notice that if $s \in T$ then $\mathbf{N}_s \cap [T] \neq \emptyset$ because T is superclosed. Thus s must have two incomparable extensions, since otherwise $\mathbf{N}_s \cap [T]$ would be a nonempty open set of weight (and size) 1. The implication (d) $\Rightarrow$ (e) follows from Theorem 1.1.17, which ensures the existence of a superclosed $T \subseteq {}^{<\kappa}\kappa$ with $[T]$ homeomorphic to X: such a T is then necessarily splitting by condition (d). Finally, for the implication (e) $\Rightarrow$ (a) notice that if T is splitting and superclosed, then for every two incomparable extensions $t, t' \in T$ of a given $s \in T$ we have $\mathbf{N}_t \cap [T] \neq \emptyset$ and $\mathbf{N}_{t'} \cap [T] \neq \emptyset$ but $\mathbf{N}_t \cap \mathbf{N}_{t'} = \emptyset$, hence $|\mathbf{N}_s \cap [T]| > 1$ for all $s \in T$. $\square$

Remark 1.2.5. Notice that if κ is inaccessible, then the splitting condition on the superclosed tree T in items (d) and (e) above can be strengthened to

$$\forall s \in T \, \forall \nu < \kappa \, \exists \alpha < \kappa \, (\alpha > \mathrm{lh}(s) \wedge |\mathrm{Lev}_\alpha(T_s)| \geq \nu). \tag{1.2.1}$$

Notice also that if $\alpha < \kappa$ witnesses (1.2.1) for given $s \in T$ and $\nu < \kappa$, then every $\alpha \leq \alpha' < \kappa$ witnesses the same fact because T is pruned.

Lemma 1.2.4 allows us to prove the following strengthening of [42, Proposition 3.1] and [71, Theorem 2.53], answering in particular [42, Question 3.2] for the case of κ-additive spaces.

Theorem 1.2.6. *Let X be a nonempty κ-additive SC_κ-space. If X is (κ-)perfect, then there is a closed*[13] *$C \subseteq X$ which is homeomorphic to ${}^\kappa 2$.*

Proof. By Lemma 1.2.4 we may assume that $X = [T]$ with $T \subseteq {}^{<\kappa}\kappa$ superclosed and splitting. Recursively define a map $\varphi \colon {}^{<\kappa}2 \to T$ by setting $\varphi(\emptyset) = \emptyset$ and then letting $\varphi(t^\frown 0)$ and $\varphi(t^\frown 1)$ be incomparable extensions in T of the sequence of $\varphi(t)$. At limit levels we set $\varphi(t) = \bigcup_{\alpha < \mathrm{lh}(t)} \varphi(t \restriction \alpha)$, which is still an element of T because the latter is $<\kappa$-closed.

By construction, φ is a tree-embedding from ${}^{<\kappa}2$ into T, i.e. φ is monotone and preserves incomparability. Moreover, $\mathrm{lh}(\varphi(t)) \geq \mathrm{lh}(t)$ for every $t \in {}^{<\kappa}2$. Let T' be the subtree of T generated by $\varphi({}^{<\kappa}2)$, that is

$$T' = \{s \in T \mid s \subseteq \varphi(t) \text{ for some } t \in {}^{<\kappa}2\}.$$

It is easy to see that T' is pruned. We now want to check that it is also $< \kappa$-closed by showing that if $s \notin T'$ for some s of limit length, then there is $\alpha < \mathrm{lh}(s)$ such that $s \restriction \alpha \notin T'$. (We present a detailed argument because the claim uses in an essential way that ${}^{<\kappa}2$ is finitely splitting, and would instead fail if e.g. ${}^{<\kappa}2$ is replaced by ${}^{<\kappa}\omega$.) Set $A = \{t \in {}^{<\kappa}2 \mid \varphi(t) \subseteq s\}$. Since φ preserves incomparability, all sequences in A are comparable and thus the sequence $\bar{t} = \bigcup\{t \mid t \in A\} \in {}^{<\kappa}2$ is

[12] This is a bit redundant: if T is splitting and $<\kappa$-closed, then it is also automatically pruned.

[13] Superclosed if $X \subseteq {}^\kappa\kappa$.

well-defined and such that $\varphi(\bar{t}) \subsetneq s$ (here we use that φ is defined in a continuous way at limit levels and $s \notin T'$). Since $s \notin T'$, the sequences $\varphi(\bar{t}^\frown 0)$ and $\varphi(\bar{t}^\frown 1)$ are both incomparable with s by the choice of $\bar{t}$, and since $\mathrm{lh}(s)$ is limit there is $\mathrm{lh}(\varphi(\bar{t})) < \alpha < \mathrm{lh}(s)$ such that the above sequences are incomparable with $s \restriction \alpha$ as well: we claim that such α is as required. Given an arbitrary $t \in {}^{<\kappa}2$, we distinguish various cases. If t is incomparable with $\bar{t}$, then $\varphi(t)$ is incomparable with $\varphi(\bar{t})$ and thus with $s \restriction \alpha$ as well because by construction $\varphi(\bar{t}) \subseteq s \restriction \alpha$. If $t \subseteq \bar{t}$, then by monotonicity of φ we have that $\varphi(t) \subseteq \varphi(\bar{t}) = s \restriction \mathrm{lh}(\varphi(\bar{t}))$ and thus $\varphi(t)$ is a proper initial segment of $s \restriction \alpha$ by $\alpha > \mathrm{lh}(\varphi(\bar{t}))$. Finally, if t properly extends $\bar{t}$, then $t \supseteq \bar{t}^\frown i$ for some $i \in \{0, 1\}$: but then $\varphi(t) \supseteq \varphi(\bar{t}^\frown i)$ is incomparable with $s \restriction \alpha$ again. So in all cases we get that $s \restriction \alpha \not\subseteq \varphi(t)$, and since t was arbitrary this entails $s \restriction \alpha \notin T'$, as required.

This shows that T' is a superclosed subtree of T. Moreover, φ canonically induces the function $f_\varphi \colon {}^\kappa 2 \to C = [T']$ where

$$f_\varphi(x) = \bigcup\nolimits_{\alpha<\kappa} \varphi(x \restriction \alpha),$$

which is well-defined by monotonicity of φ and $\mathrm{lh}(\varphi(x \restriction \alpha)) \geq \alpha$. Moreover f_φ is a bijection because φ is a tree-embedding, and by construction $f_\varphi(\mathbf{N}_t) = \mathbf{N}_{\varphi(t)} \cap C$ for all $t \in {}^{<\kappa}2$. Since $\{\mathbf{N}_{\varphi(t)} \cap C \mid t \in {}^{<\kappa}2\}$ is clearly a basis for C, this shows that f_φ is a homeomorphism between ${}^\kappa 2$ and C. □

Remark 1.2.7. Notice that the space X_0 from equation 1.1.3 is a nonempty κ-additive (κ-)perfect, $f\mathrm{SC}_\kappa$-space of size λ. Thus, Theorem 1.2.6 fails for the class of $f\mathrm{SC}_\kappa$-spaces.

The previous theorem can be turned into the following characterization: a topological space contains a closed homeomorphic copy of ${}^\kappa 2$ if and only if it contains a nonempty closed (κ-)perfect κ-additive SC_κ- subspace.

Finally, we briefly discuss κ-Lindelöf and K_κ-sets. The Alexandrov-Urysohn characterization of the Baire space (Theorem 1.2.1(2)) implicitly deals with Baire category. In fact, compact sets are closed, thus requiring that they have empty interior is equivalent to requiring that they are nowhere dense. The latter notion makes sense also in the generalized setting, but the notion of meagerness needs to be replaced with κ-meagerness, where a subset $A \subseteq X$ is called **κ-meager** if it can be written as a union of κ-many nowhere dense sets. A topological space is **κ-Baire** if no non-empty open subset of X is κ-meager. It is not difficult to see that if κ is regular then ${}^\kappa\kappa$ is κ-Baire (see e.g. [65, 6]), so the next lemma applies to it.

Lemma 1.2.8. *Suppose that X is a κ-additive κ-Baire space. Then the following are equivalent:*

(a) *all κ-Lindelöf subsets of X have empty interior;*

(b) *all K_κ subsets of X have empty interior.*

Proof. The nontrivial implication (a) ⇒ (b) follows from the fact that if we have $A = \bigcup_{\alpha<\kappa} A_\alpha \subseteq X$ with all A_α's κ-Lindelöf, then A is κ-meager because in a κ-additive space all κ-Lindelöf sets are necessarily closed and thus, by (a), the A_α's are nowhere dense; thus the interior of A, being κ-meager as well, must be the empty set. □

Finally, observe that if a space X can be partitioned into κ-many nonempty clopen sets, then it is certainly not κ-Lindelöf. The next lemma shows that the converse holds as well if X is κ-additive and of weight at most κ.

Lemma 1.2.9. *Let X be a nonempty κ-additive space of weight $\leq \kappa$. If X is not κ-Lindelöf, then it can be partitioned into κ-many nonempty clopen subsets.*

Proof. Since X is zero-dimensional and not κ-Lindelöf, there is a clopen covering $\{U_\alpha \mid \alpha < \kappa\}$ of it which does not admit a $<\kappa$-sized subcover. Without loss of generality, we may assume that $U_\alpha \not\subseteq \bigcup_{\beta<\alpha} U_\beta$. Then the sets $V_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$ form a κ-sized partition of X. Since by κ-additivity the V_α's are clopen, we are done. $\square$

We are now ready to characterize the generalized Baire space ${}^\kappa\kappa$ in the class of SC_κ-spaces (compare it with Theorem 1.2.1(2)).

Theorem 1.2.10. *Up to homeomorphism, the generalized Baire space ${}^\kappa\kappa$ is the unique nonempty κ-additive SC_κ-space for which all κ-Lindelöf subsets (equivalently: all K_κ-subsets) have empty interior.*

Proof. Clearly, ${}^\kappa\kappa$ is a κ-additive SC_κ-space. Moreover, every κ-Lindelöf subset of ${}^\kappa\kappa$ has empty interior as otherwise for some $s \in {}^{<\kappa}\kappa$ the basic clopen set $\mathbf{N}_s$ would be κ-Lindelöf as well, which is clearly false because $\{\mathbf{N}_{s^\frown\alpha} \mid \alpha < \kappa\}$ is a κ-sized clopen partition of $\mathbf{N}_s$. By Lemma 1.2.8 and the fact that ${}^\kappa\kappa$ is κ-Baire we get that also the K_κ-subsets of ${}^\kappa\kappa$ have empty interior.

Conversely, let X be any nonempty κ-additive SC_κ-space all of whose κ-Lindelöf subsets have empty interior. By Theorem 1.1.17 we may assume that $X = [T]$ for some superclosed tree $T \subseteq {}^{<\kappa}\kappa$: our aim is to define a homeomorphism between ${}^\kappa\kappa$ and $[T]$. We recursively define a map $\varphi \colon {}^{<\kappa}\kappa \to T$ by setting $\varphi(\emptyset) = \emptyset$ and $\varphi(t) = \bigcup_{\alpha<\mathrm{lh}(t)} \varphi(t \restriction \alpha)$ if $\mathrm{lh}(t)$ is limit (this is still a sequence in T because the latter is $<\kappa$-closed). For the successor step, assume that $\varphi(t)$ has already been defined. Notice that $\mathbf{N}_{\varphi(t)} \cap [T]$ is open and nonempty (because T is superclosed), hence it is not κ-Lindelöf by assumption. By Lemma 1.2.9 there is a κ-sized partition of $\mathbf{N}_{\varphi(t)} \cap [T]$ into clopen sets, which can then be refined to a partition of the form $\{\mathbf{N}_{t_\alpha} \cap [T] \mid \alpha < \kappa\}$: set $\varphi(t^\frown\alpha) = t_\alpha$. It is now easy to see that the function

$$f_\varphi \colon {}^\kappa\kappa \to X, \quad x \mapsto \bigcup_{\alpha<\kappa} \varphi(x \restriction \alpha)$$

induced by φ is a homeomorphism between ${}^\kappa\kappa$ and X. $\square$

Theorem 1.2.10 can be used to get an easy proof of the fact that ${}^\kappa 2$ is homeomorphic to ${}^\kappa\kappa$ when κ is not weakly compact, i.e. when ${}^\kappa 2$ is not κ-Lindelöf itself. Indeed, ${}^\kappa 2$ is clearly a nonempty κ-additive SC_κ-space, so it is enough to check that all its κ-Lindelöf subsets have empty interior. But for zero-dimensional spaces this is equivalent to the fact that every nonempty open subspace is not κ-Lindelöf, which in this case is true because all basic open subsets of ${}^\kappa 2$ are homeomorphic to it, and thus they are not κ-Lindelöf.

We next move to the characterization(s) of ${}^\kappa 2$. When κ is not weakly compact, Theorem 1.2.10 already does the job, but we are anyway seeking a generalization along the lines of Brouwer's characterization of ${}^{\omega}2$ from Theorem 1.2.1(1) (thus involving perfectness and suitable compactness properties). Since ${}^{\kappa}2$ is κ-Lindelöf if and only if κ is weakly compact, we distinguish between the corresponding two cases and first concentrate on the case when κ is not weakly compact. In this situation, there is no space at all sharing all (natural generalizations of) the conditions appearing in Theorem 1.2.1(1).

Proposition 1.2.11. *Let κ be a non weakly compact cardinal. Then there is no nonempty κ-additive (κ-)perfect κ-Lindelöf SC_κ-space.*

Proof. Suppose towards a contradiction that there is such a space X. By Theorem 1.2.6, we could then find a homeomorphic copy $C \subseteq X$ of ${}^{\kappa}2$ with C closed in X. But then C, and hence also ${}^{\kappa}2$, would be κ-Lindelöf, contradicting the fact that κ is not weakly compact. □

Proposition 1.2.11 seems to suggest the we already reached a dead end in our attempt to generalize Brouwer's theorem for non-weakly compact cardinals. However, this is quite not true: we are now going to show that relaxing even just one of the conditions on the space give a compatible set of requirements. For example, if we restrict the attention to κ-Lindelöf SC_κ-spaces, then κ-additivity and κ-perfectness cannot coexists by Proposition 1.2.11, but they can be satisfied separately. Indeed, the space

$$X = \{x \in {}^{\kappa}2 \mid x(\alpha) = 0 \text{ for at most one } \alpha < \kappa\}$$

is a κ-additive κ-Lindelöf SC_κ-space, while endowing ${}^{\kappa}2$ with the product topology (rather than the bounded topology) we get a κ-perfect κ-Lindelöf (in fact, compact) SC_κ-space. If instead we weaken the Choquet-like condition to being just a $f\mathrm{SC}_\kappa$-space, then we have the following example.

Proposition 1.2.12. *There exists a nonempty κ-additive (κ-)perfect κ-Lindelöf $f\mathrm{SC}_\kappa$-space.*

Proof. Consider the tree $T_0 = \{s \in {}^{<\kappa}2 \mid |\{\alpha \mid s(\alpha) = 0\}| < \omega\}$ and the space $X_0 = [T_0]$ from equation (1.1.3), which is clearly a κ-additive (κ-)perfect $f\mathrm{SC}_\kappa$-space. Suppose towards a contradiction that X_0 is not κ-Lindelöf, and let $\mathcal{F}$ be a clopen partition of X_0 of size κ (which exists by Lemma 1.2.9). Without loss of generality, we may assume that each set in $\mathcal{F}$ is of the form $\mathbf{N}_s \cap [T_0]$ for some $s \in T_0$. Set $F = \{s \in T_0 \mid \mathbf{N}_s \cap [T_0] \in \mathcal{F}\}$: then F is a maximal antichain in T_0, i.e. distinct $s, t \in F$ are incomparable and for each $x \in [T_0]$ there is $s \in F$ such that $s \subseteq x$. By definition, each sequence $s \in F$ has only a finite number of coordinates with value 0: for each $n \in \omega$, let F_n be the set of those $s \in F$ that have exactly n-many zeros. Since $|F| = \kappa$ and $\{F_n \mid n \in \omega\}$ is a partition of F, there exists some n such that $|F_n| = \kappa$: let ℓ be the smallest natural number with this property, and set $F_{<\ell} = \bigcup_{n<\ell} F_n$. Then $|F_{<\ell}| < \kappa$ and $\gamma = \sup\{\mathrm{lh}(s) \mid s \in F_{<\ell}\} < \kappa$ by regularity of κ.

We claim that there is $s \in F_\ell$ such that $s(\beta) = 0$ for some $\gamma \leq \beta < \mathrm{lh}(s)$. If not, the map $s \mapsto \{\alpha < \mathrm{lh}(s) \mid s(\alpha) = 0\}$ would be an injection (because F is an antichain) from F_ℓ to $\{A \subseteq \gamma \mid |A| = \ell\}$, contradicting $|F_\ell| = \kappa$. Given now s as above, let $x = (s \restriction \gamma)^\frown 1^{(\kappa)}$. Then $x \in X_0$ and $|\{\alpha < \kappa \mid x(\alpha) = 0\}| < \ell$, thus there is $t \in F_{<\ell}$ such that $x \in \mathbf{N}_t \cap [T_0]$. Since $t \in F_{<\ell}$ implies $\mathrm{lh}(t) \leq \gamma$, this means that $t \subseteq x \restriction \gamma = s \restriction \gamma \subseteq s$, contradicting the fact that F is an antichain. □

The remaining option is to drop the condition of being κ-Lindelöf. In a sense, this is the most promising move, as we are assuming that κ is not weakly compact and thus ${}^{\kappa}2$, the space we are trying to characterize, thus not satisfy such a property. Indeed, we are now going to show that dropping such (wrong) requirement, we already get the desired characterization.

Lemma 1.2.13. *Suppose that κ is not weakly compact and X is a κ-additive SC_κ-space. Then X is (κ-)perfect if and only if every κ-Lindelöf subsets of X has empty interior.*

Proof. It is clear that if all κ-Lindelöf subsets of X have empty interior, then X has no isolated point because if $x \in X$ is isolated then $\{x\}$ is open and trivially κ-Lindelöf. Suppose now that X is perfect but has a κ-Lindelöf subset with nonempty interior. By zero-dimensionality, it would follow that there is a nonempty clopen set $O \subseteq X$ which is κ-Lindelöf. But then O would be a nonempty κ-additive perfect κ-Lindelöf SC_κ-space, contradicting Proposition 1.2.11. □

Lemma 1.2.13 allows us to replace the last condition in the characterization of ${}^{\kappa}\kappa$ from Theorem 1.2.10 with (κ-)perfectness. Together with the fact that ${}^{\kappa}\kappa$ is homeomorphic to ${}^{\kappa}2$ when κ is not weakly compact, this leads us to the following analogue of Theorem 1.2.1(1) (which of course can also be viewed as an alternative characterization of ${}^{\kappa}\kappa$).

Theorem 1.2.14. *Let κ be a non weakly compact cardinal. Up to homeomorphism, the generalized Cantor space ${}^{\kappa}2$ (and hence also ${}^{\kappa}\kappa$) is the unique nonempty κ-additive (κ-)perfect SC_κ-space.*

We now move to the case when κ is weakly compact. In contrast to the previous situation, the condition of being κ-Lindelöf obviously becomes relevant (and necessary) because ${}^{\kappa}2$ now has such property—this is the only difference between Theorem 1.2.14 and Theorem 1.2.15.

Theorem 1.2.15. *Let κ be a weakly compact cardinal. Up to homeomorphism, the generalized Cantor space ${}^{\kappa}2$ is the unique nonempty κ-additive (κ-)perfect κ-Lindelöf SC_κ-space.*

Proof. For the nontrivial direction, let X be any nonempty perfect κ-additive κ-Lindelöf SC_κ-space. By Lemma 1.2.4(e) we may assume that $X = [T]$ for some splitting superclosed tree $T \subseteq {}^{\kappa}\kappa$. Notice that the fact that X is κ-Lindelöf entails that $|\mathrm{Lev}_\alpha(T)| < \kappa$ for all $\alpha < \kappa$: this will be used in combination with the strong form of the splitting condition from equation (1.2.1) in Remark 1.2.5 to prove the following claim.

Claim 1.2.15.1. For every $\alpha < \kappa$ there is $\beta < \kappa$ such that $|\mathrm{Lev}_{\alpha+\beta}(T_t)| = |{}^{\beta}2|$ for all $t \in \mathrm{Lev}_\alpha(T)$.

Proof. Recursively define a sequence of ordinals $(\gamma_n)_{n\in\omega}$, as follows. Set $\gamma_0 = 0$. Suppose now that the γ_i have been defined for all $i \leq n$, and set $\bar{\gamma}_n = \sum_{i\leq n} \gamma_i$. Then choose $\gamma_{n+1} < \kappa$ large enough to ensure

(1) $\gamma_{n+1} \geq \max\left\{2^{|\gamma_n|}, |\mathrm{Lev}_{\alpha+\bar{\gamma}_n}(T)|\right\}$;

(2) $|\mathrm{Lev}_{\alpha+\bar{\gamma}_n+\gamma_{n+1}}(T_s)| \geq |\gamma_n|$ for all $s \in \mathrm{Lev}_{\alpha+\bar{\gamma}_n}(T)$.

Such a γ_{n+1} exists because $|\mathrm{Lev}_{\alpha+\bar{\gamma}_n}(T)| < \kappa$ (because X is κ-Lindelöf) and $2^{|\gamma_n|} < \kappa$ (because κ is inaccessible). Set $\beta = \sum_{n\in\omega} \gamma_n = \sup_{n\in\omega} \bar{\gamma}_n$. By construction, $|{}^{\beta}2| = \prod_{n\in\omega} 2^{|\gamma_n|} = \prod_{n\in\omega} |\gamma_n|$. On the other hand for every $t \in \mathrm{Lev}_\alpha(T)$ we have

$$\prod_{n\in\omega} |\gamma_n| \leq |\mathrm{Lev}_{\alpha+\beta}(T_t)| \leq |\mathrm{Lev}_{\alpha+\beta}(T)| \leq \prod_{n\in\omega} |\gamma_n|,$$

where the first inequality follows from (2) while the last one follows from (1). □

Using Claim 1.2.15.1 we can easily construct a club $0 \in C \subseteq \kappa$ such that if $(\alpha_i)_{i<\kappa}$ is the increasing enumeration of C and β_i is such that $\alpha_{i+1} = \alpha_i + \beta_i$, then there is a bijection $\varphi_t \colon \mathrm{Lev}_{\alpha_{i+1}}(T_t) \to {}^{\beta_i}2$ for every $i < \kappa$ and $t \in \mathrm{Lev}_{\alpha_i}(T)$.

Define $\varphi \colon T \to {}^{<\kappa}2$ by recursion on $\mathrm{lh}(s)$ as follows. Set $\varphi(\emptyset) = \emptyset$. For an arbitrary $s \in T \setminus \{\emptyset\}$, let $j < \kappa$ be largest such that $\alpha_j \leq \mathrm{lh}(s)$ (here we use that C is a club). If $\alpha_j < \mathrm{lh}(s)$, set $\varphi(s) = \varphi(s \restriction \alpha_j)$. If instead $\alpha_j = \mathrm{lh}(s)$, then we distinguish two cases. If $j = i+1$ we set $\varphi(s) = \varphi(s \restriction \alpha_i) {}^\frown \varphi_{s\restriction\alpha_i}(s)$; if j is limit (whence also $\mathrm{lh}(s)$ is limit), we set $\varphi(s) = \bigcup_{\beta<\mathrm{lh}(s)} \varphi(s \restriction \beta)$.

It is clear that φ is $\subseteq$-monotone and for all $\alpha \in C$ the restriction of φ to $\mathrm{Lev}_\alpha(T)$ is a bijection with ${}^{\alpha}2$. It easily follows that

$$f_\varphi \colon [T] \to {}^{\kappa}2, \quad x \mapsto \bigcup_{\alpha<\kappa} \varphi(x \restriction \alpha)$$

is a homeomorphism, as required. □

The proof of the nontrivial direction in Theorem 1.2.15 requires κ to be just inaccessible (and not necessarily weakly compact). The stronger hypothesis on κ in the statement is indeed due to the other direction: if κ is not weakly compact, then ${}^{\kappa}2$ is not κ-Lindelöf and, indeed, by Proposition 1.2.11 there are no spaces at all as in the statement.

Remark 1.2.16. It is easy to check that the function f_φ constructed in the previous proof preserves superclosed sets, that is, it is such that $C \subseteq [T]$ is superclosed if and only if $f_\varphi(C) \subseteq {}^{\kappa}2$ is superclosed. This follows from the fact that if S is a superclosed subtree of T, then the $\subseteq$-downward closure of $\varphi(S)$ is a superclosed subtree S' of ${}^{\kappa}2$; conversely, if $S' \subseteq {}^{<\kappa}2$ is a superclosed tree, then $S = \{t \in T \mid \varphi(t) \in S'\}$ is a superclosed subtree of T.

In view of Theorem 1.1.32, most of the characterizations provided so far can equivalently be rephrased in the context of $\mathbb{G}$-Polish spaces. For example, the following is the characterization of the generalized Cantor and Baire spaces in terms of $\mathbb{G}$-metrics.

Theorem 1.2.17.

(1) *Up to homeomorphism, the generalized Cantor space ${}^{\kappa}2$ is the unique nonempty (κ-)perfect (κ-Lindelöf, if κ is weakly compact) spherically complete $\mathbb{G}$-Polish space.*

(2) *Up to homeomorphism, the generalized Baire space ${}^{\kappa}\kappa$ is the unique nonempty spherically complete $\mathbb{G}$-Polish space for which all κ-Lindelöf subsets (equivalently: all K_κ-subsets) have empty interior.*

In this section we studied in detail the κ-Lindelöf property in relation with the generalized Cantor space: it turns out that this property has important consequences for other spaces as well. For example, as it happens in the classical case, compactness always bring with itself a form of completeness.

Proposition 1.2.18. *Let X be a space of weight $\leq \kappa$. If X is κ-Lindelöf, then it is an $f\mathrm{SC}_\kappa$-space.*

Proof. Define a strategy σ for II such that when I plays a relatively open set U and a point $x \in U$, then σ answers with any relatively open set V satisfying $x \in V$ and $\mathrm{cl}(V) \subseteq U$ (such a V exists by regularity). Now suppose $\langle (U_\alpha, x_\alpha), V_\alpha \mid \alpha < \kappa \rangle$ is a run of the strong fair κ-Choquet game played accordingly to σ. If $\bigcap_{\alpha<\kappa} V_\alpha = \emptyset$, then the family $\{X \setminus \mathrm{cl}(V_\alpha) \mid \alpha < \kappa\}$ is an open cover of X because $\bigcap_{\alpha<\kappa} \mathrm{cl}(V_\alpha) = \bigcap_{\alpha<\kappa} U_\alpha = \bigcap_{\alpha<\kappa} V_\alpha = \emptyset$, and thus it has a subcover of size $< \kappa$ because X is κ-Lindelöf. But then there is $\delta < \kappa$ such that $\bigcap_{\alpha<\delta'} \mathrm{cl}(V_\alpha) = \emptyset$ for all $\delta \leq \delta' < \kappa$. Considering any limit ordinal $\delta' \geq \delta$, we then get $\bigcap_{\alpha<\delta'} V_\alpha = \bigcap_{\alpha<\delta'} \mathrm{cl}(V_\alpha) = \emptyset$, so that player II won the run of $f\mathrm{G}^s_\kappa(X)$ anyway. □

A similar argument explains the relation between compactness and SC_κ-spaces.

Proposition 1.2.19. *Let X be a space of weight $\leq \kappa$. If X is compact, then it is an SC_κ-space.*

The following is the analogue in our context of the standard fact that compact metrizable spaces are automatically Polish.

Corollary 1.2.20. *Every κ-Lindelöf $\mathbb{G}$-metrizable space is $\mathbb{G}$-Polish.*

Proof. Choose a strictly decreasing sequence $(\varepsilon_\alpha)_{\alpha<\kappa}$ that is coinitial in $\mathbb{G}^+$. By κ-Lindelöfness, for each $\alpha < \kappa$ there is a covering $\mathcal{B}_\alpha$ of X of size $< \kappa$ consisting of open balls of radius ε_α. It follows that $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$ is a basis for X of size $\leq \kappa$. By Proposition 1.2.18 the space X is then strongly fair κ-Choquet, and since $\mathbb{G}$-metrizability implies κ-additivity we get that X is $\mathbb{G}$-Polish by Theorem 1.1.21. □

Using Proposition 1.2.18, many statements of Section 1.1 can be reformulated for the special case of weakly compact cardinals and κ-Lindelöf spaces. For example, the next proposition is a reformulation of Proposition 1.1.13 in this special case.

Proposition 1.2.21. *Let X be a κ-additive κ-Lindelöf space of weight $\leq \kappa$ (in which case X is automatically an $f\mathrm{SC}_\kappa$-space by Proposition 1.2.18). Then X is homeomorphic to a closed set $C \subseteq {}^\kappa 2$. If furthermore X is an SC_κ-space, then C can be taken to be superclosed.*

Proof. First notice that if κ is not weakly compact, then the result trivially holds by Proposition 1.1.13 since in this case ${}^\kappa 2$ and ${}^\kappa \kappa$ are homeomorphic (via a homeomorphism which preserves superclosed sets). Thus we may assume that κ is weakly compact. By Theorems 1.1.16 and 1.1.17 again we can further assume that $X = [T]$ for some (superclosed, in the case of an SC_κ-space) tree $T \subseteq {}^{<\kappa}\kappa$. Since X is κ-Lindelöf, by [104, Lemma 2.6(1)] the set $[T]$ is bounded, i.e. there is $y \in {}^\kappa\kappa$ such that $x(\alpha) \leq y(\alpha)$ for all $x \in [T]$ and $\alpha < \kappa$. Consider the space $Z = \{z \in {}^\kappa\kappa \mid \forall \alpha < \kappa\, (z(\alpha) \leq y(\alpha))\}$. It is clearly a nonempty κ-additive κ-perfect SC_κ-space. Moreover, since by definition it is bounded by y and κ is weakly compact, by [104, Lemma

2.6(1)] and the fact that Z is closed in ${}^\kappa\kappa$ it follows that Z is also κ-Lindelöf. By Theorem 1.2.15 there is a homeomorphism $h\colon Z \to {}^\kappa 2$, which moreover preserves superclosed subsets of Z by Remark 1.2.16. Since by definition $X \subseteq Z$, it follows that $h(X)$ is a (super)closed subset of ${}^\kappa 2$ homeomorphic to X, as required. □

Using Proposition 1.2.21, we can restate Theorems 1.1.21 and 1.1.32 for the special case of κ-Lindelöf spaces, further refining the picture given in Figure 1 with one more dividing line, namely κ-Lindelöfness.

Theorem 1.2.22. *For any space X the following are equivalent:*

(a) *X is a κ-Lindelöf and κ-additive space of weight $\leq \kappa$;*

(b) *X is a κ-Lindelöf $\mathbb{G}$-metrizable space;*

(c) *X is a κ-Lindelöf $\mathbb{G}$-Polish space;*

(d) *X is a κ-Lindelöf κ-additive $f\mathrm{SC}_\kappa$-space.*

If κ is weakly compact, the above conditions are also equivalent to:

(e) *X is homeomorphic to a closed subset of ${}^\kappa 2$.*

Theorem 1.2.23. *For any space X the following are equivalent:*

(a) *X is a κ-Lindelöf κ-additive SC_κ-space;*

(b) *X is a κ-Lindelöf spherically $<\kappa$-complete $\mathbb{G}$-metrizable space;*

(c) *X is a κ-Lindelöf spherically complete $\mathbb{G}$-Polish space.*

If κ is weakly compact, the above conditions are also equivalent to:

(d) *X is homeomorphic to a superclosed subset of ${}^\kappa 2$.*

1.3 Characterizations of standard κ-Borel spaces

In this section we deal with the κ-Borel structure of topological spaces, and show how standard κ-Borel spaces (Definition 1.1.1) are exactly the κ-Borel spaces obtained from Polish-like spaces in any of the classes considered so far by forgetting their topology. For the sake of definiteness, throughout the section we work with $f\mathrm{SC}_\kappa$-spaces and SC_κ-spaces, but all results can be reformulated in terms of $\mathbb{G}$-Polish and spherically complete $\mathbb{G}$-Polish spaces—see Section 1.1.

We start by proving some results about changes of topology, which might be of independent interest. The next proposition shows how to change the topology of an $f\mathrm{SC}_\kappa$-space while preserving its κ-Borel structure. This generalizes [93, Theorem 13.1] to our setup.

Proposition 1.3.1. *Let $(B_\alpha)_{\alpha<\kappa}$ be a family of κ-Borel subsets of an $f\mathrm{SC}_\kappa$-space (X,τ). Then there is a topology τ' on X such that:*

(1) *τ' refines τ;*

(2) *each* B_α *is* τ'*-clopen,*

(3) $\mathsf{Bor}_\kappa(X, \tau') = \mathsf{Bor}_\kappa(X, \tau)$*, and*

(4) (X, τ') *is a* κ*-additive* $f\mathrm{SC}_\kappa$*-space.*

Proof. Let $\mathscr{A}$ be the collection of those $A \subseteq X$ for which there is a topology τ' which satisfies (1)–(4) above (where in (2) the set B_α is replaced by A). Notice that $\mathscr{A}$ is trivially closed under complementation. We first show that $\mathscr{A}$ contains all closed subsets of X.

Claim 1.3.1.1. Let C be a closed subset of an $f\mathrm{SC}_\kappa$-space (X, τ). Then there is a topology τ' which satisfies (1)–(4) above (where in (2) the set B_α is replaced by C).

Proof of the Claim. Let $\bar{\tau}$ be the smallest topology generated by $\tau \cup \{C\}$. Then (1)–(3) are trivially satisfied. Furthermore, $(X, \bar{\tau})$ is homeomorphic to the sum of the spaces C and $X \setminus C$ (endowed with the relative topologies inherited from X). Since both C and $X \setminus C$ are $f\mathrm{SC}_\kappa$-spaces by Theorem 1.1.16, and since the class of $f\mathrm{SC}_\kappa$-spaces is trivially closed under ($\leq \kappa$-sized) sums, then X is an $f\mathrm{SC}_\kappa$-space as well. Applying Proposition 1.1.35 to $(X, \bar{\tau})$ we then get a topology $\tau' \supseteq \bar{\tau} \supseteq \tau$ which satisfies all of (1)–(4). □

Claim 1.3.1.2. Let $(A_\alpha)_{\alpha<\kappa}$ be a family of sets in $\mathscr{A}$. Then there is a topology τ'_∞ simultaneously witnessing $A_\alpha \in \mathscr{A}$ for all $\alpha < \kappa$.

Proof of the Claim. For each $\alpha < \kappa$ let τ'_α be a topology witnessing $A_\alpha \in \mathscr{A}$. Define τ'_∞ as the smallest κ-additive topology containing $\bigcup_{\alpha<\kappa} \tau'_\alpha$. Then (1)–(3) are obvious, since τ'_∞ refines each $\tau'_\alpha \supseteq \tau$ and $\tau'_\infty \subseteq \mathsf{Bor}_\kappa(X, \tau)$. To prove (4), for each $\alpha < \kappa$ fix a closed $C_\alpha \subseteq {}^\kappa\kappa$ and a homeomorphism $h_\alpha \colon C_\alpha \to (X, \tau'_\alpha)$ as given by Theorem 1.1.16. Endow ${}^\kappa({}^\kappa\kappa)$ with the κ-supported product topology, i.e. the topology generated by the sets $\prod_{\alpha<\kappa} U_\alpha$, where each U_α is open in the bounded topology of ${}^\kappa\kappa$, and only $<\kappa$-many of them differ from ${}^\kappa\kappa$. Then $\prod_{\alpha<\kappa} C_\alpha$ is closed in ${}^\kappa({}^\kappa\kappa)$, and since the maps h_α are continuous, the set

$$\Delta = \left\{ (x_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} C_\alpha \;\middle|\; \forall \alpha, \beta < \kappa \left(h_\alpha(x_\alpha) = h_\beta(x_\beta) \right) \right\}$$

is closed as well. It is then easy to check that the map $h \colon \Delta \to (X, \tau'_\infty)$ sending $(x_\alpha)_{\alpha<\kappa} \in \Delta$ to $h_0(x_0)$ is a homeomorphism. Therefore the desired result follows from Theorem 1.1.16 and the fact that the spaces ${}^\kappa({}^\kappa\kappa)$ and ${}^\kappa\kappa$ are clearly homeomorphic. □

Claim 1.3.1.2 in particular reduces our task of proving the theorem for a whole family $(B_\alpha)_{\alpha<\kappa}$ to showing that $B \in \mathscr{A}$ for every single κ-Borel set $B \subseteq X$. To this aim, by Claim 1.3.1.1 and closure of $\mathscr{A}$ under complementation it is enough to show that $\mathscr{A}$ is closed under intersections of length $\leq \kappa$. So let $A = \bigcap_{\alpha<\kappa} A_\alpha$ be such that $A_\alpha \in \mathscr{A}$ for every $\alpha < \kappa$. By Claim 1.3.1.2, there is a topology τ'_∞ simultaneously witnessing $A_\alpha \in \mathscr{A}$ for all $\alpha < \kappa$. Then A is closed in the κ-additive $f\mathrm{SC}_\kappa$-space (X, τ'_∞). Therefore Claim 1.3.1.1 applied to A, viewed as a subset of (X, τ'_∞), yields the desired topology $\tau' \supseteq \tau'_\infty \supseteq \tau$. □

Proposition 1.3.1 provides an alternative proof of [105, Lemma 1.11]: Every κ-Borel subset of ${}^\kappa\kappa$ equals a continuous injective image of a closed subset of ${}^\kappa\kappa$. To see this, let $B \subseteq {}^\kappa\kappa$ be κ-Borel, and let τ' be the topology obtained by applying Proposition 1.3.1 with $B_\alpha = B$ for all $\alpha < \kappa$. Let D be a closed subset of ${}^\kappa\kappa$ and $h\colon (D, \tau_b) \to ({}^\kappa\kappa, \tau')$ be a homeomorphism as given by Proposition 1.1.13. Then $C = h^{-1}(B)$ is closed in D and hence in ${}^\kappa\kappa$. Moreover, the map $h'\colon (D, \tau_b) \to ({}^\kappa\kappa, \tau_b)$ obtained by composing h with the identity function $({}^\kappa\kappa, \tau') \to ({}^\kappa\kappa, \tau_b)$ is still a continuous bijection because $\tau' \supseteq \tau_b$. Therefore, $h' \restriction C$ is a continuous injection from the closed set $C \subseteq {}^\kappa\kappa$ onto B. Notice also that, by construction, h' is actually a κ-Borel isomorphism because $\mathsf{Bor}_\kappa({}^\kappa\kappa, \tau') = \mathsf{Bor}_\kappa({}^\kappa\kappa, \tau_b)$. More generally, the same argument shows that [105, Lemma 1.11] can be extended to arbitrary $f\mathrm{SC}_\kappa$-spaces.

Corollary 1.3.2. *For every κ-Borel subset B of an $f\mathrm{SC}_\kappa$-space there is a continuous κ-Borel isomorphism from a closed $C \subseteq {}^\kappa\kappa$ to B.*

The space C in the previous corollary is an $f\mathrm{SC}_\kappa$-space by Theorem 1.1.16, hence applying Theorem 1.1.40 we further get

Corollary 1.3.3. *Each κ-Borel subset B of an $f\mathrm{SC}_\kappa$-space is κ-Borel isomorphic to a κ-additive SC_κ-space.*

The following is the counterpart of Proposition 1.3.1 in terms of functions and can be proved by applying it to the preimages of the open sets in any $\leq \kappa$-sized basis for the topology of Y.

Corollary 1.3.4. *Let (X, τ) be an $f\mathrm{SC}_\kappa$-space and Y be any space of weight $\leq \kappa$. Then for every κ-Borel function $f\colon X \to Y$ there is a topology τ' on X such that:*

(1) *τ' refines τ;*

(2) *$f\colon (X, \tau') \to Y$ is continuous,*

(3) *$\mathsf{Bor}_\kappa(X, \tau') = \mathsf{Bor}_\kappa(X, \tau)$, and*

(4) *(X, τ') is a κ-additive $f\mathrm{SC}_\kappa$-space.*

Finally, combining the results obtained so far we get that all the proposed generalizations of (St.Bor. 1) and (St.Bor. 2) give rise to the same class of spaces. In particular, up to κ-Borel isomorphism such class coincide with any of the classes of Polish-like spaces we analyzed in the previous sections. (Notice also that Theorem 1.3.5 substantially strengthens [42, Corollary 3.4].)

Theorem 1.3.5. *A κ-Borel space $(X, \mathscr{B})$ is standard if and only if there is a topology τ on X such that*

(1) *(X, τ) is an $f\mathrm{SC}_\kappa$-space, and*

(2) *$\mathsf{Bor}_\kappa(X, \tau) = \mathscr{B}$.*

Moreover, condition (1) can equivalently be replaced by

(1′) *(X, τ) is a κ-additive SC_κ-space.*

Remark 1.3.6. Since κ-additive SC_κ-spaces and $f\mathrm{SC}_\kappa$-spaces form, respectively, the smallest and largest class of Polish-like spaces considered in this chapter, in Theorem 1.3.5 we can further replace those classes with any of the other ones: κ-additive $f\mathrm{SC}_\kappa$-spaces, SC_κ-spaces, $\mathbb{G}$-Polish spaces, spherically complete $\mathbb{G}$-Polish spaces, and so on.

A natural question is to ask whether Proposition 1.3.1 can be extended in at least some direction. As in the classical case, the answer is mostly negative and thus Proposition 1.3.1 is essentially optimal. In fact:

(a) We cannot in general consider more than κ-many (even closed, or open) subsets, since this could force τ' to have weight greater than κ—think about turning into clopen sets more than κ-many singletons.

(b) We obviously cannot turn a set which is not κ-Borel into a clopen (or even just κ-Borel) one pretending to maintain the same κ-Borel structure. Notice however that, in contrast to the classical case, one can consistently have that there are non-κ-Borel sets $B \subseteq {}^\kappa\kappa$ for which there is a κ-additive $f\mathrm{SC}_\kappa$ topology $\tau' \supseteq \tau_b$ turning B into a τ'-clopen set, so that all conditions in Proposition 1.3.1 except for (3) are satisfied with respect to such B (see Corollary 1.4.10 for more details and limitations).

(c) By Example 1.1.37, we cannot require that the topology τ' be SC_κ (instead of just $f\mathrm{SC}_\kappa$). The same remains true if we consider a single κ-Borel set B (instead of a whole family $(B_\alpha)_{\alpha<\kappa}$), we start from the stronger hypothesis that (X,τ) is already a κ-additive SC_κ-space, and we weaken the conclusions by dropping condition (3) and relaxing condition (2) to "B is τ'-open" (or "B is τ'-closed").

As it is clear from the discussion, in the last item the problem arises from the fact that there is a tension between condition (1) and our desire to strengthen condition (4) from $f\mathrm{SC}_\kappa$ to SC_κ. However, we are now going to show that if we drop the problematic condition (1), then it is possible to obtain the desired strengthening, at least when we just consider a few κ-Borel sets at a time.

Proposition 1.3.7. *For every κ-Borel subset B of an $f\mathrm{SC}_\kappa$-space (X,τ) there is a topology τ'' on X such that:*

(1) *B is τ''-clopen,*

(2) $\mathsf{Bor}_\kappa(X,\tau'') = \mathsf{Bor}_\kappa(X,\tau)$, *and*

(3) *(X,τ'') is a κ-additive SC_κ-space (hence so are its subspaces B and $X \setminus B$ because they are τ''-open).*

Proof. By Corollary 1.3.3, there are κ-additive SC_κ topologies τ_1 and τ_2 on, respectively, B and $X \setminus B$ such that $\mathsf{Bor}_\kappa(B,\tau_1) = \mathsf{Bor}_\kappa(X,\tau) \restriction B$ and $\mathsf{Bor}_\kappa(X\setminus B,\tau_2) = \mathsf{Bor}_\kappa(X,\tau) \restriction (X \setminus B)$. Let τ'' be the topology on X construed as the sum of (B,τ_1) and $(X \setminus B,\tau_2)$: then τ'' is as required. □

The proof of Proposition 1.3.7 can easily be adapted to work with κ-many *pairwise disjoint* κ-Borel subsets of X. This in turn implies that the proposition can e.g.

be extended to deal with $<\kappa$-many κ-Borel sets simultaneously, even when such sets are not pairwise disjoint. Indeed, if $(B_\alpha)_{\alpha<\nu}$ with $\nu < \kappa$ is such a family, then for each $s \in {}^\nu 2$ we can set

$$C_s = \{x \in X \mid \forall \alpha < \nu \, (x \in B_\alpha \iff s(\alpha) = 1)\}.$$

Since $2^\nu \leq \kappa^{<\kappa} = \kappa$, the family $(C_s)_{s \in {}^\nu 2}$ is a partition of X into $\leq \kappa$-many κ-Borel sets, and any topology τ'' working simultaneously for all the C_s will work for all sets in the family $(B_\alpha)_{\alpha<\nu}$ as well. In contrast, Proposition 1.3.7 cannot be extended to arbitrary κ-sized families of κ-Borel sets, even when we restrict to $X = {}^\kappa\kappa$. Indeed, let $C \subseteq {}^\kappa\kappa$ be as in Example 1.1.37 and let $(B_\alpha)_{\alpha<\kappa}$ be an enumeration of $\{C\} \cup \{\mathbf{N}_s \cap C \mid s \in {}^{<\kappa}\kappa\}$. Then $(B_\alpha)_{\alpha<\kappa}$ is a family of Borel subsets of ${}^\kappa\kappa$ such that there is no SC_κ topology τ'' on ${}^\kappa\kappa$ making each B_α a τ''-open subset of ${}^\kappa\kappa$, since otherwise $\tau'' \restriction C$ would be an SC_κ topology on C refining $\tau_b \restriction C$.

From a different perspective, it might be interesting to understand which subspaces of a Polish-like space inherit a standard κ-Borel structure form their superspace. Of course this includes all κ-Borel sets, as standard κ-Borel spaces are closed under κ-Borel subspaces, and we are now going to show that no other set has such property. We begin with a preliminary result which is of independent interest, as it shows that if a (regular Hausdorff) topology of weight $\leq \kappa$ induces a standard κ-Borel structure, then it can be refined to a Polish-like topology with the same κ-Borel sets.

Proposition 1.3.8. *Let (X, τ) be a space of weight $\leq \kappa$. We have that $(X, \mathsf{Bor}_\kappa(X, \tau))$ is a standard κ-Borel space if and only if there is a topology $\tau' \supseteq \tau$ such that (X, τ') is a κ-additive $f\mathrm{SC}_\kappa$-space and $\mathsf{Bor}_\kappa(X, \tau) = \mathsf{Bor}_\kappa(X, \tau')$.*

Proof. The backward implication follows from Theorem 1.3.5. For the forward implication, suppose that $(X, \mathsf{Bor}_\kappa(X, \tau))$ is standard κ-Borel. By Theorem 1.3.5, there is a topology $\hat{\tau}$ such that $(X, \hat{\tau})$ is an $f\mathrm{SC}_\kappa$-space, and $\mathsf{Bor}_\kappa(X, \hat{\tau}) = \mathsf{Bor}_\kappa(X, \tau)$. Then the identity function $i \colon (X, \hat{\tau}) \to (X, \tau)$ satisfies the hypothesis of Corollary 1.3.4, hence there is a κ-additive $f\mathrm{SC}_\kappa$ topology τ' such that $i \colon (X, \tau') \to (X, \tau)$ is continuous and $\mathsf{Bor}_\kappa(X, \tau') = \mathsf{Bor}_\kappa(X, \hat{\tau}) = \mathsf{Bor}_\kappa(X, \tau)$, which implies $\tau \subseteq \tau'$. ☐

Finally, we anticipate a result that will be proven in next chapter (see Theorem 2.4.15).

Theorem 1.3.9. *Let $(X, \mathscr{B})$ be a standard κ-Borel space, and let $A \subseteq X$. Then $(A, \mathscr{B} \restriction A)$ is a standard κ-Borel space if and only if $A \in \mathscr{B}$.*

Corollary 1.3.10. *Let X, Y be standard κ-Borel spaces. If $A \subseteq X$ is κ-Borel and $f \colon A \to Y$ is a κ-Borel embedding, then $f(A)$ is κ-Borel in Y.*

Corollary 1.3.10 is the analogue of the classical fact that an injective Borel image of a Borel set is still Borel (see [93, Section 15.A]). Notice however that in the generalized version the hypothesis on f is stronger: we need it to be a κ-Borel embedding, and not just an injective κ-Borel map. This is mainly due to the fact that in the generalized context we lack the analogue of Luzin's separation theorem. Indeed, one can even prove [105, Corollary 1.9] that there are non-κ-Borel sets which are continuous injective images of the whole ${}^\kappa\kappa$, hence our stronger requirement cannot be dropped.

We finally come to the problem of characterizing which topologies induce a standard κ-Borel structure. Of course this class is larger than the collection of all $f\mathrm{SC}_\kappa$ topologies, even when restricting to the κ-additive case. Indeed, the relative topology on any κ-Borel non-G^κ_δ subspace $B \subseteq {}^\kappa\kappa$ generates a standard κ-Borel structure, yet it is not $f\mathrm{SC}_\kappa$ itself because of Theorems 1.1.21 and 1.1.28. On the other hand, if a space (X, τ) is homeomorphic to a κ-Borel subset of ${}^\kappa\kappa$, then it clearly generates a standard κ-Borel structure by definition. Theorems 1.1.12 and 2.4.15 allow us to reverse the implication, yielding the desired characterization in the case of κ-additive topologies. (For the nontrivial direction, use the fact that by Theorem 1.1.12 every κ-additive space of weight $\leq \kappa$ is, up to homeomorphism, a subspace of ${}^\kappa\kappa$.)

Corollary 1.3.11. *Let (X, τ) be a κ-additive space of weight $\leq \kappa$. Then $(X, \mathsf{Bor}_\kappa(X, \tau))$ is a standard κ-Borel space if and only if (X, τ) is* homeomorphic *to a κ-Borel subset of ${}^\kappa\kappa$ (or, equivalently, of ${}^\kappa 2$).*

In [118, Definition 3.6], the author considered *topological* spaces (X, τ) with weight $\leq \kappa$ such that the induced κ-Borel structure is κ-Borel isomorphic to a κ-Borel subset of ${}^\kappa\kappa$. By Corollary 1.3.11 it turns out that when τ is regular Hausdorff and κ-additive, a space (X, τ) satisfies [118, Definition 3.6] if and only if it is homeomorphic (and not just κ-Borel isomorphic) to a κ-Borel subset of ${}^\kappa\kappa$.

1.4 Final remarks and open questions

In the classical setup, Polish spaces are closed under countable sums, countable products, and G_δ subspaces. Moving to the generalized context, all classes considered so far are trivially closed under sums of size $\leq \kappa$. However, by Theorem 1.1.32 the class of SC_κ-spaces is already lacking closure with respect to closed subspaces (even when restricting the attention to κ-additive spaces or, equivalently, to spherically complete $\mathbb{G}$-Polish spaces). In view of Proposition 1.1.10, the class of $f\mathrm{SC}_\kappa$-spaces is a more promising option. Indeed, since such class is also straightforwardly closed under $\leq \kappa$-products, where the product is naturally endowed by the $< \kappa$-supported product topology, we easily get:

Theorem 1.4.1. *The class of $f\mathrm{SC}_\kappa$-spaces is closed under G^κ_δ subspaces and $\leq \kappa$-sized sums and products.*

Moving to $\mathbb{G}$-Polish spaces, by Theorem 1.1.28 we still have closure under G^κ_δ subspaces. However, it is then not transparent how to achieve closure under $\leq \kappa$-sized products. The naïve attempt of mimicking what is done in the classical case would require to first develop a theory of convergent κ-indexed series in some suitable group $\mathbb{G}$, and then use it to try to define the complete $\mathbb{G}$-metric on the product. Theorem 1.1.21 provides an elegant bypass to these difficulties and directly leads us the the following theorem.

Theorem 1.4.2. *The class of $\mathbb{G}$-Polish spaces (equivalently: κ-additive $f\mathrm{SC}_\kappa$-spaces) is closed under G^κ_δ-subspaces and $\leq \kappa$-sized sums and products.*

Proof. For $\leq \kappa$-sized products, just notice that both the property of being κ-additive and the property of being strongly fair κ-Choquet are straightforwardly preserved by such operation. $\square$

Moreover, we also get the analogue of Sierpiński's theorem [93, Theorem 8.19]: the classes of $\mathbb{G}$-Polish spaces and $f\mathrm{SC}_\kappa$-spaces are both closed under continuous open images. (Notice that a similar result holds for SC_κ-spaces, as observed in [42, Proposition 2.7].)

Theorem 1.4.3. *Let X be $\mathbb{G}$-Polish, and Y be a space of weight $\leq \kappa$. If there is a continuous open surjection f from X onto Y, then Y is $\mathbb{G}$-Polish.*

The same is true is we replace $\mathbb{G}$-Polishness by the (weaker) property of being an $f\mathrm{SC}_\kappa$-space.

Proof. By Theorem 1.1.21, it is enough to show that the properties of being strongly fair κ-Choquet and being κ-additive are preserved by f. The former is straightforward. For the latter, let $(U_\alpha)_{\alpha<\nu}$ be a sequence of open subsets of Y, for some $\nu < \kappa$. If $\bigcap_{\alpha<\nu} U_\alpha \neq \emptyset$, let y be arbitrary in $\bigcap_{\alpha<\nu} U_\alpha$ and, using surjectivity of f, let $x \in X$ be such that $f(x) = y$. Since $x \in \bigcap_{\alpha<\nu} f^{-1}(U_\alpha)$ and the latter set is open by κ-additivity of X, there is $V \subseteq X$ open such that $x \in V \subseteq \bigcap_{\alpha<\nu} f^{-1}(U_\alpha)$. It follows that $f(V)$ is an open neighborhood of y such that $f(V) \subseteq \bigcap_{\alpha<\nu} U_\alpha$, as desired. □

There is still one interesting open question related to $f\mathrm{SC}_\kappa$-subspaces of a given space of weight $\leq \kappa$. By Corollary 1.1.27, if X is also κ-additive and $Y \subseteq X$ is an $f\mathrm{SC}_\kappa$-subspace of it, then Y is G^κ_δ in X. We do not know if the same remains true if we drop κ-additivity. The following corollary is the best result we have in this direction: it follows from Theorem 1.1.12 and the fact that by $\kappa^{<\kappa} = \kappa$ and the proof of Proposition 1.1.35, every (regular Hausdorff) topology of weight $\leq \kappa$ can be naturally refined to a κ-additive one in such a way that the new open sets are F^κ_σ (i.e. a $\leq \kappa$-sized union of closed sets or, equivalently, the complement of a G^κ_δ set) in the old topology.

Corollary 1.4.4. *Let X be a space of weight $\leq \kappa$, and $Y \subseteq X$ be an $f\mathrm{SC}_\kappa$-subspace of it. Then Y is a $\leq \kappa$-sized intersection of F^κ_σ subsets of X.*

It is then natural to ask whether the above computation can be improved.

Question 1.4.5. *In the same hypotheses of Corollary 1.4.4, is Y a G^κ_δ subset of X? What if we assume that X be $f\mathrm{SC}_\kappa$?*

In the literature on generalized descriptive set theory, the notion of an analytic set is usually generalized as follows.

Definition 1.4.6. A subset of a space[14] of weight $\leq \kappa$ is κ**-analytic** if and only if it is a continuous image of a closed subset of ${}^\kappa\kappa$. A set is κ**-coanalytic** if its complement is κ-analytic, and it is κ**-bianalytic** if it is both κ-analytic and κ-coanalytic.

Although the definition works for a larger class of spaces, in this chapter we will concentrate on subsets of $f\mathrm{SC}_\kappa$-spaces. Analogously to what happens in the classical case, one can then prove that Definition 1.4.6 is equivalent to several other variants: for example, a set $A \subseteq {}^\kappa\kappa$ is κ-analytic if and only if it is the projection of a closed

[14] Since $f\mathrm{SC}_\kappa$-spaces have been introduced in the present work, the definition of κ-analytic sets given in the literature is of course usually restricted to the spaces ${}^\kappa\kappa$ and ${}^\kappa 2$ and their powers. The only exception is [118], where it is given for all $\leq \kappa$-weighted topologies generating a standard κ-Borel structure (see [118, Definitions 3.6 and 3.8]).

$C \subseteq ({}^{\kappa}\kappa)^2$, if and only if[15] it is a κ-Borel image of some set $B \in \mathsf{Bor}_\kappa({}^{\kappa}\kappa)$ (see [6, Corollary 7.3] and [118, Proposition 3.11]). As explained in [105, Theorem 1.5], a major difference from the classical setup is instead that we cannot add among the equivalent reformulations of κ-analyticity that of being a continuous image of the whole ${}^{\kappa}\kappa$—this condition defines a properly smaller class when κ is uncountable (and, as usual, $\kappa^{<\kappa} = \kappa$).

The reason for using Definition 1.4.6 instead of directly generalizing [93, Definition 14.1] is precisely that we were still lacking an appropriate notion of generalized Polish-like space. We can now fill this gap.

Proposition 1.4.7. *Let X be an $f\mathrm{SC}_\kappa$-space. For any $A \subseteq X$, the following are equivalent:*

(a) *A is κ-analytic (i.e. a continuous image of a closed subset of ${}^{\kappa}\kappa$);*

(b) *A is a continuous image of a $\mathbb{G}$-Polish space;*

(c) *A is a continuous image of an $f\mathrm{SC}_\kappa$-space.*

Proof. The implications (a) ⇒ (b) and (b) ⇒ (c) follow from Theorem 1.1.21. For the remaining implication (c) ⇒ (a), suppose that Y is an $f\mathrm{SC}_\kappa$-space and that $g \colon Y \to X$ is continuous and onto A. Use Proposition 1.1.35 to refine the topology τ of Y to a topology τ' such that (Y, τ') is κ-additive and still $f\mathrm{SC}_\kappa$. Use Theorem 1.1.21 again to find a closed set $C \subseteq {}^{\kappa}\kappa$ and a homeomorphism $f \colon C \to (Y, \tau')$: then $g \circ f$ is a continuous surjection from C onto A. □

Clearly, in Proposition 1.4.7(c) we can equivalently consider κ-additive $f\mathrm{SC}_\kappa$-spaces. We instead cannot restrict ourselves to SC_κ-spaces, even when further requiring κ-additivity. Indeed, by Theorem 1.1.32 and [105, Proposition 1.3] every such space is a continuous image of the whole ${}^{\kappa}\kappa$: it follows that the collection of all continuous images of κ-additive SC_κ-spaces coincides with the collection of continuous images of ${}^{\kappa}\kappa$, and it is thus strictly smaller than the class of all κ-analytic sets by the mentioned [105, Theorem 1.5].

A variant of Definition 1.4.6 considered in [105] is the class I^{κ}_{cl} of continuous *injective* images of closed subsets of ${}^{\kappa}\kappa$ (clearly, all such sets are in particular κ-analytic). When $\kappa = \omega$ the class I^{κ}_{cl} coincides with Borel sets, but when $\kappa > \omega$ the class I^{κ}_{cl} is strictly larger than $\mathsf{Bor}_\kappa({}^{\kappa}\kappa)$ by [105, Corollary 1.9]. Moreover, if $\mathrm{V} = \mathrm{L}[x]$ with $x \subseteq \kappa$, then by [105, Corollary 1.14] all κ-analytic subsets of ${}^{\kappa}\kappa$ belong to I^{κ}_{cl}. This result can be extended to κ-analytic subsets of arbitrary $f\mathrm{SC}_\kappa$-spaces.

Corollary 1.4.8. *Assume that $\mathrm{V} = \mathrm{L}[x]$ with $x \subseteq \kappa$, and let X be an arbitrary $f\mathrm{SC}_\kappa$-space. Then every κ-analytic $A \subseteq X$ is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$.*

Proof. By Corollary 1.1.36 there is a closed $C \subseteq {}^{\kappa}\kappa$ and a continuous bijection $f \colon C \to X$. Notice that $f^{-1}(A)$ is κ-analytic in C because the class of κ-analytic

[15] This reformulation involves only κ-Borel sets and functions, thus the notion of a κ-analytic set is independent on the actual topology. This allows us to naturally extend this concept to subsets of arbitrary (standard) κ-Borel spaces.

sets is easily seen to be closed under continuous preimages, hence it is κ-analytic in ${}^\kappa\kappa$ as well. By [105, Corollary 1.14] there is a continuous injection from some closed $D \subseteq {}^\kappa\kappa$ onto $f^{-1}(A)$, which composed with f gives the desired result. $\square$

We are now going to show that the class I^κ_{cl} can be characterized through changes of topology.

Theorem 1.4.9. *Let (X, τ) be an $f\mathrm{SC}_\kappa$-space and $A \subseteq X$. Then the following are equivalent:*

(a) *$A \in I^\kappa_{\mathrm{cl}}$;*

(b) *there is an $f\mathrm{SC}_\kappa$ topology τ' on A such that $\tau' \supseteq \tau \restriction A$.*

Proof. Suppose first that $C \subseteq {}^\kappa\kappa$ is closed and $f \colon C \to X$ is a continuous injection with range A. Let τ' be obtained by pushing forward along f the (relative) topology of C, so that (A, τ') and C are homeomorphic. Then (A, τ') is an $f\mathrm{SC}_\kappa$-space by Theorem 1.1.21, and τ' refines $\tau \restriction A$ because f was continuous.

Conversely, if (A, τ') is an $f\mathrm{SC}_\kappa$-space then by Theorem 1.1.21 again there is a closed $C \subseteq {}^\kappa\kappa$ and a homeomorphism $f \colon C \to (A, \tau')$. Since $\tau' \supseteq \tau \restriction A$, it follows that C and f witness $A \in I^\kappa_{\mathrm{cl}}$. $\square$

This also allows us to precisely determine to what extent the technique of change of topology discussed in Section 1.3 can be applied to non-κ-Borel sets.

Corollary 1.4.10. *Let (X, τ) be an $f\mathrm{SC}_\kappa$-space.*

(1) *Let $A \subseteq X$. If there is an $f\mathrm{SC}_\kappa$ topology $\tau' \supseteq \tau$ on X such that A is τ'-clopen (or even just $A \in \mathsf{Bor}_\kappa(X, \tau')$), then A is κ-bianalytic.*

(2) *If $\mathrm{V} = \mathrm{L}[x]$ with $x \subseteq \kappa$, then for all κ-bianalytic $A \subseteq X$ there is a κ-additive $f\mathrm{SC}_\kappa$ topology $\tau' \supseteq \tau$ on X such that A is τ'-clopen.*

Proof. For part (1) observe that since A is τ'-clopen, then by Proposition 1.1.10 both A and $X \setminus A$ are $f\mathrm{SC}_\kappa$-spaces when endowed with the relativization of τ'. Therefore by Theorem 1.4.9 they are in I^κ_{cl}, and thus κ-analytic. If instead of A being τ'-clopen we just have $A \in \mathsf{Bor}_\kappa(X, \tau')$, use Proposition 1.3.1 to further refine τ' to a suitable τ'' turning A into a τ''-clopen set, and then apply the previous argument to τ'' instead of τ'.

We now move to part (2). By Corollary 1.4.8, under our assumption all κ-analytic subsets of X are in I^κ_{cl}. It follows that for every κ-bianalytic set $B \subseteq X$ there is a continuous bijection $f \colon C \to X$ with $C \subseteq {}^\kappa\kappa$ closed and $f^{-1}(B)$ clopen relatively to C: just fix $f_0 \colon C_0 \to B$ and $f_1 \colon C_1 \to X \setminus B$ witnessing $B \in I^\kappa_{\mathrm{cl}}$ and $X \setminus B \in I^\kappa_{\mathrm{cl}}$, respectively, let C be the sum of C_0 and C_1, and set $f = f_0 \cup f_1$. Pushing forward the topology of C along f we then get the desired τ' (the fact that $\tau' \supseteq \tau_b$ follows again from the continuity of f). $\square$

Corollary 1.4.10 justifies our claim that there might be non-κ-Borel sets that can be turned into clopen sets via a nice change of topology (see item (b) on page 65). Indeed, when κ is uncountable there are κ-bianalytic subsets of ${}^\kappa\kappa$ which are not κ-Borel (see e.g. [65, Theorem 18]), and Corollary 1.4.10(2) applies to them.

Having extended the notion of a κ-analytic set to arbitrary $f\mathrm{SC}_\kappa$-spaces, it is natural to ask whether the deep analysis carried out in [105] can be transferred to such wider context. Some of the results have already been explicitly extended in this work, see e.g. Corollaries 1.1.15, 1.3.2, and 1.4.8, which extend, respectively, [105, Proposition 1.3, Lemma 1.11, and Corollary 1.14]. Other results naturally transfer to our general setup using the ideas developed so far.

Question 1.4.11. *Which other results from [105] hold for κ-analytic subsets of arbitrary $f\mathrm{SC}_\kappa$-spaces? For example, for which $f\mathrm{SC}_\kappa$-spaces X is there a closed $C \subseteq X$ which is not a continuous image of the whole ${}^\kappa\kappa$, or a non-κ-Borel set $A \subseteq X$ which is an injective continuous image of ${}^\kappa\kappa$?*

Similar questions can be raised about the analogue of the Hurewicz dichotomy for κ-analytic subsets of ${}^\kappa\kappa$ studied in [104].

We now move to generalizations of the perfect set property.

Definition 1.4.12. Let X be a space. A set A has the **κ-perfect set property** (κ-PSP for short) if either $|A| \leq \kappa$ or A contains a closed set homeomorphic to ${}^\kappa 2$.

The κ-Borel version of the κ-PSP would then read as follows: either $|A| \leq \kappa$ or A contains a κ-Borel set which is κ-Borel isomorphic to ${}^\kappa 2$. However, for most applications it is convenient to consider a slightly stronger reformulation.

Definition 1.4.13. Let X be a space. A set A has the **Borel κ-perfect set property** (Bor_κ-PSP for short) if either $|A| \leq \kappa$ or there is a *continuous* κ-Borel embedding $f \colon {}^\kappa 2 \to A$ with $f({}^\kappa 2) \in \mathsf{Bor}_\kappa(X)$.

By Corollary 1.3.10, if the κ-Borel structure of X is standard then the fact that $f({}^\kappa 2) \in \mathsf{Bor}_\kappa(X)$ follows from the other conditions. Notice also that the Bor_κ-PSP is in general strictly weaker than the κ-PSP. For example, consider the space $X = {}^\kappa 2$ equipped with the *product* topology. It is a κ-perfect SC_κ-space, hence X itself and all its open subsets have the Bor_κ-PSP (see Corollary 1.4.14 below). However, X is compact: thus its clopen subsets cannot contain a closed homeomorphic copy of the generalized Cantor space, which clearly is not compact, and thus they do not have the κ-PSP.

In Definitions 1.4.12 and 1.4.13 we are of course allowing the special case $A = X$. With this terminology, Theorem 1.2.6 asserts that the κ-PSP holds for all κ-additive κ-perfect SC_κ-spaces. From this and Proposition 1.1.35, we can easily infer the following fact, which is just a more precise formulation of [42, Proposition 3.1]. (Of course here we are also using that if τ is κ-perfect, then the topology from the proof of Proposition 1.1.35 is still κ-perfect.)

Corollary 1.4.14. *If X is a nonempty κ-perfect SC_κ-space, then there is a continuous κ-Borel embedding from ${}^\kappa 2$ into X (with a κ-Borel range, necessarily). In particular, the Bor_κ-PSP holds for κ-perfect SC_κ-spaces.*

It is instead independent of ZFC whether the (Borel) κ-perfect set property holds for (κ-additive) $f\mathrm{SC}_\kappa$-spaces. Indeed, if there is a κ-Kurepa tree T with $< 2^\kappa$-many branches, then no κ-PSP-like property can hold for $[T]$ because of cardinality reasons. On the other hand, in [132] the third author constructed a model of ZFC where all "definable" subsets of ${}^\kappa\kappa$ (including e.g. all κ-analytic sets and way more) have the Bor_κ-PSP: combining Proposition 1.1.35 with Theorem 1.1.21 we then get that such property holds for arbitrary $f\mathrm{SC}_\kappa$-spaces and their "definable" subsets. Indeed, the same reasoning combined with Proposition 1.3.8 can be used to show that *if* the Bor_κ-PSP holds for all closed subsets of, say, ${}^\kappa\kappa$, then it automatically propagates to all κ-Borel subsets of all $f\mathrm{SC}_\kappa$-spaces. Moreover, we can even just start from superclosed sets (equivalently, up to homeomorphism, from κ-additive SC_κ-spaces). Indeed, if $C = [T] \subseteq {}^\kappa\kappa$ is closed, then arguing as in the proof of Lemma 1.1.38 we can construct a superclosed set $C' = [T']$ such that $C \subseteq C'$, $|C'| \leq \max\{|C|, \kappa\}$, and all points in $C' \setminus C$ are isolated in C'. It follows that if the Bor_κ-PSP holds for C' then it holds also for C because if $f \colon {}^\kappa 2 \to C'$ is a continuous injection then $f({}^\kappa 2) \subseteq C$ (use the fact that ${}^\kappa 2$ is perfect). Summing up we thus have:

Theorem 1.4.15. *The following are equivalent:*

(a) *the* Bor_κ-PSP *holds for superclosed subsets of* ${}^\kappa\kappa$*;*

(b) *the* Bor_κ-PSP *holds for closed subsets of* ${}^\kappa\kappa$*;*

(c) *the* Bor_κ-PSP *holds for all (*κ*-additive)* $f\mathrm{SC}_\kappa$*-spaces;*

(d) *the* Bor_κ-PSP *holds for all* κ*-Borel subsets of all* $f\mathrm{SC}_\kappa$*-spaces.*

The Borel κ-perfect set property for $f\mathrm{SC}_\kappa$-spaces has important consequences for their classification up to κ-Borel isomorphisms.

Corollary 1.4.16. *Suppose that the* Bor_κ-PSP *holds for (super)closed subsets of* ${}^\kappa\kappa$*. If* X *is an* $f\mathrm{SC}_\kappa$*-space with* $|X| > \kappa$*, then* X *is* κ*-Borel isomorphic to* ${}^\kappa 2$*. In particular, any two* $f\mathrm{SC}_\kappa$*-spaces* X, Y *are* κ*-Borel isomorphic if and only if* $|X| = |Y|$*.*

In particular, if the Bor_κ-PSP holds for (super)closed subsets of ${}^\kappa\kappa$ then up to κ-Borel isomorphism the generalized Cantor space ${}^\kappa 2$ is the unique $f\mathrm{SC}_\kappa$-space of size $> \kappa$.

Proof. By our assumption and Theorem 1.4.15, ${}^\kappa 2$ is κ-Borel isomorphic to a κ-Borel subsets of X. Conversely, X is κ-Borel isomorphic to a κ-Borel subset of ${}^\kappa 2$ by Theorem 1.3.5 and the fact that ${}^\kappa 2$ and ${}^\kappa\kappa$ are κ-Borel isomorphic. Thus the result follows from the natural κ-Borel version of the usual Cantor-Schröder-Bernstein argument. □

Using the same argument and Corollary 1.4.14 we also get that when restricting to κ-perfect SC_κ-spaces the conclusions of Corollary 1.4.16 hold unconditionally—see [42, Corollary 3.7].

When dealing with topological game theory, one often wonders about what kind of winning strategies the players have at disposal in the given game. In this context, one can differentiate between perfect information strategies, that need to know all previous moves in order to be able to give an answer, and tactics, that instead rely only on the last move to determine the answer. The two notions do not coincide in general: there are games where a player has a winning strategy, but not a winning tactic. For example, [46] describes a topological space where player II has a winning strategy but no winning tactic in the classical strong Choquet game (see also [109]). Debs' example can easily be adapted to show that there exists a (non-κ-additive) topological space of weight κ where player II has a winning strategy but not a winning tactic in $f\mathrm{G}^s_\kappa(X)$ (or in $\mathrm{G}^s_\kappa(X)$), or that there is a κ-additive topological space of weight $> \kappa$ with the same property. In contrast, Proposition 1.1.13 implies that for κ-additive spaces of weight $\leq \kappa$ the two notions of winning tactic and winning strategy can be used interchangeably.

Corollary 1.4.17. *Let X be a κ-additive space of weight $\leq \kappa$. Then* II *has a winning strategy in $f\mathrm{G}^s_\kappa(X)$ (resp. $\mathrm{G}^s_\kappa(X)$) if and only if she has a winning tactic.*

Proof. For the nontrivial direction, by Proposition 1.1.13 we can restrict the attention to (super)closed subsets of ${}^\kappa\kappa$, so let $X = [T]$ for some pruned tree $T \subseteq {}^{<\kappa}\kappa$. Then any function $\sigma \colon \tau \to \tau$ that associate to every nonempty open set $U \subseteq [T]$ a basic open set $\mathbf{N}_s \cap [T] \subseteq U$ for some $s \in T$ is a winning tactic for II in $f\mathrm{G}^s_\kappa([T])$. Indeed, the answers $N_{s_\alpha} \cap [T]$ of σ at every round α are such that $s_\alpha \subseteq s_\beta$ for any $\alpha < \beta < \kappa$. Hence, if the game does not stop before κ-many rounds, then the final intersection $\bigcap_{\alpha<\kappa} \mathbf{N}_{s_\alpha} \cap [T]$ is nonempty, since it contains $s = \bigcup_{\alpha<\kappa} s_\alpha$ (or any sequence extendeding s, if s has length $< \kappa$). A similar argument shows that if T is superclosed, than the tactic described above is winning also for $\mathrm{G}^s_\kappa([T])$. □

For more details about perfect information strategies and tactics, and for some interesting problems in the field, see for example [131].

In this chapter we generalized metrics by allowing values in structures different from $\mathbb{R}$. Another possible generalization of metric spaces is given by uniform spaces. In this context we have a notion of completeness as well, which is however strictly weaker than the notions we considered so far. Indeed, all $\mathbb{G}$-metrizable spaces of weight $\leq \kappa$ (that is, by Theorem 1.1.12, all subspaces of ${}^\kappa\kappa$) are paracompact and Hausdorff, and this entails that they are completely uniformizable. It follows that any non-G^κ_δ subset of ${}^\kappa\kappa$ is a completely uniformizable space of weight $\leq \kappa$ which is not $f\mathrm{SC}_\kappa$ and, more generally, that the class of completely uniformizable spaces of weight $\leq \kappa$ properly extends the class of all κ-additive spaces with weight $\leq \kappa$ (recall that we are tacitly restricting to regular Hausdorff spaces). Thus by Theorem 2.4.15 such class contains spaces which are not even κ-Borel isomorphic to an $f\mathrm{SC}_\kappa$-space (that is, they are not standard κ-Borel): this seems to rule out the possibility of developing a decent (generalized) descriptive set theory in such a generality. Nevertheless, from the topological perspective it would still be interesting to know whether this property also extends the class of non-κ-additive $f\mathrm{SC}_\kappa$-spaces or at least SC_κ-spaces.

Question 1.4.18. *Is every $f\mathrm{SC}_\kappa$-space completely uniformizable?*

Chapter 2

Generalized Polish spaces at singular cardinals

2.1 Preliminaries

Throughout the chapter, we work in ZFC. If not specified otherwise, μ and κ will always denote *regular* infinite cardinals, and λ an arbitrary infinite cardinal. In generalized descriptive set theory, usually one furthermore assume that λ is a cardinal of cofinality μ and satisfies $2^{<\lambda} = \lambda$. This assumption implies that $\lambda^{<\mu} = \lambda$, and if λ is singular, it also implies that λ is strong limit (the converse is trivial). While we will often use (some, or all) these further assumptions, we (try to) specify each time if we need to require something more from λ and μ (other then being infinite cardinals, and μ being regular). We do not assume a priori that λ is singular, but singular cardinals will be our main focus, as for regular cardinals most results just follow from Chapter 1.

2.1.1 Topology

All topological spaces in this work are assumed to be regular and Hausdorff, unless otherwise specified.

