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SPECIAL GEOMETRIC STRUCTURES IN DIMENSION SIX AND SEVEN

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Introduction

In this thesis we study the geometry induced by certain classes of stable differential forms on smooth manifolds of dimension six and seven.

Stability of k-forms is a very rigid condition and it happens in very few cases. Luckily, it is also a pointwise condition, so it can be defined simply by working on a vector space. Let V be an n-dimensional real vector space. We say that a k-form $\sigma \in \Lambda^k V^*$ is *stable* if its orbit, under the natural action of $\mathrm{GL}(V)$ on $\Lambda^k V^*$, is open. Stability does not depend on what orientation one chooses for V, but, once such an orientation is fixed, a stable form always induces a volume form, as shown in [29, 83]. The case $k = 3$ is of particular interest, as recalled for instance by Hitchin in [83]. Indeed, besides the fact that 3 is the only possible odd number for the stability of k-forms in dimension six and seven, it is interesting to note that it may occur only in dimension six, seven and eight. Moreover, the stability of 3-forms induces very interesting geometries in these dimensions. In this thesis, we shall focus on the 6- and 7-dimensional cases.

Let M be a 6-dimensional smooth manifold and let $\rho \in \Lambda^3(M)$ be a stable 3-form, i.e., $\rho(p)$ is stable on the tangent vector space T_pM, for every $p \in M$. Fix an orientation $\Omega \in \Lambda^6(M)$. The stability of ρ may allow us to define an almost complex structure $J_{\rho,\Omega} =: J$ on M, depending on both ρ and the fixed orientation Ω. This possibility is encoded in the negativity of a polynomial $\lambda(\rho)$ of degree 4 in the coefficients of ρ in every local frame. When this happens, since the pointwise stabilizer of ρ is isomorphic to the special linear group $\mathrm{SL}(3,\mathbb{C})$, we say that ρ defines an $\mathrm{SL}(3,\mathbb{C})$-*structure* on M. Pointwise, there exists a basis $(f_1,\dots,f_6)$ of T_pM, called *adapted basis*, such that

$$\rho_p = f^{135} - f^{146} - f^{236} - f^{245},$$

with respect to the dual basis $(f^1,\dots,f^6)$. We recall that a 2-form α is stable if and only if it is non-degenerate, namely if $\alpha^3 \neq 0$. Given an $\mathrm{SL}(3,\mathbb{C})$-structure ρ and a stable 2-form $\tilde{\omega}$ on M, we say that ρ is *tamed* by $\tilde{\omega}$ if the $(1,1)$-part of $\tilde{\omega}$ is positive with respect to the induced almost complex structure J. This means that M inherits a Riemannian metric g, which is J-Hermitian. Explicitly,

$$g = \omega(\cdot, J\cdot),$$

where $\omega := \tilde{\omega}^{1,1}$. Special types of these structures are given by $\mathrm{SU}(3)$-structures. Let us assume ρ to be an $\mathrm{SL}(3,\mathbb{C})$-structure on M and let ω be a non-degenerate positive $(1,1)$-form on M such that $\rho \wedge J\rho = \frac{2}{3}\omega^3$. In particular, ω is a taming form with respect to J with vanishing $(2,0)+(0,2)$-part. Since the pointwise stabilizer of the pair (ω,ρ) is isomorphic to SU(3), we say that (ω, ρ) defines an SU(3)-*structure* on M. As in the case of $\mathrm{SL}(3,\mathbb{C})$-structures, it is always possible to find an *adapted basis* $(f_1, \dots, f_6)$ of T_pM such that

$$\rho_p = f^{135} - f^{146} - f^{236} - f^{245}, \qquad \omega_p = f^{12} + f^{34} + f^{56},$$

at every point $p \in M$. The study of SU(3)-structures and, more generally, tamed $\mathrm{SL}(3,\mathbb{C})$-structures has many interactions with the stability of 3-forms in dimension seven.

Let N be a 7-dimensional smooth manifold and let φ be a stable 3-form on it. The 3-form φ endows N with a Riemannian metric g_φ and an orientation Vol_{g_φ}. Explicitly,

$$g_\varphi(X,Y)\mathrm{Vol}_{g_\varphi} = \frac{1}{6}\,\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for every pair of vector fields X, Y on N, where $\iota_V\varphi$ denotes the contraction of φ by the vector field V. We denote by ∇^{g_φ} the Levi-Civita connection of g_φ and by $*_\varphi$ the Hodge operator determined by g_φ and Vol_{g_φ}. Since the pointwise stabilizer of such a form is isomorphic to the exceptional Lie group G_2 appearing in Berger's list (see [14]), these structure are more commonly known as G_2-*structures*. An *adapted basis* for a G_2-structure φ, at a point $p \in N$, is a basis $(f_1, \dots, f_7)$ of T_pN such that

$$\varphi_p = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}.$$

We can see how tamed $\mathrm{SL}(3,\mathbb{C})$-structures in dimension six and G_2-structures in dimension seven are linked one to the other by working on a vector space. Let W be an oriented 7-dimensional real vector space endowed with a 3-form $\varphi \in \Lambda^3 W^*$. Choose a non-zero vector $z \in W$ and a complementary subspace $V \subset W$, so that $W \cong V \oplus \mathbb{R}z$. Then, we can write

$$\varphi = \widetilde{\omega} \wedge \theta + \rho,$$

where $\theta \in W^*$ is the dual of z with respect to the chosen complement V, $\widetilde{\omega} \in \Lambda^2 V^*$ and $\rho \in \Lambda^3 V^*$. The 3-form φ on W is a G_2-structure if and only if the 3-form ρ on V is an $\mathrm{SL}(3,\mathbb{C})$-structure tamed by the non-degenerate 2-form $\widetilde{\omega}$. The pair $(\widetilde{\omega}, \rho)$ defines an SU(3)-structure on V, up to a suitable normalization, if and only if $\widetilde{\omega} \wedge \rho = 0$. When this happens, the vector space V coincides with the g_φ-orthogonal complement of $\mathbb{R}z \subset W$. On the other hand, if φ defines a G_2-structure on W inducing the Riemannian metric g_φ, we can consider the 6-dimensional subspace $U \coloneqq (\mathbb{R}z)^{\perp_{g_\varphi}} \subset W$ and the g_φ-orthogonal splitting $W = U \oplus \mathbb{R}z$. It then follows that φ induces an SU(3)-structure on U.

Translating this construction into the manifold setting, we can consider a 7-dimensional manifold N endowed with a G_2-structure φ and an orientable 6-dimensional submanifold $i\colon M \hookrightarrow N$. Let X be a vector field along M such that

$$T_pM \oplus \langle X_p \rangle = T_pN, \quad p \in N. \tag{1}$$

Then, the G_2-structure φ induces an $\mathrm{SL}(3,\mathbb{C})$-structure $\rho \coloneqq i^*\varphi$ on M tamed by the non-degenerate 2-form $\widetilde{\omega} \coloneqq \iota_X\varphi$. If the direct sum in (1) is g_φ-orthogonal and X is of constant unit norm with respect to g_φ, then $(\widetilde{\omega}, \rho)$ defines an SU(3)-structure on M.

In this thesis, we study *closed* $\mathrm{SL}(3,\mathbb{C})$-*structures* in dimension six and *closed* G_2-*structures* in dimension seven, where by closed we mean that the defining form is closed with respect to the de Rham differential d of the manifold.

In the 6-dimensional case, we focus on closed $\mathrm{SL}(3,\mathbb{C})$-structures which are either *mean convex* or *tamed by a symplectic form*. Let ρ be a closed $\mathrm{SL}(3,\mathbb{C})$-structure on an oriented smooth 6-manifold M. As shown in [35], the differential $d\hat{\rho}$ of the 3-form $\hat{\rho} = J\rho$ is of type $(2,2)$ with respect to J. Therefore, one may wonder if $d\hat{\rho}$ is *semi-positive*. When this happens, we say that ρ defines a *mean convex closed* $\mathrm{SL}(3,\mathbb{C})$-*structure* on M. In a similar way, we say that an $\mathrm{SU}(3)$-structure (ω,ρ) is *closed* and *mean convex* if $d\rho = 0$ and $d\hat{\rho}$ is semi-positive. Note that J is integrable if and only if $d\hat{\rho} = 0$. A special class of mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures is given by *nearly-Kähler* structures. Indeed, a nearly-Kähler structure can be defined as an $\mathrm{SU}(3)$-structure (ω,ρ) satisfying the following conditions:

$$d\omega = -\frac{3}{2}\nu_0\,\rho, \quad d\hat{\rho} = \nu_0\,\omega^2,$$

where $\nu_0 \in \mathbb{R} - \{0\}$. Therefore, up to a change of sign of ρ, we can suppose $\nu_0 > 0$. The nearly-Kähler condition forces the induced Riemannian metric g to be Einstein and, up to now, very few examples of manifolds admitting complete nearly-Kähler structures are known (see for instance [23, 62, 77, 78, 109, 110]).

In Chapter 2, we study mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures on nilmanifolds, i.e., on compact quotients $\Gamma\backslash G$ of connected simply connected nilpotent Lie groups G by co-compact discrete subgroups (lattices) Γ. Nilmanifolds provide a large class of compact 6-manifolds admitting invariant closed $\mathrm{SL}(3,\mathbb{C})$-structures [24, 25, 26, 28, 53], where by invariant we mean induced by a left-invariant one on the nilpotent Lie group G. Note that nilmanifolds cannot admit invariant nearly Kähler structures, since by [107] the Ricci tensor of a left-invariant metric on a non-abelian nilpotent Lie group always has a strictly negative direction and a strictly positive direction.

In Section 2.3, we provide a full classification of nilmanifolds admitting an invariant mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structure and we prove that an explicit example of mean convex closed $\mathrm{SU}(3)$-structure can also be found on each of them. We then restrict to half-flat $\mathrm{SU}(3)$-structures. An $\mathrm{SU}(3)$-structure (ω,ρ) on a smooth 6-manifold M such that

$$\begin{cases} d\rho = 0, \\ d\omega^2 = 0 \end{cases}$$

is called *half-flat* and we refer to it simply as a half-flat structure (see for instance [12, 20, 24, 26, 29, 50, 70, 83, 94] for general results on this type of structures). We then study the interplay between these structures and mean convex ones and we determine nilmanifolds admitting an $\mathrm{SU}(3)$-structure which is both half-flat and mean convex.

Half-flat structures play an important role in constructing 7-dimensional manifolds endowed with a *torsion-free* G_2-structure, i.e., a stable 3-form φ which is parallel with respect to the Levi-Civita connection ∇^{g_φ} of g_φ. In particular, every oriented hypersurface of a 7-manifold endowed with a torsion-free G_2-structure is naturally endowed with a half-flat structure. Conversely, a 6-manifold with a real analytic half-flat structure can be realized as a hypersurface of a 7-manifold endowed with a torsion-free G_2-structure. This was proved by Hitchin in [83], with the introduction of the evolution equations

$$\begin{cases} \frac{\partial}{\partial t}\rho(t) = d\omega(t), \\ \frac{\partial}{\partial t}\omega(t) \wedge \omega(t) = -d\hat{\rho}(t), \end{cases} \tag{2}$$

now commonly known as *Hitchin flow equations.* A solution to (2), starting from a given SU(3)-structure (ω, ρ) at $t_0 \in \mathbb{R}$, is a one-parameter family of SU(3)-structures $(\omega(t), \rho(t))$, with t belonging to an open interval I containing t_0, which solves the Hitchin flow equations and such that $(\omega(t_0), \rho(t_0)) = (\omega, \rho)$. It is not difficult to see that, if the initial condition is half-flat, then the solution is half-flat as long as it exists. Eventually, we study the solutions to the Hitchin flow equations starting from our examples of mean convex half-flat structures given in table 6.3 in the Appendix. In particular, since the mean convexity of the initial data is preserved by the flow in all of them, we conjecture that this could always be the case.

Given a closed $\mathrm{SL}(3,\mathbb{C})$-structure ρ on a 6-manifold, another natural condition to study is the existence of a symplectic form Ω taming J. As shown in [35], a mean convex $\mathrm{SL}(3,\mathbb{C})$-structure on a compact 6-manifold cannot be tamed by any symplectic forms. If we remove the assumption of mean convexity, examples of tamed closed $\mathrm{SL}(3,\mathbb{C})$-structures are given by *symplectic half-flat structures* (ω, ρ), i.e., half-flat structures (ω, ρ) with $d\omega = 0$. In this case, ρ is tamed by the symplectic form ω. In [28], nilmanifolds admitting invariant symplectic half-flat structures were classified. Later, this classification was generalized to solvmanifolds, i.e. compact quotients $\Gamma \backslash G$ of connected simply connected solvable Lie groups G by lattices Γ (for more details, see [47]). In Chapter 2, we also classify solvmanifolds admitting invariant closed $\mathrm{SL}(3,\mathbb{C})$-structures tamed by a symplectic form. In particular, we prove that solvmanifolds admitting an invariant tamed closed $\mathrm{SL}(3,\mathbb{C})$-structure also admit an invariant symplectic half-flat SU(3)-structure. As an application, we classify 7-dimensional solvable Lie groups of the form $G \times S^1$ admitting closed G_2-structures.

In literature, many examples of closed G_2-structures have been obtained on nilmanifolds and solvmanifolds. The first one was given by Fernández on the compact quotient of a nilpotent Lie group [42]. In the solvable non-nilpotent case, various examples are currently known and many of them satisfy additional meaningful conditions that one can impose on a closed G_2-structure, see e.g. [43, 56, 64, 96, 97, 98]. In these examples, the closed G_2-structure on the Lie group G is left-invariant and thus it is determined by a G_2-structure φ on the Lie algebra $\mathfrak{g} = \mathrm{Lie(G)}$ which is closed with respect to the Chevalley-Eilenberg differential of $\mathfrak{g}$. The isomorphism classes of nilpotent and unimodular non-solvable Lie algebras admitting closed G_2-structures were determined in [27] and [55], respectively. In Chapter 3, we prove that a 7-dimensional Lie algebra $\mathfrak{g}$ with non-trivial center endowed with a closed G_2-structure φ is the central extension of a 6-dimensional Lie algebra $\mathfrak{h}$ via a closed 2-form $\omega_0 \in \Lambda^2\mathfrak{h}^*$. This allows us to reduce the classification problem from 7 dimensions to 6. In detail, $\mathfrak{g}$ admits a closed G_2-structure φ if and only if $\mathfrak{h}$ admits an $\mathrm{SL}(3,\mathbb{C})$-structure ρ tamed by a symplectic form $\widetilde{\omega}$ satisfying $d\rho = -\omega_0 \wedge \widetilde{\omega}$. As special cases of this, we consider the case $\omega_0 = 0$ and the case $\omega_0^3 \neq 0$. In the former, the central extension reduces to the Lie algebra direct sum between $\mathfrak{h}$ and the abelian Lie algebra $\mathbb{R}$. In the latter, $\mathfrak{g}$ admits a contact 1-form θ such that $d\theta = \omega_0$. In this case, we refer to the pair $(\mathfrak{g}, \theta)$ as the *contactization* of $(\mathfrak{h}, \omega_0)$.