Recall that a **topology** on a set X is a subset $\tau \subseteq \mathscr{P}(X)$ that contains both $\emptyset$ and X and that is closed under arbitrary unions and finite intersections. Sets from τ are called **open**, while sets of the form $X \setminus O$ for $O \in \tau$ are called **closed**. A set is called **clopen** if it is both closed and open. A topology is **Hausdorff** if every two distinct points $x, y \in X$ can be separated by open sets, i.e. there are $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. A topology is called **regular** if for every $x \in X$ and $U \in \tau$ open neighborhood of x there is $V \in \tau$ such that $x \in V \subseteq \mathrm{cl}(V) \subseteq U$, where $\mathrm{cl}(V) = \bigcap\{X \setminus O \mid O \in \tau, O \cap V = \emptyset\}$ denotes the **closure** of V. A **local basis** of a point $x \in X$ is a family $\mathcal{B} \subseteq \tau$ such that for every $U \in \tau$ open neighborhood of x there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Usually, we tacitly assume that $x \in B$ for every $B \in \mathcal{B}$ when we say that $\mathcal{B}$ is a local basis for x. A **basis** for the topology is a family $\mathcal{B} \subseteq \tau$ such that for every $x \in U$ and for every $U \in \tau$ open neighborhood of x there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$. The **weight** $w(X)$ a topological space (X, τ) is the smallest size of a basis for τ. A subset $Y \subseteq X$ is called **dense** (in X) if $\mathrm{cl}(Y) = X$. The **density** $\mathrm{d}(X)$ of (X, τ) is the smallest size of a dense subset $Y \subseteq X$. A topology τ is called zero-dimensional if it admits a basis made of clopen sets. It is called **μ-additive** if it is closed under $< \mu$-sized intersections. Given two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X, we say that $\mathcal{A}$ refines $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$. A family $\mathcal{U}$ of subsets of X is called a cover of X if $\bigcup \mathcal{U} = X$. $\mathcal{U}$ is called **locally finite** if for every $x \in X$ there is $U \in \tau$ such that U intersect finitely many elements of $\mathcal{U}$. A topology is called **paracompact** if every open cover $\mathcal{U}$ can be refined into a locally finite open cover. A topology is called **Lebesgue zero-dimensional**[1] if every open cover $\mathcal{U}$ can be refined into a clopen partition of X. Clearly, if a space is Lebesgue zero-dimensional, it is also paracompact.

The following notable lemma will frequently be used without giving an explicit reference to it.

Lemma 2.1.1 ([87, Lemma 2.1]). *Suppose X and Y are T_1 topological spaces with bases $\mathcal{A}$ and $\mathcal{B}$, respectively, and there is a bijection $f \colon \mathcal{A} \to \mathcal{B}$ such that $\bigcap \mathcal{F} = \emptyset$ if and only if $\bigcap f[\mathcal{F}] = \emptyset$ for every $\mathcal{F} \subseteq \mathcal{A}$. Then X and Y are homeomorphic.*

Recall that given topological spaces (X_α, τ_α) for $\alpha < \delta$, the space $\bigsqcup_{\alpha<\delta} X_\alpha$ is the disjoint sum of the X_α's equipped with the smallest topology that makes each X_α clopen: a set $U \subseteq \bigsqcup_{\alpha<\delta} X_\alpha$ is open if and only if $U \cap X_\alpha \in \tau_\alpha$ for all $\alpha < \delta$. Given a cardinal $\delta' \leq \delta$, the δ'-supported topology on $\prod_{\alpha<\delta} X_\alpha$ is the topology generated by the sets of the form $\prod_{\alpha<\delta} U_\alpha$ where each U_α is open in X_α and $U_\alpha = X_\alpha$ for all but $< \delta'$-many $\alpha < \delta$.

2.1.2 λ-Borel spaces

Given an infinite cardinal η and a nonempty set X, an **η-algebra** on X is a collection $\mathscr{B}$ of subsets of X which is closed under complements and unions (hence also intersections) of size $< \eta$. A **basis** for $\mathscr{B}$ is a collection $\mathscr{A}$ of subsets of X such that $\mathscr{B}$ is the smallest η-algebra containing $\mathscr{A}$; equivalently, $\mathscr{B}$ is obtained from $\mathscr{A}$ by closing it under complements and unions of length $< \eta$. The η-algebra $\mathscr{B}$ **separates points** if for all distinct $x, y \in X$ there is $B \in \mathscr{B}$ such that $x \in B$ and $y \notin B$.

Call a pair $(X, \mathscr{B})$ a **λ-Borel space** if $\mathscr{B}$ is a λ^+-algebra on X which separates points and admits a basis of size λ. The elements of $\mathscr{B}$ are then called **λ-Borel** sets of X. If $(X, \mathscr{B})$ is a λ-Borel space and $Y \subseteq X$, then setting $\mathscr{B} \restriction Y = \{B \cap Y \mid B \in \mathscr{B}\}$ we get that $(Y, \mathscr{B} \restriction Y)$ is again a λ-Borel space. If $(X, \mathscr{B})$ and $(X', \mathscr{B}')$ are λ-Borel spaces, we say that a function $f \colon X \to X'$ is **λ-Borel (measurable)** if $f^{-1}(B) \in \mathscr{B}$ for all $B \in \mathscr{B}'$. A **λ-Borel isomorphism** between $(X, \mathscr{B})$ and $(X', \mathscr{B}')$ is a bijection f such that both f and f^{-1} are λ-Borel; two λ-Borel spaces are then **λ-Borel isomorphic** if there is a λ-Borel isomorphism between them. Finally, a **λ-Borel embedding** $f \colon X \to X'$ is an injective function which is a λ-Borel isomorphism between $(X, \mathscr{B})$ and $(f(X), \mathscr{B}' \restriction f(X))$.

Notice that every T_0 topological space (X, τ) of weight λ can be seen as a λ-Borel space in a canonical way by pairing it with the collection

$$\mathsf{Bor}_\lambda(X, \tau)$$

[1] This notation differs from part of the literature, where a space is called Lebesgue zero-dimensional if every *finite* open cover of X can be refined to a clopen partition of X. There, spaces where every open cover of any size can be refined to a clopen partition are usually called **ultraparacompact** spaces.

of all its λ-Borel subsets, i.e. with the smallest λ^+-algebra generated by its topology. (We sometimes remove τ from this notation if clear from the context.) If not specified otherwise, we are always tacitly referring to such λ^+-Borel structure when dealing with λ-Borel isomorphisms and λ-Borel embeddings between topological spaces.

The set $\mathsf{Bor}_\lambda(X,\tau)$ of λ-Borel subsets of (X,τ) can be stratified as follows. Set $\lambda\text{-}\mathbf{\Sigma}^0_1(X,\tau)=\tau$ and for $1<\alpha<\lambda^+$ set

$$\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)=\left\{\bigcup_{i<\lambda}A_i \;\middle|\; X\setminus A_i\in\bigcup_{\beta<\alpha}\lambda\text{-}\mathbf{\Sigma}^0_\beta(X,\tau)\text{ for all } i<\lambda\right\}.$$

Set also $\lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)=\{X\setminus A\mid A\in\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)\}$ and $\lambda\text{-}\mathbf{\Delta}^0_\alpha(X,\tau)=\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)\cap\lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)$. As usual, the reference to X or to τ (or both) might be omitted if they are clear from the context. An easy computation shows that if (X,τ) is regular and has weight $\leq\lambda$ for all $1\leq\alpha<\beta<\lambda^+$

$$\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau),\lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)\subseteq\lambda\text{-}\mathbf{\Delta}^0_\beta(X,\tau)\subseteq\lambda\text{-}\mathbf{\Sigma}^0_\beta(X,\tau),\lambda\text{-}\mathbf{\Pi}^0_\beta(X,\tau)$$

and

$$\mathsf{Bor}_\lambda(X,\tau)=\bigcup_{1\leq\alpha<\lambda^+}\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)=\bigcup_{1\leq\alpha<\lambda^+}\lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)\bigcup_{1\leq\alpha<\lambda^+}\lambda\text{-}\mathbf{\Delta}^0_\alpha(X,\tau).$$

Given a set $A\in\mathsf{Bor}_\lambda(X,\tau)$, its (**$\lambda$-Borel**) **rank** is the smallest $1\leq\alpha<\lambda^+$ such that $A\in\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)\cup\lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)$, and is denoted by $\mathrm{rank}(A)$ or $\mathrm{rank}_{(X,\tau)}(A)$ if we want to specify the ambient space we are working in. Notice that if $Y\subseteq X$ is endowed with the relative topology $\tau\restriction Y$, then for all $1\leq\alpha<\lambda^+$

$$\lambda\text{-}\mathbf{\Sigma}^0_\alpha(Y,\tau\restriction Y)=\{A\cap Y\mid A\in\lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)\}$$

and

$$\lambda\text{-}\mathbf{\Pi}^0_\alpha(Y,\tau\restriction Y)=\{A\cap Y\mid A\in\lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)\}.$$

In particular, for every $A\subseteq X$ we have $\mathrm{rank}_{(Y,\tau\restriction Y)}(A\cap Y)\leq\mathrm{rank}_{(X,\tau)}(A)$.

When $\lambda=\omega$, it is customary to just speak of Borel spaces, Borel sets, Borel functions, and so on, removing λ from all the notation and terminology.

2.1.3 Trees

We recall here some of the basic notions about trees and partial orders from previous chapter and from the literature.

Let $(\mathbb{T},\leq_{\mathbb{T}})$ be a partial order, and denote by $<_{\mathbb{T}}$ the strict part of $\leq_{\mathbb{T}}$. Two elements s and t of $\mathbb{T}$ are **comparable** if $s\leq_{\mathbb{T}}t$ or $t\geq_{\mathbb{T}}s$, and **incomparable** otherwise. A **chain** is a linear suborder of $\mathbb{T}$, and a **branch** is a maximal chain. We denote by $\mathrm{Br}(\mathbb{T})$ the set of all branches of $\mathbb{T}$. For every $q\in\mathbb{T}$, we denote by $\mathrm{succ}_{\mathbb{T}}(q)=\{p\in\mathbb{T}\mid q<_{\mathbb{T}}p\}$ and $\mathrm{pred}_{\mathbb{T}}(q)=\{p\in\mathbb{T}\mid p<_{\mathbb{T}}q\}$ the set of successors and predecessors of q in $\mathbb{T}$, respectively. We say $(\mathbb{T},\leq_{\mathbb{T}})$ is wellfounded if every subset $A\subseteq\mathbb{T}$ has a minimal element, i.e. an element $t\in A$ such that $\mathrm{pred}_A(t)=\emptyset$. Given a well-founded partial order $\mathbb{T}$, recursively define $\mathrm{Lev}_\alpha(\mathbb{T})$ to be the set of minimal elements of $\mathbb{T}\setminus\bigcup_{\beta<\alpha}\mathrm{Lev}_\beta(\mathbb{T})$. The **height** $\mathrm{ht}_{\mathbb{T}}(q)$ of an element q in $\mathbb{T}$ is the unique α such that $q\in\mathrm{Lev}_\alpha(\mathbb{T})$. The height of $\mathbb{T}$ is defined as the smallest ordinal α such that $\mathrm{Lev}_\alpha(\mathbb{T}) = \emptyset$, and it is denoted by $\mathrm{ht}(\mathbb{T})$. Given a branch $b \in \mathrm{Br}(\mathbb{T})$, we can define the height of b as $\mathrm{ht}_{\mathbb{T}}(b) = \sup\{\mathrm{ht}_{\mathbb{T}}(q) + 1 \mid q \in b\}$, and denote by $\mathrm{Br}_\alpha(\mathbb{T})$ the set of branches of $\mathbb{T}$ of height α. Notice that $\mathrm{ht}(\mathbb{T}) = \sup\{\mathrm{ht}_{\mathbb{T}}(b) \mid b \in \mathrm{Br}(\mathbb{T})\}$.

We say that $(\mathbb{T}, \leq_{\mathbb{T}})$ is a **tree** if $(\mathrm{pred}(q), \leq_{\mathbb{T}} \restriction \mathrm{pred}(q))$ is a well-order for every $q \in \mathbb{T}$. In particular, this implies that $(\mathbb{T}, \leq_{\mathbb{T}})$ is wellfounded. Notice also that every sub-order of a tree is a tree as well. The elements of a tree are usually called **nodes**. Given $s \in \mathbb{T}$, the **localization of** $\mathbb{T}$ **at** s is the subtree $\mathbb{T}_s \subseteq \mathbb{T}$ with domain

$$\mathbb{T}_s = \{t \in \mathbb{T} \mid t \text{ is comparable with } s\}.$$

Given $q \in \mathbb{T}$, we denote by $\mathrm{immsucc}(q)$ the set of immediate successors of q, i.e. the set of $\leq_{\mathbb{T}}$-minimal elements of $\mathrm{succ}(q)$. An note $q \in T$ is called a **leaf** if $\mathrm{succ}(q) = \emptyset$. We say that $\mathbb{T}$ is **δ-closed** if it has no branch of height δ. We say $\mathbb{T}$ is **$<\delta$-closed** (resp., **$\leq\delta$-closed**) if it is α-closed for any $\alpha < \delta$ (resp., $\alpha \leq \delta$). We say that $\mathbb{T}$ is **weakly pruned** if it has no leaf (or, equivalently, if it has no branch of successor height). We say that $\mathbb{T}$ is **splitting** if $|\mathrm{immsucc}(q)| \neq 1$ for every $q \in \mathbb{T}$, that is, for every node $q \in \mathbb{T}$ either it is a leaf or it has at least two distinct immediate successors; given a cardinal $\delta > 1$ we also say that $\mathbb{T}$ is **δ-splitting** if $|\mathrm{immsucc}(q)| = \delta$ or $|\mathrm{immsucc}(q)| = 0$ (i.e. q is a leaf) for every $q \in \mathbb{T}$. We say that a tree $\mathbb{T}$ is **normal** if $\mathrm{pred}(q) = \mathrm{pred}(p)$ implies $p = q$ for every $p, q \in \mathbb{T}$ whose length is not a successor. (This includes the case in which the length is 0 and implies that $\mathrm{Lev}_0(\mathbb{T})$ is a singleton.) We say that $\mathbb{T}$ is **balanced** if all branches have the same height; necessarily, such height coincides with $\mathrm{ht}(\mathbb{T})$, thus being balanced is equivalent to being $<\delta$-closed for $\delta = \mathrm{ht}(T)$. We say that $\mathbb{T}$ is **pruned** if it is weakly pruned and every node of $\mathbb{T}$ belongs to a branch of height $\mathrm{ht}(\mathbb{T})$. Finally, we say that $\mathbb{T}$ is **superclosed** if it is (weakly) pruned and $<\delta$-closed for $\delta = \mathrm{ht}(T)$; equivalently, a tree is superclosed if and only if it is $<\delta$-closed for $\delta = \mathrm{ht}(T)$ and it has limit height.

When $\mathbb{T}$ is a tree, the set of branches $\mathrm{Br}(\mathbb{T})$ of $\mathbb{T}$ is also called **complete body** of $\mathbb{T}$ and denoted by $[[\mathbb{T}]]_c$. The **body** $[[\mathbb{T}]] \subseteq [[\mathbb{T}]]_c$ of $\mathbb{T}$ is the set

$$[[\mathbb{T}]] = \{b \in [[\mathbb{T}]]_c \mid \mathrm{ht}_{\mathbb{T}}(b) = \mathrm{ht}(\mathbb{T})\}.$$

Notice that while $[[\mathbb{T}]]_c \neq \emptyset$ for all nonempty trees $\mathbb{T}$, if $\mathrm{ht}(\mathbb{T})$ is limit we might have $[[\mathbb{T}]] = \emptyset$ even if $\mathbb{T} \neq \emptyset$. Notice also that a tree $\mathbb{T}$ is superclosed if and only if $[[\mathbb{T}]] = [[\mathbb{T}]]_c$ and $\mathrm{ht}(\mathbb{T})$ is limit.

Of particular interest are the so-called trees of sequences. Given an ordinal γ and a nonempty set A, we denote by ${}^\gamma A$ the set of all sequences of length γ and values in A, and we also set ${}^{<\gamma}A = \bigcup_{\alpha<\gamma} {}^\alpha A$. Given $s \in {}^{<\gamma}A$, the length $\mathrm{lh}(s)$ of s is the unique ordinal $\alpha < \gamma$ such that $s \in {}^\alpha A$. We write $s \subseteq t$ to say that s is an initial segment of t, that is, $\mathrm{lh}(s) \leq \mathrm{lh}(t)$ and $s(\beta) = t(\beta)$ for every $\beta < \mathrm{lh}(s)$. For every $t \in T$ and $\alpha \leq \mathrm{lh}(t)$, we denote by $t \restriction \alpha$ the unique $s \subseteq t$ of length α. The concatenation between two sequences s, t is denoted by $s^\frown t$, and to simplify the notation we write $s^\frown a$ and $a^\frown s$ instead of $s^\frown t$ and $t^\frown s$, respectively, when $t = \langle a \rangle$ is a sequence of length 1. By definition, every subset of $T \subseteq {}^{<\gamma}A$ is a tree when equipped with the initial segment (or inclusion) relation $\subseteq$: such trees are called **tree of sequences**. We do not require in general that T is closed under initial segments, thus length $\mathrm{lh}(s)$, which is independent of T, and height $\mathrm{ht}_T(s)$, which instead heavily depends on T, need not to coincide for a node $s \in T$.

Fix an ordinal γ and a nonempty set A. Given a tree of sequences $T \subseteq {}^{<\gamma}A$, define the **boundary** $\delta(T)$ of T as the set of all those sequences $s \in {}^{\leq\gamma}A$ such that $t \notin T$ for every $s \subseteq t \in {}^{<\gamma}A$ and T is cofinal in the set of $\subseteq$-predecessors of s, i.e. for every $\alpha < \mathrm{lh}(s)$ there is $p \in T$ with $s \restriction \alpha \subseteq p$. The complete body $[[T]]_c = \mathrm{Br}(T)$ of T may be canonically identified with the set

$$\{s \in \delta(T) \mid \mathrm{lh}(s) \text{ is a limit ordinal, or } \mathrm{lh}(s) = \alpha + 1 \text{ and } s(\alpha) = \bar{a}\}, \qquad (2.1.1)$$

where $\bar{a}$ is a fixed element of A. (When A is an ordinal as well, we canonically set $\bar{a} = 0$.) More precisely, each branch $b \in \mathrm{Br}(T)$ can be identified with $\bigcup b$ if $\mathrm{ht}_T(b)$ is limit, and with $(\bigcup b) ^\frown \bar{a}$ otherwise. With a small abuse of terminology and notation, the set in (2.1.1) will again be called complete body of T and denoted by $[[T]]_c$. Also the notion of body $[[T]]$ of T can be adapted accordingly, the advantage being that in this way both $[[T]]_c$ and $[[T]]$ consist of sequences of elements of A instead of sequences of sequences. Notice however that for arbitrary trees of sequences $T \subseteq {}^{<\gamma}A$ the body $[[T]]$ needs not to be a subset of ${}^{\gamma}A$, and might consist of sequences of different length (but same height with respect to the tree T).

A **descriptive set-theoretic** (**DST** for short) **tree** is a tree of sequences $T \subseteq {}^{<\gamma}A$ which moreover is downward closed under $\subseteq$, i.e. $s \restriction \alpha \in T$ for every $s \in T$ and $\alpha \leq \mathrm{lh}(s)$. Equivalently, a tree of a sequences T is a DST tree if and only if $\mathrm{lh}(s) = \mathrm{ht}_T(s)$ for every $s \in T$. In the case of DST trees, when writing $T \subseteq {}^{<\gamma}A$ we often tacitly assume that T has height γ. Notice that by such convention, if $T \subseteq {}^{<\gamma}A$ is a DST tree with γ limit its body is a subset of ${}^{\gamma}A$ (in contrast to what happens for an arbitrary tree of sequences) and can be described as

$$[[T]] = \{x \in {}^{\gamma}A \mid \forall \alpha < \gamma\, (x \restriction \alpha \in T)\}.$$

Thus we recover the classical notion of body for DST trees considered e.g. in [93] for $\gamma = \omega$ and in [6] for γ an arbitrary infinite cardinal. For this reason, when T is a DST tree we simply write $[T]$ instead of $[[T]]$, as customary in the literature; accordingly, we also set $[T]_c = [[T]]_c$.

The following are well-known facts showing how trees, trees of sequences and DST are closely related to each other. In particular, trees and trees of sequences are one and the same, up to isomorphism, while DST trees correspond to normal trees.

Proposition 2.1.2. *Let $(\mathbb{T}, \leq_{\mathbb{T}})$ be a tree, let $\gamma = \mathrm{ht}(\mathbb{T})$, and set $\vartheta = |\mathbb{T}|$.*

(1) *If $\mathbb{T}$ is normal, then it is isomorphic to a DST tree $T \subseteq {}^{<\gamma}\vartheta$ with $\mathrm{ht}(T) = \mathrm{ht}(\mathbb{T})$.*

(2) *If $\mathbb{T}$ is not normal and γ is limit, then $\mathbb{T}$ is isomorphic to the subtree $T' = \bigcup_{\alpha<\gamma} \mathrm{Lev}_{\alpha+1}(T)$ of a DST tree $T \subseteq {}^{<\gamma}\vartheta$ with $\mathrm{ht}(T) = \mathrm{ht}(\mathbb{T})$. If γ is successor, then a similar result holds with $T \subseteq {}^{<\gamma+1}\vartheta$ and $\mathrm{ht}(T) \leq \mathrm{ht}(\mathbb{T}) + 1$.*

Proposition 2.1.3. *Let γ be a limit ordinal. For every tree of sequences $T \subseteq {}^{<\gamma}A$ there is a normal tree of sequences $T' \subseteq {}^{<\gamma}A$ such that $\mathrm{ht}(T') \leq \mathrm{ht}(T) + 1$ and $\mathrm{ht}(T') = \mathrm{ht}(T)$ if the latter is limit, $T \subseteq T'$, T and T' are $\subseteq$-cofinal in each other (hence $[[T]]_c = [[T']]_c$ and $[[T]] = [[T']]$), and T and T' share the same properties in the following list: being δ-closed for $\delta < \gamma$, being weakly pruned, being balanced, being splitting, being pruned (if $\mathrm{ht}(T)$ is limit), being superclosed.*

2.1.4 Trees and topology

The (complete) body $[[\mathbb{T}]]_c = \mathrm{Br}(\mathbb{T})$ of a tree $\mathbb{T}$ can be given a natural topology, namely, the one generated by sets of the form

$$\mathbf{N}_q^{\mathbb{T}} = \{b \in [[\mathbb{T}]]_c \mid q \in b\}$$

for $q \in \mathbb{T}$. Subspaces X of $[[\mathbb{T}]]_c$, including the notable case $X = [[\mathbb{T}]]$, are endowed with the relative topology, which is generated by the sets $\mathbf{N}_q^X = \mathbf{N}_q^{\mathbb{T}} \cap X$. When $T \subseteq {}^{<\gamma}A$ is a tree of sequences, using our identification of $[[T]]_c$ as a subset of ${}^{\leq\gamma}A$ the sets in the above basis can be construed as $\mathbf{N}_s^T = \{t \in [[T]]_c \mid s \subseteq t\}$ for $s \in T$, and similarly for $\mathbf{N}_s^X$ with $X \subseteq [[T]]_c$. The above topology will be called **bounded topology**. This is because taking $\gamma = \kappa$ with $\kappa > \omega$ regular and either $A = \kappa$ and $T = {}^{<\kappa}\kappa$, or $A = 2 = \{0, 1\}$ and $T = {}^{<\kappa}2$ we recover the usual generalized Baire space ${}^{\kappa}\kappa$ and generalized Cantor space ${}^{\kappa}2$, whose topology is usually called "bounded topology", which so far played a central role in the literature on generalized descriptive set theory (for regular cardinals).

When λ is any cardinal of cofinality $\mathrm{cof}(\lambda) = \mu$, the rightful generalizations of the Cantor and Baire spaces become respectively:

(a) the **generalized Baire space**

$$^{\mu}\lambda = \{x \mid x \colon \mu \to \lambda\},$$

with its bounded topology;

(b) the **generalized Cantor space**

$$^{\lambda}2 = \{x \mid x \colon \lambda \to 2\},$$

with its bounded topology;

The assumption $2^{<\lambda} = \lambda$ ensures then that ${}^{\lambda}2$ has weight λ and (since it implies also the weaker assumption $\lambda^{<\mu} = \lambda$) that the space ${}^{\mu}\lambda$ has weight λ.

These spaces are particular instances of spaces of the form ${}^{\gamma}A$ with γ a cardinal, again tacitly endowed with the bounded topology: such spaces will in general be dubbed **spaces of sequences**, and they are crucially involved in the current chapter. In these particular cases we often write $\mathbf{N}_s$ instead of $\mathbf{N}_s^{{}^{<\gamma}A}$.

The following is a well-known fact (see e.g. [87] or Theorem 3.8 of [121] and the preceding paragraphs).

Fact 2.1.4. The following are equivalent:

1. λ is weakly compact.
2. ${}^{\lambda}2$ is λ-Lindelöf.
3. $2^{<\lambda} = \lambda$ and ${}^{\lambda}2$ is not homeomorphic to ${}^{\mu}\lambda$.

In particular, for all singular cardinals satisfying $2^{<\lambda} = \lambda$ the generalized Cantor space and the generalized Baire space are homeomorphic.

A simple but crucial fact which easily follows from the definitions of (complete) body and bounded topology is the following.

Fact 2.1.5. Let $\mathbb{T}_1, \mathbb{T}_2$ be trees, and suppose that $f\colon \mathbb{T}_1 \to \mathbb{T}_2$ be an isomorphism. Then the map sending each branch of $\mathbb{T}_1$ to the corresponding branch of $\mathbb{T}_2$ through f is a homeomorphism between the (complete) bodies of $\mathbb{T}_1$ and $\mathbb{T}_2$.

Notice that Fact 2.1.5 passes to subtrees: if $f\colon \mathbb{T}_1 \to \mathbb{T}_2$ and $\mathbb{T}_1'$ is a subtree of $\mathbb{T}_1$, then $f \restriction \mathbb{T}_1'$ is an isomorphism between $\mathbb{T}_1'$ and the corresponding subtree of $\mathbb{T}_2$, so it canonically induces a homeomorphism between the (complete) bodies of such subtrees.

Every set $X \subseteq {}^{\leq\gamma}A$ canonically induces a DST tree

$$T_X = \{x \restriction \alpha \mid x \in X \wedge \alpha < \mathrm{lh}(x)\}. \tag{2.1.2}$$

Notice that for every DST tree $T \subseteq {}^{<\gamma}A$ we have $T_{[T]_c} = T$, while $T_{[T]} = T$ if and only if T is pruned. Conversely, for every $X \subseteq {}^{\gamma}A$ we have that $[T_X] \subseteq {}^{\gamma}A$ is the closure of X in ${}^{\gamma}A$ (where the latter, being the body of ${}^{<\gamma}A$, is equipped with the bounded topology described above), and T_X is the unique pruned DST tree such that X is dense in $[T_X]$. In particular, for any tree $T \subseteq {}^{<\gamma}A$ and $X \subseteq {}^{\gamma}A$ set $\mathcal{B}^X_T = \left\{\mathbf{N}^X_s \mid s \in T\right\}$. Then $\mathcal{B}^X_T \subseteq \mathcal{B}^X_{T\cap T_X} \cup \{\emptyset\}$, and thus $\mathcal{B}^X_T$ is a basis for X if and only if $\mathcal{B}^X_{T\cap T_X}$ is as well.

We also notice that following key fact.

Fact 2.1.6. Let γ be limit, $X \subseteq {}^{\gamma}A$, and let $T \subseteq T_X$ be a (non necessarily DST) pruned tree. Then $\mathcal{B}_T = \left\{\mathbf{N}^X_s \mid s \in T\right\}$ is a basis for X if and only if $X \subseteq [[T]]$. If we instead consider non-pruned trees $T \subseteq T_X$, then the equivalence becomes: $\mathcal{B}_T = \left\{\mathbf{N}^X_s \mid s \in T\right\}$ is a basis for X if and only if

(i) $X' \subseteq [[T]]$, where X' is the Cantor-Bendixson derivative of X consisting of all its accumulation points, and

(ii) for every isolated point $x \in X$ there is $s \in T$ with $\mathbf{N}^X_s = \{x\}$.

If $\gamma > \omega$ is limit, not every (pruned) DST tree is $< \delta$-closed. It follows that there are closed sets $X \subseteq {}^{\gamma}A$ such that T_X is not superclosed. A concrete example when γ is an uncountable cardinal and $A = 2$ is the following:

$$X^{\gamma}_0 = \{x \in {}^{\gamma}2 \mid |\{\alpha < \gamma \mid x(\alpha) = 0\}| < \aleph_0\}. \tag{2.1.3}$$

The above discussion shows that it makes sense to call a closed subset $C \subseteq {}^{\gamma}A$ **superclosed** (in the bounded topology on ${}^{\gamma}A$) if $C = [T']$ for some superclosed DST tree $T' \subseteq {}^{<\gamma}A$.

2.1.5 Games

A **game** G of length μ played by two players I and II is a tuple (A, R, W) where A is the set of possible moves for the two players, $R \subseteq A^{<\mu}$ is a weakly pruned DST tree called set of **legal positions** (meant to be the set of positions that can be reached if both players respect the rules of the game), and $W \subseteq \delta(R)$ is the **payoff** set, i.e. the set of positions which assign the victory to I. The two players alternatively pick elements of A, with I moving first at the beginning and at all limit stages. Thus during a run of the game the players will increasingly construct sequences $p \in {}^{<\mu}A$, called

positions; a position is legal if $p \in R$. If $p = \langle p_\beta \mid \beta < 2\gamma \rangle \in {}^{<\mu}A$ is a position of even length, we will sometimes write it as $p = \langle x_\alpha, y_\alpha \mid \alpha < \gamma \rangle$ where x_α is I's move at round α (i.e. $x_\alpha = p_{2\alpha}$) and y_α is II's reply on that round (i.e. $y_\alpha = p_{2\alpha+1}$). Both players are required to respect the rules, i.e. if a position p is reached after his/her last move (in which case $\mathrm{lh}(p)$ is a successor ordinal), then $p \in R$ whenever $p \restriction \mathrm{lh}(s) - 1 \in R$. A run of the game ends when a position $p \in \delta(R)$ is reached: then I wins if and only if $p \in W$. A **strategy** for II is a function $\sigma \colon {}^{<\mu}A \to A$ which tells him/her what to play next. More precisely, given such a σ and a sequence $r = \langle r_\alpha \mid \alpha < \gamma \rangle \in {}^{<\mu}A$ (to be interpreted as a sequence of moves of I), we canonically get a position of length 2γ setting

$$r * \sigma = \langle r_\alpha, \sigma(r \restriction \alpha + 1) \mid \alpha < \gamma \rangle.$$

We say that r is **compatible with** σ if $(r*\sigma) \restriction \beta \in R$ for all $\beta < 2\gamma$; the strategy σ is **legal** if moreover $r*\sigma \in R$ whenever γ is a successor ordinal and r is compatible with σ. Notice that in order to define a legal strategy σ for II, it is enough to define it on all sequences r which are compatible with σ. (The other values of σ are totally irrelevant, as they will never be reached in an actual run of the game where both players are following the rules and II is following σ.) This will be tacitly used throughout the rest of the chapter. A strategy σ for player II is called **tactic** if its value on a sequence $r = \langle r_\alpha \mid \alpha < \gamma + 1 \rangle \in {}^{<\mu}A$ only depends on γ and r_γ (i.e. the round and the last move of I). Each tactic σ can be thus canonically identified with a function $\sigma' \colon A \times \mu \to A$ obtained by setting $\sigma'(a, \gamma) = b$ if and only if $\sigma(r) = b$ for some/any $r \in {}^{\gamma+1}A$ with $r_\gamma = a$. A strategy (or tactic) for II is **winning** if $r * \sigma \notin W$ for all $r \in {}^{<\mu}A$ compatible with σ such that $r * \sigma \in \delta(R)$. We say that II wins the given game G if (s)he has a winning strategy in it. Strategies and tactics for I are defined similarly.

2.1.6 Generalized metrics

Consider a totally ordered[2] Abelian group

$$\mathbb{G} = \langle G, +_{\mathbb{G}}, 0_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle,$$

and let $<_{\mathbb{G}}$ denote the strict part of $\leq_{\mathbb{G}}$. Given a set X, a $\mathbb{G}$**-metric** is a function $d \colon X^2 \to \mathbb{G}$ satisfying the usual rules of a distance function, i.e. for all $x, y, z \in X$

- $0_{\mathbb{G}} \leq_{\mathbb{G}} d(x, y)$ and $d(x, y) = 0_{\mathbb{G}}$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq_{\mathbb{G}} d(x, y) +_{\mathbb{G}} d(y, z)$.

A $\mathbb{G}$**-ultrametric** (or **non-Archimedean** $\mathbb{G}$**-metric**) is a $\mathbb{G}$-metric where the triangle inequality holds in the following stronger form:

- $d(x, z) \leq_{\mathbb{G}} \max_{\leq_{\mathbb{G}}} \{d(x, y), d(y, z)\}$.

[2] This means that the order $\leq_{\mathbb{G}}$ is linear and translation-invariant (on both sides).

Notice that $\mathbb{G}$-ultrametrics do not depend on the operation $+_{\mathbb{G}}$ of $\mathbb{G}$, and thus could be defined also for structures $\mathbb{G} = \langle \mathbb{G}, 0_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle$ without an operation $+_{\mathbb{G}}$ or, equivalently, replacing the operation $+_{\mathbb{G}}$ with $\max_{\leq_{\mathbb{G}}}$ (which does not result in a totally ordered group, though: we will elaborate on this later on).

Every $\mathbb{G}$-metric space (X, d) is naturally equipped with the (d-)topology generated by its open balls

$$B_d(x, \varepsilon) = \{y \in X \mid d(x, y) <_{\mathbb{G}} \varepsilon\},$$

where $x \in X$ and $\varepsilon \in \mathbb{G}^+ = \{g \in \mathbb{G} \mid 0 <_{\mathbb{G}} g\}$. If X is already a topological space, we say that the $\mathbb{G}$-metric d is compatible with the topology of X if the latter coincides with the d-topology. A topological space is called **$\mathbb{G}$-(ultra)metrizable** if it admits a compatible $\mathbb{G}$-(ultra)metric.

Call **degree of** $\mathbb{G}$ the coinitiality $\mathrm{Deg}(\mathbb{G})$ of the positive cone $\mathbb{G}^+$ with respect to $\leq_{\mathbb{G}}$. By definition, $\mathrm{Deg}(\mathbb{G})$ is a regular cardinal μ. Canonical examples of totally ordered Abelian groups $\mathbb{G}_\mu$ of degree $\mu \geq \omega$ are the real line $\mathbb{R}$ if $\mu = \omega$, and the group $\prod_{i<\mu} \mathbb{Z}$ (with pointwise sum and lexicographic order) if $\mu > \omega$. The degree of $\mathbb{G}$ determines many topological properties of a $\mathbb{G}$-metrizable space. For example, every point of a $\mathbb{G}$-metrizable space over a totally ordered Abelian group $\mathbb{G}$ of degree μ is either isolated, or it has a local basis of size $\leq \mu$ well-ordered by reverse inclusion. In particular, this implies that $\mathbb{G}$-metrizable spaces are divided into classes depending on the degree of $\mathbb{G}$, with discrete spaces being the only spaces that can be metrizable over structures of different degrees. It also implies that every $\mathbb{G}$-metrizable space is $\mathrm{Deg}(\mathbb{G})$-additive (see also Lemma 1.1.19). For this reason, a topological space is called **μ-metrizable** if it admits a compatible $\mathbb{G}$-metric for *some* totally ordered Abelian group $\mathbb{G}$ of degree $\mathrm{Deg}(\mathbb{G}) = \mu$. Similarly, a topological space is called **μ-ultrametrizable** if it admits a compatible $\mathbb{G}$-ultrametric for *some* totally ordered Abelian group $\mathbb{G}$ of degree $\mathrm{Deg}(\mathbb{G}) = \mu$.

While the definition of $\mathbb{G}$-metric could be stated for any structure $\mathbb{G}$ with a binary operation $+_{\mathbb{G}}$, a constant $0_{\mathbb{G}}$, and a binary relation $\leq_{\mathbb{G}}$, not every possible choice would give a class of topological spaces that is suitable for our purpose. For example, if we allow $\mathbb{G}$ to vary among all totally ordered Abelian monoid, we get the class of semimetrizable spaces, that is too wide for us. The problem lies in the fact that there exist totally ordered monoids $\mathbb{G}$ where the set $\{x +_{\mathbb{G}} y \mid x, y \in G\}$ is bounded away from $0_{\mathbb{G}}$, making trivial the triangle inequality. Nevertheless, if we avoid this problem we can extend the definition of metrizability even further.

Let $\mathbb{G} = \langle G, +_{\mathbb{G}}, 0_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle$ be a totally ordered pointed semigroup, i.e. $+_{\mathbb{G}}$ is a binary associative (but not necessarily commutative) operation, $0_{\mathbb{G}}$ is a singled out element of G (but, despite the notation, not necessarily the neutral element of $+_{\mathbb{G}}$, which might not exist), and $\leq_{\mathbb{G}}$ is linear and translation invariant. For example we can take any linear order $\langle G, \leq_{\mathbb{G}} \rangle$, pick any element $0_{\mathbb{G}} \in G$, and then equip the resulting structure it with the binary operation $\max_{\leq_{\mathbb{G}}}$. This setup allows us to formally define $\mathbb{G}$-(ultra)metrics and $\mathbb{G}$-(ultra)metrizable spaces as before, and also the degree $\mathrm{Deg}(\mathbb{G})$ of $\mathbb{G}$ can be defined accordingly. A totally ordered pointed semigroup $\mathbb{G}$ is said **$0_{\mathbb{G}}$-continuous** if for all sequences $\langle r_\alpha \mid \alpha < \gamma \rangle$ coinitial in $\mathbb{G}^+ = \{g \in G \mid 0_{\mathbb{G}} <_{\mathbb{G}} g\}$, we have that $\langle r_\alpha + r_\alpha \mid \alpha < \gamma \rangle$ is still coinitial in $\mathbb{G}^+$. A canonical example of $0_{\mathbb{G}}$-continuous totally ordered pointed semigroup with degree μ is given by $\langle \{0_{\mathbb{G}}\} \cup \mu, \max_{\leq_{\mathbb{G}}}, 0_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle$, where for all $\alpha, \beta \in \mu$ we set

$0_{\mathbb{G}} <_{\mathbb{G}} \alpha$ and $\alpha \leq_{\mathbb{G}} \beta$ if and only if $\alpha \geq \beta$; such a semigroup will be denoted by $\mathbb{S}_\mu$. Notice also that if $\mathbb{G}$ is actually an Abelian *group* with $0_{\mathbb{G}}$ as neutral element, then it is automatically $0_{\mathbb{G}}$-continuous, hence we are strictly enlarging the class of structures used to define generalized metrics.

Recall that we have set $\mathbb{G}_\omega = \mathbb{R}$ and $\mathbb{G}_\mu = \prod_{i<\mu} \mathbb{Z}$ (for $\mu > \omega$) as canonical examples of totally ordered Abelian groups of degree μ, and that μ is always assumed to be a regular cardinal.

Theorem 2.1.7 ([126, Theorem 2]). *Let X be a topological space. The following are equivalent:*

(1) *X is μ-metrizable, i.e. X is $\mathbb{G}$-metrizable over* some *totally ordered Abelian group $\mathbb{G}$ of degree μ;*

(2) *X is $\mathbb{G}$-metrizable over* some *$0_{\mathbb{G}}$-continuous totally ordered pointed semigroup $\mathbb{G}$ of degree μ;*

(3) *X is $\mathbb{G}_\mu$-metrizable.*

This shows that once the regular cardinal μ is fixed, there is some flexibility in choosing the structure $\mathbb{G}$ to define a corresponding (generalized) metric. In particular, if needed we can always restrict the attention to totally ordered Abelian groups (which simplifies some computations), or even just to the canonical groups $\mathbb{G}_\mu$. Notice however that sometimes it is useful to consider structures that are not necessarily groups, e.g. when considering $\mathbb{G}$-ultrametrics. Indeed, in this case it is more natural to look for semigroups: even if $\mathbb{G} = \langle G, +_{\mathbb{G}}, 0_{\mathbb{G}}, \leq_G \rangle$ were a group, a $\mathbb{G}$-ultrametric is precisely a $\mathbb{G}'$-metric for the $0_{\mathbb{G}}$-continuous pointed semigroup $\mathbb{G}' = \langle G, \max_{\leq_{\mathbb{G}}}, 0_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle$, and viceversa. The use of semigroups also allows us to distinguish μ-ultrametrizable spaces inside the class of μ-metrizable spaces as those spaces that are $\mathbb{G}$-metrizable for *every* $0_{\mathbb{G}}$-continuous totally ordered pointed semigroup of the right degree (Corollary 2.2.55). This must be contrasted with item (2) in Theorem 2.1.7, where the quantification over semigroups is only existential. Nevertheless, it must be pointed out that this distinction is relevant only in the case $\mu = \omega$, where e.g. the space $\mathbb{R}$ space is (ω-)metrizable but not $\mathbb{Q}$-metrizable and hence not (ω-)ultrametrizable; in all other cases μ-metrizability and μ-ultrametrizability actually coincide (Theorem 2.2.1).

A canonical example of μ-(ultra)metrizable space is the following.

Example 2.1.8. Let μ be an infinite regular cardinal, $\mathbb{G}$ be any $0_{\mathbb{G}}$-continuous totally ordered pointed semigroup of degree μ, and let $\langle r_\alpha \mid \alpha < \mu \rangle$ be coinitial in $\mathbb{G}^+$. Consider the space ${}^\mu\lambda$. Then the function

$$d(x,y) = \begin{cases} 0_{\mathbb{G}} & \text{if } x = y \\ r_\alpha & \text{if } x(\alpha) \neq y(\alpha) \text{ and } x \restriction \alpha = y \restriction \alpha \end{cases} \tag{2.1.4}$$

is a $\mathbb{G}$-ultrametric compatible with the bounded topology on ${}^\mu\lambda$.

Similarly, if $\mathrm{cof}(\lambda) = \mu$ and $(\lambda_\alpha)_{\alpha<\mu}$ is a strictly increasing sequence of ordinals cofinal in λ, then the function

$$d(x,y) = \begin{cases} 0_{\mathbb{G}} & \text{if } x = y \\ r_\alpha & \text{if } x \restriction \lambda_\alpha \neq y \restriction \lambda_\alpha \text{ and } x \restriction \lambda_\beta = y \restriction \lambda_\beta \text{ for all } \beta < \alpha \end{cases} \tag{2.1.5}$$

is a $\mathbb{G}$-ultrametric compatible with the bounded topology on ${}^{\lambda}A$ for any set A (and in particular, on ${}^{\lambda}2$).

2.2 Characterizations of μ-metrizable spaces

In this section we provide various characterizations of (μ-)metrizability. The literature about μ-metrizability is wide, and several notions have been proven equivalent to it. We collect here the most relevant to us, like Nagata-Smirnov bases and tree bases, and introduce some new ones, most notably the one originating from the μ-uniform local basis game.

The ultimate goal of this section is to prove the following theorem, where as usual μ is an infinite regular cardinal (countable or uncountable).

Theorem 2.2.1. *Let X be a (regular Hausdorff) topological space.*

(1) *The following are equivalent:*

  (a) *X is μ-metrizable;*

  (b) *X is a μ-additive NS^2_μ-space;*

  (c) *X is a μ-additive NS_μ-space;*

  (d) *X is a μ-additive, paracompact, μ-uniformly based space.*

(2) *The following are equivalent:*

  (a) *X is μ-metrizable (and Lebesgue zero-dimensional, if $\mu = \omega$);*

  (b) *X is μ-ultrametrizable;*

  (c) *X is a μ-additive space with a clopen $\mathrm{NS}^{(2)}_\mu$-basis;*

  (d) *X is μ-additive and μ-tree-based;*

  (e) *X is μ-additive, μ-uniformly based and Lebesgue zero-dimensional.*

  If moreover X has weight $\leq \lambda$ for some cardinal λ, then the above conditions are equivalent to

  (f) *X is homeomorphic to a subspace of ${}^{\mu}\lambda$;*

  and if $\mu = \mathrm{cof}(\lambda)$ also to

  (g) *X is homeomorphic to a subspace of ${}^{\lambda}2$.*

In particular, when $\mu > \omega$ then all the above items (a)–(d) from part (1) and (a)–(g) from part (2) (under the appropriate assumptions on λ) are equivalent to each other.

The proof will be given in Section 2.2.5. The symbol NS stands for Nagata-Smirnov: $\mathrm{NS}^{(2)}_\mu$-bases and $\mathrm{NS}^{(2)}_\mu$-spaces are studied in Section 2.2.1. Tree bases are instead studied in Section 2.2.2. Finally, the property of being μ-uniformly based, which to the best of our knowledge is new, is introduced in Section 2.2.3 through a corresponding topological game. These properties are all automatically present in every space of weight $\leq \mu$ (under the additional assumption of μ-additivity for μ-metrizability, and of μ-additivity plus Lebesgue zero-dimensionality for μ-tree-basis, unconditionally for all the other notions). Thus their presence becomes relevant only when we look at spaces of weight $> \mu$ like we do in generalized descriptive set theory on a singular cardinal $\lambda > \mathrm{cof}(\lambda) = \mu$, but they become trivial, for example, in descriptive set theory on a regular cardinal.

It is worth noticing that on the way of proving Theorem 2.2.1, we also provide a new characterization of (classical) metrizability, namely: A (regular Hausdorff) space X is metrizable if and only if it is paracompact and ω-uniformly based (Theorem 2.2.39). The relevance of this result, in comparison with the existing metrization theorems, is briefly discussed at the beginning of Section 2.2.3.

Some of the results of this section already appeared in literature, although in a very sparse way and sometimes in weaker or substantially different forms. To give some examples:

- the equivalence among items (a), (b) and (c) in Theorem 2.2.1(1) is proved in [119, 141, 21] for $\mu = \omega$ and [137, 84] for $\mu > \omega$, but in the latter case item (c) involves only $\mathrm{NS}^{\omega}_{\mu}$-spaces rather than NS^{μ}_{μ}-spaces;
- the equivalence among items (a), (b) and (c) in Theorem 2.2.1(2) is proved in [44] for $\mu = \omega$ and [137] for $\mu > \omega$, but again item (c) is limited to $\mathrm{NS}^{\omega}_{\mu}$-bases rather than NS^{2}_{μ}-bases;
- [121, Theorem 3.3] essentially proves the equivalence among items (a), (b), (d) and (f) in Theorem 2.2.1(2), but although *a posteriori* this shows that if $\mu > \omega$ item (c) of Theorem 2.2.1(1) is equivalent to item (d) of Theorem 2.2.1(2), the known proofs always artificially pass through μ-metrizability: in contrast, in Proposition 2.2.29 we provide a direct link between the two concepts for μ-metrizable spaces.

While proving the new results, namely the addition of item (d) in Theorem 2.2.1(1), of items (e) and (g) in Theorem 2.2.1(2), and the various strengthenings of the known results mentioned above, we will provide a complete self-contained presentation of the subject in a way that should be accessible to readers without a deep knowledge in general topology. In particular, we will reprove (with a few exceptions) in a simpler and more direct way the known statements from the literature that we actually need, and track back definitions and results in the literature with the appropriate references when possible.

2.2.1 Nagata-Smirnov bases

In classical descriptive set theory, one restricts the attention to second-countable topological space. For these spaces, metrizability can be characterized as follows:

Theorem 2.2.2 (Urysohn metrization theorem)**.** *Let X be a second countable topological space. Then, the following are equivalent:*

(a) *X is regular Hausdorff.*

(b) *X is metrizable.*

Analogously, in generalized descriptive set theory for uncountable regular cardinals μ the main focus is on topological spaces of weight $\leq \mu$ (see Chapter 1). In this context, the following Sikorski's Metrization Theorem 2.2.3 can be seen as an analogue of the Urysohn Metrization Theorem 2.2.2 for this class of spaces. Recall that by convention μ is a regular[3] cardinal, and notice that the requirement on μ-additivity is implicit in Urysohn's theorem, as in that case $\mu = \omega$ and ω-addivity holds for all topological spaces.

Theorem 2.2.3 ([139, Theorem (viii)-(x)])**.** *Let X be a topological space of weight $\leq \mu$. Then, the following are equivalent:*

(a) *X is regular Hausdorff and μ-additive;*

(b) *X is μ-metrizable;*

These theorems do not extend to μ-metrizable spaces of weight greater than μ, as there are regular Hausdorff spaces of uncountable weight that are not metrizable, and regular Hausdorff μ-additive spaces of weight $> \mu$ that are not μ-metrizable.

The most famous characterization of metrizable spaces holding unconditionally is Nagata-Smirnov metrization theorem (see [119, 141] and its variant by Bing [21]), and its generalization to μ-metrizable spaces is given by [137]. In those theorems, the crucial condition is the existence of a particular basis for the topology. A family $\mathcal{F}$ of subsets of a topological space X is said **locally $<\delta$-small** if every point $x \in X$ has a neighborhood U that intersects less than δ-many elements of $\mathcal{F}$. This way, a locally $<\omega$-small family is precisely a **locally finite family**. A locally <2-small family is usually called **discrete family**.

Let $\mathcal{B}$ be a basis of a topological space X, and δ, γ be cardinals. We say that $\mathcal{B}$ is a (δ, γ)**-Nagata-Smirnov basis** (or $\mathrm{NS}^\delta_\gamma$**-basis**) if it can be written as $\mathcal{B} = \bigcup_{i<\gamma} \mathcal{B}_i$ where each $\mathcal{B}_i$ is locally $<\delta$-small. The family $\{\mathcal{B}_i \mid i \in \gamma\}$ is called a (δ, γ)**-Nagata-Smirnov cover** (or $\mathrm{NS}^\delta_\gamma$**-cover**) for $\mathcal{B}$. Notice that if $\mathcal{B}$ is a $\mathrm{NS}^\delta_\gamma$-basis with $\mathrm{NS}^\delta_\gamma$-cover $\{\mathcal{B}_i \mid i \in \gamma\}$ and $\mathcal{B}' \subseteq \mathcal{B}$ is a another basis, then $\mathcal{B}'$ is still a $\mathrm{NS}^\delta_\gamma$-basis as witnessed by the $\mathrm{NS}^\delta_\gamma$-cover $\{\mathcal{B}_i \cap \mathcal{B}' \mid i \in \gamma\}$. Thus if $\mathcal{B}$ is a $\mathrm{NS}^\delta_\gamma$-basis for a space X of weight $\leq \lambda$, without loss of generality we can assume that $|\mathcal{B}| \leq \lambda$.

Definition 2.2.4. Let δ, γ be cardinals. A (regular Hausdorff) topological space is called a (δ, γ)**-Nagata-Smirnov space** (briefly, $\mathrm{NS}^\delta_\gamma$**-space**) if it has a $\mathrm{NS}^\delta_\gamma$-basis.

For ease of notation and terminology, when $\delta = \gamma$ we simply speak of γ-Nagata-Smirnov spaces (or NS_γ-spaces), NS_γ-bases, and NS_γ-covers. Notice that if $\gamma' \geq \gamma$ and $\delta' \geq \delta$, then every $\mathrm{NS}^\delta_\gamma$-space is also a $\mathrm{NS}^{\delta'}_{\gamma'}$-space. We are now ready to state the Bing-Nagata-Smirnov Metrization Theorem.

Theorem 2.2.5 ([119, 141, 21])**.** *For any topological space X the following are equivalent:*

(a) *X is a regular Hausdorff NS^2_ω-space;*

(b) *X is a regular Hausdorff $\mathrm{NS}^\omega_\omega$-space;*

[3] The restriction to regular cardinals is due to the fact that if η is a singular cardinal, then η-additivity coincides with η^+-additivity, and thus the only η-additive spaces of weight $\leq \eta$ are the discrete ones.

(c) *X is metrizable.*

This result was later generalized to μ-metrizability: again, the key ingredient to be added when $\mu > \omega$ is μ-additivity. (Again, the case $\mu = \omega$ is already covered by Theorem 2.2.5 because ω-additivity comes for free: the genuinely new result is about the case when μ is uncountable.)

Theorem 2.2.6 ([137, Theorem 6] and [84, Theorem 2.2])**.** *For any topological space X the following are equivalent:*

(a) *X is a regular Hausdorff μ-additive NS^2_μ-space;*

(b) *X is a regular Hausdorff μ-additive NS^ω_μ-space;*

(c) *X is μ-metrizable.*

Notice that every basis $\mathcal{B} = \{B_\alpha \mid \alpha < \mu\}$ *of size* μ trivially admits $\{\{B_\alpha\} \mid \alpha < \mu\}$ as a NS^2_μ-cover. Thus, all spaces of weight $\leq \mu$ are in particular NS^2_μ-spaces.

Remark 2.2.7. Every (regular Hausdorff) space of weight $\leq \mu$ is a NS^2_μ-space.

In particular, Theorems 2.2.5 and 2.2.6 extend Theorems 2.2.2 and 2.2.3, respectively, to spaces of weight $> \mu$.

In classical descriptive set theory, an important dividing line in the class of Polish spaces is given by zero-dimensionality (in the sense of small inductive dimension). Recall that a topological space X is **zero-dimensional** if it admits a basis consisting of clopen sets. Since Polish spaces are second-countable, this is equivalent to requiring that X be **Lebesgue zero-dimensional**, i.e. every open cover of X can be refined[4] to a clopen partition of X. However, when we move to spaces of uncountable weight the two concepts are no longer equivalent: Lebesgue zero-dimensionality implies zero-dimensionality, but the converse, in general, is not true, as there are even complete metric spaces that are zero-dimensional but have Lebesgue covering dimension 1 (see Roy's space [129]). As observed e.g. in [52], in order to have a decent (generalized) descriptive set theory the correct notion to be considered is the stronger one, namely Lebesgue zero-dimensionality. This is due to the following phenomenon. In view of Theorem 2.2.5, a zero-dimensional metrizable space has a $\mathrm{NS}^\omega_\omega$-basis as well as a clopen basis: it is thus natural to ask if one can have a single basis which is both (ω, ω)-Nagata-Smirnov and consists of clopen sets. It turns out that this precisely corresponds to being Lebesgue zero-dimensional.

Theorem 2.2.8 ([44])**.** *Let X be a topological space. The following are equivalent:*

(1) *X is metrizable and Lebesgue zero-dimensional.*

(2) *X is ultrametrizable.*

(3) *X has a $\mathrm{NS}^{(2)}_\omega$-basis consisting of clopen sets.*

In particular, Roy's space [129] is an example of an (ω-additive) zero-dimensional $\mathrm{NS}^\omega_\omega$-space which does not admit a $\mathrm{NS}^\omega_\omega$-basis of clopen sets (since it has Lebesgue covering dimension 1).

[4] Recall that a family $\mathcal{A}$ refines a family $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

Moving to μ-metrizability for an uncountable μ, the situation radically changes. Since μ-metrizable spaces are μ-additive and the latter implies zero-dimensionality when $\mathrm{cof}(\mu) > \omega$, every μ-metrizable space is zero-dimensional; but actually this can be improved to Lebesgue zero-dimensionality using the following notion and results.

Definition 2.2.9. Let δ, γ be cardinals. A topological space X is (δ, γ)**-paracompact** if every open cover of X can be refined into a subcover which is the union of γ-many locally $<\delta$-small families. As a shortcut, we use the term γ**-paracompact** for $(\gamma, 1)$-paracompact.

Thus ω-paracompactness is just the usual notion of paracompactness, which is a notable conseguence of metrizability. Clearly, every $\mathrm{NS}^{\delta}_{\gamma}$-space is (δ, γ)-paracompact. Notice also that if X is (δ, γ)-paracompact and $\delta' \geq \delta$, $\gamma' \geq \gamma$, then X is also (δ', γ')-paracompact.

It is well known that a space is (ω, ω)-paracompact if and only if it is paracompact. In the uncountable case, even more is true.

Proposition 2.2.10 ([10, Theorem 3.4])**.** *Let δ be a regular uncountable cardinal, and let X be a δ-additive space. The following are equivalent:*

(1) *X is (δ, δ)-paracompact.*

(2) *X is paracompact.*

(3) *X is Lebesgue zero-dimensional.*

Proposition 2.2.11. *Let $\mu > \omega$. Every μ-metrizable space is Lebesgue zero-dimensional.*

Proof. Let X be μ-metrizable. By Theorem 2.2.6, X is a (μ-additive) $\mathrm{NS}^{\omega}_{\mu}$-space, hence it is (ω, μ)-paracompact, and thus also (μ, μ)-paracompact. Setting $\delta = \mu$ in Proposition 2.2.10 we are done. □

As for the possibility of having a basis that is simultaneously Nagata-Smirnov (for the appropriate parameters) and made of clopen sets, Shu-Tang [137] showed that if $\mu > \omega$ then every μ-metrizable space has a $\mathrm{NS}^{\omega}_{\mu}$-basis consisting of clopen sets. In what follows, we are going to prove in a very direct way a slightly improved version of this statement. We will use the following simple observations.

Fact 2.2.12. Let δ and γ be cardinals.

(1) The union of finitely many locally $<\delta$-small families of (open) subsets of X is again locally $<\delta$-small. Similarly, if $\gamma \leq \mathrm{cof}(\delta)$ and X is a γ-additive topological space, then the union of $<\gamma$-many locally $<\delta$-small families of (open) subsets of X is again locally $<\delta$-small.

(2) Every $\mathrm{NS}^{\delta}_{\gamma}$-basis $\mathcal{B}$ admits a $\mathrm{NS}^{\delta}_{\gamma}$-cover $\{\mathcal{B}_\alpha \mid \alpha < \gamma\}$ with the property that for every finite subset $F \subseteq \gamma$ there exists $j < \gamma$ such that $\bigcup_{i \in F} \mathcal{B}_i \subseteq \mathcal{B}_j$. Indeed, let $\{F_\alpha \mid \alpha < \gamma\}$ be an enumeration of all finite subsets of γ, and fix any $\mathrm{NS}^{\delta}_{\gamma}$-cover $\{\mathcal{B}'_\alpha \mid \alpha < \gamma\}$ of $\mathcal{B}$. Then each $\bigcup_{i \in F_\alpha} \mathcal{B}'_i$ is again locally $<\delta$-small, so that we can define a new $\mathrm{NS}^{\delta}_{\gamma}$-cover of $\mathcal{B}$ with the desired property by setting $\mathcal{B}_\alpha = \bigcup_{i \in F_\alpha} \mathcal{B}'_i$ for each $\alpha < \gamma$.

(3) Similarly, if $\rho < \mathrm{cof}(\delta)$ is a cardinal such that $\gamma^\rho = \gamma$, and X is ρ^+-additive, we can assume that the same is true for all subsets of γ of size ρ, that is: Every $\mathrm{NS}^\delta_\gamma$-basis $\mathcal{B}$ of X admits a $\mathrm{NS}^\delta_\gamma$-cover $\{\mathcal{B}_\alpha \mid \alpha < \gamma\}$ such that for every $F \subseteq \gamma$ of size ρ we have $\bigcup_{i \in F} \mathcal{B}_i \subseteq \mathcal{B}_j$ for some $j < \gamma$.

(4) Finally, if $\delta = \gamma$ is regular and X is δ-additive, then every $\mathrm{NS}^\delta_\delta$-basis of X admits a $\mathrm{NS}^\delta_\delta$-cover $\{\mathcal{B}_\alpha \mid \alpha < \delta\}$ such that $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ for all $\alpha \leq \beta < \delta$. (If not, just replace each $\mathcal{B}_\alpha$ with $\bigcup_{i \leq \alpha} \mathcal{B}_i$.)

Definition 2.2.13. Given a family $\mathcal{A}$ of open subsets of a topological space (X, τ) and a point $x \in X$, define the set $\mathrm{CN}(\mathcal{A}, x)$ of **complemented $\mathcal{A}$-neighborhoods of** x by setting

$$\mathrm{CN}(\mathcal{A}, x) = \{A \in \mathcal{A} \mid x \in A\} \cup \{X \setminus \mathrm{cl}(A) \mid A \in \mathcal{A} \wedge x \notin \mathrm{cl}(A)\}.$$

When needed, we write $\mathrm{CN}_\tau(\mathcal{A}, x)$ to make explicit that closures in the definition of $\mathrm{CN}(\mathcal{A}, x)$ are computed relatively to the topology τ.

Lemma 2.2.14. *Let δ be a cardinal. Let X be a δ-additive space, and let $\mathcal{A}$ be a locally $<\delta$-small family of open subsets of X. Then for every $x \in X$ the set $\bigcap \mathrm{CN}(\mathcal{A}, x)$ is open.*

Proof. Fix $x \in X$. Given $y \in \bigcap \mathrm{CN}(\mathcal{A}, x)$, we want to find some open set O such that $y \in O \subseteq \bigcap \mathrm{CN}(\mathcal{A}, x)$. Let U be an open neighborhood of y such that $\mathcal{U} = \{A \in \mathcal{A} \mid A \cap U \neq \emptyset\}$ has size $< \delta$. Define

$$\mathcal{B} = \{A \in \mathcal{U} \mid x \in A\} \cup \{X \setminus \mathrm{cl}(A) \mid A \in \mathcal{U} \wedge x \notin \mathrm{cl}(A)\} \subseteq \mathrm{CN}(\mathcal{A}, x).$$

We claim that $O = U \cap \bigcap \mathcal{B}$ works. Since $|\mathcal{B}| \leq |\mathcal{U}| < \delta$ and X is δ-additive, the set O is open, and clearly $y \in U \cap \bigcap \mathcal{B}$ because $\bigcap \mathrm{CN}(\mathcal{A}, x) \subseteq \bigcap \mathcal{B}$: thus we only need to show that $O \subseteq \bigcap \mathrm{CN}(\mathcal{A}, x)$. Since $y \in \bigcap \mathrm{CN}(\mathcal{A}, x)$, for every $A \in \mathcal{A}$ we have that either $A \in \mathcal{U}$, or else $x \notin A$ and $U \subseteq X \setminus \mathrm{cl}(A)$. Given an arbitrary $C \in \mathrm{CN}(\mathcal{A}, x)$, we have two cases. Either $C = A$ for some $A \in \mathcal{A}$ with $x \in A$, in which case by the previous observation we can conclude $A \in \mathcal{U}$ and hence $O \subseteq \bigcap \mathcal{B} \subseteq C$. Or else $C = X \setminus \mathrm{cl}(A)$ for some $A \in \mathcal{A}$ with $x \notin \mathrm{cl}(A)$, and hence by the previous observation either $A \in \mathcal{U}$ and $O \subseteq \bigcap \mathcal{B} \subseteq C$ again, or else $O \subseteq U \subseteq X \setminus \mathrm{cl}(A) = C$. □

Lemma 2.2.15. *Let X be a topological space, and let $\mathcal{A}$ be a family of clopen sets. Then $\{\bigcap \mathrm{CN}(\mathcal{A}, x) \mid x \in X\}$ is a partition of X.*

Proof. Observe that[5] $x \in \bigcap \mathrm{CN}(\mathcal{A}, x)$ for every $x \in X$, hence such sets form a cover of X consisting of nonempty sets. If for some $x, y \in X$ we have $\bigcap \mathrm{CN}(\mathcal{A}, x) \neq \bigcap \mathrm{CN}(A, y)$, then $\mathrm{CN}(\mathcal{A}, x) \neq \mathrm{CN}(\mathcal{A}, y)$, and in turn $\{A \in \mathcal{A} \mid x \in A\} \neq \{A \in \mathcal{A} \mid y \in A\}$ because $\mathcal{A}$ consists of clopen sets. Suppose without loss of generality that there is $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$: since A is clopen, $X \setminus \mathrm{cl}(A) = X \setminus A \in \mathrm{CN}(\mathcal{A}, y)$, and so $\bigcap \mathrm{CN}(\mathcal{A}, x) \cap \bigcap \mathrm{CN}(\mathcal{A}, y) = \emptyset$. □

Lemma 2.2.16. *Let $\delta, \gamma \geq 2$ be cardinals and let X be a δ-additive $\mathrm{NS}^\delta_\gamma$-space. The following are equivalent:*

[5] When $\mathcal{A} = \emptyset$ we have $\mathrm{CN}(\mathcal{A}, x) = \emptyset$ as well, hence $\bigcap \mathrm{CN}(\mathcal{A}, x) = \bigcap \emptyset = X$ by convention.

(a) *X has a $\mathrm{NS}^\delta_\gamma$-basis consisting of clopen sets.*

(b) *X has a NS^2_γ-basis consisting of clopen sets.*

(c) *X has a basis which is a γ-sized union of clopen partitions $\{\mathcal{B}_\alpha \mid \alpha < \gamma\}$.*

Furthermore, if $\delta = \gamma$ is regular, then the above conditions are also equivalent to the following:

(d) *X has a basis which is a union of clopen partitions $\{\mathcal{B}_\alpha \mid \alpha < \gamma\}$ such that $\mathcal{B}_\beta$ refines $\mathcal{B}_\alpha$ for every $\alpha \leq \beta < \gamma$.*

Proof. The implications (d) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), and (b) $\Rightarrow$ (a) are obvious. We simultaneously prove (a) $\Rightarrow$ (c) and (a) $\Rightarrow$ (d) (under the extra cardinal assumption). Let $\mathcal{B}'$ be a $\mathrm{NS}^\delta_\gamma$-basis of clopen sets with cover $\{\mathcal{B}'_\alpha \mid \alpha < \gamma\}$. If $\delta = \gamma$ is regular, we may further assume that $\mathcal{B}'_\alpha \subseteq \mathcal{B}'_\beta$ for every $\alpha \leq \beta < \gamma$ by Fact 2.2.12, which implies that $\bigcap \mathrm{CN}(\mathcal{B}'_\beta, x) \subseteq \bigcap \mathrm{CN}(\mathcal{B}'_\alpha, x)$ for every $x \in X$. By Lemmas 2.2.14 and 2.2.15, setting $\mathcal{B}_\alpha = \{\bigcap \mathrm{CN}(\mathcal{B}'_\alpha, x) \mid x \in X\}$ and $\mathcal{B} = \bigcup_{\alpha<\gamma} \mathcal{B}_\alpha$ we get the desired basis. □

Proposition 2.2.17. *Let δ, γ be a uncountable cardinals with δ regular. Suppose that at least one of $\delta = \gamma$ and $\gamma^\omega = \gamma$ holds. Let X be a δ-additive $\mathrm{NS}^\delta_\gamma$-space. Then X has a NS^2_γ-basis $\mathcal{D}$ consisting of clopen sets.*

Proof. Let $\mathcal{B}$ be a $\mathrm{NS}^\delta_\gamma$-basis for X with $\mathrm{NS}^\delta_\gamma$-cover $\{\mathcal{B}_\alpha \mid \alpha < \gamma\}$. By Fact 2.2.12 and our assumptions on δ and γ, without loss of generality we may assume that for every family $\mathcal{A} \subseteq \mathcal{B}$ of size ω there is $\alpha < \gamma$ such that $\mathcal{A} \subseteq \mathcal{B}_\alpha$. Let

$$\mathcal{C}_\alpha = \left\{ \bigcap \mathcal{A} \;\middle|\; \mathcal{A} \subseteq \mathcal{B}_\alpha \wedge |\mathcal{A}| = \omega \wedge \bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} \mathrm{cl}(A) \right\},$$

and set $\mathcal{D}_\alpha = \{\bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) \mid x \in X\}$ and $\mathcal{D} = \bigcup_{\alpha<\gamma} \mathcal{D}_\alpha$. We claim that each $\mathcal{D}_\alpha$ is a clopen partition of X and that $\mathcal{D}$ is a (NS^2_γ-)basis.

Since X is at least ω_1-additive, by definition each set in $\mathcal{C}_\alpha$ is clopen, hence by Lemma 2.2.15 each $\mathcal{D}_\alpha$ is a partition of X and its elements are closed: we want to show that they are open as well.[6] Since each $\mathcal{B}_\alpha$ is locally $<\delta$-small, for every $\alpha < \gamma$ and $x \in X$ the set $\bigcap \mathrm{CN}(\mathcal{B}_\alpha, x)$ is open by Lemma 2.2.14.

Claim 2.2.17.1. $\bigcap \mathrm{CN}(\mathcal{B}_\alpha, x) \subseteq \bigcap \mathrm{CN}(\mathcal{C}_\alpha, x)$.

Proof of the claim. Indeed, fix any $B \in \mathcal{C}_\alpha$ and a countable family $\mathcal{A} \subseteq \mathcal{B}_\alpha$ witnessing this, and recall that B is clopen. We distinguish two cases, according to the definition of $\mathrm{CN}(\mathcal{C}_\alpha, x)$. If $x \in B = \bigcap \mathcal{A}$, then we have $x \in A$ for every $A \in \mathcal{A}$ and so $\bigcap \mathrm{CN}(\mathcal{B}_\alpha, x) \subseteq \bigcap \mathcal{A} = B$ by construction. If instead $x \notin \mathrm{cl}(B) = B$, then there is $A \in \mathcal{A}$ such that $x \notin \mathrm{cl}(A)$ because $B = \bigcap_{A \in \mathcal{A}} \mathrm{cl}(A)$, hence $X \setminus \mathrm{cl}(A) \in \mathrm{CN}(\mathcal{B}_\alpha, x)$ for such A and

$$\bigcap \mathrm{CN}(\mathcal{B}_\alpha, x) \subseteq \bigcup_{A \in \mathcal{A}} (X \setminus \mathrm{cl}(A)) = X \setminus \bigcap_{A \in \mathcal{A}} \mathrm{cl}(A) = X \setminus B.$$

This finishes the proof of the claim. □

[6] Notice that we cannot directly apply Lemma 2.2.14 because the family $\mathcal{C}_\alpha$ might not be locally $<\delta$-small if there is $\rho < \delta$ such that $\rho^\omega \geq \delta$.

Fix any $\bigcap \mathrm{CN}(\mathcal{C}_\alpha, x)$ and a point y in it. Since $y \in \bigcap \mathrm{CN}(\mathcal{C}_\alpha, y)$ and $\mathcal{D}_\alpha$ is a partition of X, it follows that $\bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) = \bigcap \mathrm{CN}(\mathcal{C}_\alpha, y)$. By the claim, the open set $O = \bigcap \mathrm{CN}(\mathcal{B}_\alpha, y)$ is such that $y \in O \subseteq \bigcap \mathrm{CN}(\mathcal{C}_\alpha, y) = \mathrm{CN}(\mathcal{C}_\alpha, x)$. Since y was arbitrary, this shows that $\bigcap \mathrm{CN}(\mathcal{C}_\alpha, x)$ is an open set, as desired.

So it remains to prove that $\mathcal{D}$ is a basis for X (and hence, in particular, a NS^2_μ-basis for X). Consider any nonempty open set $O \subseteq X$ and an arbitrary point $x \in O$. We want to find $\alpha < \gamma$ such that $x \in \bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) \subseteq O$. Using the regularity of the space X, define a family $\mathcal{A} = \{U_i \mid i < \omega\} \subseteq \mathcal{B}$ of basic open sets such that $x \in \mathrm{cl}(U_{i+1}) \subseteq U_i \subseteq O$ for every $i \in \omega$. By assumption, since $|\mathcal{A}| = \omega$ there is $\alpha < \gamma$ such that $\mathcal{A} \subseteq \mathcal{B}_\alpha$, and furthermore $\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} \mathrm{cl}(A)$ by construction. Hence, $\bigcap \mathcal{A} \in \mathcal{C}_\alpha$, and since $x \in \bigcap \mathcal{A} \subseteq O$ then $x \in \bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) \subseteq O$ as well and we are done. □

In general, if $\delta < \delta' \leq \mu$ then being a $\mathrm{NS}^{\delta'}_\mu$-space is (strictly) weaker than being a NS^{δ}_μ-space. In particular, considering NS_μ-spaces is more general than considering just NS^ω_μ or NS^2_μ-spaces. In contrast, the previous result shows that if $\mu > \omega$ is regular, then for μ-additive spaces all these notions coincide and we can thus get rid of the parameter δ.

Corollary 2.2.18. *Let X be a μ-additive topological space. Then X is a NS_μ-space if and only if it is a NS^δ_μ-space for some/any $2 \leq \delta \leq \mu$.*

This slightly improves Theorem 2.2.6, where the parameter δ was allowed to vary only between 2 and ω, and thus provides a better analogue of Theorem 2.2.5 for an uncountable μ. Another interesting consequence is that when $\mu > \omega$ we always have a NS^2_μ-basis consisting of clopen sets as soon as the space is μ-metrizable, which gives the desired strengthening of another theorem of Shu-Tang from [137], the difference being that now we can get a clopen NS^2_μ-basis instead of a clopen NS^ω_μ-basis. This should be contrasted with the case $\mu = \omega$, where a (ω-)metrizable space has a NS_ω-basis of clopen sets if and only if it is Lebesgue zero-dimensional (Theorem 2.2.8), which in such setup is a nontrivial requirement.

Corollary 2.2.19. *Let $\mu > \omega$ and X be a μ-metrizable space. Then X has a $\mathrm{NS}^{(2)}_\mu$-basis consisting of clopen sets.*

Notice that Proposition 2.2.17 strengthens the mentioned Shu-Tangs's theorem [137] also in the direction of requiring potentially less additivity. For example, suppose that $\mu^\omega = \mu$ and that the space X is at least ω_1-additive: then X has a *clopen* NS^2_μ-basis as soon as it has a NS^ω_μ-basis. (Notice that together with μ-additivity the latter yields to μ-metrizability, the original Shu-Tang's hypothesis, by Theorem 2.2.6: however, here the space needs not to be μ-additive if $\mu > \omega_1$.)

Finally, using Lemma 2.2.16 we further get the following strengthening of Corollary 2.2.19.

Corollary 2.2.20. *Let $\mu > \omega$ and let X be a μ-metrizable space. Then X has a basis which is a μ-sized union of clopen partitions $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$ such that $\mathcal{B}_\beta$ refines $\mathcal{B}_\alpha$ for every $\alpha \leq \beta < \mu$.*

Of course, a similar result hold for $\mu = \omega$ if we further require that X be Lebesgue zero-dimensional.

2.2.2 Tree bases

Topological spaces admitting a basis that form a tree under the reverse inclusion relation $\supseteq$ have been introduced and studied for the first time by Kurepa [99, 100]. Later, this notion has been recovered and studied thanks to its deep connections with non-Archimedean spaces and ultrametrics (see e.g. [121]).

Definition 2.2.21. A **tree basis** for a topological space X is a basis $\mathcal{B}$ such that $\emptyset \notin \mathcal{B}$ and $(\mathcal{B}, \supseteq)$ is a tree; the height of $\mathcal{B}$ is the height of $(\mathcal{B}, \supseteq)$ as a tree. We say that X is **tree-based** if (it is regular Hausdorff and) it admits a tree basis. Moreover, if γ is an ordinal we say that X is γ**-tree-based** if it admits a tree basis of height $\leq \gamma$.

Notice that a basis contained in a tree basis is still a tree basis. It follows that if X has a tree basis $\mathcal{B}$ and has weight $\leq \lambda$, then there is tree basis $\mathcal{B}' \subseteq \mathcal{B}$ with $|\mathcal{B}'| \leq \lambda$. With a little abuse of notation, we often denote the tree $(\mathcal{B}, \supseteq)$ by $\mathcal{B}$ because in this context the tree-relation is fixed and there is no danger of confusion.

Proposition 2.2.22. *Let X be a Hausdorff topological space, and let $\mathcal{B}$ be a tree basis for X. Then*

(1) *the tree $\mathcal{B}$ is splitting;*

(2) *if $U, V \in \mathcal{B}$ are $\supseteq$-incomparable, then $U \cap V = \emptyset$;*

(3) *every $U \in \mathcal{B}$ is clopen, hence X is regular;*

(4) *for every $x \in X$ the family $\mathcal{B}(x) = \{B \in \mathcal{B} \mid x \in B\}$ is a branch through $\mathcal{B}$ and is a local basis of x, hence $\bigcap \mathcal{B}(x) = \{x\}$;*

(5) *conversely, if $\mathcal{A} \subseteq \mathcal{B}$ is a branch through $\mathcal{B}$ and $\bigcap \mathcal{A} \neq \emptyset$, then $\mathcal{A} = \mathcal{B}(x)$ for a (necessarily unique) $x \in X$.*

Proof. (1) Pick any $U \in \mathcal{B}$. If U is a singleton, then it is obviously a leaf in the tree $\mathcal{B}$. If instead $|U| > 1$, then it must have at least one $\supseteq$-immediate successor $\emptyset \neq V \in \mathcal{B}$ because otherwise X would not even be T_0. Pick $y \in U \setminus V$. Consider a $\supseteq$-minimal element in $\{W \in \mathcal{B} \mid W \supsetneq U \wedge y \in W\}$, which is nonempty because X is Hausdorff: it is a $\supseteq$-immediate successor of U different from V, hence we are done.

(2) Suppose that $U \cap V \neq \emptyset$. Since $\mathcal{B}$ is a basis and $U \cap V$ is open, there is $W \in \mathcal{B}$ such that $W \subseteq U \cap V$, hence both U and V belong to $\mathrm{pred}(W)$ in the tree $\mathcal{B}$: it follows that they must be $\supseteq$-comparable by definition of tree.

(3) If $U \in \mathcal{B}$ is a singleton, then it is trivially closed because X is Hausdorff and we are done. Assume now that $|U| > 1$, and pick any $x \in \mathrm{cl}(U)$: we want to show that $x \in U$. By case assumption, there is $y \in U$ with $y \neq x$. Fix $V \in \mathcal{B}$ such that $x \in V$ but $y \notin V$. Since x is a closure point of U, it follows that $V \cap U \neq \emptyset$ and hence either $U \subseteq V$ or $V \subseteq U$ by item (2). But the first alternative is violated by y, hence we get $x \in V \subseteq U$, as desired.