Using these characterizations, we prove the following:

Theorem. *Let $\mathfrak{g}$ be a 7-dimensional unimodular solvable non-nilpotent Lie algebra with non-trivial center. Then, $\mathfrak{g}$ admits closed G_2-structures if and only if it is isomorphic to one of the following:*

$$
\begin{aligned}
\mathfrak{s}_1 &= (e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 0),\\
\mathfrak{s}_2 &= (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, 0),\\
\mathfrak{s}_3 &= (-e^{16} + e^{25}, -e^{15} - e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0, 0),\\
\mathfrak{s}_4 &= (0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}, 0),\\
\mathfrak{s}_5 &= (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0, 0),\\
\mathfrak{s}_6 &= (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0, 0),\ \ \alpha > 0,\\
\mathfrak{s}_7 &= (e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0, 0),\\
\mathfrak{s}_8 &= (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, e^{34}),\\
\mathfrak{s}_9 &= (-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0, e^{34}),\\
\mathfrak{s}_{10} &= \left(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 2\,e^{16} + e^{25} - e^{34} + \sqrt{3}\,e^{24} + \sqrt{3}\,e^{35}\right),\\
\mathfrak{s}_{11} &= \left(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 2\,e^{16} + e^{25} - e^{34} - \sqrt{3}\,e^{24} - \sqrt{3}\,e^{35}\right).
\end{aligned}
$$

In particular, $\mathfrak{g}$ is the contactization of a symplectic Lie algebra if and only if it is isomorphic either to $\mathfrak{s}_{10}$ or to $\mathfrak{s}_{11}$.

Moreover, the simply connected Lie groups corresponding to some of these Lie algebras admit lattices. Recall that, a necessary condition for the existence of lattices is the unimodularity of G, or, equivalently, of its Lie algebra $\mathfrak{g}$ (see [107]). We use the results of [15] to construct a lattice for two of them (see Remark 3.18). In this way, we obtain new locally homogeneous examples of compact 7-manifolds with a closed G_2-structure. Finally, as a corollary of the classification result, we show that the abelian Lie algebra and a certain 2-step solvable Lie algebra are the only unimodular Lie algebras with non-trivial center admitting torsion-free G_2-structures.

A torsion-free G_2-structure can be equivalently defined as a G_2-structure φ wich is both closed and coclosed, i.e.,

$$
\begin{cases} d\varphi = 0,\\ d *_\varphi \varphi = 0. \end{cases}
$$

In particular, the induced metric g_φ is Ricci-flat and the Riemannian holonomy group $\mathrm{Hol}(g_\varphi)$ is isomorphic to a subgroup of G_2. The first complete examples of torsion-free G_2-structures were constructed in [21]. In the compact case, 7-manifolds admitting torsion-free G_2-structures were constructed first in [88] and, later, in [30, 91]. In [88], Joyce proved that under certain conditions, a closed G_2-structure on a compact 7-manifold can be deformed into a torsion-free G_2-structure. In [19], Bryant introduced the so-called *Laplacian flow*, a geometric flow evolving a closed G_2-structure φ in the direction of its Hodge Laplacian. Short-time existence was proved in [22]. As shown in [101], the stationary points of the flow are exactly torsion-free G_2-structures. In [102], the authors proved a result of dynamical stability, stating that, on a compact 7-manifold, if the initial data is close enough to being torsion-free, the solution exists for all positive times and converges to a torsion-free G_2-structure. Therefore, closed G_2-structures provide an important tool in the study of torsion-free G_2-structures on compact 7-manifolds.

A special class of closed G_2-structures that has attracted a lot of attention in recent years is given by Laplacian solitons. A closed G_2-structure φ on a 7-manifold N is said to be a *Laplacian soliton* if it satisfies the equation

$$\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X \varphi, \tag{3}$$

for some real constant λ and some complete vector field X on N, where $\Delta_\varphi = d \circ d^*_\varphi + d^*_\varphi \circ d$ denotes the Hodge Laplacian of the induced metric g_φ. Here, $d^*_\varphi = - *_\varphi \circ d \circ *_\varphi$ denotes the codifferential of d induced by φ. These G_2-structures give rise to self-similar solutions of the Laplacian flow, i.e., solutions which differ from the initial data only by a uniform rescaling and time-dependent diffeomorphisms. Depending on the sign of λ, a Laplacian soliton is called *expanding* ($\lambda > 0$), *steady* ($\lambda = 0$), or *shrinking* ($\lambda < 0$). On a compact manifold, every Laplacian soliton which is not torsion-free must be expanding and satisfy (3) with $\mathcal{L}_X\varphi \neq 0$ (see [99, 101]). The existence of non-torsion-free Laplacian solitons on compact manifolds is still an open problem. In the non-compact setting, examples of Laplacian solitons of every type are known, see e.g. [9, 56, 57, 63, 96, 97, 98, 111]. In particular, the steady Laplacian solitons in [9] and the shrinking Laplacian soliton in [63] are inhomogeneous and of *gradient type*, i.e., X is a gradient vector field. As for the known homogeneous examples, they consist of simply connected Lie groups G endowed with a left-invariant closed G_2-structure satisfying the equation (3) with respect to a vector field X defined by a one-parameter group of automorphisms induced by a derivation D of the Lie algebra $\mathfrak{g}$ of G. According to [96], these Laplacian solitons are called *semi-algebraic*.

In Chapter 4, we consider semi-algebraic Laplacian solitons on unimodular Lie algebras with non-trivial center. Under a natural assumption on the derivation D, we are able to relate the constant λ in (3) to a certain eigenvalue of D and to the norm of the intrinsic torsion form of the semi-algebraic Laplacian soliton φ, namely the unique 2-form τ such that

$$d *_\varphi \varphi = \tau \wedge \varphi = - *_\varphi \tau.$$

Moreover, we show that λ coincides with the squared norm of τ whenever the Lie algebra is the contactization of a symplectic one. In this last case, the semi-algebraic Laplacian soliton must be expanding. We also prove the non-existence of semi-algebraic Laplacian solitons on certain Lie algebras with 1-dimensional center and we obtain the classification of all unimodular Lie algebras with center of dimension at least two that admit semi-algebraic Laplacian solitons, up to isomorphism.

Let φ be a coclosed G_2-structure on a smooth 7-manifold N. We say that φ is *of pure type* if $d\varphi \wedge \varphi = 0$. It is well known that a coclosed G_2-structure φ on a 7-manifold N induces half-flat SU(3)-structures on all orientable 6-dimensional submanifolds of N via the construction described earlier. Moreover, if (ω, ρ) is balanced in the sense of [49], then φ is a purely coclosed G_2-structure. We say that a half-flat structure (ω, ρ) is *balanced* if $d\hat{\rho} = 0$. Such structures arise as a generalization of torsion-free SU(3)-structures, i.e., SU(3)-structures satifying

$$d\omega = 0, \quad d\rho = 0, \quad d\hat{\rho} = 0,$$

in the non-Kählerian case $d\omega \neq 0$. In 1986, Hull and Strominger [87, 125], independently, introduced a system of PDEs, now known as the *Hull–Strominger system*, to formalize certain properties of the inner space model used in string theory. Let J be a complex structure on a smooth 6-manifold M and let $\Psi = \rho + i\hat{\rho}$ be a nowhere-vanishing holomorphic $(3,0)$-form on M. We denote by E a holomorphic vector bundle on M endowed with the Chern connection. The Hull–Strominger system consists of a set of partial differential equations involving a pair of Hermitian metrics (g, h) on (M, E). One of these equations dictates the metric g on M to be conformally balanced, more precisely $d\left(\|\Psi\|_\omega \omega^2\right) = 0$, where $\omega = g(J\cdot, \cdot)$ is the *fundamental form* associated with (g, J) and $\|\Psi\|_\omega$ is the norm of Ψ given explicitly by $\Psi \wedge \overline{\Psi} = -\frac{i}{6}\|\Psi\|_\omega^2 \omega^3$. When one assumes all structures to be invariant under the smooth action of a certain Lie group G, the aforementioned condition reduces to the *balanced equation* $d\omega^2 = 0$, since the norm of Ψ is constant. Notice that in these cases the pair (ω, ρ) defines a balanced $\mathrm{SU}(3)$-structure on M, up to a suitable uniform scaling of ρ.

The issue of the existence and uniqueness of solutions to the Hull–Strominger system is understood only in some special cases. In literature, particular focus is placed on the non-Kählerian case (see for instance [39, 40, 41, 67, 68, 69, 75, 113]). In particular, in [41], a class of invariant solutions to the Hull–Strominger system on complex Lie groups was provided. These solutions extend to solutions on all compact complex parallelizable manifolds, by Wang's classification theorem [130]. Moreover, in [52], it was shown that a compact complex homogeneous space with invariant complex volume admitting a balanced metric is necessarily a complex parallelizable manifold. Therefore, the invariant solutions given in [41] exhaust the complex compact homogeneous case. If one allows the Lie group acting on the homogeneous space to be real, many other solutions to the Hull–Strominger system are known in literature, see for instance [75, 118, 119, 128]. Then, one may wonder what happens in the cohomogeneity one case. A cohomogeneity one manifold M is a connected smooth manifold with an action of a compact Lie group G having an orbit of codimension one. In Chapter 4, we prove the following non-existence result:

Theorem. *Let M be a 6-dimensional simply connected cohomogeneity one manifold under the almost effective action of a compact connected Lie group G. Then, M admits no G-invariant balanced non-Kähler $\mathrm{SU}(3)$-structures.*

Chapter 1

Preliminaries

In this chapter, we review the basic notions and fix some notations on the main topics of interest of this thesis. After recalling some classical results on Lie groups and their Lie algebras, we give a quick overview on the theory of group actions on smooth manifolds, a bridge towards G-structures, the final goal of our preliminary discussion.

We start by fixing some notations. We shall denote smooth manifolds with capital Latin letters like M and N and points of a smooth manifold M with lowercase Latin letters such as p and q. The tangent and the cotangent bundles of M will be denoted by TM and T^*M, respectively, with vector fields denoted by capital Latin letters such as X and Y and specific vectors denoted by vector fields evaluated at a point (for example $X_p \in T_pM$ for a vector field X) or lowercase Latin letters such as v and w. If E is the total space of a fiber bundle $\pi\colon E \to M$ we shall denote by $\Gamma(E)$ the $C^\infty(M)$-module of its sections. $\Lambda^k(M)$ will denote the space of k-forms on M, that is, the space $\Gamma(\Lambda^k(T^*M))$ of sections of the bundle of k-forms. Its elements, k-forms, will be denoted by lowercase Greek letters such as α, β and γ. Given a point p of an n-dimensional smooth manifold M, a basis for the tangent space T_pM will be denoted by the n-tuple $(e_1, \dots, e_n)$, with its dual basis for T_p^*M denoted by $(e^1, \dots, e^n)$. The wedge products of 1-forms $e^i \wedge e^j \wedge \dots\dots \wedge e^k$ will be shortened as $e^{ij\cdots k}$. In a similar way, β^k will be a shorthand for the wedge product $\beta \wedge \beta \wedge \dots \wedge \beta$ of k copies of a differential form β. Lie groups and Lie algebras will usually be denoted by capital Latin letters such as G or H and by gothic letters like $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Similarly to the case of tangent spaces, a basis of vectors for an n-dimensional Lie algebra $\mathfrak{g}$ and its dual basis for $\mathfrak{g}^*$ will be denoted by n-tuples $(e_1, \dots, e_n)$ and $(e^1, \dots, e^n)$, respectively.

1.1 Lie group actions

Our first aim is to recall some facts about Lie groups and Lie algebras and to describe how they can be exploited to impose symmetry on geometric structures on smooth manifolds, via smooth actions. For more details, we refer for instance to [3].

1.1.1 Lie groups and Lie algebras

Definition 1.1. A Lie group G is a smooth manifold endowed with a group structure $(G, \cdot)$ such that the map

$$\begin{aligned} G \times G &\to G, \\ (g, h) &\mapsto gh^{-1} \end{aligned}$$

is smooth.

Let G be a connected Lie group. Each element $g \in G$ defines three automorphisms of G,

$$\begin{aligned} L_g &: G \to G, \quad h \mapsto g \cdot h, \\ R_g &: G \to G, \quad h \mapsto h \cdot g, \\ C_g &: G \to G, \quad h \mapsto g \cdot h \cdot g^{-1}, \end{aligned}$$

called *left translation* by g, *right translation* by g and *conjugation* by g, respectively.

Focusing on left translations, we say that a tensor field ψ on a Lie group G is *left-invariant* if, for every $g \in G$,

$$L_g^* \psi = \psi,$$

i.e., if it is invariant under the action induced by the left translations on the corresponding space of tensor fields.

Recall that a (real) Lie algebra $\mathfrak{g}$ is defined as a (real) vector space endowed with a Lie bracket, namely a $\mathfrak{g}$-valued $\mathbb{R}$-bilinear skew-symmetric form satisfying the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad X, Y, Z \in \mathfrak{g}.$$

Focusing on left-invariant vector fields, we recall the following.

Definition 1.2. Given a Lie group G, its Lie algebra $\mathfrak{g}$, or $\mathrm{Lie}(G)$, is the Lie algebra of left-invariant vector fields of G, with Lie bracket induced by the Lie brackets of vector fields.

The Lie algebra $\mathfrak{g}$ of a Lie group G is naturally isomorphic to the tangent space T_eG of G at the identity element $e \in G$, via the map

$$\begin{aligned} \mathfrak{g} &\to T_eG, \\ X &\mapsto X_e. \end{aligned}$$

By a well known result, given a Lie algebra $\mathfrak{g}$, there exists a unique (up to isomorphism) connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$.

The importance of the concept of Lie algebra $\mathfrak{g}$ of a Lie group G lies in the fact that left-invariant tensor fields on G can be identified with tensors of the same type defined on $\mathfrak{g}$ and vice versa.

Every Lie group G acts on its Lie algebra $\mathfrak{g}$ by the *adjoint action*

$$\begin{aligned} \mathrm{Ad} &: G \to \mathrm{GL}(\mathfrak{g}), \\ &g \mapsto (dC_g)_e : \mathfrak{g} \to \mathfrak{g}. \end{aligned}$$

Its differential at the identity element $e \in G$,

$$\mathrm{ad} := (d\mathrm{Ad})_e \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

is such that

$$\mathrm{ad}_X Y = [X, Y], \quad X, Y \in \mathfrak{g},$$

where $[\cdot, \cdot]$ denotes the Lie bracket of $\mathfrak{g}$. We say that G is *unimodular* if

$$|\det(\mathrm{Ad}(g))| = 1, \quad g \in G.$$

When G is simply connected, this is equivalent to the unimodularity of its Lie algebra $\mathfrak{g}$, i.e.,

$$\mathrm{tr}(\mathrm{ad}_X) = 0, \quad X \in \mathfrak{g}.$$

Every compact Lie group is unimodular. We recall that the Lie algebras of compact Lie groups are called *compact*.

The *Killing Cartan form* of $\mathfrak{g}$ is the symmetric bilinear form

$$\mathcal{B}(X, Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y), \quad X, Y \in \mathfrak{g}.$$

If $\mathcal{B}$ is non-degenerate, then $\mathfrak{g}$ is said to be *semisimple*. A Lie group G is called semisimple if its Lie algebra is semisimple. The Killing Cartan form is an important tool for characterizing the structure of a Lie algebra, as the next results show. For further details, see for instance [3, Section 2.3].

Proposition 1.3. *The Killing Cartan form is* Ad*-invariant, that is, for all $X, Y \in \mathfrak{g}$ and $g \in G$, one has that*

$$\mathcal{B}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = \mathcal{B}(X, Y).$$

Theorem 1.4. *Let G be an n-dimensional semisimple connected Lie group. Then, G is compact if and only if its Killing Cartan form $\mathcal{B}$ is negative-definite.*

Theorem 1.5. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is the direct sum of simple Lie algebras $\mathfrak{g}_i$, that is,*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k,$$

where each $\mathfrak{g}_i$ is non-abelian and contains no non-zero proper ideals.