(4) The fact that $\mathcal{B}(x)$ is a chain follows from item (2). Maximality follows instead from item (3). Indeed, let $V \in \mathcal{B} \setminus \mathcal{B}(X)$. Since V is clopen, by regularity there

is $U \in \mathcal{B}$ such that $x \in U$ and $U \cap V = \emptyset$, so that $U \in \mathcal{B}(x)$ and U and V are $\supseteq$-incomparable. This means that V cannot be added to $\mathcal{B}(x)$ in order to build a larger chain. The fact that $\mathcal{B}(x)$ is a local basis of x trivially follows from its definition.

(5) Let $x \in \bigcap \mathcal{A}$. Then by definition $\mathcal{A} \subseteq \mathcal{B}(x)$, and since $\mathcal{A}$ is a maximal chain then $\mathcal{A} = \mathcal{B}(x)$. □

It turns out that the converse to Proposition 2.2.22(2) is true as well. By [121, Theorem 2.9], X is a tree-based topological space if and only if X has a basis where every two basic open sets are either disjoint or comparable with respect to inclusion (i.e. $\mathcal{B}$ is a so-called non-Archimedean basis). Furthermore, Proposition 2.2.22 easily allows us to relate μ-metrizability and tree bases.

Proposition 2.2.23. *A space X is μ-ultrametrizable if and only if it is μ-additive and μ-tree-based.*

Proof. If d is a $\mathbb{G}$-ultrametric on X, then all points in an open ball $B_d(x, \varepsilon)$ are centers of it, and hence any two given open balls that do not intersect have to be comparable with respect to inclusion. It follows that if $\mathrm{Deg}(\mathbb{G}) = \mu$ as witnessed by a strictly decreasing sequence $\langle r_\alpha \mid \alpha < \mu \rangle$, then $\mathcal{B} = \{B_d(x, r_\alpha) \mid x \in X \wedge \alpha < \mu\}$ is a tree basis for X of height $\leq \mu$. Moreover, as already observed in Section 2.1.6, μ-additivity follows from μ-(ultra)metrizability.

Conversely, assume that X is μ-additive and that $\mathcal{B}$ is a tree basis of height $\leq \mu$. Given $\alpha < \mu$ and $x \in X$, let $B_\alpha(x)$ be the unique $B \in \mathrm{Lev}_\alpha(\mathcal{B})$ such that $x \in B$ if such a B exists, otherwise $B_\alpha(x)$ is undefined. By μ-additivity, a point x is isolated in X if and only if some $B_\alpha(x)$ is undefined, and in this case there is a maximal $\alpha < \mu$ such that $B_\alpha(x)$ is defined. Let $\mathbb{G}$ be any totally ordered Abelian group with $\mathrm{Deg}(\mathbb{G}) = \mu$, and let $\langle r_\alpha \mid \alpha < \mu \rangle$ be strictly decreasing and cofinal in $\mathbb{G}^+$. Given distinct $x, y \in X$, set $d(x, y) = r_\alpha$ if and only if $\alpha < \mu$ is smallest such that both $B_\alpha(x)$ and $B_\alpha(y)$ are defined and $B_\alpha(x) \neq B_\alpha(y)$. Using Proposition 2.2.22 and the observation above, it is easy to verify that d is a well-defined $\mathbb{G}$-ultrametric on X. Moreover $B_d(x, r_\alpha) = B_\alpha(x)$ if the latter is defined, while if $B_\alpha(x)$ is undefined then x is isolated and $B_d(x, r_\alpha) = \{x\}$. □

Proposition 2.2.22(3) also implies that a tree-based space is zero-dimensional. This can be strengthened as follows.

Proposition 2.2.24. *Let X be a Hausdorff space with a tree basis $\mathcal{B}$. Then X is Lebesgue zero-dimensional in the following strong sense: Every open cover $\mathcal{A}$ of X can be refined to a clopen partition $\mathcal{C}$ consisting of* basic *open sets (i.e. $\mathcal{C} \subseteq \mathcal{B}$).*

Proof. Without loss of generality, we may assume $\mathcal{A} \subseteq \mathcal{B}$, so that its elements are clopen by Proposition 2.2.22(3). Let $x \in X$. Let $A_x \in \mathcal{A}$ be the $\supseteq$-minimum element of $\mathcal{A}$ such that $x \in A_x$: we claim that $\mathcal{C} = \{A_x \mid x \in X\} \subseteq \mathcal{A}$ works. The only nontrivial part is showing that the elements of $\mathcal{C}$ are pairwise disjoint. But if $A_x, A_y \in \mathcal{C}$ are such that $A_x \cap A_y \neq \emptyset$, then $A_x \subseteq A_y$ or $A_y \subseteq A_x$ by Proposition 2.2.22(2). Suppose the former: if the inclusion were proper, then A_y would contradict the minimality of A_x because $x \in A_x \subsetneq A_y \in \mathcal{A}$. The other case is similar. □

Proposition 2.2.22(4) easily yields to the following useful lemma from [121], which may be viewed as a very weak form of additivity within the tree bases.

Lemma 2.2.25 ([121, Theorem 2.3]). *Let $\mathcal{B}$ be a tree basis for a Hausdorff space X, and let $\mathcal{A} \subseteq \mathcal{B}$ be arbitrary. Then either $\bigcap \mathcal{A}$ is open, or $\bigcap \mathcal{A} = \{x\}$ for some $x \in X$ and $\mathcal{A}$ is a local basis of x (i.e. $\mathcal{B}$ is an* ***ortho-base****).*

Proof. If $\bigcap \mathcal{A} = \emptyset$ we are done, so suppose this is not the case. By definition, for every $x \in \bigcap \mathcal{A}$ we have $\mathcal{A} \subseteq \mathcal{B}(x)$. If $\mathcal{A}$ is $\supseteq$-cofinal in $\mathcal{B}(x)$, then $\bigcap \mathcal{A} = \bigcap \mathcal{B}(x) = \{x\}$ by Proposition 2.2.22(4) and we are done. The remaining case is when $\mathcal{A}$ is not $\supseteq$-cofinal in $\mathcal{B}(x)$ for every $x \in \bigcap \mathcal{A}$. But then by linearity of $\mathcal{B}(x)$ for every such x there is $U \in \mathcal{B}(x)$ such that $U \subsetneq V$ for every $V \in \mathcal{A}$, hence $\bigcap \mathcal{A}$ is open. $\square$

Among tree-based spaces, we find all spaces of the form ${}^{\gamma}A$ and their subspaces. Indeed, let $X \subseteq {}^{\gamma}A$ and $T \subseteq {}^{<\gamma}A$ be a tree: if $\mathcal{B}_T = \{\mathbf{N}_s^X \mid s \in T\}$ is a basis for X and $\emptyset \notin \mathcal{B}_T$ (see Fact 2.1.6 and previous paragraph), then is actually a tree basis of height $\leq \gamma$. Moreover, in this case T is isomorphic to $\mathcal{B}_T$ if and only if T is splitting.

Fact 2.2.26. For every tree $\mathbb{T}$, the complete body $[[\mathbb{T}]]_c$ is a tree-based space, while the body $[[\mathbb{T}]]$ is a δ-additive δ-tree-based space for $\delta = \mathrm{cof}(\mathrm{ht}(\mathbb{T}))$.

In particular, for every (limit) ordinal γ and non-empty set A, the space ${}^{\gamma}A$ with bounded topology is a δ-additive, δ-tree-based for $\delta = \mathrm{cof}(\mathrm{ht}(\mathbb{T}))$.

Conversely, we are now going to observe that tree-based spaces can always be construed as subspaces of $[[T]]_c$ for some tree of sequences $T \subseteq {}^{<\gamma}A$. First, notice that Proposition 2.2.22(4)–(5) easily yields to the following.

Lemma 2.2.27. *Let $\mathcal{B}$ be a tree basis for a Hausdorff space X. Then the map $x \mapsto \mathcal{B}(x)$ is a homeomorphism between X and the subspace $Y \subseteq [[\mathcal{B}]]_c$ given by*

$$Y = \{\mathcal{A} \in [[\mathcal{B}]]_c \mid \bigcap \mathcal{A} \neq \emptyset\}.$$

Indeed, by definition of bounded topology the above homeomorphism maps each $B \in \mathcal{B}$ to the basic open set $\mathbf{N}_B^Y$ and viceversa, so it is an isomorphism between $\mathcal{B}$ and the canonical basis for the bounded topology on $Y \subseteq [[\mathcal{B}]]_c$ (viewed as trees). Further applying Proposition 2.1.2 to the tree $(\mathcal{B}, \supseteq)$ we then get:

Corollary 2.2.28. *Let γ be a limit ordinal. Let X be a space with weight $\leq \lambda$, and assume that X is γ-tree-based. Then there is a tree $T \subseteq {}^{<\gamma}\lambda$ (isomorphic to $(\mathcal{B}, \supseteq)$) such that X embeds into $[[T]]_c$.*

When X is μ-additive for an uncountable μ, admitting a tree basis of height $\leq \mu$ provides yet another reformulation of being a NS_μ-space, and hence of μ-metrizability.

Proposition 2.2.29. *Let X be a μ-additive space with $\mu > \omega$. Then X is a NS_μ-space if and only if it is μ-tree-based.*

Proof. One direction easily follows from Proposition 2.2.17 and Lemma 2.2.16. For the other direction, let $\mathcal{B}$ be a tree basis of height $\leq \mu$. Setting $X_\alpha = \bigcup \mathrm{Lev}_\alpha(\mathcal{B})$, every point $x \in X \setminus X_\alpha$ is isolated because the branch $\mathcal{B}(x)$ has length $\leq \alpha < \mu$

and hence $\bigcap \mathcal{B}(x) = \{x\}$ is open by μ-additivity. Thus by Proposition 2.2.22(2) the family

$$\mathcal{B}_\alpha = \{\{x\} \mid x \in X \setminus X_\alpha\} \cup \mathrm{Lev}_\alpha(\mathcal{B}),$$

is a clopen partition of X, and $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$ is a $\mathrm{NS}^{(2)}_\mu$-cover of $\mathcal{B}$. □

Corollary 2.2.30. *Let $\mu > \omega$ and X be a topological space. Then X is μ-metrizable if and only if it is μ-additive and μ-tree-based.*

As usual, similar results hold if $\mu = \omega$ and X is assumed to be Lebesgue zero-dimensional. In particular, by Remark 2.2.7 we get also the following.

Corollary 2.2.31. *Every regular Hausdorff (Lebesgue zero-dimensional if $\mu = \omega$) μ-additive space of weight $\leq \mu$ is μ-tree-based.*

Remark 2.2.32. In the above results, both μ-additivity and the fact that the tree basis has height $\leq \mu$ are necessary: if $\mu^{<\mu} = \mu$, the lexicographic topology on ${}^{\mu}2$ gives a zero-dimensional NS_μ-space which is not tree-based (cf. Proposition 3.3.2), while Propositions 2.5.1 and 2.5.2 show tree-based spaces (either not μ-additive or with tree bases of height $> \mu$) that are not NS_μ-spaces.

Proposition 2.2.33. *Let X be a μ-additive μ-tree-based space of weight $\leq \lambda$. Then X is homeomorphic to a subspace Y of ${}^{\mu}\lambda$.*

Proof. By Corollary 2.2.28 we can assume that $X \subseteq [[T]]_c$ for some tree $T \subseteq {}^{<\mu}\lambda$. By μ-additivity, all elements of $X \cap {}^{<\mu}\lambda$ are isolated in X. Given $s \in {}^{<\mu}\lambda$, let $s^\frown 0^{(\mu)}$ denote the unique sequence $x \in {}^{\mu}\lambda$ such that $s \subseteq x$ and $x(\alpha) = 0$ for all $\alpha \geq \mathrm{lh}(s)$. Then $Y = (X \cap {}^{\mu}\lambda) \cup \{s^\frown 0^{(\mu)} \mid s \in X \cap {}^{<\mu}\lambda\}$, viewed as a subspace of ${}^{\mu}\lambda$, is homeomorphic to X. □

By regularity of μ, Proposition 2.2.33 can obviously be reversed: every subspace of ${}^{\mu}\lambda$ is μ-additive, μ-tree-based, and has weight $\leq \lambda^{<\mu}$.

2.2.3 Games and μ-uniformly based spaces

Two of the most important consequences of metrizability (besides regularity and being Hausdorff) are paracompactness and first-countability. Many theorems true for metrizable spaces need only these two conditions (or even just one of them). However, these two conditions alone are not enough to grant full metrizability, as shown e.g. by the Sorgenfrey line. We are going to show that what is missing is just a *uniform* version of first-countability (Definition 2.2.34), a condition defined through a suitable topological game that is hidden in metrizability and its equivalent reformulations considered so far (existence of NS_ω-bases or of tree bases). Interestingly enough, all definitions and results work well also for μ-metrizable spaces, so we will not distinguish between the cases $\mu = \omega$ and $\mu > \omega$ in the discussion below.

Let X be a topological space. The **μ-uniform local basis game** is a game of length μ (see Section 2.1.5) of the form

I	x_0		x_1		...	x_γ		...
II		V_0		V_1	...		V_γ	...

At each round $\alpha < \mu$, player I picks a point $x_\alpha \in X$, and player II replies with an open set V_α containing x_α. At the end of the run, player II wins if either $\bigcap_{\alpha<\mu} V_\alpha = \emptyset$, or $\{V_\alpha \mid \alpha < \mu\}$ is a local basis of a point of X; otherwise I wins. Notice that when X is at least T_1, if II has won the run of the game and $\bigcap_{\alpha<\mu} V_\alpha \neq \emptyset$, then $\bigcap_{\alpha<\mu} V_\alpha = \{x\}$ for the (necessarily unique) $x \in X$ of which $\{V_\alpha \mid \alpha < \mu\}$ is a local basis. Notice also that if at some round $\alpha < \mu$ player I plays $x_\alpha \notin \mathrm{cl}(\bigcap_{\beta<\alpha} V_\beta)$, then II can easily wins by playing $X \setminus \mathrm{cl}(\bigcap_{\beta<\alpha} V_\beta)$.

Definition 2.2.34. A topological space is μ-**uniformly based** if player II has a winning strategy in the corresponding μ-uniform local basis game.

The following easy observation shows that we can always assume that the strategy for II in the μ-uniform local basis game only picks basic open sets from any prescribed basis.

Lemma 2.2.35. *Let X be μ-uniformly based and $\mathcal{B}$ be any basis for X. Then player* II *has a winning strategy σ in the μ-uniform local basis game G on X such that* $\mathrm{ran}(\sigma) \subseteq \mathcal{B}$.

Proof. Let σ' be a winning strategy for II in G. For every $r \in {}^{<\mu}X$ of successor length with last element $x \in X$, pick any $B \in \mathcal{B}$ such that $x \in B \subseteq \sigma'(r)$ and set $\sigma(r) = B$. We claim that σ is still winning. Pick any $b \in {}^{\mu}X$. If the intersection $\bigcap_{\alpha<\mu} \sigma(b \restriction (\alpha+1)) \neq \emptyset$ is non-empty, then also $\bigcap_{\alpha<\mu} \sigma'(b \restriction (\alpha+1)) \neq \emptyset$ because $\sigma(b \restriction (\alpha+1)) \subseteq \sigma'(b \restriction (\alpha+1))$. Since σ' is winning, then $\{\sigma'(b \restriction (\alpha+1)) \mid \alpha < \mu\}$ is a neighborhood basis of some point $x \in X$, and $\bigcap_{\alpha<\mu} \sigma'(b \restriction (\alpha+1)) = \{x\}$. It follows that $\bigcap_{\alpha<\mu} \sigma(b \restriction (\alpha+1)) = \{x\}$ too, and that $\{\sigma(b \restriction (\alpha+1)) \mid \alpha < \mu\}$ is a local basis of x as well. □

It is clear that If X is μ-uniformly based, then every point of X has a local basis of size at most μ. But the μ-uniform local basis game is introduced to mimic a stronger property that is common to all μ-metrizable space, namely, the fact that every μ-sequence of open sets with vanishing diameter is a local basis of some point of the space, if its intersection is not empty. (In particular, all μ-metrizable spaces are μ-uniformly based.) The latter property is strictly stronger than just having a local basis of size μ at every point, see for example Proposition 2.5.1.

We first show that the existence of NS_μ-bases or of tree bases is enough to ensure that the space is μ-uniformly based.

Proposition 2.2.36. *Suppose that X has a NS_μ-basis $\mathcal{B}$. Then X is μ-uniformly based and (μ, μ)-paracompact.*

Proof. The fact that the NS_μ-space X is (μ, μ)-paracompact follows from the comment after Definition 2.2.9, so let us prove that it is also μ-uniformly based. Let $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$ be a NS_μ-cover for $\mathcal{B}$, and for every $x \in X$ and $\alpha < \mu$ fix a canonical neighborhood $U(x, \alpha)$ of x such that $\{B \in \mathcal{B}_\alpha \mid B \cap U(x, \alpha) \neq \emptyset\}$ has size $< \mu$. By Fact 2.2.12(4), without loss of generality we may assume that $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ for every $\alpha < \beta < \mu$. For ease of notation, for every $B \in \mathcal{B}$ set $B^0 = B$ and $B^1 = X \setminus \mathrm{cl}(B)$.

We want to define a strategy σ for player II. Let $\langle V_\alpha \mid \alpha < \gamma\rangle$ be the sets played by II up to a certain round γ, and suppose that I just played a point x_γ. Without loss of generality, we can assume that $x_\alpha \in \mathrm{cl}(\bigcap_{\alpha<\gamma} V_\alpha)$, as otherwise II would win by playing $X \setminus \mathrm{cl}(\bigcap_{\alpha<\gamma} V_\alpha)$ on round γ. In particular, we can assume that $V \neq \emptyset$. Moreover, if $\{V_\alpha \mid \alpha < \gamma\}$ is already a local basis for $x_\gamma \in X$, then II could just play X from that point on and win the run, so we can assume that this is not the case. This means that there is $B \in \mathcal{B}$ such that $x_\gamma \in B$ and $V_\alpha \nsubseteq B$ for every $\alpha < \gamma$. Let $\delta(\gamma) < \mu$ be the smallest ordinal for which there are $i_\gamma \in \{0, 1\}$ and $B_\gamma \in \mathcal{B}_{\delta(\gamma)}$ satisfying $x_\gamma \in B_\gamma^{i_\gamma}$ and $V_\alpha \nsubseteq B_\gamma^{i_\gamma}$ for every $\alpha < \gamma$ (this is well defined by case assumption), and let II reply with $V_\gamma = B_\gamma^{i_\gamma} \cap U_\gamma$ where $U_\gamma = U(x_\gamma, \delta(\gamma))$. This concludes the definition of σ.

We claim that σ is a winning strategy for II. Let $\langle x_\alpha, V_\alpha \mid \alpha < \mu\rangle$ be a run in the μ-uniform local basis game where II followed σ. Suppose towards a contradiction that $\langle V_\alpha \mid \alpha < \mu\rangle$ is not winning for II, so that in particular $V = \bigcap_{\alpha<\mu} V_\alpha \neq \emptyset$. Then, by definition of σ we must have $x_\gamma \in \mathrm{cl}(\bigcap_{\alpha<\gamma} V_\alpha)$ and $\{V_\alpha \mid \alpha < \gamma\}$ is not a local basis for x_γ, for every $\gamma < \mu$. So let $\delta(\gamma)$, $i_\gamma \in \{0, 1\}$, $B_\gamma \in \mathcal{B}_{\delta(\gamma)}$, and U_γ be as in the definition of σ, and recall that $V_\gamma = B_\gamma^{i_\gamma} \cap U_\gamma$.

Claim 2.2.36.1. $B_\beta \neq B_\gamma$ and $B_\gamma \cap U_\beta \neq \emptyset$ for all $\beta < \gamma < \mu$.

Proof of the claim. First assume, towards a contradiction, that $B_\beta = B_\gamma$. If $i_\beta = i_\gamma$, then $B_\beta^{i_\beta} = B_\gamma^{i_\gamma}$, and thus $V_\beta = B_\beta^{i_\beta} \cap U_\beta \subseteq B_\gamma^{i_\gamma}$, contradicting the choice of $B_\gamma^{i_\gamma}$ in defining σ. If instead $i_\beta \neq i_\gamma$, then $B_\beta^{i_\beta} \cap B_\gamma^{i_\gamma} = \emptyset$, and thus also $V_\beta \cap V_\gamma = \emptyset$, contradicting $V \neq \emptyset$. This shows that $B_\beta \neq B_\gamma$.

Now suppose towards a contradiction that $B_\gamma \cap U_\beta = \emptyset$, so that $\mathrm{cl}(B_\gamma) \cap U_\beta = \emptyset$ because U_β is open, and hence $U_\beta \subseteq X \setminus \mathrm{cl}(B_\gamma)$. If $i_\gamma = 1$, then we would again have $V_\beta = B_\beta^{i_\beta} \cap U_\beta \subseteq B_\gamma^{i_\gamma}$, in contrast with the definition of σ. If instead $i_\gamma = 0$ then $U_\beta \cap B_\gamma^{i_\gamma} = \emptyset$, hence we have again $V_\beta \cap V_\gamma = \emptyset$, contradicting $V \neq \emptyset$. Thus $B_\gamma \cap U_\beta \neq \emptyset$, as desired. □

Pick any $x \in V = \bigcap_{\alpha<\mu} V_\alpha$. Since σ is not winning, there is a basic open neighborhood $O \in \mathcal{B}$ of x such that $V_\alpha \nsubseteq O$ for every $\alpha < \mu$. Since the space is regular, we may find a basic $O' \in \mathcal{B}$ such that $x \in O' \subseteq \mathrm{cl}(O') \subseteq O$. Let $\bar{\alpha}$ be such that both O and O' belong to $\mathcal{B}_{\bar{\alpha}}$.

Claim 2.2.36.2. $\delta(\gamma) \leq \bar{\alpha}$ for every $\gamma < \mu$.

Proof of the claim. The sets O and $X \setminus \mathrm{cl}(O')$ form a (open) cover of X, and thus x_γ belongs to one of the two sets. Furthermore, for every $\alpha < \mu$ we have that $V_\alpha \nsubseteq X \setminus \mathrm{cl}(O')$ (since $x \in V_\alpha$ while $x \notin X \setminus \mathrm{cl}(O')$) and $V_\alpha \nsubseteq O$ (by the choice of O). Therefore, independently of I's choice of x_γ at round γ, by definition of $\delta(\gamma)$ we must have $\delta(\gamma) \leq \bar{\alpha}$. □

For every $\beta \leq \bar{\alpha}$, let $I_\beta = \{\gamma < \mu \mid \delta(\gamma) = \beta\}$. By Claim 2.2.36.2, $\mu = \bigcup_{\beta\leq\bar{\alpha}} I_\beta$. Since μ is regular and $\bar{\alpha} < \mu$, there is $\bar{\beta} \leq \bar{\alpha}$ such that $|I_{\bar{\beta}}| = \mu$: let $\bar{\gamma}$ be the smallest ordinal in $I_{\bar{\beta}}$. By Claim 2.2.36.1 and the fact that $U_{\bar{\gamma}} = U(x_{\bar{\gamma}}, \delta(\bar{\gamma})) = U(x_{\bar{\gamma}}, \bar{\beta})$ intersects less than μ elements of $\mathcal{B}_{\bar{\beta}}$, we have

$$\mu = |I_{\bar{\beta}}| = |\{B_\gamma \mid \gamma \in I_{\bar{\beta}}\}| \leq 1 + |\{B \in \mathcal{B}_{\bar{\beta}} \mid B \cap U_{\bar{\gamma}} \neq \emptyset\}| < \mu,$$

a contradiction. □

As a corollary, by Remark 2.2.7 we get the following.

Corollary 2.2.37. *Every (regular Hausdorff) space of weight $\leq \mu$ is μ-uniformly based and (μ, μ)-paracompact.*

Proposition 2.2.38. *Let X be a space with a tree basis $\mathcal{B}$ of height $\leq \mu$. Then X is μ-uniformly based and ((μ, μ)-)paracompact.*

Proof. By Proposition 2.2.24 the tree-based space X is Lebesgue zero-dimensional, hence paracompact; let us prove that it is also μ-uniformly based. Without loss of generality, we may assume that $X \in \mathcal{B}$. We define a strategy σ for II as follows. Let $\langle x_\alpha, V_\alpha \mid \alpha < \gamma \rangle$ be a partial play in the μ-uniform local basis game on X such that $V_\alpha \in \mathcal{B}$ for every $\alpha < \gamma$ (this will be granted by our definition of σ), and suppose that player I has played a point x_γ on the next round. Notice that the set $V = \bigcap_{\alpha<\gamma} V_\gamma$ is closed because each set in $\mathcal{B}$ is clopen by Proposition 2.2.22(3), thus we can assume that $x_\gamma \in V$ (otherwise II wins by playing $X \setminus V$). We distinguish two cases.

Case 1 if there is $B \in \mathcal{B}$ such that $x_\gamma \in B \subsetneq V$, set $\sigma(\langle x_\alpha \mid \alpha \leq \gamma \rangle) = B$;

Case 2 otherwise set $\sigma(\langle x_\alpha \mid \alpha \leq \gamma \rangle) = X$.

This concludes the definition of σ.

We claim that σ is a winning strategy for player II. Indeed, assume that we have $\bigcap_{\alpha<\mu} V_\alpha \neq \emptyset$, where the V_α's are II's moves at the end of a run in which (s)he followed σ. If Case 2 never occurred along the run, then by definition of σ the V_α's form a strictly $\subseteq$-decreasing chain of basic open sets in $\mathcal{B}$, and by definition they all belong to $\mathcal{B}(x)$ for some/any $x \in \bigcap_{\alpha<\mu} V_\alpha$. By regularity of μ and the fact that $\mathcal{B}$ has height $\leq \mu$, this implies that $\mathcal{B}(x)$, which by Proposition 2.2.22(4) is a local basis for x, has length μ, and that the V_α's are $\supseteq$-cofinal in $\mathcal{B}(x)$. It follows that $\{V_\alpha \mid \alpha < \mu\}$ is a local basis for x, as desired. Suppose now that $\gamma < \mu$ is least such that Case 2 occurs, and let $V = \bigcap_{\alpha<\gamma} V_\alpha$: recall that V is non-empty by assumption. First, if $V = \{x\}$ is open of size 1, then there is $\beta < \gamma$ such that $V_\beta = \{x\}$ (since we are in Case 2), and so $\{V_\beta\}$ and thus $\{V_\alpha \mid \alpha < \mu\}$ are already a local basis of the point x. If this is not the case, then V can not be open, as otherwise, we could find $U \in \mathcal{B}$ with $x_\gamma \in U \subsetneq V$ because X is Hausdorff, contradicting the fact that we are in Case 2. Thus, by Lemma 2.2.25 applied to $\mathcal{A} = \{V_\alpha \mid \alpha < \gamma\}$ we can conclude that $\mathcal{A}$ is already a local basis of x_γ, and hence so is $\{V_\alpha \mid \alpha < \mu\}$. □

Notice that in tree-based spaces, player II need not to have a winning tactic in the μ-uniform local basis game: see Proposition 2.5.2 and the ensuing corollary.

We are now ready to prove our new metrization theorem. Because of its relevance, we single out the classical case $\mu = \omega$, and deal with the uncountable case $\mu > \omega$ in a separate theorem.

Theorem 2.2.39. *A topological space X is metrizable if and only if it is regular Hausdorff, paracompact and ω-uniformly based.*

Proof. It is well known that every metrizable space is (regular Hausdorff) paracompact, and it is ω-uniformly based by Theorem 2.2.5 and Proposition 2.2.36 (or directly by letting II play open balls of vanishing diameter with respect to a compatible metric).

For the reverse implication, in view of Theorem 2.2.5 it is enough to show that if X is (regular Hausdorff) paracompact and ω-uniformly based, then it has a NS_ω-basis. Suppose σ is a winning strategy for player II in the ω-uniform local basis game G on X. We recursively define locally finite open covers $\{\mathcal{B}_n \mid n < \omega\}$ of X, a DST tree $T \subseteq X^{<\omega}$ (all of whose branches are to be intended as sequences of moves of player I in a run of G), and a surjection

$$f\colon \bigcup_{n<\omega} \mathcal{B}_n \to T \setminus \emptyset, \quad B \mapsto r_B$$

satisfying the following properties (for all $n < \omega$):

(1) $f[\mathcal{B}_n] = T \cap {}^{n+1}X = \mathrm{Lev}_{n+1}(T)$;

(2) for every $B \in \mathcal{B}_n$, the sequence $r_B \in T \cap {}^{n+1}X$ is such that $B \subseteq A \cap \sigma(r_B)$ for some $A \in \mathcal{B}_{n-1}$ with $f(A) = r_B \restriction n$, where if $n = 0$ we set $\mathcal{B}_{-1} = \{X\}$ and $f(X) = \emptyset$.

Notice that the last condition implies that for every $B \in \mathcal{B}_n$ and for every $i < n+1$ there is $A \in \mathcal{B}_{i-1}$ such that $f(A) = r_B \restriction i$ and $B \subseteq A$.

Start with $n = 0$. The family $\mathcal{B}'_0 = \{\sigma(\langle x \rangle) \mid x \in X\}$ covers X because $x \in \sigma(\langle x \rangle)$ by the rules of G and $x \in X$ is arbitrary. Using paracompacteness, find a locally finite open refinement $\mathcal{B}_0$ of $\mathcal{B}'_0$. For every $B \in \mathcal{B}_0$, choose some x_B such that $B \subseteq \sigma(\langle x_B \rangle)$, and set $r_B = \langle x_B \rangle$. Finally, let $\mathrm{Lev}_1(T) = \{r_B \mid B \in \mathcal{B}_0\}$.

Given now an arbitrary $n < \omega$, suppose that $\mathcal{B}_i$, $\mathrm{Lev}_{i+1}(T)$, and $f \restriction \mathcal{B}_i\colon \mathcal{B}_i \to \mathrm{Lev}_{i+1}(T)$ have been defined for every $i \leq n$ in accordance with our constraints. Let

$$\mathcal{B}'_{n+1} = \{A \cap \sigma(r_A {}^\frown x) \mid A \in \mathcal{B}_n \wedge x \in A\}.$$

It is a cover of X because $\mathcal{B}_n$ is a cover of X, and for every $x \in X$ and $A \in \mathcal{B}_n$ such that $x \in A$ we have $x \in A \cap \sigma(r_A {}^\frown x)$ by the rules of G. As before, let $\mathcal{B}_{n+1}$ be a locally finite open refinement of $\mathcal{B}'_{n+1}$. For each $B \in \mathcal{B}_{n+1}$ choose $A \in \mathcal{B}_n$ and $x \in A$ such that $B \subseteq A \cap \sigma(r_A {}^\frown x)$, set $r_B = r_A {}^\frown x$, and let $\mathrm{Lev}_{n+2}(T) = \{r_B \mid B \in \mathcal{B}_{n+1}\}$. All desired conditions are trivially met by the previous construction.

We claim that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a basis for X (and hence a NS_ω-basis, as desired). Given any $x \in X$ and an open neighborhood V of it, define

$$T(x) = \{\emptyset\} \cup \{r \in T \setminus \{\emptyset\} \mid x \in B \text{ for some } B \in \mathcal{B}_{\mathrm{lh}(r)-1} \text{ with } f(B) = r\}.$$

The set $T(x)$ is actually a DST subtree of T, since if $r \in \mathrm{Lev}_{n+1}(T(x))$ and $B \in \mathcal{B}_n$ witness this (so that in particular $x \in B$), then by the comment after condition (2) we have that $B \subseteq A$ for some $A \in \mathcal{B}_{i-1}$ with $r_A = r \restriction i$, hence $x \in A$ and A itself witnesses that $r \restriction i \in T(x)$. Every level of $T(x)$ is finite because $f^{-1}[\mathrm{Lev}_{n+1}(T(x))] = \{B \in \mathcal{B}_n \mid x \in B\}$, which is finite because $\mathcal{B}_n$ is locally finite. Moreover, $\mathrm{ht}(T(x)) = \omega$ because each $\mathcal{B}_n$ covers of X. By König's lemma there is an infinite branch $b = \langle x_n \mid n < \omega \rangle \in [T(x)]$. By construction and condition (2), $x \in \bigcap_{n<\omega} B_n \subseteq \bigcap_{n<\omega} \sigma(b \restriction (n+1))$, where each $B_n \in \mathcal{B}_n$ is a witness of $b \restriction (n+1) \in T(x)$. Since σ is winning in G , the family $\{\sigma(b \restriction (n+1)) \mid n < \omega\}$ is a local basis for a point, which necessarily is x itself. Thus so is $\{B_n \mid n < \omega\} \subseteq \mathcal{B}$, which means that $x \in B_n \subseteq V$ for some $n < \omega$ and we are done. □

As a simple consequence of Theorem 2.2.39, it might be worth noticing that being ω-uniformly bases is enough (together with being regular Hausdorff) to characterize metrizability within the class of Lebesgue zero-dimensional spaces.

Corollary 2.2.40. *Let X be a (regular Haudorff) Lebesgue zero-dimensional space. Then X is metrizable if and only if it is ω-uniformly based.*

We now move to the generalized case and prove an analogous result when μ is uncountable. In this setup, the proof can be slightly simplified because for μ-additive spaces the notion of (μ, μ)-paracompacteness is equivalent to the apparently stronger conditions of being paracompact, or even being Lebesgue zero-dimensional.

Theorem 2.2.41. *Let X be a topological space and $\mu > \omega$. Then the following are equivalent:*

(1) *X is μ-metrizable;*

(2) *X is (regular Hausdorff) μ-additive, μ-uniformly based and (μ, μ)-paracompact.*

Moreover, in item (2) we can equivalently replace (μ, μ)-paracompactness with paracompactness or with Lebesgue zero-dimensionality.

Proof. Recall that a μ-metrizable space is always μ-additive, and since we assumed $\mu > \omega$ it is also Lebesgue zero-dimensional (Proposition 2.2.11), and hence paracompact and (μ, μ)-paracompact. Moreover, it is μ-uniformly based: this can easily be seen directly, or by passing through Theorem 2.2.6 and Proposition 2.2.36.

We now show that (2) implies (1). By Proposition 2.2.10 we can assume that X is Lebesgue zero-dimensional. (Indeed, the same proposition also justifies the additional statement in the theorem.) Let σ be a winning strategy for player II in the μ-uniform local basis game G on X. By Corollary 2.2.30 and the fact that we already have μ-additivity by hypothesis, it is enough to show that X has a tree basis $\mathcal{B}$ of height $\leq \mu$. The basis $\mathcal{B}$ will be defined by recursively constructing clopen partitions $\mathcal{B}_\alpha$ of X, for $\alpha < \mu$, so that $\mathcal{B}_\beta$ refines $\mathcal{B}_\alpha$ for every $\alpha \leq \beta < \mu$: then, once we ensure that $\mathcal{B} = \bigcup_{\alpha<\mu} \mathcal{B}_\alpha$ is a basis for X one can see that it is actually a tree basis with respect to the relation $\supseteq$. The $\mathcal{B}_\alpha$'s are defined recursively, simultaneously constructing a function $f \colon \mathcal{B} \to X$ such that for every $\gamma < \mu$ and $B \in \mathcal{B}_\gamma$

$$B \subseteq \sigma(\langle f(B_\alpha) \mid \alpha \leq \gamma \rangle), \tag{2.2.1}$$

where $B_\gamma = B$ and B_α is the unique set in $\mathcal{B}_\alpha$ satisfying $B \subseteq B_\alpha$.

First define $\mathcal{B}'_0 = \{\sigma(\langle x \rangle) \mid x \in X\}$, and let $\mathcal{B}_0$ be a clopen partition refining $\mathcal{B}'_0$ given by Lebesgue zero-dimensionality. For every $B \in \mathcal{B}_0$, choose $x \in X$ such that $B \subseteq \sigma(\langle x \rangle)$, and set $f(B) = x$.

Now suppose we have defined $\mathcal{B}_\alpha$ and $f \restriction \mathcal{B}_\alpha$ for all $\alpha < \gamma$ so that (2.2.1) is satisfied up to that point. For every $x \in X$, let $B_\alpha(x)$ be the unique element of $\mathcal{B}_\alpha$ containing x, and let $r_x = \langle f(B_\alpha(x)) \mid \alpha < \gamma \rangle ^\frown x$. Let $\mathcal{B}'_\gamma = \{\sigma(r_x) \cap \bigcap_{\alpha<\gamma} B_\alpha(x) \mid x \in X\}$: it is an open cover of X by μ-additivity. Let $\mathcal{B}_\alpha$ be a clopen partition refining $\mathcal{B}'_\alpha$ given by Lebesgue zero-dimensionality. For every $B \in \mathcal{B}_\alpha$, choose $x \in X$ such that $B \subseteq \sigma(r_x) \cap \bigcap_{\alpha<\gamma} B_\alpha(x)$ and set $f(B) = x$. This concludes the construction of $\mathcal{B} = \bigcup_{\alpha<\mu} \mathcal{B}_\alpha$ and f.

It remains to show that $\mathcal{B}$ is a basis for X. Pick an arbitrary $x \in X$, and set $\vec{r}_x = \langle f(B_\gamma(x)) \mid \gamma < \mu \rangle$, where $B_\gamma(x)$ is again the unique element of $\mathcal{B}_\gamma$ containing x. Since $B_\gamma(x) \subseteq \sigma(\vec{r}_x \restriction (\gamma+1))$ by construction and since σ was winning in G, from $x \in \bigcap_{\gamma<\mu} B_\gamma(x) \subseteq \bigcap_{\gamma<\mu} \sigma(\vec{r}_x \restriction (\gamma+1)) \neq \emptyset$ we conclude that the family $\{\sigma(\vec{r}_x \restriction (\gamma+1)) \mid \gamma < \mu\}$ is a local basis of x, and hence so is $\{B_\gamma(x) \mid \gamma < \mu\}$, proving that the entire $\mathcal{B}$ is a basis. □

Remark 2.2.42. The proof of Theorem 2.2.41 works also in the case $\mu = \omega$, in which case μ-additivity comes from free and can be dropped, if we restrict the attention to Lebesgue zero-dimensional spaces. Thus it provides an alternative and direct way to prove Corollary 2.2.40.

Theorem 2.2.41 crucially requires the space to be μ-additive. Nevertheless, for spaces that lack μ-additivity related results can be obtained if we require that player II has a winning tactic instead of just a strategy. (Notice that when dropping μ-additivity, some strengthening of the other conditions is in order because there are non-μ-additive spaces with a strategy in the μ-uniformly basis game that are not NS_μ-spaces: see Proposition 2.5.2 and the ensuing corollary.) Our analysis reveals that in μ-metrizable spaces player II always has a winning tactic (Proposition 2.2.44), and that the existence of such a tactic is indeed equivalent, modulo some form of a para-compactness, to the existence of a suitable Nagata-Smirnov basis (Theorem 2.2.45). These results hold both in the countable and uncountable case, i.e. we can indifferently take $\mu = \omega$ or $\mu > \omega$.

Proposition 2.2.43. *Let X be (δ,μ)-paracompact space such that player* II *has a winning tactic σ in the μ-uniform local basis game G on X. Then X is a NS^δ_μ-space.*

Proof. For every $\alpha < \mu$, let $\mathcal{U}_\alpha = \{\sigma(x,\alpha) \mid x \in X\}$: it is an open cover of X by the rules of G. Using (δ,μ)-paracompactness, let $\mathcal{A}_\alpha$ be an open cover of X refining $\mathcal{U}_\alpha$ such that $\mathcal{A}_\alpha = \bigcup_{\beta<\mu} \mathcal{B}^\beta_\alpha$ for suitable locally $<\delta$-small families $\mathcal{B}^\beta_\alpha$. We claim that $\mathcal{B} = \bigcup_{\alpha,\beta<\mu} \mathcal{B}^\beta_\alpha$ is a basis for X, and hence an NS^δ_μ-basis (as witnessed by the NS^δ_μ-cover $\{\mathcal{B}^\beta_\alpha \mid \alpha,\beta < \mu\}$).

Let $x \in X$. Since each $\mathcal{A}_\alpha$ is an open cover of X, for every $\alpha < \mu$ there is $\beta_\alpha < \mu$ and $B_\alpha \in \mathcal{B}^{\beta_\alpha}_\alpha$ such that $x \in B_\alpha$. Let $x_\alpha \in X$ be such that $B_\alpha \subseteq \sigma(x_\alpha,\alpha)$. Then $\langle x_\alpha, \sigma(x_\alpha,\alpha) \mid \alpha < \mu \rangle$ is a run in the μ-uniform local basis game on X in which II followed σ. Since $x \in \bigcap_{\alpha<\mu} B_\alpha \subseteq \bigcap_{\alpha<\mu} \sigma(x_\alpha,\alpha) \neq \emptyset$, it follows that $\{\sigma(x_\alpha,\alpha) \mid \alpha < \mu\}$ is a local basis for a point of X, which necessarily is x itself Hence the same is true of $\{B_\alpha \mid \alpha < \mu\}$ and we are done. □

Proposition 2.2.44. *Let X be a δ-additive NS^δ_μ-space. Then player* II *has a winning tactic σ in the μ-uniform local basis game G on X.*

Proof. Let $\mathcal{B}$ be a NS^δ_μ-basis for X with NS^δ_μ-cover $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$. Without loss of generality, by Fact 2.2.12(2) we may assume that for every $\alpha,\beta < \mu$ there is $\gamma < \mu$ such that $\mathcal{B}_\alpha \cup \mathcal{B}_\beta \subseteq \mathcal{B}_\gamma$. Define σ by setting, for $x \in X$ and $\alpha < \mu$,

$$\sigma(x,\alpha) = \bigcap \mathrm{CN}(\mathcal{B}_\alpha, x),$$

where $\mathrm{CN}(\mathcal{B}_\alpha, x)$ is as in Definition 2.2.13. Since $\mathcal{B}_\alpha$ is locally $<\delta$-small and X is δ-additive, the set $\sigma(x,\alpha)$ is open by Lemma 2.2.14, and thus if player II follows σ his/her moves are legal.

We claim that the tactic σ is winning. Indeed, suppose that $\langle x_\alpha, V_\alpha \mid \alpha < \mu\rangle$ is a run of G in which $V_\alpha = \sigma(x_\alpha, \alpha)$, i.e. II followed σ. If $\bigcap_{\alpha<\mu} V_\alpha = \emptyset$ then II won the run, so let us assume that there is $x \in \bigcap_{\alpha<\mu} V_\alpha$: we are going to show that $\{V_\alpha \mid \alpha < \mu\}$ is a local basis for x, so that II has won again. Let $B \in \mathcal{B}$ be any basic open neighborhood of x, and use regularity of X to find some basic open set $B' \in \mathcal{B}$ such that $x \in B' \subseteq \mathrm{cl}(B') \subseteq B$. By the assumptions on $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$, there is $\gamma < \mu$ such that B' and B are both in $\mathcal{B}_\gamma$. Notice that $x_\gamma \in B$, as otherwise we would have $x_\gamma \in X \setminus B \subseteq X \setminus \mathrm{cl}(B')$, and hence $V_\gamma = \sigma(x_\alpha, \gamma) \subseteq X \setminus \mathrm{cl}(B')$ by definition of σ: but this would contradict the fact that $x \in V_\gamma \cap B'$. Since $x_\gamma \in B$, by definition of σ we have $x \in V_\gamma = \sigma(x_\gamma, \gamma) \subseteq B$ and we are done. □

In particular, it turns out that spaces with a winning tactic for II in the μ-uniform local basis game on X (and the right degree of paracompactness), precisely characterize the NS^ω_μ-spaces, which in view of Theorem 2.2.6 is a very natural class: in a sense, it is the one obtained by dropping μ-addivity from μ-metrizability (if $\mu > \omega$).

Theorem 2.2.45. *A (regular Hausdorff) space is a* NS^ω_μ*-space if and only if it is* (ω, μ)*-paracompact and player* II *has a winning tactic in the* μ*-uniform local basis game on* X.

When $\mu = \omega$, Theorem 2.2.45 and Theorem 2.2.5 together show that we can require a stronger form of winning strategy for II in the relevant game.

Corollary 2.2.46. *A topological space* X *is metrizable if and only if it is regular Hausdorff, paracompact, and player* II *has a winning* tactic *in the* μ*-uniform local basis game on* X.

2.2.4 G^μ_δ sets

In this section we briefly consider spaces of sequences like ${}^\mu\lambda$ and ${}^\lambda 2$, and the complexity of their subsets in relation to (complete) bodies of trees.

Definition 2.2.47. Let X be a topological space and η be an infinite cardinal. A subset $A \subseteq X$ is called G^η_δ if it can be written as η-sized intersection of open subsets of X.

Notice that when $\eta = \omega$ we recover the classical notion of a G_δ set. If the space X is sufficiently well behaved, and in particular if it is μ-metrizable, one can easily show that the collection of G^μ_δ sets includes all open and closed sets.

Lemma 2.2.48. *Let* X *be a* δ*-additive* NS^δ_μ*-space. Then every closed* $C \subseteq X$ *is a* G^μ_δ *set.*

Proof. Let $\mathcal{B}$ be a NS^δ_μ-basis for X with NS^δ_μ-cover $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$. For every $\alpha < \mu$, define $\mathcal{U}_\alpha = \{U \in \mathcal{B}_\alpha \mid \mathrm{cl}(U) \cap C = \emptyset\}$ and consider the open set $U_\alpha = \bigcup \mathcal{U}_\alpha$. Since each $\mathcal{B}_\alpha$ is locally $<\delta$-small and X is δ-additive, by e.g. [10, Lemma 3.1] we have $\mathrm{cl}(U_\alpha) = \bigcup\{\mathrm{cl}(U) \mid U \in \mathcal{U}_\alpha\}$, hence $\mathrm{cl}(U_\alpha) \cap C = \emptyset$. This means that $X \setminus \bigcup_{\alpha<\mu} \mathrm{cl}(U_\alpha) \subseteq C$. Conversely, by regularity of X for every $x \in X \setminus C$ there is $U \in \mathcal{B}$ such that $x \in U$ and $\mathrm{cl}(U) \cap C = \emptyset$, hence $U \in \mathcal{U}_\alpha$ for some $\alpha < \mu$ and $x \in U_\alpha \subseteq \mathrm{cl}(U_\alpha) \subseteq \bigcup_{\alpha<\mu} \mathrm{cl}(U_\alpha)$. Thus $X \setminus C = \bigcup_{\alpha<\mu} \mathrm{cl}(U_\alpha)$ and so C is G^μ_δ because $C = \bigcap_{\alpha<\mu}(X \setminus \mathrm{cl}(U_\alpha))$. □

We now move to the space $X = {}^{\mu}\lambda$. Recall that a set $C \subseteq {}^{\mu}\lambda$ is closed if and only if there is a (pruned) *DST* tree $T \subseteq {}^{<\mu}\lambda$ such that $C = [T]$: indeed, the tree T can be canonically defined as the tree $T = T_C$ from (2.1.2). Dropping all hypothesis on T, we get a characterization of G^{μ}_{δ} sets of ${}^{\mu}\lambda$.

Lemma 2.2.49. *Let γ be a limit ordinal and A be nonempty. Let $B \subseteq {}^{\gamma}A$ be such that*[7] *$B = [[T]]$ for some tree $T \subseteq {}^{<\gamma}A$ of height $\eta = |\operatorname{ht}(T)|$. Then B is G^{η}_{δ} in ${}^{\gamma}A$.*

Proof. The condition $[[T]] = B \subseteq {}^{\gamma}A$ entails that the branches of T of height $\operatorname{ht}(T)$, when viewed as sequences in ${}^{\leq\gamma}A$, have length γ, which in turn implies that $\operatorname{ht}(T)$ is limit. For every $\alpha < \operatorname{ht}(T)$ define $U_\alpha = \bigcup_{s \in \operatorname{Lev}_\alpha(T)} \mathbf{N}_s^{{}^{\gamma}A}$. We now claim that $\bigcap_{\alpha<\eta} U_\alpha = [[T]] = B$. Using $[[T]] \subseteq {}^{\gamma}A$, the inclusion $[[T]] \subseteq \bigcap_{\alpha<\eta} U_\alpha$ is obvious. Conversely, if $x \in \bigcap_{\alpha<\eta} U_\alpha$ then for each $\alpha < \eta = \operatorname{ht}(T)$ there is $i_\alpha < \gamma$ such that $x \restriction i_\alpha \in \operatorname{Lev}_\alpha(T)$. It follows that $b = \bigcup_{\alpha<\operatorname{ht}(T)} x \restriction i_\alpha = x \restriction \sup_{\alpha<\operatorname{ht}(T)} i_\alpha \in [[T]]$. But since $[[T]] \subseteq {}^{\gamma}A$, it follows that $\sup_{\alpha<\operatorname{ht}(T)} i_\alpha = \gamma$ and $x = b \in [[T]]$. □

Proposition 2.2.50. *A set $B \subseteq {}^{\mu}\lambda$ is G^{μ}_{δ} in ${}^{\mu}\lambda$ if and only if there is a (not necessarily DST) tree $T \subseteq {}^{<\mu}\lambda$ such that $B = [[T]]$.*

Proof. We may assume $B \neq \emptyset$, as otherwise the statement is trivial.

Assume first that $B = [[T]]$ for some tree $T \subseteq {}^{<\mu}\lambda$. Notice that this implies that $\operatorname{ht}(T) = \mu$ because μ is regular. Hence B is G^{μ}_{δ} by Lemma 2.2.49.

Conversely, assume that B is G^{μ}_{δ} in ${}^{\mu}\lambda$, and let $\{U_\alpha \mid \alpha < \mu\}$ be a family of open subsets of ${}^{\mu}\lambda$ such that $B = \bigcap_{\alpha<\mu} U_\alpha$. Without loss of generality, we may assume that $B \neq \emptyset$ and that $U_\beta \subseteq U_\alpha$ for every $\alpha \leq \beta < \mu$ because the space ${}^{\mu}\lambda$ is μ-additive by regularity of μ. For every $\alpha < \mu$, let

$$\mathcal{A}'_\alpha = \{s \in {}^{<\mu}\lambda \mid \operatorname{lh}(s) \geq \alpha \wedge \mathbf{N}_s \subseteq U_\alpha\},$$

and let $\mathcal{A}_\alpha$ be the set of minimal elements of $\mathcal{A}'_\alpha$. This implies that the sequences in $\mathcal{A}_\alpha$ are pairwise incomparable and $U_\alpha = \bigcup_{s \in \mathcal{A}_\alpha} \mathbf{N}_s$. Consider the tree $T = \bigcup_{\alpha<\mu} \mathcal{A}_\alpha$: we claim that $[[T]] = B$.

Suppose first that $x \in B = \bigcap_{\alpha<\mu} U_\alpha$, and let $I = \{i < \mu \mid s \restriction i \in T\}$. Then I is unbounded in μ: for every $\alpha < \mu$ we have that $x \in U_{\alpha+1}$, which means that $x \restriction i \in \mathcal{A}_{\alpha+1}$ for some $i > \alpha$ and hence $i \in I$. Since μ is regular, it follows that $b_x = \{x \restriction i \mid i \in I\}$ is a chain of length μ. This implies that $\operatorname{ht}(T) = \mu$ and that b_x, by its definition and its length, is a branch of T. Thus $x = \bigcup b_x \in [[T]]$, and this proves $B \subseteq [[T]]$.

Conversely, let us prove that $[[T]] \subseteq \bigcap_{\alpha<\mu} U_\alpha = B$. Since we showed that $\operatorname{ht}(T) = \mu$, each $x \in [[T]]$ is an element of ${}^{\mu}\lambda$. Moreover, for every $x \in [[T]]$ there is an unbounded set $I \subseteq \mu$ such that $x \restriction i \in T$ for every $i \in I$. For every $i \in I$, let α_i be smallest ordinal such that $x \restriction i \in \mathcal{A}_{\alpha_i}$: by minimality of the elements in the $\mathcal{A}_\alpha$'s, it follows that the α_i's are all distinct, hence $\{\alpha_i \mid i \in I\}$ is still unbounded in μ. Since the sequence of the U_α's is decreasing with respect to inclusion, it follows that $x \in \bigcap_{i \in I} U_{\alpha_i} = \bigcap_{\alpha<\mu} U_\alpha$, and we are done. □

[7]Recall that, as discussed in Section 2.1.3, since T is a tree of sequences we canonically identify its body with a subset of ${}^{\leq\gamma}A$ (see equation (2.1.1)), and in particular branches b of limit height are identified with $\bigcup b$.

Remark 2.2.51. Let $B \subseteq {}^{\mu}\lambda$ be G^{μ}_{δ} and T be such that $B = [[T]]$, as in the proposition above. Then $T' = T \cap T_B$ still satisfies $B = [[T']]$. This shows that without loss of generality, when picking a tree T such that $B = [[T]]$ we may assume that $T \subseteq T_B$. Notice also that from $B = [[T]]$ and $T \subseteq T_B$ we can conclude that T is necessarily pruned.

Notice that by Propositions 2.2.29 and 2.2.33, the above Proposition 2.2.50 provides an alternative proof of Lemma 2.2.48 when $\delta = \mu$. Moreover, it easily generalizes the following fact, which is well-known if $\mu = \lambda = \omega$.

Corollary 2.2.52. *Every G^{μ}_{δ} subset of ${}^{\mu}\lambda$ is homeomorphic to a closed subset of ${}^{\mu}\lambda$. In particular, in ${}^{\mu}\lambda$ the classes of closed sets and G^{μ}_{δ} sets coincide up to homeomorphism.*

Proof. Given a G^{μ}_{δ} set $B \subseteq {}^{\mu}\lambda$, let $T \subseteq {}^{<\mu}\lambda$ be a tree such that $B = [[T]]$ (Proposition 2.2.50), so that in particular $\mathrm{ht}(T) = \mu$ is limit. By Proposition 2.1.3, we may assume that T is a normal tree. By Proposition 2.1.2(1), there is an isomorphism between T and a DST tree $T' \subseteq {}^{<\mu}\lambda$ (of height μ). Then $B = [[T]]$ is homeomorphic to $[[T']] = [T']$ by Fact 2.1.5. Since $[T']$ is closed, we are done. □

To complete the proof of Theorem 2.2.1 we need one last ingredient, namely, that ${}^{\mu}\lambda$ can be embedded in ${}^{\lambda}2$ if $\mathrm{cof}(\lambda) = \mu$. The following result can easily be proved with ad hoc constructions, but using Lemma 2.2.49 we can use a very short and elegant argument.

Proposition 2.2.53. *Assume that* $\mathrm{cof}(\lambda) = \mu$. *Then ${}^{\lambda}2$ contains a G^{μ}_{δ} homeomorphic copy of ${}^{\mu}\lambda$.*

Proof. Let $\langle \lambda_\alpha \mid \alpha < \mu \rangle$ be an increasing sequence of ordinals cofinal in λ. For an ordinal β, let $0^{(\beta)}$ be the constant sequence of length β with value 0. Recursively define the function $\phi \colon {}^{<\mu}\lambda \to {}^{<\lambda}2$ by setting $\phi(\emptyset) = \emptyset$,

$$\phi(s \frown \gamma) = \phi(s) \frown 0^{(\lambda_{\mathrm{lh}(s)})} \frown 0^{(\gamma)} \frown 1,$$

and $\phi(s) = \bigcup_{\alpha < \mathrm{lh}(s)} \phi(s \restriction \alpha)$ for s of limit length. It is easy to see that ϕ is an isomorphism between the whole ${}^{<\mu}\lambda$ and the tree $T' = \phi[{}^{<\mu}\lambda] \subseteq {}^{<\lambda}2$, which has height μ by construction: hence ${}^{\mu}\lambda = [[{}^{<\mu}\lambda]]$ is homeomorphic to $[[T']]$ by Fact 2.1.5. Notice that ϕ maps branches of ${}^{<\mu}\lambda$ into elements of ${}^{\lambda}2$, since we have $\mathrm{lh}(\phi(s)) \geq \lambda_{\mathrm{lh}(s)-1}$ for every $s \in {}^{<\mu}\lambda$ of successor length. It follows that $[[T']] \subseteq {}^{\lambda}2$. Then, $[[T']]$ is G^{μ}_{δ} in ${}^{\lambda}2$ by Lemma 2.2.49 and we are done. □

For the sake of completeness, we also observe that under further assumptions on λ the space ${}^{\lambda}2$ can be embedded into ${}^{\mu}\lambda$, this time even as a superclosed set.

Proposition 2.2.54. *Suppose that* $\mathrm{cof}(\lambda) = \mu$ *and that* $2^{<\lambda} = \lambda$. *Then ${}^{\mu}\lambda$ contains a superclosed set homeomorphic to ${}^{\lambda}2$.*

Proof. Let $\langle \lambda_\alpha \mid \alpha < \mu \rangle$ be an increasing sequence of ordinals cofinal in λ. For every $\alpha < \mu$, let $\delta_\alpha = |{}^{\lambda_\alpha}2|$, so that $\delta_\alpha \leq \lambda$ by $2^{<\lambda} = \lambda$. Then ${}^{\lambda}2$ is obviously homeomorphic to the superclosed set $\prod_{\alpha<\mu} \delta_\alpha \subseteq {}^{\mu}\lambda$. □

2.2.5 Proof of Theorem 2.2.1

Combining together the results obtained in the previous sections, we finally get the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. We start with part (1). If $\mu = \omega$, then (a) $\iff$ (b) $\iff$ (c) by the classical Theorem 2.2.5. If instead $\mu > \omega$, then (a) $\iff$ (b) are equivalent by the well-known Theorem 2.2.6, and obviously (b) $\Rightarrow$ (c). But (c) $\Rightarrow$ (b) by Corollary 2.2.18, hence we get again the equivalence among the first three items. Finally, (a) $\iff$ (d) by Theorem 2.2.39 (if $\mu = \omega$) or by Theorem 2.2.41 (if $\mu > \omega$).

Let us move to part (2). Assume first that $\mu = \omega$. Then (a) $\iff$ (b) $\iff$ (c) by the well-known Theorem 2.2.8. Also, (a) $\iff$ (e) by Corollary 2.2.40, while (b) $\iff$ (d) by Proposition 2.2.23. If X has weight $\leq \lambda$, then (d) $\Rightarrow$ (f) by Proposition 2.2.33, and if moreover $\mu = \operatorname{cof}(\lambda)$ then (f) $\Rightarrow$ (g) by Proposition 2.2.53. Finally, both (f) $\Rightarrow$ (b) and (g) $\Rightarrow$ (b) by Example 2.1.8. This concludes the proof of the countable case.

We now move to the case $\mu > \omega$. Now (a) $\Rightarrow$ (c) by Corollary 2.2.19 (recalling that μ-additivity readily follows from μ-metrizability), (c) $\Rightarrow$ (d) by Proposition 2.2.29, (d) $\iff$ (a) by Corollary 2.2.30, while (a) $\iff$ (e) by Theorem 2.2.41. All remaining (bi-)implications (b) $\iff$ (d), (d) $\Rightarrow$ (f), (f) $\Rightarrow$ (g), (f) $\Rightarrow$ (b), and (g) $\Rightarrow$ (b) (under the appropriate hypotheses on λ) are proved as in the countable case, hence we are done. □

In view of the fact that ${}^{\mu}\lambda$ is $\mathbb{G}$-ultrametrizable for every $0_{\mathbb{G}}$-continuous totally ordered (Abelian) semigroup (Example 2.1.8), and that if $\mathbb{G}$ is of the form $\mathbb{G} = (G, \max_{\leq_{\mathbb{G}}}, 0_{\mathbb{G}}, \leq_{\mathbb{G}})$ with $\operatorname{Deg}(\mathbb{G}) = \mu$ then every $\mathbb{G}$-metric is actually a $\mathbb{G}$-ultrametric by definition, the equivalence between (b) and (f) in Theorem 2.2.1(2) yields to the following. (Recall the definition of $\mathbb{S}_\mu$ from page 83.)

Corollary 2.2.55 (cf. Theorem 2.1.7)**.** *For any topological space X, the following are equivalent:*

(1) *X is μ-ultrametrizable, i.e. X admits a compatible $\mathbb{G}$-ultrametric for* some *totally ordered Abelian group $\mathbb{G}$ of degree μ;*

(2) *X is $\mathbb{G}$-metrizable over* every *$0_{\mathbb{G}}$-continuous totally ordered pointed semigroup $\mathbb{G}$ of degree μ;*

(3) *X is $\mathbb{S}_\mu$-(ultra)metrizable.*

2.3 Variations on completeness

In this section, we investigate various notions of (generalized) completeness for our classes of topological spaces. The first two, strictly related to each other, are based on Choquet-like games and mirror what has been done in Chapter 1 for regular cardinals κ (Section 2.3.1). They are motivated by the fact that in classical descriptive set theory, by a theorem of Choquet a (regular Hausdorff) space is Polish if and only if it is second-countable and strong Choquet [93, Theorem 8.18]. On a different approach, if we are already dealing with a (μ-)metrizable space, then it is natural to introduce completeness as convergence of (long) Cauchy sequences, as in the classical definition of a Polish space (Section 2.3.2). For regular cardinals, this possibility was already exploited in Chapter 1 together with a stronger completeness notion, still based on the presence of a (μ-)metric, called spherically completeness. Besides adapting the above concepts to embrace singular cardinals as well, in this section we introduce two other completeness notions leading to compact-based spaces and μ-Lindelöf-based spaces (Section 2.3.3). They show that, just as μ-metrizability can be characterized through the existence of particular bases like Nagata-Smirnov-basis and tree-bases, also completeness can be characterized through the existence of bases having certain compactness-like properties. Besides being interesting on their own, these new notions also allow us to better clarify the relationships among the previous ones, and in a sense simplify some of the proofs from Chapter 1.

In each of the three groups of definitions above (Choquet-like completeness, metric-related completeness, and completeness given by certain bases), we always have a weaker version and a stronger one. In this section, we are going to show that for μ-metrizable spaces[8] all the three weak notions coincide with each other, and the same for the three stronger notions.

Theorem 2.3.1. *Let X be a (regular Hausdorff) space of weight $\leq \lambda$, and further assume[9] that X be Lebesgue zero-dimensional if $\mu = \omega$. The following are equivalent:*

(1) *X is (λ, μ)-Polish;*

(2) *X is a spherically μ-complete μ-metrizable space;*

(3) *X is a μ-metrizable $f\mathrm{SC}^{\lambda}_{\mu}$-space;*

(4) *X is a μ-Lindelöf-based μ-metrizable space;*

(5) *X is homeomorphic to a closed subset of ${}^{\mu}\lambda$;*

(6) *X is homeomorphic to a G^{μ}_{δ} subset of ${}^{\mu}\lambda$;*

If furthermore $\mu = \mathrm{cof}(\lambda)$, then they are also equivalent to the following:

(7) *X is homeomorphic to a G^{μ}_{δ} subset of ${}^{\lambda}2$.*

Theorem 2.3.2. *Let X be a (regular Hausdorff) space of weight $\leq \lambda$, and further assume that X be Lebesgue zero-dimensional if $\mu = \omega$. The following are equivalent:*

(1) *X is a spherically $<\mu$-complete (λ, μ)-Polish space;*

(2) *X is a spherically complete μ-metrizable space;*

(3) *X is a μ-metrizable $\mathrm{SC}^{\lambda}_{\mu}$-space;*

(4) *X is a compact-based μ-metrizable space;*

[8] Given that (generalized) Cauchy-completeness, which is arguably the most natural one, can be defined only in presence of a μ-metric, this setup appears quite natural for this analysis.

[9] Formally, this additional requirement must be made explicit only in the countable case. However, recall that when $\mu > \omega$ all μ-metrizable spaces are Lebesgue zero-dimensional by Proposition 2.2.11, so this condition is always implicitly present.

(5) *X is homeomorphic to a superclosed subset of ${}^{\mu}\lambda$.*

In particular, when $\mu = \omega$ then all the above items (1)–(7) from Theorem 2.3.1 and (1)–(5) from Theorem 2.3.2 (under the appropriate assumptions on λ) are equivalent to each other, because, for example, in this case the classes of $f\mathrm{SC}^{\lambda}_{\omega}$ and $\mathrm{SC}^{\lambda}_{\omega}$-spaces coincide (Fact 2.3.8).

The proof of the theorem will be given in Subsection 2.3.4.

Obviously, in Theorems 2.3.1 and 2.3.2 we could systematically replace μ-metrizability with any of the equivalent conditions from Theorem 2.2.1. Moreover, by Corollary 2.3.33 we get that if X is a (λ, μ)-Polish space (or a space satisfying any of the conditions in Theorem 2.3.1) and $Y \subseteq X$ is a closed subspace satisfying some/any of the stronger completeness notions from Theorem 2.3.2, then Y is a retract of X. (This is relevant because when $\mu > \omega$ it is no longer true that every closed subset of ${}^{\mu}\lambda$ is a retract of it.)

Corollary 2.3.3. *Let X be a (Lebesgue zero-dimensional, if $\mu = \omega$) (λ, μ)-Polish space. Then every* spherically complete *closed subspace $Y \subseteq X$ is a retract of X.*

If we drop μ-metrizability (and we restrict ourselves to spaces of weight $\leq \lambda$), the largest class of spaces among all the possibilities considered here is the one of strong fair μ-Choquet spaces. Theorem 2.2.1 shows that a first important dividing line within such class is given by μ-metrizability (and its equivalent characterizations). Theorem 2.3.2 shows instead that a second important dividing line is given by the stronger forms of completeness, in particular strong μ-Choquetness or equivalently, if we are in the class of μ-metrizable spaces, spherically completeness. Dropping μ-metrizability, the largest class of spaces among those from Theorem 2.3.2 is given by strong μ-Choquet spaces. When developing generalized descriptive set-theory for *regular* cardinals as in Chapter 1, this neat picture is quite exhaustive and gives us a well-delimited setup to work with. (Recall that when μ equals the weight of the space, properties like having a NS^2_{μ}-basis are automatic: hence μ-metrizability amounts to being μ-additive and there are no other distinctions to be made in that respect.) When moving to singular cardinals, the situation is more graded because we already have various non-equivalent possibilities to extend (weak forms of) μ-metrizability beyond the realm of μ-additive spaces: being a NS_{μ}-space, having a tree-basis of height $\leq \mu$, being (μ, μ)-paracompact and μ-uniformly based, and so on. Moreover, the addition of concepts like being μ-Lindelöf-based and compact-based opens up the possibility of exploring different classes of "complete" spaces.

2.3.1 Choquet-like games

Let δ be an infinite cardinal. Recall from Chapter 1 that the strong δ-Choquet game $\mathrm{G}^{s}_{\delta}(X)$ on a topological space X is the game played by two players I and II, where for every round $\alpha < \delta$, player I pick a set U_α and a point x_α, and player II replies with a set V_α

I	(U_0, x_0)		(U_1, x_1)		$\dots$	(U_α, x_α)		$\dots$
II		V_0		V_1	$\dots$		V_α	$\dots$

so that

(1) $U_{\alpha+1} \subseteq V_\alpha \subseteq U_\alpha$;

(2) U_α and V_α are open relatively to the intersection of all previous moves;

(3) if possible, i.e. if $\bigcap_{\beta<\alpha} V_\beta \neq \emptyset$, the sets U_α and V_α are non-empty and $x_\alpha \in V_\alpha \subseteq U_\alpha$.

Player II wins the run of $\mathrm{G}^s_\delta(X)$ if the set $\bigcap_{\alpha<\delta} U_\alpha = \bigcap_{\alpha<\delta} V_\alpha$ is nonempty.

Definition 2.3.4. A (regular Hausdorff) space X is called **strong μ-Choquet** (or **SC_μ-space**) if player II has a winning strategy in $\mathrm{G}^s_\mu(X)$.

The space X is called **strong (λ, μ)-Choquet** (or **SC^λ_μ-space**) if it is a SC_μ-space and it has[10] weight $\leq \lambda$.

The strong *fair* μ-Choquet game $f\mathrm{G}^s_\delta(X)$ is the variant of $\mathrm{G}^s_\delta(X)$ where the rules are the same, but the winning condition for II is modified so that (s)he wins if either the set $\bigcap_{\alpha<\delta} U_\alpha = \bigcap_{\alpha<\delta} V_\alpha$ is nonempty (as before), or else there is a (necessarily limit) ordinal $\gamma < \delta$ such that the set $\bigcap_{\alpha<\gamma} U_\alpha = \bigcap_{\alpha<\gamma} V_\alpha$ is empty.

Definition 2.3.5. A (regular Hausdorff) space X is called **strong fair μ-Choquet** (or **$f\mathrm{SC}_\mu$-space**) if player II has a winning strategy in $f\mathrm{G}^s_\mu(X)$.

The space X is called **strong fair (λ, μ)-Choquet** (or **$f\mathrm{SC}^\lambda_\mu$-space**) if it a $f\mathrm{SC}_\mu$ space and has weight $\leq \lambda$.

Of course these variants are relevant only if $\mu > \omega$, as otherwise the two games $\mathrm{G}^s_\mu(X)$ and $f\mathrm{G}^s_\mu(X)$ coincide and we have a unique class of topological spaces. Moreover, in Corollary 2.3.37 we will show that for (Lebesgue zero-dimensional) μ-metrizable spaces there is no difference between having a winning strategy for II or having a winning *tactic* for the same player in both G^s_μ and $f\mathrm{G}^s_\mu$.

By (the proof of) Proposition 1.1.10, we get that strong fair μ-Choquet spaces are closed under G^μ_δ spaces.

Proposition 2.3.6. *Let X be an $f\mathrm{SC}_\mu$-space and $Y \subseteq X$ be G^μ_δ in X. Then Y is an $f\mathrm{SC}_\mu$-space as well.*

The class of spaces of strong (fair) μ-Choquet spaces is also closed under products and sums of any size.

Fact 2.3.7. Let $\{X_\alpha \mid \alpha < \delta\}$ be a family of SC_μ-spaces (respectively, $f\mathrm{SC}_\mu$-spaces). Then both $\bigsqcup_{\alpha<\delta} X_\alpha$ and, for any cardinal $\delta' \leq \delta$, the product $\prod_{\alpha<\delta} X_\alpha$ endowed with the δ'-supported topology are SC_μ-spaces (respectively, $f\mathrm{SC}_\mu$-spaces).

This is obvious for sums. For products $\prod_{\alpha<\delta} X_\alpha$, it is enough to pick for each $\alpha < \delta$ a winning strategy σ_α in the strong (fair) μ-Choquet game on X_α with the additional property that for every $\gamma < \mu$

$$\sigma_\alpha(\langle (U_\beta, x_\beta) \mid \beta < \gamma \rangle) = X_\alpha$$

[10]Notice that we are deliberately allowing our spaces to have weight strictly smaller than λ. Although this might sound unnatural at first glance, it allows us to state some of our results in a more elegant form and is perfectly coherent with what is done in the classical setting, where one includes among Polish spaces also those of finite weight.

for every sequence of moves (U_β, x_β) of player I in which $U_\beta = X_\alpha$ for all $\beta < \gamma$. (This is to ensure that moves of II in the strategy below are legal, i.e. they are open in the relevant product topology.) Then it is not difficult to show that the coordinate-wise product function $\prod_{\alpha<\delta} \sigma_\alpha$ is a winning strategy for player II in the strong (fair) μ-Choquet on the space $\prod_{\alpha<\delta} X_\alpha$, no matter which δ' was chosen as support for the product topology.

Finally, notice when λ has countable cofinality, the two classes of $f\mathrm{SC}^\lambda_\omega$-spaces and $\mathrm{SC}^\lambda_\omega$-spaces coincide. If we add any of the metrizability conditions from Section 2.2, these classes coincide also with the one of λ-Polish spaces.

Fact 2.3.8. Suppose λ is a cardinal of countable cofinality. Then, a space is $f\mathrm{SC}^\lambda_\omega$ if and only if it is $\mathrm{SC}^\lambda_\omega$. Furthermore, the following classes coincide:

(a) $f\mathrm{SC}^\lambda_\omega$ paracompact ω-uniformly based-spaces.

(b) $f\mathrm{SC}^\lambda_\omega$ $\mathrm{NS}^\omega_\omega$-spaces.

(c) $f\mathrm{SC}^\lambda_\omega$ NS^2_ω-spaces.

(d) λ-Polish spaces, i.e. completely metrizable spaces of weight $\leq \lambda$.

Proof. By Theorem 1.1.12 all these spaces are metrizable. The result then follows from [36] (or replacing second countability with paracompactness in [93, Theorem 8.17] - see also [52, 53]). □

2.3.2 Generalized Cauchy-completeness

Recall that a μ-**metric** is a $\mathbb{G}$-metric d for some totally ordered Abelian group $\mathbb{G} = \langle G, +_\mathbb{G}, 0_\mathbb{G}, \leq_\mathbb{G} \rangle$ of degree $\mathrm{Deg}(\mathbb{G}) = \mu$, and μ-**metric space** is a $\mathbb{G}$-metric space (X, d) for some some totally ordered Abelian group $\mathbb{G}$ of degree $\mathrm{Deg}(\mathbb{G}) = \mu$. A sequence $(x_i)_{i<\mu}$ of points from a μ-metric space (X, d) is **(**d**-)Cauchy** if

$$\forall \varepsilon \in \mathbb{G}^+ \, \exists \alpha < \mu \, \forall \beta, \gamma \geq \alpha \, (d(x_\beta, x_\gamma) <_\mathbb{G} \varepsilon).$$

The space (X, d) (or the $\mathbb{G}$-metric d) is **Cauchy-complete** if every Cauchy sequence $(x_i)_{i<\mu}$ converges to some (necessarily unique) $x \in X$, that is,

$$\forall \varepsilon \in \mathbb{G}^+ \, \exists \alpha < \mu \, \forall \beta \geq \alpha \, (d(x_\beta, x) <_\mathbb{G} \varepsilon).$$

Definition 2.3.9. A (Hausdorff regular) space is called (λ, μ)**-Polish** if it has weight $\leq \lambda$ and admits a compatible Cauchy-complete μ-metric.

Notice that if $\mu \leq \lambda$ and X is μ-metrizable, then having weight $\leq \lambda$ is equivalent to having density character $\leq \lambda$.

Definition 2.3.9 generalizes both the notion of λ-Polish space as introduced in [52], which corresponds to the case $\mu = \omega$, and the notion of a $\mathbb{G}$-Polish space from Chapter 1, which instead corresponds to the case where $\lambda = \mu$ is regular (see Fact 2.3.16).

The proof of Proposition 1.1.26 shows that the following holds.

Proposition 2.3.10. *Let X be a μ-metrizable space, and (Y, d) be Cauchy-complete μ-metric space. Let $A \subseteq X$ and $f \colon A \to Y$ be continuous. Then, there is a G^μ_δ set $B \subseteq X$ and a continuous function $g \colon B \to Y$ such that $A \subseteq B \subseteq \mathrm{cl}(A)$ and g extends f, i.e. $g \restriction A = f$.*

Corollary 2.3.11. *Let X be a μ-metrizable space and let Y be a (λ, μ)-Polish subspace of X, where λ is the weight of X. Then Y is G^{μ}_{δ} in X.*

Moreover, arguing as in Lemma 1.1.20 and recalling that μ-metrizable spaces are μ-additive, we also get:

Proposition 2.3.12. *If X is (λ, μ)-Polish, then it is also a μ-additive $f\mathrm{SC}^{\lambda}_{\mu}$-space.*

We are now going to show that requiring a stronger form of completeness for $\mathbb{G}$-metrics, we obtain a special class of (λ, μ)-Polish spaces which is related to μ-additive $\mathrm{SC}^{\lambda}_{\mu}$-spaces.

Recall that a μ-metric d is called **spherically complete** if the intersection of every *decreasing* (with respect to inclusion) sequence of open balls of X is nonempty[11]. If we instead consider only sequences of order type μ (respectively, $<\mu$) we say that d is **spherically μ-complete** (respectively, **spherically $<\mu$-complete**).

Definition 2.3.13. A μ-metrizable space X is said to me **spherically complete** if it admits a compatible spherically complete μ-metric d. Spherically $(<)\mu$-complete μ-metrizable spaces are defined analogously.

First, the same argument of Cantor's intersection theorem shows that weak forms of spherical completeness imply the usual Cauchy-completeness.

Proposition 2.3.14. *Every spherically μ-complete μ-metric is Cauchy-complete.*

The following is the analogue of Proposition 2.3.12.

Proposition 2.3.15. *Every spherically $<\mu$-complete (λ, μ)-Polish space X is a μ-additive $\mathrm{SC}^{\lambda}_{\mu}$-space.*

Proof. Let d_1 be a compatible spherically $<\mu$-complete μ-metric on X, and fix a compatible Cauchy-complete $\mathbb{G}$-metric d_2 on X, where $\mathbb{G}$ is a totally ordered Abelian group with $\mathrm{Deg}(\mathbb{G}) = \mu$. Let $\langle r_\alpha \mid \alpha < \mu \rangle$ be a coinitial sequence in $\mathbb{G}^+$. Consider the following tactic σ for II in $\mathrm{G}^s_\mu(X)$. Suppose I has played an open set U and a point $x \in U$ at round α. Choose $\beta > \alpha$ such that $W_\alpha = B_{d_2}(x, r_\beta)$ satisfies $\mathrm{cl}(W_\alpha) \subseteq U$, and then choose an d_1-open ball V_α such that $x \in V_\alpha \subseteq W_\alpha$: set $\sigma((U, x), \alpha) = V_\alpha$. We claim that σ is winning. Indeed, for any (partial) run $\langle (U_\alpha, x_\alpha), V_\alpha \mid \alpha < \gamma \rangle$ in $\mathrm{G}^s_\mu(X)$ in which II followed σ, we have that $\bigcap V_\alpha \neq \emptyset$: if $\gamma < \mu$, this is due to the spherically $<\mu$-completeness of d_1, since all the V_α's are open balls of d_1; if instead $\gamma = \mu$, this is due to the Cauchy-completeness of d_2, since $\bigcap_{\alpha<\mu} V_\alpha = \bigcap_{\alpha<\mu} W_\alpha = \bigcap_{\alpha<\mu} \mathrm{cl}(W_\alpha)$ and the W_α's have vanishing diameters with respect to d_2. □

When λ has countable cofinality, we recover complete (classical) metrizability. For uncountable cofinality instead, we recover the notion of $\mathbb{G}$-Polish spaces.

Fact 2.3.16. Let X be a topological space, and let λ be a cardinal of cofinality μ.

(a) If $\mu = \omega$ is countable, then X is (λ, ω)-Polish if and only if it is λ-Polish spaces, i.e. completely metrizable spaces of weight $\leq \lambda$.

[11] Once again: different definitions are possible, see Definition 1.1.29.

(b) If $\lambda = \mu > \omega$ is regular and uncountable, then X is (resp., spherically complete) (μ, μ)-Polish if and only if it is (resp., spherically complete) $\mathbb{G}$-Polish for some/every totally ordered Abelian group $\mathbb{G}$ of degree μ.

Proof. By [126] every $\mathbb{G}$-metrizable space is also metrizable when $\mathrm{Deg}(\mathbb{G}) = \omega$, and thus point (a) follows from Proposition 2.3.12 and Fact 2.3.8. Point (b) follows from Corollary 1.1.22. □

Fact 2.3.16(a) hold for spherically complete (λ, μ)-Polish spaces as well if we assume furthermore that X is Lebesgue zero-dimensional, but this is already contained in Theorem 2.3.2.

Finally, it is easy to show that all closed subsets of ${}^{\mu}\lambda$ have a natural $\mathbb{G}$-

Proposition 2.3.17. *Every closed subset $X \subseteq {}^{\mu}\lambda$ admit a compatible spherically μ-complete μ-metric.*

If X is furthermore superclosed, the μ-metric can be taken spherically complete.

Proof. Let $X = [T]$ be the body of a DST tree $T \subseteq {}^{<\mu}\lambda$. Let $\mathbb{G}$ be a totally ordered Abelian group of degree μ, and let $\langle r_\alpha \mid \alpha < \mu \rangle$ be coinitial in $\mathbb{G}^+$. Recall the metric from Example 2.1.8

$$d(x, y) = \begin{cases} 0_{\mathbb{G}} & \text{if } x = y \\ r_\alpha & \text{if } x(\alpha) \neq y(\alpha) \text{ and } x \restriction \alpha = y \restriction \alpha \end{cases}$$

It is easy to see then that $d \restriction X$ is a spherically μ-complete $\mathbb{G}$-ultrametric compatible with the topology of X, and that if furthermore T is superclosed, then $d \restriction X$ is also spherically complete. □

2.3.3 Completeness via bases

Recall that given a regular cardinal δ, a space is said δ-**Lindelöf** if every open cover of the space has a subcover of size $< \delta$. With this terminology, a space X is compact if and only if it is ω-Lindelöf. Every form of compactness brings with itself a form of completeness: for example, it is well-known that every compact metrizable space is completely metrizable, and analogous statements hold for compact (respectively, μ-Lindelöf) spaces and $\mathrm{SC}^{\lambda}_{\mu}$-spaces (respectively, $f\mathrm{SC}^{\lambda}_{\mu}$-spaces) when $\mu > \omega$ is regular (see Chapter 1). Compactness and δ-Lindelöfness can be restated to make this connection more explicit.

Definition 2.3.18. Let δ be an infinite cardinal, X be a topological space, and $\mathcal{D}$ be a family of closed subsets of X. We say that $\mathcal{D}$ is a δ-**Lindelöf family** if for every $\mathcal{C} \subseteq \mathcal{D}$ we have $\bigcap \mathcal{C} \neq \emptyset$ whenever $\bigcap \mathcal{A} \neq \emptyset$ for every $\mathcal{A} \subseteq \mathcal{C}$ of size $< \delta$.

Then, a topological space is δ-Lindelöf if and only if every family of closed subsets of it is δ-Lindelöf. Notice that if $\mathcal{D}$ is a δ-Lindelöf family, $\mathcal{C} \subseteq \mathcal{D}$, and $\delta' \geq \delta$, then $\mathcal{C}$ is a δ'-Lindelöf family.

Being fully μ-Lindelöf is a very restrictive condition. In fact, even the generalized Baire space ${}^{\mu}\mu$ is not μ-Lindelöf, while the generalized Cantor space ${}^{\mu}2$ is μ-Lindelöf if and only if $\mu = \omega$ or μ is weakly compact. Thus it makes sense to weaken such requirements as follows.

Definition 2.3.19. Let δ be an infinite cardinal. A topological space X is called **δ-Lindelöf-based** if it is regular Hausdorff and it admits a **δ-Lindelöf basis** $\mathcal{B}$, i.e. a basis $\mathcal{B}$ for the topology such that $\mathrm{cl}[\mathcal{B}] = \{\mathrm{cl}(B) \mid B \in \mathcal{B}\}$ is a δ-Lindelöf family in X; when $\delta = \omega$, we simply speak of **compact-based** spaces and **compact basis**.

Remark 2.3.20. The idea to define a notion of completeness by looking at intersections of families of subsets of a basis for the topology has been considered before — see e.g. the notion of $\mathcal{B}$-compactness from [87] and of $\mathcal{B}$-completeness from [7] (which correspond to $\mathcal{B}$ being a compact basis in our terminology), or the theory of s-complete spaces and d-closed subsets in [9].

If $\mathcal{B}$ is a tree basis for a space X, properties like being δ-Lindelöf-based have a strong effect on the tree-structure of $\mathcal{B}$. This is because by Fact 2.2.22 the elements of $\mathcal{B}$ are clopen, hence $\mathcal{B}$ is a δ-Lindelöf basis if and only if $\mathcal{B}$ itself is a δ-Lindelöf family. Moreover, given $\mathcal{C} \subseteq \mathcal{B}$ we have that if $\bigcap \mathcal{C} = \emptyset$, then either $\mathcal{C}$ is not a chain, or else it is cofinal in a branch of $\mathcal{B}$. In the former case, taking $\subseteq$-incomparable elements $C_0, C_1 \in \mathcal{C}$ we get that $\mathcal{C}' = \{C_0, C_1\}$ is such that $\bigcap \mathcal{C}' = \emptyset$. So in order to have that $\mathcal{B}$ is a δ-Lindelöf basis it is enough to consider the branches of it. This yields the following criterion:

Fact 2.3.21. Let δ be an infinite cardinal. A tree basis $\mathcal{B}$ for a space X is δ-Lindelöf if and only if every branch of $\mathcal{B}$ of cofinality $\geq \delta$ has nonempty intersection.

In particular, this shows that the complete body of every tree (and thus in particular every space of the form ${}^{\gamma}A$) with the bounded topology is compact-based. This extends further Fact 2.2.26.

Fact 2.3.22. For every tree $\mathbb{T}$, the complete body $[[\mathbb{T}]]_c$ is a compact-based (tree-based) space, while the body $[[\mathbb{T}]]$ is a δ-additive, δ-Lindelöf-based and δ-tree-based space for $\delta = \mathrm{cof}(\mathrm{ht}(\mathbb{T}))$.

In particular, for every (limit) ordinal γ and non-empty set A, the space ${}^{\gamma}A$ with bounded topology is a δ-additive, compact-based δ-tree-based for $\delta = \mathrm{cof}(\mathrm{ht}(\mathbb{T}))$.

Specializing this more to the case where $X \subseteq {}^{\mu}\lambda$ and $\mathcal{B} = \mathcal{B}_T$ for some tree $T \subseteq T_X$, we get a tight connection between Linedlöf-like properties of $\mathcal{B}_T$ and the behaviour of the (complete) body of T. (Compare with Fact 2.1.6.)

Fact 2.3.23. Let λ be any cardinal. If $X \subseteq {}^{\mu}\lambda$ and $T \subseteq T_X$ is pruned, then the family $\mathcal{B}_T = \{\mathbf{N}_s^X \mid s \in T\}$ is a μ-Lindelöf basis for X if and only if $X = [[T]]$. Similarly, $\mathcal{B}_T$ it is a compact basis for X if and only if $X = [[T]]_c$, in which case $[[T]]_c \subseteq {}^{\mu}\lambda$, and hence $[[T]] = [[T]]_c$ and T is superclosed (since μ is a limit ordinal).

Remark 2.3.24. Setting $T = T_X$ in Fact 2.3.23, we get that the canonical basis

$$\{\mathbf{N}_s^X \mid s \in {}^{<\mu}\lambda \wedge \mathbf{N}_s^X \neq \emptyset\}$$

of a space $X \subseteq {}^{\mu}\lambda$ is μ-Lindelöf if and only if X is closed, and it is a compact basis if and only if X is superclosed.

Proposition 2.3.25. *Let δ be an infinite cardinal. Let X be a δ-Lindelöf-based tree-based space. Then every tree basis $\mathcal{B}$ contains a δ-Lindelöf tree basis $\mathcal{B}' \subseteq \mathcal{B}$.*

Proof. Let $\mathcal{B}$ be a tree basis for X and let $\mathcal{A}$ be a δ-Lindelöf basis. We are going to build a basis $\mathcal{B}' \subseteq \mathcal{B}$ such that if $B_\alpha \in \mathrm{Lev}_\alpha(\mathcal{B}')$ and $B_{\alpha+1} \in \mathrm{Lev}_{\alpha+1}(\mathcal{B}')$ are such that $B_{\alpha+1} \subseteq B_\alpha$, then there is $A_\alpha \in \mathcal{A}$ with $B_{\alpha+1} \subseteq \mathrm{cl}(A_\alpha) \subseteq B_\alpha$. In this way, if $\langle \mathcal{B}_\alpha \mid \alpha < \gamma \rangle \in \mathrm{Br}(\mathcal{B}')$ and $\mathrm{cof}(\gamma) \geq \delta$ we get $\bigcap_{\alpha<\gamma} B_\alpha = \bigcap_{\alpha<\gamma} \mathrm{cl}(A_\alpha) \neq \emptyset$ (because $\mathcal{A}$ was a δ-Lindelöf basis), hence $\mathcal{B}' \subseteq \mathcal{B}$ is a δ-Lindelöf basis too by Fact 2.3.21.

Given an open set U with at least two points, let

$$\mathcal{B}'_U = \{B \in \mathcal{B} \mid \exists A \in \mathcal{A}\, (B \subseteq \mathrm{cl}(A) \subsetneq U)\}.$$

and let $\mathcal{B}_U \subseteq \mathcal{B}$ be the set of minimal elements of $\mathcal{B}'_U$ (with respect to the order of the tree $\mathcal{B}$, i.e. the superset relation $\supseteq$). Since X is regular Hausdorff, then $\bigcup \mathcal{B}'_U = \bigcup \mathcal{B}_U = U$. Moreover, by Proposition 2.2.22(2) the sets in $\mathcal{B}_U$ are pairwise disjoint, hence $\mathcal{B}_U$ is a clopen partition of U. Furthermore, $B \subsetneq U$ for every $B \in \mathcal{B}_U$. If U is open and contains just one point, set instead $\mathcal{B}_U = \{U\}$.

We construct $\mathcal{B}'$ by induction on its levels. Let $\mathrm{Lev}_0(\mathcal{B}') = \mathrm{Lev}_0(\mathcal{B})$. Suppose that $\gamma > 0$ and that $\mathrm{Lev}_\alpha(\mathcal{B}')$ has been defined for every $\alpha < \gamma$. If $\gamma = \beta + 1$ is a successor ordinal, set

$$\mathrm{Lev}_\gamma(\mathcal{B}') = \bigcup \{\mathcal{B}_U \mid U \in \mathrm{Lev}_\beta(\mathcal{B}') \wedge |U| > 1\}.$$

If instead γ is limit, let $X_\gamma = \{x \in X \mid \forall \alpha < \gamma\, (x \in \bigcup \mathrm{Lev}_\alpha(\mathcal{B}'))\}$. For each $x \in X_\gamma$ and $\alpha < \gamma$, let $B_\alpha(x)$ be the (unique, by Proposition 2.2.22(2)) $B \in \mathrm{Lev}_\alpha(\mathcal{B}')$ such that $x \in B$. By Lemma 2.2.25, either $\{B_\alpha(x) \mid \alpha < \gamma\}$ is a local basis for $x \in X_\gamma$, or else $\bigcap_{\alpha<\gamma} B_\alpha(x)$ is a nonempty open set. Let

$$\mathcal{U}_\gamma = \left\{ \bigcap_{\alpha<\gamma} B_\alpha(x) \;\middle|\; x \in X \wedge \bigcap_{\alpha<\gamma} B_\alpha(x) \text{ is open} \right\}$$

and

$$\mathrm{Lev}_\gamma(\mathcal{B}') = \bigcup \{\mathcal{B}_U \mid U \in \mathcal{U}_\gamma\}.$$

It is easy to verify that $\mathcal{B}'$ is a basis with the desired property. □

Corollary 2.3.26. *Let δ be an infinite cardinal. If $X \subseteq {}^\mu\lambda$ is δ-Lindelöf-based, then there is a pruned tree $T \subseteq T_X$ such that $\mathcal{B}_T = \{\mathbf{N}^X_s \mid s \in T\}$ is a δ-Lindelöf basis for X.*

Proof. Consider the tree T_X, so that $\mathcal{B}_{T_X} = \{\mathbf{N}^X_s \mid s \in T_X\}$ is a tree basis for X. By Proposition 2.3.25 there is $\mathcal{B}' \subseteq \mathcal{B}_{T_X}$ such that $\mathcal{B}'$ is a μ-Lindelöf basis for X. For each $B \in \mathcal{B}'$ let $s_B \in T_X$ be the shortest sequence satisfying $B = \mathbf{N}^X_{s_B}$ and let $T' = \{s_B \mid B \in \mathcal{B}'\}$, so that $\mathcal{B}_{T'} = \mathcal{B}'$. The tree T' might fail to be pruned, but this can be fixed as follows. If $s \in T'$ is such that $\mathbf{N}^X_s = \{x\}$ for some $x \in X$, add to T' all sequences of the form $x \restriction (\mathrm{lh}(s) + \alpha)$ for $\alpha < \mu$: the resulting tree T is then pruned and $\mathcal{B}_T = \mathcal{B}_{T'} = \mathcal{B}'$. Hence T is as required. □

Together with the equivalence between (a) and (d) in Theorem 2.2.1(2), Proposition 2.3.25 and Corollary 2.3.26 are the key ingredients which allow us to prove analogues of Theorem 1.1.28, Propositions 1.1.13 and 1.1.14, and Corollary 1.1.15 in the context of μ-Lindelöf-based and compact-based μ-metrizable spaces.

The following theorem characterizes the concept of being μ-Lindelöf-based in terms of descriptive set-theoretical complexity.

Theorem 2.3.27. *Let X be a (Lebesgue zero-dimensional, if $\mu = \omega$) μ-Lindelöf-based μ-metrizable space. Then a subspace $Y \subseteq X$ is μ-Lindelöf-based if and only if it is G^μ_δ in X.*

Proof. By Theorem 2.2.1 we can assume that $X \subseteq {}^\mu\lambda$, where λ is the weight of X.

Assume first that $Y \subseteq X \subseteq {}^\mu\lambda$ is μ-Lindelöf-based. Setting $\delta = \mu$ in Corollary 2.3.26, we get a pruned tree $T \subseteq T_Y$ such that $\mathcal{B}_T$ is a μ-Lindelöf basis for Y. Then $Y = [[T]]$ by Fact 2.3.23, hence Y is G^μ_δ in ${}^\mu\lambda$ by Proposition 2.2.50, and thus it is also G^μ_δ in X.

Setting $Y = X$ in the previous argument, we actually get that X is G^μ_δ in the whole ${}^\mu\lambda$. Since the class of G^μ_δ is closed under ($\leq \mu$-sized) intersections by definition, if Y is G^μ_δ in X then it is also G^μ_δ in ${}^\mu\lambda$. By Proposition 2.2.50 there is a tree $T \subseteq {}^{<\mu}\lambda$ such that $Y = [[T]]$. By Remark 2.2.51 we can assume that $T \subseteq T_Y$ and thus that T is pruned. By Fact 2.3.23, this means that $\mathcal{B}_T$ is a μ-Lindelöf basis for Y, witnessing that Y is μ-Lindelöf-based. □

Thanks to Example 2.1.8 and Fact 2.3.22, every space of the form ${}^\lambda A$ is both μ-metrizable and compact-based (so also μ-Lindelöf-based). Thus, applying Theorem 2.3.27 to it we get the following.

Corollary 2.3.28. *Assume $\mu = \mathrm{cof}(\lambda)$ and let A be non-empty. A subset $X \subseteq {}^\lambda A$ is μ-Lindelöf-based if and only if it is G^μ_δ in ${}^\lambda A$.*

Combining Theorem 2.3.27 with Corollary 2.2.52, we also get another kind of characterization of being μ-Lindelöf-based.

Corollary 2.3.29. *A (Lebesgue zero-dimensional, if $\mu = \omega$) μ-metrizable space of weight $\leq \lambda$ is μ-Lindelöf-based if and only if it is homeomorphic to a closed subset of ${}^\mu\lambda$.*

We now move to compact-based spaces. Unfortunately, we cannot have a characterization of this notion just in terms of complexity, since we do not have an analogue of Proposition 2.2.50 for the bodies of superclosed trees. However, we can still obtain a characterization along the lines of Corollary 2.3.29 (see Proposition 2.3.31). First a useful lemma.

Lemma 2.3.30. *A subspace $X \subseteq {}^\mu\lambda$ is compact-based if and only if $X = [[T]]$ for some superclosed tree $T \subseteq {}^{<\mu}\lambda$.*

Proof. If $T \subseteq {}^\mu\lambda$ is superclosed then $[[T]] = [[T]]_c$. Hence if $X = [[T]]$, then $\mathcal{B}_T = \{\mathbf{N}^X_s \mid s \in T\}$ is compact by Fact 2.3.23.

Conversely, assume that X is compact-based. By Corollary 2.3.26 applied with $\delta = \omega$ there exists a pruned tree $T \subseteq T_X$ such that $\mathcal{B}_T$ is a compact basis for X. By Fact 2.3.23 this implies that $X = [[T]]_c = [[T]]$ and hence that T superclosed, as desired. □

A simple corollary of Lemma 2.3.30 is that a (Lebesgue zero-dimensional, if $\mu = \omega$) μ-metrizable space of weight $\leq \lambda$ is compact-based if and only if it is homeomorphic to $[[T]]$ for some superclosed tree $T \subseteq {}^{<\mu}\lambda$. This can be slightly improved by requiring that T be a DST tree, so that $[[T]]$ is a superclosed subset of ${}^\mu\lambda$.

Proposition 2.3.31. *A (Lebesgue zero-dimensional, if $\mu = \omega$) μ-metrizable space of weight $\leq \lambda$ is compact-based if and only if it is homeomorphic to a superclosed subset of ${}^{\mu}\lambda$.*

Proof. As usual, we can assume to work inside a subspace $X \subseteq {}^{\mu}\lambda$. Every superclosed set in ${}^{\mu}\lambda$ is compact-based by Lemma 2.3.30, hence so is X if it is homeomorphic to a superclosed set.

Conversely, suppose that X is compact-based. By Lemma 2.3.30 again, there is a superclosed (not necessarily DST) tree T' such that $X = [[T']]$, and by Proposition 2.1.3 we may assume that T' is normal. Then by Proposition 2.1.2 there exists a DST tree $T \subseteq {}^{<\mu}\lambda$ isomorphic to T', which is superclosed because the latter is a property which is clearly preserved under isomorphisms. By Fact 2.1.5, $[[T']]$ is homeomorphic to $[[T]] = [T]$, thus X is homeomorphic to the superclosed set $C = [T]$. □

If we are in the situation where X is μ-Lindelöf-based and $Y \subseteq X$ is compact-based, we would like to simultaneously realize Corollary 2.3.29 (applied to X) and Proposition 2.3.31 (applied to Y) so that the corresponding superclosed set is a subset of the closed subset of ${}^{\mu}\lambda$ coming from X. The following result, which is the analogue of Proposition 1.1.14 for μ-Lindelöf-based and compact-based spaces, shows that this is possible.

Proposition 2.3.32. *Suppose that X is a (Lebesgue zero-dimensional, if $\mu = \omega$) μ-Lindelöf-based μ-metrizable space of weight $\leq \lambda$, and let Y be a compact-based* closed *subspace of X. Then there exist a closed $C \subseteq {}^{\mu}\lambda$ and a homeomorphism $\phi\colon X \to C$ which maps Y into a superclosed $C' \subseteq C$.*

Proof. By Corollary 2.3.29, we may assume that $X \subseteq {}^{\mu}\lambda$ is closed. Since Y is closed in X (and hence in ${}^{\mu}\lambda$), we have $X = [T_X]$ and $Y = [T_Y]$. Notice that $T_Y \subseteq T_X$ and they are both pruned and that $\mathcal{B}_{T_X} = \{\mathbf{N}^X_s \mid s \in T_X\}$ is a μ-Lindelöf basis for X by Fact 2.3.23. By the proof of Lemma 2.3.30, there exists a pruned superclosed (not necessarily DST) subtree $T'_Y \subseteq T_Y$ such that $Y = [[T'_Y]]$. By Proposition 2.1.3 we can assume that T'_Y is normal. Define $T'_X = (T_X \setminus T_Y) \cup T'_Y$: since T_Y was a DST tree and both T_X and T'_Y are pruned, then T'_X is pruned and $X = [[T'_X]]$, so that $\mathcal{B}_{T'_X} = \{\mathbf{N}^X_s \mid s \in T'_X\}$ is still a μ-Lindelöf basis for X by Fact 2.3.23. Furthermore, the fact that T_Y was a DST tree entails that $T_X \setminus T_Y$ is upward closed in T_X, hence $T'_Y \subseteq T_Y$ is a downward closed subset of T'_X. By Proposition 2.1.2 there exists a tree isomorphism $f\colon T'_X \to T$ mapping T'_X to a DST tree $T \subseteq {}^{<\mu}\lambda$, and by Fact 2.1.5 and the comment following it this isomorphism canonically induces a homeomorphism ϕ_f between $X = [[T'_X]]$ and the closed set $[[T]] = [T]$. Moreover, the restriction of ϕ_f to $Y = [[T'_Y]]$ is a homeomoprhism between Y and the body C of the tree $f[T'_Y]$. But since f was a tree isomorphism, $f[T'_1]$ is superclosed, pruned, and downward closed in T, thus in particular it is also a DST tree and C is superclosed. Setting $\phi = \phi_f$ we are done. □

Besides its technical content, Proposition 2.3.32 is useful in that it shows, together with [105, Proposition 1.3], that compact-based subspaces of μ-Lindelöf-based μ-metrizable spaces are always a retract of the ambient space (see also [9, Theorem 4.6]).

Corollary 2.3.33. *Let X be a (Lebesgue zero-dimensional, if $\mu = \omega$) μ-Lindelöf-based μ-metrizable space. Then every compact-based closed subspace $Y \subseteq X$ is a retract of X.*

The notions of being μ-Lindelöf-based and compact-based are closely related to the other completeness notions that we considered in the previous sections. For example, by Proposition 2.3.6 and Theorem 2.3.27 every (Lebesgue zero-dimensional, if $\mu = \omega$) μ-Lindelöf-based μ-metrizable space of weight $\leq \lambda$ (which by Theorem 2.2.1(2) might be conceived as a subspace of the μ-Lindelöf-based μ-metrizable $f\mathrm{SC}^{\lambda}_{\mu}$-space ${}^{\mu}\lambda$) is an $f\mathrm{SC}^{\lambda}_{\mu}$-space. More generally:

Proposition 2.3.34. *If X is μ-Lindelöf-based, then player* II *has a winning strategy (and even a winning tactic, if X is μ-additive) in $f\mathrm{G}^{s}_{\mu}(X)$.*

Moreover, if X is compact-based then player II *has a winning strategy (and even a winning tactic, if X is μ-additive) in $\mathrm{G}^{s}_{\delta}(X)$ for every infinite cardinal δ.*

Proof. Let $\mathcal{B}$ be a μ-Lindelöf basis (or a compact basis, if X is compact-based). We define a winning strategy σ for II in the relevant game as follows. If player I just played (U_α, x_α) at round α, let II pick any open set $U'_\alpha \subseteq X$ such that $U_\alpha = U'_\alpha \cap \bigcap_{\beta<\alpha} V_\beta$, and then reply by playing $V_\alpha = V'_\alpha \cap U_\alpha$ for some $V'_\alpha \in \mathcal{B}$ such that $x \in V'_\alpha \subseteq \mathrm{cl}(V'_\alpha) \subseteq U'_\alpha$. Notice that if X is μ-additive then we can set $U'_\alpha = U_\alpha$ and hence σ is actually a tactic.

Suppose first we are in the case of a μ-Lindelöf-based space, and let $\langle (U_\alpha, x_\alpha), V_\alpha \mid \alpha < \mu \rangle$ be a run of the game in which II followed σ. Using the notation from the previous paragraph and arguing by induction on γ, for any $\gamma \leq \mu$ limit we have $\bigcap_{\beta<\gamma} U_\beta = \bigcap_{\beta<\gamma} U'_\beta = \bigcap_{\beta<\gamma} \mathrm{cl}(V'_\beta)$: hence if such sets are nonempty for every $\gamma < \mu$, then also the intersection corresponding to $\gamma = \mu$ is nonempty because the V'_β's belong to the μ-Lindelöf basis $\mathcal{B}$. Similarly, if X were compact-based and μ is replaced by any infinite cardinal δ, then all intersections $\bigcap_{\beta<\gamma} U_\beta = \bigcap_{\beta<\gamma} U'_\beta = \bigcap_{\beta<\gamma} \mathrm{cl}(V'_\beta)$ are nonempty because the V'_β's belong to the compact basis $\mathcal{B}$. □

Corollary 2.3.35. *Every μ-Lindelöf-based space of weight $\leq \lambda$ is an $f\mathrm{SC}^{\lambda}_{\mu}$-space, and every compact-based space of weight $\leq \lambda$ is an $\mathrm{SC}^{\lambda}_{\mu}$-space.*

For μ-tree-based spaces, the converse is true as well.

Proposition 2.3.36. *Suppose that X is μ-tree-based and has weight $\leq \lambda$. Then X is $f\mathrm{SC}^{\lambda}_{\mu}$ if and only if it is μ-Lindelöf-based, and X is $\mathrm{SC}^{\lambda}_{\mu}$ if and only if it is compact-based.*

Proof. One direction is given by Corollary 2.3.35, so let us simultaneously prove the two forward implications. Let $\mathcal{B}$ be a tree basis for X of height $\leq \mu$, and let σ be a winning strategy for II in the strong (fair) μ-Choquet game. Without loss of generality, we may assume that σ has range contained in $\mathcal{B}$. We want to define a new tree basis $\mathcal{A} \subseteq \mathcal{B}$ together with a function $f \colon \bigcup_{\alpha<\mu} \mathrm{Lev}_{\alpha+1}(\mathcal{A}) \to X$ such that for every branch $\langle A_\alpha \mid \alpha < \gamma \rangle$ of $\mathcal{A}$ of limit height, the sequence of moves $\langle (A_\alpha, f(A_{\alpha+1})) \mid \alpha < \gamma \rangle$ of player I is compatible with σ. Since elements of $\mathcal{B}$, and hence of $\mathcal{A}$, are clopen by Proposition 2.2.22, this last condition ensures that $\mathcal{A}$ is a μ-Lindelöf tree basis (if σ was winning in $f\mathrm{G}^{s}_{\mu}(X)$) or a compact tree basis (if σ was winning in $\mathrm{G}^{s}_{\mu}(X)$) by Fact 2.3.21.

We recursively define the levels of $\mathcal{A}$ and corresponding restrictions of f. More precisely, for every $\gamma < \mu$ we define a family $\mathcal{A}_\gamma \subseteq \mathcal{B}$ and, if γ is successor, a function $f_\gamma \colon \mathcal{A}_\gamma \to X$ such that letting $\mathcal{A}_{\leq\gamma} = \bigcup_{\alpha\leq\gamma} \mathcal{A}_\alpha$ and $f_{<\gamma} = \bigcup_{\alpha<\gamma} f_{\alpha+1}$ we have:

(1) $\mathrm{Lev}_\gamma(\mathcal{A}_{\leq\gamma}) = \mathcal{A}_\gamma$.

(2) For every $B \in \mathrm{Lev}_\gamma(\mathcal{B})$ and $x \in B$ there is $A \in \mathcal{A}_{\leq\gamma}$ such that $x \in A \subseteq B$.

(3) For every branch $\langle A_\alpha \mid \alpha \leq \gamma\rangle$ of $\mathcal{A}_{\leq\gamma}$ of length $\gamma + 1$ and every $x \in A_\gamma$, the sequence of moves $\langle (A_\alpha, f_{<\gamma}(A_{\alpha+1})) \mid \alpha < \gamma\rangle^\frown(A_\gamma, x)$ is compatible with σ .

Then $\mathcal{A} = \bigcup_{\alpha<\mu} \mathcal{A}_\alpha \subseteq \mathcal{B}$ is a (tree) basis by item (2), and by item (3) it satisfies the additional requirement discussed above.

Start by setting $\mathcal{A}_0 = \mathrm{Lev}_0(\mathcal{B})$. This trivially satisfies items (1)–(3), so we can move to the inductive step. Given $0 < \gamma < \mu$, suppose that $f_{\alpha+1}$ and $\mathcal{A}_\beta$ have been defined for every $\alpha < \beta < \gamma$. Let $\mathcal{A}_{<\gamma} = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$. Define X_γ to be the set of those $x \in X$ such that $\{A \in \mathcal{A}_{<\gamma} \mid x \in A\}$ is a local basis for x, and notice that if $\gamma' \leq \gamma$ then $X_{\gamma'} \subseteq X_\gamma$. For elements of X_γ we have nothing more to add to $A_{<\gamma}$ to make it a local basis for x, so we will concentrate on $X \setminus X_\gamma$: indeed, our construction will ensure that $\mathcal{A}_\gamma$ is a clopen partition of $X \setminus X_\gamma$. For every $x \in X \setminus X_\gamma \subseteq \bigcap_{\gamma'<\gamma}(X \setminus X_{\gamma'})$, the set $A_\gamma(x) = \bigcap\{A \in \mathcal{A}_{<\gamma} \mid x \in A\}$ is open by Lemma 2.2.25 and $x \notin X_\gamma$. Moreover, $A_\gamma(x) \subseteq X \setminus X_\gamma$ because if $y \in A_\gamma(x)$ then $y \in \bigcap_{\gamma'<\gamma}(X \setminus X_{\gamma'})$ and $\{A \in \mathcal{A}_{<\gamma} \mid y \in A\} = \{A \in \mathcal{A}_{<\gamma} \mid x \in A\}$ by the fact that $\mathcal{A}_{\gamma'}$ is a partition of $X \setminus X_{\gamma'}$; hence if $y \neq x$ then $\{A \in \mathcal{A}_{<\gamma} \mid y \in A\}$ does not separate y from x and thus cannot be a local basis for y. This shows that $\{A_\gamma(x) \mid x \in X \setminus X_\gamma\}$ is an open cover of $X \setminus X_\gamma$. For every $x \in X \setminus X_\gamma$, let $B_\gamma(x)$ be the minimum (with respect to the tree order $\supseteq$) $B \in \mathcal{B}$ such that $x \in B \subsetneq A$ for every $A \in \mathcal{A}_{<\gamma}$ with $x \in A$, so that in particular $B_\gamma(x) \subseteq A_\gamma(x)$. The family $\{B_\gamma(x) \mid x \in X \setminus X_\gamma\}$ is a clopen partition of $X \setminus X_\gamma$ by $A_\gamma(x) \subseteq X \setminus X_\gamma$, Proposition 2.2.22(2) and minimality of the $B_\gamma(x)$'s.

Assume first that γ is limit, so that $f_{<\gamma}$ is already entirely defined, and set $\mathcal{A}_\gamma = \{B_\gamma(x) \mid x \in X \setminus X_\gamma\}$. Since $X \setminus X_\gamma \subseteq \bigcap_{\gamma'<\gamma}(X \setminus X_{\gamma'})$ and we are ensuring that each $\mathcal{A}_{\gamma'}$ is a clopen partition of $X \setminus X_{\gamma'}$, for every $x \in X \setminus X_\gamma$ and $\gamma' < \gamma$ there is $A \in \mathcal{A}_{\gamma'}$ with $x \in A$: by item (1) applied to such ordinals $\gamma' < \gamma$ and the choice of $B_\gamma(x)$ we then have that $B_\gamma(x) \in \mathrm{Lev}_\gamma(\mathcal{A}_{\leq\gamma})$ and item (1) is satisfied. As for item (2), pick any $x \in B \in \mathrm{Lev}_\gamma(\mathcal{B})$. Notice that every predecessor B' of B in $\mathcal{B}$ belongs to $\mathrm{Lev}_\alpha(\mathcal{B})$ for some $\alpha < \gamma$, and thus by induction hypothesis there is $A \in \mathcal{A}_{<\gamma}$ such that $x \in A \subseteq B'$. In particular, if there is no $A \in \mathcal{A}_{<\gamma}$ such that $x \in A \subseteq B$ (so that in particular $x \in X \setminus X_\gamma$), then by Proposition 2.2.22(2) B is minimal such that $x \in B \subsetneq A$ for all $A \in \mathcal{A}_{<\gamma}$ with $x \in A$, and thus $B_\gamma(x) = B$ by definition of $B_\gamma(x)$. Finally, item (3) is satisfied by induction and $B_\gamma(x) \subseteq A_\gamma(x)$.

Assume now that $\gamma = \beta + 1$ is successor. In this case $\mathcal{A}_\gamma$ will be a partition refining $\{B_\gamma(x) \mid x \in X \setminus X_\gamma\}$. Granting this, the argument in the previous paragraph shows that items (1) and (2) will be satisfied by such an $\mathcal{A}_\gamma$. As for item (3), we first need to define f_γ. For $x \in X \setminus X_\gamma$ and $\alpha \leq \beta$, let $A_\alpha \in \mathcal{A}_\alpha$ be (the unique set) such that $x \in A_\alpha$. Define

$$r_x = \langle (A_\alpha, f_{<\gamma}(A_{\alpha+1})) \mid \alpha < \beta\rangle^\frown\langle (A_\beta, x)\rangle.$$

Then r_x is compatible with σ by item (3) applied to β. Let

$$\mathcal{C}_\gamma = \{\sigma(r_x) \cap B_\gamma(x) \mid x \in X \setminus X_\gamma\},$$

and notice that it covers $X \setminus X_\gamma$. Since both $\sigma(r_x) \in \mathcal{B}$ and $B_\gamma(x) \in \mathcal{B}$ and they have nonempty intersection, as witnessed by x itself, we must have that they are comparable with respect to inclusion by Proposition 2.2.22(2), hence either $\sigma(r_x) \cap B_\gamma(x) = \sigma(r_x)$ or $\sigma(r_x) \cap B_\gamma(x) = B_\gamma(x)$. It follows that $\mathcal{C}_\gamma \subseteq \mathcal{B}$. Let $\mathcal{A}_\gamma$ be the set of minimal elements of $\mathcal{C}_\gamma$ (with respect to the tree relation $\supseteq$ of $\mathcal{B}$): then $\mathcal{A}_\gamma$ is a partition of $X \setminus X_\gamma$ by Proposition 2.2.22(2) again. For every $A \in \mathcal{A}_\gamma$, choose $x \in X \setminus X_\gamma$ such that $A = \sigma(r_x) \cap B_\gamma(x)$, and set $f_\gamma(A) = x$. It is easy to check that item (3) is then satisfied by construction. □

This means that for μ-additive μ-tree-based spaces, having a winning strategy in the relevant Choquet-like games is the same as having a winning tactic in those games. In particular, since every (Lebesgue zero-dimensional) μ-metrizable space is μ-additive and μ-tree-based, we have the following:

Corollary 2.3.37. *Let X be a (Lebesgue zero-dimensional, if $\mu = \omega$) μ-metrizable space. Then player* II *has a winning strategy in the strong fair μ-Choquet game $f\mathrm{G}^s_\mu(X)$ if and only if it has a winning tactic in it, and the same is true for the strong μ-Choquet game $\mathrm{G}^s_\mu(X)$.*

2.3.4 Proof of Theorems 2.3.1 and 2.3.2

Combining together all the results obtained, we can prove the two theorems announced at the beginning of the section.

Proof of Theorem 2.3.1. First, every (λ, μ)-Polish space is μ-metrizable, thus (1) $\Rightarrow$ (3) by Proposition 2.3.12. Also, (2) $\Rightarrow$ (1) by Proposition 2.3.14. Every μ-metrizable space is μ-tree-based by Theorem 2.2.1, thus (3) $\iff$ (4) by Proposition 2.3.36. The equivalence (4) $\iff$ (5) is Corollary 2.3.29, while (5) $\iff$ (6) is given by Corollary 2.2.52. The implication (6) $\Rightarrow$ (7) follows from Proposition 2.2.53, while (7) $\Rightarrow$ (4) by Corollary 2.3.28. Finally, (5) $\Rightarrow$ (2) by Proposition 2.3.17. □

Proof of Theorem 2.3.2. First, (2) is a strict strengthening of (1), so (2) $\Rightarrow$ (1). Every spherically-complete (λ, μ)-Polish space is μ-metrizable, thus (1) $\Rightarrow$ (3) by Proposition 2.3.15. Every μ-metrizable space is μ-tree-based by Theorem 2.2.1, thus (3) $\iff$ (4) by Proposition 2.3.36, and the equivalence (4) $\iff$ (5) is Proposition 2.3.31. Finally, (5) $\Rightarrow$ (2) by Proposition 2.3.17. □

2.4 Standard λ-Borel spaces

Sections 2.2 and 2.3 provide a large number of classes of spaces which could claim to be the "right" generalization of Polish spaces. To determine which are the better classes to work with if we aim at developing a decent (generalized) descriptive set theory is a quite challenging task and certainly requires a deeper analysis. It might well happen that we will have to accept a blurry situation, in which different results will rely on different classes, or that we will instead have to restrict to the class of μ-metrizable spaces, where all the notions collapse into a simple and well-delimited setup (although this would then show that when $\mu > \omega$, generalized descriptive set theory would just be a theory of *Lebesgue zero-dimensional* Polish-like spaces). Whatever the answer to this question will be, we are now going to show that also in the generalized context there is a unique notion of standard λ-Borel space (Theorem 2.4.13), thus providing a solid ground for the development of descriptive set theory from λ-Borel sets onward. Moreover, all the different classes of topological spaces considered so far are the same up to λ-Borel isomorphism (Theorem 2.4.12), and actually in most cases the only differences concern the *finite* levels of their λ-Borel hierarchy (even though λ might be very large in the cardinal hierarchy). This shows that even if the situation might be chaotic from the topological viewpoint, there is a chance that a better picture can be obtained from the point of view of (generalized) descriptive set theory.

2.4.1 Standard λ-Borel spaces

Recall from Section 2.1.2 the notion of (λ-)Borel space and related concepts. A well-known fact in classical descriptive set theory concerning Borel spaces is the following.

Theorem 2.4.1 ([93]). *Let $(X, \mathscr{B})$ be a Borel space. Then the following are equivalent:*

(1) *there is a Polish topology τ on X such that $\mathscr{B} = \mathsf{Bor}(X, \tau)$;*

(2) *there is a (Lebesgue) zero-dimensional Polish topology on X such that $\mathscr{B} = \mathsf{Bor}(X, \tau)$;*

(3) *there is a Borel set $A \subseteq {}^{\omega}\omega$ such that $(X, \mathscr{B})$ is Borel isomorphic to A, where the latter is equipped with the Borel structure $\mathsf{Bor}({}^{\omega}\omega) \restriction A$ inherited from ${}^{\omega}\omega$.*

A Borel space is called **standard** if it satisfies the equivalent conditions above. Theorem 2.4.1 conveys two distinct (although related) pieces of information, namely, that the class of Polish spaces and the class of *(Lebesgue) zero-dimensional* Polish spaces are the same up to Borel isomorphism, and that it is equivalent to saying that the Borel structure of a Borel space X is generated by a nice (i.e. Polish, or zero-dimensional Polish) topology and that, up to Borel isomorphism, X is a Borel subset of the Baire space ${}^{\omega}\omega$ (or any other uncountable Polish space, including the Cantor space ${}^{\omega}2$). These results were extended in Chapter 1 to the context of κ-Borel spaces for regular cardinals κ satisfying $2^{<\kappa} = \kappa$. In this section we further extend this to all cardinals λ, including the case where λ is singular, still under the assumption $2^{<\lambda} = \lambda$. Except for Proposition 2.4.3, in the rest of the section we assume this hypothesis and that $\mathrm{cof}(\lambda) = \mu$ and $(\lambda_i)_{i<\mu}$ is a strictly increasing sequence of limit ordinals cofinal in λ.

When standard λ-Borel spaces (for $\lambda > \omega$) were first introduced in [118], no reasonable notion of a Polish-like space for generalized descriptive set theory was known, hence the following definition was adopted[12] (compare it with Theorem 2.4.1(1)).

[12] To be precise, the definition in [118] is slightly different, yet equivalent, to the present one—see Chapter 1 for more on this.

Definition 2.4.2. A λ-Borel space $(X, \mathscr{B})$ is **standard** if it is λ-Borel isomorphic to a λ-Borel subset of ${}^{\mu}\lambda$.

As a first result, we show that for various important classes of topological spaces considered in this chapter, we can enrich their topology (in a minimal way) to make them μ-additive without destroying their relevant properties and without altering their λ-Borel structure. (Here we are not requiring that $\mu = \mathrm{cof}(\lambda)$, although the assumption $\lambda^{<\mu} = \lambda$ implies that $\mu \leq \mathrm{cof}(\lambda)$.)

Proposition 2.4.3. *Assume that $\lambda^{<\mu} = \lambda$. For any (regular Hausdorff) space (X, τ) of weight $\leq \lambda$, the smallest μ-additive topology $\tau' \supseteq \tau$ is still regular Hausdorff and of weight $\leq \lambda$, it is such that $\mathsf{Bor}_\lambda(X, \tau') = \mathsf{Bor}_\lambda(X, \tau)$, and moreover:*

(1) *if (X, τ) is a NS_μ-space, then so is (X, τ');*

(2) *if (X, τ) is a μ-tree-based space, then so is (X, τ');*

(3) *if (X, τ) is a μ-uniformly based space, then so is (X, τ');*

(4) *if (X, τ) is an $f\mathrm{SC}^\lambda_\mu$-space, then so is (X, τ');*

(5) *if (X, τ) is an SC^λ_μ-space, then so is (X, τ').*

Proof. If $\mu = \omega$ we have $\tau' = \tau$ and there is nothing to prove, so let us assume that μ is uncountable. Let $\mathcal{B}$ be a basis for (X, τ) of size $\leq \lambda$, and let τ' be the smallest μ-additive topology refining τ. Notice that $\tilde{\mathcal{B}} = \{\bigcap \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B} \wedge |\mathcal{A}| < \mu\}$ is a basis for (X, τ') of size $\lambda^{<\mu} = \lambda$. Since $\tilde{\mathcal{B}} \subseteq \mathsf{Bor}_\lambda(X, \tau)$ and $\tilde{\mathcal{B}}$ has size λ, we have that $\tau \subseteq \tau' \subseteq \mathsf{Bor}_\lambda(X, \tau)$, and thus $\mathsf{Bor}_\lambda(X, \tau') = \mathsf{Bor}_\lambda(X, \tau)$. Also, notice that since (X, τ) is Hausdorff, then (X, τ') is Hausdorff as well, and since (X, τ) is ω_1-additive, then it is in particular zero-dimensional and thus regular.

We now prove item (1). Assume that $\mathcal{B}$ is a NS_μ-basis for (X, τ) with a NS_μ-cover $\{\mathcal{B}_\alpha \mid \alpha < \mu\}$. By Fact 2.2.12(4) we can assume that $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ for every $\alpha \leq \beta < \mu$. Analogously to what we did in the proof of Proposition 2.2.17 (but using families $\mathcal{A}$ of size $< \mu$ instead of countable families), we set

$$\mathcal{C}_\alpha = \left\{ \bigcap \mathcal{A} \;\middle|\; \mathcal{A} \subseteq \mathcal{B}_\alpha \wedge |\mathcal{A}| < \mu \wedge \bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} \mathrm{cl}(A) \right\},$$

$\mathcal{D}_\alpha = \{\bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) \mid x \in X\}$, and $\mathcal{D} = \bigcup_{\alpha < \gamma} \mathcal{D}_\alpha$. Arguing as in Proposition 2.2.17 and using the fact that τ' is μ-additive, we get that each $\mathcal{D}_\alpha$ is a τ'-clopen partition of X. We need to prove that $\mathcal{D}$ is a basis for (X, τ') (and so in particular a NS_μ-basis for it, since $\{\mathcal{D}_\alpha \mid \alpha < \mu\}$ is a NS^2_μ-cover of $\mathcal{D}$). Consider any $O \in \tilde{\mathcal{B}}$ and $x \in O$. Let $\mathcal{A} \subseteq \mathcal{B}$ be such that $|\mathcal{A}| < \mu$ and $O = \bigcap \mathcal{A}$. We want to find $\alpha < \mu$ such that $x \in \bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) \subseteq O$. For every $A \in \mathcal{A}$, using the regularity of X find a family $\{U^A_i \mid i < \omega\} \subseteq \mathcal{B}$ of basic τ-open sets such that $x \in \mathrm{cl}_\tau(U^A_{i+1}) \subseteq U^A_i \subseteq A$ for every $i \in \omega$, where cl_τ denotes closure with respect to τ. Define $\mathcal{A}' = \{U^A_i \mid i < \omega, A \in \mathcal{A}\}$. Then $|\mathcal{A}'| < \mu$, and since we assumed that $\bigcup_{\beta < \alpha} \mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$ for every $\alpha < \mu$, there exists $\alpha < \mu$ such that $\mathcal{A}' \subseteq \mathcal{B}_\alpha$. Furthermore,

$$\bigcap \mathcal{A}' = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{i \in \omega} U^A_i \right) = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{i \in \omega} \mathrm{cl}(U^A_i) \right) = \bigcap_{A' \in \mathcal{A}'} \mathrm{cl}(A').$$

Hence $\bigcap \mathcal{A}' \in \mathcal{C}_\alpha$, and since $x \in \bigcap \mathcal{A}'$ then $x \in \bigcap \mathrm{CN}(\mathcal{C}_\alpha, x) \subseteq \bigcap \mathcal{A}' \subseteq \bigcap \mathcal{A} = O$ and we are done.

Next we move to item (2). Assume that (X, τ) has a tree basis $\mathcal{B}$ of height μ. Define

$$\mathcal{B}' = \left\{ \bigcap \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B} \wedge |\mathcal{A}| < \mu \wedge \bigcap \mathcal{A} \neq \emptyset \right\}.$$

Then $\mathcal{B}' \supseteq \mathcal{B}$ is by definition a basis for τ', and it is a tree basis of height $\leq \mu$ because by Proposition 2.2.22 the condition $\bigcap \mathcal{A} \neq \emptyset$ implies that $\mathcal{A}$ is contained in some (not necessarily unique) branch of $\mathcal{B}$.

As for item (3), assume that σ is a winning strategy for player II in the μ-uniform local basis game on (X, τ). Recall that in such a game there are no restrictions on the possible moves of player I, so there is no distinction between (legal) sequences of moves for I in the game on (X, τ) and (legal) sequences of moves for the same player in the game on (X, τ'). We define a strategy σ' for II in the μ-uniform local basis game on (X, τ') as follows, ensuring in particular that each move of player II is actually a τ'-clopen set. Consider any sequence of moves $r = \langle x_\alpha \mid \alpha \leq \gamma \rangle$ for player I of successor length. We distinguish two cases. If $x_\gamma \in \sigma'(r \restriction (\gamma' + 1))$ for all $\gamma' < \gamma$, then let $\sigma'(r)$ be any τ'-clopen set such that

$$x_\gamma \in \sigma'(r) \subseteq \sigma(r) \cap \bigcap_{\gamma' < \gamma} \sigma'(r \restriction (\gamma' + 1)).$$

Such a set exists because $x_\gamma \in \sigma(r)$ (since σ was legal) and τ' is μ-additive, hence also zero-dimensional because we are in the case $\mu > \omega$. If instead $x_\gamma \notin \sigma'(r \restriction (\gamma' + 1))$ for some $\gamma' < \gamma$, then set $\sigma'(r) = X \setminus \sigma'(r \restriction (\gamma' + 1))$. It is clear that in all cases $\sigma'(r)$ is a τ'-clopen set which contains x_γ, hence it is in particular a legal move for II in the game on (X, τ') and σ' is a legal strategy: we claim that is also winning. Let $\langle x_\alpha, V'_\alpha \mid \alpha < \mu \rangle$ be a run of the game on (X, τ') in which II followed σ', and set $V_\alpha = \sigma(\langle x_\beta \mid \beta \leq \alpha \rangle)$ for all $\alpha < \mu$. If the second case above occurred at some round γ, then $\bigcap_{\alpha < \mu} V'_\alpha = \emptyset$ because $V'_\gamma = X \setminus V'_{\gamma'}$ for some $\gamma' < \gamma$, so without loss of generality we can assume that only the first case occurred along the run. Then for every $\alpha < \mu$ we have $V'_\alpha \subseteq V_\alpha$ by construction. If $x \in \bigcap_{\alpha < \mu} V'_\alpha \neq \emptyset$, then $x \in \bigcap_{\alpha < \mu} V_\alpha \neq \emptyset$ as well, and hence $\{V_\alpha \mid \alpha < \mu\}$ is a local basis of x with respect to τ because σ was winning. Let O be a τ'-open neighborhood of x: without loss of generality, $O = \bigcap_{\beta < \gamma} U_\beta$ for some $\gamma < \mu$ and τ-open sets U_β. For each $\beta < \gamma$ there is $\alpha_\beta < \mu$ such that $x \in V_{\alpha_\beta} \subseteq U_\beta$. Let $\delta = \sup\{\alpha_\beta + 1 \mid \beta < \gamma\}$. Then $\delta < \mu$ because μ is regular, and by construction and case assumption

$$x \in V'_\delta \subseteq \bigcap_{\gamma' < \delta} V'_{\gamma'} \subseteq \bigcap_{\gamma' < \delta} V_{\gamma'} \subseteq \bigcap_{\beta < \gamma} V_{\alpha_\beta} \subseteq \bigcap_{\beta < \gamma} U_\beta = O.$$

This shows that $\{V'_\alpha \mid \alpha < \mu\}$ is a local basis for x with respect to τ', as desired.

Finally, if player II has a winning strategy in $f\mathrm{G}^s_\mu(X, \tau)$ (respectively, $\mathrm{G}^s_\mu(X, \tau)$), then the same argument from [42, Proposition 4.3 and Lemma 4.4] shows that II also has a winning strategy in $f\mathrm{G}^s_\mu(X, \tau')$ (respectively, $\mathrm{G}^s_\mu(X, \tau')$). This proves items (4) and (5) and concludes the proof. □

Items (5) and (4) in Proposition 2.4.3 cannot be reversed. For SC^λ_μ spaces, this is easy to see, as for example, the space of rational numbers $\mathbb{Q}$ is not strong Choquet, but the smallest μ-additive topology refining it is SC^λ_μ when $\mu > \omega$ (since it is discrete). See also [42] for more examples. As for $f\mathrm{SC}^\lambda_\mu$-spaces, Proposition 2.5.4 shows that there are spaces (X,τ) that are not $f\mathrm{SC}^\lambda_\mu$, yet the smallest μ-additive topology generated by τ is $f\mathrm{SC}^\lambda_\mu$.

2.4.2 Changes of topology

A fundamental technique to prove Theorem 2.4.1 is the possibility of changing the topology of a given Polish space to turn a given Borel set into a clopen set, without losing Polishness and without modifying the Borel structure of the space. In Chapter 1 this technique was extended to the generalized setting for regular cardinals: by adapting that argument to the singular context, we are now going to prove the same result in full generality.

Theorem 2.4.4. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ and let (X,τ) be an $f\mathrm{SC}^\lambda_\mu$-space. Let $\mathcal{B}' \subseteq \mathsf{Bor}_\lambda(X,\tau)$ be of size $\leq \lambda$. Then there is a topology τ' on X such that:*

(1) *τ' refines τ;*

(2) *each $B \in \mathcal{B}'$ is τ'-clopen,*

(3) $\mathsf{Bor}_\lambda(X,\tau') = \mathsf{Bor}_\lambda(X,\tau)$, *and*

(4) *(X,τ') is a μ-additive $f\mathrm{SC}^\lambda_\mu$-space with a NS^2_μ-basis of clopens (hence it is also Lebesgue zero-dimensional and (λ,μ)-Polish).*

We split the proof of Theorem 2.4.4 into three technical lemmas that will also be used to prove Theorem 2.4.15. To simplify the terminology, say that a collection $\mathcal{B} \subseteq \mathsf{Bor}_\lambda(X,\tau)$ of λ-Borel sets is **downward closed** if for every $1 < \alpha < \lambda^+$ and $B \in \mathcal{B}$ with $\mathrm{rank}(B) = \alpha$ there is $\mathcal{F} \subseteq \mathcal{B}$ such that $|\mathcal{F}| \leq \lambda$, each element of $\mathcal{F}$ has λ-Borel rank $< \alpha$, and $B = \bigcup \mathcal{F}$ (if $B \in \lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)$) or $B = \bigcap \mathcal{F}$ (if $B \in \lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)$). In other words, $\mathcal{B}$ is downward closed if every time that a set $B \in \mathcal{B}$ has rank $\leq \alpha$ for some $1 < \alpha < \lambda^+$, this is witnessed within the family $\mathcal{B}$ itself.

Lemma 2.4.5. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ and let (X,τ) be an $f\mathrm{SC}^\lambda_\mu$-space. Let $\mathcal{B}' \subseteq \mathsf{Bor}_\lambda(X,\tau)$ be of size $\leq \lambda$. Then there is a family $\mathcal{B} = (B_\alpha)_{\alpha<\lambda}$ of λ-Borel sets with $\mathcal{B} \supseteq \mathcal{B}'$ satisfying the following conditions:*

(a) *$\mathcal{B}$ contains a basis for the topology τ.*

(b) *$\mathcal{B}_i = \{B_\alpha \mid \alpha < \lambda_i\}$ is closed under complements for every $i < \mu$.*

(c) *$\mathcal{B}$ is downward closed.*

Proof. We recursively define families $\mathcal{B}_n$, $n \in \omega$, of size at most λ as follows. Given a $\leq \lambda$-sized basis $\mathcal{A}$ for τ, let $\mathcal{B}_0$ be the closure under complements of $\mathcal{B}' \cup \mathcal{A}$. In the inductive step, for every $B \in \mathcal{B}_n$ of rank > 1 choose a family $\mathcal{F}_B \subseteq \mathsf{Bor}_\lambda(X,\tau)$ witnessing this. More precisely, if $\mathrm{rank}(B) = \alpha > 1$ and $B \in \lambda\text{-}\mathbf{\Sigma}^0_\alpha(X,\tau)$ (respectively, $B \in \lambda\text{-}\mathbf{\Pi}^0_\alpha(X,\tau)$) pick $\mathcal{F}_B \subseteq \bigcup_{1\leq\beta<\alpha} \lambda\text{-}\mathbf{\Pi}^0_\beta(X,\tau)$ (respectively, $\mathcal{F}_B \subseteq \bigcup_{1\leq\beta<\alpha} \lambda\text{-}\mathbf{\Sigma}^0_\beta(X,\tau)$) of size $\leq \lambda$ such that $B = \bigcup \mathcal{F}_B$ (respectively, $B = \bigcup \mathcal{F}_B$).

Then let $\mathcal{B}_{n+1}$ be the closure under complement of $\mathcal{B}_n \cup \bigcup_{B \in \mathcal{B}_n} \mathcal{F}_B$. By construction, $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$ has size $\leq \lambda$, satisfies (a) and (c), and is closed under complements. To get (b) it is enough to enumerate $\mathcal{B}$ as $(B_\alpha)_{\alpha<\lambda}$ in such a way that $B_{2\beta+1} = X \setminus B_{2\beta}$ for every $\beta < \lambda$: since the λ_i's are limit ordinals, this works. $\square$

Lemma 2.4.6. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ and let (X, τ) be an $f\mathrm{SC}^\lambda_\mu$-space. Let $\mathcal{B} \subseteq \mathsf{Bor}_\lambda(X, \tau)$ be a family of size $\leq \lambda$ which contains a basis $\mathcal{A}$ for τ and it is both closed under complements and downward closed. Then the smallest μ-additive topology $\bar{\tau} \supseteq \mathcal{B}$ on X is such that:*

(1) *$\bar{\tau}$ refines τ;*

(2) *each $B \in \mathcal{B}$ is $\bar{\tau}$-clopen,*

(3) *$\mathsf{Bor}_\lambda(X, \bar{\tau}) = \mathsf{Bor}_\lambda(X, \tau)$, and*

(4) *$(X, \bar{\tau})$ is a μ-additive $f\mathrm{SC}^\lambda_\mu$-space.*

Proof. By definition, the topology $\bar{\tau}$ refines τ and is zero-dimensional by $\mathcal{A} \subseteq \mathcal{B}$ and closure of $\mathcal{B}$ under complements. For the same reason, each $B \in \mathcal{B}$ is $\bar{\tau}$-clopen. It is easy to check that $(X, \bar{\tau})$ is Hausdorff and regular since $\bar{\tau} \supseteq \tau$ and $\bar{\tau}$ is zero-dimensional. A basis for $\bar{\tau}$ is given by the family of all intersections of size $< \mu$ of elements of $\mathcal{B}$, and since $\lambda^{<\mu} = \lambda$ (because we assumed $2^{<\lambda} = \lambda$) the space $(X, \bar{\tau})$ has weight $\leq \lambda$. Moreover, $\mathsf{Bor}_\lambda(X, \bar{\tau}) = \mathsf{Bor}_\lambda(X, \tau)$ because $\mathcal{B}$ has size λ and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathsf{Bor}_\lambda(X, \tau)$.

It remains to show that $(X, \bar{\tau})$ is also $f\mathrm{SC}^\lambda_\mu$, and for this we proceed similarly to Proposition 1.3.1. For every $A \in \mathcal{B}$, let

$$\mathcal{B} \restriction A = \{A, X \setminus A\} \cup \{B \in \mathcal{B} \mid \mathrm{rank}(B) < \mathrm{rank}(A)\}.$$

Let $\mathscr{A}$ be the collection of those $A \in \mathcal{B}$ for which the smallest μ-additive topology generated by $\mathcal{B} \restriction A$ is $f\mathrm{SC}^\lambda_\mu$. Notice that $\mathscr{A}$ is trivially closed under complementation.

Claim 2.4.6.1. Let $(A_i)_{i<\lambda}$ be a family of sets in $\mathscr{A}$. Let $\bar{\tau}_\infty$ be the smallest μ-additive topology generated by $\bigcup_{i<\lambda}(\mathcal{B} \restriction A_i)$. Then $(X, \bar{\tau}_\infty)$ is an $f\mathrm{SC}^\lambda_\mu$-space.

Proof of the Claim. Let $\bar{\tau}_i$ be the smallest μ-additive topology generated by $\mathcal{B} \restriction A_i$, and let $\tilde{X} = \prod_{i<\lambda}(X, \bar{\tau}_i)$ be endowed with the μ-supported product topology. Since $A_i \in \mathscr{A}$ the space $(X, \bar{\tau}_i)$ is Hausdorff for every $i < \lambda$, hence the diagonal

$$\Delta = \left\{ (x_i)_{i<\lambda} \in \tilde{X} \;\middle|\; \forall i, j < \lambda \left(x_i = x_j \right) \right\}$$

is closed in $\tilde{X}$. It is then easy to check that the map $h \colon \Delta \to (X, \bar{\tau}_\infty)$ sending $(x_i)_{i<\lambda} \in \Delta$ to x_0 is a homeomorphism. By Fact 2.3.7, player II has a winning strategy in the strong fair μ-Choquet game on $\tilde{X}$. Since Δ is closed in $\tilde{X}$, we get that Δ (and thus $(X, \bar{\tau}_\infty)$) is an $f\mathrm{SC}^\lambda_\mu$-space by Proposition 2.3.6. $\square$

Notice that if $(A_i)_{i<\lambda}$ is an enumeration of $\mathcal{B}$, then the corresponding $\bar{\tau}_\infty$ from Claim 2.4.6.1 coincides with $\bar{\tau}$ because $\bigcup_{i<\lambda}(\mathcal{B} \restriction A_i) = \mathcal{B}$, hence to complete our proof it is enough to show that $A \in \mathscr{A}$ for every $A \in \mathcal{B}$. This is done by induction on $\mathrm{rank}(A)$, using the following two claims and the fact that $\mathscr{A}$ is closed under complements.

Claim 2.4.6.2. Let C be an open or closed set of an $f\mathrm{SC}^\lambda_\mu$-space (Y, τ_Y), and let $\bar{\tau}_Y$ be the smallest μ-additive topology generated by $\tau_Y \cup \{C, Y \setminus C\}$. Then $(Y, \bar{\tau}_Y)$ is an $f\mathrm{SC}^\lambda_\mu$-space.

Proof of the Claim. Let $\bar{\bar{\tau}}$ be the smallest topology generated by $\tau_Y \cup \{C, Y \setminus C\}$. Then $(Y, \bar{\bar{\tau}})$ is homeomorphic to the sum of the spaces C and $Y \setminus C$ (endowed with the relative topologies inherited from Y). Since both C and $Y \setminus C$ are $f\mathrm{SC}^\lambda_\mu$-spaces by Theorem 2.3.6, and since the class of $f\mathrm{SC}^\lambda_\mu$-spaces is trivially closed under $\leq \lambda$-sized sums, then $(Y, \bar{\bar{\tau}})$ is an $f\mathrm{SC}^\lambda_\mu$-space as well. By Proposition 2.4.3 applied to $(Y, \bar{\bar{\tau}})$, the smallest μ-additive topology generated by $\bar{\bar{\tau}}$ is strongly fair μ-Choquet, and since such topology coincides with the smallest μ-additive topology generated by $\tau_Y \cup \{C, Y \setminus C\}$ we are done. □

In particular, setting $(Y, \tau_Y) = (X, \tau)$ we get that if $C \in \mathcal{B}$ is closed or open, then $C \in \mathscr{A}$, i.e. that all sets in $\mathcal{B}$ of λ-Borel rank 1 belong to $\mathscr{A}$.

Claim 2.4.6.3. Let $1 < \alpha < \lambda^+$ be such that $\mathcal{B}_{<\alpha} = \{B \in \mathcal{B} \mid \mathrm{rank}(B) < \alpha\} \subseteq \mathscr{A}$. Let $A \in \mathcal{B}$ be such that $A = \bigcap_{i<\lambda} A_i$ with $A_i \in \mathcal{B}_{<\alpha}$ for all $i < \lambda$. Then $A \in \mathscr{A}$.

Proof. By Claim 2.4.6.1, the smallest μ-additive topology $\bar{\tau}_\infty$ generated by $\bigcup_{B \in \mathcal{B}_{<\alpha}} \mathcal{B} \restriction B$ is strongly fair μ-Choquet, and since $\bigcup_{B \in \mathcal{B}_{<\alpha}} \mathcal{B} \restriction B$ is closed under complements then each A_i is $\bar{\tau}_\infty$-clopen and $A = \bigcap_{i<\lambda} A_i$ is $\bar{\tau}_\infty$-closed. The smallest μ-additive topology generated by $\bar{\tau}_\infty \cup \{A, X \setminus A\}$ coincide with the smallest μ-additive topology generated by $\mathcal{B} \restriction A$. Therefore setting $(Y, \tau_Y) = (X, \bar{\tau}_\infty)$ and $C = A$ in Claim 2.4.6.2 we get the desired result. □

Since $\mathcal{B}$ is downward closed, Claim 2.4.6.3 can be used to show that if $\mathcal{B}_{<\alpha} \subseteq \mathscr{A}$, then every set in $\mathcal{B} \cap \lambda$-$\mathbf{\Pi}^0_\alpha(X, \tau)$, and hence also every set in $\mathcal{B} \cap \lambda$-$\mathbf{\Sigma}^0_\alpha(X, \tau)$, belongs to $\mathscr{A}$. This concludes the proof of Lemma 2.4.6. □

Lemma 2.4.7. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ and let (X, τ) be an $f\mathrm{SC}^\lambda_\mu$-space. Let $\mathcal{B} = (B_\alpha)_{\alpha<\lambda}$ be a family of λ-Borel subsets of X satisfying conditions (a)–(c) in the conclusion of Lemma 2.4.5. For $i < \mu$ set*

$$\tilde{\mathcal{B}}_i = \mathcal{B}_i \cup \left\{ \bigcup \mathcal{F} \;\middle|\; \mathcal{F} \subseteq \mathcal{B}_i \right\} \cup \left\{ \bigcap \mathcal{F} \;\middle|\; \mathcal{F} \subseteq \mathcal{B}_i \right\},$$

where as in Lemma 2.4.5 the family $\mathcal{B}_i$ is defined by $\mathcal{B}_i = \{B_\alpha \mid \alpha < \lambda_i\}$. Let $\tilde{\mathcal{B}} = \bigcup_{i<\mu} \tilde{\mathcal{B}}_i$.

Then the smallest μ-additive topology τ' generated by $\tilde{\mathcal{B}}$ on X is such that:

(1) *τ' refines τ;*

(2) *each $B \in \tilde{\mathcal{B}}$, and hence in particular each $B \in \mathcal{B}$, is τ'-clopen,*

(3) $\mathsf{Bor}_\lambda(X, \tau') = \mathsf{Bor}_\lambda(X, \tau)$, *and*

(4) *(X, τ') is a μ-additive $f\mathrm{SC}^\lambda_\mu$-space with a NS^2_μ-basis of clopens (hence it is also Lebesgue zero-dimensional and (λ, μ)-Polish).*

Proof. The smallest μ-additive topology $\bar{\tau} \supseteq \mathcal{B}$ satisfies the conclusion of Lemma 2.4.6. Since $2^{<\lambda} = \lambda$, we have that $|\tilde{\mathcal{B}}| \leq \lambda$: indeed, $|\tilde{\mathcal{B}}_i| \leq |\mathscr{P}(\mathcal{B}_i)| \leq 2^{\lambda_i} \leq \lambda$ for every $i < \mu$. Moreover, since each set in $\mathcal{B}$ is $\bar{\tau}$-clopen, then every element of $\tilde{\mathcal{B}}$ is either open or closed in $\bar{\tau}$, and since $\mathcal{B}_i$ is closed under complements then so is each $\tilde{\mathcal{B}}_i$ and the entire $\tilde{\mathcal{B}}$. Furthermore, $\tilde{\mathcal{B}}$ is downward closed by definition and the fact that $\mathcal{B}$ was downward closed. Thus we can apply once again Lemma 2.4.6 to the space $(X, \bar{\tau})$ and the family $\tilde{\mathcal{B}}$: we claim that the smallest μ-additive topology containing $\tilde{\mathcal{B}}$ from that lemma is the desired τ'. Indeed, $\tau' \supseteq \bar{\tau} \supseteq \tau$, each $B \in \tilde{\mathcal{B}}$ is τ'-clopen, $\mathsf{Bor}_\lambda(X, \tau') = \mathsf{Bor}_\lambda(X, \bar{\tau}) = \mathsf{Bor}_\lambda(X, \tau)$, and (X, τ') is a μ-additive $f\mathrm{SC}^\lambda_\mu$-space.

Fix any $i < \mu$. Since each element of $\mathcal{B}_i$ is clopen in $\bar{\tau} \subseteq \tau'$ and $\mathcal{B}_i$ is closed under complements, we have that

$$\mathrm{CN}_{\tau'}(\mathcal{B}_i, x) = \mathrm{CN}_{\bar{\tau}}(\mathcal{B}_i, x) = \{B \in \mathcal{B}_i \mid x \in B\}$$

Set $\mathcal{D}_i = \{\bigcap \mathrm{CN}_{\tau'}(\mathcal{B}_i, x) \mid x \in X\}$ and $\mathcal{D} = \bigcup_{i<\mu} \mathcal{D}_i$. Each $\mathcal{D}_i$ is a partition by Lemma 2.2.15, and it consists of τ'-clopen sets because by definition $\tilde{\mathcal{B}}$ contains every intersection of sets from $\mathcal{B}_i$ and elements of $\tilde{\mathcal{B}}$ are τ'-clopen. In particular, this shows that $\{\mathcal{D}_i \mid i < \mu\}$ is a NS^2_μ-cover of $\mathcal{D}$ consisting of τ'-clopen sets.

It remains to show that $\mathcal{D}$ is a basis for τ'. For every $O \in \tau'$ and $x \in O$ let $\mathcal{A} \subseteq \tilde{\mathcal{B}}$ be a family of size $< \mu$ such that $x \in \bigcap \mathcal{A} \subseteq O$: we may find $\mathcal{A}$ since τ' is the smallest μ-additive topology containing $\tilde{\mathcal{B}}$. Then by definition of $\tilde{\mathcal{B}}$ for every $A \in \mathcal{A}$ we may find i_A and $\mathcal{F}_A \subseteq \mathcal{B}_{i_A}$ such that $A = \bigcup \mathcal{F}_A$ or $A = \bigcap \mathcal{F}_A$. (Notice that this includes the case in which $A \in \mathcal{B}_{i_A}$, as then $A = \bigcup \mathcal{F}_A = \bigcap \mathcal{F}_A$ for $\mathcal{F}_A = \{A\}$.) In the first case, set $\mathcal{F}'_A = \{F\}$ for some $F \in \mathcal{F}_A$ such that $x \in F$, otherwise set $\mathcal{F}'_A = \mathcal{F}_A$. Let $\mathcal{A}' = \bigcup_{A \in \mathcal{A}} \mathcal{F}'_A$ and let $i = \sup_{A \in \mathcal{A}} i_A$. Then $i < \mu$ because $|\mathcal{A}| < \mu$ and μ is regular, and thus $\mathcal{A}' \subseteq \mathcal{B}_i$. Then we have

$$x \in \bigcap \mathrm{CN}_{\tau'}(\mathcal{B}_i, x) = \{B \in \mathcal{B}_i \mid x \in B\} \subseteq \bigcap \mathcal{A}' \subseteq \bigcap \mathcal{A} \subseteq O,$$

and since $\bigcap \mathrm{CN}_{\tau'}(\mathcal{B}_i, x) \in \mathcal{D}_i \subseteq \mathcal{D}$ we are done. □

To prove Theorem 2.4.4, it is now enough to apply Lemma 2.4.5 and Lemma 2.4.7 one after the other.

Once the change-of-topology technique is available, we can derive a number of interesting and useful consequences. Naturally adapting the arguments in Corollary 1.3.2 we get the following corollaries of Theorem 2.4.4.

Corollary 2.4.8. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. For every λ-Borel subset B of an $f\mathrm{SC}^\lambda_\mu$-space there is a continuous λ-Borel isomorphism from a closed $C \subseteq {}^\mu\lambda$ to B.*

Corollary 2.4.9. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Each λ-Borel subset B of an $f\mathrm{SC}^\lambda_\mu$-space is λ-Borel isomorphic to a superclosed subset of ${}^\mu\lambda$ (and thus to a μ-additive SC^λ_μ-space with a NS^2_μ-basis of clopens).*

Proof. By Corollary 2.4.8 the set B is λ-Borel isomorphic to a a closed $C \subseteq {}^\mu\lambda$. Then the same argument of Lemma 1.1.38 gives the desired result. □

Corollary 2.4.8 gives in particular that every $f\mathrm{SC}^\lambda_\mu$-space is a continuous injective image of a closed subset of ${}^\mu\lambda$. In the case of SC^λ_μ-spaces, by Corollary 2.3.33 we can also obtain the following related result.

Corollary 2.4.10. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Every SC^λ_μ-space is a continuous image of ${}^\mu\lambda$.*

The following is the counterpart of Theorem 2.4.4 in terms of functions and can be proved by applying it to the preimages of the basic open sets in any $\leq \lambda$-sized basis for the topology of Y.

Corollary 2.4.11. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Let (X, τ) be an $f\mathrm{SC}^\lambda_\mu$-space and Y be any space of weight $\leq \lambda$. Then for every λ-Borel function $f : (X, \tau) \to Y$ there is a topology τ' on X such that:*

(1) *τ' refines τ;*

(2) *$f : (X, \tau') \to Y$ is continuous; each B_α is τ'-clopen,*

(3) *$\mathsf{Bor}_\lambda(X, \tau') = \mathsf{Bor}_\lambda(X, \tau)$, and*

(4) *(X, τ') is a μ-additive $f\mathrm{SC}^\lambda_\mu$-space with a NS^2_μ-basis of clopens (hence it is also Lebesgue zero-dimensional and (λ, μ)-Polish).*

2.4.3 Main results

We are now ready to prove the analogue of Theorem 2.4.1 in our generalized context. First of all, Theorem 2.4.4 and Corollary 2.4.9 show that all classes of Polish-like spaces naturally arisen in this chapter coincide up to λ-Borel isomorphism: once we have the weakest notion of completeness, i.e. being an $f\mathrm{SC}^\lambda_\mu$-space, all the rest comes for free if we are interested in results depending just on the λ-Borel structure of the space (and not on its actual topology). The classes of spaces listed in the next theorem are just a sample of the variations on the theme that might be considered.

Theorem 2.4.12. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Up to λ-Borel isomorphism, the following classes of (regular Hausdorff) topological spaces coincide:*

(1) *$f\mathrm{SC}^\lambda_\mu$-spaces;*

(2) *SC^λ_μ-spaces;*

(3) *(λ, μ)-Polish spaces;*

(4) *(Lebesgue zero-dimensional) spherically complete (λ, μ)-Polish spaces;*

(5) *μ-Lindelöf-based spaces;*

(6) *compact-based spaces.*

Moreover, items (3) and (4) can be replaced by any of their reformulations from Theorems 2.3.1 and 2.3.2.

Using the same results, we can also show that standard λ-Borel spaces can equivalently be defined in terms of Polish-like topologies generating them.

Theorem 2.4.13. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. A λ-Borel space $(X, \mathscr{B})$ is standard if and only if there is a topology τ' on X such that $\mathsf{Bor}_\lambda(X, \tau') = \mathscr{B}$ and the following condition holds:*

(1) *(X, τ') is a $f\mathrm{SC}^\lambda_\mu$-space.*

Moreover, condition (1) might be replaced by any of the following ones:

(2) *(X, τ') is a μ-additive $f\mathrm{SC}^\lambda_\mu$-space with a NS^2_μ-basis of clopens (hence it is also Lebesgue zero-dimensional (λ, μ)-Polish).*

(3) *(X, τ') is a μ-additive SC^λ_μ-space with a NS^2_μ-basis of clopens (hence it is also Lebesgue zero-dimensional and spherically complete (λ, μ)-Polish).*

Furthermore, if $\mathscr{B} = \mathsf{Bor}_\lambda(X, \tau)$ for some topology τ of weight $\leq \lambda$, in conditions (1) and (2) we can additionally require that $\tau \subseteq \tau'$.

Remark 2.4.14. Since μ-additive SC^λ_μ-space with a NS_μ-basis of clopens and $f\mathrm{SC}^\lambda_\mu$-spaces form, respectively, the smallest and largest class of Polish-like spaces considered in this chapter, in Theorem 2.4.13 we can further replace those classes with any of the other ones like μ-additive $f\mathrm{SC}^\lambda_\mu$-spaces or μ-tree-based SC^λ_μ-spaces, and so on.