Let $\mathfrak{g}$ be the Lie algebra of an n-dimensional Lie group G. Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{g}$, with associated dual basis $(e^1, \ldots, e^n)$, which can be thought of as a basis for left-invariant 1-forms on G. We denote by

$$[e_i, e_j] = c_{ij}^k \, e_k$$

the *structure equations* of $\mathfrak{g}$ with respect to the chosen basis. The real numbers $c_{ij}^k = -c_{ji}^k$ are called *structure constants*. Differentiating the 1-forms $e^1, \ldots, e^n$ with respect to the differential d of G, one gets

$$de^i(e_j, e_k) = e_j(e^i(e_k)) - e_k(e^i(e_j)) - e^i([e_j, e_k]) = -c_{jk}^i,$$

Therefore, the structure equations of $\mathfrak{g}$ can be equivalently defined as

$$de^i = -\sum_{j<k} c^i_{jk}\, e^{jk},$$

where d is now seen as a linear map from $\Lambda^1\mathfrak{g}^*$ to $\Lambda^2\mathfrak{g}^*$. A Lie algebra $\mathfrak{g}$ only depends on its structure equations with respect to some basis, so we can indicate it via the n-tuple

$$\left(de^1 = -\sum_{j<k} c^1_{jk}\, e^{jk}, \dots, de^n = -\sum_{j<k} c^n_{jk}\, e^{jk}\right).$$

We can extend d to a linear map $d\colon \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ by requiring that

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta,$$

for every k-form α and every r-form β. At the Lie algebra level, the differential d is known as the *Chevalley-Eilenberg differential* of $\mathfrak{g}$ and it satisfies the identity $d^2 = d\circ d = 0$. Thus, $(\Lambda^\bullet\mathfrak{g}^*, d)$ is a differential complex. We define the spaces

$$\mathcal{Z}^k\mathfrak{g} = \ker d\big|_{\Lambda^k(\mathfrak{g})^*}, \quad \mathcal{B}^k\mathfrak{g} = \operatorname{Im} d|_{\Lambda^{k-1}\mathfrak{g}},$$

satisfying $\mathcal{B}^k\mathfrak{g} \subset \mathcal{Z}^k\mathfrak{g}$. Their quotient defines the *k-th cohomology group*

$$\mathcal{H}^k\mathfrak{g} := \frac{\mathcal{Z}^k\mathfrak{g}}{\mathcal{B}^k\mathfrak{g}}.$$

The dimension $b_k := \dim \mathcal{H}^k\mathfrak{g}^*$ is called *k-th Betti number* of $\mathfrak{g}$. We denote by $\mathfrak{z}(\mathfrak{g})$ the center of the Lie algebra $\mathfrak{g}$, i.e.,

$$\mathfrak{z}(\mathfrak{g}) = \{X\in\mathfrak{g} \mid [X,Y]=0,\, Y\in\mathfrak{g}\},$$

which is an ideal of $\mathfrak{g}$. Other relevant ideals of a Lie algebra $\mathfrak{g}$ are the terms of its *lower central series*, which we denote as follows:

$$\begin{aligned}\mathcal{C}^0(\mathfrak{g}) &= \mathfrak{g},\\ \mathcal{C}^{r+1}(\mathfrak{g}) &= [\mathfrak{g}, \mathcal{C}^r(\mathfrak{g})], \quad r\geq 0.\end{aligned}$$

By construction, $\mathcal{C}^{r+1}(\mathfrak{g})$ is an ideal of $\mathcal{C}^r(\mathfrak{g})$, for every $r\geq 0$. The ideal $\mathcal{C}^1(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]$ is known as the *commutator ideal* of $\mathfrak{g}$.

Definition 1.6. We say that $\mathfrak{g}$ is *(k-step) nilpotent* if there exists an integer k such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ and $\mathcal{C}^i(\mathfrak{g}) \neq \{0\}$ for every $i<k$. We say that a Lie group G is (k-step) *nilpotent* if its Lie algebra is.

The *derived series* of $\mathfrak{g}$ is defined by:

$$\begin{aligned}\mathcal{D}^0(\mathfrak{g}) &= \mathfrak{g},\\ \mathcal{D}^{r+1}(\mathfrak{g}) &= [\mathcal{D}^r(\mathfrak{g}), \mathcal{D}^r(\mathfrak{g})], \quad r\geq 0,\end{aligned}$$

By construction, $\mathcal{D}^{r+1}(\mathfrak{g})$ is an ideal of $\mathcal{D}^r(\mathfrak{g})$, for every $r\geq 0$.

Definition 1.7. We say that $\mathfrak{g}$ is *(k-step) solvable* if there exists an integer k such that $\mathcal{D}^k(\mathfrak{g}) = \{0\}$ and $\mathcal{D}^i(\mathfrak{g}) \neq \{0\}$ for every $i < k$. We say that a Lie group G is (k-step) *solvable* if its Lie algebra is.

Equivalently, we can say that a Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing Cartan form $\mathcal{B}$ is such that

$$\mathcal{B}(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = \{0\}.$$

In particular, if $\mathcal{B}$ vanishes identically, then $\mathfrak{g}$ is solvable.

Remark 1.8. Since one has $\mathcal{D}^r(\mathfrak{g}) \subset \mathcal{C}^r(\mathfrak{g})$ for every $r \geq 0$, every (k-step) nilpotent Lie algebra is also trivially (at most k-step) solvable.

Lie algebras have been classified in low dimensions only. Focusing on the nilpotent and solvable ones, a full classification has been achieved up to dimension six only. According to [73, 105], there are 34 isomorphism classes of 6-dimensional real nilpotent Lie algebras $\mathfrak{g}_i$, $i = 1, \ldots, 34$, listed in Table 6.1. Solvable (non-nilpotent) Lie algebras of dimension six have been classified in [108, 123, 124, 127]. In higher dimension the problem is still open and only some partial results are known: in dimension seven, we recall Gong's classification of indecomposable real nilpotent Lie algebras [71]. This list contains 140 Lie algebras and 9 one-parameter families. In addition, there are 35 isomorphism classes of 7-dimensional decomposable nilpotent Lie algebras [105, 121].

1.1.2 Smooth actions on manifolds and G-structures

Let M be a smooth manifold and $\mathrm{Diff}(M)$ be its group of diffeomorphisms.

Definition 1.9. A *left action* of a Lie group G on M is defined by a map

$$\alpha \colon G \times M \to M$$

such that

$$\alpha(e, p) = p, \quad \alpha(g, \alpha(h, p)) = \alpha(gh, p),$$

for every $g, h \in G$ and $p \in M$. For every $g \in G$, we denote

$$\alpha^g := \alpha(g, \cdot) \in \mathrm{Diff}(M)$$

and, when no confusion arises, $g \cdot p := \alpha(g, p)$, for every $p \in M$.

In a similar way, we give the following.

Definition 1.10. A *right action* of a Lie group G on M is defined by a map

$$\sigma \colon M \times G \to M$$

such that

$$\sigma(p, e) = p, \quad \sigma(\sigma(p, g), h) = \sigma(p, gh)$$

for every $g, h \in G$ and $p \in M$. When no confusion arises, we denote $p \cdot g := \sigma(p, g)$ for every $g \in G$, $p \in M$.

In what follows, we consider a left action $\alpha\colon G \times M \to M$, but the same constructions also hold in the case of right actions, with the due adjustments. As usual, $\mathfrak{g}$ denotes the Lie algebra of the Lie group G acting on M.

Definition 1.11. For every $X \in \mathfrak{g}$, the *induced vector field* on M associated with X is

$$\hat{X}_p := \frac{d}{dt}\Big|_{t=0} (\exp(tX) \cdot p), \quad p \in M,$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map of G. The map

$$\begin{aligned} \mathfrak{g} &\to \Gamma(TM), \\ X &\mapsto \hat{X} \end{aligned}$$

is called the *infinitesimal action* of $\mathfrak{g}$ on M.

The *orbit* $G \cdot p$, through $p \in M$, is defined as the image of the map

$$\begin{aligned} \alpha_p\colon G &\to M \\ g &\mapsto g \cdot p, \end{aligned}$$

while the *isotropy* (or *stabilizer group*) at $p \in M$, is the set

$$G_p := \alpha_p^{-1}(p) = \{g \in G \mid g \cdot p = p\}.$$

We say that α is *proper* if the inverse image of each compact set in $M \times M$ via the map

$$\begin{aligned} G \times M &\to M \times M, \\ (g, p) &\mapsto (p, g \cdot p) \end{aligned}$$

is also compact in $G \times M$. We say that α is *almost effective* if the kernel of the action, namely $\bigcap_{p\in M} G_p$, is a discrete subset of G, *effective* if $\bigcap_{p\in M} G_p = \{e\}$. Moreover, if $G_p = \{e\}$ for all $p \in M$, we say that α is *free*. We recall that the isotropy group G_p changes by conjugation as p moves along its orbit $G \cdot p$, namely

$$G_{g\cdot p} = g\, G_p\, g^{-1}, \quad g \in G,\, p \in M.$$

Let us assume that G acts properly on a smooth manifold M. Then, each isotropy group G_p is a compact Lie subgroup of G. Moreover, each quotient space G/G_p is a smooth manifold, the orbit $G \cdot p$ is the image of an embedding of G/G_p into M and the tangent space $T_pG \cdot p$ is given by

$$T_pG \cdot p = (d\alpha_p)_e\,(\mathfrak{g}).$$

We denote by M/G the set of G-orbits in M and equip M/G with the quotient topology relative to the canonical projection

$$\begin{aligned} \pi\colon M &\to M/G, \\ p &\mapsto G \cdot p. \end{aligned}$$

Exploiting the theory of right actions, we can define two special classes of fiber bundles. Let N be a smooth manifold and G be a Lie group. A *principal G-bundle* on N is the data of a smooth manifold P, a locally trivial surjection

$$\pi\colon P \to N$$

and a right and free action of G on P which preserves the fibers of π and is transitive on each of them. The fibers of π can then be seen as the orbits of the G-action on P, which implies that they are diffeomorphic to G. Moreover, one has that N is diffeomorphic to the orbit space P/G. The group G is known as the *structure group* of the principal bundle.

Example 1.12. Let N be an n-dimensional smooth manifold. A *linear frame* u at a point $p \in N$ is an ordered basis $(v_1, \ldots, v_n)$ of the tangent space T_pN. Let $\mathrm{GL}(N)$ be the set of all linear frames at all points of N and denote by $\pi\colon \mathrm{GL}(N) \to N$ the map sending a linear frame u at p into p. The natural right action of the general linear group $\mathrm{GL}(n, \mathbb{R})$ on $\mathrm{GL}(N)$ makes $\pi\colon \mathrm{GL}(N) \to N$ a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

A *reduction* of the structure group G to a closed subgroup $H \subseteq G$ is a submanifold Q of P which is invariant under the G-action restricted to H and such that

$$\pi|_Q\colon Q \to N,$$

together with the H-action on Q, defines a principal H-bundle.

Definition 1.13. A *G-structure* on a smooth manifold N is a reduction of the bundle of linear frames $\mathrm{GL}(N)$ to a closed subgroup $G \subseteq \mathrm{GL}(n, \mathbb{R})$.

Let P be the total space of a principal G-bundle over a smooth manifold N and suppose that the structure group G also acts on a smooth manifold M on the left. Therefore, one gets a right action of G on the product manifold $P \times M$, given by

$$(u, p) \cdot g = (u \cdot g, g^{-1} \cdot p), \quad u \in P,\ p \in M,\ g \in G.$$

We denote by $P \times_G M$ the quotient space of $P \times M$ via this action, which defines the total space of a fiber bundle on N with fiber M and structure group G, which is called the *fiber bundle associated with P with standard fiber M*. Note that, if we replace M with a vector space V on which G acts linearly, then the previous construction yields a vector bundle on N with standard fiber the vector space V.

Going back to the theory of left actions, we now analyze the properties induced by proper ones. In particular, proper actions induce Riemannian metrics which are invariant with respect to the action. We start with the following.

Definition 1.14. Let (M, g) be a Riemannian manifold. An action of a Lie group G on M is said to be *isometric* if $\alpha^h \in \mathrm{Diff}(M)$ is an isometry of (M, g), for all $h \in G$. In this case the metric g is said to be G-invariant and $\alpha^G := \{\alpha^h,\ h \in G\}$ is a subgroup of the isometry group $I\,(M, g)$ of (M, g).

The action fields induced by the action of the isometry group $I(M, g)$ on a Riemannian manifold (M, g) are called *Killing vector fields* and are characterized by the fact that the Lie derivative of the metric g with respect to them is identically zero. Therefore, the action fields induced by an isometric action are Killing.

For a proof of the following theorem, see for instance [3, Theorem 3.65].

Theorem 1.15. *Let $\alpha\colon G \times M \to M$ be a proper action. Then, there exists a G-invariant metric g on M such that α is isometric on (M, g).*

In the theory of left actions, great importance lies in the study of special submanifolds.

Definition 1.16. Let $\alpha\colon G \times M \to M$ be a left action. A *slice* at $p_0 \in M$ for the action α is an embedded submanifold S_{p_0} of M through p_0 such that

(i) $T_{p_0}M = \left(d\alpha_{p_0}\right)_e(\mathfrak{g}) \oplus T_{p_0}S_{p_0}$ and $T_pM = \left(d\alpha_p\right)_e(\mathfrak{g}) + T_pS_{p_0}$, $\forall p \in S_{p_0}$;

(ii) S_{p_0} is G_{p_0}-invariant;

(iii) if $p \in S_{p_0}$, $g \in G$ and $\alpha(g, p) \in S_{p_0}$, then $g \in G_{p_0}$.

If g is a Riemannian metric on M, a slice at p_0 is said to be *normal* with respect to g if the direct sum in point (i) is orthogonal.

Theorem 1.17. *Let $\alpha\colon G \times M \to M$ be a proper left action. Then, for every $p_0 \in M$ there exists a slice S_{p_0} at p_0 which is normal with respect to the induced G-invariant metric g.*

A slice S_{p_0} through p_0 can be defined by

$$S_{p_0} \coloneqq \exp_{p_0}\left(B_\varepsilon(0)\right),$$

where $B_\varepsilon(0)$ is an open ball of radius $\varepsilon > 0$ around the origin in the normal space $T_{p_0}(G \cdot p_0)^\perp$ to the tangent space $T_{p_0}(G \cdot p_0)$ inside $T_{p_0}M$. In particular, $T_{p_0}M = \left(d\alpha_{p_0}\right)_e(\mathfrak{g}) \overset{\perp}{\oplus} T_{p_0}S_{p_0}$.

Now, for every $p \in M$, the isotropy group G_p acts on T_pM by

$$\begin{aligned} G_p \times T_pM &\to T_pM, \\ (g, X) &\mapsto \left(d\alpha^g\right)_p(X). \end{aligned}$$

Since $g \in G_p$ leaves $G \cdot p$ invariant, this action leaves the tangent space $T_p(G \cdot p)$ and any its complement invariant.

The restriction

$$\chi_p\colon G_p \times T_p(G \cdot p) \to T_p(G \cdot p)$$

is called the *isotropy representation* of the action at p and, for every slice S_{p_0} at p, the restriction

$$\sigma_p\colon G_p \times T_pS \to T_pS_{p_0}$$

is called the *slice representation* of the action at p.