We conclude this section by solving two natural problems concerning standard λ-Borel spaces. All results below again extend corresponding theorems in Chapter 1 (which deals only with regular cardinals) to arbitrary λ's satisfying $2^{<\lambda} = \lambda$; we refer to that paper for a more thorough discussion on their relevance, in particular in relation to what happens in the classical case $\lambda = \omega$.

First, we want to characterize those subspaces of a Polish-like space that inherit a standard λ-Borel structure from it.

Theorem 2.4.15. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Let $(X, \mathscr{B})$ be a standard λ-Borel space, and let $A \subseteq X$. Then $(A, \mathscr{B} \restriction A)$ is a standard λ-Borel space if and only if $A \in \mathscr{B}$.*

Proof. Since $(X, \mathscr{B})$ is standard λ-Borel, by definition we can find some $B \in \mathsf{Bor}_\lambda({}^\mu\lambda)$ which is λ-Borel isomorphic to $(X, \mathscr{B})$.

If $A \in \mathscr{B}$, then the subspace $(A, \mathscr{B} \restriction A)$ is λ-Borel isomorphic to a set in $\mathsf{Bor}_\lambda(B) \subseteq \mathsf{Bor}_\lambda({}^\mu\lambda)$, hence $(A, \mathscr{B} \restriction A)$ is standard λ-Borel by definition.

Conversely, assume that $(A, \mathscr{B} \restriction A)$ is standard λ-Borel. Let τ_X be a μ-additive $f\mathrm{SC}^\lambda_\mu$ topology on X with $\mathscr{B} = \mathsf{Bor}_\lambda(X, \tau_X)$, whose existence is granted by Theorem 2.4.13. Using the same theorem, together with the fact that the topology induced by τ_X on A has weight $\leq \lambda$ and generates $\mathscr{B} \restriction A$, find a μ-additive $f\mathrm{SC}^\lambda_\mu$ topology τ_A on A such that $\tau_X \restriction A \subseteq \tau_A$ and $\mathscr{B} \restriction A = \mathsf{Bor}_\lambda(A, \tau_A)$.

Claim 2.4.15.1. There are families $\mathcal{C}_X = (B_\alpha)_{\alpha<\lambda}$ and $\mathcal{C}_A = (C_\alpha)_{\alpha<\lambda}$ of λ-Borel subsets of, respectively, (X, τ_X) and (A, τ_A) which simultaneously satisfy (the analogues of) items (a)–(c) from Lemma 2.4.5 (with respect to the corresponding ambient spaces), and moreover $C_\alpha = B_\alpha \cap A$ for all $\alpha < \lambda$.

Proof of the Claim. Let $\mathcal{D}_A$ be a basis for (A, τ_A) of size $\leq \lambda$, and let $\mathcal{C}^0_A$ be the family obtained by applying Lemma 2.4.5 to $\mathcal{B}' = \mathcal{D}_A$ (with respect to the space (A, τ_A)). Since $\mathcal{C}^0_A \subseteq \mathsf{Bor}_\lambda(A, \tau_A) = \mathsf{Bor}_\lambda(X, \tau) \restriction A$, for every $C \in \mathcal{C}^0_A$ we can find $B_C \in \mathsf{Bor}_\lambda(X, \tau)$ such that $C = B_C \cap A$: let $\mathcal{C}^0_X = \{B_C \mid C \in \mathcal{C}^0_A\}$. Now let $\mathcal{C}^1_X$ be the family of λ-Borel sets given by Lemma 2.4.5 applied to $\mathcal{B}' = \mathcal{C}^0_X$ (with respect to the space (X, τ_X)), and let $\mathcal{C}^1_A = \{B \cap A \mid B \in \mathcal{C}^1_X\}$. Recursively proceed in the same fashion for ω-many steps, that is: at even stages $n > 0$ let $\mathcal{C}^n_A$ be the family of λ-Borel sets of (A, τ_A) given by Lemma 2.4.5 applied to $\mathcal{B}' = \mathcal{C}^{n-1}_A$, and let $\mathcal{C}^n_X$ be a family of λ-Borel sets in (X, τ_X) whose traces are exactly the elements of $\mathcal{C}^n_A$; at odd stages, apply Lemma 2.4.5 to $\mathcal{C}^{n-1}_X$ to get $\mathcal{C}^n_X$, and let $\mathcal{C}^n_A = \{B \cap A \mid B \in \mathcal{C}^n_X\}$. Finally, set $\mathcal{C}_X = \bigcup_{n\in\omega} \mathcal{C}^n_X$ and $\mathcal{C}_A = \bigcup_{n\in\omega} \mathcal{C}^n_A$, and notice that $\mathcal{C}_A = \{B \cap A \mid B \in \mathcal{C}_X\}$ by construction.

The family $\mathcal{C}^0_A \subseteq \mathcal{C}_A$ contains the basis $\mathcal{D}_A$ for τ_A, while $\mathcal{C}^1_X \subseteq \mathcal{C}_X$ contains a basis for τ_X because we applied Lemma 2.4.5. Moreover, for all even $n \in \omega$ the family $\mathcal{C}^n_A$ is closed under complements and downward closed in (A, τ_A) because we applied Lemma 2.4.5, and since if $n > 0$ then $\mathcal{C}^{n-1}_A \subseteq \mathcal{C}^n_A$ by construction it follows that $\mathcal{C}_A$ is closed under complements and downward closed as well. Similarly, for any odd $n \in \omega$ the family $\mathcal{C}^n_X$ is closed under complements and downward closed in (X, τ_X), and since $\mathcal{C}^{n-1}_X \subseteq \mathcal{C}^n_X$ also $\mathcal{C}_X$ is closed under complements and downward closed. Enumerate $\mathcal{C}_X = (B_\alpha)_{\alpha<\lambda}$ so that $B_{2\beta+1} = X \setminus B_{2\beta}$ for all $\beta < \lambda$, and set $C_\alpha = B_\alpha \cap A$ for all $\alpha < \lambda$. By construction $\mathcal{C}_A = (C_\alpha)_{\alpha<\lambda}$, and

$$C_{2\beta+1} = B_{2\beta+1} \cap A = (X \setminus B_{2\beta}) \cap A = A \setminus (B_{2\beta} \cap A) = A \setminus C_{2\beta},$$

hence the families $\mathcal{C}_X = (B_\alpha)_{\alpha<\lambda}$ and $\mathcal{C}_A = (C_\alpha)_{\alpha<\lambda}$ are as required. □

Let τ'_X be the topology on X constructed as in Lemma 2.4.7 starting from $\mathcal{B} = \mathcal{C}_X$ and, similarly, let τ'_A be the topology on A constructed in the same way but starting from $\mathcal{B} = \mathcal{C}_A$. By Lemma 2.4.7 both (X, τ'_X) and (A, τ'_A) are Lebesgue zero-dimensional (λ, μ)-Polish spaces, and $\mathsf{Bor}_\lambda(X, \tau'_X) = \mathsf{Bor}_\lambda(X, \tau_X)$ Moreover, since $C_\alpha = B_\alpha \cap A$ for all $\alpha < \lambda$ we have that $\tau'_A = \tau'_X \restriction A$ by construction, i.e. (A, τ'_A) is a subspace of (X, τ'_X). Therefore by Corollary 2.3.11 we have that A is a G^μ_δ subset of (X, τ'_X), and so in particular $A \in \mathsf{Bor}_\lambda(X, \tau'_X) = \mathsf{Bor}_\lambda(X, \tau_X) = \mathscr{B}$. □

Corollary 2.4.16. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Let X, Y be standard λ-Borel spaces. If $A \subseteq X$ is λ-Borel and $f \colon A \to Y$ is a λ-Borel embedding, then $f(A)$ is λ-Borel in Y.*

We finally come to the problem of understanding which topologies generate a standard λ-Borel structure. Using the results of this chapter, it can be shown that this class is larger than the collection of e.g. all (λ, μ)-Polish spaces (see also Chapter 1 for more details). On the other hand, if a space (X, τ) is homeomorphic to a λ-Borel subset of ${}^\mu\lambda$, then it clearly generates a standard λ-Borel structure by definition. Theorems 2.2.1(2) and 2.3.6 together with Corollary 2.4.16 allow us to reverse the implication, yielding the desired characterization in the case of μ-metrizable topologies.

Corollary 2.4.17. *Assume that $2^{<\lambda} = \lambda$ has cofinality μ. Let (X, τ) be a μ-metrizable space of weight $\leq \lambda$, and if $\mu = \omega$ further assume that X be Lebesgue zero-dimensional. Then $(X, \mathsf{Bor}_\lambda(X, \tau))$ is a standard λ-Borel space if and only if (X, τ) is* homeomorphic *to a λ-Borel subset of ${}^\mu\lambda$ (or, equivalently, of ${}^\lambda 2$).*

2.5 Examples and counterexamples

In this section, we analyze some spaces that share some but not all of the properties described in the chapter, and thus serve as a counterexample for many possible conjectures about how much the hypothesis of the theorems stated can be weakened.

Proposition 2.5.1. *Suppose $2^{<\lambda} = \lambda > \mathrm{cof}(\lambda) = \mu$. There is a (regular Hausdorff) μ-additive, (paracompact, Lebesgue zero-dimensional,) tree-based, compact-based (and thus SC^λ_μ) space X of weight λ such that every point $x \in X$ has a local basis of size μ, but X is not μ-uniformly based (thus, not μ-metrizable nor a NS_μ-space).*

Proof. Consider the set $A = \{s \in {}^\mu 2 \mid s(\alpha) = 0 \text{ for } < \mu \text{ many } \alpha\}$, and define the DST tree $T = \{s \in {}^{<\lambda}2 \mid s \restriction \mu \notin A\}$, where $s \restriction \mu = s$ if $\mathrm{lh}(s) \leq \mu$. Let $X = [T]_c$ with the usual bounded topology: we claim X is as wanted.

First, X has weight $2^{<\lambda} = \lambda$, and by Fact 2.3.22 X is a compact-based tree-based (regular, Hausdorff) space. This implies that X is also Lebesgue zero-dimensional (by Proposition 2.2.24, and this implies also paracompact) and an SC^λ_μ-space (by Corollary 2.3.35). Finally, X is μ-additive and every point $x \in X$ has a local basis of size μ because every branch of T has cofinality μ.

Finally, we show that I has a winning strategy in the μ-uniform local basis on X, and hence X is not μ-metrizable (nor a NS_μ-space) by Theorem 2.2.1(1). By Lemma 2.2.35 we may assume that player II plays only open sets in the canonical base $\mathcal{B}_T = \{\mathbf{N}_s \mid s \in T\}$. Let $r = \langle \mathbf{N}_{s_\gamma} \mid \gamma < \delta\rangle$ be a sequence of moves of player II, and let $t_r = \bigcup_{\gamma<\delta} s_\gamma$, where we set $\bigcup \emptyset = \emptyset$. Define $\sigma(r) = t_r{}^\frown 0^\frown 1^{(\mu)}$ if $\mathrm{lh}(t_r) < \mu$, and define $\sigma(r) = x$ for a fixed point $x \in X$ otherwise. First, notice that $\sigma(r) \in [T]_c = X$ for any r, so σ is well-defined. We claim this σ is a winning strategy for I. Indeed, suppose that $\langle x_\alpha, \mathbf{N}_{s_\alpha} \mid \alpha < \mu\rangle$ is a run of the μ-uniformly based game on X played by I accordingly to σ. Then, an easy induction on $r \restriction \alpha$ shows that $\mathrm{lh}(s_\alpha) < \mu$ for every $\alpha < \mu$, and furthermore

$$|\{\alpha < \mathrm{lh}(s_\beta) \mid s_\beta(\alpha) = 0\}| < |\{\alpha < \mathrm{lh}(s_\gamma) \mid s_\gamma(\alpha) = 0\}|$$

for every $\beta < \gamma < \mu$. Hence, if $s = \sup\{s_\alpha \mid \alpha < \mu\}$, then we have $\mathrm{lh}(s) = \mu$ and $s \notin A$, thus $s \in T$ and $\mathbf{N}_s = \bigcap_{\alpha<\mu} \mathbf{N}_{s_\alpha} \neq \emptyset$ (as for example $s^\frown 1^{(\lambda)} \in \mathbf{N}_s$). Furthermore, for every $x \in \mathbf{N}_s$ and for every $\alpha < \mu$ we have $x \in \mathbf{N}_s \subsetneq \mathbf{N}_{s_\alpha}$, so particular $\{\mathbf{N}_{s_\alpha} \mid \alpha < \mu\}$ is not a local basis of any point. □

Proposition 2.5.2. *Suppose $\lambda^{<\mu} = \lambda > \mathrm{cof}(\lambda) = \mu > \omega$, and there is $\gamma < \mu$ such that ${}^\gamma\lambda$ contains a subset which is not G^μ_δ in it.*

Then there is a (regular, Hausdorff, paracompact, Lebesgue zero-dimensional,) μ-uniformly based μ-tree-based compact-based (and thus SC^λ_μ) space X of weight λ that is not a NS_μ-space.

Proof. Let γ and $A \subseteq {}^\gamma\lambda$ be such that A is not G^μ_δ in ${}^\gamma\lambda$. Define

$$T = \{s \in {}^{<\mu}\lambda \mid s \restriction \gamma \notin A\},$$

where $s \restriction \gamma = s$ if $\mathrm{lh}(s) \leq \gamma$. Let $X = [T]_c$ with the usual bounded topology. We claim X is as wanted.

First, X has weight $\lambda^{<\mu} = \lambda$, and by Fact 2.3.22 X is a compact-based tree-based (regular, Hausdorff) space. This implies that X is also Lebesgue zero-dimensional (by Proposition 2.2.24, and this implies also paracompact), that X is μ-uniformly based (by Proposition 2.2.38), and that X is SC^λ_μ (by Corollary 2.3.35).

It remains to prove that X is not NS_μ. Notice that X can be partitioned into two sets $X_1 = X \cap {}^{\gamma}\lambda = A$ and $X_2 = X \cap {}^{\mu}\lambda = [T]$. Suppose by contradiction that X is also a NS_μ-space, and let $\mathcal{B} = \bigcup_{\alpha<\mu} \mathcal{B}_\alpha$ be a NS_μ-basis with $\mathcal{B}_\alpha$ locally $<\mu$-small for every $\alpha < \mu$. For every $x \in X_1$ and $\alpha < \mu$, let $s(\alpha, x) \in {}^{<\gamma}\lambda$ be given by definition of NS_μ-cover (and by the fact that $\{\mathbf{N}^X_s \mid s \in T\}$ is a basis for X) such that

$$|\{B \in \mathcal{B}_\alpha \mid B \cap \mathbf{N}^X_{s(\alpha,x)} \neq \emptyset\}| < \mu.$$

Let $Y = {}^{\gamma}\lambda$, and define also $U_\alpha = \bigcup_{x\in A} \mathbf{N}^Y_{s(\alpha,x)}$. Then, every U_α is an open set in ${}^{\gamma}\lambda$, and $A \subseteq U_\alpha$. Since A is not G^μ_δ in ${}^{\gamma}\lambda$, there is $s \in \bigcap_{\alpha<\mu} U_\alpha \setminus A$. Since $s \notin A$, in particular we have that $\mathbf{N}^{{}^\mu\lambda}_s = \mathbf{N}^X_s \subseteq X$ is non-empty. For every $\alpha < \mu$, let $x_\alpha \in X_1$ be such that $s \in \mathbf{N}^Y_{s(\alpha,x_\alpha)}$, i.e. $s(\alpha, x_\alpha) \subseteq s$. Notice that the family $\mathcal{B}' = \{B \cap \mathbf{N}^X_s \mid B \in \mathcal{B}, B \cap \mathbf{N}^X_s \neq \emptyset\}$ is a basis for the subspace $\mathbf{N}^X_s \subseteq X$, and furthermore $\{B \in \mathcal{B}_\alpha \mid B \cap \mathbf{N}^X_s \neq \emptyset\} \subseteq \{B \in \mathcal{B}_\alpha \mid B \cap \mathbf{N}^X_{s(\alpha,x_\alpha)} \neq \emptyset\}$ for every $\alpha < \mu$. Hence, we have

$$w(\mathbf{N}^X_s) \leq |\mathcal{B}'| \leq |\bigcup_{\alpha<\mu} \{B \in \mathcal{B}_\alpha \mid B \cap \mathbf{N}^X_{s(\alpha,x_\alpha)} \neq \emptyset\}| \leq \sum_{\alpha<\mu} \mu = \mu.$$

However, the subspace $\mathbf{N}^X_s$ is homeomorphic to ${}^{\mu}\lambda$, and in particular it has weight $\lambda^{<\mu} = \lambda > \mu$. □

Corollary 2.5.3. *Suppose $\lambda^{<\mu} = \lambda > \mathrm{cof}(\lambda) = \mu > \omega$, and there is $\gamma < \mu$ such that ${}^{\gamma}\lambda$ contains a subset which is not G^μ_δ in it.*

Then there is a (regular, Hausdorff, paracompact, Lebesgue zero-dimensional,) μ-tree-based compact-based (and thus SC^λ_μ) space X of weight λ where player II *has a winning strategy but not a winning tactic in the μ-uniform local basis game.*

Proposition 2.5.4. *Suppose $\mu > \omega$ and $2^{<\mu} \leq \lambda$. There is a μ-tree-based space (X, τ) of size and weight $2^{<\mu}$ such that (X, τ) is not $f\mathrm{SC}^\lambda_\mu$, but (X, τ') is $f\mathrm{SC}^\lambda_\mu$, for τ' be the smallest μ-additive topology generated by τ.*

In particular, if $2^{<\mu} = \mu$, then (X, τ) is also a NS_μ-space.

Proof. Let $f, g : {}^{<\mu}2 \to \mu$ be two functions defined respectively by

$$f(s) = \mathrm{ot}(\{\alpha \in \mathrm{lh}(s) \mid s(\alpha) = 0\}), \qquad g(s) = \mathrm{ot}(\{\alpha \in \mathrm{lh}(s) \mid s(\alpha) = 1\})$$

for every $s \in {}^{<\mu}2$.

Let T be the tree of those $s \in {}^{<\mu}2$ such that $f(s \restriction \beta) \leq g(s \restriction \beta)$ for every limit ordinal $\beta < \mathrm{lh}(s)$. Then T is trivial closed under initial segment (and thus it is a DST tree). Define $X = [T]_c \setminus [T]$, with τ being the bounded topology inherited from $[T]_c$. First, X is μ-tree-based, by Fact 2.2.26, and it has weight and size $2^{<\mu}$ since $X \subseteq {}^{<\mu}2$.

We claim that player I has a winning strategy in the strong fair μ-Choquet game on X. Without loss of generality, by Remark 1.1.4 we may assume player II plays only open sets in the canonical basis $\{\mathbf{N}_s \mid s \in T\}$. Let $\langle V_\alpha \mid \alpha < \gamma\rangle$ be the sequence of moves of player II of a partial run of the game up to round $\gamma < \mu$. If $\bigcap_{\alpha<\gamma} V_\alpha$ has size ≤ 1, then player I has only one forced move to do. Otherwise, we have $\bigcap_{\alpha<\gamma} V_\alpha = \mathbf{N}_s$ for some $s \in T$, by Lemma 2.2.25 (and since T is closed under initial segment). Let α be the minimum ordinal such that $s^\frown 1^{(\omega)} {}^\frown 0^{(\alpha)} \notin T$. Define $\sigma(\mathbf{N}_s) = (\mathbf{N}_{s^\frown 1^{(\omega)}}, s^\frown 1^{(\omega)} {}^\frown 0^{(\alpha)})$. Then, given a partial run of the game $\langle (N_{s_\alpha}, x_\alpha), V_\alpha \mid \alpha < \gamma \rangle$, we have $f(s_\alpha) \leq g(s_\alpha)$ for every $\alpha < \gamma$ by construction, and so $f(s) \leq g(s)$ as well for $s = (\bigcup_{\alpha<\gamma} s_\alpha)$. Then, $\bigcap_{\alpha<\gamma} \mathbf{N}_{s_\alpha}$ is empty if and only if $\gamma = \mu$, and thus σ is winning for I. However, player II has a winning strategy in (X, τ'), since this space is discrete. $\square$

Chapter 3

Examples and classification

In this section, we are going to study some examples of $f\mathrm{SC}_\kappa$ or SC_κ-spaces and show that these classes are rich, as they contain as many distinct spaces as possible up to homeomorphism.

For sake of simplicity, we work with a regular uncountable cardinal κ satisfying $2^{<\kappa} = \kappa$. However, it is not difficult to see that many constructions can be adapted to the singular case as well.

3.1 LOTS, GO-spaces and their relation with the Choquet games

One of the greatest sources of examples of non κ-additive $f\mathrm{SC}_\kappa$ or SC_κ-spaces comes from linearly ordered sets. First, we recall some basic facts about these spaces that we are going to use throughout all other sections, and we prove some more that show their relationship with the Choquet games. We refer to [120] for notation and for a good introduction to the topic.

Given a linear order $(\mathbb{L}, <)$, we usually denote with 0 the minimum and with 1 the maximum of $\mathbb{L}$, if $\mathbb{L}$ has any of the two. An **extreme point** or **endpoint** of $\mathbb{L}$ is a point that is either the minimum or the maximum. An **open interval** of $\mathbb{L}$ is a set of the form $(s, t) = \{x \in \mathbb{L} \mid s < x < t\}$ for some $a, b \in \mathbb{L} \cup \{-\infty, +\infty\}$, where $+\infty$ (resp., $-\infty$) is an element outside $\mathbb{L}$ that is assumed greater (resp., smaller) than any element of $\mathbb{L}$. Similarly we define **intervals** of other forms $(a, b] = (a, b) \cup \{b\}$, and $[a, b) = (a, b) \cup \{a\}$, and $[a, b] = (a, b) \cup \{a, b\}$. The **order topology** $\tau^{\mathbb{L}}_<$ on $\mathbb{L}$ is the smallest topology generated by the open intervals of the order. A topological space with a linear order $(\mathbb{L}, <_\mathbb{L}, \tau)$ is called a **linearly ordered topological space**, or **LOTS** if the topology coincide with the order topology $\tau = \tau^{\mathbb{L}}_<$. A topological space with a linear order $(X, <_X, \tau)$ is called a **GO-space** or **generalized ordered space**[1] if there is a LOTS $(\mathbb{L}, <_\mathbb{L}, \tau_\mathbb{L})$ such that X is both a subspace and a suborder of $\mathbb{L}$. Given two GO-spaces $(X, <_X, \tau_X)$ and $(Y, <_Y, \tau_Y)$, we say that X and Y are **isomorphic** if there is a function $f : X \to Y$ that is both an isomorphism of orders and a homeomorphism of topological spaces. When dealing with more ordered sets

[1] An interval that is open in the topology of a GO-space may not be an open interval of the form (a, b): for this reason, we try to stick to the term *τ-open interval* or *interval open in X* to denote intervals that are open in the topology of X, and leave the term *open interval* for sets of the form (a, b).

$\mathbb{L}_1 \subseteq \mathbb{L}_2$ at the same time, we use the notation $(x, y)_{\mathbb{L}_i}$ to specify in which set an interval is calculated (and similarly for other types of intervals). Given a linear order $(\mathbb{L}, <)$, a subset $A \subseteq \mathbb{L}$ is called **convex** if and only if $[a, b] \subseteq A$ for every $a, b \in A$. Every convex set A can be written as union of intervals $A = \bigcup_{i<\alpha, j<\beta}[a_i, b_j]$ for a decreasing sequence $(a_i)_{i\in\alpha}$ and an increasing sequence $(b_i)_{i\in\beta}$, and A is open in the order topology if and only if we may choose the sequences $(a_i)_{i\in\alpha}$ and $(b_i)_{i\in\beta}$ so that $A = \bigcup_{i<\alpha, j<\beta}(a_i, b_j)$. Notice that any convex GO-subspace X of a LOTS $(\mathbb{L}, <, \tau)$ is again a LOTS with topology and order inherited from $\mathbb{L}$.

GO-spaces can be characterized in terms of the relationship between the order and the topology.

Proposition 3.1.1 ([120, Proposition VIII.A])**.** *Let $(X, <, \tau)$ be a linear order with a topology. Then X is a GO-space if and only if:*

- *τ refines the order topology $\tau_<^X$ of $(X, <)$.*
- *τ has a basis made of convex subsets of $(X, <)$.*

Notice that if $(X, <, \tau)$ is a GO-space and A is τ-open convex set, then for every $x \in A$ we may find a τ-open interval I of one of the forms $[x, x]$, $[x, b)$, $(a, x]$ or (a, b) such that $x \in I \subseteq A$. Indeed, if $A = \{x\}$ we are done. Otherwise, if x is the minimum of A and there is $b \in A$ with $b > x$, then $[x, b) = A \cap (-\infty, b)$ is τ-open, since τ refines the order topology and $(-\infty, b)$ is open in the order topology. Furthermore, $x \in [x, b) \subseteq [x, b] \subseteq A$ by definition of convex set. Similarly if x is the maximum of A. And if x is neither the maximum nor the minimum of A, then it means that there are $a, b \in A$ such that $a < x < b$, and thus $x \in (a, b) \subseteq [a, b] \subseteq A$.

Thus, we can always work with intervals instead of convex sets, if needed.

Corollary 3.1.2. *A topological space with a linear order $(X, <, \tau)$ is a GO-space if and only if τ refines the order topology and it admits a basis made of intervals of $(X, <)$.*

Notice that the intersection of convex sets is again a convex set. Thus, we get the following.

Corollary 3.1.3. *Let $(X, <, \tau)$ be a GO-space, and let τ_κ be smallest κ-additive topology refining τ. Then, $(X, <, \tau_\kappa)$ is again a GO-space.*

For a topological space X, the density $\mathrm{d}(X)$ (i.e. the smallest size of a dense subspace) is always less than or equal to the weight $w(X)$ (i.e. the smallest size of a basis for the topology). In general it is possible that $\mathrm{d}(X) < w(X)$. For LOTS, however, these two cardinals are always equal.

Remark 3.1.4. Let $(\mathbb{L}, <)$ be a complete linear order. Then the order topology has weight κ if and only if $\mathbb{L}$ contains a dense subset of size κ.

The same is not true for GO-spaces, as, for example, the Sorgenfrey line has density ω but weight continuum $\mathfrak{c}$.

Given a linear order $(\mathbb{L}, <_{\mathbb{L}})$, a **cut** is a convex partition (C, D) of $\mathbb{L}$. We call a cut (C, D): a **jump** or a **trivial cut** if both C has a maximum and D has a minimum; **left-cut**, if C has no maximum; **right-cut**, if D has no minimum. A cut (C, D) is called a **gap** if it is both a left and right cut. A cut (or gap) (C, D) is called an **end cut** (or end gap) if $C = \emptyset$ or $D = \emptyset$. Notice that if $(\mathbb{L}, <_{\mathbb{L}}, \tau_{<}^{\mathbb{L}})$ is furthermore a LOTS, then a non-trivial cut (C, D) is a gap if and only if $\{C, D\}$ is a clopen partition in the order topology. For a GO-space $(X, <_X, \tau)$ instead, we may have convex clopen partitions that are not gaps. Since these partitions play a key role in many topological properties, we define a further notion. A **(right or left) pseudo-gap** in a GO-space $(X, <_X, \tau)$ is a cut where C has a maximum if and only if D has no minimum and where additionally C and D are both clopen and non-empty in X. In other words, a pseudo-gap is a clopen right or left cut that is not a gap. Notice that while the concept of gap depends only on the order, the concept of pseudo-gap depends strongly from the topology. Also, LOTS have no pseudo-gaps.

Given a subset of a linear order $C \subseteq (\mathbb{L}, <)$, the **downward closure** of C is the set $\downarrow C = \{x \in \mathbb{L} \mid \exists c \in C[x \leq c]\}$. Similarly, the **upward closure** of C is the set $\uparrow C = \{x \in \mathbb{L} \mid \exists c \in C[x \geq c]\}$. The **cofinality** of $\mathbb{L}$ is the smallest size of a subset $C \subseteq \mathbb{L}$ such that $\downarrow C = \mathbb{L}$, and the **coinitiality** of $\mathbb{L}$ is the smallest size of a subset $C \subseteq \mathbb{L}$ such that $\uparrow C = \mathbb{L}$. Notice that we may define the cofinality also on any subset $X \subseteq \mathbb{L}$ using the order inherited from $\mathbb{L}$, and the cofinality of X is always either 0 if X is empty, 1 if X has a maximum or an infinite regular cardinal otherwise. The same holds for the coinitiality[2].

A (μ, ν)**-cut** is a cut (C, D) such that the cofinality of $(C, <)$ is μ and the coinitiality of $(D, <)$ is ν. We call (μ, ν) the **type** of the cut. We define (μ, ν)**-gaps** and (μ, ν)**-pseudo-gaps** as (μ, ν)-cuts which are, respectively, also gaps or pseudo-gaps. Notice then that a (μ, ν)-cut is an end gap if and only if one between μ and ν is 0 and the other is an infinite regular cardinal. It is a gap if and only if μ and ν are both infinite regular cardinals; it is a pseudo-gap (and not a gap) if and only if C and D are open and one between μ and ν is 1 and the other is an infinite regular cardinal.

With abuse of notation, given two subsets $C, D \subseteq \mathbb{L}$ and $x \in \mathbb{L}$, we write $C < D$ to denote $c < d$ for every $c \in C$ and $d \in D$, and similarly we use the notations $C < x$ and $x < C$. Notice that if C, D are two subsets of $\mathbb{L}$ such that $C < D$ and there is no $x \in \mathbb{L}$ such that $C < x < D$, then (C, D) uniquely identifies a cut $(\downarrow C, \uparrow D)$. In this case, we say that (C, D) form a cut (or a gap/pseudo-gap, if $(\downarrow C, \uparrow D)$ is a gap/pseudo-gap). With abuse of notation, we often confuse a gap (C, D) with two sets that generate it.

Given a linear order $(\mathbb{L}, <)$, a subset $C \subseteq \mathbb{L}$ is called **upward bounded** if there is $x \in \mathbb{L}$ such that $C \leq x$. It is called **downward bounded** if there is $x \in \mathbb{L}$ such that $C \geq x$. The order $\mathbb{L}$ is called **boundedly complete** if every upward bounded subset has a supremum and every downward bounded subset has an infimum. Equivalently, $\mathbb{L}$ is boundedly complete if and only if it has no gaps other than end gaps, or if and only if every convex set is an interval. $\mathbb{L}$ is called **complete**[3] if it is boundedly complete and it has both maximum and minimum. The order $(\mathbb{L}, <)$ is called **(order) dense** if between any two points there is a third distinct point. We say that $\mathbb{L}' \subseteq \mathbb{L}$ is a **dense suborder** of $\mathbb{L}$ if for every $a, b \in \mathbb{L}$ with $a < b$, either $a, b \in \mathbb{L}'$ or there is $c \in \mathbb{L}'$ such that $a < c < b$. If $\mathbb{L}$ is an infinite linear order, this implies that $\mathbb{L}'$ is dense in the order topology of $\mathbb{L}$. If $\mathbb{L}$ furthermore is order dense, then the converse

[2]This differs from part of the literature where, for example, the cofinality and coinitiality of linear orders with endpoints are set to ∞.

[3]In literature often the term complete linear order is used to refer to what we call here boundedly complete linear order. For this reason, in order to avoid confusion in this chapter, we always try to avoid using the term *complete* alone and write instead *complete with endpoints* as a reminder.

is true as well.

Remark 3.1.5 (Completion through Dedekind cuts). For any linear order $(\mathbb{L}, <)$ there is a complete order with endpoints $(\hat{\mathbb{L}}, <)$ that contains $\mathbb{L}$ as a dense suborder.
This is given by the set of all trivial and left cuts of $\mathbb{L}$, ordered by the relation $(A, B) \leq_{\mathbb{L}} (C, D)$ if and only if $A \subseteq C$, identifying each $x \in \mathbb{L}$ with the cut $((-\infty, x), [x, +\infty))$.

Lemma 3.1.6 ([120, VIII.C]). *Every open set U of a LOTS $(\mathbb{L}, <, \tau)$ has a convex open partition.*

Proof. Let U be open. Define an equivalence relation $\sim$ on U by setting $x \sim y$ if and only if there is an open interval $I \subseteq U$ such that $x, y \in I$. Then, it is immediate to verify that $\sim$ is an equivalence relation and that the equivalence classes $[x]_\sim$ are convex and open for every $x \in U$. Thus, $\{[x]_\sim \mid x \in U\}$ is as wanted. □

Remark 3.1.7. Let $(\mathbb{L}, <, \tau)$ be a linearly ordered topological space. An interval (s, t) is clopen if and only if s has a successor and t has a predecessor. Let $\langle s_i \mid i < \alpha\rangle$ and $\langle t_i \mid i < \beta\rangle$ be respectively a strictly increasing and a strictly decreasing family of distinct elements of $\mathbb{L}$, for α, β limit ordinals. The open set $U = \bigcup_{i<\alpha, j<\beta}(s_i, t_i)$ is clopen if and only if $\{s_i \mid i < \alpha\}$ has no infimum and $\{t_i \mid i < \beta\}$ has no supremum. A similar argument can be used for intervals and the union of intervals of other forms.

The previous remark has the following corollaries. Recall that a topological space is called **connected** if it contains no proper clopen subset.

Proposition 3.1.8. *Let $(\mathbb{L}, <, \tau)$ be a linearly order space. Then, the order topology is connected if and only if the order is boundedly complete and dense.*

Proof. First, if $\mathbb{L}$ has a (non-end) gap or two consecutive points, then it contains a proper clopen set by Remark 3.1.7.

Conversely, assume there is a clopen set U. Then, by Lemma 3.1.6 we may find a convex clopen partition $\mathcal{P}$ of X refining $\{U, X \setminus U\}$, and then the result follows from Remark 3.1.7. □

Notice that GO-topologies instead are never connected.

Remark 3.1.9. Given a linear order $(X, <)$, any GO-topology other than the order topology on X is not connected.

In fact, any interval of the form $(-\infty, b]$, $[a, +\infty)$ or $[a, b]$ that is open in a GO-topology is clopen, and by Corollary 3.1.2 any GO-topology either coincide with the order topology or contains an interval of one of the form above.

Recall that space is said zero-dimensional if it has a basis of clopen sets.

Corollary 3.1.10. *Let $(\mathbb{L}, <)$ be a linear order. Then the order topology is not zero-dimensional if and only if there is an open interval I such that $(I, <)$ is a dense boundedly complete linear order with at least two points.*

Proof. ($\Leftarrow$) Let $I = (a, b)$ be an open interval such that $(I, <)$ is dense and boundedly complete. Then it is connected by Proposition 3.1.8, and for every point $x \in (a, b)$ there is no clopen set U such that $x \in U \subsetneq (a, b)$.

($\Rightarrow$) Assume every interval is not dense or not boundedly complete. Consider $x \in (-\infty, t)$, we want to find $U \subseteq (-\infty, t)$ clopen such that $x \in U$. First, if $(x, t) = \emptyset$ then t is a successor of x and thus $(-\infty, x]$ is clopen. Otherwise, the interval (x, t) is non-empty, and thus it is either not dense or not boundedly complete. In the first case, if $y < z$ are two consecutive points, then we are done since $(-\infty, z)$ would be clopen. Suppose conversely (x, t) is dense but not boundedly complete, and we may find $\langle y_\alpha \mid \alpha < \gamma \rangle \subseteq (x, t)$ bounded in (x, t) that has no supremum. Then $U = \bigcup_{\alpha<\gamma}(-\infty, y_\alpha)$ is clopen and $U \subseteq (-\infty, t)$ as required. The same argument can be used for intervals of the form $(s, +\infty)$, and so also for intervals of the form $(s, t) = (-\infty, t) \cap (s, +\infty)$. □

The compactness properties of a GO-space can be described in terms of existence or non-existence of certain types of gaps.

Proposition 3.1.11 ([120, Theorem VIII.2])**.** *A GO-space $(\mathbb{L}, <, \tau)$ is compact if and only if it has no gaps nor pseudo-gaps.*

In particular, for LOTS it is enough to check the completeness of the order (since they do not contains pseudo-gaps).

Corollary 3.1.12. *A LOTS $(\mathbb{L}, <, \tau)$ is compact if and only if $(\mathbb{L}, <)$ is a complete linear order (with endpoints).*

[120, Theorem VIII.2] can be extended to κ-Lindelöf spaces (this may be folklore, but we give an explicit proof for the reader's convenience).

Proposition 3.1.13. *Let κ be a regular cardinal. A GO-space $(X, <, \tau)$ is κ-Lindelöf if and only if it has no (μ, ν)-gaps nor (μ, ν)-pseudo-gaps with $\max(\mu, \nu) \geq \kappa$, nor clopen convex partition of size $\geq \kappa$.*

Proof. First, it is clear that if (C, D) is a convex clopen partition and C contains a cofinal strictly increasing sequence $\langle y_i \mid i < \mu \rangle$ of length $\mu \geq \kappa$, then the family $\{D\} \cup \{(-\infty, y_i) \mid i < \mu\}$ is a cover of size μ without smaller refinement, and thus X is not κ-Lindelöf. The same argument works for the coinitiality of D. Finally, it is clear that if X is κ-Lindelöf then it can not contain a clopen convex partition of size $\geq \kappa$.

Conversely, suppose X has no gaps nor pseugo-gaps of type (μ, ν) or (ν, μ) with $\kappa \leq \mu < \infty$ and no clopen convex partition of size $\geq \kappa$.

Let $\mathcal{U}$ be an open cover of X. Define an equivalence relation on X by saying that $x \sim y$ if there is a family $\mathcal{U}'$ of convex open sets refining $\mathcal{U}$ and of size $|\mathcal{U}'| < \kappa$ such that $\bigcup \mathcal{U}'$ is a convex open neighborhood of both x and y. Notice that $\sim$ is symmetric, and it is reflexive by Lemma 3.1.6, and it is transitive since if $\mathcal{U}'$ witness $x \sim y$ and $\mathcal{U}''$ witness $y \sim z$, then $\mathcal{U}' \cup \mathcal{U}''$ witness $x \sim z$. Hence, $\sim$ is an equivalence relation. Furthermore, if $\mathcal{U}'$ witness $x \sim y$, then it witness also that $x \sim z$ for any $z \in \bigcup \mathcal{U}'$: thus each equivalence class is convex and open. Then, $\mathcal{P} = \{[x]_\sim \mid x \in X\}$ is a clopen convex partition of X. By assumption, $\mathcal{P}$ has size $|\mathcal{P}| < \kappa$. Also, every convex set $C \in P$ is clopen, and thus $\uparrow C$ and $\downarrow C$ are clopen as well. In particular, $(X \setminus (\uparrow C), \uparrow C)$ and $(\downarrow C, X \setminus (\downarrow C))$ are gaps or pseudo-gaps, which means that C must have coinitiality and cofinality $< \kappa$ (by assumption on the types of gap or pseudo-gap of X). For every $C \in P$, fix a decreasing sequence $\langle y_i \mid i < \alpha \rangle$ of length

$\alpha < \kappa$ coinitial in C and an increasing sequence $\langle z_i \mid i < \beta \rangle$ of length $\beta < \kappa$ cofinal in C. Choose a point $p \in C$ and let $\mathcal{A}_i$ witness that $y_i \sim p$ for every $i < \alpha$ and let $\mathcal{A}'_i$ witness that $z_i \sim p$ for every $i < \beta$. Then $\mathcal{A}_C = \bigcup_{i<\alpha} \mathcal{A}_i \cup \bigcup_{i<\beta} \mathcal{A}'_i$ is a refinement of $\mathcal{U}$ of size $< \kappa$ that covers C. Thus, $\mathcal{A} = \bigcup_{C \in P} \mathcal{A}_C$ is a refinement of $\mathcal{U}$ of size $< \kappa$ that covers X, as wanted. □

Notice that, unlike the countable case, here we need to require explicitly that X contains no clopen convex partition of size $\geq \kappa$. This is due mostly to one fact: ω is a weakly compact cardinal, while κ, in general, is not. When κ is not weakly compact, in fact, we can not remove this additional requirement, as for examples the κ-Cantor set ${}^{\kappa}2$ with lexicographic order and bounded topology is a GO-space that has no gaps nor pseudo-gaps of type (μ, ν) or (ν, μ) with $\kappa \leq \mu < \infty$, and yet it is not κ-Lindelöf (see Fact 2.1.4). The same remains true if we restrict the attention to LOTS, as $({}^{\kappa}2, \tau_b)$ is homeomorphic to $({}^{\kappa}\mathbb{Z}, \tau_b)$, which together with lexicographic order is a LOTS (Proposition 3.2.2).

However, for weakly compact cardinals we can reestablish the previous stronger version of the theorem.

Proposition 3.1.14. *Let κ be weakly compact. A GO-space $(X, <, \tau)$ is κ-Lindelöf if and only if it has no (μ, ν)-gaps nor (μ, ν)-pseudo-gaps with $\max(\mu, \nu) \geq \kappa$.*

Proof. By Proposition 3.1.13, it is enough to show that if X has no gaps nor pseudo-gaps of type (μ, ν) or (ν, μ) with $\kappa \geq \mu$, than it has no clopen convex partition of size $\geq \kappa$ either.

Suppose not, and let $\mathcal{P}$ be a convex clopen partition of X of size $|\mathcal{P}| = \mu \geq \kappa$. Let $\mathcal{P} = \{P_i \mid i < \mu\}$ be an enumeration of $\mathcal{P}$.

Notice that for every $P, Q \in \mathcal{P}$, since P and Q are convex and disjoint, then either $P < Q$ (i.e. $p < q$ for every $p \in P$ and $q \in Q$) or $Q < P$. Color each set $\{i, j\} \in [k]^2$ by red when $P_i < P_j$ holds if and only if $i < j$ holds, otherwise color $\{i, j\}$ by blue. Since κ is weakly compact, there is a set $H \subseteq \kappa$ of size $|H| = \kappa$ such that $[H]^2$ is monochromatic. Then, picking $p_i \in P_i$ for every $i \in H$ we get an infinite increasing or decreasing sequence of length κ (suppose increasing, and the other case is similar). Let $C = \bigcup_{i \in H}(-\infty, p_i)$: then $C = \bigcup\{P \in \mathcal{P} \mid P < p_i \text{ for some } i \in H\}$ is clopen, and thus (C, D) is a gap or a pseudo-gap with type (μ, ν) with $\mu \geq \kappa$, contradiction. □

We now study the relation between strong Choquet games and GO-spaces. We start by proving a very general result that allows us to obtain some interesting corollaries.

Theorem 3.1.15. *Let $(X, <, \tau)$ be a GO-space of weight $\leq \kappa$. Then X is SC_κ (resp., $f\mathrm{SC}_\kappa$) if and only if player II has a strategy winning every match of the strong (resp., fair) κ-Choquet game on (X, τ) where player I plays only open intervals*[4].

Proof. We prove the statement here for the strong κ-Choquet game. The proof for strong fair κ-Choquet games follows similarly.

[4]Notice that we are not allowing to play *the intersection of an open interval with all previous moves*, but we allow to play only open intervals that are already fully contained in the intersection of all previous moves. Notice also that II is allowed to play any τ-open set instead.

First, it is clear that if II has a winning strategy σ that wins every match of the strong κ-Choquet game, then this strategy wins in particular those matches where player I use only open intervals (recall that open intervals are τ-open sets by Proposition 3.1.1).

Conversely, suppose II has a strategy σ in the strong κ-Choquet game that wins the matches where player I use only open intervals.

We want to define a winning strategy $\tilde{\sigma}$ for player II that is suitable for every match of the strong κ-Choquet game on $(X, <, \tau')$.

First, by Corollary 3.1.2 (and preceding paragraph), for every convex set U and for every $x \in U$ we may find a interval $\tilde{I}$ such that $x \in \tilde{I} \subseteq U$, and furthermore

1. $\tilde{I} = [x, x]$ if and only if $U = \{x\}$;
2. $\tilde{I} = [x, b)$ if and only if x is the minimum of U;
3. $\tilde{I} = (a, x]$ if and only if x is the minimum of U;
4. otherwise $\tilde{I} = (a, b)$ is an open interval.

Given a convex set U, a point $x \in U$, and an open interval $I = (a, b)$, we call I an approximation of (U, x) if $\tilde{I} = I \cup \{x\}$ is defined as above (and $x \in \tilde{I} \subseteq U$).

Notice in particular that an interval I approximate $(I \cup \{x\}, x)$ if and only if $x \in \mathrm{cl}(I)$.

Notice that for every convex $Y \subseteq X$, for every convex $U \subseteq Y$ that is τ-open in Y, for every $x \in U$ and for every approximation I of (U, x), then the set $\{x\} \cup I$ is τ-open in Y. Indeed, if I is empty we are done since in this case $U = \{x\}$ by definition, and if $I = (a, b)$ is an open interval we are done since every GO-topology refines the order topology. Otherwise, I is the intersection of U with an open interval $(-\infty, b)$ or $(a, +\infty)$, and thus τ-open once again (in Y).

Now $(X, <, \tau')$ has a basis made of convex sets, by Proposition 3.1.1. Since the intersection of convex sets is again convex, by Remark 1.1.4 we may assume that player I always plays only convex sets. Then, by previous argument we may assume that player I plays only couples of the form $(I \cup \{x\}, x)$ for I an open interval with $x \in \mathrm{cl}(I)$: if not, replace the move (U, x) of player I with $(I \cup \{x\}, x)$ for I an approximation of (U, x).

Let $r = \langle I_\alpha \cup \{x_\alpha\}, x_\alpha \mid \alpha \leq \delta \rangle$ be a sequence of moves played by player I until a certain round δ, where I_α is an open interval and $x_\alpha \in \mathrm{cl}(I_\alpha)$. We say that r is:

- **good** if $x_\delta \in I_\delta$.
- **ok** if $x_\delta \notin I_\delta$ and there is $\beta < \delta$ such that $x_\beta \notin I_\beta$ and $x_\beta = x_\gamma \neq x_\delta$ for every γ with $\beta \leq \gamma < \delta$.
- **bad** otherwise.

Define also G_r, O_r and B_r to be, respectively, the sets of all $\alpha \leq \delta$ for which $r \restriction (\alpha + 1)$ is, respectively, good, ok or bad.

If r is good, define $\tilde{\sigma}(r) = \sigma(r \restriction G_r)$, where $r \restriction G_r = \langle I_\alpha, x_\alpha \mid \alpha \in G_r \rangle$.

If r is bad, we let II just copy the last move of I, i.e. we set $\tilde{\sigma}(r) = I_\delta \cup \{x_\delta\}$.

If finally r is ok, we associate to r a sequence r' with only open intervals as sets. Let $A_r = \{\alpha \in O_r \cup B_r \mid x_\beta \neq x_\alpha$ for cofinally many $\beta < \alpha\}$. (Recall that we say that $\{\beta\}$ is cofinal in $\alpha = \beta + 1$.) Let $\gamma = \mathrm{ot}(A_r)$ and let $A_r = \{\alpha(\epsilon)\}_{\epsilon<\gamma}$ be an increasing enumeration of A_r. Notice that by construction $\gamma = \gamma' + 2$ and $\alpha(\gamma' + 1) = \delta$ since r is ok. Define

$$r' = \langle I_{\alpha(\epsilon)}, x_{\alpha(\epsilon+1)} \mid \epsilon \leq \gamma' \rangle.$$

Notice that if $r \subseteq s$ are two partial matches that are ok, then the associated sequences r', s' also satisfies $r' \subseteq s'$. Also, r' is a legal sequence of moves of player I using only open intervals, since if $x_{\alpha(\epsilon+1)} \neq x_{\alpha(\epsilon)}$ and $x_{\alpha(\epsilon+1)} \in \{x_{\alpha(\epsilon)}\} \cup I_{\alpha(\epsilon)}$, then $x_{\alpha(\epsilon+1)} \in I_{\alpha(\epsilon)}$. Define $\tilde{\sigma}(r) = (I_\delta \cup \{x_\delta\}) \cap \sigma(r')$: it is a legal answer, since $x_\delta = x_{\alpha(\gamma'+1)} \in \sigma(r')$.

Now let $r = \langle I_\alpha, x_\alpha, V_\alpha \mid \alpha < \delta \rangle$ be a match of the strong κ-Choquet game up to a limit round $\delta \leq \kappa$ played by II accordingly to $\tilde{\sigma}$. Assume that $\bigcap_{\alpha<\beta} I_\alpha \neq \emptyset$ for every $\beta < \delta$. We claim $\bigcap_{\alpha<\delta} I_\alpha \neq \emptyset$.

First, if there are cofinally many rounds α such that $r \restriction (\alpha + 1)$ is good, then we are done since $\tilde{\sigma} = \sigma$ on those rounds, and σ is winning for II.

Second, suppose there is $\alpha < \delta$ such that $x_\beta = x_\alpha$ for every $\alpha \leq \beta < \delta$. we are done since $x_\alpha \in \bigcap_{\gamma<\delta} I_\gamma$.

Hence, suppose we are not in those cases: then there are cofinally many α such that $r \restriction (\alpha + 1)$ is ok. Indeed, there can not be cofianlly many $\alpha < \delta$ such that $r \restriction (\alpha + 1)$ is good (as otherwise we would be in the first case); and if there is $\alpha < \delta$ such that $r \restriction (\beta + 1)$ is bad for every for every $\alpha \leq \beta < \delta$, this would imply $x_\beta = x_\alpha$ for every $\alpha \leq \beta < \delta$, and we should be in the second case. Define as before $A_r = \{\alpha \in O_r \cup B_r \mid x_\beta \neq x_\alpha$ for cofinally many $\beta < \alpha\}$. Let also $\gamma = \mathrm{ot}(A)$ and let $A = \{\alpha(\epsilon)\}_{\epsilon<\gamma}$ be an increasing enumeration of A. Notice that γ is limit, since we assumed that there is no $\alpha < \delta$ such that $x_\beta = x_\alpha$ for every $\alpha \leq \beta < \delta$. Also, given $\epsilon < \gamma$ we have $\alpha(\epsilon) \in B_r$ if and only if ϵ is limit. Thus, for every $\epsilon < \gamma$ we have that $r \restriction (\alpha(\epsilon+1)+1)$ is ok and we may define $r'_{\epsilon+1} = \langle I_{\alpha(\epsilon')}, x_{\alpha(\epsilon'+1)} \mid \epsilon' \leq \epsilon \rangle$ as in the definition of the strategy $\tilde{\sigma}$ on ok sequences.

Then, by construction we have that $I_{\alpha(\epsilon+2)} \cup \{x_{\alpha(\epsilon+2)}\} \subseteq \sigma(r'_{\epsilon+1}) \subseteq I_{\alpha(\epsilon)}$ for every $\epsilon < \gamma$, and $r' = \bigcup_{\epsilon<\gamma} r'_{\epsilon+1}$ is a legal sequence of moves in the strong κ-Choquet game where player I plays only open intervals and II replies accordingly to σ. Therefore, $\bigcap_{\alpha<\delta} I_\alpha = \bigcap_{\epsilon<\gamma} \sigma(r'_{\epsilon+1}) \neq \emptyset$ since σ is winning. □

An interesting consequence is that in order to check whether a GO-space is strong (fair) κ-Choquet it is enough to check whether any weaker GO-topology is strong (fair) κ-Choquet. In particular this applies to the order topology, which is coarser than any GO-topology (Proposition 3.1.1).

Corollary 3.1.16. *Given a linear order* $(\mathbb{L}, <)$*, if a GO-topology* τ *is* $f\mathrm{SC}_\kappa$ *(resp.,* SC_κ*), then any other GO-topology of weight* $\leq \kappa$ *refining* τ *is* $f\mathrm{SC}_\kappa$ *(resp.,* SC_κ*).*

In particular, if the order topology is $f\mathrm{SC}_\kappa$ *(resp.,* SC_κ*), then any GO-topology of weight* $\leq \kappa$ *is* $f\mathrm{SC}_\kappa$ *(resp.,* SC_κ*).*

We can not expect to obtain the reverse implication in general, as for example for every linear order $(\mathbb{L}, <)$ (of size $\leq \kappa$), the discrete topology is a GO-topology that makes the space SC_κ, but there are linear orders (like $\mathbb{Q}$) for which the order topology is not SC_κ.

In Remark 1.1.4, we said that in the Choquet games we can always assume that the two players play only basic open sets intersected with all previous moves. Theorem 3.1.15 implies that for linearly ordered spaces and GO-spaces we may even ask that the sets played are not intersected with previous moves.

Corollary 3.1.17. *Let $(X, <, \tau)$ be a GO-space of weight $\leq \kappa$ with a basis $\mathcal{B}$. Then X is $f\mathrm{SC}_\kappa$ (resp., SC_κ) if and only if player* II *has a strategy in the strong (fair) κ-Choquet game where she plays only sets in $\mathcal{B}$ and that is winning for every match where* I *plays only sets in $\mathcal{B}$.*

Theorem 3.1.15 gives a nice characterization of Choquet games in term of gaps. Let us define the following game.

Definition 3.1.18. Given a non-empty set $\Omega_\kappa \subseteq \kappa^+$, the **$\Omega_\kappa$-gap game** on a linear order $(\mathbb{L}, <)$ is the game played by two players

I	x_0, y_0, z_0		x_1, y_1, z_1		$\dots$	$x_\alpha, y_\alpha, z_\alpha$		$\dots$
II		x'_0, z'_0		x'_0, z'_0	$\dots$		x'_α, z'_α	$\dots$

where at each round $\alpha < \kappa$, player I chooses points $x_\alpha, y_\alpha, z_\alpha$ in $\mathbb{L}$ that satisfies $x'_\epsilon \leq x_\alpha < y_\alpha < z_\alpha \leq z'_\epsilon$ for every $\epsilon < \alpha$, and then player II chooses $x'_\alpha, z'_\alpha \in \mathbb{L}$ such that $x_\alpha \leq x'_\alpha < y_\alpha < z'_\alpha \leq z_\alpha$. The game stops after κ-many rounds or when it is not possible anymore to play accordingly to the rules. Then, player I wins if the two sequences obtained $\langle (x_\alpha, z_\alpha) \mid \alpha < \delta \rangle$ form a gap of type (δ, δ) for some $\delta \in \Omega_\kappa$ (and player II otherwise).

Proposition 3.1.19. *A LOTS $(\mathbb{L}, <, \tau)$ is $f\mathrm{SC}_\kappa$ (resp., SC_κ) if and only if player* II *has a winning strategy in the Ω_κ-gap game for $\Omega_\kappa = \{\kappa\}$ (resp., $\Omega_\kappa = \kappa^+$) and $\mathbb{L}$ has weight $\leq \kappa$.*

Proof. Every open interval (s, t) (and every point $x \in (s, t)$) identify a couple $s < t$ (resp., a triple $s < x < t$) that is a legal move of player II (resp., I) in the Ω_κ-gap game, and vice-versa. Then, the result follows from Corollary 3.1.17. □

Combining this result and Corollary 3.1.16, we get the following.

Corollary 3.1.20. *Let $(X, <, \tau)$ be a GO-space of weight $\leq \kappa$. If $(X, <)$ has no (κ, κ)-gaps, then $(X, <, \tau)$ is strong fair κ-Choquet. If furthermore $(X, <)$ has no (δ, δ)-gap for any $\delta < \kappa$, then $(X, <, \tau)$ is also strong κ-Choquet.*

In particular, every boundedly complete linear order with any GO-topology is an SC_κ-space.

LOTS have nice separation properties.

First, every GO-space is normal (and even collectionwise normal, see [120, Theorem VIII.1]), and thus in particular Hausdorff and regular.

However, paracompactness of GO-spaces is a more difficult subject, since not every LOTS is paracompact (see e.g. ω_1 with order topology). The following is a characterization of the GO-spaces which have this property.

Proposition 3.1.21 ([120, Theorem VIII.4])**.** *A GO-space $(X, <, \tau)$ is paracompact if and only if every gap or pseudo-gap (C, D) has a closed discrete subspace $C' \subseteq C$ cofinal in C and a closed discrete subspace $D' \subseteq D$ cofinal in D.*

3.2 Lexicographic topologies and their properties

In this section, we study a particular class of LOTS that is closely related to the κ-Cantor and κ-Baire spaces with bounded topology.

Given a linear order $(\mathbb{L}, <)$ and given a pair of sequences $s, t \in {}^{\kappa}\mathbb{L}$ of length κ, define $d(x, y) = \min\{\alpha < \kappa \mid x(\alpha) \neq y(\alpha)\}$. The **lexicographic order** on ${}^{\kappa}\mathbb{L}$ is the linear order defined by $x <_{\text{lex}} y$ if $x(d(x, y)) <_{\mathbb{L}} y(d(x, y))$. We call **lexicographic topology** the order topology induced by the lexicographic order.

Remark 3.2.1. The bounded topology on ${}^{\kappa}\mathbb{L}$ is finer than the smallest κ-additive topology refining the lexicographic topology on ${}^{\kappa}\mathbb{L}$.

Indeed, the bounded topology is κ-additive, and for every $x, y, z \in {}^{\kappa}\mathbb{L}$ such that $y \in (x, z)$, if $\alpha = \max(d(x, y), d(y, z))$, then $y \in \mathbf{N}_{y \restriction (\alpha+1)} \subseteq (x, z)$.

In particular, if the bounded topology on ${}^{\kappa}\mathbb{L}$ is κ-perfect, then also the lexicographic topology on ${}^{\kappa}\mathbb{L}$ is.

Depending on the order, it may happen both that the bounded topology is exactly the smallest κ-additive topology refining the lexicographic topology on ${}^{\kappa}\mathbb{L}$, or that it is strictly finer. Similarly to gaps, we say that a point $l \in \mathbb{L}$ has **type** (μ, ν) if $(-\infty, l)$ has cofinality μ and $(l, +\infty)$ has coinitiality ν. A (μ, ν)**-point** is a point of type (μ, ν).

Proposition 3.2.2. *Let* $(\mathbb{L}, <)$ *be a linear order with at least two points. Then:*

1. *Suppose* $\mathbb{L}$ *has no maximum nor minimum. Then the lexicographic topology on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is* κ*-additive and coincide with the bounded topology.*

2. *Suppose* $\mathbb{L}$ *is dense, and if* $\mathbb{L}$ *has a minimum, then it has no* (μ, ν)*-point with* $\mu \geq \kappa$*, and if* $\mathbb{L}$ *has a maximum, then it has no* (μ, ν)*-point with* $\nu \geq \kappa$*. Then, the bounded topology on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is the smallest* κ*-additive topology refining the lexicographic one.*

3. *Suppose* $\mathbb{L}$ *is not dense, and it has coinitiality and cofinality* $< \kappa$*. Suppose also that if* $\mathbb{L}$ *has a minimum, then it has no* (μ, ν)*-point with* $\mu \geq \kappa$*, and if* $\mathbb{L}$ *has a maximum, then it has no* (μ, ν)*-point with* $\nu \geq \kappa$*. Then, the bounded topology on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is the smallest* κ*-additive topology refining the lexicographic one.*

4. *Otherwise, the bounded topology on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is strictly finer than the smallest* κ*-additive topology refining the lexicographic one.*

Proof. Recall that we denote with 0 and 1 respectively the minimum and maximum of $\mathbb{L}$, if $\mathbb{L}$ has any of the two.

Suppose first that $\mathbb{L}$ has no maximum nor minimum. Then, for every $s \in {}^{<\kappa}\mathbb{L}$ and $x \in \mathbf{N}_s$ there are points $y, z \in \mathbf{N}_s$ such that $x \in (y, z)_{\text{lex}} \subseteq \mathbf{N}_s$, thus $\mathbf{N}_s$ is open in the lexicographic topology. Since the lexicographic topology is coarser than the bounded topology, then 1 follows.

Next, we want to prove 2 and 3 together, showing that if either 2 or 3 holds then for every $s \in {}^{<\kappa}\mathbb{L}$ and $x \in \mathbf{N}_s$ we may find a set U such that $x \in U \subseteq \mathbf{N}_s$ and U is open in the smallest κ-additive topology refining the lexicographic one.

If $x \neq s^\frown 0^\kappa$ and $x \neq s^\frown 1^\kappa$, then as before we may find $y, z \in \mathbf{N}_s$ such that $x \in (y, z)_{\text{lex}} \subseteq \mathbf{N}_s$ and we are done.

Thus, assume $\mathbb{L}$ has a minimum 0 and $x = s^\frown 0^\kappa$. The case where $\mathbb{L}$ has a maximum and $x = s^\frown 1^\kappa$ follows symmetrically.

Let $z \in \mathbf{N}_s$ be such that $x <_{\text{lex}} z$ and let α be the smallest ordinal such that $x(\beta) = 0$ for every $\beta \geq \alpha$.

If α is limit, then there is a cofinal set $I \subseteq \alpha$ such that $x(\beta) \neq 0$ for every $\beta \in I$. Then, we have $x \in \bigcap_{\beta \in I}(x \restriction \beta^\frown 0^{(\kappa)}, z)_{\text{lex}} \subseteq \mathbf{N}_s$, as wanted.

If instead $\alpha = \beta + 1$, let $l = x(\beta) > 0$. If either 2 or 3 holds, then $(-\infty, l)$ has cofinality $< \kappa$. If there is an infinite cofinal sequence $(p_i)_{i<\gamma}$ of size $\gamma < \kappa$ in $(-\infty, l)_{\mathbb{L}}$, then $x \in \bigcap_{i<\gamma}(x \restriction \beta^\frown p_i^{(\kappa)}, z)_{\text{lex}} \subseteq \mathbf{N}_s$ as wanted. This is the only possible case if $\mathbb{L}$ is dense, thus this proves 2.

Otherwise, l must have a predecessor l'. If 3 holds, then $\mathbb{L}$ has cofinality $< \kappa$ and we may find a set $D \subseteq \mathbb{L}$ cofinal in $\mathbb{L}$ of size $< \kappa$ (possibly $D = \{1\}$ if $\mathbb{L}$ has a maximum). Then, $x \in \bigcap_{d \in D}(x \restriction \beta^\frown l'^\frown d^{(\kappa)}, z)_{\text{lex}} \subseteq \mathbf{N}_s$ as wanted.

In order to prove 4, suppose first $\mathbb{L}$ has minimum 0 and a (μ, ν)-point l with $\mu \geq \kappa$. Then, given $s = \langle l \rangle$ and $x = l^\frown 0^{(\kappa)}$, for every family of lexicographic-open intervals $\mathcal{A}$ of size $|\mathcal{A}| < \kappa$, if $x \in \bigcap \mathcal{A}$ then $\bigcap \mathcal{A} \nsubseteq \mathbf{N}_s$. Similarly, if $\mathbb{L}$ has a maximum and a (μ, ν)-point l with $\nu \geq \kappa$. Finally, suppose $\mathbb{L}$ is not dense and it has a minimum and cofinality $\geq \kappa$. Let $l' < l$ be two consecutive points. Then, $s = \langle l \rangle$ and $x = l^\frown 0^{(\kappa)}$ once again proves that for every family of lexicographic-open intervals $\mathcal{A}$ of size $|\mathcal{A}| < \kappa$, if $x \in \bigcap \mathcal{A}$ then $\bigcap \mathcal{A} \nsubseteq \mathbf{N}_s$. Similarly if $\mathbb{L}$ is not dense and it has a maximum and coinitiality $\geq \kappa$. □

Since the bounded topology on the space of sequences ${}^\kappa X$ on a set X with at least two points is always κ-perfect, we get the following.

Corollary 3.2.3. *If $(\mathbb{L}, <)$ is a linear order with at least two points, then the lexicographic space $({}^\kappa\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ is κ-perfect.*

A topological space (X, τ) is called **orderable** if it is homeomorphic to a LOTS, and **suborderable** if it is homeomorphic to a GO-space. Equivalently: X is orderable if and only if $(X, <, \tau)$ is a LOTS for some linear order $<$ on X, and suborderable if $(X, <, \tau)$ is a GO-space for some linear order $<$ on X.

Corollary 3.2.4. *For every X, the space ${}^\kappa X$ with bounded topology is orderable.*

If furthermore X is infinite, then there is an order $<$ on X such that the space $({}^\kappa X, <_{\text{lex}}, \tau_b)$ with lexicographic order and bounded topology is a LOTS.

Proof. First, if X is infinite, then let $<$ be any linear order on X such that $(X, <)$ has no minimum and no maximum. Then the result follows from Proposition 3.2.2. If X is finite, then ${}^\kappa X$ is (homeomorphic to) the κ-Cantor space, which is homeomorphic to ${}^\kappa\omega$, and the result follows again by the previous case. □

When $\mathbb{L}$ has a maximum or a minimum, the lexicographic topology has fewer properties than the bounded one.

Proposition 3.2.5. *Suppose $\mathbb{L}$ has a maximum or a minimum and contains at least two points. Then the lexicographic topology on ${}^\kappa\mathbb{L}$ is not ω_1-additive.*

Proof. Without loss of generality, suppose $\mathbb{L}$ has a minimum 0 and let $l > 0$ be another element of $\mathbb{L}$. Consider the increasing sequence $\langle x_n \mid n \leq \omega \rangle$ where $x_n =$

$l^{(n)} \frown 0^{(\kappa)}$ for every $n \leq \omega$, and let $x = l^{(\omega)} \frown 0^{(\kappa)}$. Then $\bigcap_{n<\omega}(x_n, \infty) = [x, +\infty)$, but $[x, +\infty)$ can not be expressed as union of intervals, since if $x \in (y, z)$, then there is $x_n \in (y, x) = (y, z) \setminus [x, +\infty)$ and so $(y, x) \not\subseteq [x, +\infty)$. □

However, every lexicographic space contains a dense subset where the lexicographic topology coincides with the bounded topology. This is the analogue of what happens in the classical case for the Baire space and the irrational numbers of $\mathbb{R}$. Given a linear order $(\mathbb{L}, <)$ and a subset $X \subseteq {}^{\kappa}\mathbb{L}$, define the **rationals of** X as the set

$$\mathbb{Q}(X) = \{s \frown l^{(\kappa)} \in X \mid s \in {}^{<\kappa}\mathbb{L}, l \text{ is an endpoint of } \mathbb{L}\},$$

and let the **irrationals of** X be the set $\mathbb{I}(X) = X \setminus \mathbb{Q}(X)$.

It is easy to see that for every linear order $(\mathbb{L}, <)$ of size $\leq \kappa$ and with at least two points and an endpoint, the rationals $\mathbb{Q}({}^{\kappa}\mathbb{L})$ are a dense $\mathcal{F}_\sigma$ subspace of ${}^{\kappa}\mathbb{L}$ of size κ (we assumed $\kappa^{<\kappa} = \kappa$). As in the classical case, the irrationals instead are homeomorphic to the κ-Baire space.

Proposition 3.2.6. *Suppose $(\mathbb{L}, <)$ is a linear order of size $2 \leq |\mathbb{L}| \leq \kappa$ with at least one end point. Then, the order topology on the irrationals $(\mathbb{I}({}^{\kappa}\mathbb{L}), <_{\text{lex}}, \tau_{\text{lex}})$ coincide with the topology inherited from ${}^{\kappa}\mathbb{L}$, and this space is a G^{κ}_{δ} subspace of ${}^{\kappa}\mathbb{L}$ homeomorphic to the κ-Baire space with bounded topology $({}^{\kappa}\kappa, \tau_b)$.*

Proof. First, we prove that the topology of $\mathbb{I}({}^{\kappa}\mathbb{L})$ inherited from $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ coincide with both the bounded topology on $\mathbb{I}({}^{\kappa}\mathbb{L})$ and with the order topology induced by the lexicographic order on it. Indeed, the order topology is coarser than the topology inherited from ${}^{\kappa}\mathbb{L}$, by Proposition 3.1.1, and this GO-topology is coarser than the bounded topology by Remark 3.2.1. Thus we just need to prove that for every $s \in {}^{<\kappa}\mathbb{L}$, the set $\mathbf{N}_s \cap \mathbb{I}({}^{\kappa}\mathbb{L})$ is open in the order topology. Let $\langle p_\alpha \mid \alpha < \gamma\rangle$ be coinitial in $\mathbb{L}$, and let $\langle q_\beta \mid \beta < \delta\rangle$ be cofinal in $\mathbb{L}$. Choose also $y \in \mathbb{I}({}^{\kappa}\mathbb{L})$. Then, we have

$$\mathbf{N}_s \cap \mathbb{I}({}^{\kappa}\mathbb{L}) = \bigcup\{(s \frown (p_\alpha)^{(\mu)} \frown y, s \frown (q_\beta)^{(\nu)} \frown y) \cap \mathbb{I}({}^{\kappa}\mathbb{L}) \mid \alpha < \gamma, \beta < \delta, \mu, \nu < \kappa\}.$$

Thus, $\mathbb{I}({}^{\kappa}\mathbb{L})$ is κ-additive. Also, it is easy to see that $\mathbb{I}({}^{\kappa}\mathbb{L})$ is SC_κ. In fact, the set $\mathbf{N}_s \cap \mathbb{I}({}^{\kappa}\mathbb{L}) \neq \emptyset$ is non-empty for every $s \in {}^{<\kappa}\mathbb{L}$. Thus, it is enough that at each round α, if player I plays a couple (U, x), then player II choose an $s \in {}^{<\kappa}\mathbb{L}$ such that $x \in \mathbf{N}_s \cap \mathbb{I}({}^{\kappa}\mathbb{L}) \subseteq U$ and plays $\mathbf{N}_{s \frown \langle l_1, l_2\rangle} \cap \mathbb{I}({}^{\kappa}\mathbb{L})$ for l_1, l_2 two distinct points of $\mathbb{L}$ to ensure that the sequence created is not constant. Finally, given a subset $X \subseteq \mathbb{I}({}^{\kappa}\mathbb{L})$ with non-empty interior, then we may find $s \in {}^{<\kappa}\mathbb{L}$ such that $\mathbf{N}_s \cap \mathbb{I}({}^{\kappa}\mathbb{L}) \subseteq X$. Then, if 0 is the minimum of $\mathbb{L}$ and a is another point of $\mathbb{L}$, the cover $\mathcal{U} = \{(-\infty, s \frown 0^{(\kappa)}) \cap X\} \cup \{(s \frown 0^{(\alpha)} \frown a^{(\kappa)}, +\infty) \cap X \mid \alpha < \kappa\}$ has no subcover of size $< \kappa$. An analogue construction can be done if $\mathbb{L}$ has a maximum 1. Thus, every κ-Lindelöf subset of $\mathbb{I}({}^{\kappa}\mathbb{L})$ has empty interior, and the result follows from Theorem 1.2.10. □

Among all GO-topologies, some play a special role as they appear naturally in many situations. The **lower-limit topology** on a linear order $(\mathbb{L}, <)$ is the GO-topology generated by the intervals of the form $[a, b)$ for $a \in \mathbb{L}$ and $b \in \mathbb{L} \cup \{+\infty\}$. The **upper-limit topology** is the GO-topology generated by the intervals of the form $(a, b]$ for $a \in \mathbb{L} \cup \{-\infty\}$ and $b \in \mathbb{L}$.

Proposition 3.2.7. *Let* $(\mathbb{L}, <)$ *be a linear order (of size* $\leq \kappa$*).*

- *Suppose* $\mathbb{L}$ *has no maximum nor minimum. Then, every GO-topology of weight* $\leq \kappa$ *on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is* SC_{κ}*.*
- *Suppose* $\mathbb{L}$ *has a maximum but no minimum. Then, every GO-topology of weight* $\leq \kappa$ *on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is* $f\mathrm{SC}_{\kappa}$ *(resp.,* SC_{κ}*) if and only if the lower-limit topology on* $(\mathbb{L}, <)$ *is* $f\mathrm{SC}_{\kappa}$ *(resp.,* SC_{κ}*).*
- *Suppose* $\mathbb{L}$ *has a minimum but no maximum. Then, every GO-topology of weight* $\leq \kappa$ *on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is* $f\mathrm{SC}_{\kappa}$ *(resp.,* SC_{κ}*) if and only if the upper-limit topology on* $(\mathbb{L}, <)$ *is* $f\mathrm{SC}_{\kappa}$ *(resp.,* SC_{κ}*).*
- *Suppose* $\mathbb{L}$ *has both maximum and minimum. Then, every GO-topology of weight* $\leq \kappa$ *on* $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ *is* $f\mathrm{SC}_{\kappa}$ *(resp.,* SC_{κ}*) if and only if both the upper-limit and lower-limit topologies on* $(\mathbb{L}, <)$ *are* $f\mathrm{SC}_{\kappa}$ *(resp.,* SC_{κ}*).*

Proof. First, by Corollary 3.1.16 we may work using only the lexicographic topology on $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$.

If $\mathbb{L}$ has size 1 then the result is trivial, so assume $\mathbb{L}$ has size at least 2. Also, if $\mathbb{L}$ has no maximum nor minimum, then the result follows already from Proposition 3.2.2 since the bounded topology is always SC_{κ}.

So suppose $\mathbb{L}$ has a minimum 0 but no maximum. The other cases follow similarly.

Assume first that $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ is $f\mathrm{SC}_{\kappa}$, and let σ be a winning strategy for player II in this game. By Remark 1.1.4, we may assume that σ answers only with open intervals of $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$. We want to find a winning strategy $\tilde{\sigma}$ for the strong fair κ-Choquet game on $\mathbb{L}$ with upper-limit topology. By Theorem 3.1.15, it is enough if $\tilde{\sigma}$ wins every game where player I plays only open intervals of the form (a, b). So let $r = \langle (a_i, b_i), c_i \mid i \leq \gamma \rangle$ be a legal sequence of moves for player I on the strong κ-Choquet game on $\mathbb{L}$. For every $i \leq \gamma$, define $x_i = c_i {}^\frown 0^{(\kappa)}$ and $y_i = a_i {}^\frown 0^{(\kappa)}$ and $z_i = b_i {}^\frown 0^{(\kappa)}$. Then, we have $x_i \in (y_i, z_i)$ in ${}^{\kappa}\mathbb{L}$, and $r' = \langle (y_i, z_i), x_i \mid i \leq \gamma \rangle$ is a legal sequence of moves of player I in the strong κ-Choquet game on ${}^{\kappa}\mathbb{L}$. So let $(y', z') = \sigma(r')$. Let $a' = y'(0)$ and let $b' = z'(0)$. Notice that by definition of x_i we must have $a' < c_\gamma \leq b'$. Thus, $(a', b']$ is a legal answer for II on the strong κ-Choquet game on $\mathbb{L}$ and we may define $\tilde{\sigma}(r) = (a', b']$. Then, it is easy to see that for every sequence of moves $r = \langle (a_i, b_i), c_i \mid i < \delta \rangle$ of player I played accordingly to $\tilde{\sigma}$, we have that $\bigcap_{\alpha<\delta} \tilde{\sigma}(r \restriction \alpha)$ is empty if and only if $\bigcap_{\alpha<\delta} \sigma(r' \restriction \alpha)$ is empty, and thus $\tilde{\sigma}$ is winning in the strong fair κ-Choquet game on $\mathbb{L}$ with upper-limit topology. The same argument shows that if σ is winning also in the strong κ-Choquet game on ${}^{\kappa}\mathbb{L}$, then $\tilde{\sigma}$ is winning in the strong κ-Choquet game on $\mathbb{L}$ with upper-limit topology.

Conversely, assume player II has a strategy $\tilde{\sigma}$ that is winning in the strong fair κ-Choquet game on $\mathbb{L}$ with upper-limit topology. We may assume $\tilde{\sigma}$ answer only with intervals of the form $(a, b]$ for some $a, b \in \mathbb{L}$, by Remark 1.1.4 and Corollary 3.1.2. By Theorem 3.1.15, it is enough to show that II has a strategy in the strong fair κ-Choquet game on ${}^{\kappa}\mathbb{L}$ that wins every match where player I plays only open intervals. Suppose $r = \langle (y_i, z_i), x_i \mid i \leq \gamma \rangle$ is a legal sequences of moves of player I in the strong fair κ-Choquet game on ${}^{\kappa}\mathbb{L}$ up to a certain round $\gamma < \kappa$ with $x_i \in (y_i, z_i)$ for every $i \leq \gamma$. Let $\nu > \max(d(x_\gamma, z_\gamma), \gamma)$ and choose $l_2 \in \mathbb{L}$ such that $l_2 > x_\gamma(\nu)$.