Theorem 1.18. *Let $\alpha\colon G \times M \to M$ be a left action of a compact Lie group G on a smooth manifold M and let $p_0 \in M$ be such that $\alpha(g, p_0) = p_0$, for all $g \in G$. Then, there exists a G-invariant open neighborhood U of p_0 in M and a diffeomorphism f from U onto an open neighborhood V of 0 in $T_{p_0}M$, such that*

$$f(p_0) = 0, \quad df_{p_0} = \mathrm{Id}_{T_{p_0}M},$$

and

$$\alpha(g, p) = f^{-1}\left(\left(d\alpha^g\right)_{p_0}(f(p))\right), \quad g \in G,\ p \in U.$$

For a proof, see [36, pages 96-97].

Definition 1.19. Let M and N be smooth manifolds. We say that a G-action α on M and a G'-action β on N are equivalent if there exists a Lie group isomorphism $\Phi\colon G \to G'$ and a diffeomorphism F from M to N which is Φ-equivariant, namely such that

$$F \circ \alpha^g = \beta^{\Phi(g)} \circ F, \quad g \in G.$$

Definition 1.20. Let $\alpha\colon G \times M \to M$ be a proper left action. Given $p_0 \in M$, let S_{p_0} be a slice at p_0. We define a *tubular neighborhood* of the orbit $G \cdot p_0$ as the image of S_{p_0} under the G-action, namely

$$\mathrm{Tub}(G \cdot p_0) \coloneqq \alpha(G, S_{p_0}).$$

The next theorem gives $\mathrm{Tub}(G \cdot p_0)$ the structure of an associated fiber bundle.

Theorem 1.21. *Let $\alpha\colon G \times M \to M$ be a proper left action. Then, for every $p_0 \in M$, there exists a G-equivariant diffeomorphism between $\mathrm{Tub}(G \cdot p_0)$ and $G \times_{G_{p_0}} B$. Here, B is an open G_{p_0}-invariant neighborhood of 0 in $T_{p_0}(G \cdot p_0)^{\perp}$ and $G \times_{G_{p_0}} B$ is the total space of the fiber bundle over $U \subset M$ associated with the G_{p_0}-principal bundle $G \to G/G_{p_0}$ with fiber B and G_{p_0}-action on B defined by*

$$\begin{aligned} G_{p_0} \times B &\to B, \\ (g, X) &\mapsto \left(d\alpha^g\right)_{p_0}(X). \end{aligned}$$

Proof. One first proves that there exists a unique G-equivariant diffeomorphism

$$\Phi\colon G \times_{G_{p_0}} S_{p_0} \to U \subset M,$$

where S_{p_0} is a normal slice through p_0. Then, the claim follows from Theorem 1.18, by observing that the G_{p_0}-action on S_{p_0} is equivalent to the tangent action of G_{p_0} on an open G_{p_0}-invariant neighborhood B of 0 in $T_{p_0}S_{p_0} = T_{p_0}(G \cdot p_0)^{\perp}$. For more details, see [36, pages 102-103]. □

Definition 1.22. Let $\alpha\colon \times M \to M$ be a proper left action and let p be a point of M. The orbit $G \cdot p$ through p is called *principal* if there exists a neighborhood V of p in M such that, for each $q \in V$, $G_p \subset G_{g \cdot q}$ for some $g \in G$.

Proposition 1.23. *Let $\alpha\colon G \times M \to M$ be a proper left action and let p be a point of M. Then, the following are equivalent:*

(i) *$G \cdot p$ is a principal orbit;*

(ii) *If S_p is a slice at p, then $G_p = G_q$ for all $q \in S_p$.*

For a proof of these properties, see [3, Proposition 3.74].

Proposition 1.24. *Let $\alpha\colon G \times M \to M$ be an isometric proper left action and let S_p be a slice at p. Then, the orbit $G \cdot p$ is principal if and only if the slice representation of G_p is trivial.*

See for instance [3, Exercise 3.77]. We denote by M^{princ} the set of points of M contained in principal orbits.

Theorem 1.25. *Let $\alpha\colon G \times M \to M$ be a proper left action, where M is connected. Then,*

- *M^{princ} is open and dense in M;*
- *M^{princ}/G is a connected submanifold of M/G;*
- *If $G \cdot p$ and $G \cdot q$ are principal orbits, there exists $g \in G$ such that $G_p = gG_q g^{-1}$.*

For a proof, see [3, pages 75-76].

Definition 1.26 (Orbit types)**.** Let $\alpha\colon G \times M \to M$ be a proper left action.

(i) The orbit $G \cdot p$ has a *larger orbit type* than $G \cdot q$ if there is $g \in G$ such that $G_p \subset G_{g \cdot q}$;

(ii) The orbits $G \cdot p$ and $G \cdot q$ have the *same orbit type* if there is $g \in G$ such that $G_p = G_{g \cdot q}$;

(iii) An orbit $G \cdot p$ is said to be regular if the dimension of $G \cdot p$ coincides with the dimension of principal orbits;

(iv) A non-principal regular orbit is called *exceptional*;

(v) A non-regular orbit is called *singular*.

1.2 Stable forms

In this section, we review the properties of the geometric structures defined by *stable* differential forms. As a reference, see for instance [82, 83].

Definition 1.27. Let V be an n-dimensional real vector space. A k-form $\alpha \in \Lambda^k V^*$ is *stable* if its orbit under the natural action of $\mathrm{GL}(V)$ is open in $\Lambda^k V^*$.

Stability occurs only in even dimension and in dimension seven. More precisely, in even dimension $n = 2m \neq 6, 8$, a k-form may be stable only if $k = 2, 2m - 2$. The cases where n is 6 or 8 are richer. Indeed, besides the previous values of k, stability may occur also for $k = 3, 2m - 3$. In odd dimension $n = 7$, instead, a k-form may be stable only if k is equal to 3 or 4. Moreover, depending on both n and k, stability can be characterized in the following ways.

Proposition 1.28. [82] *A 2-form ω a $2m$-dimensional real vector space V is stable if and only if $\omega^m \neq 0$, i.e., if ω is non-degenerate.*

The previous proposition follows from a dimensional argument. Let ω be a non-degenerate 2-form on V. The orbit $\mathrm{GL}(V)/\mathrm{Sp}(V,\omega)$ of ω has the same dimension of $\Lambda^2 V^*$, so it has to be open in $\Lambda^2 V^*$. On the other hand, since $\Lambda^2 V^*$ contains only one open orbit, it has to coincide with the orbit of a non-degenerate 2-form. Given a stable 2-form ω, we shall call $\frac{1}{m!}\omega^m$ the *Liouville volume form* defined by ω.

Stability of 3-forms in dimension six was first characterized in Reichel's thesis from 1907 [120]. The result was later riformulated by Hitchin [82]. Let V be a 6-dimensional real vector space and let ρ be a 3-form on it. Choose an orientation $\Omega \in \Lambda^6 V^*$ and consider the endomorphism $S_\rho \colon V \to V$ defined via the identity

$$\iota_v \rho \wedge \rho \wedge \eta = \eta(S_\rho(v))\Omega, \quad \eta \in V^*,\ v \in V,$$

where $\iota_v\rho$ denotes the contraction of ρ by $v \in V$. One has $S_\rho^2 = \lambda(\rho)\mathrm{Id}_V$ for some quartic polynomial $\lambda(\rho)$ in the coefficients of ρ. Moreover, ρ is stable if and only if $\lambda(\rho) \neq 0$ and the quartic hypersurface defined by $\lambda(\rho) = 0$ is invariant under the $\mathrm{GL}(V)$-action and divides $\Lambda^3 V^*$ into two open sets

$$\begin{aligned}\mathcal{O}^+ &\coloneqq \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) > 0\},\\ \mathcal{O}^- &\coloneqq \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) < 0\}.\end{aligned}$$

The identity component of the stabilizer of a 3-form lying in the former is conjugate to $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ and in the latter to $\mathrm{SL}(3,\mathbb{C})$. This motivates the following.

Proposition 1.29. [82] *A stable 3-form $\rho \in \mathcal{O}^-$ defines a complex structure and a complex volume form on V.*

The complex structure also depends on the choice of $\Omega \in \Lambda^6 V^*$. Explicitly,

$$J_{\rho,\Omega} \coloneqq (-\lambda(\rho))^{-1/2}\, S_\rho.$$

Notice that the 3-form $\hat{\rho} \coloneqq J_\rho \rho$ is stable, as well. Here, we are using the convention $J_{\rho,\Omega}\gamma(v_1,\ldots,v_k) = \gamma(J_{\rho,\Omega}v_1,\ldots,J_{\rho,\Omega}v_k)$, for every k-form γ and $v_1,\ldots,v_k \in V$. The complex volume form is then given by

$$\Psi \coloneqq \rho + i\hat{\rho}.$$

Remark 1.30. A simple computation shows that J_ρ does not change if ρ is rescaled by a non-zero real constant, i.e., $J_\rho = J_{s\rho}$ for every $s \in \mathbb{R} - \{0\}$.

Following [35], we introduce the next definition.

Definition 1.31. A *definite* 3-form on an oriented 6-dimensional real vector space V is a stable 3-form $\rho \in \mathcal{O}^-$.

Now, let us focus on the concept of stability of 3-forms on 7-dimensional real vector spaces. Let W be a 7-dimensional real vector space, let φ be a 3-form on it and consider the symmetric bilinear map

$$b_\varphi \colon W \times W \to \Lambda^7 W^*, \quad b_\varphi(v,w) = \frac{1}{6}\, \iota_v\varphi \wedge \iota_w\varphi \wedge \varphi, \quad v,w \in W. \tag{1.1}$$

By [83], φ is stable if and only if $\det(b_\varphi)^{1/9} \in \Lambda^7 W^*$ is non-zero. Therefore, the symmetric bilinear map

$$g_\varphi := \det(b_\varphi)^{-1/9}\, b_\varphi$$

is well defined and $\mathrm{Vol}_{g_\varphi} := \det(b_\varphi)^{1/9}$ defines a volume form on W. By [18, 80], there are exactly two open orbits of a stable 3-form φ, one corresponding to the case where g_φ is positive definite, the other to the case where g_φ is of signature $(3,4)$. We denote by $\Lambda^3_+ W^*$ the orbit of φ corresponding to the former case.

As for the 6-dimensional case, we introduce the following.

Definition 1.32. [35] A *definite* 3-form on a 7-dimensional real vector space W is a stable 3-form $\varphi \in \Lambda^3_+ W^*$.

1.3 $\mathrm{SL}(3,\mathbb{C})$-structures

In this section, M always denotes a 6-dimensional oriented smooth manifold.

An $\mathrm{SL}(3,\mathbb{C})$-*structure* on M is a reduction of the frame bundle of M to $\mathrm{SL}(3,\mathbb{C})$. This is equivalent to the existence of a *definite* 3-form ρ on M, i.e., such that $\rho(p) \in \Lambda^3 T_p^* M$ is definite for every $p \in M$. Since the identity component of the pointwise stabilizer of ρ is isomorphic to $\mathrm{SL}(3,\mathbb{C})$, we introduce the following.

Definition 1.33. An $\mathrm{SL}(3,\mathbb{C})$-structure on M (respectively, on T_pM) is a definite 3-form $\rho \in \Lambda^3(M)$ (respectively, a definite 3-form $\rho(p)$ on T_pM).

Fix $\Omega \in \Lambda^6(M)$ and let $\rho \in \Lambda^3(M)$ be an $\mathrm{SL}(3,\mathbb{C})$-structure on M. By Section 1.2, ρ induces an almost complex structure $J_{\rho,\Omega}$ on M which, at a pointwise level, coincides with the linear complex structure on T_pM induced by ρ_p and Ω_p, at every point $p \in M$. We shall denote it by J_ρ or, even more simply, by J, when no confusion arises. We denote by $\hat\rho = J\rho$ the image of ρ via J and by $\Psi = \rho + i\hat\rho$ the complex $(3,0)$-form induced by ρ. Notice that $\hat\rho$ is stable, as well. Moreover, the wedge product between ρ and $\hat\rho$ defines the same orientation as Ω. By Section 1.2, at each point p of M, there exists a basis $(e_1,\dots,e_6)$ of T_pM with dual basis $(e^1,\dots,e^6)$ such that $\Omega_p = e^{123456}$ and

$$\rho_p = e^{135} - e^{146} - e^{236} - e^{245}. \tag{1.2}$$

In particular,

$$J_p(e_1) = e_2, \quad J_p(e_3) = e_4, \quad J_p(e_5) = e_6. \tag{1.3}$$

Definition 1.34. A basis $(e_1,\dots,e_6)$ of T_pM with dual basis satisfying (1.2) is called an *adapted basis* for the $\mathrm{SL}(3,\mathbb{C})$-structure ρ at $p \in M$.

From a Riemannian point of view, a special type of $\mathrm{SL}(3,\mathbb{C})$-structures we are interested in is the following.

Definition 1.35. An $\mathrm{SL}(3,\mathbb{C})$-structure $\rho \in \Lambda^3(M)$ is *tamed* by a stable 2-form $\tilde{\omega} \in \Lambda^2(M)$ if the $(1,1)$-part $\tilde{\omega}^{1,1}$ of $\tilde{\omega}$ is positive in the standard sense. The 2-form $\tilde{\omega}$ is also called a *taming form* with respect to J_ρ.

The positivity in the standard sense of $\tilde{\omega}^{1,1}$ implies that the symmetric tensor

$$g := \tilde{\omega}^{1,1}(\cdot, J_\rho \cdot).$$

is positive-definite. This implies that g defines an *almost Hermitian metric* on M, namely a Riemannian metric satisfying

$$g(J_\rho \cdot, J_\rho \cdot) = g(\cdot, \cdot).$$

We shall refer to (g, J_ρ) as the induced *almost Hermitian structure* on M. Special types of these structures are given by $\mathrm{SU}(3)$-structures.

Definition 1.36. An $\mathrm{SU}(3)$-*structure* (ω, ρ) on M is the data of an $\mathrm{SL}(3,\mathbb{C})$-structure ρ tamed by a stable 2-form ω satisfying the compatibility conditions

$$\omega \wedge \rho = 0 \tag{1.4}$$

and the normalization condition

$$\rho \wedge \hat{\rho} = \frac{2}{3}\omega^3. \tag{1.5}$$

Remark 1.37. In the language of principal bundles, an $\mathrm{SU}(3)$-*structure* on M is a reduction of the frame bundle of M to $\mathrm{SU}(3)$.

Remark 1.38. An $\mathrm{SU}(3)$-structure can be equivalent defined as the data of a Riemannian metric g, a g-orthogonal almost complex structure J and a nowhere-vanishing $(3,0)$-form Ψ. If we denote by ρ the real part of Ψ and by ω the 2-form

$$\omega(\cdot, \cdot) := g(J \cdot, \cdot),$$

then the pair (ω, ρ) defines an $\mathrm{SU}(3)$-structure in the sense of Definition 1.36, inducing the almost complex structure J and the Riemannian metric g. The 2-form ω is also called the *fundamental form* associated with the almost Hermitian structure (g, J).

Let (ω, ρ) be an $\mathrm{SU}(3)$-structure on M. As we have seen in the previous section, we can always find a pointwise coframe $\left(e^1, \dots, e^6\right)$ such that

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \rho = e^{135} - e^{146} - e^{236} - e^{245}. \tag{1.6}$$

Definition 1.39. A basis $(e_1, \dots, e_6)$ of T_pM with dual basis satisfying (1.6) is called an *adapted basis* for the $\mathrm{SU}(3)$-structure (ω, ρ) at $p \in M$.

We denote by $g = \omega(J_\rho\cdot, \cdot)$ the induced almost Hermitian metric and by $*_g$ the Hodge operator associated with g and the Liouville volume form $\frac{\omega^3}{6}$ on M. Then, $\hat{\rho} = *_g\rho$. Moreover, with respect to an adapted basis $(e_1, \dots, e_6)$ at $p \in M$, we may write

$$g_p = \sum_{i=1}^{6}(e^i)^2,$$

and

$$\hat{\rho}_p = e^{136} + e^{145} + e^{235} - e^{246}.$$

Let d be the de Rham differential of M. We then give the following.

Definition 1.40. An $\mathrm{SL}(3,\mathbb{C})$-structure ρ (resp. an $\mathrm{SU}(3)$-structure (ω, ρ)) is *closed* if $d\rho = 0$.

Closed $\mathrm{SL}(3,\mathbb{C})$-structures have been recently studied in [35, 65]. In Chapter 2, we shall study two special classes of closed $\mathrm{SL}(3,\mathbb{C})$-structures, both characterized by the (semi-)positivity of certain differential forms. They are called *mean convex* and *tamed by a symplectic form*, respectively. In particular, we shall also analyze the compatibility of mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures with the half-flat condition for $\mathrm{SU}(3)$-structures.

Definition 1.41. An $\mathrm{SU}(3)$-structure (ω, ρ) is called *half-flat* if

$$\begin{cases} d\rho = 0, \\ d\omega^2 = 0. \end{cases} \tag{1.7}$$

Half-flat $\mathrm{SU}(3)$-structures have been extensively studied in literature (see for instance [25, 26, 28, 83, 115, 116]). Special focus was placed, in particular, on *symplectic half-flat* and *balanced* $\mathrm{SU}(3)$-structures.

Definition 1.42. An $\mathrm{SU}(3)$-structure (ω, ρ) is called *symplectic half-flat* if

$$\begin{cases} d\rho = 0, \\ d\omega = 0. \end{cases} \tag{1.8}$$

Definition 1.43. An $\mathrm{SU}(3)$-structure (ω, ρ) is called *(non-Kähler) balanced* if

$$\begin{cases} d\rho = 0, \\ d\omega^2 = 0, \qquad (d\omega \neq 0), \\ d\hat{\rho} = 0. \end{cases} \tag{1.9}$$

In Chapter 5, we shall study the existence of balanced $\mathrm{SU}(3)$-structures on simply connected manifolds of cohomogeneity one, namely manifolds admitting an action of a compact Lie group having an orbit of codimension one.

Finally, we have the most restrictive case in which all the defining differential forms are closed.

Definition 1.44. An SU(3)-structure (ω, ρ) is called *torsion-free* if

$$\begin{cases} d\rho = 0, \\ d\omega = 0, \\ d\hat{\rho} = 0. \end{cases} \tag{1.10}$$

A torsion-free SU(3)-structure (ω, ρ) is equivalent characterized by asking that the defining tensors ω and $\Psi = \rho + i\hat{\rho}$ are both parallel with respect to the Levi-Civita connection ∇^g of the induced metric g, i.e.,

$$\begin{cases} \nabla^g \omega = 0, \\ \nabla^g \Psi = 0. \end{cases} \tag{1.11}$$

For this reason, torsion-free SU(3)-structures are also called *parallel*. More commonly, they are referred to as *integrable* SU(3)-structures or *Calabi-Yau* structures in the compact case. By the holonomy principle, an SU(3)-structure is integrable if and only if $\mathrm{Hol}(g)$ is isomorphic to a subgroup of SU(3), where $\mathrm{Hol}(g)$ is the holonomy group of ∇^g. In particular, the induced metric g is Ricci-flat, i.e., the Ricci tensor $\mathrm{Ric}(g)$ associated with g vanishes identically (see, for instance, [89, Proposition 7.1.1]).

The obstruction for $\mathrm{Hol}(g)$ to be isomorphic to a subgroup of SU(3), hence for (ω, ρ) to be torsion-free, is represented by the so-called *intrinsic torsion* τ of the SU(3)-structure. The intrinsic torsion of (ω, ρ) is a section of the vector bundle $T^*M \otimes \mathfrak{su}(3)^\perp$, which can be identified with the pair $(\nabla^g \omega, \nabla^g \Psi)$. Here, $\mathfrak{su}(3)^\perp \subset \mathfrak{so}(6)$ is the orthogonal complement of $\mathfrak{su}(3)$ with respect to the Killing Cartan form $\mathcal{B}$ of $\mathfrak{so}(6)$. Moreover, by [24, Theorem 1.1], the intrinsic torsion of (ω, ρ) is completely determined by the differentials $d\omega$, $d\rho$ and $d\hat{\rho}$. More precisely, the natural action of SU(3) on the space of k-forms $\Lambda^k(M)$, for $k = 2, 3$, gives rise to the following splittings:

$$\begin{aligned} \Lambda^2(M) &= C^\infty(M)\,\omega \,\oplus\, \Lambda^2_6(M) \,\oplus\, \Lambda^2_8(M), \\ \Lambda^3(M) &= C^\infty(M)\,\rho \oplus C^\infty(M)\,\hat{\rho} \,\oplus\, \Lambda^3_6(M) \,\oplus\, \Lambda^3_{12}(M), \end{aligned}$$

where the irreducible r-dimensional SU(3)-modules $\Lambda^k_r(M)$ are defined as follows

$$\Lambda^2_6(M) = \left\{\sigma \in \Lambda^2(M) \,|\, J\sigma = -\sigma\right\}, \quad \Lambda^2_8(M) = \left\{\beta \in \Lambda^2(M) \,|\, J\beta = \beta,\ \beta \wedge \omega^2 = 0\right\}$$

and

$$\Lambda^3_6(M) = \left\{\alpha \wedge \omega \,|\, \alpha \in \Lambda^1(M)\right\}, \quad \Lambda^3_{12}(M) = \left\{\gamma \in \Lambda^3(M) \,|\, \gamma \wedge \omega = 0,\ \gamma \wedge \rho = 0,\ \gamma \wedge \hat{\rho} = 0\right\}.$$

Therefore, there exist unique $w_0^+, w_0^- \in C^\infty(M)$, $\nu_1, w_1 \in \Lambda^1(M)$, $w_2^+, w_2^- \in \Lambda^2_8(M)$, $w_3 \in \Lambda^3_{12}(M)$, such that

$$\begin{aligned} d\omega &= -\frac{3}{2} w_0^- \rho + \frac{3}{2} w_0^+ \hat{\rho} + w_3 + \nu_1 \wedge \omega, \\ d\rho &= w_0^+ \omega^2 + w_2^+ \wedge \omega + w_1 \wedge \rho, \\ d\hat{\rho} &= w_0^- \omega^2 + w_2^- \wedge \omega + J w_1 \wedge \rho. \end{aligned} \tag{1.12}$$

Notice that every 2-form $\sigma \in \Lambda^2_6(M)$ satisfies the identity $\sigma \wedge \omega = *_g \sigma$, while every 2-form $\beta \in \Lambda^2_8(M)$ satisfies the identity $\beta \wedge \omega = - *_g \beta$.

Definition 1.45. The differential forms $w_0^{\pm}, w_1, \nu_1, w_2^{\pm}, w_3$ uniquely defined by (1.12) are called *intrinsic torsion forms* of the SU(3)-structure.

An SU(3)-structure is torsion-free if and only if its intrinsic torsion forms vanish identically.

1.4 G_2-structures

Let $\mathbb{R}^7 = \langle e_1, \dots, e_7 \rangle$ and consider the 3-form

$$\varphi_0 \coloneqq e^{135} - e^{146} - e^{236} - e^{245} + e^{127} + e^{347} + e^{567}.$$

The exceptional Lie group G_2 is the stabilizer of φ_0 under the natural action of $\mathrm{GL}(7,\mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. Explicitly,

$$\mathrm{G}_2 \coloneqq \{a \in \mathrm{GL}(7,\mathbb{R}) \,|\, a^*(\varphi_0) = \varphi_0\} \subset \mathrm{SO}(7).$$

G_2 is a compact, connected, simply connected, simple Lie group of dimension 14. Therefore, the orbit

$$\mathrm{GL}(7,\mathbb{R}) \cdot \varphi_0 \cong \mathrm{GL}(7,\mathbb{R})/\mathrm{G}_2$$

is open in $\Lambda^3(\mathbb{R}^7)^*$. Hence, φ_0 is stable in the sense of Definition 1.27.

Let N be a 7-dimensional smooth manifold. A G_2-structure on N is a reduction of the structure group of its frame bundle to the exceptional Lie group G_2. This reduction exists if and only if N is orientable and spin (see [76]) and it is equivalent to the existence of a *definite* 3-form $\varphi \in \Omega^3(N)$, i.e., such that $\varphi(p) \in \Lambda^3 T_p^*N$ is definite for every $p \in N$. Since the identity component of the pointwise stabilizer of φ isomorphic to G_2, we introduce the following.

Definition 1.46. A G_2-structure on N (respectively, on T_pN) is a definite 3-form $\varphi \in \Lambda^3(N)$ (respectively, a definite 3-form φ on T_pN).

A G_2-structure φ endows N with a Riemannian metric g_φ and an orientation Vol_{g_φ}. Explicitly,

$$g_\varphi(X,Y)\mathrm{Vol}_{g_\varphi} = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for every pair of vector fields X, Y on N. We denote by ∇^{g_φ} the Levi-Civita connection of g_φ and by $*_\varphi$ the Hodge operator determined by g_φ and Vol_{g_φ}. Moreover, the 4-form $\hat{\varphi} \coloneqq *_\varphi\varphi$ is stable, as well.

Let $\varphi \in \Lambda^3(N)$ be a G_2-structure on N. As we have recalled in Section 1.2, at each point p of N, there exists a basis $(e_1, \dots, e_7)$ of T_pN with dual basis $(e^1, \dots, e^7)$ such that

$$\varphi_p = e^{135} - e^{146} - e^{236} - e^{245} + e^{127} + e^{347} + e^{567}. \tag{1.13}$$

Definition 1.47. A basis $(e_1, \dots, e_7)$ of T_pN satisfying (1.13) is called an *adapted basis* for the G_2-structure φ at $p \in N$.

Moreover, with respect to an adapted basis $(e_1, \dots, e_7)$ at $p \in N$, we may write

$$(g_\varphi)_p = \sum_{i=1}^{7} (e^i)^2, \qquad \mathrm{Vol}_{g_\varphi} = e^{1234567}$$

and

$$\hat{\varphi}_p = e^{1234} + e^{3456} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

In particular, this basis is orthonormal with respect to g_φ.

Definition 1.48. A G_2-structure φ is *closed* if $d\varphi = 0$.

Definition 1.49. A G_2-structure φ is *coclosed* if $d *_\varphi \varphi = 0$.

Closed and coclosed G_2-structures exist on every open 7-manifold admitting G_2-structures [37]. In [31], the authors extended this result by proving that a compact 7-manifold admitting G_2-structures always admits coclosed ones. The same is no longer true for closed G_2-structures. Examples of compact 7-manifolds with a closed G_2-structure can be obtained by taking products of lower dimensional manifolds endowed with suitable geometric structures (see for instance [46, 51, 56, 93]). In Chapter 3, we shall investigate their existence on certain classes of compact 7-manifolds.

Definition 1.50. A G_2-structure φ is *torsion-free* if

$$\begin{cases} d\varphi = 0, \\ d *_\varphi \varphi = 0, \end{cases} \tag{1.14}$$

i.e., if it is both closed and coclosed.

A torsion-free G_2-structure φ is equivalently characterized by asking that the 3-form φ is parallel with respect to the Levi-Civita connection ∇^{g_φ} (see [46]), i.e.,

$$\nabla^{g_\varphi} \varphi = 0. \tag{1.15}$$

For this reason, torsion-free G_2-structure are also called *parallel*. More commonly, they are also referred to as *integrable* G_2-structures. By the holonomy principle, a G_2-structure φ is integrable if and only if $\mathrm{Hol}(g_\varphi)$ is isomorphic to a subgroup of G_2. In this case the induced metric g_φ is Ricci-flat (see [16]).

The first complete examples of torsion-free G_2-structures were constructed in [21]. In the compact case, 7-manifolds admitting torsion-free G_2-structures were constructed first in [88] and, later, in [30, 90, 91]. Up to now, all known and potentially effective methods to obtain new examples of torsion-free G_2-structures on compact 7-manifolds involve closed ones: in [88], Joyce proved that a closed G_2-structure on a compact 7-manifold with small torsion, in a suitable sense, can be deformed into a torsion-free G_2-structure. In [19], Bryant introduced the *Laplacian flow*, a geometric flow for closed G_2-structures (for more details, see Section 4.1). Since the stationary points of this flow are torsion-free G_2-structures, one may expect that, starting from a closed initial data, the flow could possibly converge to a torsion-free G_2-structure.

The obstruction for $\mathrm{Hol}(g_\varphi)$ to be isomorphic to a subgroup of G_2 is represented by the so-called *intrinsic torsion* τ_φ of the G_2-structure. The intrinsic torsion of a G_2-structure is a section of the vector bungle $T^*N \otimes \mathfrak{g}_2^\perp$ and it can be identified with the covariant derivative $\nabla^{g_\varphi}\varphi$. Here $\mathfrak{g}_2^\perp \subset \mathfrak{so}(7)$ is the orthogonal complement of $\mathfrak{g}_2$ with respect to the Killing Cartan form $\mathcal{B}$ of $\mathfrak{so}(7)$. In a similar way to what happens for SU(3)-structures, the components of τ_φ are completely determined by those of $d\varphi$ and $d *_\varphi \varphi$ (see [19, Proposition 1]). The decompositions of the spaces $\Lambda^k(N)$, for $k = 2, 3$, into irreducible G_2-modules induce the following decompositions:

$$
\begin{aligned}
\Lambda^2(N) &= \Lambda^2_7(N) \,\oplus\, \Lambda^2_{14}(N),\\
\Lambda^3(N) &= C^\infty(N)\,\varphi \oplus \Lambda^3_7(N) \,\oplus\, \Lambda^3_{27}(N),
\end{aligned}
$$

where the irreducible r-dimensional G_2-modules $\Lambda^k_r(N)$ are defined as follows

$$
\Lambda^2_7(N) = \{\kappa \in \Lambda^2(N) \mid *_\varphi (\kappa \wedge \varphi) = 2\kappa\}, \quad \Lambda^2_{14}(N) = \{\kappa \in \Lambda^2(N) \mid \kappa \wedge *_\varphi\varphi = 0\},
$$

and

$$
\Lambda^3_7(N) = \{*_\varphi(\alpha \wedge \varphi) \mid \alpha \in \Lambda^1(N)\}, \quad \Lambda^3_{27}(N) = \{\beta \in \Lambda^3(N) \mid \beta \wedge \varphi = 0,\ \beta \wedge *_\varphi\varphi = 0\}.
$$

Therefore, there exist unique $\tau_0 \in C^\infty(N)$, $\tau_1 \in \Lambda^1(N)$, $\tau_2 \in \Lambda^2_{14}(N)$ and $\tau_3 \in \Lambda^3_{27}(N)$ such that

$$
\begin{aligned}
d\varphi &= \tau_0 *_\varphi \varphi + 3\tau_1 \wedge \varphi + *_\varphi\tau_3,\\
d *_\varphi \varphi &= 4\tau_1 \wedge *_\varphi\varphi + \tau_2 \wedge \varphi.
\end{aligned}
\tag{1.16}
$$

Definition 1.51. The differential forms $\tau_0, \tau_1, \tau_2, \tau_3$ uniquely defined by (1.16) are called *intrinsic torsion forms* of the G_2-structure.