Let α be the minimum ordinal such that $x_\gamma(\beta) = 0$ for every β with $\alpha \leq \beta < \kappa$, where $\min(\emptyset) = \infty$. Notice that $d(x_\gamma, y_\gamma) < \alpha$. First, if $\alpha \neq d(x_\gamma, y_\gamma)+1$, choose β such that $d(x_\gamma, y_\gamma) < \beta < \alpha$ and $x_\gamma(\beta) \neq 0$ and define $\sigma(r) = (x_\gamma \restriction \beta ^\frown 0^{(\kappa)}, x_\gamma \restriction \nu ^\frown l_1^{(\kappa)})$. Otherwise, let $d(x_\gamma, y_\gamma) = \delta = \alpha - 1$ and $A = \{\beta \leq \gamma \mid d(y_\beta, x_\gamma) = \delta\}$. Notice that A is a final segment of γ. For every $\beta \in A$, define $a_\beta = y_\beta(\delta)$, and let $b_\beta = x_\beta(\delta)$. Then, $r' = \langle (a_\beta, b_\beta], b_\beta \mid \beta \in A\rangle$ is a legal sequence of moves of player I in the strong fair κ-Choquet game on $\mathbb{L}$. Let $(c, d] = \tilde{\sigma}(r')$ be the answer of the strategy $\tilde{\sigma}$ to r'. Let also $l_1 \in \mathbb{L}$ be such that $y_\gamma <_{\text{lex}} x_\gamma \restriction \delta ^\frown c ^\frown l_2^{(\kappa)}$. Define $\sigma(r) = (x_\gamma \restriction \delta ^\frown c ^\frown l_2^{(\kappa)}, x_\gamma \restriction \nu ^\frown l_1^{(\kappa)})$.

Now suppose $r = \langle (y_i, z_i), x_i \mid i < \gamma\rangle$ is a legal sequences of moves of player I in the in the strong fair κ-Choquet game on ${}^\kappa\mathbb{L}$ up to a certain limit round γ with $x_i \in (y_i, z_i)$ for every $i \leq \gamma$. First, suppose there is a cofinal subset $I \subseteq \gamma$ such that $d(x_\alpha, y_\alpha) < d(x_\beta, y_\beta)$ for every $\alpha, \beta \in I$ with $\alpha < \beta$. Let $s_\alpha = x_\alpha \restriction d(x_\alpha, y_\alpha)$, and let $s = \bigcup_{\alpha \in I} s_\alpha$. Let also $w = s ^\frown 0^{(\kappa)}$ if $\text{lh}(s) < \kappa$, or $w = s$ if $\text{lh}(s) = \kappa$. Then, we must have $y_\alpha <_{\text{lex}} w \leq_{\text{lex}} x_\alpha <_{\text{lex}} z_\alpha$ for every $\alpha \in I$, and thus $\bigcap_{i<\gamma}(y_i, z_i) \neq \emptyset$. Otherwise, let $\epsilon < \gamma$ be the minimum ordinal such that $d(x_\alpha, y_\alpha) = d(x_\beta, y_\beta) = \delta$ for every $\alpha < \beta < \gamma$ with $\epsilon \leq \alpha$. This implies $x_\alpha = x_\epsilon \restriction (\delta + 1) ^\frown 0^\kappa$ for every $\alpha \geq \epsilon$, by definition of σ. Define as above $A = [\epsilon, \gamma) = \{\beta \leq \gamma \mid d(x_\beta, y_\gamma) = \delta\}$ and $a_\beta = y_\beta(\delta)$ and $b_\beta = x_\beta(\delta)$ for every $\beta \in A$: then, $r' = \langle (a_\beta, b_\beta], b_\beta \mid \beta \in A\rangle$ is compatible with $\tilde{\sigma}$ by definition of σ. If $\gamma = \kappa$, then r' also has length κ and thus $\bigcap_{\beta \in A}(a_\beta, b_\beta] \neq \emptyset$ and we may find $c \in \bigcap_{\beta \in A}(a_\beta, b_\beta]$. Then, we have $y_\alpha <_{\text{lex}} x_\epsilon \restriction \epsilon ^\frown c ^\frown 0^{(\kappa)} \leq_{\text{lex}} x_\alpha <_{\text{lex}} z_\alpha$ and thus $\bigcap_{i<\gamma}(y_i, z_i) \neq \emptyset$. The same proof gives the wanted result for the strong κ-Choquet game. □

Thanks to Corollary 3.1.16, instead of checking the upper-limit and lower-limit topologies we can check only that the order topology is $f\text{SC}_\kappa$ or SC_κ.

Corollary 3.2.8. *If a LOTS $(\mathbb{L}, <, \tau)$ is $f\text{SC}_\kappa$ (resp., SC_κ), then every GO-space on the lexicographic order $({}^\kappa\mathbb{L}, <_{\text{lex}}, \tau')$ is $f\text{SC}_\kappa$ (resp., SC_κ).*

Finally, we collect some properties of the lexicographic orders. These facts are probably folklore, but we reprove them here for the reader's convenience.

Lemma 3.2.9. *Let $(\mathbb{L}, <)$ be a linear order. Then, the lexicographic order ${}^\kappa\mathbb{L}$ is dense if and only if $\mathbb{L}$ is dense or $\mathbb{L}$ has no maximum or no minimum.*

Proof. Assume $\mathbb{L}$ is dense or it has no minimum or it has no maximum. Choose any pair of sequences $x, y \in {}^\kappa\mathbb{L}$. First, if $\mathbb{L}$ is dense, then there is an element $l \in \mathbb{L}$ such that $x(d(x, y)) < l < y(d(x, y))$, and thus $x <_{\text{lex}} (x \restriction d(x, y)) ^\frown l^{(\kappa)} <_{\text{lex}} y$. Otherwise, let $\alpha = d(x, y) + 1$. Suppose $\mathbb{L}$ has no maximum. Then, we have for example $x <_{\text{lex}} (x \restriction \alpha) ^\frown l^{(\kappa)} <_{\text{lex}} y$ for some $l > x(\alpha)$. Similarly, if $\mathbb{L}$ has no minimum we have for example $x <_{\text{lex}} (y \restriction \alpha) ^\frown l^{(\kappa)} <_{\text{lex}} y$ for some $l < y(\alpha)$.

Conversely, if $\mathbb{L}$ has minimum 0 and maximum 1 and two consecutive points $l_1 < l_2$, then there is no $z \in {}^\kappa\mathbb{L}$ such that $l_1 ^\frown 1^{(\kappa)} <_{\text{lex}} z <_{\text{lex}} l_2 ^\frown 0^{(\kappa)}$. □

For GO-spaces and LOTS, we have a homogeneous property similar to the one obtained for the bounded topology. Indeed, given a linear order $\mathbb{L}$, for every $s \in {}^{<\kappa}\mathbb{L}$ the set $\mathbf{N}_s$ with order and topology inherited from $({}^\kappa\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ is isomorphic to the whole $({}^\kappa\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$. Since the bounded topology refines the lexicographic one, we may find sets of the form $\mathbf{N}_s$ in every non-empty open set.

Remark 3.2.10. Let $(\mathbb{L}, <)$ be a linear order. Then, every non-empty open set U of the lexicographic space $({}^{\kappa}\mathbb{L}, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ contains a subspace and suborder $V \subseteq U$ isomorphic to the whole space $({}^{\kappa}\mathbb{L}, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$.

The gaps in the lexicographic topology depend strongly on the gaps and cofinality of the starting order.

Lemma 3.2.11. *Let $\mathbb{L}$ be a linear order and let $I \subseteq {}^{\kappa}\mathbb{L}$ be a non-empty open interval in the lexicographic order. Then, I contains a (γ, δ)-gap for some cardinals δ and γ if and only if one of the following conditions hold:*

1. *$\mathbb{L}$ has a (γ, δ)-gap.*
2. *$\mathbb{L}$ has cofinality γ and $\delta < \kappa$.*
3. *$\mathbb{L}$ has coinitiality δ and $\gamma < \kappa$.*

We prove this result in a more technical lemma that helps us also in other situations.

Lemma 3.2.12. *Let $\mathbb{L}$ be a linear order, let also $\langle p_i \mid i < \mu\rangle$ be an increasing sequence and $\langle q_i \mid i < \nu\rangle$ be a decreasing sequences in $\mathbb{L}$ such that $p_\alpha < q_\beta$ for every $\alpha < \mu$ and $\beta < \nu$. Let $s \in {}^{<\kappa}\mathbb{L}$ and for every $\alpha < \mathrm{lh}(s)$ choose $l_\alpha \in \mathbb{L}$.*

1. *If $\langle p_i \mid i < \mu\rangle$ and $\langle q_i \mid i < \nu\rangle$ form a gaps in $\mathbb{L}$, then $C = \langle s^\frown(p_i)^{(\kappa)} \mid i < \mu\rangle$ and $D = \langle s^\frown(q_i)^{(\kappa)} \mid i < \nu\rangle$ form a (μ, ν)-gap in ${}^{\kappa}\mathbb{L}$.*
2. *If $\langle p_i \mid i < \mu\rangle$ is cofinal in $\mathbb{L}$, $\mathrm{lh}(s)$ is limit and $s(\alpha) < l_\alpha$ for every $\alpha < \mathrm{lh}(s)$, then $C = \langle s^\frown(p_i)^{(\kappa)} \mid i < \mu\rangle$ and $D = \langle s \upharpoonright \alpha^\frown(l_\alpha)^{(\kappa)} \mid \alpha < \mathrm{lh}(s)\rangle$ form a $(\mu, \mathrm{cof}(\mathrm{lh}(s)))$-gap in ${}^{\kappa}\mathbb{L}$.*
3. *If $\langle q_i \mid i < \nu\rangle$ is coinitial in $\mathbb{L}$, $\mathrm{lh}(s)$ is limit and $s(\alpha) > l_\alpha$ for every $\alpha < \mathrm{lh}(s)$, then $C = \langle s \upharpoonright \alpha^\frown(l_\alpha)^{(\kappa)} \mid \alpha < \mathrm{lh}(s)\rangle$ and $D = \langle s^\frown(q_i)^{(\kappa)} \mid i < \nu\rangle$ form a $(\mathrm{cof}(\mathrm{lh}(s)), \nu)$-gap in ${}^{\kappa}\mathbb{L}$.*

Furthermore, every gap of ${}^{<\kappa}\mathbb{L}$ is generated this way for some choices of $\langle p_i \mid i < \mu\rangle$, $\langle q_i \mid i < \nu\rangle$, s and $\langle l_\alpha \mid \alpha < \mathrm{lh}(s)\rangle$.

Proof. Let $C = \langle s^\frown(p_i)^{(\kappa)} \mid i < \mu\rangle$ and $D = \langle s^\frown(q_i)^{(\kappa)} \mid i < \nu\rangle$. If there is $x \in {}^{\kappa}\mathbb{L}$ such that $c \leq x \leq d$ for every $c \in C$ and $d \in D$, then $p_\alpha < x(\mathrm{lh}(s)) \leq q_\beta$ for every $\alpha < \mu$ and $\beta < \nu$.

Let instead $C = \langle s^\frown(p_i)^{(\kappa)} \mid i < \mu\rangle$ and $D = \langle s \upharpoonright \alpha^\frown(l_\alpha)^{(\kappa)} \mid \alpha < \mathrm{lh}(s)\rangle$. Consider $x \in {}^{\kappa}\mathbb{L}$ such that $c \leq x \leq d$ for every $c \in C$ and $d \in D$. Then, $s \subseteq x$, as otherwise $s \upharpoonright (d(s, x) + 1)^\frown(l_\alpha)^{(\kappa)} < x$. But then $p_\alpha < x(\mathrm{lh}(s))$ for every $\alpha < \mu$ and thus $\langle p_i \mid i < \mu\rangle$ is not cofinal. The last case follows symmetrically.

Conversely, let $\langle y_\alpha \mid \alpha < \gamma\rangle$ and $\langle z_\beta \mid \beta < \delta\rangle$ be respectively an increasing and a decreasing sequence in ${}^{\kappa}\mathbb{L}$ that form a gap (possibly an end gap) for some (regular) cardinals γ, δ. Let $\mu = \sup\{d(y_\alpha, z_\beta) \mid \alpha < \gamma, \beta < \delta\}$ (where we assume $\sup(\emptyset) = 0$). If $\mu = 0$ let $s = \emptyset$, and otherwise let $s = \bigcup_{\alpha<\mu} s_\alpha$, where the element $s_\alpha = y_\beta \upharpoonright \alpha = z_{\beta'} \upharpoonright \alpha$ witness that there are $\beta < \gamma$ and $\beta' < \delta$ such that $d(y_\beta, z_{\beta'}) > \alpha$ (since $\langle y_\alpha \mid \alpha < \gamma\rangle$ is increasing and $\langle z_\beta \mid \beta < \delta\rangle$ is decreasing, s_α is well-defined and does not depend on the choice of β and β'). Notice that $\mu < \kappa$, as otherwise $s \in {}^{\kappa}\mathbb{L}$ and we would have $y_\alpha <_{\text{lex}} s <_{\text{lex}} z_\beta$ for every $\alpha < \gamma$ and $\beta < \delta$. Let

$$Y = \{y_\beta(\mu) \mid s \subseteq y_\beta, \beta < \gamma\}, \qquad Z = \{z_\beta(\mu) \mid s \subseteq z_\beta, \beta < \delta\}$$

Then, (Y, Z) must form a gap in $\mathbb{L}$, as otherwise if $Y < l < Z$ we would have $y_\alpha <_{\text{lex}} s^\frown l^{(\kappa)} <_{\text{lex}} z_\beta$ for every $\alpha < \gamma$ and $\beta < \delta$. Then, it is easy to see that there are choices of sequences $\langle p_i \mid i < \mu\rangle$ and $\langle q_i \mid i < \nu\rangle$ and $\langle l_\alpha \mid \alpha < \text{lh}(s)\rangle$ in $\mathbb{L}$ such that the couple of sets (C, D) defined as in point 1 (if (Y, Z) is not an end gap), point 2 (if $Z = \emptyset$) or point 3 (if $Y = \emptyset$) form the same gap of $\langle y_\alpha \mid \alpha < \gamma\rangle$ and $\langle z_\beta \mid \beta < \delta\rangle$. □

Notice that Lemma 3.2.11, we have that the lexicographic order $({}^{\kappa}\mathbb{L}, <_{\text{lex}})$ is complete if and only if it is boundedly complete if and only if the starting linear order $(\mathbb{L}, <)$ is complete and has endpoints. Thus, thanks to Corollary 3.1.12 we get the following.

Corollary 3.2.13. *Let $\mathbb{L}$ be a linear order. Then, $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau)$ is compact if and only if $\mathbb{L}$ is a complete linear order (with endpoints).*

Similarly, thanks to Proposition 3.1.13, we get the following. Recall that a linear order has endpoints if and only if it has no $(\gamma, 0)$-gap and no $(0, \gamma)$-gap for some infinite γ, and ω is weakly compact, so for $\kappa = \omega$ we recover the previous statement.

Corollary 3.2.14. *Let $\mathbb{L}$ be a linear order and let ν be a regular cardinal.*

- *If $\nu < \kappa$ and $\mathbb{L}$ has not both endpoints, then $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ is not ν-Lindelöf.*
- *If there are cardinals γ and δ with $\max(\gamma, \delta) \geq \nu$ such that $\mathbb{L}$ contains a (δ, γ)-gap, then $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ is not ν-Lindelöf.*
- *If κ is weakly compact and $\mathbb{L}$ contains no (δ, γ)-gap with $\max(\gamma, \delta) \geq \kappa$, then $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ is κ-Lindelöf.*

Proof. First, if $\nu < \kappa$ and $\mathbb{L}$ has no maximum or no minimum, then ${}^{\kappa}\mathbb{L}$ contains a (γ, δ)-gap or (δ, γ)-gap for any regular cardinal $\delta < \kappa$ by Lemma 3.2.11, thus it can not be ν-Lindelöf by Proposition 3.1.13.

Similarly, if there are cardinals γ and δ with $\max(\gamma, \delta) \geq \nu$ such that $\mathbb{L}$ contains a (δ, γ)-gap, the result follows from Lemma 3.2.11 and Proposition 3.1.13.

If instead κ is weakly compact and $\mathbb{L}$ contains no (δ, γ)-gap nor (γ, δ)-gap for (regular) cardinals γ and δ with $\delta \geq \kappa$, then the result follows from Proposition 3.1.14 and Lemma 3.2.11. □

Thanks to Proposition 3.1.8 and Corollary 3.1.10, if we additional require that $\mathbb{L}$ is also dense, we can characterize when a lexicographic space is connected.

Corollary 3.2.15. *Let $(\mathbb{L}, <)$ be a linear order. Then, the following are equivalent:*

1. *$(\mathbb{L}, <)$ is a complete dense linear order with endpoints.*
2. *$({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau)$ is compact and $(\mathbb{L}, <)$ is dense.*
3. *$({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau)$ is connected.*

4. $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau)$ *is not zero-dimensional.*

Finally, we characterize when lexicographic spaces are paracompact.

Proposition 3.2.16. *Given a linear order* $(\mathbb{L}, <)$*, the space* $({}^{\kappa}\mathbb{L}, <_{\text{lex}}, \tau_{\text{lex}})$ *is paracompact if and only if one of the following holds:*

1. $\mathbb{L}$ *has no endpoint.*

2. $\mathbb{L}$ *has (a minimum but) no maximum and for every gap* (C, D) *there is a cofinal closed discrete subset* $C' \subseteq C$.

3. $\mathbb{L}$ *has (a maximum but) no minimum and for every gap* (C, D) *there is a coinitial closed discrete subset* $D' \subseteq D$.

4. $\mathbb{L}$ *(has both endpoints, but it) is paracompact.*

Proof. We prove the statement for the case where $\mathbb{L}$ has a minimum 0 but no maximum, and the other cases follow from a similar argument.

Assume first that for every gap (X, Y) of $\mathbb{L}$ there is a discrete closed cofinal subset $X' \subseteq X$. We want to show that for every gap (C, D) of ${}^{\kappa}\mathbb{L}$ there are a closed discrete subspace $C' \subseteq C$ cofinal in C, and a closed discrete subspace $D' \subseteq D$ cofinal in D: then ${}^{\kappa}\mathbb{L}$ is paracompact by Proposition 3.1.21.

By Lemma 3.2.12, every gap of ${}^{\kappa}\mathbb{L}$ is generated by a gap of $\mathbb{L}$. Let $\langle p_i \mid i < \mu\rangle$ and $\langle q_i \mid i < \nu\rangle$ form a gap in $\mathbb{L}$. Then, $\langle p_i \mid i < \mu\rangle$ is non-empty (since $\mathbb{L}$ has a minimum) and we may assume that $\langle p_i \mid i < \mu\rangle$ is closed and discrete by assumption on $\mathbb{L}$.

First, suppose $\langle q_i \mid i < \nu\rangle$ is non-empty and $C = \langle s^\frown (p_i)^{(\kappa)} \mid i < \mu\rangle$ and $D = \langle s^\frown (q_i)^{(\kappa)} \mid i < \nu\rangle$ be a gap of ${}^{\kappa}\mathbb{L}$ for some $s \in {}^{<\kappa}\mathbb{L}$. For every $l \in \mathbb{L}$, let (a_l, b_l) be an interval in $\mathbb{L}$ that contains at most one point of $\langle p_i \mid i < \mu\rangle$. Then, the intervals $(s^\frown (a_l)^{(\kappa)}, s^\frown (b_l)^{(\kappa)})$ forms an open covering of ${}^{\kappa}\mathbb{L}$ and each of them contains at most one point of C, proving that C is closed and discrete. For every $i < \nu$, let instead $l_i \in \mathbb{L}$ be such that $q_i < l$: we can find such an element since $\mathbb{L}$ has no maximum. Then, the intervals $(s^\frown (q_{i+1})^{(\kappa)}, s^\frown q_i{}^\frown (l_i)^{(\kappa)})$ form an open covering of the clopen set $\uparrow D = \{x \in {}^{\kappa}\mathbb{L} \mid s^\frown (q_i)^{(\kappa)} \leq x \text{ for some } i < \mu\}$ and each of them contains at most one element of D, proving that D is closed and discrete.

Second, suppose instead $\langle q_i \mid i < \nu\rangle = \emptyset$ and $\langle p_i \mid i < \mu\rangle$ is cofinal in $\mathbb{L}$. Let $C = \langle s^\frown (p_i)^{(\kappa)} \mid i < \mu\rangle$ and $D = \langle s \restriction \alpha^\frown (l_\alpha)^{(\kappa)} \mid \alpha < \text{lh}(s)\rangle$ be a gap of ${}^{\kappa}\mathbb{L}$ for some $s \in {}^{<\kappa}\mathbb{L}$ of limit length and $\langle l_\alpha \mid \alpha < \text{lh}(s)\rangle$ such that $s(\alpha) < l_\alpha$ for every $\alpha < \text{lh}(s)$. Then, the same proof of the previous case shows that C is closed and discrete. For every $\alpha < \text{lh}(s)$, let $l'_\alpha \in \mathbb{L}$ be such that $l_\alpha < l'_\alpha$. Then, the intervals $(s \restriction (\alpha + 1)^\frown (l_{\alpha+1})^{(\kappa)}, s \restriction \alpha^\frown (l'_\alpha)^{(\kappa)})$ shows that D is closed and discrete.

To prove the converse, suppose that $\mathbb{L}$ has a gap (C, D) for which there are no closed discrete subsets $C' \subseteq C$. Let $\langle p_i \mid i < \mu\rangle$ and $\langle q_i \mid i < \nu\rangle$ be two sequences identifying (C, D). We claim that the associated gap $(\tilde{C}, \tilde{D})$ generated by the pair $\langle (p_i)^{(\kappa)} \mid i < \mu\rangle$ and $\langle (q_i)^{(\kappa)} \mid i < \nu\rangle$ of ${}^{\kappa}\mathbb{L}$ proves that ${}^{\kappa}\mathbb{L}$ is not paracompact as well (using Proposition 3.2.16). So let $A \subseteq \tilde{C}$ be cofinal, we want to prove that A is not closed or not discrete. Let $X = \{x(0) \mid x \in A\}$. Without loss of generality (passing to a subset of A and X if necessary) we may assume that X is a well-ordered subset of $\mathbb{L}$. Also, X is cofinal in C, hence it can not be closed and discrete. This implies that there is $z \in \mathbb{L}$ such that for every $a \in \mathbb{L}$ the set $(a, z] \cap X$ is infinite. But then every open interval (y, w) in ${}^{\kappa}\mathbb{L}$ containing $z^\frown 0^{(\kappa)}$ must contain infinitely many elements of A, proving that A is either not closed or not discrete. □

3.3 Examples of LOTS and GO-spaces

We now apply all the results obtained so far to provide concrete examples of SC_κ-spaces with different properties. We start studying the properties of spaces of the form ${}^{\kappa}\alpha$ for an ordinal α.

Proposition 3.3.1. *For every ordinal $1 < \alpha < \kappa^+$, the space $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is a κ-perfect, zero-dimensional, not ω_1-additive, SC_κ-space. Furthermore:*

1. *If α is successor, then $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is compact.*
2. *If $\mathrm{cof}(\alpha) = \omega$, then $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is paracompact. If furthermore κ is weakly compact, then $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is κ-Lindelöf.*
3. *If $\omega < \mathrm{cof}(\alpha) < \kappa$, then $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is not paracompact. If furthermore κ is weakly compact, then $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is κ-Lindelöf.*
4. *If α is limit of cofinality $\mathrm{cof}(\alpha) \geq \kappa$, then $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is not paracompact nor κ-Lindelöf.*

Proof. First, the bounded topology on ${}^{\kappa}\alpha$ is κ-perfect, thus the lexicographic topology is as well (see Remark 3.2.1). Also, if $\alpha > 1$ then the order on α is not dense, thus ${}^{\kappa}\alpha$ is zero-dimensional by Corollary 3.2.15. Now: every ordinal α has no gaps other than end gaps, thus the order topology on α is SC_κ by Corollary 3.1.20. Hence, the lexicographic topology on ${}^{\kappa}\alpha$ is also SC_κ by Proposition 3.2.7. Also, $({}^{\kappa}\alpha, <_{\mathrm{lex}}, \tau_{\mathrm{lex}})$ is not ω_1-additive by Proposition 3.2.5. The relations with compactness and paracompactness follow from Corollary 3.2.13 and Proposition 3.2.16, noticing that if $\mathrm{cof}(\alpha) = \delta > \omega$, then for any cofinal increasing sequence $\langle \alpha_i \mid i < \delta \rangle$ in α, the ordinal $\beta = \sup_{i<\omega}\{\alpha_i\}$ proves that $\langle \alpha_i \mid i < \delta \rangle$ is not closed or not discrete. □

The previous proposition in particular applies to the κ-Cantor space and the κ-Baire space.

Corollary 3.3.2. *The κ-Cantor space ${}^{\kappa}2$ with the lexicographic topology is a κ-perfect, zero-dimensional, not ω_1-additive, compact SC_κ-space.*

Corollary 3.3.3. *The κ-Baire space ${}^{\kappa}\kappa$ with the lexicographic topology is a κ-perfect, zero-dimensional, not ω_1-additive, not κ-Lindelöf SC_κ-space.*

Proposition 3.2.2 shows that if we refine the lexicographic topology on the Cantor space, we recover directly the bounded topology. For the Baire space, we still get a topology that is homeomorphic to the bounded one, although it is coarser than the bounded topology.

Proposition 3.3.4. *The smallest κ-additive topology that refines the lexicographic topology on the κ-Baire space is strictly coarser than the bounded topology of the Generalized Baire space, yet it is homeomorphic to it.*

Proof. Let τ be the smallest κ-additive topology refining the lexicographic topology τ_{lex} on the κ-Baire space. First, by Proposition 3.2.2, τ is strictly coarser than the bounded topology τ_b. By Corollary 3.3.3, $({}^\kappa\kappa, \tau)$ is a κ-additive, zero-dimensional, κ-perfect, SC_κ-space. Also, it is not κ-Lindelöf, and by Remark 3.2.10 every κ-Lindelöf subset of it must have empty interior. Thus, the result follows from Theorem 1.2.10. □

When dealing with order topologies, one should be careful that when a linear order $\mathbb{L}$ can be embedded into another one $\mathbb{L}'$, usually the order topology on $\mathbb{L}$ is different from the GO-topology it would inherit from $\mathbb{L}'$. This is the case for example of the Cantor set ${}^\kappa 2$: the GO-topology it inherits from the κ-Baire set is slightly different from its own lexicographic topology.

Proposition 3.3.5. *The κ-Cantor set with GO-topology inherited from the lexicographic order on the κ-Baire set is a κ-perfect, zero-dimensional, not ω_1-additive, κ-Lindelöf (but not δ-Lindelöf for $\delta < \kappa$), SC_κ-space.*

Proof. Let $({}^\kappa\kappa, <_{\text{lex}}, \tau_{\text{lex}})$ be the κ-Baire set with lexicographic topology, and let τ be the topology induced by the restriction of τ_{lex} on ${}^\kappa 2$.

By Corollary 3.2.3 we have that $({}^\kappa 2, <_{\text{lex}}, \tau)$ is κ-perfect. Furthermore, it is zero-dimensional since it is a subset of $({}^\kappa\kappa, <_{\text{lex}}, \tau_{\text{lex}})$. Also, τ refines the lexicographic topology on ${}^\kappa 2$ (which is SC_κ), and thus $({}^\kappa 2, <_{\text{lex}}, \tau)$ is SC_κ by Corollary 3.1.16. Also, the same argument of Proposition 3.2.5 shows that this space is not ω_1-additive. Finally, $({}^\kappa 2, <_{\text{lex}}, \tau)$ is not δ-Lindelöf for any regular cardinal $\delta < \kappa$ since the sets $\{(-\infty, 0^{(\delta)} {}^\frown 1^{(\kappa)}]\} \cup \{(0^{(\alpha)} {}^\frown 1^{(\kappa)}, +\infty) \mid \alpha < \delta\}$ form a τ-open cover of size δ without smaller refinements. Thus, it remains only to prove that $({}^\kappa 2, <_{\text{lex}}, \tau)$ is κ-lindelöf.

First, notice that $({}^\kappa 2, <_{\text{lex}})$ has no gaps since it is a complete linear order (and this property depends only on the order, not on the topology). Notice also that every τ-open interval is either open also in the lexicographic topology on the Cantor set, or it is of the form $(x, s {}^\frown 1^{(\kappa)}]$ for some $x \in {}^\kappa 2$ and $s \in {}^{<\kappa}2$. In particular, for every convex τ-open set A, either A is open also in the lexicographic topology on the Cantor set, or A has a maximum that is not $1^{(\kappa)}$ and $A \setminus \{\max(A)\}$ is open in the lexicographic topology on the Cantor set.

Then, $({}^\kappa 2, <_{\text{lex}}, \tau)$ has also no pseudo-gaps of types (μ, ν) with $\max(\mu, \nu) \geq \kappa$: indeed, let (C, D) be a non-trivial clopen cut. Then, (C, D) must be of the form $((-\infty, x], (x, +\infty))$ for some $x \in {}^\kappa 2$ of the form $x = s {}^\frown 1^{(\kappa)}$, since otherwise (C, D) would be a gap, which is impossible since $({}^\kappa 2, <_{\text{lex}})$ is complete. Thus, (C, D) has type $(1, \operatorname{cof}(\operatorname{lh}(s)))$, and $\operatorname{cof}(\operatorname{lh}(s)) < \kappa$ as wanted.

Then, it is enough that we check that $({}^\kappa 2, <_{\text{lex}}, \tau)$ has no clopen convex partition of size $\geq \kappa$: the result will follow from Proposition 3.1.13.

So let $\mathcal{U}$ be a clopen convex partition of $({}^\kappa 2, <_{\text{lex}}, \tau)$. Define $\mathcal{A}'$ be the set of those $A \in \mathcal{A}$ that are open also in the lexicographic topology on ${}^\kappa 2$, and let $\mathcal{A}'' = \mathcal{A} \setminus \mathcal{A}'$. Define also $C = \{\max(A) \mid A \in \mathcal{A}''\}$.

Notice that $C = {}^\kappa 2 \setminus \bigcup\{A \setminus \sup(A) \mid A \in \mathcal{A}\}$ is a closed (and thus compact) subspace of ${}^\kappa 2$ with the lexicographic topology. In particular, this implies that the lexicographic order on it $(C, <_{\text{lex}})$ is complete and with endpoints, by Proposition 3.1.11. Then, $(C, <_{\text{lex}})$ is anti-well-ordered: suppose not and suppose there is a strictly increasing sequence $\langle p_i \mid i < \omega \rangle$ in C, and let $p = \sup_{^{\kappa}2}\{p_i \mid i < \omega\}$. Since C is closed, we must have $p \in C$, and by definition of C there exist $U \in \mathcal{A}''$ such that $p = \max(U)$ and $U \setminus \{p\}$ open in the lexicographic topology on ${}^{\kappa}2$. Since $\mathcal{A}$ is a partition of ${}^{\kappa}2$, then we must have $p_i \notin U$ for every $i < \omega$, contradiction.

Now notice that for every $c \in C$, the coinitiality of $(c, +\infty)$ is $< \kappa$, as there is $s \in {}^{<\kappa}2$ such that $c = s^\frown 1^{(\kappa)}$. Since C is closed, this implies C must have size $< \kappa$. Also, it must have a maximum. So let $C = \{c_i \mid i \leq \alpha\}$ be an enumeration of C such that $c_j <_{\text{lex}} c_i$ for every $i < j \leq \alpha < \kappa$. Let also $c_{\alpha+1} = -\infty$ for simplicity, and notice that $c_0 = 1^{(\kappa)}$ by definition of C.

Notice that for every $i \leq \alpha$, the set $(c_{i+1}, c_i]$ with the topology inherited from $({}^{\kappa}2, \tau_{\text{lex}})$ is κ-Lindelöf. Furthermore, by definition of C each set $A \in \mathcal{A}$ is either disjoint from $(c_{i+1}, c_i]$ or it is contained in it and it is open in the topology that $(c_{i+1}, c_i]$ inherits from $({}^{\kappa}2, \tau_{\text{lex}})$. Thus, $\mathcal{A} \restriction (c_{i+1}, c_i] = \{A \in \mathcal{A} \mid A \subseteq (c_{i+1}, c_i]\}$ must have size $< \kappa$ for each $i \leq \alpha$, and so $\mathcal{A} = \bigcup_{i \leq \alpha} \mathcal{A}_i \restriction (c_{i+1}, c_i]$ has size $< \kappa$ as well as wanted. □

We now look at the order-completion $(\hat{\mathcal{B}}, <_{\text{lex}})$ of the lexicographic order on the generalized Baire set $({}^{\kappa}\kappa, <_{\text{lex}})$ as given by Remark 3.1.5.

Proposition 3.3.6. *The completion $(\hat{\mathcal{B}}, <_{\text{lex}}, \tau_{\text{lex}})$ of the κ-Baire space, with its order topology, is a κ-perfect, connected, compact, (not ω_1-additive,) SC_κ-space.*

Proof. First, notice that the κ-Baire set with lexicographic order has only (κ, δ)-gaps by Lemma 3.2.11. Thus, no point in $\hat{\mathcal{B}} \setminus {}^{\kappa}\kappa$ is κ-isolated, and $({}^{\kappa}\kappa, <_{\text{lex}}, \tau_{\text{lex}})$ is κ-perfect, thus $(\hat{\mathcal{B}}, <_{\text{lex}}, \tau_{\text{lex}})$ is κ-perfect as well. Second, the lexicographic order on ${}^{\kappa}\kappa$ is dense (by Lemma 3.2.9), and thus the order on the completion $(\hat{\mathcal{B}}, <_{\text{lex}})$ is dense as well. Then by Proposition 3.1.8 and Corollary 3.1.12, the topology is connected and compact. The space is not ω_1-additive since it is not zero-dimensional. Finally, since the order is complete, the space is strong κ-Choquet by Corollary 3.1.20. □

Other classical (and very useful) examples of Polish spaces in the classical case come from manifolds. The arguably most important example in this sense is the unit circle S^1_1. This space can be constructed starting from the unit interval $[0, 1]$ of the real line and identifying its endpoints. The same procedure can be used on any linearly ordered topological space with endpoints, and the resulting space shares many similarities with both the (classical) unit circle S^1_1 and the starting linear order used to define it.

Example 3.3.7. Given a linear order $\mathbb{L}$ with endpoints, define $S^1_1(\mathbb{L})$ to be the **circle made from** $\mathbb{L}$ by joining the maximum and minimum.

Formally, if $(\mathbb{L}, \leq)$ is the linear order and $0, 1$ are respectively the maximum and minimum, define $S^1_1(\mathbb{L}) = \mathbb{L}/\!\sim$ where $\sim\, = \{(x, x) \mid x \in \mathbb{L}\} \cup \{(0, 1), (1, 0)\}$ is the relation that identifies 0 and 1.

The set $S^1_1(\mathbb{L})$ can be made into a topological space passing the topology $\tau^{\mathbb{L}}_{<}$ through the quotient. Since we are identifying only two points, it is straightforward that most topological properties of $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ pass to $S^1_1(\mathbb{L})$.

Proposition 3.3.8. *Let $(\mathbb{L}, <, \tau)$ be a LOTS with endpoints.*

1. *If $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ is $f\mathrm{SC}_\kappa$, then $S^1_1(\mathbb{L})$ is $f\mathrm{SC}_\kappa$.*

2. *If* $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ *is* SC_κ*, then* $S^1_1(\mathbb{L})$ *is* SC_κ*.*

3. *If* $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ *is zero-dimensional, then* $S^1_1(\mathbb{L})$ *is zero-dimensional.*

4. *If* $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ *is* κ*-perfect, then* $S^1_1(\mathbb{L})$ *is* κ*-perfect.*

5. *If* $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ *is* κ*-additive, then* $S^1_1(\mathbb{L})$ *is* κ*-additive.*

6. *For any cardinal* λ*, if* $(\mathbb{L}, \tau^{\mathbb{L}}_{<})$ *is* λ*-Lindelöf, then* $S^1_1(\mathbb{L})$ *is* λ*-Lindelöf.*

7. *If* $(\mathbb{L}, <, \tau)$ *is connected, then* $S^1_1(\mathbb{L}) \setminus \{x\}$ *is connected for every* $x \in S^1_1(\mathbb{L})$*. In particular, if* $(\mathbb{L}, <, \tau)$ *is connected there is no embedding from* $S^1_1(\mathbb{L})$ *into* $(\mathbb{L}, <, \tau)$*.*

3.4 Classification up to homeomorphism

In this section, we study the size of homeomorphism types of certain classes of SC_κ, $f\mathrm{SC}_\kappa$ and $\mathbb{G}$-Polish. The basic proof idea is similar to that of [42, Proposition 6.5], with some changes for each class: we report each case in detail for the reader's convenience.

First, we want to find κ-many spaces without homeomorphic open subsets.

Definition 3.4.1. Given a subset A of κ, let T_A denote the tree of all $t \in {}^{<\kappa}\kappa$ such that for all $\beta \in A$ the set $\{\alpha < \beta \mid t(\alpha) = 0\}$ is bounded below β. Let $X_A = [T_A]$.

Then, each X_A is a κ-additive $f\mathrm{SC}_\kappa$ space. Furthermore, each X_A is perfect, and $|X_A| = 2^\kappa$ for every $A \subseteq \kappa$.

Lemma 3.4.2. *Suppose* A *and* B *are disjoint subsets of* κ *and* A *is stationary. Then no nonempty open subset of* X_A *is homeomorphic to an open subset of* X_B*.*

Proof. Suppose by contradiction $\phi : U \to V$ is a homeomorphism with U open subset of X_A and V open subset of X_B. Without loss of generality, passing to restrictions of ϕ if needed, we may assume $U = \mathbf{N}_s$ for some $s \in T_a$.

Proceeding recursively, we construct two families $\langle s_\alpha \mid \alpha < \kappa\rangle \subseteq T_A$ and $\langle t_\alpha \mid \alpha < \kappa\rangle \subseteq T_B$ such that

(a) $s_0 = s$ and $s_i{}^\frown 0 \subseteq s_j$ for any $i < j < \kappa$.

(b) $t_i \subseteq t_j$ for any $i < j < \kappa$.

(c) $\mathrm{lh}(s_i) \leq \mathrm{lh}(t_i) < \mathrm{lh}(s_{i+1})$ for every $i < \kappa$.

(d) $\phi[\mathbf{N}_{s_{i+1}} \cap X_A] \subseteq \mathbf{N}_{t_i} \cap X_B \subseteq \phi[\mathbf{N}_{s_i{}^\frown 0} \cap X_A]$ for every $i < \kappa$.

Given s_α, we may find $t_\alpha \in T_B$ and $s_{\alpha+1} \in T_A$ satisfying the requirement using the fact that ϕ is a homeomorphism on an open subset of X_B.

For limit steps, suppose $s_\varepsilon, t_\varepsilon$ has been defined for every $\varepsilon < \alpha$ and define $s_\alpha = \bigcup_{\varepsilon<\alpha} s_\varepsilon$. We claim that $\mathbf{N}_{s_\alpha} \cap X_A \neq \emptyset$ and $s \in T_A$ as required. Call $\gamma_\alpha = \mathrm{lh}(s_\alpha)$: then $\mathbf{N}_{s_\alpha} \cap X_A$ may be empty only if $\gamma_\alpha \in A$. So suppose $\gamma_\alpha \in A$. Let $t = \bigcup_{\varepsilon<\alpha} t_\varepsilon$, then by point (c) we have $\mathrm{lh}(t) = \gamma_\alpha$ so $\mathrm{lh}(t) \notin B$ and $t \in T_B$ since $A \cap B = \emptyset$.

Now $\phi[\mathbf{N}_{s_\alpha} \cap X_A] = \bigcap_{\varepsilon<\alpha} \phi[\mathbf{N}_{s_\varepsilon} \cap X_A] = \bigcap_{\varepsilon<\alpha} \mathbf{N}_{t_\varepsilon} \cap X_B = \mathbf{N}_t \cap X_B \neq \emptyset$ hence $\mathbf{N}_{s_\alpha} \cap X_A \neq \emptyset$ and we are done.

So consider $\langle s_\alpha \mid \alpha < \kappa \rangle \subseteq T_A$ constructed as above, and let $x = \bigcup_{\alpha<\kappa} s_\alpha$. Define Lim_x to be the set of those $\alpha < \kappa$ such that $\{\beta < \alpha \mid x(\beta) = 0\}$ is unbounded below α. Since $\{\beta < \kappa \mid x(\beta) = 0\}$ is unbounded in κ, then Lim_x is a club of κ and there is $\alpha \in \mathrm{Lim}_x \cap A$ which contradicts the definition of A. □

We are now ready to classify homeomorphism types of $\mathbb{G}$-Polish and $f\mathrm{SC}_\kappa$ spaces.

Theorem 3.4.3. *There are 2^κ many pairwise not homeomorphic perfect $\mathbb{G}$-Polish spaces of size 2^κ.*

Proof. Fix a sequence $\vec{S} = \langle S_\alpha \mid \alpha < \kappa \rangle$ of pairwise disjoint stationary subsets of κ. For any subset A of κ, let $X^A = \bigsqcup_{i \in A} X_{S_i}$ be the disjoint union of those X_{S_i} for which $i \in A$. Since each X_{S_i} is a closed subset of κ^κ and hence a $\mathbb{G}$-Polish space and this class of spaces is closed under the disjoint union of size $\leq \kappa$, then X^A is again a $\mathbb{G}$-Polish space for each $A \subseteq \kappa$.

It suffices to show that X^A and X^B are not homeomorphic if $A \neq B$.

Assume that $\alpha \in A \setminus B$ and $f \colon X^A \to X^B$ is a homeomorphism: then for each $\beta \in B$ such that $X_{S_\alpha} \cap f^{-1}(X_{S_\beta})$ is nonempty, the restriction of f to the open set $X_{S_\alpha} \cap f^{-1}(X_{S_\beta})$ would be an homeomorphism contradicting Lemma 3.4.2. □

We now turn our attention to SC_κ spaces.

If κ is not weakly compact, then there is only one homeomorphism type of perfect κ-additive SC_κ spaces by Theorem 1.2.10. Giving up on perfectness however we get a result similar to Theorem 3.4.3.

Recall that for a tree $T \subseteq {}^{<\kappa}\kappa$ we denote with $\delta(T)$ the boundary of the tree.

Definition 3.4.4. Given $A \subseteq \kappa$, define $\hat{T}_A = T_A \cup \{s ^\frown 0^{(\alpha)} \mid s \in \delta(T_A) \text{ and } \alpha < \kappa\}$ and let $Y_A = [\hat{T}_A]$.

Notice that $\hat{T}_A$ is a superclosed subtree of ${}^{<\kappa}\kappa$, hence Y_A is a κ-additive SC_κ space, X_A is exactly Y_A minus its isolated points, and $|Y_A| = |X_A| = 2^\kappa$ for every $A \subseteq \kappa$.

We need the following corollary of Lemma 3.4.2

Corollary 3.4.5. *Suppose A and B are disjoint subsets of κ and A is stationary. Then there is no homeomorphism between open subsets of Y_A and Y_B of size 2^κ.*

Proof. Suppose $\phi : U \to V$ is a homeomorphism with U open subset of Y_A of size 2^κ and V open subset of Y_B. Since $|Y_A \setminus X_A| \leq \kappa$ then $U \cap X_A \neq \emptyset$. Since a homeomorphism sends isolated points in isolated points, we have $\phi[X_A \cap U] \subseteq X_B$ and $\phi[(Y_A \setminus X_A) \cap U] \subseteq Y_B \setminus X_B$, hence $\phi \restriction (U \cap X_A) : U \cap X_A \to X_B$ contradicts Lemma 3.4.2. □

Theorem 3.4.6. *There are 2^κ many pairwise not homeomorphic κ-additive SC_κ spaces of size 2^κ.*

Proof. Fix a sequence $\vec{S} = \langle S_\alpha \mid \alpha < \kappa\rangle$ of pairwise disjoint stationary subsets of κ. For any subset A of κ, let $Y^A = \bigsqcup_{i\in A} Y_{S_i}$ be the disjoint union of those Y_{S_i} for which $i \in A$: it is again an SC_κ space.

It suffices to show that Y^A and Y^B are not homeomorphic if $A \neq B$.

Assume that $\alpha \in A \setminus B$ and $f\colon Y^A \to Y^B$ is a homeomorphism. Then there is $\beta \in B$ such that the set $U = Y_{S_\alpha} \cap f^{-1}(Y_{S_\beta})$ is an open subset of Y^A of size 2^κ. The restriction of f to U contradicts Corollary 3.4.5. □

Assume now κ is weakly compact instead.

Definition 3.4.7. Given $A \subseteq \kappa$, define $\tilde{T}_A = T_A \cup \{s^\frown t \mid s \in \delta(T_A) \text{ and } t \in {}^{<\kappa}2\}$ and let $Z_A = [\tilde{T}_A]$.

Once again, $\tilde{T}_A$ is a superclosed splitting subtree of ${}^{<\kappa}\kappa$, hence Z_A is a perfect κ-additive SC_κ space. Furthermore, X_A is equal to Z_A minus its κ-Lindelöf open subsets, hence every open set $U \subseteq Z_A \setminus X_A$ is K_κ, and $|Z_A| = 2^\kappa$ for every $A \subseteq \kappa$.

Corollary 3.4.8. *Suppose A and B are disjoint subsets of κ and A is stationary. Then there is no homeomorphism between non-K_κ open subsets $U \subseteq Z_A$ and $V \subseteq Z_B$.*

Proof. Suppose $\phi : U \to V$ is a homeomorphism with U open subset of Z_A and V open subset of Z_B and U and V are not K_κ. Then $U \cap X_A \neq \emptyset$. Since ϕ is a homeomorphism, then a point $x \in U$ has a κ-Lindelöf neighborhood if and only if $\phi(x) \in V$ has a κ-Lindelöf neighborhood. Since for every $C \subseteq \kappa$, X_C is exactly the set of points of Z_C that does not have a κ-Lindelöf neighborhood, then we have $\phi[X_A \cap U] \subseteq X_B$ and $\phi[(Z_A \setminus X_A) \cap U] \subseteq Z_B \setminus X_B$. Then $\phi \restriction (U \cap X_A) : U \cap X_A \to X_B$ contradicts Lemma 3.4.2. □

Theorem 3.4.9. *Suppose that κ is weakly compact. Then there are 2^κ many pairwise not homeomorphic perfect κ-additive SC_κ spaces of size 2^κ.*

Proof. Fix a sequence $\vec{S} = \langle S_\alpha \mid \alpha < \kappa\rangle$ of pairwise disjoint stationary subsets of κ. For any subset A of κ, let $Z^A = \bigsqcup_{i\in A} Z_{S_i}$ be the disjoint union of those Z_{S_i} for which $i \in A$: it is again a SC_κ space.

It suffices to show that Z^A and Z^B are not homeomorphic if $A \neq B$.

Assume that $\alpha \in A \setminus B$ and $f\colon Z^A \to Z^B$ is a homeomorphism. Since Z_{S_α} is not K_κ then there is $\beta \in B$ such that the set $U = Z_{S_\alpha} \cap f^{-1}(Z_{S_\beta})$ is an open subset of Z^A which is not K_κ. The restriction of f to U contradicts Corollary 3.4.8. □

Finally, we turn to κ-Lindelöf spaces.

If κ is weakly compact, then there is only one homeomorphism type of κ-Lindelöf perfect κ-additive SC_κ spaces by Theorem 1.2.15.

For $f\mathrm{SC}_\kappa$ spaces, we have the following.

Definition 3.4.10. Given a subset A of κ, let $T'_A = T_A \cap {}^{<\kappa}3$. Let $X'_A = [T'_A]$.

If κ is weakly compact, then X'_A is the κ-Lindelöf version of X_A. The choice of ${}^{<\kappa}3$ instead of ${}^{<\kappa}2$ is to ensure that the space X'_A has size 2^κ. We have for this class as well a lemma similar to 3.4.2.

Lemma 3.4.11. *Suppose A and B are disjoint subsets of κ and A is stationary. Then no nonempty open subset of X'_A is homeomorphic to an open subset of X'_B.*

The proof is the same as Lemma 3.4.2.

Theorem 3.4.12. *Suppose κ is weakly compact. There are 2^κ many pairwise not homeomorphic perfect, κ-Lindelöf, $\mathbb{G}$-Polish spaces of size 2^κ.*

Giving up on perfectness instead we have the following.

Definition 3.4.13. Define $\hat{T}'_A = T'_A \cup \{s^\frown 0^{(\alpha)} \mid s \in \delta(T'_A)$ and $\alpha < \kappa\}$ and let $Y'_A = [\hat{T}'_A]$ for every $A \subseteq \kappa$.

Proposition 3.4.14. *The space Y'_A is a κ-Lindelöf, SC_κ-space of size 2^κ.*

Proof. It is an SC_κ-space since $\hat{T}'_A$ is a superclosed subset of ${}^{<\kappa}3$, and it has size 2^κ. To prove it is κ-Lindelöf, firse notice that ${}^\kappa 3$ is homeomorphic to ${}^\kappa 2$, and so ${}^\kappa 3$ is κ-Lindelöf if κ is weakly compact. Let $\mathcal{A} \subseteq \mathcal{B}$ be an open cover of Y'_A. Without loss of generality, we may assume that for every $A \in \mathcal{A}$ we have $A = \mathbf{N}_s \cap Y'_A$ for some $s \in T'_A \cup \delta(T'_A)$. Define $\mathcal{A}' = \{\mathbf{N}_s \mid \mathbf{N}_s \cap Y'_A \in \mathcal{A}\}$, it is an open cover of ${}^\kappa 3$. Since this space is κ-Lindelöf we may find an open subcover $\mathcal{F} \subseteq \mathcal{A}'$ of size $|\mathcal{F}| < \kappa$. Then $\{A \cap Y'_A \mid A \in \mathcal{F}\}$ is an open subcover of $\mathcal{A}$ of size $< \kappa$. □

From the same arguments of Lemma 3.4.5 and Theorem 3.4.6 we get the following.

Lemma 3.4.15. *Suppose A and B are disjoint subsets of κ and A is stationary. Then no nonempty open subset of Y'_A of size 2^κ is homeomorphic to an open subset of Y'_B.*

Theorem 3.4.16. *Suppose κ is weakly compact. There are 2^κ many pairwise not homeomorphic κ-Lindelöf, κ-additive SC_κ spaces of size 2^κ.*

Part II

Ramsey theory and combinatorics

Chapter 4

Finite monoids in combinatorics

4.1 Introduction to Ramsey and $\mathbb{Y}$-controllable monoids

First, we recall the main combinatorial notions that we use throughout this (and next) chapter. Then, we try to give a better insight into these definitions by showing how these concepts can be formulated in an equivalent way.

First, a semigroup $(S, \cdot)$ is a set S with $\cdot$ a binary associative operation. A partial semigroup $(S, \cdot)$ is a set S with a binary associative operation $\cdot$ such that $(a \cdot b) \cdot c$ is defined if and only if $a \cdot (b \cdot c)$ is defined. In particular, every partial semigroup $(S, \cdot)$ can be seen as a subset of a semigroup $(S \cup \{\perp\}, \cdot)$ for some element $\perp \notin S$ with $\perp \cdot a = a \cdot \perp = \perp$ for every $a \in M$ (however, not every subset of a semigroup is a partial semigroup). A monoid is a semigroup that has a (unique) identity element 1 such that $a \cdot 1 = 1 \cdot a = a$ for every $a \in M$.

Given a partial semigroup $(S, \cdot)$, an endomorphism of S is a function $f : S \to S$ such that if $x \cdot y$ is defined, then $f(x) \cdot f(y)$ is as well and $f(x \cdot y) = f(x) \cdot f(y)$. We say that a monoid M acts on a set X if for every $m \in M$ there is a function $f_m : X \to X$ satisfying $f_a(f_b(x)) = f_{a \cdot b}(x)$ and $f_1(x) = x$ for every $x \in X$ and $a, b \in M$ and 1 the identity of the monoid. With abuse of notation, we always write $m(x)$ to denote $f_m(x)$. If X is a partial semigroup, we say that the action of M is by endomorphism if every function is an endomorphism.

We need to introduce a few semigroups that we will use abundantly in the following sections.

Example 4.1.1. Let X be a set. The **semigroup of words** W_X over the alphabet X is the free semigroup generated by X. In other words, $W_X = (X^{<\omega}, ^\frown)$ is the set of all finite sequences of elements of X with operation given by the concatenation of sequences $(a_0, \ldots, a_n)^\frown(b_0, \ldots, b_m) = (a_0, \ldots, a_n, b_0, \ldots, b_m)$ for any $(a_0, \ldots, a_n), (b_0, \ldots, b_m) \in X^{<\omega}$.

Example 4.1.2. Let X be a set, and let $Y = \bigcup_{n \in \omega} \{n\} \times X$. The **partial semigroup of located words** $(\mathrm{FIN}_X, ^\frown)$ on the alphabet X is the partial subsemigroup of W_Y consisting of those words $((n_0, a_0), \ldots, (n_i, a_i))$ such that $n_0 < \cdots < n_i$.

Notice that both the semigroup of words and the partial semigroup of located words have an identity (the empty sequence), so they are (partial) monoids. Both above examples can be seen just as instances of a more general situation.

Example 4.1.3. Let $(X_n)_{n\in\omega}$ be a family of sets, and let $Y = \bigcup_{n\in\omega} X_n$. The **partial semigroup of located words** $\langle (X_n)_{n\in\omega}\rangle$ on the family of alphabets $(X_n)_{n\in\omega}$ is the partial subsemigroup of W_Y consisting of all those sequences $x_1 {}^\frown ... {}^\frown x_n \in W_Y$ for which there exists $i_1 < ... < i_n \in \omega$ such that $x_k \in X_{i_k}$.

We often omit the adjective *located* for words of FIN_X and $\langle (X_n)_{n\in\omega}\rangle$.

If a monoid M acts on X, we say that X is an **M-set**. Given a family of M-sets $(X_n)_{n\in\omega}$, we say that the action of M on $(X_n)_{n\in\omega}$ is **uniform** if for every $k, n \in \omega$, $m \in M$ and $x \in X_k \cap X_n$, if $m_k x$ and $m_n x$ are the results of the action of m on x respectively in X_k and in X_n, then $m_k x = m_n x$. Notice that the action of M on $(X_n)_{n\in\omega}$ is uniform if and only if it extends to $X = \bigcup_{n\in\omega} X_n$.

Notice that if X is an M-set, then M acts coordinate-wise also on W_X and FIN_X. If M acts uniformly on $(X_n)_{n\in\omega}$, then M acts coordinate-wise also on $\langle (X_n)_{n\in\omega}\rangle$.

A **distinguished point** or **variable** of an M-set X is an element $x \in X$ such that $Mx = \{mx \mid m \in M\} = X$. A **pointed M-set** is an M-set X together with a fixed distinguished point $x \in X$. Given a pointed M-set (X, x), we say that a word $w \in W_X$ is a **variable word** if w contains x. Similarly, we define **variable located words** for FIN_X. If $(X_n, x_n)_{n\in\omega}$ is a sequence of pointed M-sets, we say that a word $w \in \langle (X_n)_{n\in\omega}\rangle$ is a **variable located word**[1] if it contains x_n for some $n \leq \omega$. Notice that variable words form a both-sided ideal of the semigroup W_X, and the same is true for variable located words in FIN_X or $\langle (X_n)_{n\in\omega}\rangle$.

Notice that if M is a monoid, then $(M, 1)$ is a pointed M-set with identity as the distinguished point. In this case, variable words and variable located words are exactly the words containing the identity of the monoids 1.

Notice that all coordinate-wise actions of a monoid on W_X, FIN_X and $\langle (X_n)_{n\in\omega}\rangle$ are by endomorphism.

Given a monoid M acting on a partial semigroup S, we say that an infinite sequence $\bar{s} = (s_i)_{i<\omega} \in S^\omega$ is **basic** if the product $m_0 s_{i_0} {}^\frown \dots {}^\frown m_n s_{i_n}$ is defined for every $i_0 < \dots < i_n$ and $m_0, ..., m_n \in M$.

Given a monoid M acting by endomorphism on a partial semigroup S, for every $w = (n_i, a_i)_{i\leq h} \in \mathrm{FIN}_M$ and for every basic sequence $\bar{s} = (s_i)_{i<\alpha} \in S^{\leq\omega}$ with $\mathrm{lh}(\bar{s}) > n_h$, we can define $w(\bar{s}) = a_0 s_{n_0} \cdots a_h s_{n_h}$. This operation satisfies $(w {}^\frown w')(\bar{s}) = w(\bar{s}) {}^\frown w'(\bar{s})$ for every $w, w' \in \mathrm{FIN}_M$.

Definition 4.1.4. Let M be a monoid acting by endomorphisms on a partial semigroup S, and let $\bar{s}$ be an infinite sequence of elements of S. Given a family $C \subseteq \mathrm{FIN}_M$, we define the (combinatorial) **C-span** of $\bar{s}$ as the set

$$\langle \bar{s}\rangle_C = C\bar{s} = \big\{m_0\, s_{i_0} \cdots m_n\, s_{i_n} \mid ((i_0, m_0), \dots, (i_n, m_n)) \in C\big\}.$$

For ease of notation, we define also the C-span for $C \subseteq W_M$ as the C'-span for $C' = \{(n_i, a_i)_{i<h} \in \mathrm{FIN}_M \mid (a_i)_{i<h} \in C\}$. We define also the **$M$-span** as $\langle \bar{s}\rangle_M = \langle \bar{s}\rangle_{\mathcal{V}_M}$ for $\mathcal{V}_M$ the set of variable words of FIN_M (i.e. those words of FIN_M containing the identity). Since the identity of M acts as the identity function on S, the M-span is precisely the set of those elements obtained from $\bar{s}$ where one coordinate is not changed by the action of M. Among all possible C-spans, the M-span is

[1] This definition is tricky: see also the definition of strongly variable word and the following discussion.

special: it is the largest span granting that if all elements of $\bar{s}$ belong to a both-sided ideal $I \subseteq S$, then $\langle \bar{s} \rangle_M \subseteq I$ as well.

Definition 4.1.5 ([142])**.** A monoid M is said **Ramsey** if for all sequences of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for all finite colorings of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence of variable words $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ with monochromatic M-span $\langle \bar{s} \rangle_M$.

The class of partial semigroups of the form $\langle (X_n)_{n\in\omega} \rangle$ is general enough to cover most cases of interest, like W_X and FIN_X for any X, and yet sufficiently well-behaved to allow to talk about variable words (a concept that does not exist in every semigroup). However, many other choices are possible, and in what follow we discuss briefly different choices that would bring to equivalent definitions.

First, notice that if M is a monoid acting by endomorphisms on a partial semigroup S, if $\bar{s} \in S^\omega$ is basic, then for every partial subsemigroup $C \subseteq M^{<\omega}$, the C-span $\langle \bar{s} \rangle_C$ of $\bar{s}$ is a partial subsemigroup of S. In particular, semigroups of located words can be seen just as the full span of a specific sequence.

Remark 4.1.6. Given a uniform family of pointed M-sets $(X_n, x_n)_{n\in\omega}$, we have that $\bar{t} = (x_n)_{n\in\omega}$ is basic and $\langle (X_n)_{n\in\omega} \rangle = \langle \bar{t} \rangle_{\mathrm{FIN}_M}$. Conversely, given any M-set X and a basic sequence $\bar{t} \in X^\omega$, we have $\langle \bar{t} \rangle_{\mathrm{FIN}_M} = \langle (X_n)_{n\in\omega} \rangle$ for $X_n = Mt_n$.

Since in a semigroup in general we do not have a concept of variable words, we need a different way to control that the sequence $\bar{s}$ with monochromatic span is "nice". Given a partial semigroup S on which a monoid M is acting by endomorphism, and given two sequences $\bar{s}$ and $\bar{t}$ in S^ω, we say that $\bar{s}$ is **extracted** from $\bar{t}$ by M, or $\bar{s} \leq_M \bar{t}$, if there is an increasing sequence $(i_n)_{n\in\omega}$ of natural numbers such that $s_n \in \langle t_{i_n}, \ldots, t_{(i_{n+1})-1} \rangle_M$.

The notion of extraction of sequences allows us to get a very strong form of control. For example, if $I \subseteq S$ is a both-ideal and $\bar{t} \in I^\omega$ is a sequences of elements of S all inside the ideal I, then every sequence $\bar{s} \leq_M \bar{t}$ belongs to I^ω as well. Since variable (located) words form a both-sided ideal of W_M (and of FIN_M or $\langle (X_n)_{n\in\omega} \rangle$), then every sequence extracted from a sequence of variable words is again of variable words. However, this kind of control allows us to get the same result for other ideals as well. Even for variable words, asking that a sequence $\bar{s}$ is extracted from a given sequence $\bar{t}$ allows getting something stronger than just $\bar{s}$ being a sequence of variable words. Given a sequence of pointed M-sets $(X_n, x_n)_{n\in\omega}$, call a located word $w = a_0 ^\frown \cdots ^\frown a_n \in \langle (X_n)_{n\in\omega} \rangle$ **strongly variable** if there exists $i_0 < \cdots < i_n < \omega$ such that $a_{i_h} \in X_{i_h}$ for every $h < n$ and $a_{i_k} = x_{i_k}$ for some $k < n$. Not every variable word is a strongly variable word. For example: if $X_0 = \{x_0, y\} \subseteq X_i$ and $x_0 \neq x_i$ for every $i \in \omega \setminus \{0\}$, the word (y, x_0) is a variable word (as it contains a distinguished point), but not a strongly variable word since this distinguished point can not be obtained from the set in which it is distinguished. And in some scenarios, the difference is even more evident. For example: for every sequence of (finite) pointed M-sets $(X_n, x_n)_{n\in\omega}$, adding the pointed M-sets (My, y) for every $y \in \bigcup_{n\in\omega} X_n$ to the starting sequence, we get another sequence of pointed M sets $(X'_n, x'_n)_{n\in\omega}$ such that $\langle (X_n)_{n\in\omega} \rangle \subseteq \langle (X'_n)_{n\in\omega} \rangle$, and every word of $\langle (X'_n)_{n\in\omega} \rangle$ is a variable word (since every element of any X'_n is the distinguished point of some other X'_m). If for example the original sets X_n were disjoint, every word $w \in \langle (X_n)_{n\in\omega}\rangle$ is a strongly variable word in $\langle (X'_n)_{n\in\omega}\rangle$ if and only if it was already a strongly variable word in $\langle (X_n)_{n\in\omega}\rangle$.

Strongly variable words, however, has the disadvantage that they do not form a both-sided ideal of $\langle (X_n)_{n\in\omega}\rangle$, as in general they are not even closed under concatenation (e.g. if $X_0 = \{x_0\}$ and $X_1 = \{x_1\}$ and $x_0 \neq x_1$, and $x_0 \in X_i$ and $x_0 \neq x_i$ for all $i > 1$, then (x_0) is a strongly variable word while (x_0, x_0) is not).

Remark 4.1.7. Given a uniform family of pointed M-sets $(X_n, x_n)_{n\in\omega}$, a sequence $\bar{s} \in \langle (X_n)_{n\in\omega}\rangle$ is a basic sequence of strongly variable words if and only if $\bar{s} \leq_M \bar{t}$ for $\bar{t} = (x_n)_{n\in\omega}$.

Notice, however, that it is possible (e.g. in FIN_X) that every variable word is also a strongly variable word.

The situation thus seems very complicated. Luckily, the next result shows that in reality, things work in a much smoother way, as everything gives equivalent combinatorial statements. Hence, in practice, we may work with any of the notions described above without having to think too much about it.

Definition 4.1.8. Given a monoid M acting by endomorphism on a semigroup S, a sequence $\bar{t} \in S^\omega$ is a **free sequence** if for every $x \in \langle \bar{t}\rangle_{W_M}$ there are unique $m_0, \dots, m_n \in M$ and $i_0, \dots, i_n \in \omega$ such that $x = m_0 s_{i_0} \cdot \dots \cdot m_n s_{i_n}$.

The next remark is saying that if we consider the category of partial semigroups of the form $\langle \bar{s}\rangle_M$ with surjective homomorphism sending generating sequences into generating sequences as arrows, then the elements generated by basic free sequences are (weakly) initial in the category.

Remark 4.1.9. Suppose M is a monoid acting by endomorphism on partial semigroups S and S', and let $\bar{t} \in S^\omega$ and $\bar{s} \in (S')^\omega$. If $\bar{t}$ is basic and free, then the function $f(m_0 t_{i_0} \cdot \dots \cdot m_n t_{i_n}) = m_0 s_{i_0} \cdot \dots \cdot m_n s_{i_n}$ is a well-defined surjective homomorphism from $\langle \bar{t}\rangle_C$ to $\langle \bar{s}\rangle_C$ for every choice of $C \subseteq W_M$.

Lemma 4.1.10. *The following are equivalent for a monoid M and $C \subseteq \mathrm{FIN}_M$:*

1. *There is a partial semigroup S and an action by endomorphism of M on S and a basic free sequence $\bar{t} \in S^\omega$ such that for every finite coloring of S there is $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s}\rangle_C$ monochromatic.*

2. *For every action by endomorphism of M on any partial semigroup S, for every sequence $\bar{t} \in S^\omega$ and for every finite coloring of S there is $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s}\rangle_C$ monochromatic.*

Proof. Let $f : \langle \bar{x}\rangle_C \to \langle \bar{y}\rangle_C$ be the surjective homomorphism given in Remark 4.1.9. Then every coloring c of $\langle \bar{y}\rangle_C$ induces a coloring $c' = c \circ f$ of $\langle \bar{x}\rangle_C$. Furthermore, it is easy to see that if $\bar{s} \leq_M \bar{x}$ and $f(\bar{s}) = (f(s_n))_{n\in\omega}$, then we have $f(\bar{s}) \leq_M \bar{y}$ and $f \restriction \langle \bar{s}\rangle_C = \langle f(\bar{s})\rangle_C$, and the result follows. □

Thus, we may define Ramsey monoids in many different possible ways.

Given a set X, an infinite sequence $\bar{t} \in (W_X)^\omega$ is said **rapidly increasing** if $|t_n| > \sum_{i=0}^{n-1} |t_i|$ for all $n \in \omega$.

Proposition 4.1.11. *A monoid M is Ramsey if and only if one of the following holds:*

(a) *For every (partial) semigroup S on which M acts by endomorphisms, for every (basic) sequence $\bar{t} \in S^\omega$, for every finite coloring of S there is an infinite sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle_M$ is monochromatic.*

(b) *For every M-set X, for every $\bar{t} \in X^\omega$ and for every finite coloring of W_X there is an infinite basic sequence $\bar{s} \leq_M \bar{t}$ in $(W_X)^\omega$ with monochromatic M-span.*

(c) *For all sequences of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for all finite colorings of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence of **strongly** variable words $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ with monochromatic M-span $\langle \bar{s} \rangle_M$.*

(d) *For all finite coloring of FIN_M there is a basic sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that $\langle \bar{s} \rangle_M$ is monochromatic.*

(e) *There is a rapidly increasing sequence of variable words $\bar{t} \in (W_M)^\omega$ such that for all finite coloring of W_M there is an infinite sequence $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s} \rangle_M$ monochromatic.*

Proof. It is easy to see that (a) (with or without parenthesis) implies (b), and the latter is equivalent to (c) by Remarks 4.1.6 and 4.1.7. Point (c) is a strictly stronger version of the definition of Ramsey monoid, and being Ramsey implies (d) and (e).

Finally, (d) and (e) each implies (a) by Lemma 4.1.10, since every rapidly increasing sequence of variable words is (basic and) free in W_M, while $\bar{s} \in (\mathrm{FIN}_M)^\omega$ is a basic sequence of variable located words if and only if $\bar{s} \leq_M \bar{t}$ for $\bar{t} = (n, 1)_{n\in\omega}$, which is a basic and free sequence in FIN_M. □

Other choices of (classes of) partial semigroups are also possible, as long as they admit at least one free basic sequence. Also, notice that other conditions are possible as well, but for more complicated reasons: see e.g. Theorem 5.2.13.

We move to the next main definition we are going to work with.

Given a monoid M, define $\mathbb{X}(M)= \{aM \mid a \in M\}$. It is a partial order under inclusion. Define $\mathbb{Y}(M)\subseteq \mathscr{P}(\mathbb{X}(M))$ as the family of all non-empty chains of $(\mathbb{X}(M), \subseteq)$, i.e. the subsets of $\mathbb{X}(M)$ which are linearly ordered by inclusion. Given $x, y \in \mathbb{Y}(M)$, define $x \leq_{\mathbb{Y}} y$ if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x. If $\mathbb{X}(M)$ contains no infinite descending chains (e.g. if it is finite), then $\mathbb{Y}(M)$ is a forest. Notice that M acts by endomorphism on $(\mathbb{X}(M), \subseteq)$, and thus also on $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$.

Given a preorder $(\mathbb{P}, \leq_{\mathbb{P}})$, let $\langle \mathbb{P} \rangle$, with operation $\vee$, be the semigroup freely generated by $\mathbb{P}$ modulo the relations

$$p \vee q = q = q \vee p \text{ for } p \leq_{\mathbb{P}} q.$$

In other words, $(\langle \mathbb{P} \rangle, \leq)$ is the semigroup of words $\mathbb{P}^{<\omega}$ over the alphabet $\mathbb{P}$ modulo the smallest congruence relation extending the relations above. In particular, $\langle \mathbb{P} \rangle$ is a monoid, with identity given by (the class of) the empty sequence.

We will be interested in the semigroup $(\langle \mathbb{Y}(M) \rangle, \vee)$, and in particular in its subsemigroups $(\langle M\mathbf{y} \rangle, \vee)$ for $\mathbf{y} \in \mathbb{Y}(M)$, where $M\mathbf{y} = \{m\mathbf{y} \mid m \in M\}$ is the smallest suborder of $\mathbb{Y}(M)$ generated by $\mathbf{y}$ under the action of M. Notice that an element $\mathbf{y} \in \mathbb{Y}(M)$ is maximal if and only if it contains $1M$.

The following definition is also borrowed from [142], although there it does not have an explicit name.

Definition 4.1.12. We say that M is **$\mathbb{Y}$-controllable** if for every $\mathbf{y}$ maximal element in $\mathbb{Y}(M)$, for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for every finite coloring of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and such that for every $m, n \in \omega$ and for every $a_i, b_j \in M$ if $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} \in F$ and $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_m\mathbf{y}$, then $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$.

Once again, there are several other ways to formulate this definition in an equivalent way. Let us discuss for a moment some alternatives that might be useful to better grasp the idea behind this notion and the similarities and differences with other notions and theorems in combinatorics.

First, notice that we may always assume that $F \subseteq \langle M\mathbf{y} \rangle$, since the elements of F that are not in $\langle M\mathbf{y} \rangle$ play no role in the definition. The monochromatic set obtained in the definition can be described in another way that closely resembles the one used for Ramsey monoids. Indeed, for every $\mathbf{y} \in \mathbb{Y}(M)$, the order $\leq_{\mathbb{Y}}$ on $M\mathbf{y}$ induces a preorder $\leq_{\mathbf{y}}$ on M given by $a \leq_{\mathbf{y}} b$ if $a\mathbf{y} \leq_{\mathbb{Y}} b\mathbf{y}$. It is not difficult to see that the monoid $(\langle M \rangle, \vee)$ generated by $(M, \leq_{\mathbf{y}})$ is isomorphic to the monoid $\langle M\mathbf{y} \rangle$ generated by $(M\mathbf{y}, \leq_{\mathbb{Y}})$. This gives us an advantage: every element $f \in \langle M\mathbf{y} \rangle$ can be seen as a family (more precisely, an equivalence class) of located words of $W_M^{<\omega}$ (and thus of FIN_M, since W_M is the quotient of FIN_M). In other words: we may think of any $f \in F$ as the equivalence class $f = \{(n_i, m_i)_{i \leq h} \in \mathrm{FIN}_M \mid m_0\mathbf{y} \vee \cdots \vee m_h\mathbf{y} = f\}$.

Thus, if M acts by endomorphism on a semigroup S, given a sequence $\bar{s} \in S^\omega$ and an element $f \in \langle M\mathbf{y} \rangle$, we may denote with $\langle \bar{s} \rangle_f$ the f-span (where f is seen as an equivalence class of words of FIN_M) of $\bar{s}$, i.e. the set of all elements of the form $m_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ for $m_0\mathbf{y} \vee \cdots \vee m_n\mathbf{y} = f$.

This notation allows us to rewrite the previous definition.

Remark 4.1.13. A monoid M is $\mathbb{Y}$-controllable if and only if for every $\mathbf{y}$ maximal element in $\mathbb{Y}(M)$, for every finite $F \subseteq \langle M\mathbf{y} \rangle$, for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for every finite coloring of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence of variable words $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ with monochromatic f-span $\langle \bar{s} \rangle_f$ for every $f \in F$.

Once again, thanks to Lemma 4.1.10, we can restate the definition of $\mathbb{Y}$-controllable monoid in many equivalent ways using different classes of partial semigroups.

Proposition 4.1.14. *A monoid M is $\mathbb{Y}$-controllable if and only if for every $\mathbf{y}$ maximal element in $\mathbb{Y}(M)$ and for every finite $F \subseteq \langle M\mathbf{y} \rangle$, one of the following hold:*

(a) For every (partial) semigroup S on which M acts by endomorphisms, for every (basic) sequence $\bar{t} \in S^\omega$, for every finite coloring of S there is an infinite sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle_f$ is monochromatic for every $f \in F$.

(b) For every M-set X, for every $\bar{t} \in X^\omega$ and for every finite coloring of W_X there is $\bar{s} \leq_M \bar{t}$ infinite with monochromatic f-span $\langle \bar{s} \rangle_f$ for every $f \in F$.

*(c) For every uniform sequence of pointed M-sets $(X_n)_{n\in\omega}$, for every finite coloring of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence of **strongly** variable words $\bar{s}$ in $(\langle (X_n)_{n\in\omega} \rangle)^\omega$ with monochromatic f-span $\langle \bar{s} \rangle_f$ for every $f \in F$.*

(d) For all finite coloring of FIN_M *there is a basic sequence* $\bar{s} \in (\mathrm{FIN}_M)^\omega$ *of variable words such that* $\langle \bar{s} \rangle_f$ *is monochromatic for every* $f \in F$.

(e) There is a rapidly increasing sequence of variable words $\bar{t} \in (W_M)^\omega$ *such that for all finite coloring of* W_M *there is an infinite sequence* $\bar{s} \leq_M \bar{t}$ *with* $\langle \bar{s} \rangle_f$ *monochromatic for every* $f \in F$.

4.2 Basic notions of semigroup and monoid theory

In this section, we introduce the basic algebraic notions and definitions we are going to use throughout this and the next chapters.

A monoid is said **aperiodic** if for all $a \in M$ there exists $n \in \omega$ such that $a^n = a^{n+1}$. The class of finite aperiodic monoids has been widely studied, as it is involved in one of the most important theorems in finite automata theory, due to Schützenberger [134]. It states that star-free languages are exactly those languages whose syntactic monoid is finite and aperiodic. By a result of McNaughton and Papert, these also correspond to the languages definable in FO[$<$], i.e. first-order logic with signature $<$ [112].

A warning: there is another notion that is closely related to aperiodic monoids and goes by a similar name. A monoid is said **periodic** if for all $a \in M$ there exist distinct $i, j \in \omega$ such that $a^i = a^j$. Obviously, every aperiodic monoid is periodic. This should not cause confusion, as here “aperiodic” does not mean “not periodic”, but rather “cycle free”. In a similar way, we say that an element $a \in M$ has **finite period** k for $k \in \omega$, $k > 0$ if $a^{k+1} = a$ and $a^i \neq a$ for every $0 < i < k+1$. We say that an element $a \in M$ has **infinite period** if $a^i \neq a^j$ for every $i < j < \omega$.

One of the best ways to describe monoids and semigroups is using Green’s relations. They were first introduced by Green in his doctoral thesis and in [76]. The **Green’s relations** $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ on a monoid M are the equivalence relations defined by, respectively, $a \mathcal{R} b$ if $aM = bM$, $a \mathcal{L} b$ if $Ma = Mb$ and $a \mathcal{J} b$ if $MaM = MbM$. The Green’s relation $\mathcal{H}$ is the intersection of $\mathcal{R}$ and $\mathcal{L}$, while the Green’s relation $\mathcal{D}$ is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$. In every finite monoid, we have $\mathcal{D} = \mathcal{J}$. The same is true for aperiodic monoids (see e.g. [79]). The Green’s relations induce quasi-orders on the monoid. Given two element $a, b \in M$, define $a \leq_{\mathcal{R}} b$ if $aM \subseteq bM$, $a \leq_{\mathcal{L}} b$ if $Ma \subseteq Mb$, $a \leq_{\mathcal{J}} b$ if $MaM \subseteq MbM$, and finally $a \leq_{\mathcal{H}} b$ if both $a \leq_{\mathcal{R}} b$ and $a \leq_{\mathcal{L}} b$ hold. If $\mathcal{K}$ is an equivalence relation, we say that an equivalence class $[a]_{\mathcal{K}}$ is **trivial** if it contains exactly one element, and we say that a monoid M is $\mathcal{K}$**-trivial** if every $\mathcal{K}$-class is trivial. For more information about Green’s relations, see e.g. [37].

Among finite monoids, the class of aperiodic monoids can be characterized in many ways. We report here some of the most famous options used in literature. Among all possibilities, we isolate the notion of $\mathcal{R}$-rigid monoid as the operative definition we are going to use in the proofs of the next section.

Definition 4.2.1. A monoid is said $\mathcal{R}$**-rigid** if for every $a, b \in M$, if $ab \mathcal{R} b$, then $ab = b$.