A G_2-structure is torsion-free if and only if its intrinsic torsion forms vanish identically.

1.5 The link between $\mathrm{SL}(3,\mathbb{C})$-structures and G_2-structures

In this section, we recall the interplay between G_2-structures in dimension seven and the geometry of $\mathrm{SL}(3,\mathbb{C})$-structures and SU(3)-structures in dimension six. Consider a 7-dimensional real vector space W and let φ be a 3-form on it. Choose a non-zero vector $z \in W$ and a complementary subspace $V \subset W$ so that $W \cong V \oplus \mathbb{R}z$. Then, we can write

$$
\varphi = \widetilde{\omega} \wedge \theta + \rho,
$$

where $\theta \in W^*$ is the dual of z with respect to the chosen subspace V, $\widetilde{\omega} \in \Lambda^2 V^*$ and $\rho \in \Lambda^3 V^*$. The 3-form φ on W is definite if and only if the 3-form ρ on V is definite and $\widetilde{\omega}$ is a taming form for the complex structure J induced by ρ and one of the two orientations of V.

Remark 1.52. If $\varphi = \widetilde{\omega} \wedge \theta + \rho$ is definite, then the 2-form $\widetilde{\omega} = \iota_z\varphi|_V$ on V has rank 6, namely it is a non-degenerate 2-form. The pair $(\widetilde{\omega}, \rho)$ defines an SU(3)-structure on V, up to a suitable normalization, if and only if $\widetilde{\omega} \wedge \rho = 0$. When this happens, the vector space V coincides with the g_φ-orthogonal complement of $\mathbb{R}z \subset W$.

On the other hand, if φ defines a G_2-structure on W, we can consider the 6-dimensional subspace $U := (\mathbb{R}z)^\perp \subset W$ and the g_φ-orthogonal splitting $W = U \oplus \mathbb{R}z$. Then, if we let $u := |z|_\varphi = g_\varphi(z,z)^{1/2}$ and $\eta := u^{-2}z^\flat$, so that $\eta(z) = 1$, we have

$$\varphi = u\,\omega \wedge \eta + \psi_+,$$

for some $(\omega, \psi_+) \in \Lambda^2 U^* \times \Lambda^3 U^*$. Therefore, the pair (ω, ψ_+) defines an SU(3)-structure on U inducing the metric g such that

$$g_\varphi = g + u^2 \eta \otimes \eta.$$

Moreover,

$$*_\varphi \varphi = \frac{1}{2}\,\omega \wedge \omega + u\,\psi_- \wedge \eta,$$

where $\psi_- := J_{\psi_+}\psi_+$, J_{ψ_+} being the almost complex structure induced by ψ_+. In particular,

$$\mathrm{Vol}_{g_\varphi} = u\,\mathrm{Vol}_g \wedge \eta.$$

Since the vector subspaces V and U are isomorphic, there exists an SU(3)-structure on V corresponding to (ω, ψ_+) via the identification $V \cong U$. We shall denote this SU(3)-structure using the same symbols. It follows from the discussion in Remark 1.52 that V and U coincide if and only if $\widetilde{\omega} \wedge \rho = 0$. In such a case, η and θ coincide, too.

Remark 1.53.

1. The structures $(\widetilde{\omega}, \rho)$ and (ω, ψ_+) on V are related as follows. On $W = V \oplus \mathbb{R}z$, we have $\varphi = \widetilde{\omega} \wedge \theta + \rho$ and $\varphi = u\,\omega \wedge \eta + \psi_+$. Thus, $\widetilde{\omega} = \iota_z \varphi = u\,\omega$. Moreover, since $\eta(z) = 1$, we can consider the decomposition $\eta = \eta_V + \theta$, where $\eta_V \in V^*$ and see that

$$\rho = u\,\omega \wedge \eta_V + \psi_+.$$

2. Let $(e_1, \dots, e_7)$ be a basis of $W = V \oplus \mathbb{R}z$ with $V = \langle e_1, \dots, e_6 \rangle$ and $e_7 = z$. Then, a basis for $U = (\mathbb{R}e_7)^\perp$ is given by $\left(e_k - \frac{g_\varphi(e_k, e_7)}{u^2} e_7\right)_{k=1,\dots,6}$. Consequently, $(e^1, \dots, e^6)$ is a basis of U^*.

The following identities will be useful in the sequel. The reader may refer to [56, Lemma 3.7] for a proof.

Lemma 1.54. *Let (ω, ψ_+) be an* SU(3)*-structure on a 6-dimensional real vector space V and let $\alpha \in V^*$. We denote $\psi_- = J\psi_+$, J being the almost complex structure induced by ψ_+. Then,*

(1) $*_g(\alpha \wedge \psi_-) \wedge \omega = J\alpha \wedge \psi_+ = \alpha \wedge \psi_-$,

(2) $*_g(\alpha \wedge \psi_-) \wedge \omega^2 = 0$,

(3) $*_g(\alpha \wedge \psi_-) \wedge \psi_+ = - *_g (\alpha \wedge \psi_+) \wedge \psi_- = \alpha \wedge \omega^2 = 2 *_g (J\alpha)$,

(4) $*_g(\alpha \wedge \psi_-) \wedge \psi_- = *_g(\alpha \wedge \psi_+) \wedge \psi_+ = -J\alpha \wedge \omega^2 = 2 *_g \alpha$.

At the level of smooth manifolds, the previous construction hold by working on the tangent space of a 7-dimensional smooth manifold N endowed with a G_2-structure φ. Let $i\colon M \hookrightarrow N$ be an orientable 6-dimensional submanifold of N and let X be a vector field along M which, at every point $p \in M$, is complementary to the the tangent space of M at p, i.e., such that

$$T_pM \oplus \langle X_p \rangle = T_pN.$$

The 3-form $\rho \coloneqq i^*\varphi$ is an $\mathrm{SL}(3,\mathbb{C})$-structure tamed by the non-degenerate 2-form $\widetilde{\omega} \coloneqq \iota_X\varphi$. Note that when $X = \nu$ is a normal vector field of unit norm with respect to g_φ, then the pair $(i^*\varphi, \iota_\nu\varphi)$ defines an $\mathrm{SU}(3)$-structure on M.

Conversely, if ρ is an $\mathrm{SL}(3,\mathbb{C})$-structure tamed by a non-degenerate 2-form $\widetilde{\omega}$ on an oriented smooth 6-manifold M, we may define a G_2-structure φ on the cartesian product between M and a smooth 1-fold I; explicitly,

$$\varphi = \widetilde{\omega} \wedge \theta + \rho,$$

where θ is a nowhere-vanishing global 1-form on I. In particular, φ induces the product metric if and only if $\widetilde{\omega} \wedge \rho = 0$.

Remark 1.55. Let φ be a G_2-structure on a smooth 7-manifold N and let $(e_1, \dots, e_7)$ be an adapted basis of T_pN at a point $p \in N$. The sextuple $(e_1, \dots, e_6)$ is an adapted basis for the $\mathrm{SU}(3)$-structure (ω, ρ) induced on $\mathbb{R}e_7^\perp \subset T_pN$. In particular,

$$\varphi_p = \omega \wedge e^7 + \rho, \qquad \hat{\varphi}_p = \frac{1}{2}\omega^2 + \hat{\rho} \wedge e^7.$$

Chapter 2

Closed $\mathrm{SL}(3, \mathbb{C})$-structures

As remarked in [35], closed $\mathrm{SL}(3, \mathbb{C})$-structures obey an h-principle, since every hypersurface in $\mathbb{R}^7$ acquires a closed $\mathrm{SL}(3, \mathbb{C})$-structure. In this chapter, we consider closed $\mathrm{SL}(3, \mathbb{C})$-structures which are either mean convex or tamed by a symplectic form. These notions were introduced by Donaldson in relation to G_2-manifolds with boundary. Since both definitions rely on concepts of semi-positivity for (p, p)-forms, we start by recalling some general results on these structures. Then, we frame them within the theory of closed $\mathrm{SL}(3, \mathbb{C})$-structures and $\mathrm{SU}(3)$-structures, providing some characterizations in terms of the intrinsic torsion forms of the $\mathrm{SU}(3)$-structure. We then focus on nilmanifolds and solvmanifolds. In particular, we classify nilmanifolds which carry an invariant mean convex closed $\mathrm{SL}(3, \mathbb{C})$-structure and those which admit an invariant mean convex half-flat $\mathrm{SU}(3)$-structure. Finally, we prove that, if a solvmanifold admits an invariant tamed closed $\mathrm{SL}(3, \mathbb{C})$-structure, then it also has an invariant symplectic half-flat $\mathrm{SU}(3)$-structure. The main contents and results of this chapter were published in [59].

2.1 Semi-positive (p, p)-forms

In this section we review the definitions and main results regarding semi-positive (p, p)-forms on complex vector spaces. For more details, we refer for instance to [33, 81].

Let V be a complex vector space of complex dimension n, with coordinates $(z_1, \dots, z_n)$. Notice that V can be considered also as a real vector space of dimension $2n$ endowed with the complex structure J given by the multiplication by i. Consider the exterior algebra

$$\Lambda V^* \otimes \mathbb{C} = \bigoplus \Lambda^{p,q} V^*,$$

where $\Lambda^{p,q} V^*$ is a shorthand for $\Lambda^p V^* \otimes \Lambda^q \overline{V}^*$. A canonical orientation for V is given by the (n, n)-form

$$\tau(z) \coloneqq \frac{1}{2^n} i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_n \wedge d\overline{z}_n = dx_1 \wedge dy_1 \wedge dx_n \wedge \ldots \wedge dy_n, \tag{2.1}$$

where $z_j = x_j + i y_j$. We say that a (p, p)-form γ is real if $\gamma = \overline{\gamma}$. One may introduce a natural notion of positivity for real (p, p)-forms.

Definition 2.1. A real (p,p)-form $\gamma \in \Lambda^{p,p}V^*$ is said to be *semi-positive* (resp. *positive*) if, for all α_j of $\Lambda^{1,0}V^*$, $1 \le j \le n-p$,

$$\gamma \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_{n-p} \wedge \overline{\alpha}_{n-p} = \lambda\tau(z),$$

where $\lambda \ge 0$ (resp. $\lambda > 0$ when $\alpha_1, \ldots, \alpha_{n-p}$ are linearly indipendent).

We now focus on the case $n = 3$ and we provide equivalent definitions for semi-positive real forms of type $(1,1)$ and $(2,2)$. For a more general discussion we refer the reader to [33, Chapter III].

Proposition 2.2. *Let $\alpha = \frac{i}{2}\sum_{j,k} a_{j\bar{k}}\, dz_j \wedge d\overline{z}_k$ be a real $(1,1)$-form on V. Then, the following are equivalent:*

(i) *α is semi-positive (resp. positive);*

(ii) *the Hermitian matrix of coefficients $(a_{j\bar{k}})$ is positive semi-definite (resp. positive definite);*

(iii) *there exist coordinates $(w_1, \ldots w_n)$ on V such that*

$$\alpha = \frac{i}{2}\sum_{k=1}^{n} \tilde{a}_{k\bar{k}}\, dw_k \wedge d\overline{w}_k,$$

with $\tilde{a}_{k\bar{k}} \ge 0$ (resp. $\tilde{a}_{k\bar{k}} > 0$), $k = 1, \ldots, n$.

Proof.
(i) $\Longleftrightarrow$ (ii) follows from [33, Chapter III, Corollary 1.7] and its straightforward generalization for the case of positive $(1,1)$-forms;
(ii) $\Longleftrightarrow$ (iii) is achieved by diagonalizing the Hermitian matrix of coefficients $(a_{j\bar{k}})$. □

The next result follows from [33, Chapter III, Corollary 1.9 and Proposition 1.11].

Proposition 2.3. *If α_1, α_2 are semi-positive real $(1,1)$-forms, then $\alpha_1 \wedge \alpha_2$ is semi-positive.*

We want to characterize the semi-positivity of real $(2,2)$-forms. Let γ be a real $(2,2)$-form on a 3-dimensional complex vector space V. We may write

$$\gamma = -\frac{1}{4}\sum_{\substack{i<k \\ j<l}} \gamma_{i\bar{j}k\bar{l}}\, dz_i \wedge d\overline{z}_j \wedge dz_k \wedge d\overline{z}_l, \tag{2.2}$$

with respect to some coordinates (z_1, z_2, z_3) on V. We may associate with γ the real $(1,1)$-form

$$\beta = \frac{i}{2}\sum_{m,n} \beta_{m\bar{n}}\, dz_m \wedge d\overline{z}_n,$$

whose coefficients are determined by those of γ following

$$\beta_{m\bar{n}} := \frac{1}{4}\sum_{i,j,k,l} \gamma_{i\bar{j}k\bar{l}}\varepsilon_{ikm}\varepsilon_{jln}. \tag{2.3}$$

Here ε_{abc} is the Levi-Civita symbol, with $\varepsilon_{123}=1$. Using a change of basis $dz_i=\sum_p A_i^p dw_p$, the matrix $(\beta_{m\overline{n}})$ changes by congruence via the matrix $\tilde{A}=\det(A)(A^t)^{-1}$, where $A=(A_i^p)$. Consequently, the semi-positivity of β does not depend on the choice of coordinates on V. Notice that the matrix $(\beta_{m\overline{n}})$ is Hermitian, since $\gamma=\overline{\gamma}$ implies $\gamma_{i\overline{j}k\overline{l}}=\overline{\gamma_{j\overline{i}l\overline{k}}}$.