Proposition 4.2.2. *Let* M *be a monoid. Then, each statement implies the followings:*

1. M *is aperiodic.*

2. *For every $g, a, g' \in M$, if $gag' = a$, then $ga = ag' = a$.*

3. *M is $\mathcal{R}$-rigid.*

4. *M is $\mathcal{H}$-trivial.*

5. *M contains no non-trivial subgroup.*

6. *Every element $a \in M$ of finite period is idempotent.*

Furthermore, if M is periodic (i.e. for every $a \in M$ the set $\{a^n \mid n \in \omega\}$ is finite) all above statements are equivalent.

For the ease of the reader, we report also a short proof of the equivalence.

Proof. First, assume 1, and let $g, a, g' \in M$ be such that $gag' = a$. By induction this implies $g^n a(g')^n = a$ for every $n \in \omega$. Choose n such that $g^{n+1} = g^n$ and $(g')^{n+1} = (g')^n$. Then, 2 holds since

$$ga = g(g^n a(g')^n) = g^{n+1}a(g')^n = g^n a(g')^n = a$$

and thus also $ag' = (ga)g' = a$.

If $ab \mathcal{R} b$, then by definition of $\mathcal{R}$ there exists $a' \in M$ such that $b = aba'$, hence 2 implies 3.

Notice that if $a, b \in M$ are such that $a \mathcal{H} b$, then in particular $a \mathcal{R} b$ and there is $x \in M$ such that $xb = a$. Thus $xb \mathcal{R} b$, and 3 implies 4.

Now if $G \subseteq M$ is a subgroup of M, then for every $a, b \in G$ there are x, y such that $ax = ya = b$, and symmetrically there are x', y' such that $bx' = y'b = a$, hence $a \mathcal{H} b$ and G is contained inside one single $\mathcal{H}$-class. Therefore, 4 implies 5.

If $a \in M$ is an element of finite period k, then $\{a, a^2, \dots, a^{k-1}\}$ is a cyclic subgroup of M, and it is trivial if and only if $k = 2$ and a is idempotent, hence 5 implies 6.

Finally, if M is periodic, then for every $a \in M$ there are minimal $n, k \in \omega$ such that $a^{n+k} = a^n$. Hence a^n has finite period $k + 1$, and 6 implies $k = 1$ and thus 1 hold. □

These conditions do not coincide in general for monoids that are not periodic. For example, $(\mathbb{N}, +, 0)$ is not aperiodic, but satisfies all other conditions. Also, the bicyclic monoid $M = \langle p, q \mid pq = \emptyset \rangle$ (i.e. the free monoid of words over $\{p, q\}$ quotient the relation $pq \sim \emptyset$) is $\mathcal{H}$-trivial, but for example $a = qp^2$ and $b = qp$ shows that it is not $\mathcal{R}$-rigid. Or again, $(\mathbb{Z}, +)$ shows that point 6 does not imply point 5.

However, we will work mostly with monoids that satisfies one condition that implies periodicity: for these monoids, all statements are equivalent. We anticipate this result, even if it uses

Remark 4.2.3. If $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq a\mathbf{y}\}$ is finite for every $a \in M$ and for every $\mathbf{y} \in \mathbb{Y}(M)$, then M is periodic and thus all statements of Proposition 4.2.2 are equivalent.

The class of aperiodic monoids is closed under most basic operations. For example, the following holds:

Proposition 4.2.4. *Let $(S_1, *_1), \dots, (S_n, *_n)$ be aperiodic semigroups. Then, the following are aperiodic:*

1. *The product monoid $S_1 \times \cdots \times S_n$ with coordinate-wise operation.*
2. *The disjoint union $S_1 \sqcup \cdots \sqcup S_n$ with any operation $*$ such that $a * b = a *_i b$ when $a, b \in S_i$.*

For more information about aperiodic monoids and their relations with languages and automata, see for example [102] or [125].

Let us move to the next class. Given a monoid M, recall that $\mathbb{X}(M)$ is the set of all principal right ideals generated by M. Notice that $\mathbb{X}(M)$ is linearly ordered by inclusion if and only if $\leq_{\mathcal{R}}$ is a total quasi-order. The existence of a total quasi-order on M affects the behaviour of Green's relations, and having that $\leq_{\mathcal{R}}$ is total has even stronger consequences. The next proposition collects some well-known properties of monoids where $\leq_{\mathcal{R}}$ is total (see [85, Proposition 3.18-3.20]).

Proposition 4.2.5. *Let M be a finite monoid with linear $\mathbb{X}(M)$. Then, the following hold:*

1. *For every $a \in M$, the principal right ideal aM is a both-sided ideal.*
2. *$\mathcal{J} = \mathcal{D} = \mathcal{R}$ and $\mathcal{L} = \mathcal{H}$, while $\leq_{\mathcal{R}} = \leq_{\mathcal{J}}$ and $\leq_{\mathcal{L}} = \leq_{\mathcal{H}}$.*
3. *$\mathcal{R}$ is a congruence relation.*
4. *$\leq_{\mathcal{R}}$ is translation-invariant on both sides.*

For more information about monoids with linear $\mathbb{X}(M)$, see for example [85].

Combining results about aperiodic monoids with results about monoids with linear $\mathbb{X}(M)$, one can obtain further properties and characterizations of the class of finite aperiodic monoids with linear $\mathbb{X}(M)$. For example, a finite monoid with linear $\mathbb{X}(M)$ is aperiodic if and only if it is $\mathcal{L}$-trivial. Notice that in light of Theorem 4.4.7 (and later on, Theorem 5.2.11), every property of this class of monoids will give a necessary condition for a monoid to be Ramsey.

Finally, let us introduce a seemingly new class of monoids, the class of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$.

Let $\mathbb{X}_{\mathcal{R}}(M)$ be the subset of $\mathbb{X}(M)$ of those aM such that $[a]_{\mathcal{R}}$ is non-trivial. We say that $\mathbb{X}_{\mathcal{R}}(M)$ is linear if it is linearly ordered by inclusion. Recall also that M is called **almost $\mathcal{R}$-trivial** if for every non-trivial $\mathcal{R}$-class $[a]_{\mathcal{R}}$ we have $Ma = \{a\}$ (see [142] and [96]).

Proposition 4.2.6. *Every finite almost $\mathcal{R}$-trivial monoid M is aperiodic and has linear $\mathbb{X}_{\mathcal{R}}(M)$.*

Proof. Let M be a finite almost $\mathcal{R}$-trivial monoid. First, we want to show that $\mathbb{X}_{\mathcal{R}}(M)$ has at most one element that is the minimum of $\mathbb{X}(M)$ (and so $\mathbb{X}_{\mathcal{R}}(M)$ is in particular linearly ordered by inclusion). If $[a]_{\mathcal{R}}$ is a non-trivial $\mathcal{R}$-class then, for every $m \in M$ we have $ma = a$, that means $a \in mM$ and $aM \subseteq mM$. Hence, if $[a]_{\mathcal{R}}$ and $[b]_{\mathcal{R}}$ are non-trivial $\mathcal{R}$-classes, then we have $aM = bM$. Now let us prove that M is aperiodic. Since M almost $\mathcal{R}$-trivial, then $Mb = \{b\}$ holds for every non-trivial $\mathcal{R}$-class. This in particular implies that $(Mb) \cap [b]_{\mathcal{R}} = \{b\}$ holds for every $\mathcal{R}$-class, and this is just a rephrasing of the $\mathcal{R}$-rigid condition. Then the claim follows from Proposition 4.2.2. □

Notice that the converse does not hold, as it is easy to show that there are aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ that have more than one non-trivial $\mathcal{R}$-class (for example, by combining almost $\mathcal{R}$-trivial monoids with point 2 of Proposition 4.2.4; see also Example 4.2.7). Also, there are aperiodic monoids that have non-trivial $\mathcal{R}$-classes $[a]_{\mathcal{R}}$ such that a is not idempotent (a minimal example is given by the monoid in Table 4.1, see also Example 4.2.7). These conditions are impossible for almost $\mathcal{R}$-trivial monoids, as shown in the proof of Proposition 4.2.6. Finally, there are examples of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ that do not have linear $\mathbb{X}(M)$ (the easiest examples coming from $\mathcal{R}$-trivial monoids). Thus, the class of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ properly extends both the class of almost $\mathcal{R}$-trivial monoids and the class of aperiodic monoids with linear $\mathbb{X}(M)$.

Example 4.2.7. Consider the Gowers' monoid $G_k = (\{0, \dots, k-1\}, \bar{+})$ with operation $i \bar{+} j = \min(i+j, k-1)$. Consider also the Carlson's semigroup $C_A = (A, *)$, i.e. a finite set A with operation $a * b = b$ for every $a, b \in A$. Let $C_A^1 = C_A \cup \{1_{C_A^1}\}$ be the corresponding monoid. Then, for every k and A the monoid $M = (G_k \times C_A^1)$ is aperiodic and has linear $\mathbb{X}_{\mathcal{R}}(M)$, while $\tilde{M} = (G_k \times C_A) \cup \{1_{\tilde{M}}\}$ is aperiodic, has linear $\mathbb{X}(\tilde{M})$ and all its $\mathcal{R}$-classes other than $[1_{\tilde{M}}]_{\mathcal{R}}$ are non-trivial. If $k \geq 2$, neither of these monoids is almost $\mathcal{R}$-trivial.

For those familiar with finite automata theory, Schützenberger's Theorem provides a wonderful way to produce examples of aperiodic monoids. Starting from a star-free language S, or from a formula in FO[<], we always generate a finite aperiodic syntactic monoid. For example, the monoid from Table 4.1 is the syntactic monoid of the star-free language S in the alphabet $A = \{a, g, h\}$ defined as

$$S = \{g, h\}^* h \cup \{g, h\}^* a \{g, h\}^* g \cup A^* a A^* a A^*$$

or, equivalently, defined by the formula in FO[<] that says *"the word is non-empty, and if it does not contain the letter a, then the word ends with h, and if there is exactly one letter a, then the word ends with g"*.

1	0	a	b	g	h
0	0	0	0	0	0
a	0	0	0	b	a
b	0	0	0	b	a
g	0	a	b	g	h
h	0	a	b	g	h

Table 4.1: Syntactic monoid of the language S.

4.3 Dynamic theory

In this section, we study the actions of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ on compact right topological semigroups. The main objective is to prove Theorem 4.3.5.

This result reveals the relation between aperiodic monoids and dynamic theory and it will be the key point to sufficient conditions for a monoid to be Ramsey or $\mathbb{Y}$-controllable. The advantage to work with compact right topological semigroups is that they are the common ground for many different techniques, either from logic or ergodic theory (see e.g. [18], [68], [108], [142], [154]).

Let us recall some notions. A semigroup $(U, \cdot)$ with a topology τ is a right topological semigroup if the map $x \mapsto x \cdot u$ is continuous from U to U for every $u \in U$. It is called compact if τ is compact. A element u in a semigroup $(U, \cdot)$ is called idempotent if $u \cdot u = u$. The set of idempotents of U is denoted by $E(U)$. We define a partial order $\leq_U$ in $E(U)$ by

$$u \leq_U v \iff uv = u = vu.$$

Finally, let $I(U)$ be the smallest compact both-sided ideal of U. It exists by compactness of U.

We report some facts about idempotents, corresponding to [154, Lemma 2.1, Lemma 2.3, Corollary 2.5].

Proposition 4.3.1. *Let U be a compact right topological semigroup. Then,*

1. *$E(U)$ is non-empty.*
2. *For every idempotent v there is a $\leq_U$-minimal idempotent u such that $u \leq_U v$.*
3. *Any both-sided ideal of U contains all the minimal idempotents of U.*

Fact 4.3.2. Let M be a monoid, let U be any set, and fix a left action of M on U. Then, for every $a, b \in M$ such that $aM \subseteq bM$ we have $a(U) \subseteq b(U)$.

Proof. In fact, if $bm = a$ for some $m \in M$, then $a(U) = b(m(U)) \subseteq b(U)$. □

In particular, if $a \,\mathcal{R}\, b$, then $a(U) = b(U)$.

Lemma 4.3.3. *Let M be an aperiodic monoid such that $X_{\mathcal{R}}(M)$ is linear. Then, for every distinct $a, b \in M$ with $a \,\mathcal{R}\, b$ there are two distinct $g, h \in M$ such that $ag = b$, $bh = a$ and $gh = h$, $hg = g$. This in particular implies $gM = hM$.*

Proof. Fix a non-trivial $\mathcal{R}$-class $[c]_{\mathcal{R}}$ and let $a, b \in [c]_{\mathcal{R}}$ with $a \neq b$.

For every $y, z \in M$, define

$$G_{y,z} = \{g_{y,z} \in M : yg_{y,z} = z\}.$$

Notice that if $y \,\mathcal{R}\, z$, then $G_{y,z}$ is non-empty. Let $\tilde{g} \in G_{a,b}$ and $\tilde{h} \in G_{b,a}$. Since M is aperiodic, there is $n \in \omega$ such that $(\tilde{g}\tilde{h})^n = (\tilde{g}\tilde{h})^{n+1}$ and $(\tilde{h}\tilde{g})^n = (\tilde{h}\tilde{g})^{n+1}$. Define $g = (\tilde{g}\tilde{h})^n\tilde{g}$ and $h = (\tilde{h}\tilde{g})^n\tilde{h}$. Then, we have

$$hgh = (\tilde{h}\tilde{g})^n\tilde{h}(\tilde{g}\tilde{h})^n\tilde{g}(\tilde{h}\tilde{g})^n\tilde{h} = (\tilde{h}\tilde{g})^{3n+1}\tilde{h} = (\tilde{h}\tilde{g})^n\tilde{h} = h,$$

and similarly $ghg = g$.

Notice that $\tilde{h}\tilde{g} \in G_{b,b}$, since $b\tilde{h}\tilde{g} = a\tilde{g} = b$, and so also $(\tilde{h}\tilde{g})^n \in G_{b,b}$. Thus, $h \in G_{b,a}$, since $bh = b(\tilde{h}\tilde{g})^n\tilde{h} = b\tilde{h} = a$. Similarly, $g \in G_{a,b}$.

Since $hg \in G_{b,b}$ and $G_{b,a} \cap G_{b,b} = \emptyset$, we have that $h \neq hg$. However, we have $hM = hghM \subseteq hgM \subseteq hM$, thus $hg \mathcal{R} h$ and the class $[h]_{\mathcal{R}}$ is non-trivial. Similarly, $g \mathcal{R} gh$ and $g \neq gh$, so the class $[g]_{\mathcal{R}}$ is non-trivial.

Since $\mathbb{X}_{\mathcal{R}}(M)$ is linear either $gM \subseteq hM$ or $hM \subseteq gM$. Suppose for example $gM \subseteq hM = hgM$. Then, $g = hgm$ for some $m \in M$, which implies $g = hg$, by Proposition 4.2.2. Hence, $gM = hgM = hM$, and $gh = h$ by Proposition 4.2.2. □

Lemma 4.3.4. *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear. Then, for every $a \in M$, if there are $b, c \in [a]_{\mathcal{R}}$ such that $bc = c$, then for every $b, c \in [a]_{\mathcal{R}}$ we have $bc = c$.*

Proof. First, notice that if $xy = y$ for some $x, y \in M$, then $xz = z$ for every $z \in [y]_{\mathcal{R}}$ since $xzM = xyM = yM = zM$ and since M is $\mathcal{R}$-rigid by Proposition 4.2.2.

Therefore, we just need to prove that given a non-trivial $\mathcal{R}$-class $[a]_{\mathcal{R}}$ such that $ax = x$ for every $x \in [a]_{\mathcal{R}}$, and given an element $b \in [a]_{\mathcal{R}}$ with $b \neq a$, then we have $ba = a$.

Let h be such that $bh = a$. Notice that $ha \mathcal{R} hb$ since $haM = hbM$, and also $ha \neq hb$ since $bha = a \neq b = bhb$. Then $haM \in \mathbb{X}_{\mathcal{R}}(M)$ and so $haM \subseteq aM$ or $aM \subseteq haM$.

If $haM \subseteq aM$ then

$$aM = aaM = bhaM \subseteq baM \subseteq bM = aM.$$

Hence, $ba \mathcal{R} a$ and $ba = a$.

If $aM \subseteq haM$ then $a = ham$ for some m, and by Proposition 4.2.2, $a = ha$. Hence, $ba = bha = a$. □

Theorem 4.3.5. *Let M be an aperiodic monoid. Let U be a compact right topological semigroup on which M acts by continuous endomorphisms. If $\mathbb{X}_{\mathcal{R}}(M)$ is linear and finite, then there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \mathcal{R} b$.*

Proof. Let $a_0M \subsetneq \ldots \subsetneq a_nM$ be an increasing enumeration of $\mathbb{X}_{\mathcal{R}}(M)$ and define $a_{n+1} = 1$. Every $a_i(U)$ is a semigroup, since $a_i(u_1) \cdot a_i(u_2) = a_i(u_1 \cdot u_2)$, and it is compact because it is a continuous image of a compact space. Then, $a_i(U)$ is a compact subsemigroup of the compact semigroup $a_{i+1}(U)$. We want to find a chain of idempotents u_i such that $u_{i+1} \leq_U u_i$ and such that u_i is minimal in $E(a_i(U))$ with respect to $\leq_{a_i(U)}$, for every $i \leq n+1$.

First, by points 1 and 2 of Proposition 4.3.1, we can find $u_0 \in a_0(U)$ satisfying the requirement. Then, suppose we have $u_i \in a_i(U)$ idempotent. Since $a_i(U) \subseteq a_{i+1}(U)$ we may apply point 2 of Proposition 4.3.1 to find $u_{i+1} \in a_{i+1}(U)$ idempotent such that $u_{i+1} \leq_{a_{i+1}(U)} u_i$ and u_{i+1} is minimal in $E(a_{i+1}(U))$, and this concludes the construction. Since $a_{n+1} = 1$ and $E(a_{n+1}(U)) = E(U)$, by point 3 of Proposition 4.3.1 we also know that $u_{n+1} \in I(U)$.

We claim that $u = u_{n+1}$ satisfies the requirements of the thesis.

First, we want to show that for each $\mathcal{R}$-class $[a_i]_{\mathcal{R}}$ with $a_ia_i = a_i$ we have

$$b(u) = u_i \text{ for all } b \in [a_i]_{\mathcal{R}}.$$

By Lemma 4.3.4, for every $b \in [a_i]_{\mathcal{R}}$ we have $ba_i = a_i$, and this implies that for every $v \in a_i(U)$, say $v = a_i(u_v)$, we have $b(v) = b(a_i(u_v)) = a_i(u_v) = v$. In particular for $v = u_i$, we have $b(u_i) = u_i$. Notice that the action of M is order-preserving on $(U, \leq_U)$ since it is by endomorphisms. Since $u \leq_U u_i$ we get

$$b(u) \leq_U b(u_i) = u_i.$$

Thus, $b(u) \leq_{a_i(U)} u_i$, and since u_i is minimal in $a_i(U)$, we get $b(u) = u_i$.

Now consider a non trivial $\mathcal{R}$-class $[a_i]_{\mathcal{R}}$ such that $a_i a_i \notin [a_i]_{\mathcal{R}}$, and let $a, b \in [a_i]_{\mathcal{R}}$. Let g, h be given as in Lemma 4.3.3 such that $ag = b$ and $bh = a$ and $hg = g$. Notice that this implies $bg = bhg = ag = b$ and also $h(u) = g(u)$, since $[g]_{\mathcal{R}}$ belongs to the previous case. Then, $a(u) = bh(u) = bg(u) = b(u)$. □

Notice that the proof of Theorem 4.3.5 does not rely on M being aperiodic: this hypothesis is used only to obtain the thesis of Lemmas 4.3.3 and 4.3.4. Also, it is possible to obtain idempotents of this form even for (aperiodic) monoids that do not satisfy the thesis of the two lemmas: see Example 5.4.8.

We take the opportunity to state a corollary of Lemma 4.3.3.

Corollary 4.3.6. *Let M be a finite aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear, let U be a set and fix a left action of M on U. Then, for every $a, b \in M$ with $a \,\mathcal{R}\, b$ and for every $u \in a(U)$ we have $a(u) = b(u)$.*

Proof. Let $a, b \in M$ be such that $a \,\mathcal{R}\, b$ and $a \neq b$, and let $g, h \in M$ be given by Lemma 4.3.3 such that $ag = b$ and $bh = a$, and $gh = h$ and $hg = g$. This in particular implies $gg = ghg = hg = g$, and $bg = bhg = ag = b$. Notice that by linearity of $\mathbb{X}_{\mathcal{R}}(M)$ either $a(M) \subseteq g(M)$ or $g(M) \subseteq a(M)$ holds, since both $[a]_{\mathcal{R}}$ and $[g]_{\mathcal{R}}$ are non-trivial. Then, we have $a(M) \subseteq g(M)$, since $|aM| = |bhM| \leq |hM| = |gM|$, and also $a(U) \subseteq g(U)$, by Fact 4.3.2. Fix $u \in a(U)$ and find $v \in U$ such that $u = g(v)$. We have

$$a(u) = a(g(v)) = a((gg)(v))) = ag(g(v))) = b(g(v)) = b(u).$$ □

4.4 Coloring theorems and aperiodic monoids

In this section, we discuss how to obtain sufficient conditions for a monoid to be Ramsey or $\mathbb{Y}$-controllable starting from Theorem 4.3.5 and by following ideas from Solecki's paper.

Let us recall some relevant notions for this section. Given a monoid M, the set $\mathbb{Y}(M) \subseteq \mathscr{P}(\mathbb{X}(M))$ consists of the non-empty subsets of $\mathbb{X}(M)$ which are linearly ordered by inclusion. Define $x \leq_{\mathbb{Y}} y$, for $x, y \in \mathbb{Y}(M)$, if and only if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x.

Notice that every partial order $(\mathbb{P}, \leq)$ generates a semigroup $\langle \mathbb{P} \rangle$, with operation $\vee$, defined as the semigroup freely generated by $\mathbb{P}$ modulo the relations

$$p \vee q = q = q \vee p \qquad \text{for all } p, q \in \mathbb{P} \text{ with } p \leq_{\mathbb{Y}} q$$

In particular, we will be interested in the semigroup $(\langle \mathbb{Y}(M) \rangle, \vee)$.

Notice that the action of M on itself by multiplication extends to a natural left action of M on $\mathbb{Y}(M)$ defined as $m(x) = \{maM \mid aM \in x\}$ for every $m \in M$ and

$x \in \mathbb{Y}(M)$. It is easy to check that this action is order-preserving, thus it extends to a left action by endomorphism of M on $(\langle \mathbb{Y}(M) \rangle, \vee)$ defined as

$$m(p_0 \vee \cdots \vee p_n) = m(p_0) \vee \cdots \vee m(p_n).$$

Recall that an embedding of a semigroup $(U, \cdot)$ into a semigroup $(U', *)$ is a map $f : U \to U'$ that is an isomorphism on the image, i.e. f is injective and for every $x, y \in U$ we have $f(x \cdot y) = f(x) * f(y)$. Given two left actions of M on U and U', a map $f : U \to U'$ is said M**-equivariant** if it preserves the action of M, i.e. $f(ma) = mf(a)$.

Definition 4.4.1. A monoid M is called **good** if for every left action of M by continuous endomorphisms on a compact right topological semigroup U there exists a function $h\colon \mathbb{Y}(M) \to E(U)$ such that

(i) h is M-equivariant;

(ii) h is order reversing with respect to $\leq_{\mathbb{Y}}$ and $\leq_U$;

(iii) h maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$.

In other words, a monoid M is good if whenever M is acting by continuous endomorphisms on a compact right topological semigroup U, then there exists also an M-equivariant embedding $h : \langle \mathbb{Y}(M) \rangle \to U$ such that

$$h[\{p_0 \vee \cdots \vee p_n \in \langle \mathbb{Y}(M) \rangle \mid p_i \text{ maximal in } \mathbb{Y}(M) \text{ for some } i \leq n\}] \subseteq I(U).$$

The notion of good monoids was first used by Solecki in [142]. We borrow here three results that are contained or essentially proved therein.

The following useful lemma has the same function as two other lemmas by Lupini [107, Lemma 2.2] and Barrett [13, Theorem 5.8], i.e. to get stronger conclusions from results like Theorem 4.3.5.

Lemma 4.4.2 ([142, Lemma 2.5])**.** *Let M be a finite monoid. Assume that for every left action of M by continuous endomorphisms on a compact right topological semigroup U there is a M-equivariant f from $\mathbb{Y}(M)$ to U such that f maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$. Then, M is good.*

We isolate the following lemma from the proof of [142, Theorem 2.4] since it gives a sufficient condition for a monoid to be good.

Lemma 4.4.3. *Let M be a finite monoid and assume that for every action by continuous endomorphisms of M on a compact right topological semigroup U there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \,\mathcal{R}\, b$. Then, M is good.*

Proof. Let $\pi : \mathbb{Y}(M) \to \mathbb{X}(M)$ be the function that maps a set $x \in \mathbb{Y}(M)$ to the maximal element in x with respect to $\subseteq$. Let $u \in E(U) \cap I(U)$ be given by hypothesis. The function $f : \mathbb{X}(M) \to E(U)$ that maps aM to $a(u)$ is well-defined, and maps $1M$ to $u \in E(U) \cap I(U)$. Also, notice that if y is a maximal element of $\mathbb{Y}(M)$, then $1M \in y$ and so $\pi \circ f(y) = u \in E(U) \cap I(U)$. Since both f and π are M-equivariant the map $f \circ \pi : \mathbb{Y}(M) \to E(U)$ satisfies the assumptions of Lemma 4.4.2, from which we get that M is good. □

Solecki in [142, Corollary 4.3] states that every finite almost $\mathcal{R}$-trivial monoid is $\mathbb{Y}$-controllable, but in the proof, he shows something stronger. The hypothesis that M is almost $\mathcal{R}$-trivial is used only to apply [142, Theorem 2.4], which states that every finite almost $\mathcal{R}$-trivial monoid is good. The remaining part of the proof never uses this hypothesis again and relies instead on the fact that M is good. In other words, from the proof of [142, Corollary 4.3] one can derive also the following result.

Theorem 4.4.4. *Let M be a finite monoid. If M is good, then it is $\mathbb{Y}$-controllable.*

However, the reader can find a short model-theoretic proof of this result in Section 4.5.

Finally, the following is a restatement of part of the proof of [142, Corollary 4.5 (i)].

Fact 4.4.5. If M is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear, then M is Ramsey.

Proof. Notice that $\mathbb{X}(M)$ is linear if and only if $\mathbb{X}(M) \in \mathbb{Y}(M)$. We want to use the definition of $\mathbb{Y}$-controllable with $\mathbf{y} = \mathbb{X}(M)$ and $F = \{\mathbf{y}\}$. It is enough to notice that for every $a \in M$ we have

$$a\mathbb{X}(M) = \{amM \mid mM \in \mathbb{X}(M)\} = \{xM \mid xM \subseteq aM\}.$$

Hence, if $aM \subseteq bM$, then $a\mathbf{y} \vee b\mathbf{y} = b\mathbf{y} = b\mathbf{y} \vee a\mathbf{y}$, and so $a_1\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = \mathbf{y} \in F$ for every $a_1, .., a_n \in M$ with at least one i such that $a_i = 1$. □

Theorem 4.4.6. *Let M be a finite monoid.*

1. *If M is aperiodic and has a linear $\mathbb{X}_{\mathcal{R}}(M)$, then it is $\mathbb{Y}$-controllable.*
2. *If M is $\mathbb{Y}$-controllable, then it is aperiodic.*

Proof. First, let M be a finite aperiodic monoid with linear $\mathbb{X}_{\mathcal{R}}(M)$. By Theorem 4.3.5 and Lemma 4.4.3, we get that M is good. Hence, Theorem 4.4.4 implies that M is $\mathbb{Y}$-controllable, and statement 1 holds.

In order to prove 2, let $(M^{<\omega}, ^\frown)$ be the free semigroup over M, with coordinatewise action. Notice that $(M^{<\omega}, ^\frown)$ can be seen as $\langle (X_n)_{n<\omega}\rangle$ setting all $X_n = M$, with 1 as distinguished point, and a word w has a distinguished point if and only if $1 \in \operatorname{ran} w$ (in which case we call w a variable word).

Suppose M is not aperiodic, and let $a \in M$ be such that $a^{n+1} \neq a^n$ for every $n \in \omega$. Let $A = \{a^n \mid n \in \omega\}$, and let $C = \{m \in M \mid a^n m \in A \text{ for some } n \in \omega\}$. Then, we have $ac \neq c$ for every $c \in C$, and $ac \in C$ if and only if $c \in C$.

Let $\mathbf{y} = \{a^n M \mid n \in \omega\}$, where we set $a^0 = 1$, and let $F = \{\mathbf{y}\}$. Then, $\mathbf{y}$ is a maximal element of $\mathbb{Y}(M)$, and $\mathbf{y} \vee \mathbf{y} = \mathbf{y} = a\mathbf{y} \vee \mathbf{y}$.

Let $C \cup \{\bot\}$ be the set of colors. Given a word $w \in M^{<\omega}$, let m be the first letter of w in C, if any. If there is such m, color w by m. Otherwise, color w by $\bot$. Consider any sequence of variable words $\bar{y} \in (M^{<\omega})^\omega$, and consider the words $y_0{}^\frown y_1$ and $a(y_0){}^\frown y_1$ with colors c_1 and c_2 respectively. Then, $c_1 \in C$, since y_0 is a variable word and $1 \in \operatorname{ran}(y_0)$. Hence, by definition of C we have $c_2 = ac_1$. Therefore, $c_2 = ac_1 \neq c_1$, contradicting the fact that M is $\mathbb{Y}$-controllable. □

Theorem 4.4.7. *Let M be a finite monoid. The following are equivalent:*

1. *M is Ramsey.*

2. *M is aperiodic and $\mathbb{X}(M)$ is linear.*

Proof. Proof of point 2 of Theorem 4.4.6 also shows that if M is Ramsey then it is aperiodic. If M is Ramsey, then $\mathbb{X}(M)$ is linear by [142, Corollary 4.5 (ii)]. Theorem 4.4.6 and Fact 4.4.5 prove that 2 implies 1. □

Corollary 4.4.8. *Let M be a finite monoid. Then, M is Ramsey if and only if it is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear.*

4.5 A model-theoretic approach

In this section, we give a short explicit proof of Theorem 4.4.4 using model theory as an alternative approach to ultrafilters.

First, let us recall some basic definitions and facts of model theory. From now on, we consider a semigroup G on which M acts by endomorphisms. Let $L = (\cdot, \prec)$ be the first-order language with one single function $(- \cdot -)$ of ariety 2 denoting the operation the partial semigroup, and a binary relation $\prec$. Let $L^+ = L \cup \{A \mid A \subseteq G\} \cup \{m \mid m \in M\}$ be the expansion of L where we add one unary relation symbol for each subset of G and one function of ariety 1 for each element of our monoid (much of the language is redundant, but it is added for sake of simplicity). Then G is a L^+-structure, where each symbol $A(x)$ for $A \subseteq G$ is interpreted in G as $x \in A$, every $m \in M$ is interpreted as the unary function given by the action of m over G. We denote with $\mathcal{G}$ a monster model for G in the language L^+, i.e. $\mathcal{G} \supset G$ is an elementary extension of G that is also a $< \kappa$-saturated and $< \kappa$-homogeneous for some cardinal κ large enough. For our purpose, it is enough to take $k > |\mathscr{P}(G)|$.

Given $A \subseteq \mathcal{G}$, we denote with $L^+(A)$ the language $L^+ \cup \{a \mid a \in A\}$ where we added a constant symbol for each element of A. A **partial 1-type** over A is a set of formulas in the language $L^+(A)$ with one free variable. A **(complete) 1-type** $p(x)$ over A is a maximal partial 1-type over A, i.e. a type such that for each formula $\phi(x)$ with parameters in A, either $\phi(x)$ or $\neg\phi(x)$ belongs to $p(x)$. Given an element $a \in \mathcal{G}$, we denote with $\mathrm{tp}(a/A)$ the complete 1-type over A generated by a, i.e. the set of all formulas with one free variable and with parameters in A that are satisfied by a. We write $a \vDash p(x)$ to say that every formula of $p(a)$ holds in $\mathcal{G}$. Given $a, b \in \mathcal{G}$, we write $a \equiv_A b$ for $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. We say that a 1-type $p(x)$ over A is **finitely satisfied** in G if every finite conjunction of formulas in $p(x)$ has a solution in G.

We define $S(G)$ to be the space of all complete 1-types over G in the free variable x that are finitely satisfied in G. Notice that since $\mathcal{G}$ is a model of the theory of G, the type over G of every element of the monster $\mathcal{G}$ is finitely satisfiable in G. In other words, we have $S(G) = \{\mathrm{tp}(a/G) \mid a \in \mathcal{G}\}$. Also, taking parameters in G is redundant (since we can already refer to elements of G in the language L^+ without parameters), but we prefer to stick to this notation as a reminder of this.

It is easy to see that the space of types $S(G)$ defined this way is just another version of Stone–Čech compactification βG. Indeed, since we have a unary relation $A(x)$ for every $A \subseteq G$, a type $p(x) \in S(G)$ identifies a unique ultrafilter $\{A \subseteq G \mid A(x) \in p(x)\}$, and conversely every ultrafilter U over G identifies a unique type $\{\phi(x) \mid [G \vDash \phi(x) \leftrightarrow A(x)] \wedge A \in U\}$. However, by using types we can benefit from the language and tools of logic and model theory, which allows us to significantly simplify the notation and the proofs.

As it happens for example in $\beta\mathbb{N}$, we can introduce an operation on $S(G)$ that turns it into a compact right topological semigroup. Given an element $a \in \mathcal{G}$ and $A \subseteq \mathcal{G}$, we write $a \downfree_G A$ if $\mathrm{tp}(a/G \cup A)$ is finitely satisfied in G[2]. In other words, the relation $a \downfree_G A$ is saying that the type of a over $G \cup A$ is uniquely determined by the type of a over G: indeed, for every formula $\phi(x, y_0, \ldots, y_n)$ in $L^+(G)$ and for every tuple $(a_0, \ldots, a_n) \in A^{n+1}$, let $R = \{g \in G \mid \mathcal{G} \vDash \phi(g, a_0, \ldots, a_n)\}$. Then $\neg[\phi(x, a_0, \ldots, a_n) \leftrightarrow R(x)]$ is not satisfiable in G, hence $R(x) \in \mathrm{tp}(a/G)$ if and only if $\phi(x, a_0, \ldots, a_n) \in \mathrm{tp}(a/G \cup \{b\})$.

When $A = \{b\}$ is a singleton, we write $a \downfree_G b$. Given two types $p, q \in S(G)$, we define $p \cdot_G q$ as $\mathrm{tp}(a \cdot b/G)$, for any $a, b \in \mathcal{G}$ such that $a \vDash p(x)$, $b \vDash q(x)$ and $a \downfree_G b$.

As usual, we will consider the compact topology τ on $S(G)$ generated by the basic open sets $\{t \in S(G) \mid \phi(x) \in t\}$, for $\phi(x) \in L^+(G)$.

Notice that we can identify the semigroup $(G, \cdot)$ with an open discrete subsemigroup of $(S(G), \cdot_G, \tau)$: indeed, for every $g \in G$, the atomic formula $x = g$ defines a unique type $p_g(x)$ that is finitely consistent in G (corresponding to a principal ultrafilter), and thus $G \cong \{p_g(x) \in S(G) \mid g \in G\}$.

With this operation and topology,

Proposition 4.5.1. *$(S(G), \cdot_G, \tau)$ is a compact right topological semigroup on which M acts by continuous endomorphisms by $m(\mathrm{tp}(a/G)) = \mathrm{tp}(m(a)/G)$.*

Proof. Most of the required properties follow immediately from [39] (see e.g. Remark 2.7 and Propositions 4.4 and 6.3). However, for sake of completeness, we prefer to give a complete and self-contained proof of the statement.

First, in order to prove that the operation is well defined it is enough to show that for every formula $\phi(x, y)$ in $L^+(G)$ and for every $a, a', b, b' \in \mathcal{G}$, if we have $a \downfree_G b$ and $a' \downfree_G b'$ and $a \equiv_G a'$, and $b \equiv_G b'$, then $\phi(a, b)$ holds in $\mathcal{G}$ if and only if $\phi(a', b')$ holds in $\mathcal{G}$. Since $b \equiv_G b'$, for every $g \in G$ we have that $\phi(g, b)$ holds in $\mathcal{G}$ if and only if $\phi(g, b')$ holds in $\mathcal{G}$. Let

$$A = \{g \in G \mid \mathcal{G} \vDash \phi(g, b)\} = \{g \in G \mid \mathcal{G} \vDash \phi(g, b')\}.$$

Then $\neg[A(x) \leftrightarrow \phi(x, b)]$ is not satisfiable in G, and since $\mathrm{tp}(a/G \cup \{b\})$ is finitely satisfied in G, we have that $\phi(a, b)$ holds in $\mathcal{G}$ if and only if $A(a)$ hold in $\mathcal{G}$. Similarly $\phi(a', b')$ holds in $\mathcal{G}$ if and only if $A(a')$ hold in $\mathcal{G}$. But $a \equiv_G a'$, hence $A(a)$ holds in $\mathcal{G}$ if and only if $A(a')$ and holds in $\mathcal{G}$.

Next, we want to show the associativity of the operation. First, notice that for every type $p(x) \in S(G)$ and $A \subseteq \mathcal{G}$ of small size (i.e. less than the saturation degree of the monster model), there is always an element $a \in \mathcal{G}$ such that $a \vDash p(x)$ and $a \downfree_G A$: it is enough to extend $p(x)$ to a complete type $p'(x)$ over $G \cup A$ finitely satisfiable in G. This type will be realized by some element a of the monster model (by saturation), and a will have all the desired properties. Secondly, notice that for every $a, b \in \mathcal{G}$ and $A \subseteq \mathcal{G}$ of small size (with $a, b \notin A$), if $a \downfree_G \{b\} \cup A$ and $b \downfree_G A$, then $a \cdot b \downfree_G A$. In fact, let $\phi(x)$ be a formula in $L^+(G \cup A)$ satisfied by

[2]In literature this relation is also denoted by $a \downfree_G A$.

$a \cdot b$. Then $\psi_1(x) = \phi(x \cdot b)$ is a formula in $L^+(G \cup A \cup \{b\})$ satisfied by a, and since $a \mathop{\smile\!\!\!\!\!\mid}_G A \cup \{b\}$ there is $g \in G$ such that $\psi_1(g) = \phi(g \cdot b)$ hold in $\mathcal{G}$. Now let $\psi_2(x) = \phi(g \cdot x)$: since $b \mathop{\smile\!\!\!\!\!\mid}_G A$, there is $g' \in G$ such that $\psi(g') = \phi(g \cdot g')$ hold in $\mathcal{G}$, as wanted.

Therefore, given $p, q, r \in S(G)$, consider $a, b, c \in \mathcal{G}$ such that $b \mathop{\smile\!\!\!\!\!\mid}_G c$ and $a \mathop{\smile\!\!\!\!\!\mid}_G \{b, c, b \cdot c\}$. Then we also have $ab \mathop{\smile\!\!\!\!\!\mid}_G c$, and thus

$$p \cdot_G (q \cdot_G r) = \mathrm{tp}(a \cdot (b \cdot c)/G) = \mathrm{tp}((a \cdot b) \cdot c/G) = (p \cdot_G q) \cdot_G r.$$

The action of M on $S(G)$ is well-defined and continuous. Indeed, for every formula $\phi(x)$ in $L^+(G)$, let $\psi(x) = \phi(m(x))$: it is still a formula in $L^+(G)$. This implies first that if $a \equiv_G b$ then $m(a) \equiv_G m(b)$, and thus action is well-defined, and secondly that $m^{-1}[\{p(x) \in S(G) \mid \phi(x) \in p(x)\}] = \{p(x) \in S(G) \mid \psi(x) \in p(x)\}$ and thus m continuous. Furthermore, if $a \mathop{\smile\!\!\!\!\!\mid}_G b$ then also $m(a) \mathop{\smile\!\!\!\!\!\mid}_G m(b)$, since for every $\psi(x) \in \mathrm{tp}(m(a)/G \cup \{m(b)\})$ there is $\phi(x) \in \mathrm{tp}(a/G \cup \{b\})$ such that $\psi(x) = \phi(m(x))$. Thus M acts by endomorphism, since

$$m(p \cdot_G q) = m(\mathrm{tp}(a \cdot b/G)) = \mathrm{tp}(m(a \cdot b)/G) = \mathrm{tp}(m(a) \cdot m(b)/G) = m(p) \cdot_G m(q).$$

Finally, we need to prove that for every $r \in S(G)$, the map $f_r : S(G) \to S(G)$, $f_r(p) = p \cdot_G r$ is continuous. Let $\phi(x)$ be a formula in $L^+(G)$. Choose an element $b \in \mathcal{G}$ such that $b \models r(x)$, and define $A_b = \{g \in G \mid \mathcal{G} \models \phi(g \cdot b)\}$. Notice that for every other $b' \in \mathcal{G}$ satisfying $b' \models r(x)$, we have $b' \equiv_G b$ and thus $A_{b'} = A_b$. Then $\neg[A_b(x) \leftrightarrow \phi(x \cdot b)]$ is a formula in $L^+(G \cup \{b\})$ that is not satisfiable in G. Hence, for every type $p(x) \in S(G)$, and for every $a, b \in \mathcal{G}$ with $a \models p(x)$ and $b \models r(x)$, if $a \mathop{\smile\!\!\!\!\!\mid}_G b$ we must have that $A_b(a)$ holds in $\mathcal{G}$ if and only if $\phi(a \cdot b)$ holds in $\mathcal{G}$. This implies that

$$f_r^{-1}[\{p(x) \in S(G) \mid \phi(x) \in p(x)\}] = \{p(x) \in S(G) \mid A_b(x) \in p(x)\}$$

and thus f_r is continuous. □

We say that the infinite tuple $\bar{c} = (c_i)_{i<\omega}$ is a **coheir sequence** of $p(x)$ over G if $c_n \models p(x)$ and $c_n \mathop{\smile\!\!\!\!\!\mid}_G \{c_i \mid i < n\}$ and $c_{n+1} \equiv_{G \cup \{c_0, \dots, c_{n-1}\}} c_n$ for every $n < \omega$. We say that $\bar{c}$ is **indiscernible** over G if for any formula $\phi(x_0, \dots, x_n)$ in $L^+(G)$ and for any natural numbers $i_0 > \cdots > i_n$ and $k_0 > \cdots > k_n$ we have that $\phi(c_{i_0}, \dots, c_{i_n})$ holds in $\mathcal{G}$ if and only if $\phi(c_{k_0}, \dots, c_{k_n})$ holds in $\mathcal{G}$. The following is a well-known fact but we prove it here for the reader's convenience.

Fact 4.5.2. Every coheir sequence over G is also indiscernible over G.

Proof. Suppose not and let n be minimal such that there exist a formula $\phi(x_0, \dots, x_n)$ in $L^+(G)$ and $i_0 > \cdots > i_n$ and $k_0 > \cdots > k_n$ such that $\phi(c_{i_0}, \dots, c_{i_n})$ and $\neg\phi(c_{k_0}, \dots, c_{k_n})$ hold in $\mathcal{G}$. Assume without loss of generality that $i_0 \geq k_0$. Since $c_{i_0} \equiv_{G \cup \{c_0, \dots, c_{k_0 - 1}\}} c_{k_0}$, we have that also $\phi(c_{i_0}, \dots, c_{i_n}) \wedge \neg\phi(c_{i_0}, c_{k_1}, \dots, c_{k_n})$ hold in $\mathcal{G}$. But $\mathrm{tp}(c_{i_0}/G \cup \{c_0, \dots, c_{i_0 - 1}\})$ is finitely satisfied in G, hence we may find $g \in G$ such that both $\phi(g, c_{i_1}, \dots, c_{i_n})$ and $\neg\phi(g, c_{k_1}, \dots, c_{k_n})$ hold in $\mathcal{G}$, contradicting the minimality of n. □

The following is an easy well-known fact.

Fact 4.5.3. For every type $p(x) \in S(G)$ there is a coheir sequence of $p(x)$.

Proof. We proceed by induction. First, by saturation of $\mathcal{G}$ and since $p(x)$ is finitely satisfiable in G, we may find $c_0 \in \mathcal{G}$ satisfying $p(x)$. Now given $(c_i)_{i<n+1}$ coheir sequence, let $p'_n(x) = \mathrm{tp}(c_n/G \cup \{c_0, \dots, c_{n-1}\})$, and then extend $p'_n(x)$ to a complete type $p_n(x)$ over $G \cup \{c_0, \dots, c_n\}$ that is finitely satisfiable in G. Again, since $G \cup \{c_0, \dots, c_n\}$ has small size and since $p_n(x)$ is finitely satisfiable in G, by saturation we may find $c_{n+1} \in \mathcal{G}$ such that $c_{n+1} \vDash p_n(x)$. Now: since $p_n(x)$ is a complete type over $G \cup \{c_0, \dots, c_n\}$, we have $\mathrm{tp}(c_{n+1}/G \cup \{c_0, \dots, c_n\}) = p_n(x)$, which is finitely satisfiable in G by construction, and thus $c_{n+1} \mathop{\smile\!\!\!\!\!\mid}_G \{c_0, \dots, c_n\}$. Since $p(x) \subseteq p_n(x)$, we have $c_{n+1} \vDash p(x)$. Finally, since $\mathrm{tp}(c_n/G \cup \{c_0, \dots, c_{n-1}\}) \subseteq p_n(x)$, we have $c_{n+1} \equiv_{G \cup \{c_0, \dots, c_{n-1}\}} c_n$, as wanted. □

We are ready to prove Theorem 4.4.4. Let us introduce the following auxiliary definition to simplify the notation of the next proof.

Definition 4.5.4. Let F be a finite subset of the semigroup $\langle \mathbb{Y}(M) \rangle$, let $\mathbf{y}$ be a maximal element in $\mathbb{Y}(M)$, and let c be a finite coloring of a semigroup S on which M acts. We say that a sequence $\bar{s} \in S^{\leq \omega}$ is $(F, \mathbf{y}, c)$**-controllable** if for every $m, n \leq |\bar{s}|$ and for every $a_i, b_j \in M$ <u>if</u> $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y}$ belongs to F and $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_m\mathbf{y}$, <u>then</u> $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$.

Notice that we can always assume that $F \subseteq \langle M\mathbf{y} \rangle$, since elements of F that are not of the form $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y}$ have no influence in the above definition.

Proof of Theorem 4.4.4. We want to prove that for every finite subset F of $\langle \mathbb{Y}(M) \rangle$, for every maximal element $\mathbf{y}$ in $\mathbb{Y}(M)$, for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for every c finite coloring of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ that is $(F, \mathbf{y}, c)$-controllable and such that s_n has a distinguished point for every $n \in \omega$.

So let $(X_n)_{n\in\omega}$ be a sequence of pointed M-sets on which M acts uniformly, and let $\perp$ be not in $\bigcup_{n\in\omega} X_n$. Define $G = (\langle (X_n)_{n\in\omega} \rangle \cup \{\perp\}, \cdot)$ to be the semigroup extending $(\langle (X_n)_{n\in\omega} \rangle, ^\frown)$ defining $x \cdot y = \perp$ if $x ^\frown y$ is not defined in the partial semigroup $\langle (X_n)_{n\in\omega} \rangle$, and $x \cdot y = x ^\frown y$ otherwise. In particular, we have $x \cdot \perp = \perp \cdot x = \perp$. We write $x \prec y$ if and only if $x \cdot y \neq \perp$. We may also extend the coordinate-wise action of M over $\langle (X_n)_{n\in\omega} \rangle$ to the whole G by letting $m(\perp) = \perp$ for every $m \in M$. By definition of endomorphism, $m(a \cdot b)$ is defined if and only if $m(a) \cdot m(b)$ is defined for every $a, b \in \langle (X_n)_{n\in\omega} \rangle$, so the action of M on G is still well defined and by endomorphism.

Then $(G, \cdot, \prec)$ is a semigroup as wanted, and we can define the language L^+, a monster model $\mathcal{G}$ for G and the compact right topological semigroup $(S(G), \cdot_G, \tau)$ as described above.

Let $e(x) = \{\text{“}g \prec x\text{”} \mid g \in G \setminus \{\perp\}\}$: it is a partial type. Notice that $e(x)$ is finitely satisfiable, since for every finite set $A \subseteq G \setminus \{\perp\}$ there is a $b \in G$ such that $a \prec b$ for every $a \in A$. Define

$$U = \{p(x) \in S(G) \mid e(x) \subseteq p(x)\}$$

to be the set of all complete types in $S(G)$ extending $e(x)$.

Claim. U is a non-empty compact right topological subsemigroup of $(S(G), \cdot_G, \tau)$ closed under the action of M.

Proof. First, since $e(x)$ is finitely satisfiable, then U is non-empty. It is also clear that U is a closed (and thus compact) subset of $S(G)$, since U is type definable and thus it is the intersection of sets defined by a single formula, which are clopen by definition. Next, notice that for every $p(x) \in U$, we have $\perp \not\models p(x)$, and thus "$x \neq \perp$" $\in p(x)$ by completeness of the type. Given two elements $p(x), q(x) \in U$, let $a, b \in \mathcal{G}$ be such that $a \models p(x)$ and $b \models q(x)$ and $a \mathop{\smile\!\!\!\!\!\!\!\;\,|}_G b$. Then, we must have that $a \prec b$ and $g \cdot a \prec b$ for every $g \in G \setminus \{\perp\}$, since $b \models e(x)$ and the formulae $\neg[x \prec b] \wedge [x \neq \perp]$ and $\neg[g \cdot x \prec b] \wedge [g \prec x]$ are not satisfiable in G. Hence, for every $g \in G \setminus \{\perp\}$ we have $g \cdot (a \cdot b) = (g \cdot a) \cdot b \neq \perp$, which implies $g \prec (a \cdot b)$ and thus $e(x) \subseteq p(x) \cdot_G q(x) \in U$, so U is closed under the operation of $S(G)$. Finally, notice that since M acts coordinate-wise on $\langle (X_n)_{n\in\omega} \rangle = G \setminus \{\perp\}$, the formula $g \prec x \wedge \neg[g \prec m(x)]$ is not satisfiable in G. Hence, for every $g \in G$ and $a \in \mathcal{G}$ we have that $g \prec a$ implies $g \prec m(a)$, and thus $m(p) \in U$ for every $p \in U$ and $m \in M$. □

Now fix a maximal element $\mathbf{y} \in \mathbb{Y}(M)$ and let $u(x) = h(\mathbf{y}) \in E(U) \cap I(U)$, where $h : \langle \mathbb{Y}(M) \rangle \to U$ is the M-equivariant embedding given by definition of good monoid. Let $\mathrm{DP} \subseteq G$ be the set of elements of $\langle (X_n)_{n\in\omega} \rangle$ that have at least one distinguished point. Notice that DP is a both-sided ideal in G. Since being an ideal is expressible in $L^+(G)$ by the formula $\forall x, y[\mathrm{DP}(y) \to \mathrm{DP}(x \cdot y) \wedge \mathrm{DP}(y \cdot x)]$, and since $\mathcal{G}$ models the theory of G, then DP defines a both-sided ideal of the monster model $\mathrm{DP}(\mathcal{G}) = \{a \in \mathcal{G} \mid \mathcal{G} \models \mathrm{DP}(a)\}$. Thus,

$$J = \{p(x) \in U \mid \text{“}\mathrm{DP}(x)\text{”} \in p(x)\}$$

is also a both-sided ideal of U. It is also non-empty, since $e(x) \cup \{\mathrm{DP}(x)\}$ is finitely satisfiable in G. Since $u(x) \in I(U)$ and I contains all non-empty both-sided ideals of U, we have that $u(x)$ is in J.

Fix also a finite coloring (i.e. partition) $c' = \{C_i \mid i < r\}$ of $\langle (X_n)_{n\in\omega} \rangle$. We can extend c' to a coloring c of the whole G by adding $\{\perp\}$ to c'. Notice that since the coloring c is finite and since $L^+(G)$ contains a relation symbol for every subset of G, then c is definable in $L^+(G)$, and thus c extends to a finite coloring of $\mathcal{G}$

$$\tilde{c} = \{\{a \in \mathcal{G} \mid \mathcal{G} \models C_i(a)\} \mid i < r\}.$$

Let $(u_n)_{n\in\omega}$ be a coheir sequence of $u(x)$. We write $\overleftarrow{u}_{\restriction i}$ for the tuple $u_{i-1}, \ldots, u_0$.

Claim. For every $i \in \omega$ and $F \subseteq \langle M\mathbf{y} \rangle$, the sequence $\overleftarrow{u}_{\restriction i}$ is $(F, \mathbf{y}, \tilde{c})$-controllable.

Proof. Since the map h is an M-equivariant embedding of $\langle \mathbb{Y}(M) \rangle$ into $E(U)$, for every $a_0, \ldots, a_n, b_0, \ldots, b_m \in M$ we have

$a_n\mathbf{y} \vee \cdots \vee a_0\mathbf{y} = b_m\mathbf{y} \vee \cdots \vee b_0\mathbf{y}$ if and only if $a_n u \cdot_G \ldots \cdot_G a_0 u = b_m u \cdot_G \ldots \cdot_G b_0 u$.

We just need to check that $a_k(u_k) \mathop{\smile\!\!\!\!\!\!\!\;\,|}_G \{a_j(u_j) \mid j < k\}$ for every $k \leq n$. Indeed, since $L^+(G)$ contains a function symbol for every element of M, then for every $a \in \mathcal{G}$, for every $A \subseteq \mathcal{G}$ of small size we have

$$\mathrm{tp}(a/G \cup A) = \mathrm{tp}(a/G \cup \{m(b) \mid m \in M, b \in A\}).$$

Furthermore, for every formula $\phi(x)$ there is a formula $\psi(x) = \phi(m(x))$ such that $\phi(m(a))$ holds if and only if $\psi(a)$ holds. Thus, if $\mathrm{tp}(a/G \cup A)$ is finitely satisfiable over G, then $\mathrm{tp}(m(a)/G \cup \{m(b) \mid m \in M, b \in A\})$ is as well.

Therefore, $a_n u_n \cdot \ldots \cdot a_0 u_0$ satisfies $a_n u(x) \cdot_G \ldots \cdot_G a_0 u(x)$ and $b_m u_m \cdot \ldots \cdot b_0 u_0$ satisfies $b_m u(x) \cdot_G \ldots \cdot_G b_0 u(x)$. Since the color of an element of $\mathcal{G}$ is determined by its type over G, this implies immediately that for every choice of $F \subseteq \langle M\mathbf{y} \rangle$ (not even necessarily finite) and $i \in \omega$, we have that $\overleftarrow{u}_{\restriction i}$ is $(F, \mathbf{y}, \tilde{c})$-controllable. □

Now fix a finite subset $F \subseteq \langle M\mathbf{y} \rangle$. Every element $f \in \langle \mathbb{Y}(M) \rangle$ has a (unique) representation of minimal length $f = p_0 \vee \cdots \vee p_j$ for some $p_0, \ldots, p_j \in \mathbb{Y}(M)$. We write $|f|$ the denote the length $j \in \omega$ of this minimal representation. Since $\mathbf{y}$ is a linearly ordered subset of $\mathbb{X}(M)$, if $f, f', f'' \in \langle M\mathbf{y} \rangle$ are such that $f' \vee f'' = f$, then we must have $|f'| \leq |f|$ and $|f''| \leq |f|$.

Therefore, there exists $k \in \omega$ such that for every $f' \in \langle M\mathbf{y} \rangle$, if $f \vee f' \in F$ for some $f \in \langle M\mathbf{y} \rangle$, then $|f'| < k$.

Claim. For every $l > k$ and for every sequence $\bar{g} = (g_0, \ldots, g_n)$ in G, the sequence $\bar{g}^\frown \overleftarrow{u}_{\restriction l}$ is $(F, \mathbf{y}, \tilde{c})$-controllable if and only if $\bar{g}^\frown \overleftarrow{u}_{\restriction k}$ is $(F, \mathbf{y}, \tilde{c})$-controllable.

Proof. It is clear that if $\bar{g}^\frown \overleftarrow{u}_{\restriction l}$ is $(F, \mathbf{y}, \tilde{c})$-controllable then so is $\bar{g}^\frown \overleftarrow{u}_{\restriction k}$.

Let $f_g = a_0\mathbf{y} \vee \cdots \vee a_h\mathbf{y}$ and $f_u = a_{h+1}\mathbf{y} \vee \cdots \vee a_m\mathbf{y}$, and $f'_g = b_0\mathbf{y} \vee \cdots \vee b_{h'}\mathbf{y}$ and $f'_u = b_{h'+1}\mathbf{y} \vee \cdots \vee b_{m'}\mathbf{y}$. Suppose

$$f_g \vee f_u = a_0\mathbf{y} \vee \cdots \vee a_m\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_{m'}\mathbf{y} = f'_g \vee f'_u \in F$$

Let $i_0 < \cdots < i_h \leq n$, and $i_m < \cdots < i_{h+1} < l$, and $i'_0 < \cdots < i'_{h'} \leq n$, and $i'_{m'} < \cdots < i'_{h'+1} < l$, and let $a_g = a_0 g_{i_0} \cdot \ldots \cdot a_h g_{i_h}$, and $a_u = a_{h+1} u_{i_{h+1}} \cdot \ldots \cdot a_m u_{i_m}$, and $b_g = b_0 g_{i'_0} \cdot \ldots \cdot b_{h'} g_{i'_{h'}}$ and $b_u = b_{h'+1} u_{i'_{h'+1}} \cdot \ldots \cdot b_{m'} u_{i'_{m'}}$.

We want to show that

$$\tilde{c}(a_g \cdot a_u) = \tilde{c}(b_g \cdot b_u).$$

First, since $f_g \vee f_u \in F$, we have $j = |f_u| < k$. Let $d_0, \ldots, d_j \in M$ be such that $f_u = d_j\mathbf{y} \vee \cdots \vee d_0\mathbf{y}$. Let $a_d = d_j u_j \cdot \ldots \cdot d_0 u_0$: then, as shown in previous claim, a_u and a_d satisfies the same type over G. In particular, for every color $C \in \tilde{c}$, we have that $C(a_g \cdot x)$ is in the type of a_u if and only if it is in the type of a_d. Thus, $\tilde{c}(a_g \cdot a_u) = \tilde{c}(a_g \cdot a_d)$.

In the same way, $\tilde{c}(b_g \cdot b_u) = \tilde{c}(b_g \cdot b_d)$ for some $b_d = d'_{j'} u_{j'} \cdot \ldots \cdot d'_0 u_0$ such that $j' < k$ and $f'_u = d'_{j'}\mathbf{y} \vee \cdots \vee d'_0\mathbf{y}$.

However, since $\bar{g}^\frown \overleftarrow{u}_{\restriction k}$ is $(F, \mathbf{y}, \tilde{c})$-controllable and since

$$f_g \vee f_u = a_0\mathbf{y} \vee \cdots \vee a_h\mathbf{y} \vee d_j\mathbf{y} \vee \cdots \vee d_0\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_{h'} \vee d'_{j'}\mathbf{y} \vee \cdots \vee d'_0\mathbf{y} = f'_g \vee f'_u,$$

then we have $\tilde{c}(a_g \cdot a_d) = \tilde{c}(b_g \cdot b_d)$. □

Notice that up to now we never used the fact that M is finite.

Now, to conclude the proof, we want to pass from the sequence $(u_n)_{n \in \omega}$ in the monster model to a basic $(F, \mathbf{y}, c)$-controllable sequence $\bar{s}$ in $\langle (X_n)_{n \in \omega} \rangle$.

First, since F, M and c are finite and since we added all subsets of G to the language L^+, for every $n \in \omega$ there are a formula $\phi_n(x_0, \ldots, x_n)$ saying that $(x_0, \ldots, x_n)$

is $(F, \mathbf{y}, c)$-controllable, a formula $\psi_n(x_0, \ldots, x_n)$ saying that $(x_0, \ldots, x_n)$ is a basic sequence, and a formula $\bigwedge_{i \le n} \mathrm{DP}(x_i)$ saying that each word has a distinguished point. (In this context, we say that a sequence $(s_0, \ldots, s_n) \in \mathcal{G}^n$ is basic if $s_i \prec s_j$ for every $i < j \le n$, i.e. if its n-type is satisfiable only by basic sequences of $\langle (X_n)_{n\in\omega} \rangle$.) Let

$$\xi_n(x_0, \ldots, x_n) = \phi_n(x_0, \ldots, x_n) \wedge \psi_n(x_0, \ldots, x_n) \wedge \bigwedge_{i \le n} \mathrm{DP}(x_i).$$

We proceed recursively. Assume that we have a (possibly empty) basic sequence $\bar{s}_{\restriction i} \in \mathrm{DP}^i$ of words with a distinguished point such that $\bar{s}_{\restriction i} {}^\frown \bar{u}_{\restriction k}$ is $(F, \mathbf{y}, c)$-controllable (i.e., $\xi_{i+k}(\bar{s}_{\restriction i}, \bar{u}_{\restriction k})$ hold in $\mathcal{G}$). The empty sequence satisfies this by the previous claim, so the base case is ok. Our goal is to find $s_i \in \langle (X_n)_{n\in\omega} \rangle$ such that the same properties hold for $\bar{s}_{\restriction i+1}$.

By induction hypothesis $\bar{s}_{\restriction i} {}^\frown \bar{u}_{\restriction k}$ is $(F, \mathbf{y}, c)$-controllable, and so by previous claim $\bar{s}_{\restriction i} {}^\frown \bar{u}_{\restriction k+1}$ is as well. Notice that $\bar{u}_{\restriction i}$ is a basic sequence, since U is closed under products and $\perp \notin U$. Also, since $e(x) \subseteq u(x)$ and $u_h \vDash u(x)$ for every $h \le k$, we have $s_j \prec u_i$ for every $j < i$ and $h \le k$. These and the fact that $\bar{s}_{\restriction i}$ is basic by induction hypothesis imply that $\bar{s}_{\restriction i}, \bar{u}_{\restriction k+1}$ is basic as well. Finally, $u_k \vDash u(x)$ and $u(x) \in J$, thus $u_k \vDash \mathrm{DP}(x)$, and other coordinates satisfies $\mathrm{DP}(x)$ as well by induction hypothesis. Therefore, $\mathcal{G} \vDash \xi_{i+k+1}(\bar{s}_{\restriction i}, u_k, \bar{u}_{\restriction k})$. Since $u_k \mathop{\smile\!\!\!\!|}_G \{u_0, \ldots, u_{k-1}\}$, the type $\mathrm{tp}(u_k / G \cup \{u_0, \ldots, u_{k-1}\})$ is finitely satisfiable in G, hence we may find $s_i \in G$ such that $\xi_{i+k+1}(\bar{s}_{\restriction i}, s_i, \bar{u}_{\restriction k})$ hold in $\mathcal{G}$, and we are done.

Notice that $\perp \notin \mathrm{DP}$, thus $(s_i)_{i\in\omega} \in \langle (X_n)_{n\in\omega} \rangle$ as wanted. □

4.6 Final remarks and open problems

We conclude with some open questions and remarks concerning the work done so far.

In the definition of M-span, we ask that at least one element of the basic sequence is moved by 1. Thanks to 4.4.8 (and Proposition 5.1.3), we can relax this condition.

Proposition 4.6.1. *Let M be a finite Ramsey monoid. Then, for any partial semigroup S, for any finite coloring of S and for every sequence $\bar{t} \in S^\omega$ there is $\bar{s} \le_M \bar{t}$ such that for every $a \in M$ the set*

$$\{m_0\, s_{i_0} \cdots m_n\, s_{i_n} \mid i_0 < \cdots < i_n, m_i \in aM, m_i \,\mathcal{R}\, a \text{ for at least one } i\}$$

is monochromatic.

Proof. By Corollary 4.4.8, M is $\mathbb{Y}$-controllable. Then, the thesis follows from the definition of $\mathbb{Y}$-controllable monoid applied to the maximal element $\mathbf{y} = \mathbb{X}(M)$ and to $F = \{a\mathbf{y} \mid a \in M\}$, and by Proposition 5.1.3. □

In the previous corollary, the action of M can be controlled with $|\mathbb{X}(M)|$-many colors. This is optimal, as in general, it is not possible to get less than $|\mathbb{X}(M)|$-many monochromatic sets. For example, choose $\mathbb{X}(M)$ as set of colors, and color each word $w \in M^{<\omega}$ by the minimum aM such that $\mathrm{ran}(w) \subseteq aM$: then, if $\bar{t}$ is a sequence of variable words, for any $\bar{s} \le_M \bar{t}$ each set defined above has a different color.

When instead M is $\mathbb{Y}$-controllable but $\mathbb{X}(M)$ is not linear, it is not difficult to see that for any $k \in \omega$ there are $\mathbf{y}$ and $F \subseteq \mathbb{Y}(M)$ and colorings of, say, $M^{<\omega}$ such that for every sequence of variable words $\bar{s}$ there are more than k-many $f \in F$ such that the sets

$$\langle \bar{s} \rangle_f = \{a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_m} \mid a_i \in M, i_0 < \cdots < i_m, a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = f\}$$

have different colors.

The next theorem is a generalization of both Theorem 4.4.7 and Milliken-Taylor theorem [115], [151]. It is a combination of Ramsey's theorem and Theorem 4.4.7, in the same way as the Milliken-Taylor Theorem is a combination of Ramsey's theorem and Hindman's theorem. For a sequence $\bar{s} \in S^\omega$ let $\bar{s}^{(n)}$ be the collection of n-subsets of $\{a \in S \mid a = s_i \text{ for some } i \in \omega\}$. Notice that for $n = 1$ the following is the content of Theorem 4.4.7.

Theorem 4.6.2. *Let M be a finite Ramsey monoid. Then, for any $n \geq 1$, for all sequences of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly, for any finite coloring of n-subsets of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and such that $\bigcup_{\bar{r} \leq_M \bar{s}} \bar{r}^{(n)}$ is monochromatic.*

Proof. The proof goes as in Theorem 4.4.4, in section 4.5. Let G and $u = g(\mathbf{y})$ be defined as in Theorem 4.4.4, with $\mathbf{y} = \mathbb{X}(M)$, and let $(u_n)_{n\in\omega}$ be a coheir sequence of u. It is straightforward to check that all elements of the span of $\overleftarrow{u}_{\restriction i}$ satisfy the type u for every $i \in \omega$. Also, notice that with signature L^+, for every $a, a', b \in \mathcal{G}^{<\omega}$ if $a \equiv_G a'$, $a' \mathop{\smile\!\!\!\!\!\!\;|}_G b$, and $a \mathop{\smile\!\!\!\!\!\!\;|}_G b$, then $a \equiv_{Mb} a'$. Then, for any $\overleftarrow{h} \leq_M \overleftarrow{u}$ we have $\overleftarrow{h} \equiv_G \overleftarrow{u}$, by the remark above and the definition of coheir sequence. All the n-subsets of an indiscernible sequence have the same color, for any $n \in \omega$. The rest of the proof is the same as in Theorem 4.4.4. ☐

Notice that here as well, thanks to Lemmar 4.1.10 we can extend this result to any partial semigroup.

It can be easily seen that if a monoid satisfies the conclusions of Proposition 4.6.1 or the conclusions of Theorem 4.6.2, then it is Ramsey. Conversely, Proposition 4.6.1 and Theorem 4.6.2 hold for all finite Ramsey monoids. Hence, their conclusions hold for a finite monoid if and only if it is Ramsey.

Our main theorems suggest a possible connection between Ramsey theory and automata theory, passing through Schützenberger's Theorem. Any result in that direction would be of the highest interest.

Limiting ourselves to Ramsey theory, there are still several challenging open questions in the context of monoid actions on semigroups.

Theorem 4.4.6 provides a sufficient condition for a monoid to be $\mathbb{Y}$-controllable. This condition is not necessary, as there are $\mathbb{Y}$-controllable monoids for which $\mathbb{X}_{\mathcal{R}}(M)$ is not linear.

Proposition 4.6.3. *Let M be a finite aperiodic monoid such that for every distinct $a, b \in M$ with $a \,\mathcal{R}\, b$, we have $a^2 = a$ and $ax = bx$ for every $x \in M \setminus \{1\}$. Then, M is $\mathbb{Y}$-controllable.*

Proof. To show that M is $\mathbb{Y}$-controllable is enough to work with $M^{<\omega}$, thanks to Proposition 5.1.3.

Consider the monoid $\tilde{M} = (M, *)$ where $x * y = y$ for all $x, y \neq 1$. It acts coordinate-wise on $M^{<\omega}$, considered as $\tilde{M}^{<\omega}$.

Let G be $M^{<\omega}$ with the signature L^+ used in the proof of Theorem 4.4.4, plus an unary function $\tilde{a}$ for any $a \in M$, which is interpreted in G as the action of $\tilde{M}$. Since $\tilde{M}$ is Ramsey and since every element in $\tilde{M}$ different from 1 is in the same $\mathcal{R}$-class, one can find an idempotent u in the space of types $S(G)$ such that $\tilde{a}u = \tilde{b}u$ for every $a, b \neq 1$. Let v be an element of the monster model satisfying u. Then, if $a \,\mathcal{R}\, b$, we have

$$av = a\tilde{a}v \equiv_G a\tilde{b}v = bv,$$

where we use the fact that for every $x \in M^{<\omega}$, and hence for every x in the monster model, we have $a\tilde{a}x = ax$ and $a\tilde{b}x = bx$, by hypothesis. Hence, we can conclude that M is $\mathbb{Y}$-controllable, by the arguments of Theorem 4.4.4. □

An example of a monoid satisfying the hypothesis of Proposition 4.6.3 for which $\mathbb{X}_{\mathcal{R}}(M)$ is not linear is given by the following Cayley table.

1	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Table 4.2: Example of (aperiodic) $\mathbb{Y}$-controllable monoid M such that $\mathbb{X}_{\mathcal{R}}(M)$ is not linear.

On the other hand, it seems possible that the necessary condition of Theorem 4.4.6 is also sufficient, and that a finite monoid is aperiodic if and only if it is Y-controllable. If true, this would suggest an even stronger connection between Ramsey theory and Schützenberger's Theorem.

Open Problem 4.6.4. *Find an algebraic characterization of* $\mathbb{Y}$*-controllable monoids.*

If M is a Ramsey monoid, then *for every action of* M *on every partial semigroup* you have a monochromatic set as described in the definition. Lupini's in [107] gave examples of non-Ramsey monoids where the same statement holds for *certain actions on certain partial semigroups* (actually, he proved a stronger statement that can be seen as the analogue of Proposition 4.6.1).

Define I_k to be the set of functions f from k to k such that $f(0) = 0$ and such that if $f(i) = j$ then either $f(i+1) = j$ or $f(i+1) = j+1$. Then, I_k is a monoid with composition of functions as operation, and k is an I_k-set with distinguished point $k-1$, where the action is defined by $fi = f(i)$. This action induces a coordinate-wise action on $\mathrm{FIN}_k = \langle(\{n\} \times k)_{n\in\omega}\rangle$ (i.e. the set of all partial functions with finite domain from $\mathbb{N}$ to k). Lupini in [107] showed that for every $k \in \omega$ and for every finite coloring of FIN_k there is an infinite sequence of words in FIN_k each containing $k-1$ such that its I_k-span is monochromatic. Notice that this result implies that every $\mathcal{R}$-trivial monoid is Ramsey. In fact, let N be a $\mathcal{R}$-trivial monoid with linear $\mathbb{X}(N)$. Without loss of generality, we may assume that $N = \{0, \ldots, k-1\}$

and that $0N \subseteq \cdots \subseteq (k-1)N$ is an increasing enumeration of $\mathbb{X}(N)$. Then, the coordinate-wise action of N on FIN_k coincides with the action of a submonoid of I_k, by Proposition 4.2.5, and Lupini's theorem implies point (d) of Proposition 4.1.11.

All monoids I_k are $\mathcal{R}$-trivial, but $\mathbb{X}(I_k)$ is linear if and only if $k \leq 3$ (see [142, Section 4 .4]). In particular, if $k > 3$ these monoids are not Ramsey, and Lupini's result does not follow from the theory of Ramsey monoids. It would be interesting to see if a similar statement holds for other (non-Ramsey) monoids.

Open Problem 4.6.5. *Classify the couples (M, k) such that $k \in \omega$ is a pointed M-set and for every finite coloring of FIN_k there is a basic sequence $\bar{s}$ in FIN_k such that s_n has a distinguished point for every $n \in \omega$ and such that the M-span of $\bar{s}$ is monochromatic.*

In the same direction, the following seems a challenging problem.

Open Problem 4.6.6. *Characterize the class of triples $(S, M, \bar{t})$, where S is a partial semigroup, M is a monoid acting on S by endomorphisms and $\bar{t}$ is a basic sequence in S, for which for every finite coloring of S there is a sequence $\bar{s} \leq_M \bar{t}$ in S such that its M-span is monochromatic.*

One can check that every finite Ramsey monoid generates examples of Ramsey spaces. However, an even nicer property might be true: there are topological Ramsey spaces that induce a collection of projected spaces such that every metrically Baire set has the Ramsey property. A sufficient condition for the latter has been found by Dobrinen and Mijares in [54]. An example of a space of this form is Carlson-Simpson space, see [32] and [154, section 5.6]). See also [54, section 4] for generalizations of the latter.

Open Problem 4.6.7. *Which topological Ramsey spaces given by finite Ramsey monoids meet the sufficient conditions given in [54]?*

Hales-Jewett theorem [77] is a corollary of Proposition 4.6.1 for the special case of monoids M such that $ab = b$ for every $a, b \in M \setminus \{1\}$. In Ramsey theory, two of the strongest known results are a polynomial generalization [20] and a density generalization [69] of Hales-Jewett theorem for these monoids.

Open Problem 4.6.8. *Do polynomial or density results hold for other finite Ramsey monoids?*

Ojeda-Aristizabal in [123] obtained upper bounds for the finite version of Gowers' FIN_k theorem, giving a constructive proof. It would be interesting to know if these upper bounds hold for other Ramsey monoids.

The work of Gowers on FIN_k and the related space $\mathrm{FIN}_{\pm k}$ was the key to his solution of an old problem in Banach spaces [74]. Also, the aforementioned example of Bartošova and Kwiatkowska found applications in metric spaces. Finally, a discussion about the connection between Ramsey spaces and Banach spaces can be found in Todorcevic's monograph. In this chapter, we found new Ramsey monoids, and consequently new Ramsey spaces. Hence, it might be possible to find applications of these new results to metric spaces.

Recently various papers have found different common generalizations of Carlson's and Gowers' theorems, see [13], [91], [108]. Of particular interest is the context of adequate layered semigroups, introduced by Farah, Hindman, and McLeod [61] and recently studied by Lupini [108] and Barrett [13]. Barrett's paper [13] describes a framework which seems well suited for a connection between Ramsey monoids and layered semigroups. His work and ours are independent from each other and were written concurrently, so we did not investigate this research line. Nevertheless, in Example 4.6.9 and in the following paragraph we show a possible connection.

Example 4.6.9. Let M be a monoid with linear $\mathbb{X}(M)$, and let $a_0M \subseteq \cdots \subseteq a_nM$ be an increasing enumeration of $\mathbb{X}(M)$. Define $\ell : \mathrm{FIN}_M \to n+1$ by

$$\ell(w) = \min\{i \mid \mathrm{ran}(w) \subseteq a_iM\}.$$

Then, (FIN_M, ℓ) is an adequate partial layered semigroup as defined in [13, Definition 3.7]. Furthermore, the canonical action $\mathcal{F}_{\mathrm{cw}}$ of M on FIN_M is made of regressive maps, by Proposition 4.2.5.

This example shows that every monoid with linear $\mathbb{X}(M)$ generates an adequate partial layered semigroup, FIN_M, and a family of regressive functions $\mathcal{F}_{\mathrm{cw}}$. On the other hand, every family of regressive functions $\mathcal{F}$ on an adequate partial layered semigroup generates a monoid $M_{\mathcal{F}}$ with composition, acting on S by endomorphisms.

Chapter 5

Infinite monoids in combinatorics

5.1 Introduction to locally Ramsey and locally $\mathbb{Y}$-controllable monoids

In the previous chapter, we gave an algebraic characterization of finite Ramsey monoids and provided necessary and sufficient conditions for a finite monoid to be $\mathbb{Y}$-controllable. In this chapter, we extend this study to infinite monoids. What we aim to achieve is a bit more general than this. We study two bigger classes of monoids, locally Ramsey and locally $\mathbb{Y}$-controllable, which are more suited for infinite monoids, and allow us to extend other theorems in combinatorics, like the infinite Carlson's Theorem for words on an infinite alphabet [31].