Proposition 2.4. *Let γ be a non-zero real $(2,2)$-form on V. Then, the following are equivalent:*

(i) γ *is semi-positive,*

(ii) $\gamma\wedge\alpha>0$ *for every positive real* $(1,1)$*-form* α*, i.e.,* $\gamma\wedge\alpha=\lambda\tau(z)$*, where* $\lambda>0$*,*

(iii) *the associated* $(1,1)$*-form* β *is semi-positive.*

Proof.
(i) $\iff$ (iii) Let γ be a real $(2,2)$ form on V. Then, γ can be written as in (2.2) with respect to a basis (dz_1,dz_2,dz_3) of $\Lambda^{1,0}V^*$. By Definition 2.1, γ is semi-positive if for all $\eta\in\Lambda^{1,0}V^*$ one has $\frac{i}{2}\gamma\wedge\eta\wedge\overline{\eta}\geq 0$. Set $\eta=\sum_m\eta_m dz_m$; then

$$\frac{i}{2}\gamma\wedge\eta\wedge\overline{\eta}=\sum_{m,n}\beta_{m\overline{n}}\eta_m\overline{\eta}_n\tau(z),$$

where the coefficients $\beta_{m\overline{n}}$ are defined in (2.3). Therefore, since η is arbitrary, γ is semi-positive if and only if the matrix $(\beta_{m\overline{n}})$ is positive semi-definite.
(i) $\implies$ (ii) Let α be a positive $(1,1)$-form on V. Then, there exists a basis (dz_1,dz_2,dz_3) of $\Lambda^{1,0}V^*$ such that,

$$\alpha=\frac{i}{2}\sum_k a_{k\overline{k}}dz_k\wedge d\overline{z}_k$$

with $a_{k\overline{k}}>0$. Let γ be a semi-positive $(2,2)$-form on V. We may write

$$\gamma=-\frac{1}{4}\sum_{\substack{i<k\\ j<l}}\gamma_{i\overline{j}k\overline{l}}dz_i\wedge d\overline{z}_j\wedge dz_k\wedge d\overline{z}_l.$$

Then,

$$\gamma\wedge\alpha=\sum_r a_{r\overline{r}}\beta_{r\overline{r}}\tau(z).$$

Since γ is semi-positive, by (iii) we have that $\beta_{r\overline{r}}\geq 0$ for all $r=1,2,3$ with at least one of them being strictly positive. Therefore, since $a_{r\overline{r}}>0$, for every r, the claim follows.
(ii) $\implies$ (i) Let $(\alpha_1,\alpha_2,\alpha_3)$ be a basis of $\Lambda^{1,0}V^*$. We define

$$\alpha_\varepsilon\coloneqq\frac{i}{2}(\alpha_1\wedge\overline{\alpha}_1+\varepsilon(\alpha_2\wedge\overline{\alpha}_2+\alpha_3\wedge\overline{\alpha}_3)).$$

We notice that, for every $\varepsilon>0$, α_ε is a positive $(1,1)$-form. Then, by hypothesis, $\gamma\wedge\alpha_\varepsilon>0$. The claim follows by continuity since $\frac{i}{2}\gamma\wedge\alpha_1\wedge\overline{\alpha}_1=\lim_{\varepsilon\to 0}(\gamma\wedge\alpha_\varepsilon)\geq 0$. $\square$

As shown in [81, Theorem 1.2], a real $(2,2)$-form γ is always diagonalizable, i.e., there exist coordinates (w_1, w_2, w_3) of V such that

$$\gamma = -\frac{1}{4}\sum_{i<k} \gamma_{i\bar{i}k\bar{k}} dw_i \wedge d\overline{w}_i \wedge dw_k \wedge d\overline{w}_k.$$

By Proposition 2.4, γ is semi-positive if and only if $\gamma_{i\bar{i}k\bar{k}} \geq 0$, for every $i < k$. In particular, the diagonal matrix $(\beta_{m\bar{n}})$ associated with γ in these coordinates is positive semi-definite. Moreover, γ is positive if and only if $\gamma_{i\bar{i}k\bar{k}} > 0$, for every $i < k$.

Remark 2.5. [106, formula (4.8)] A real $(2,2)$-form γ on V is positive if and only if $\gamma = \alpha^2$, where α is a positive $(1,1)$-form.

2.2 Mean convexity and intrinsic torsion of closed SU(3)-structures

In this section, we study the mean convex property in the context of closed SU(3)-structures and provide necessary and sufficient conditions in terms of the intrinsic torsion of the SU(3)-structure.

Let M be an oriented smooth 6-manifold endowed with an SU(3)-structure (ω, ρ). According to [35], the differential $d\hat{\rho}$ of the $\mathrm{SL}(3,\mathbb{C})$-structure $\hat{\rho} = J\rho$ is a real $(2,2)$-form, so that it could be semi-positive in the sense of Definition 2.1.

Definition 2.6. A closed $\mathrm{SL}(3,\mathbb{C})$-structure ρ on M is *(strictly) mean convex* if $d\hat{\rho}$ is a non-zero semi-positive (resp. positive) $(2,2)$-form at every point of M.

Definition 2.7. A closed SU(3)-structure (ω, ρ) on a 6-manifold M is *(strictly) mean convex* if the $\mathrm{SL}(3,\mathbb{C})$-structure ρ is (strictly) mean convex.

As a consequence of (1.12), if ρ is closed we have $d\hat{\rho} = \theta \wedge \omega$, where θ is the $(1,1)$-form defined by $\theta := w_0^- \omega + w_2^-$.

We recall that, given a real $(1,1)$-form α, the trace $\mathrm{tr}(\alpha)$ of α is given by $3\alpha\wedge\omega^2 = \mathrm{tr}(\alpha)\omega^3$. Then, in terms of w_0^- and the $(1,1)$-form θ, we can prove the following.

Proposition 2.8. *Let (ω, ρ) be a closed* SU(3)*-structure on M. Then,*

(i) *if (ω, ρ) is mean convex, the intrinsic torsion form w_0^- is strictly positive and the $(1,1)$-form θ is not negative (semi-)definite. Moreover, its trace $\mathrm{tr}(\theta)$ is strictly positive;*

(ii) *if θ is semi-positive, the* SU(3)*-structure is mean convex.*

Proof. Let us assume that (ω, ρ) is a mean convex closed SU(3)-structure on M. By (1.12), we have $d\hat{\rho} = \theta\wedge\omega$. Now, Proposition 2.4 implies $d\hat{\rho}\wedge\alpha > 0$ for every positive real $(1,1)$-form α. Then, (i) follows by choosing $\alpha = \omega$; indeed, $d\hat{\rho} \wedge \omega = w_0^-\omega^3$, since $w_2^- \in \Lambda^2_8(M)$. In particular $\mathrm{tr}(\theta) = 3w_0^- > 0$. (ii) follows from Proposition 2.3 and the positivity of ω. □

Let N be a 7-dimensional smooth manifold endowed with a torsion-free G_2-structure $\varphi \in \Lambda^3(N)$. By Section 1.5, φ induces an SU(3)-structure (ω, ρ) on each oriented hypersurface $i\colon M \hookrightarrow N$. By a quick computation, one can see that the resulting SU(3)-structure is half-flat. As a matter fact, one has

$$\rho = i^*\varphi, \quad \omega^2 = 2\, i^*(*_\varphi \varphi),$$

which are closed when φ is both closed and coclosed. Moreover, the intrinsic torsion of the half-flat structure can be identified with the second fundamental form $B \in \Gamma(S^2(T^*M))$ of M with respect to a fixed unit normal vector field ν. As in [35], with respect to J_ρ, we can write $B = B_{1,1} + B_C$, where $B_{1,1}$ is the real part of a Hermitian form and B_C is the real part of a complex quadratic form. If we denote by $\beta_{1,1} = B_{1,1}(J_\rho \cdot, \cdot)$ the corresponding $(1,1)$-form on M, we have $\beta_{1,1} \wedge \omega = \frac{1}{2} d\hat{\rho}$, from which it follows that, if (ω, ρ) is mean convex, then the mean curvature μ given explicitly by $\frac{1}{4}\mu\rho \wedge \hat{\rho} = \frac{1}{2} d\hat{\rho} \wedge \omega$ is positive with respect to the normal direction (for more details see [35, Proposition 1]). Moreover, since the wedge product with ω defines an injective map on 2-forms, comparing this with (1.12) yields $\theta = 2\beta_{1,1}$. Then, by Proposition 2.8, if $B_{1,1}$ defines a positive semi-definite Hermitian product, then the half-flat structure (ω, ρ) is mean convex.

Special types of half-flat structures (ω, ρ) are called *coupled*, when $d\omega = -\frac{3}{2} w_0^- \rho$, and *double*, when $d\hat{\rho} = w_0^- \omega^2$. Notice that, by Proposition 2.8, double structures (ω, ρ) are trivially mean convex as long as $w_0^- > 0$. However, it is straightforward to check that, if (ω, ρ) is a double structure such that $w_0^- < 0$, then $(\omega, -\rho)$ is mean convex. In [25, Theorem 4.11], a classification of 6-dimensional nilpotent Lie algebras endowed with a double structure was given. Other examples of double structures on $S^3 \times S^3$ were found in [104, 122].

For a general Lie algebra, we can show the following.

Proposition 2.9. *If a Lie algebra $\mathfrak{g}$ has a strictly mean convex closed* $\mathrm{SL}(3,\mathbb{C})$*-structure, then $\mathfrak{g}$ admits a double structure.*

Proof. Let ρ be a strictly mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structure on $\mathfrak{g}$. Then, $d\hat{\rho}$ is a positive $(2,2)$-form and, as shown in [106] (see Remark 2.5), there exists a positive $(1,1)$-form α such that $d\hat{\rho} = \alpha^2$. Moreover, since α is positive with respect to J_ρ, α^3 is a positive multiple of the volume form $\rho \wedge \hat{\rho}$. Since J_ρ does not change for a non-zero rescaling of ρ, this implies that there exists $b \neq 0$ such that $(b\rho, \alpha)$ is a double structure on $\mathfrak{g}$. $\square$

As a consequence, the classification of nilpotent Lie algebras admitting strictly mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures reduces to Theorem 4.11 in [25]. Therefore, in the next section we weaken the condition asking for the existence of (non-strictly) mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures.

2.3 Mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures on nilpotent Lie algebras

Using the classification result in [73, 105], we can prove the following.

Theorem 2.10. *Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold. Then, M admits invariant mean convex closed* $\mathrm{SL}(3,\mathbb{C})$*-structures if and only if the Lie algebra $\mathfrak{g}$ of G is not isomorphic to any of the six Lie algebras $\mathfrak{g}_i$, $i = 1, 2, 4, 9, 12, 34$, as listed in Table 6.1.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. We recall that every invariant $\mathrm{SL}(3,\mathbb{C})$-structure on M is determined by an $\mathrm{SL}(3,\mathbb{C})$-structure on $\mathfrak{g}$ and vice versa. First notice that the possibility that $\mathfrak{g}$ is abelian is precluded by Definition 2.7. Then, in order to prove the first part of the theorem, we first show the non-existence result for the five Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_4$, $\mathfrak{g}_9$ and $\mathfrak{g}_{12}$. For each of these Lie algebras, let us consider a generic closed 3-form

$$\rho = \sum_{i<j<k} p_{ijk}\, e^{ijk}, \quad p_{ijk} \in \mathbb{R}.$$

Let us assume that ρ is definite, i.e., stable with $\lambda(\rho) < 0$. Then, ρ induces an almost complex structure J_ρ and we may ask if the induced $(2,2)$-form $d\hat{\rho}$ is semi-positive. Notice that the 1-forms $\zeta^k = e^k - iJ_\rho e^k$, for $k = 1, \dots, 6$, generate the space $\Lambda^{1,0}\mathfrak{g}_i^*$ of $(1,0)$-forms with respect to J_ρ on $\mathfrak{g}_i$, $i = 1, 2, 4, 9, 12$. So, for every closed definite 3-form ρ, we extract a basis (ξ^1, ξ^2, ξ^3) for $\Lambda^{1,0}\mathfrak{g}_i^*$, where $\xi^j = \zeta^{k_j}$ for some $k_j \in \{1, \dots, 6\}$ and $j = 1, 2, 3$. Then, $(\xi^1, \xi^2, \xi^3, \overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3)$ is a complex basis for $\mathfrak{g}_i^* \otimes \mathbb{C}$ and we can write $d\hat{\rho}$ in this new basis as

$$d\hat{\rho} = -\frac{1}{4} \sum_{\substack{i<k \\ j<l}} \gamma_{i\bar{j}k\bar{l}}\, \xi^i \overline{\xi}^j \xi^k \overline{\xi}^l,$$

for some $\gamma_{i\bar{j}k\bar{l}} \in \mathbb{C}$. We note that the real 1-forms

$$e^{k_j} = \frac{1}{2}(\xi^j + \overline{\xi}^j), \quad J_\rho(e^{k_j}) = \frac{i}{2}(\xi^j - \overline{\xi}^j), \quad j = 1, 2, 3,$$

define a new real basis for $\mathfrak{g}_i^*$. Now, following Section 2.1, we consider the real $(1,1)$-form β associated with $d\hat{\rho}$, given explicitly by

$$\beta = \frac{i}{2} \sum_{m,n} \beta_{m\bar{n}}\, \xi^m \overline{\xi}^n, \quad \beta_{m\bar{n}} = \frac{1}{4} \sum_{i,j,k,l} \gamma_{i\bar{j}k\bar{l}} \varepsilon_{ikm} \varepsilon_{jln}, \tag{2.4}$$

and we compute the expression of $\beta_{m\bar{n}}$ in terms of p_{ijk}. Therefore, $d\hat{\rho}$ is semi-positive (non-zero) if and only if the Hermitian matrix $(\beta_{m\bar{n}})$ is positive semi-definite, which occurs if and only if

$$\begin{cases} \beta_{k\bar{k}} \geq 0, & k = 1, 2, 3, \\ \beta_{r\bar{r}}\beta_{k\bar{k}} - |\beta_{r\bar{k}}|^2 \geq 0, & r < k,\ r, k = 1, 2, 3, \\ \det(\beta_{m\bar{n}}) \geq 0, & \end{cases} \tag{2.5}$$

with $(\beta_{m\bar{n}})$ different from the zero matrix. Then, it can be shown that, for every closed 3-form ρ such that $\lambda(\rho) < 0$, the system (2.5) in the variables p_{ijk} has no solutions. The computations have been made with the aid of Maple 2021 (see Appendix 2 for the list of instructions used). Let us see this explicitly for $\mathfrak{g}_i$, $i = 1, 2$. By a direct computation, for the generic closed 3-form ρ on $\mathfrak{g}_1$ we have

$$\lambda(\rho) = \left[(p_{145} + 2p_{235})p_{146} + p_{145}p_{236} + p_{245}^2\right]^2 + 4p_{146}p_{236}\,(p_{126} - p_{145}p_{235} + p_{135}p_{245})$$

and, for the generic closed 3-form ρ on $\mathfrak{g}_2$, we get

$$\lambda(\rho) = \left(p_{245}^2 + p_{145}p_{236} + 2p_{146}p_{235}\right)^2 + 4p_{146}p_{236}\left(-p_{145}p_{235} + p_{135}p_{245} + p_{125}p_{146}\right).$$

Notice that, if at least one between p_{146} and p_{236} is equal to zero, then $\lambda(\rho) \geq 0$. So let us assume that both p_{146} and p_{236} are non-zero. Then, $(e^1, J_\rho e^1, e^2, J_\rho e^2, e^5, J_\rho e^5)$ defines a basis of $\mathfrak{g}_i^*$, for $i = 1, 2$, hence $(\xi^1 = e^1 - iJ_\rho e^1, \xi^2 = e^2 - iJ_\rho e^2, \xi^3 = e^5 - iJ_\rho e^5)$ is a basis of $(1,0)$-forms on $\mathfrak{g}_i$, $i = 1, 2$. By a direct computation, it can be shown that in these cases the matrix coefficient $\beta_{1\bar{1}}$ vanishes and so $\beta_{1\bar{1}}\beta_{3\bar{3}} - |\beta_{1\bar{3}}|^2 = -|\beta_{1\bar{3}}|^2 \leq 0$, but $\beta_{1\bar{3}} = 0$ implies $\lambda(\rho) = 0$, which is a contradiction.

By a very similar discussion, we may discard cases $\mathfrak{g}_4$, $\mathfrak{g}_9$ and $\mathfrak{g}_{12}$ as well. In order to prove the second part of the theorem, we construct an explicit mean convex closed SU(3)-structure (ω, ρ) on the remaining nilpotent Lie algebras (see Table 6.3). □

Using the classification of half-flat nilpotent Lie algebras (see [26]), we can then prove the following.