First, let us briefly introduce the main definitions we will use throughout the chapter. Notice that for every set M and for every sequence $(M_i)_{i\in\omega}$ of finite subsets of M, we have that $\langle(\{n\}\times M_n)_{n\in\omega}\rangle$ is a partial subsemigroup of FIN_M. Let M be a monoid acting by endomorphisms on a partial semigroup S, let $\bar{s}$ be a sequence of elements of S, and let $(M_i)_{i\in\omega}$ be a sequence of finite subsets of M. Given a family $C\subseteq \mathrm{FIN}^{<\omega}$, define $C'=C\cap\langle(\{i\}\times M_i)_{i\in\omega}\rangle$. Then, we define the $C,(M_i)_{i\in\omega}$**-span** as $\langle\bar{s}\rangle^C_{(M_i)_{i\in\omega}}=\langle\bar{s}\rangle_{C'}$. In other words:

$$\langle\bar{s}\rangle^C_{(M_i)_{i\in\omega}}=\big\{m_0\,s_{i_0}\cdots m_n\,s_{i_n}\mid m_h\in M_{i_h},((i_0,m_0),\dots,(i_n,m_n))\in C\big\}.$$

Similarly to previous chapter, we define the $(M_i)_{i\in\omega}$**-span** $\langle\bar{s}\rangle_{(M_i)_{i\in\omega}}$ of $\bar{s}$ as the set $\langle\bar{s}\rangle^{\mathcal{V}_M}_{(M_i)_{i\in\omega}}$ for $\mathcal{V}_M$ the set of variable words of W_M. In other words, a word $m_0\,s_{i_0}\cdots m_n\,s_{i_n}$ is in $\langle\bar{s}\rangle_{(M_i)_{i\in\omega}}$ if and only if $i_0<\cdots<i_n<\omega$ and $m_h\in M_{i_h}$ for every $h<n$ and $m_h=1$ for at least one $h\le n$.

Once again, thanks to Lemma 4.1.10 we have that different combinatorial properties define the same class of monoids (see also the proof of Proposition 4.1.11).

Proposition 5.1.1. *Let M be a monoid, and let $(M_i)_{i\in\omega}$ be a family of finite subsets of M. Then, the following are equivalent:*

(a) For every (partial) semigroup S on which M acts by endomorphisms, for every (basic) sequence $\bar{t}\in S^\omega$, for every finite coloring of S there is a sequence $\bar{s}\le_M\bar{t}$ such that $\langle\bar{s}\rangle_{(M_i)_{i<\omega}}$ is monochromatic.

(b) For every M-set X, for every $\bar{t}\in X^\omega$ and for every finite coloring of W_X there is an infinite $\bar{s}\le_M\bar{t}$ such that $\langle\bar{s}\rangle_{(M_i)_{i<\omega}}$ is monochromatic.

(c) For every uniform sequence of pointed M-sets $(X_n)_{n\in\omega}$, for every finite coloring of $\langle (X_n)_{n\in\omega}\rangle$ there is a basic sequence of (strongly) variable words $\bar{s}$ in $(\langle (X_n)_{n\in\omega}\rangle)^\omega$ such that $\langle \bar{s}\rangle_{(M_i)_{i<\omega}}$ is monochromatic.

(d) For all finite coloring of FIN_M there is a basic sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that $\langle \bar{s}\rangle_{(M_i)_{i<\omega}}$ is monochromatic.

(e) There is a rapidly increasing sequence of variable words $\bar{t} \in (W_M)^\omega$ such that for all finite coloring of W_M there is an infinite sequence $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s}\rangle_{(M_i)_{i<\omega}}$ monochromatic.

Definition 5.1.2. A monoid M is called **locally Ramsey** if it satisfies one of the equivalent conditions of Proposition 5.1.1.

Similarly, we define a local version of $\mathbb{Y}$-controllable monoids as well.

Proposition 5.1.3. *Let M be a monoid, let $\mathbf{y}$ be a maximal element of $\mathbb{Y}(M)$, let $F \subseteq \langle M\mathbf{y}\rangle$ be finite and let $(M_i)_{i\in\omega}$ be a family of finite subsets of M. Then, the following are equivalent:*

(a) For every (partial) semigroup S on which M acts by endomorphisms, for every (basic) sequence $\bar{t} \in S^\omega$, for every finite coloring of S there is a sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s}\rangle^f_{(M_i)_{i<\omega}}$ is monochromatic for every $f \in F$.

(b) For every M-set X, for every $\bar{t} \in X^\omega$ and for every finite coloring of W_X there is an infinite $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s}\rangle^f_{(M_i)_{i<\omega}}$ is monochromatic for every $f \in F$.

(c) For every uniform sequence of pointed M-sets $(X_n)_{n\in\omega}$, for every finite coloring of $\langle (X_n)_{n\in\omega}\rangle$ there is a basic sequence of (strongly) variable words $\bar{s}$ in $(\langle (X_n)_{n\in\omega}\rangle)^\omega$ such that $\langle \bar{s}\rangle^f_{(M_i)_{i<\omega}}$ is monochromatic for every $f \in F$.

(d) For all finite coloring of FIN_M there is a basic sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that $\langle \bar{s}\rangle^f_{(M_i)_{i<\omega}}$ is monochromatic for every $f \in F$.

(e) There is a rapidly increasing sequence of variable words $\bar{t} \in (W_M)^\omega$ such that for all finite coloring of W_M there is an infinite sequence $\bar{s} \leq_M \bar{t}$ with $\langle \bar{s}\rangle^f_{(M_i)_{i<\omega}}$ monochromatic for every $f \in F$.

In order to give more precise results, we state the notion in a slightly different way.

Definition 5.1.4. Let M be a monoid, let $\mathbf{y}$ be a maximal element in $\mathbb{Y}(M)$, let $F \subseteq \langle M\mathbf{y}\rangle$ be finite, and let $(M_i)_{i\in\omega}$ be a family of subsets of M. We say that M is $(F, \mathbf{y}, (M_i)_{i\in\omega})$**-controllable** if it satisfies one of the equivalent conditions of Proposition 5.1.3.

We say that M is $(F, \mathbf{y})$**-controllable** if it is $(F, \mathbf{y}, (M_i)_{i<\omega})$-controllable for every family $(M_i)_{i<\omega}$ of subsets of M (equivalently: for $(M_i)_{i<\omega} = (M)_{i<\omega}$). M is **locally** $(F, \mathbf{y})$**-controllable** if it is $(F, \mathbf{y}, (M_i)_{i<\omega})$-controllable for every family $(M_i)_{i<\omega}$ of *finite* subsets of M.

Similarly, we say that M is **(locally) y-controllable** if it is (locally) $(F, \mathbf{y})$-controllable for every finite $F \subseteq \langle M\mathbf{y}\rangle$. It is clear then that $\mathbb{Y}$-controllable monoids are exactly $\mathbf{y}$-controllable monoids for every maximal $\mathbf{y} \in \langle \mathbb{Y}(M) \rangle$. We define **locally $\mathbb{Y}$-controllable** monoids as those are locally $\mathbf{y}$-controllable for every maximal $\mathbf{y} \in \langle \mathbb{Y}(M) \rangle$.

5.2 Necessary conditions for Ramsey classes

First, we start by studying which properties are necessary to be locally Ramsey or locally $\mathbb{Y}$-controllable.

The following is a straightforward generalization of [142, Corollary 4.5(ii)] to locally Ramsey monoids.

Proposition 5.2.1. *If M is locally Ramsey then $\mathbb{X}(M)$ is linear.*

Proof. Suppose $\mathbb{X}(M)$ is not linear, and let aM and bM such that $aM \nsubseteq bM$ and $bM \nsubseteq aM$. By $aM \nsubseteq bM$ we have $a \notin bM$, and by $bM \nsubseteq aM$ we have $b \notin aM$. Consider the semigroup of words $(M^{<\omega}, ^\frown)$ and color each word $w \in M^{<\omega}$ with red if the letter a is in w and it appears before the first appearance of b. Otherwise, color w with blue. Then, any sequence $(M_i)_{i\in\omega}$ of finite subsets of M each containing $\{1, a, b\}$ witness that for every sequence $\bar{s} \in (M^{<\omega})^\omega$ we have that $a(s_1)^\frown s_2$ has color red and $b(s_1)^\frown s_2$ has color blue. Hence, the (M_i)-span of $\bar{s}$ is not monochromatic and M is not locally Ramsey. □

Notice that the definition of (locally) Ramsey monoids coincide exactly with the definition of (locally) $(F, \mathbf{y})$-controllable for $\mathbf{y} = \mathbb{X}(M)$ and $F = \{\mathbf{y}\}$. The latter definition requires that $\mathbb{X}(M)$ is linear. By the previous proposition, however, this is always true, and hence we get the following.

Corollary 5.2.2. *A monoid M is (locally) Ramsey if and only if it has has linear $\mathbb{X}(M)$ and it is (locally) $(F, \mathbf{y})$-controllable for $\mathbf{y} = \mathbb{X}(M)$ and $F = \{\mathbf{y}\}$.*

Next, we show that being aperiodic is necessary for this context as well.

Proposition 5.2.3. *Let M be a locally $\mathbb{Y}$-controllable or locally Ramsey monoid. Then, M is aperiodic.*

Proof. Suppose M is not aperiodic, and let $a \in M$ be such that $a^n \neq a^{n+1}$ for every $n \in \omega$. Let $\mathbf{y}$ be such that $a\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$ (e.g. $\mathbf{y} = \{a^n M \mid n \in \omega\}$ if M is locally $\mathbb{Y}$-controllable, or $\mathbf{y} = \mathbb{X}(M)$ if M is locally Ramsey) and let $F = \{\mathbf{y}\}$. We want to prove that M is not locally $(F, \mathbf{y})$-controllable.

Let $A = \{a^n \mid n \in \omega\}$, let $C = \{m \in M \mid a^n m \in A \text{ for some } n \in \omega\}$. We convene that $a^0 = 1$ is the identity of the monoid. Notice that $c \in C$ if and only if $ac \in C$, and furthermore, $ac \neq c$ for every $a \in M$ and $c \in C$.

Let $(M_i)_{i\in\omega}$ be a sequence of finite subsets of M such that $\{a^n \mid n \leq i\} \subseteq M_i$ for every $i \in \omega$. Notice that $a^n\mathbf{y} \vee \mathbf{y} = \mathbf{y}$ for every $n \in \omega$.

Consider the semigroup of words $(M^{<\omega}, ^\frown)$, and suppose first A is finite. Let $A \cup \{\bot\}$ be the set of colors, for $\bot$ some element not in A. Given a word $w \in M^{<\omega}$, color w with $\bot$ if w contains no letter in A. Otherwise, color w by m, for m be the first letter of A that appears in w. Then, given a sequence of variable words $\bar{s} \in (M^{<\omega})^\omega$, let $n \in \omega$ be such that $A \subseteq M_n$, let c be the first letter of s_n in C and let $k \in \omega$ be such that $a^k c \in A$: then $a^{k+1}c = a(a^k c) \neq a^k c$. Then, $a^k(s_n) ^\frown s_{n+1}$ has color $a^k c$ while $a^{k+1}(s_n) ^\frown s_{n+1}$ has color $a^{k+1}c$.

If instead A is infinite, then $a^i \neq a^j$ for every $i \neq j$. Given a word $w \in M^{<\omega}$, color w with black if w contains no letter in C. Otherwise, let c be the first letter of w in C. Let $n = \min\{i \in \omega \mid a^i c \in A\}$ and let k be such that $a^n c = a^k$. Color w with red if $n - k$ is odd, and with blue if $n - k$ is even. Then, for every sequence of variable words $\bar{s} \in (M^{<\omega})^\omega$ it is clear that $s_0 ^\frown s_1$ and $a(s_0) ^\frown s_1$ have different colors.

In any case, M is not locally $\mathbb{Y}$-controllable nor locally Ramsey. □

When in the previous chapter we restricted our attention only to finite monoids, the two conditions above (being aperiodic and having linear $\mathbb{X}(M)$) were the only necessary conditions we obtained. For infinite monoids, the situation is quite different, and a key role in obtaining the desired combinatorial statements is often played by the presence/absence of infinite chains in $\mathbb{X}(M)$. In this direction, we obtain the following necessary conditions.

Proposition 5.2.4. *Let M be a monoid, let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let $F = \{a\mathbf{y}\}$ for some $a \in M$. If M is locally $(F,\mathbf{y})$-controllable, then $\{a'\mathbf{y} \mid a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite.*

Proof. Towards a contradiction, assume $A = \{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is infinite. Notice that if $a' \mathcal{R} a''$ and $\{a'\mathbf{y}, a''\mathbf{y}\} \subseteq A$, then $a'\mathbf{y} = a''\mathbf{y}$: in fact $a'\mathbf{y}$ and $a''\mathbf{y}$ are $\leq_{\mathbb{Y}}$-comparable and $a'M = a''M$ is their top element. Hence,

$$A(M) = \{cM \mid c \in M, c\mathbf{y} \in A\}$$

is infinite. Also, $A(M) = \{cM \mid c\mathbf{y} \in A\}$ is linearly ordered by inclusion since it is isomorphic as an order to a subset of A, which is linear.

We want to show that $A(M)$ contains no infinite ascending chain. Suppose not, and let $\{a_i M\}$ be ascending in $A(M)$ with $a_i M \subsetneq a_{i+1}M$ and $a_i\mathbf{y} \in A$ for every $i \in \omega$. For each word $w \in M^{<\omega}$, define $i(w)$ to be the highest natural number such that a letter of w is in $[a_{i(w)}]_{\mathcal{R}}$, if there is such a letter in w. Color w with black if it contains no such letter, otherwise color w with red if $i(w)$ is odd and with blue if $i(w)$ is even. Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a, a_0, \dots, a_i\}$. Let $\bar{s} = (s_n)_{n\in\omega}$ be any sequence of variable words. If as_0 has color black, then $as_0 ^\frown a_1 s_1$ has color blue or red, since s_1 has at least a letter 1. Otherwise, let $k = i(as_0)$. Then,

$$i(as_0 ^\frown a_{k+1}s_{k+1}) = k + 1,$$

since $as_0 ^\frown a_{k+1}s_{k+1}$ contains a_{k+1} as a letter, and each other letter of $a_{k+1}s_{k+1}$ belongs to $a_{k+1}M$. This contradicts that M is locally $(F, \mathbf{y})$-controllable, since

$$a\mathbf{y} \vee a_1\mathbf{y} = a\mathbf{y} \vee a_{k+1}\mathbf{y} = a\mathbf{y}.$$

Now we want to show that $A(M)$ contains no infinite descending chain. Suppose not, and let $\{a_i M\}$ be an infinite descending chain in $A(M)$, with $a_i M \supsetneq a_{i+1}M$ and $a_i\mathbf{y} \in A$ for every $i \in \omega$. Without loss of generality, assume also that $a_0 = a$. For each word $w \in M^{<\omega}$, let H_w be the set of $i \in \omega$ such that there is a letter b in w such that $b \in [a_i]_{\mathcal{R}}$, there is no letter c in w such that $c \in [a_{i+1}]_{\mathcal{R}}$, and there exist $k \geq 2$ and a letter d in w such that $d \in [a_{i+k}]_{\mathcal{R}}$. We can think of H_w as a set counting "holes". Color w by red if H_w has even cardinality, by blue otherwise. Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a_0, \dots, a_i\}$. Let $\bar{s} = (s_n)_{n\in\omega}$ be any sequence of variable words. Given a word $w \in M^{<\omega}$ such that w has a letter in $\bigcup_{i\in\omega}[a_i]_{\mathcal{R}}$, let $l(w)$ be the maximum i such that w has a letter in $[a_i]_{\mathcal{R}}$. Let $i = l(as_0)$ and let $k = l(as_0 \frown a_{i+1}s_{i+1})$. If as_0 has the same color of $as_0 \frown a_{i+1}s_{i+1}$ then the cardinality of $H_{a_{i+1}s_{i+1}}$ is even. In this case, $as_0 \frown a_{i+1}s_{i+1} \frown a_{k+1}s_{k+1}$ and $as_0 \frown a_{k+1}s_{k+1}$ have different colors. This contradicts that M is locally $(F, \mathbf{y})$-controllable, since

$$a\mathbf{y} \vee a_{i+1}\mathbf{y} = a\mathbf{y} \vee a_{i+1}\mathbf{y} \vee a_{k+1}\mathbf{y} = a\mathbf{y} \vee a_{k+1}\mathbf{y} = a\mathbf{y}.$$

Hence, $A(M)$ must be finite, contradiction. □

Notice that when $\mathbf{y} = \mathbb{X}(M)$ is linear, then we have $a\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$ for every $a \in M$. Hence,by Corollary 5.2.2, we get the following.

Corollary 5.2.5. . *Let M be a locally Ramsey monoid. Then $\mathbb{X}(M)$ is finite.*

Finally, we have one last necessary condition for infinite $\mathbb{Y}$-controllable monoids.

Proposition 5.2.6. *Suppose M is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear. Then $[a]_{\mathcal{R}}$ is finite for every $a \in M$.*

Proof. Suppose not. By Proposition 5.2.4, we have that $\mathbb{X}(M)$ is finite. Let $[a]_{\mathcal{R}}$ be the maximal infinite $\mathcal{R}$-class. Let $B = M \setminus aM$: by maximality of $[a]_{\mathcal{R}}$, it is finite. Hence, we may find a well-order $\leq_a$ of $[a]_{\mathcal{R}}$ such that for infinitely many $m \in [a]_{\mathcal{R}}$ we have $mB \leq_a m$ (i.e. $mb \leq_a m$ for every $b \in B$). Let

$$C = \big\{c \in [a]_{\mathcal{R}} \mid cB \leq c\big\},$$

it is an infinite suborder of $([a]_{\mathcal{R}}, \leq_a)$, and thus is wellordered as well by $\leq_a$. Let $\alpha \geq \omega$ be its order type, and let $\{c_i \mid i \in \alpha\}$ be an increasing enumeration of C. By Proposition 5.2.3 and 4.2.2, M is $\mathcal{R}$-rigid, and thus for every $c \in C$ and for every $b \in M \setminus \{b \in [a]_R \mid b > c\}$ we have either $cb \notin [a]_{\mathcal{R}}$ or $cb \leq c$.

Let $\mathbf{y} = \mathbb{X}(M)$, let $F = \{\mathbf{y}\} \subseteq \langle \mathbb{Y}(M) \rangle$. Notice that $\mathbf{y}$ is a maximal in $\mathbb{Y}(M)$ and $a\mathbf{y} \vee \mathbf{y} = \mathbf{y}$ for every $a \in M$. Consider the semigroup of words $(M^{<\omega}, \frown)$ and color a word $w \in M^{<\omega}$ by black if it has no letter in C. Otherwise, let $i(w)$ be the highest index among the letters of C that appear in w, and color w by red if $i(w)$ is odd, and by blue if it is even.

Let $(y_i)_{i\in\omega}$ be a sequence of variable words. If $y_0 \frown y_1$ has color black, then $c(y_0) \frown y_1$ has color red or blue for any $c \in C$. Otherwise, let n be the maximal index of a letter in C occurring in $y_0 \frown y_1$. Then, $c_{n+1}(y_0) \frown y_1$ and $y_0 \frown y_1$ have different colors. Either case, this shows that M is not $\mathbb{Y}$-controllable. □

Remark 5.2.7. The same proof actually shows something stronger: suppose M is a monoid containing an infinite $\mathcal{R}$-class $[a]_{\mathcal{R}}$ and an infinite set $I \subseteq [a]_{\mathcal{R}}$ such that for every $c \in I$ we have that $c(M \setminus [a]_{\mathcal{R}}) \cap I = \{cm \in I \mid m \notin [a]_{\mathcal{R}}\}$ is finite. Then M is not $\mathbb{Y}$-controllable.

By Proposition 5.2.4, if $\mathbb{X}(M)$ is linear than it must be finite. Thus, we get the following.

Corollary 5.2.8. *If M is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear, then M is finite.*

Since in locally Ramsey monoids $\mathbb{X}(M)$ is always linear (Proposition 5.2.1), we get the following.

Corollary 5.2.9. *If M is locally Ramsey and $\mathbb{Y}$-controllable then it is finite.*

Proposition 5.2.6 works also for Ramsey monoids, thanks to Corollary 5.2.2. This together with Corollary 5.2.5 gives the following.

Corollary 5.2.10. *If M is Ramsey, then it is finite.*

These necessary conditions allow us to improve the algebraic characterization of Ramsey monoids given in the previous chapter.

Theorem 5.2.11 (cf. Theorem 4.4.7)**.** *A monoid is Ramsey if and only if it is finite, aperiodic, and $\mathbb{X}(M)$ is linear.*

Hence, we may extend Corollary 4.4.8 also to infinite monoids.

Corollary 5.2.12. *A monoid is Ramsey if and only if it is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear.*

Finally, the new characterization provided of Ramsey monoids allows us to show that this notion is equivalent to another (apparently) weaker notion, that corresponds roughly to Schur's Lemma for monoids.

Theorem 5.2.13. *A monoid M is Ramsey if and only if for every finite coloring of W_M there are variable words $a, b \in W_M$ such that $\langle(a, b)\rangle_M$ is monochromatic.*

Proof. It is clear that if M is Ramsey, then the first two elements of an infinite sequence of variable words are as wanted.

Conversely, assume that for every finite coloring of W_M there are variable words $a, b \in W_M$ such that $Ma^\frown b = \{ma^\frown b \mid m \in M\}$ is monochromatic. We want to show that M is finite and aperiodic and $\mathbb{X}(M)$ is linear. First, the proofs of Propositions 5.2.1, 5.2.3, 5.2.4 and 5.2.6 immediately shows that M is aperiodic, that it contains no infinite $\mathcal{R}$-class and $(\mathbb{X}(M), \supseteq)$ is a well-order (linear and wellfounded). It remains to prove that $(\mathbb{X}(M), \subseteq)$ is wellfounded as well using only two words.

Suppose $\mathbb{X}(M)$ is not finite, then $(\mathbb{X}(M), \supseteq)$ has order type α for some $\alpha \geq \omega$. Let $\mathcal{D}$ to be the set of the first ω elements of $\mathbb{X}(M)$. We claim that $a^2 = a$ for an infinite number of $aM \in \mathcal{D}$.

First, notice that for each $a \in M$ with $aM \in \mathcal{D}$ we have $aaM \in \mathcal{D}$. Suppose not: then $abM \notin \mathcal{D}$ for every $b \in aM$, since $ab \in aaM$. Also, $ax\,\mathcal{R}\,ay$ whenever $x\,\mathcal{R}\,y$. Given $a \in M$, define $\mathbb{X}(aM) = \{cM \in \mathbb{X}(M) : cM \subseteq aM\}$. Then, $|\mathcal{D} \cap \mathbb{X}(aM)| \leq |\mathbb{X}(M) \setminus \mathbb{X}(aM)|$, but the set $\mathbb{X}(M) \setminus \mathbb{X}(aM)$ is finite, while $\mathcal{D}$ and so $\mathcal{D} \cap \mathbb{X}(aM)$ are infinite, contradiction.

Let $\mathcal{C} = \{[a]_\mathcal{R} : aM \in \mathcal{D}, [a]_\mathcal{R} \cdot [a]_\mathcal{R} = [a]_\mathcal{R}\}$. Since M is aperiodic, for every $a \in M$ with $aM \in \mathcal{D}$ there is $n \in \omega$ such that $a^n = a^{n+1}$ and $a^n M = a^{2n} M \in \mathcal{D}$, hence $\mathcal{C}$ has size ω. Let $\mathcal{C} = \{[c_i]_\mathcal{R}\}_{i<\omega}$ be an enumeration of $\mathcal{C}$ such that for every $i < j$ we have $c_j M \subseteq c_i M$.

For every word $w = (a_0, \dots, a_h) \in M^{<\omega}$, define n_w as the minimum $i < \omega$ such that for every $j \leq h$, either $c_i M \subseteq a_j M$ or $a_j M \notin \mathcal{D}$.

First, notice that for every $a, b \in M$, if $bM \notin \mathcal{D}$, then $abM \notin \mathcal{D}$: if not, then $bM \subseteq abM$ and thus $b = abx$ for some $x \in M$ and thus $ab = b$ by Proposition 4.2.2, contradiction.

Also, by Proposition 4.2.2 M is $\mathcal{R}$-rigid, and thus for every $[a]_{\mathcal{R}} \in \mathcal{C}$ we have $aa = a$. Hence, if $aM \subseteq bM$ and $aM \in \mathcal{C}$, then $aM = aaM \subseteq abM$.

These two conditions together implies that if $w \in M^{<\omega}$ is a variable word and $i \geq n_w$, then $n_{c_i(w)} = i$.

Color each word $w \in M^{<\omega}$ by red if n_w is odd, and by blue if n_w is even. Then, given two variable words y_0, y_1, if $w = y_0 {}^\frown y_1$ and $n = n_w$, the words $c_{n+1}(y_0) {}^\frown y_1$ and $y_0 {}^\frown y_1$ have different colors. □

5.3 Sufficient conditions for Ramsey classes

Next, we analyze some sufficient conditions for a monoid to be (locally) Ramsey or (locally) $\mathbb{Y}$-controllable.

We start by proving a technical lemma. In Proposition 5.2.4, we showed that if M is locally $\mathbb{Y}$-controllable, then for every $a \in M$ and maximal $\mathbf{y} \in \mathbb{Y}(M)$, the set $\{a'\mathbf{y} \mid a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite. This property turns out to be crucial also in providing sufficient conditions, but in an equivalent form.

For every $F \subseteq \mathbb{Y}(M)$, let

$$\mathrm{Gen}(F) = \{\mathbf{x} \in \mathbb{Y}(M) \mid f \vee \mathbf{x} \vee f' \in F \text{ for some } f, f' \in \langle \mathbb{Y}(M) \rangle\}.$$

We also write $\mathrm{Gen}(f)$ for $\mathrm{Gen}(\{f\})$ when f is an element of $\langle \mathbb{Y}(M) \rangle$.

Lemma 5.3.1. *Let M be a monoid, let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let $A \subseteq M$. Then, the following are equivalent:*

1. *For every $a \in A$, the set $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite.*
2. *For every finite $F \subseteq \langle A\mathbf{y} \rangle$, the set $\mathrm{Gen}(F)$ is finite.*

Proof. First, assume that 1 does not hold and $B = \{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is infinite for some $a \in A$. It is enough to choose $F = \{a\mathbf{y}\}$ to show that 2 does not hold, since in this case $\mathrm{Gen}(F) = B$.

Conversely, suppose $\mathrm{Gen}(F) = \bigcup_{f \in F} \mathrm{Gen}(f)$ is infinite for some finite $F \subseteq \langle A\mathbf{y} \rangle$. Then, by pigeonhole principle there is $f \in F$ such that $\mathrm{Gen}(f)$ is infinite. Let $f = a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y}$ be the unique representation of f of minimal length. Notice that since $F \subseteq \langle A\mathbf{y} \rangle$, we must have $a_i \in A$ for every $i \leq n$. Then, for every $b_i \in M$ such that $f = b_0\mathbf{y} \vee \cdots \vee b_k\mathbf{y}$ and for every $j \leq k$ there is $i \leq n$ such that $b_j\mathbf{y} \leq_{\mathbb{Y}} a_i\mathbf{y}$. Therefore, applying the pigeonhole principle once again we may find $i \leq n$ such that $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a_i\mathbf{y}\}$ is infinite. □

Once again, we introduce a new notion to simplify the proof of the next theorem.

Definition 5.3.2. Let F be a finite subset of the semigroup $\langle \mathbb{Y}(M) \rangle$, let $\mathbf{y}$ be a maximal element in $\mathbb{Y}(M)$, let $(M_i)_{i \in \omega}$ be a sequence of finite subset of M, and let c be a finite coloring of a semigroup S on which M acts. We say that a sequence $\bar{s} \in S^{\leq \omega}$ is $(F, \mathbf{y}, (M_i)_{i \in \omega}, c)$**-controllable** if for every $m, n \leq |\bar{s}|$ and for every $a_i, b_j \in M$ if $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y}$ belongs to F and $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_m\mathbf{y}$, then $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$

has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$ such that $a_k \in M_{i_k}$ and $b_k \in M_{j_k}$.

In other words, a sequence $\bar{s} \in S^{\leq\omega}$ is $(F, \mathbf{y}, (M_i)_{i\in\omega}, c)$**-controllable** if $\langle \bar{s} \rangle^f_{(M_i)_{i\in\omega}}$ is monochromatic for every $f \in F$.

Theorem 5.3.3. *Let M be a monoid, let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let $A \subseteq M$ be the set of those $a \in M$ such that the set $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite.*

Suppose also that for every action of M by continuous endomorphisms on a compact right topological semigroup U there exists an M-equivariant embedding $h : \langle M\mathbf{y} \rangle \to E(U)$ such that $h(\mathbf{y}) \in I(U)$.

Then, M is locally $(F, \mathbf{y})$-controllable for every finite $F \subseteq \langle A\mathbf{y} \rangle$.

Proof. The first part of the proof proceeds in the same way as the proof of Theorem 4.4.4, so we report here only the main steps without all details.

We want to show that for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly, for every sequence $(M_i)_{i\in\omega}$ of finite subset of M and for every finite coloring c of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ that is $(F, \mathbf{y}, (M_i)_{i\in\omega}, c)$-controllable and such that s_n has a distinguished point for every $n \in \omega$.

Given $(X_n)_{n\in\omega}$ be a sequence of pointed M-sets on which M acts uniformly, let $G = (\langle (X_n)_{n\in\omega} \rangle \cup \{\bot\}, \cdot)$ be the semigroup extending $\langle (X_n)_{n\in\omega} \rangle$, and let $\prec$ be the binary relation on G such that $x \prec y$ if and only if $x, y \in G \setminus \{\bot\}$ and $x ^\frown y$ is defined in $\langle (X_n)_{n\in\omega} \rangle$ (as in the proof of Theorem 4.4.4). Define in the same way the language L^+, a monster model $\mathcal{G}$ for G and the compact right topological semigroup $(S(G), \cdot_G, \tau)$.

Let $e(x) = \{\text{“}g \prec x\text{”} \mid g \in G \setminus \{\bot\}\}$, and define

$$U = \{p(x) \in S(G) \mid e(x) \subseteq p(x)\}$$

to be the set of all complete types in $S(G)$ extending $e(x)$.

Then U is a non-empty compact right topological subsemigroup of $(S(G), \cdot_G, \tau)$ closed under the action of M (see again Theorem 4.4.4).

Let $u(x) = h(\mathbf{y}) \in E(U) \cap I(U)$ be given by hypothesis. Let $\mathrm{DP} \subseteq G$ be the set of elements of $\langle (X_n)_{n\in\omega} \rangle$ that have at least one distinguished point. Notice that DP is a both-sided ideal in G, and thus,

$$J = \{p(x) \in U \mid \text{“}\mathrm{DP}(x)\text{”} \in p(x)\}$$

is also a both-sided ideal of U. It is also non-empty, since $e(x) \cup \{\mathrm{DP}(x)\}$ is finitely satisfiable in G. Hence, $u(x) \in I(U)$ implies $u(x) \in J$.

Fixed a finite coloring $c' = \{C_i \mid i < r\}$ of $\langle (X_n)_{n\in\omega} \rangle$, let c be the coloring of the whole G obtained by adding $\{\bot\}$ to c'. Since c is definable in $L^+(G)$, it extends to a finite coloring of $\mathcal{G}$

$$\tilde{c} = \{\{a \in \mathcal{G} \mid \mathcal{G} \vDash C_i(a)\} \mid i < r\}.$$

Let $(u_n)_{n\in\omega}$ be a coheir sequence of $u(x)$. We write $\overleftarrow{u}_{\restriction i}$ for the tuple $u_{i-1}, \ldots, u_0$.

As shown in the proof of Theorem 4.4.4, we get that $\overleftarrow{u}$ is $\mathbf{y}$-controllable.

Claim. For every $i \in \omega$ and $(M_j)_{j<\omega}$ (possibly infinite) subsets of M, the sequence $\bar{u}_{\restriction i}$ is $(F, \mathbf{y}, (M_j)_{j<\omega}, \tilde{c})$-controllable.

Proof. The proof is similar to the one of Theorem 4.4.4 provided in Section 4.5. Let $a_0, \ldots, a_n, b_0, \ldots, b_m \in M$ be such that $a_n\mathbf{y} \vee \cdots \vee a_0\mathbf{y} = b_m\mathbf{y} \vee \cdots \vee b_0\mathbf{y} \in F$. Then, by definition of h we have also $a_n u(x) \cdot_G \ldots \cdot_G a_0 u(x) = b_m u(x) \cdot_G \ldots \cdot_G b_0 u(x)$.

Since $u_k \mathrel{\perp\!\!\!\!\!\!\;\smile}_G \{u_j \mid j < k\}$ implies $a_k(u_k) \mathrel{\perp\!\!\!\!\!\!\;\smile}_G \{a_j(u_j) \mid j < k\}$ for every $k \leq n$, we have that $a_n u_n \cdot \ldots \cdot a_0 u_0$ satisfies $a_n u(x) \cdot_G \ldots \cdot_G a_0 u(x)$ and $b_m u_m \cdot \ldots \cdot b_0 u_0$ satisfies $b_m u(x) \cdot_G \ldots \cdot_G b_0 u(x)$. Since the color of an element of $\mathcal{G}$ is determined by its type over G, this implies immediately that for every choice of $i \in \omega$, we have that $\bar{u}_{\restriction i}$ is $(F, \mathbf{y}, (M_i)_{i<\omega}, \tilde{c})$-controllable. □

Now fix a finite subset $F \subseteq \langle A\mathbf{y} \rangle$, and fix a natural number $k \in \omega$ such that for every $f, f' \in \langle M\mathbf{y} \rangle$ with $f \vee f' \in F$ there are $c_0, \ldots c_j \in M$ with $j < k$ such that $f' = c_0\mathbf{y} \vee \cdots \vee c_j\mathbf{y}$. We write $|f|$ the denote the length $j \in \omega$ of this minimal representation.

By Lemma 5.3.1, we have that $\mathrm{Gen}(F)$ is finite. Notice that this implies also that if $f' = a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y}$ and $f \vee f' \in F$ for some $f \in \langle M\mathbf{y} \rangle$, then $a_i\mathbf{y} \in \mathrm{Gen}(F)$ for every $i \leq n$. For every $\mathbf{x} \in \mathrm{Gen}(F)$ fix an element $a_x \in M$ such that $\mathbf{x} = a_x\mathbf{y}$. Let $K = \{a_x \mid \mathbf{x} \in \mathrm{Gen}(F)\}$: it is finite.

Now fix a sequence $(M_i)_{i<\omega}$ of finite subsets of M. Notice that if a sequence is $(F, \mathbf{y}, (M_i \cup K)_{i<\omega}, \tilde{c})$-controllable, then it is also $(F, \mathbf{y}, (M_i)_{i<\omega}, \tilde{c})$-controllable, so without loss of generality we can assume $K \subseteq M_i$ for every $i < \omega$.

Claim. Let $g_0, \ldots, g_n \in G$. If the sequence $(g_0, \ldots, g_n)^\frown \bar{u}_{\restriction k}$ is $(F, \mathbf{y}, (M_i)_{i<\omega}, \tilde{c})$-controllable, then for any $l > k$ the sequence $(g_0, \ldots, g_n)^\frown \bar{u}_{\restriction l}$ is $(F, \mathbf{y}, (M_i)_{i<\omega}, \tilde{c})$-controllable as well.

Proof. Let $f_g = a_0\mathbf{y} \vee \cdots \vee a_h\mathbf{y}$ and $f_u = a_{h+1}\mathbf{y} \vee \cdots \vee a_m\mathbf{y}$, and $f'_g = b_0\mathbf{y} \vee \cdots \vee b_{h'}\mathbf{y}$ and $f'_u = b_{h'+1}\mathbf{y} \vee \cdots \vee b_{m'}\mathbf{y}$. Suppose

$$f_g \vee f_u = a_0\mathbf{y} \vee \cdots \vee a_m\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_{m'}\mathbf{y} = f'_g \vee f'_u \in F$$

Let $i_0 < \cdots < i_h \leq n$, and $i_m < \cdots < i_{h+1} < l$, and $i'_0 < \cdots < i'_{h'} \leq n$, and $i'_{m'} < \cdots < i'_{h'+1} < l$, and let $a_g = a_0 g_{i_0} \cdot \ldots \cdot a_h g_{i_h}$, and $a_u = a_{h+1} u_{i_{h+1}} \cdot \ldots \cdot a_m u_{i_m}$, and $b_g = b_0 g_{i'_0} \cdot \ldots \cdot b_{h'} g_{i'_{h'}}$ and $b_u = b_{h'+1} u_{i'_{h'+1}} \cdot \ldots \cdot b_{m'} u_{i'_{m'}}$.

Now assume $a_r \in M_{i_r}$ for every $r \leq h$ and $b_r \in M_{i'_r}$ for every $r \leq h'$. We want to show that

$$\tilde{c}(a_g \cdot a_u) = \tilde{c}(b_g \cdot b_u).$$

First, since $f_g \vee f_u \in F$, we have $j = |f_u| < k$. Let $d_0, \ldots, d_j \in K$ be such that $f_u = d_j\mathbf{y} \vee \cdots \vee d_0\mathbf{y}$. Let $a_d = d_j u_j \cdot \ldots \cdot d_0 u_0$: then, as shown in previous claim, a_u and a_d satisfies the same type over G. In particular, for every color $C \in \tilde{c}$, we have that $C(a_g \cdot x)$ is in the type of a_u if and only if it is in the type of a_d. Thus, $\tilde{c}(a_g \cdot a_u) = \tilde{c}(a_g \cdot a_d)$.

In the same way, $\tilde{c}(b_g \cdot b_u) = \tilde{c}(b_g \cdot b_d)$ for some $b_d = d'_{j'} u_{j'} \cdot \ldots \cdot d'_0 u_0$ such that $j' < k$ and $f'_u = d'_{j'}\mathbf{y} \vee \cdots \vee d'_0\mathbf{y}$ and $d'_r \in K$ for every $r \leq j'$.

However, since $\bar{g}^\frown \bar{u}_{\restriction k}$ is $(F, \mathbf{y}, (M_i)_{i<\omega}, \tilde{c})$-controllable, since

$$f_g \vee f_u = a_0\mathbf{y} \vee \cdots \vee a_h\mathbf{y} \vee d_j\mathbf{y} \vee \cdots \vee d_0\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_{h'} \vee d'_{j'}\mathbf{y} \vee \cdots \vee d'_0\mathbf{y} = f'_g \vee f'_u,$$

and since each element of M acting on $\mathbf{y}$ in the above equation belongs to the right piece M_i (recall $K \subseteq M_i$ for every i), then we have $\tilde{c}(a_g \cdot a_d) = \tilde{c}(b_g \cdot b_d)$. □

Finally, we want to pass from the sequence $(u_n)_{n\in\omega}$ in the monster model to a basic $(F, \mathbf{y}, (M_i)_{i<\omega}, c)$-controllable sequence $\bar{s}$ in $\langle (X_n)_{n\in\omega}\rangle$.

We say that a sequence $(s_0, \dots, s_n) \in \mathcal{G}^n$ is basic if $s_i \prec s_j$ for every $i < j \leq n$. Notice that since F and c and all M_i are finite, being $(F, \mathbf{y}, (M_i)_{i<\omega}, c)$-controllable is definable in $L^+(G)$. So let $\xi_n(x_0, \dots, x_n)$ be a formula in $L^+(G)$ that says that the sequence $(x_0, \dots, x_n)$ is basic and $(F, \mathbf{y}, (M_i)_{i<\omega}, c)$-controllable and that each x_i has a distinguished point.

We proceed recursively. Assume that we have a (possibly empty) basic sequence $\bar{s}_{\restriction i} \in \mathrm{DP}^i$ such that $\bar{s}_{\restriction i} {}^\frown \overleftarrow{u}_{\restriction k}$ satisfies $\xi_{i+k}(x_0, \dots, x_{i+k})$ (i.e. $\bar{s}_{\restriction i} {}^\frown \overleftarrow{u}_{\restriction k}$ is basic and $(F, \mathbf{y}, (M_j)_{j<\omega}, c)$-controllable and each coordinate has a distinguished point). The empty sequence satisfies this by previous claims, so the base case is fine. Our goal is to find $\bar{s}_i \in \langle (X_n)_{n\in\omega}\rangle$ such that $\bar{s}_{\restriction i+1} {}^\frown \overleftarrow{u}_{\restriction k}$ satisfies $\xi_{i+k+1}(x_0, \dots, x_{i+k+1})$.

By induction hypothesis $\bar{s}_{\restriction i} {}^\frown \overleftarrow{u}_{\restriction k}$ is $(F, \mathbf{y}, (M_j)_{j<\omega}, c)$-controllable, and so by previous claim $\bar{s}_{\restriction i} {}^\frown \overleftarrow{u}_{\restriction k+1}$ is as well. Then, $\mathcal{G} \vDash \xi_{i+k+1}(\bar{s}_{\restriction i}, u_k, \overleftarrow{u}_{\restriction k})$. Furthermore, since $u_k \mathop{\smile\!\!\!\!|}_G \{u_0, \dots, u_{k-1}\}$, the type $\mathrm{tp}(u_k/G \cup \{u_0, \dots, u_{k-1}\})$ is finitely satisfiable in G, hence we may find $s_i \in G$ such that $\xi_{i+k+1}(\bar{s}_{\restriction i}, s_i, \overleftarrow{u}_{\restriction k})$ hold in $\mathcal{G}$. Notice that $\perp \notin \mathrm{DP}$, thus $(s_i)_{i\in\omega} \in \langle (X_n)_{n\in\omega}\rangle$ as wanted. □

In Proposition 5.2.6, we proved that if M is a monoid with linear $\mathbb{X}(M)$, then being $\mathbb{Y}$-controllable implies that each $\mathcal{R}$-class $[a]_R$ is finite. Here, we show that this condition is sufficient to be $\mathbb{Y}$-controllable if the monoid is already locally $\mathbb{Y}$-controllable.

Theorem 5.3.4. *Let M be a monoid, let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let F, A be finite sets such that $A \subseteq M\mathbf{y}$ and $A \subseteq F \subseteq \langle A\rangle$. If M is locally $(F, \mathbf{y})$-controllable and $[a]_{\mathcal{R}}$ is finite for every $a \in \mathrm{Gen}(F)$, then M is $(F, \mathbf{y})$-controllable.*

Proof. Indeed, by Proposition 5.2.4 and Lemma 5.3.1 we have that $\mathrm{Gen}(F)$ is finite. Let $K' = \{a \in M \mid a\mathbf{y} \in \mathrm{Gen}(F)\}$: it is finite, since $\mathrm{Gen}(F)$ and $[a]_{\mathcal{R}}$ are finite for every $a \in \mathrm{Gen}(F)$.

Furthermore, K' satisfies that for every $a_0\mathbf{y} \vee \dots \vee a_n\mathbf{y} \in F$, then $a_i \in K'$ for every $i \leq n$. Hence applying the definition of locally $(F, \mathbf{y})$-controllable monoid by choosing $M_i = K'$ for every $i < \omega$ shows that M is also $(F, \mathbf{y})$-controllable. □

Notice that for every finite $F \subseteq \mathbb{Y}(M)$ there are finite sets A and F' such that $A \subseteq M\mathbf{y}$ and $A \subseteq F \subseteq \langle A\rangle$. Indeed, it is enough for every $f \in F$ to choose $\mathbf{x}_i^f \in \mathbb{Y}(M)$ such that $f = \mathbf{x}_0^f \vee \dots \vee \mathbf{x}_{n_f}^f$ and define $F' = F \cup \{\mathbf{x}_i^f \mid f \in F, i \leq n_f\}$. Since being $(F', \mathbf{y})$-controllable implies being $(F, \mathbf{y})$-controllable for every $F \subseteq F'$, we get the following.

Corollary 5.3.5. *Let M be a locally $\mathbb{Y}$-controllable monoid, and assume $[a]_{\mathcal{R}}$ is finite for every $a \in M$. Then M is $\mathbb{Y}$-controllable.*

By Theorem 5.3.3, we get also the following result.

Corollary 5.3.6. *Let M be a monoid, let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let $A \subseteq M$ be the set of those $a \in M$ such that the set $\{a' \in M \mid a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite.*

Suppose also that for every action of M by continuous endomorphisms on a compact right topological semigroup U there exists an M-equivariant embedding $h : \langle M\mathbf{y}\rangle \to E(U)$ such that $h(\mathbf{y}) \in I(U)$.

Then, M is $(F, \mathbf{y})$-controllable for every finite $F \subseteq \langle A\mathbf{y}\rangle$ such that $\mathrm{Gen}(F) \subseteq A$.

Up to this moment, we have been able to work under almost optimal hypotheses. The assumption on the finiteness of $\{a' \in M \mid a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is also a necessary condition for being locally $\mathbb{Y}$-controllable (by Proposition 5.2.4). The assumption that every $\mathcal{R}$-class $[a]_{\mathcal{R}}$ is finite is sometimes necessary for $\mathbb{Y}$-controllable monoids (by Proposition 5.2.6), and it seems possible that it turns out to be a necessary condition for every $\mathbb{Y}$-controllable monoid.

To obtain the existence of the function h, however, we need to work under stronger assumptions that are far from optimal. While proof of Theorem 5.3.3 could work under weaker hypotheses than having such a function h, this would make it much more complex.

Remark 5.3.7. Let M be a monoid, let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let $A \subseteq M$ be the set of those $a \in M$ such that the set $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite. Let $F \subseteq \langle A\mathbf{y}\rangle$ be finite and such that $f \vee f' \in F$ implies $f' \in F$ for every $f \in \mathbb{Y}(M)$. Let S be a (partial) semigroup of reader's choice with a free sequence $\bar{t} \in S^{\omega}$, and let $G = \langle \bar{t}\rangle_{W_M}$ (e.g. $G = \mathrm{FIN}_M$). Let $(U, \cdot_G)$ be the compact right topological semigroup of types $p(x)$ over G such that "$g \prec x$" $\in p(x)$ for every $g \in G$.

Suppose that for every finite clopen partition (coloring) c of U there exists an idempotent $u \in I(U) \cap E(U)$ such that if $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_n\mathbf{y} \in F$, then $\hat{c}(a_0 u \cdot_G \ldots \cdot_G a_n u) = \hat{c}(b_0 u \cdot_G \ldots \cdot_G b_m u)$, where $\hat{c}$ is the closed partition of U refining the clopen partition $\{\{x \in U \mid g \cdot_G x \in C\} \mid C \in c\}$ for every $g \in G$.

Then M is locally $(F, \mathbf{y})$-controllable (and the same proof of Theorem 5.3.3 proves it).

The authors do not know whether this weaker hypothesis can be achieved more easily than the ones from Theorem 5.3.3.

Instead, we focus on finding algebraic sufficient conditions for obtaining a function h as described in Theorem 5.3.3.

Once again, [142, Lemma 2.5] (or Lupini's [107, Lemma 2.2]) provides a wonderful way to achieve this. Recall that a forest $(\mathbb{P}, \leq)$ is a partial order such that for every $p \in \mathbb{P}$, the set of predecessors $\mathrm{pred}(p) = \{x \leq p\}$ is a well-order. Recall also that $\mathbb{P}$ has height $\leq \omega$ if for every $p \in \mathbb{P}$, the set of predecessors $\mathrm{pred}(p) = \{x \leq p\}$ is a finite linear order. We say that a partial order $(\mathbb{P}, \leq)$ is a strong M-partial order if M acts by endomorphism on $\mathbb{P}$ and furthermore $\mathrm{pred}(m(p)) = m[\mathrm{pred}(p)]$ for every $m \in M$ and $p \in \mathbb{P}$. Although in [142] the author works with finite objects, the proof of [142, Lemma 2.5] holds for every forest of height $\leq \omega$.

Lemma 5.3.8 ([142, Lemma 2.5])**.** *Let M be a monoid acting by continuous endomorphisms on a compact right topological semigroup U. Let $(\mathbb{P}, \leq)$ be a strong M-partial order and a forest of height $\leq \omega$. For every M-equivariant function $f : \mathbb{P} \to U$ there exists an M-equivariant embedding $h : \langle \mathbb{P}\rangle \to E(U)$ such that $f^{-1}(I(U)) \subseteq h^{-1}(I(U))$.*

We apply this lemma in two distinct situations to give two different sufficient algebraic conditions for a monoid to be locally $\mathbb{Y}$-controllable.

First, we want to work locally using $M\mathbf{y}$. Notice that in our cases of interest, this set is always a forest of height $\leq \omega$.

Remark 5.3.9. Given a monoid M and a (maximal) element $\mathbf{y} \in \mathbb{Y}(M)$, the suborder $(M\mathbf{y}, \leq_{\mathbb{Y}})$ of $\mathbb{Y}(M)$ is a forest of height $\leq \omega$ if and only if $\{a'\mathbf{y} \mid a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite for every $a \in M$.

It is not clear to the authors whether $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is always a strong M-partial order too. However, in some situations, we can get the desired result.

Lemma 5.3.10. *Let M be a monoid and let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal. Suppose that for every $aM \in \mathbf{y}$ there is $b \in [a]_{\mathcal{R}}$ such that $b\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$. Then $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a strong M-partial order.*

Proof. Indeed, for every $a, m \in M$, since m acts by endomorphism on $\mathbb{Y}(M)$ we have $m[\mathrm{pred}(a\mathbf{y})] \subseteq \mathrm{pred}(ma\mathbf{y})$. Now consider $a', a \in M$ such that $a'\mathbf{y} \leq a\mathbf{y}$. Then, there is $cM \in \mathbf{y}$ such that if $\mathbf{y}' = \{xM \in \mathbf{y} \mid xM \subseteq cM\}$, we have $a'\mathbf{y} = a\mathbf{y}'$. By hypothesis there is $b \in [c]_{\mathcal{R}}$ such that $bM = cM$ and $b\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$. Then, we must have $b\mathbf{y} = \mathbf{y}'$, and thus $a'\mathbf{y} = ab\mathbf{y}$ as wanted. □

Whenever we have that $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a strong M-partial order and a forest of height $\leq \omega$, we can apply Lemma 5.3.8 in order to obtain the desired M-equivariant embedding h from any idempotent of U sufficiently regular under the action of M.

Lemma 5.3.11. *Let M be a monoid acting by continuous endomorphisms on a compact right topological semigroup U. Let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal such that $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a strong M-partial order that is also a forest of height $\leq \omega$.*

Assume also that there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a\mathbf{y} = b\mathbf{y}$.

Then, there exists an M-equivariant embedding $h : \langle M\mathbf{y}\rangle \to E(U)$ such that $h(\mathbf{y}) \in I(U)$.

Proof. First, notice that every element $a\mathbf{y}$ in $M\mathbf{y}$ has a maximum aM, as $\mathbf{y}$ is maximal and $M \in \mathbf{y}$. Let $u \in E(U) \cap I(U)$ be given by hypothesis and let $f : M\mathbf{y} \to U$ be the function that maps each $a\mathbf{y} \in M\mathbf{y}$ to $a(u)$. Then, f is well-defined by assumption on u) and M-equivariant and maps $\mathbf{y}$ to $u \in E(U) \cap I(U)$. Thus, the result follows from Lemma 5.3.8. □

Notice that if $\mathbf{y} \in \mathbb{Y}(M)$ is maximal, then $a\mathbf{y} = b\mathbf{y}$ implies $a \,\mathcal{R}\, b$, as $M \in \mathbf{y}$. Thus, from Theorem 4.3.5, we get the following.

Theorem 5.3.12. *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear and finite. Let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal and let $A \subseteq M$ be the set of those $a \in M$ such that the set $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite and for every $aM \in \mathbf{y}$ there is $b \in M$ such that $bM = aM$ and $b\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$.*

Then, M is locally $(F, \mathbf{y})$-controllable for every finite $F \subseteq \langle A\mathbf{y}\rangle$.

Thanks to Theorem 5.3.4, we get a similar theorem for $\mathbb{Y}$-controllable monoids as well.

Theorem 5.3.13. *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear and finite. Let $\mathbf{y} \in \mathbb{Y}(M)$ be maximal such that for every $aM \in \mathbf{y}$ there is $b \in M$ such that $bM = aM$ and $b\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$. Let $A \subseteq M$ be the set of those $a \in M$ such that the set $\{a' \in M \mid a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite.*

Then, M is $(F, \mathbf{y})$-controllable for every finite $F \subseteq \langle A\mathbf{y}\rangle$ with $\mathrm{Gen}(F) \subseteq A$.

The next sufficient condition we are going to state is a global one. Working with $M\mathbf{y}$ we had the advantage that this partial order is always a forest of height $\leq \omega$ (for the monoid we consider), but we need to prove that it is also a strong M-partial order. Now we work with $\mathbb{Y}(M)$, for which the situation is the opposite. Indeed, it is easy to check that $\mathbb{Y}(M)$ is always a strong M partial order. Indeed, for every $\mathbf{x}, \mathbf{z} \in \mathbb{Y}(M)$ and $a \in M$ with $\mathbf{x} \leq_{\mathbb{Y}} a\mathbf{z}$ for some $\mathbf{x}, \mathbf{z} \in \mathbb{Y}(M)$ and $a \in M$, let $\mathbf{z} = \{m_0M \subseteq \ldots \subseteq m_nM\}$. Then $\mathbf{x} = \{am_0M \subseteq \ldots \subseteq am_hM\}$ for some $h \leq n$, and it is clear that $\mathbf{x}' \leq \mathbf{z}$ and $a\mathbf{x}' \leq a\mathbf{z}$ for $\mathbf{x}' = \{m_0M \subseteq \ldots \subseteq m_hM\}$.

Remark 5.3.14. Let M be a monoid, then $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$ is a strong M partial order.

However, in general, $\mathbb{Y}(M)$ may not be a forest of height $\leq \omega$. This condition is closely related however to the presence of infinite chains in $\mathbb{X}(M)$.

Lemma 5.3.15. *Consider a monoid M. Then $\mathbb{X}(M)$ contains no infinite chain if and only if $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$ is a forest of height $\leq \omega$.*

Proof. Indeed, if $\mathbb{X}(M)$ contains no infinite chain then every $\mathbf{x} \in \mathbb{Y}(M)$ is finite (since it is a linearly ordered subset of $\mathbb{X}(M)$), and thus $\mathrm{pred}(\mathbf{x})$ is finite as well.

Conversely, the set $\mathrm{pred}(\mathbf{x})$ is a linear order for every $\mathbf{x} \in \mathbb{Y}(M)$. Thus, if $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$ is not a forest of height $\leq \omega$ this means that there is $\mathbf{x} \in \mathbb{Y}(M)$ such that $\mathrm{pred}(\mathbf{x})$ is infinite, and thus $\mathbf{x} \subseteq \mathbb{X}(M)$ is an infinite chain. ☐

When $\mathbb{X}(M)$ is linear, we have $\mathbb{Y}(M) = M\mathbf{y}$ for $\mathbf{y} = \mathbb{X}(M)$ and all different partial order we are considering collapses to one.

Corollary 5.3.16. *Let M be a monoid with linear $\mathbb{X}(M)$. Then $\mathbb{X}(M)$ contains no infinite chain if and only if $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a forest of height ω for every $\mathbf{y} \in \mathbb{Y}(M)$.*

The above corollary would solve many problems if it holds without the assumption of the linearity of $\mathbb{X}(M)$. It is immediate to check that if $\mathbb{X}(M)$ contains no infinite chain, then for every maximal $\mathbf{y} \in \mathbb{Y}(M)$, the set $\{a'\mathbf{y} \mid a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is finite. Unfortunately, the authors do not know whether the converse is true as well for every monoid.

Lemma 5.3.17. *Let M be a monoid acting by continuous endomorphisms on a compact right topological semigroup U. Suppose M is aperiodic, $\mathbb{X}(M)$ contains no infinite chain and $\mathbb{X}_{\mathcal{R}}(M)$ is linear.*

Then, there exists an M-equivariant embedding $h : \langle \mathbb{Y}(M)\rangle \to E(U)$ such that $h(\mathbf{y}) \in I(U)$.

Proof. Since $\mathbb{X}_{\mathcal{R}}(M)$ is linear and $\mathbb{X}(M)$ contains no infinite linear order, then $\mathbb{X}_{\mathcal{R}}(M)$ is finite as well. By Theorem 4.3.5, we can find an element $u \in U$ such that $a(u) = b(u)$ for every $a, b \in M$ with $a \mathcal{R} b$. Since every element $\mathbf{x} \in \mathbb{Y}(M)$ is finite, it has a maximum $a_{\mathbf{x}}M$: let $f : \mathbb{Y}(M) \to U$ be the function that maps each $\mathbf{x} \in \mathbb{Y}(M)$ to $a_{\mathbf{x}}(u)$. Then, f is well-defined, since $a_{\mathbf{x}}M = bM$ implies $a_{\mathbf{x}} \mathcal{R} b$ and

$a_{\mathbf{x}}(u) = b(u)$, and it is M-equivariant by definition. Furthermore, for every maximal $\mathbf{y} \in \mathbb{Y}(M)$ we have $f(\mathbf{y}) = u \in E(U) \cap I(U)$. Thus, the result follows from Lemma 5.3.8. □

As a corollary, we get the following sufficient conditions for a monoid to be locally $\mathbb{Y}$-controllable.

Theorem 5.3.18. *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear and $\mathbb{X}(M)$ contains no infinite chains.*
Then, M is locally $\mathbb{Y}$-controllable.

As a corollary, we obtain the following results.

Theorem 5.3.19. *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear, $\mathbb{X}(M)$ contains no infinite chains and each $\mathcal{R}$-class is finite.*
Then, M is $\mathbb{Y}$-controllable.

Theorem 5.3.20. *Let M be an aperiodic monoid such that $\mathbb{X}(M)$ is linear and finite.*
Then, M is locally Ramsey (and the converse is true as well).

5.4 Conclusions: some final remarks and a synthesis

We conclude this chapter by making a bit of order and schematizing all the known relationships between the four classes of monoids we worked with (Ramsey, locally Ramsey, $\mathbb{Y}$-controllable, locally $\mathbb{Y}$-controllable).

First, we proved two characterizations of Ramsey and locally Ramsey monoids.

Theorem. *A monoid M is **Ramsey** if and only if it is **finite**, **aperiodic** and $\mathbb{X}(M)$ is **linear**.*

Theorem. *A monoid M is **locally Ramsey** if and only if it is **aperiodic** and $\mathbb{X}(M)$ is **finite** and **linear**.*

The two algebraic characterizations reveal also the connection between the two classes of Ramsey and locally Ramsey monoids.

Corollary. *A monoid M is **Ramsey** if and only if it is **locally Ramsey** and all **$\mathcal{R}$-classes are finite**.*

About $\mathbb{Y}$-controllable monoids, we proved the following.

Theorem. *Let M be a monoid.*

1. *If M is $\mathbb{Y}$**-controllable**, then it is **aperiodic** and $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a **forest of height** $\leq \omega$ for every $\mathbf{y} \in \mathbb{Y}(M)$.*
2. *Conversely, if M is **aperiodic**, $\mathbb{X}_{\mathcal{R}}(M)$ is **linear**, all **$\mathcal{R}$-classes are finite** and $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$ is a **forest of height** $\leq \omega$, then M is $\mathbb{Y}$**-controllable**.*
3. *Also, if M is **aperiodic**, $\mathbb{X}_{\mathcal{R}}(M)$ is **linear and finite**, all **$\mathcal{R}$-classes are finite** and $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a **forest of height** $\leq \omega$ and a **strong** M**-partial order** for every $\mathbf{y} \in \mathbb{Y}(M)$, then M is $\mathbb{Y}$**-controllable**.*

The following statement collects the known relationships between $\mathbb{Y}$-controllable monoids and the previous two classes of Ramsey and locally Ramsey monoids.

Proposition. *Let M be a monoid. The following are equivalent:*

(a) *M is $\mathbb{Y}$-**controllable** and $\mathbb{X}(M)$ is **linear**.*

(b) *M is $\mathbb{Y}$-**controllable** and **locally Ramsey**.*

(c) *M is **Ramsey**.*

Finally, regarding locally $\mathbb{Y}$-controllable monoids, we have the following.

Theorem. *Let M be a monoid.*

1. *If M is **locally** $\mathbb{Y}$-**controllable**, then it is **aperiodic** and $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a **forest of height** $\leq \omega$ for every $\mathbf{y} \in \mathbb{Y}(M)$.*

2. *Conversely, if M is **aperiodic**, $\mathbb{X}_{\mathcal{R}}(M)$ is **linear** and $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$ is a **forest of height** $\leq \omega$, then M is **locally** $\mathbb{Y}$-**controllable**.*

3. *Also, if M is **aperiodic**, $\mathbb{X}_{\mathcal{R}}(M)$ is **linear and finite**, and $(M\mathbf{y}, \leq_{\mathbb{Y}})$ is a **forest of height** $\leq \omega$ and a **strong** M-**partial order** for every $\mathbf{y} \in \mathbb{Y}(M)$, then M is **locally** $\mathbb{Y}$-**controllable**.*

Recall also that one condition that grants that $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$ is a strong M-partial order is given by Lemma 5.3.10.

We have the following relations between locally $\mathbb{Y}$-controllable monoids and previous classes.

First, thanks to the characterization of locally Ramsey monoids and the necessary and sufficient conditions given above, we get the

Proposition. *A monoid M is **locally Ramsey** if and only if it is **locally** $\mathbb{Y}$-**controllable** and $\mathbb{X}(M)$ is **linear**.*

Although we do not have a characterization, the size of $\mathcal{R}$-classes plays a crucial role in the relationship between $\mathbb{Y}$-controllable and locally $\mathbb{Y}$-controllable monoids, thanks to Proposition 5.2.6 and Theorem 5.3.4.

Proposition. *Let M be a monoid.*

1. *If M is $\mathbb{Y}$-**controllable**, then it is **locally** $\mathbb{Y}$-**controllable**.*

2. *Conversely, if M is **locally** $\mathbb{Y}$-**controllable** and all $\mathcal{R}$-**classes are finite**, then it is $\mathbb{Y}$-**controllable**.*

In order to show that some classes are non-trivial and some hypotheses non-optimal, we provide some examples of monoids.

Proposition 5.4.1. *There exists an infinite $\mathbb{Y}$-controllable monoid.*

Proof. Let α be an ordinal and let $(M_i, *_i)_{i\in\alpha}$ be a family of disjoint aperiodic finite semigroups such that M_0 has linear $\mathbb{X}_{\mathcal{R}}(M)$ and all other M_i are $\mathcal{R}$-trivial. Let $1, 0$ be two elements outside $\bigcup_{i\in I} M_i$ and let $M = \bigcup_{i\in I} M_i \cup \{1, 0\}$, with operation defined by letting $1a = a1 = a$ for every $a \in M$, $ab = a *_i b$ if $a, b \in M_i$ and $ab = 0$ otherwise. Then, M is $\mathbb{Y}$-controllable.

It is straightforward to check that if $a \in M_i$ then $aM = aM_i \cup \{0\}$. Hence, all $\mathcal{R}$-classes are finite, $\mathbb{X}_{\mathcal{R}}(M)$ is linear (as it is isomorphic to $\mathbb{X}_{\mathcal{R}}(M_0)$, since all other M_i are $\mathcal{R}$-trivial) and furthermore it is aperiodic by Proposition 4.2.4. Also, if $b \in M_j$ then $\{aM \in \mathbb{X}(M) \mid aM \subseteq bM\} \subseteq M_j \cup \{0\}$; Thus every chain of $\mathbb{X}(M)$ must be contained in one single M_i and hence it must be finite.

Then the result follows from Theorem 5.3.19. □

Recall also that by Proposition 4.6.3 there exist $\mathbb{Y}$-controllable monoids such that $\mathbb{X}_{\mathcal{R}}(M)$ is not linear.

The following example shows that the infinite Carlson's monoid is locally Ramsey but not $\mathbb{Y}$-controllable

Proposition 5.4.2. *There exists a locally Ramsey monoid which is not $\mathbb{Y}$-controllable.*

Proof. Let X be an infinite set such that $1 \notin X$, and let $M = (X \cup \{1\}, *, 1)$ be the monoid with operation $ab = b$ for every $a, b \in M \setminus 1$. It is enough to check that M is aperiodic, that $M\setminus\{1\}$ is an infinite $\mathcal{R}$-class and that $\mathbb{X}(M) = \{M\setminus\{1\}, M\}$ is linear and finite. Then the result follows from Theorem 5.3.20 and Proposition 5.2.6. □

Similarly, one can prove the following.

Proposition 5.4.3. *There exist locally $\mathbb{Y}$-controllable monoids which are not locally Ramsey nor $\mathbb{Y}$-controllable.*

Proof. Let $(X_i)_{i<n}$ be a finite sequence of infinite sets such that $1 \notin \bigcup_{i\in\alpha} X_i$, and let $F \subseteq n$ be non-empty. Define an operation $*_i$ on every X_i by setting for every $a, b \in X_i$ either $a *_i b = b$ if $i \in F$ or $a *_i b = a$ otherwise. Let $(M, *, 1)$ be the disjoint union of the X_is, i.e. $M = \bigcup_{i\in I} X_i \cup \{1\}$ is the monoid with identity 1 and operation defined $a * b = a *_i b$ if $a, b \in X_i$, while $a * b = b * a = a$ if $a \in X_i$ and $b \in X_j$ with $i < j$. Then, the monoid is aperiodic, as $a^2 = a$ for every $a \in M$. Furthermore, if $a \in X_i$ then either $aM = \bigcup_{j<i} X_j \cup \{a\}$ if $i \notin F$, or $aM = \bigcup_{j\leq i} X_j$ if $i \in F$. In particular, $aM = bM$ if and only if $a = b$ or $a, b \in X_i$ for $i \in F$, so there are infinite $\mathcal{R}$-classes and $\mathbb{X}_{\mathcal{R}}(M) = \{\bigcup_{j\leq i} X_j \mid i \in F\}$ is finite and linear. Also, $aM \subseteq bM$ if and only if $aM = bM$ or $a \in X_j$ and $b \in X_i$ for $j < i$, thus $\mathbb{X}(M) = \{\bigcup_{j\leq i} X_j \mid i \in F\} \cup \{\bigcup_{j<i} X_j \cup \{a\} \mid a \in X_i, i \notin F\} \cup \{M\}$ contains only chain of size $\leq n$, but it is not linear. Hence, M is locally $\mathbb{Y}$-controllable by Theorem 5.3.18, but it is not locally Ramsey by Proposition 5.2.1. Finally, it is easy to see that for every $i \in F$ and $a \in X_i$, then $[a]_{\mathcal{R}}$ is an infinite $\mathcal{R}$-class satisfying that for every $c \in [a]_{\mathcal{R}}$ we have $\{cm \mid m \notin [a]_{\mathcal{R}}\} \cap [a]_{\mathcal{R}} = \{c\}$, thus M is not $\mathbb{Y}$-controllable by Remark 5.2.7.

Notice that these monoids are a generalization of Furstenberg-Katznelson monoids (which are the case of $n = 2$ and $F = \{0\}$), and they are almost $\mathcal{R}$-trivial if and only if $F = \{0\}$. □

In this work, we tried to do an in-depth study of (locally) Ramsey and (locally) $\mathbb{Y}$-controllable monoids. However, there are several other variants that we did not consider and that could reveal to be very interesting for other aspects. So let us provide some examples of other classes of monoids and show a few easy results that follow from what we proved so far.

In [142], given a finite monoid M, the set $\mathbb{Y}(M)$ is defined as the forest generated by $\mathbb{X}(M)$ by taking the set of all chains of $\mathbb{X}(M)$ ordered by initial segment. This gave us two distinct possibilities on how to extend this notion to infinite monoids: either as the forest generated by $\mathbb{X}(M)$, or as the set of all chains of $\mathbb{X}(M)$ ordered by initial segment (which does need not to be a forest in this case). We decided to go for the second option. However, one may be curious about the possibility we left behind. Define $\mathbb{W}(M)$ be the forest generated by $\mathbb{X}(M)$. This set is usually constructed by taking the set of all *increasing sequences* $\bar{s} \in (\mathbb{X}(M))^{<\infty}$ of elements of $(\mathbb{X}(M), \subseteq)$ with initial segment relation $\sqsubseteq$ as order. However, $\mathbb{W}(M)$ can be identified also as the suborder of $\mathbb{Y}(M)$ of the wellordered subsets of $\mathbb{X}(M)$

$$\mathbb{W}(M) = \{\mathbf{y} \in \mathbb{Y}(M) \mid \mathbf{y} \text{ is a wellorder}\}$$

Then, we can define a monoid to be **(locally) $\mathbb{W}$-controllable** if it is (locally) $\mathbf{y}$-controllable for every maximal $\mathbf{y} \in \mathbb{W}(M)$.

Some of the results we presented work for this class of monoids as well. For example, Proposition 5.2.4 tell us already a necessary condition for locally $\mathbb{W}$-controllable monoids. Other results can be adapted with some changes. For example, the proof of Proposition 5.2.3 can be split into two parts, the first proving that "If M is locally $\mathbb{Y}$-controllable, then every element of finite period is idempotent" and the second proving that "If M is locally $\mathbb{Y}$-controllable, then M is periodic". It is easy to see that the first part still works for locally $\mathbb{W}$-controllable monoids, as it uses only finite elements $\mathbf{y} \in \mathbb{Y}(M)$ (which are also in $\mathbb{W}(M)$).

Proposition 5.4.4. *Let M be a locally $\mathbb{W}$-controllable. Then, every element of finite period is idempotent.*

This condition corresponds to point 6 of Proposition 4.2.2, the weakest of all statements listed there. This can be improved a bit. It is not difficult for example to see that the same argument of Proposition 5.2.3 shows that M does not contains non-trivial subgroups (using that if G is a non-trivial subgroup of M and $a \in G$ is not the identity of G, then $a^2 \neq a$ and $a^n M = a^m M$ for every $n, m > 0$. Then applying the same argument of Proposition 5.2.3 to $\mathbf{y} = \{aM, M\}$ gives the wanted result). However, we can not expect to obtain that M is aperiodic, as for example the natural numbers are $\mathbb{W}$-controllable (in a trivial way, since for every wellfounded $\mathbf{y} \in \mathbb{W}(M)$ and $a, b \in \mathbb{N}$ we have $a\mathbf{y} \leq_{\mathbb{Y}} b\mathbf{y}$ if and only if $a = b$) but not aperiodic.

On the other side, from Theorem 5.3.3 and from the same arguments of Lemmas 5.3.15 and 5.3.17 applied to $\mathbb{W}(M)$ instead of $\mathbb{Y}(M)$, we can derive the following sufficient conditions for being $\mathbb{W}$-controllable.

Theorem 5.4.5. *Let M be a monoid such that $(\mathbb{X}(M), \subseteq)$ contains no infinite increasing chain (i.e. $(\mathbb{X}(M), \supseteq)$ is wellfounded). Suppose that for action by continuous endomorphisms of M on a compact right topological semigroup U there is a minimal idempotent $u \in U$ such that for every $a, b \in M$, if $a \mathcal{R} b$ then $a(u) = b(u)$.*

Then, M is locally $\mathbb{W}$-controllable.

Corollary 5.4.6. *Let M be a monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear and finite, and suppose M satisfies the thesis of Lemmas 4.3.3 and 4.3.4 (i.e., for every $a, b \in M$ there are $g, h \in M$ such that $ag = b$, $bh = a$, $hg = gg = g$ and $gh = hh = h$, and $ab = b$ and $a \mathcal{R} b$ implies $ba = a$ for every $a, b \in M$). Suppose also that $(\mathbb{X}(M), \subseteq)$ contains no infinite increasing chain (i.e. $(\mathbb{X}(M), \supseteq)$ is wellfounded).*

Then, M is locally $\mathbb{W}$-controllable.

Finally, locally $\mathbb{Y}$-controllable monoids can be seen as just a particular case of a more general notion.

Definition 5.4.7. Let $(M, 0, *, \tau)$ be a monoid with a topology[1]. Let $\mathbf{y}$ be a maximal element in $\mathbb{Y}(M)$, let $F \subseteq \langle M\mathbf{y} \rangle$ be finite. We say that (M, τ) is **topologically** $(F, \mathbf{y})$**-controllable** if it is $(F, \mathbf{y}, (M_i)_{i \in \omega})$-controllable for every family $(M_i)_{i \in \omega}$ of **compact** subsets of M.

This notion naturally extends both the one of locally $\mathbb{Y}$-controllable and the one of $\mathbb{Y}$-controllable monoids, the first by giving for example the discrete topology to M and the second by giving the indiscrete topology to it.

We conclude with a remark. While reading this chapter, a careful reader may notice a section seems missing. He would be right. The study of finite monoids (Chapter 4) had three main parts (roughly): one about conditions that are necessary to be Ramsey/$\mathbb{Y}$-controllable, one for finding idempotents in compact right topological semigroups, and one for obtaining sufficient conditions to be Ramsey/$\mathbb{Y}$-controllable knowing the existence of these idempotents. In this chapter, we focused only on the first and third part and said nothing about the second (while this work would enjoy a new section with stronger results on idempotents in compact right topological semigroups, since the ones provided in Theorem 4.3.5 are far from optimal). This is partially due to time, as the work to do is a lot and the PhD lasts a fixed bounded amount of time, but the plan for the future is to complete the work of this chapter with one more section on topological dynamics.

However, we anticipate here a first easy result that hints how Theorem 4.3.5 could be improved to obtain idempotents relying on weaker hypotheses. We do so by studying an example of a monoid that does not satisfy most of the sufficient conditions we provided, and yet is $\mathbb{Y}$-controllable.

Example 5.4.8. Consider the monoid M freely generated by the set $\{0, g, h, 1\}$ modulo the relations

- $0g = g0 = gg = 0$ and $0h = h0 = hh = 0$.
- $1x = x1 = x$ for every $x \in M$.
- $ghg = g$ and $hgh = h$.

In other words, M satisfies the Caley following table:

Then, $\mathbb{X}(M) = \{\{0\}, \{g, gh\}, \{h, hg\}, M\}$ and $\mathbb{X}_{\mathcal{R}}(M) = \{\{g, gh\}, \{h, hg\}\}$. Thus $\mathbb{X}_{\mathcal{R}}(M)$ is not linear, and M does not satisfy the theses of Lemmas 4.3.3 and 4.3.4. Also, M does not satisfy the hypothesis of Proposition 4.6.3 either.

[1] The definition makes sense for every monoid with a topology, although in a further study of the notion it would probably be the best to restrict the attention to topological monoids where the operation is compatible with the topology.

1	g	gh	h	hg	0
g	0	0	gh	g	0
gh	g	gh	0	0	0
h	hg	h	0	0	0
hg	0	0	h	hg	0
0	0	0	0	0	0

However, M is $\mathbb{Y}$-controllable. Indeed, let U be a compact right topological semigroup on which M acts by continuous endomorphisms. We proceed as in Theorem 4.3.5 to find a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \mathcal{R} b$.

First, let u_0 be a minimal idempotent in $0(U)$, and let u_1 be a minimal idempotent in $g(U)$ such that $u_1 \leq_U u_0$ as given by Proposition 4.3.1. Notice that $(gh)^2 = gh$ and $(hg)^2 = hg$, hence the two functions $g : h(U) \to gh(U) = g(U)$ and $h : g(U) \to hg(U) = h(U)$ are the inverse of each other and thus they are homeomorphisms. Hence, $h(u_1)$ is minimal in $h(U)$ and $h(u_1) \leq_U u_0$ (as otherwise, $u_0 h(u_1) \neq h(u_1)$ would imply $g(u_0 h(u_1)) = u_0 u_1 \neq gh(u_1) = u_1$, since g injective). Notice that since $u_1 \in g(U)$, we have $g(u_1) \in gg(U) = 0(U)$, and $u_1 \leq_U u_0$ implies $g(u_1) \leq_U g(u_0) = u_0$, thus $g(u_1) = u_0$ by minimality of u_0. Similarly, $h(h(u_1)) = u_0$.

Let $v = u_1 h(u_1)$. Then, $g(v) = g(u_1) gh(u_1) = u_0 u_1 = u_1$, and similarly, $h(v) = h(u_1)$. Thus, $U' = g^{-1}(u_1) \cap h^{-1}(h(u_1))$ is a non-empty compact right topological subsemigroup of U, and we may find an idempotent $u_2 \in U'$ and a minimal idempotent $u \in I(U) \cap E(U)$ such that $u \leq_U u_2$ by Proposition 4.3.1.

We claim that u is as wanted. Indeed, $g(u_2) = u_1$ and $h(u_2) = h(u_1)$ by definition of U'. Hence, by minimality $g(u) \leq_U g(u_2) = u_1$ implies $g(u) = u_1$, and similarly $h(u) = h(u_1)$. Also, $gh(u) = gh(u_1) = u_1$ and $hg(u) = h(u_1)$. Thus, u satisfies that $a(u) = b(u)$ for every $a, b \in M$ with $a \mathcal{R} b$, as wanted.

Then, M is $\mathbb{Y}$-controllable by (for example) Lemma 5.3.11.

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