Theorem 2.11. *A 6-dimensional nilmanifold $M = \Gamma\backslash G$ has an invariant mean convex half-flat structure if and only if the Lie algebra $\mathfrak{g}$ of G is isomorphic to one of the Lie algebras $\mathfrak{g}_i$, $i = 6, 7, 8, 10, 13, 15, 16, 22, 24, 25, 28, 29, 30, 31, 32, 33$, as listed in Table 6.1.*

In [26], a classification up to isomorphism of 6-dimensional real nilpotent Lie algebras admitting half-flat structures was given. They are twenty-four and they are listed in Table 6.3. So, in order to classify nilpotent Lie algebras admitting a mean convex half-flat structure, we restrict our attention to this list. An explicit example of mean convex half-flat structure on $\mathfrak{g}_i$, $i = 6, 7, 8, 10, 13, 15, 16, 22, 24, 25, 28, 29, 30, 31, 32, 33$, is provided in Table 6.3. Therefore, we only need to prove non-existence of mean convex half-flat structures on the remaining Lie algebras $\mathfrak{g}_i$, $i = 4, 9, 11, 12, 14, 21, 27$. By Theorem 2.10, we may immediately exclude the Lie algebras $\mathfrak{g}_i$, $i = 4, 9, 12$, since mean convex half-flat structures are mean convex closed SL(3, ℂ)-structures, in particular.

For the remaining Lie algebras $\mathfrak{g}_i$, $i = 11, 14, 21, 27$, whose first Betti number is 3 or 4, we first collect some necessary conditions for the existence of mean convex closed SU(3)-structures (ω, ρ) in terms of a filtration of J_ρ-invariant subspaces U_i of $\mathfrak{g}^*$. Then, by working in an SU(3)-adapted basis, we exhibit further obstructions.

Let us start by defining the filtration $\{U_j\}$ as in [25]. Let (ω, ρ) be an SU(3)-structure on a 6-dimensional real nilpotent Lie algebra $\mathfrak{g}$ and let (g, J_ρ) be the induced almost Hermitian structure. By nilpotency, there exists a basis $\left(\alpha^1, \dots, \alpha^6\right)$ of $\mathfrak{g}^*$ such that, if we denote $V_j \coloneqq \left\langle \alpha^1, \dots, \alpha^j \right\rangle$, then $dV_j \subset \Lambda^2 V_{j-1}$ and, by construction, $0 \subset V_1 \subset \ldots \subset V_5 \subset V_6 = \mathfrak{g}^*$. We notice that the basis $(e^1, \dots, e^6)$, whose corresponding structure equations are given in Table 6.1, satisfies the previous conditions and $V_j = \ker d$ when $b_1(\mathfrak{g}) = j$. In the following, we consider $V_j = \left\langle e^1, \dots, e^j \right\rangle$. As in [25], let $U_j \coloneqq V_j \cap J_\rho V_j$ be the maximal J_ρ-invariant subspace of V_j for each j. Then, since J_ρ is an automorphism of the vector space $\mathfrak{g}$, a simple dimensional computation shows that $\dim_{\mathbb{R}} U_2$, $\dim_{\mathbb{R}} U_3 \in \{0, 2\}$, $\dim_{\mathbb{R}} U_4 \in \{2, 4\}$ and $\dim_{\mathbb{R}} U_5 = 4$. Notice that the filtration $\{U_j\}$ depends on V_j and the almost complex structure J_ρ.

We can prove the following.

Lemma 2.12. *Let ρ be a mean convex closed* $\mathrm{SL}(3,\mathbb{C})$*-structure on a nilpotent Lie algebra* $\mathfrak{g}$*. If* $\mathfrak{g}$ *is isomorphic to*

$$\mathfrak{g}_{11} = (0,0,0,e^{12},e^{14},e^{15}+e^{23}+e^{24}) \quad or \quad \mathfrak{g}_{14} = (0,0,0,e^{12},e^{13},e^{14}+e^{35}),$$

then $U_3 = U_4$*. If* $\mathfrak{g}$ *is isomorphic to*

$$\mathfrak{g}_{21} = (0,0,0,e^{12},e^{13},e^{14}+e^{23}) \quad or \quad \mathfrak{g}_{27} = (0,0,0,0,e^{12},e^{14}+e^{25}),$$

then $\dim_{\mathbb{R}} U_2 = 2$*, or equivalently* $\langle e^1, e^2\rangle$ *is* J_ρ*-invariant. Moreover, on* $\mathfrak{g}_{21}$ *(up to isomorphism) we also have* $\dim_{\mathbb{R}} U_4 = 4$*.*

Proof. On each Lie algebra $\mathfrak{g}_i$, $i = 11, 14, 21, 27$, we consider the generic closed 3-form

$$\rho = \sum_{i<j<k} p_{ijk}\, e^{ijk}, \quad p_{ijk} \in \mathbb{R},$$

and we impose $\lambda(\rho) < 0$ and the mean convex condition. First, by a direct computation on each Lie algebra, we determine the expression of $\lambda(\rho)$ in terms of the coefficients p_{ijk} and a basis of $(1,0)$-forms with respect to J_ρ. Then, we exclude the cases where either $\lambda(\rho) \geq 0$ or the matrix $(\beta_{m\overline{n}})$ associated with $d\hat{\rho}$ is not positive semi-definite. As in the proof of Theorem 2.10, we first extract a basis of $(1,0)$-forms from the set of generators (ζ^i) and we use (2.4) to compute $(\beta_{m\overline{n}})$ in terms of p_{ijk}. We shall give all the details for the Lie algebra $\mathfrak{g}_{11}$. For the other cases the computations are similar and we only report the necessary conditions on p_{ijk}. The generic closed 3-form ρ on the Lie algebra $\mathfrak{g}_{11}$ has

$$\begin{aligned}\lambda(\rho) =&(p_{126}p_{236} - p_{126}p_{146} - p_{135}p_{246} + p_{145}p_{236} + p_{146}p_{235} - p_{146}p_{245} + p_{234}p_{246}\\ &- p_{235}p_{245})^2 + 4p_{246}(p_{123}p_{236}p_{246} - p_{123}p_{246}^2 - p_{124}p_{236}^2 + p_{124}p_{236}p_{246}\\ &+ 2p_{125}p_{146}p_{236} - p_{125}p_{146}p_{246} + p_{125}p_{235}p_{236} - p_{125}p_{235}p_{246} - p_{134}p_{235}p_{246}\\ &+ p_{134}p_{236}p_{245} - p_{125}p_{146}p_{246} + p_{135}p_{234}p_{246} - p_{135}p_{235}p_{245} + p_{145}p_{146}p_{235}\\ &+ p_{145}p_{235}^2 - p_{145}p_{234}p_{236}) + 4p_{146}p_{236}(-p_{125}p_{236} + p_{135}p_{235} - p_{145}p_{235}).\end{aligned}$$

Then, we have the following possibilities:

(a) $p_{246} \neq 0, p_{246} \neq p_{236}$. Then, $\left(e^1 - iJ_\rho e^1, e^2 - iJ_\rho e^2, e^3 - iJ_\rho e^3\right)$ is a basis for $\Lambda^{1,0}\mathfrak{g}_{11}^*$, but $(\beta_{m\overline{n}})$ being positive semi-definite implies $\lambda(\rho) = 0$, a contradiction.

(b) $p_{246} = 0, p_{236} \neq 0, p_{146} \neq 0$. Taking $\left(e^1 - iJ_\rho e^1, e^2 - iJ_\rho e^2, e^5 - iJ_\rho e^5\right)$ as a basis for $\Lambda^{1,0}\mathfrak{g}_{11}^*$, again we find that $(\beta_{m\overline{n}})$ being positive semi-definite implies $\lambda(\rho) = 0$.

(c) $p_{246} = p_{236} = 0$, or $p_{246} = p_{146} = 0$. These would force $\lambda(\rho) \geq 0$.

(d) $p_{236} = p_{246} \neq 0$. In particular, this implies that $V_2 = \langle e^1, e^2\rangle$ is J_ρ-invariant, i.e., $\dim_{\mathbb{R}} U_2 = 2$. Notice also that, since $J_\rho e^3(e_6) = 0$ if and only if $p_{236} = 0$, we also have that $V_4 = \langle e^1, e^2, e^3, e^4\rangle$ is not J_ρ-invariant, hence $U_2 = U_3 = U_4$.

By a very similar discussion, one can show that a generic mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structure ρ on $\mathfrak{g}_{14}$ must have $p_{245}=0$ and $p_{356}\neq 0$. In particular, since $J_\rho e^1, J_\rho e^3 \in \langle e^1, e^3\rangle$, we have $\dim_{\mathbb{R}} U_3 = 2$. Moreover, $J_\rho e^2(e_6)\neq 0$, hence $\dim_{\mathbb{R}} U_2 = 0$ and $U_3 = U_4$.

Analogously, every mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structure ρ on $\mathfrak{g}_{21}$ must have $p_{345}=0$. This implies that V_2 and V_4 are J_ρ-invariant, so that $\dim_{\mathbb{R}} U_2 = 2$, $\dim_{\mathbb{R}} U_4 = 4$ and $U_2 = U_3$.

Finally, a mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structure ρ on $\mathfrak{g}_{27}$ must have $p_{345}=0$. In particular, this implies that V_2 is J_ρ-invariant, so that $U_2 = U_3$. □

Now we can prove Theorem 2.11.

Proof of Theorem 2.11. Starting from the classification of half-flat nilpotent Lie algebras given in [26], we divide the discussion depending on the first Betti number b_1 of $\mathfrak{g}$.

When $b_1(\mathfrak{g}) = 2$, the claim follows directly by Theorem 2.10. In particular, we have seen that $\mathfrak{g}_4$ cannot admit mean convex closed $\mathrm{SL}(3,\mathbb{C})$-structures and, for the remaining Lie algebras $\mathfrak{g}_6$, $\mathfrak{g}_7$ and $\mathfrak{g}_8$ from Table 6.1, we provide an explicit example in Table 6.3 on the respective Lie algebras.

Analogously, when $b_1(\mathfrak{g}) = 3$, an explicit example of mean convex half-flat structure on $\mathfrak{g}_i$, $i = 10, 13, 15, 16, 22, 24$, is given in Table 6.3. By Theorem 2.10, we may exclude the existence of mean convex half-flat structures on $\mathfrak{g}_9$ and $\mathfrak{g}_{12}$. For the remaining Lie algebras $\mathfrak{g}_i$, $i = 11, 14, 21$, let (ω,ρ) be a mean convex half-flat structure on $\mathfrak{g}_i$. Then, by Lemma 2.12, with respect to the fixed nilpotent filtration $V_j = \langle e^1,\dots,e^j\rangle$, we may assume $\dim_{\mathbb{R}} U_3 = 2$. Using this and the information on U_4 we also collected in Lemma 2.12, we shall show that on the three Lie algebras there exists an adapted basis (f_i) with dual basis (f^i) such that $df^1 = df^2 = 0$ and $f_6 \in \mathfrak{z}(\mathfrak{g}_i)$.

Let us consider the case of $\mathfrak{g}_{21}$, first. Then, we may assume $\dim_{\mathbb{R}} U_4 = 4$, which occurs if and only if $V_4 = J_\rho V_4$. In particular, we may choose a g-orthonormal basis $\left(f^1, f^2\right)$ of U_3 such that $J_\rho f^1 = -f^2$, take $f^3, f^4 \in U_3^\perp \cap U_4$ of unit norm such that $J_\rho f^3 = -f^4$ and complete it to a basis for $\mathfrak{g}_{21}^*$ by choosing $f^5 \in U_4^\perp \cap V_5$ and $f^6 \in U_4^\perp \cap J_\rho V_5$ of unit norm such that $J_\rho f^5 = -f^6$. Then, by construction, $(f_1,\dots,f_6)$ is an adapted basis for the $\mathrm{SU}(3)$-structure (ω,ρ). In particular, since $V_5 = \langle f^1, f^2, f^3, f^4, f^5\rangle$, the inclusion $dV_j \subset \Lambda^2(V_{j-1})$ implies $f_6 \in \mathfrak{z}(\mathfrak{g}_{21})$. Therefore, since $f^1, f^2 \in V_3 = \ker d$, we have $df^1 = df^2 = 0$.

Cases $\mathfrak{g}_{11}$ and $\mathfrak{g}_{14}$ may be discussed in the same way. By Lemma 2.12, we can assume $\dim_{\mathbb{R}} U_4 = 2$ for both Lie algebras. As shown in [25], since $U_4, V_3 \subset V_4$, we have $\dim_{\mathbb{R}}(U_4 \cap V_3) \geq 1$ and we may take $\left(f^1, f^2\right)$ to be a unitary basis of U_4 with $f^1 \in V_3$. Then, since $U_3 \subset V_3 = \ker d$, we may suppose $df^1 = df^2 = 0$. Analogously, since $\dim_{\mathbb{R}}(V_4 \cap J_\rho V_5) \geq 3$ and $U_5 \cap V_4 = V_5 \cap J_\rho V_5 \cap V_4 = V_4 \cap J_\rho V_5$, then $\dim_{\mathbb{R}}(U_5 \cap V_4) \geq 3$, from which $\dim_{\mathbb{R}}(U_5 \cap V_4 \cap U_4^\perp) \geq 1$ follows. Then, we may take $\left(f^3, f^4\right)$ to be a unitary basis of $U_4^\perp \cap U_5$ with $f^3 \in V_4$. Finally, since $\dim_{\mathbb{R}}(U_5^\perp \cap V_5) \geq 1$, we may take a unitary basis $\left(f^5, f^6\right)$ of $U_5^\perp$ with $f^5 \in V_5$. By construction, $\left(f^1, f^2, \dots, f^6\right)$ is the dual basis of an adapted basis for (ω,ρ). In particular, since $U_5 \subset V_5$, we also have $V_5 = \langle f^1, f^2, f^3, f^4, f^5\rangle$, which implies $f_6 \in \mathfrak{z}(\mathfrak{g}_{\mathrm{i}})$, for $i = 11, 14$. This proves our claim.

Using this, we prove that the three Lie algebras $\mathfrak{g}_i$, $i = 11, 14, 21$, do not admit any mean convex half-flat structures. By contradiction, let us suppose there exists a nilpotent Lie algebra $\mathfrak{g}$ endowed with a mean convex half-flat structure (ω,ρ) which is isomorphic to $\mathfrak{g}_{11}$, $\mathfrak{g}_{14}$ or $\mathfrak{g}_{21}$. Without loss of generality, we may assume that there exists an adapted basis (f_i)

for the SU(3)-structure such that $df^1 = df^2 = 0$, $f_6 \in \mathfrak{z}(\mathfrak{g})$. In this basis,

$$\omega = f^{12} + f^{34} + f^{56}, \quad \rho = f^{135} - f^{146} - f^{236} - f^{245}, \quad \hat{\rho} = f^{136} + f^{145} + f^{235} - f^{246}.$$

Therefore, $\mathfrak{g}$ has structure equations

$$df^1 = df^2 = 0, \quad df^k = -\sum_{\substack{i<j\\ i,j=1}}^{5} c_{ij}^k f^{ij}, \quad k = 3,4,5,6.$$

By imposing the unimodularity condition $\sum_j c_{ij}^j = 0$, for all $i = 1,\dots,6$, and the half-flat equations

$$d\rho = 0, \quad d\omega^2 = 0,$$

a direct computation shows that, if $c_{34}^5 \neq 0$, then the Jacobi identities $d^2 f^i = 0$, $i = 3,\dots,6$, are equivalent to the conditions

$$c_{15}^4 = c_{25}^4 = c_{25}^3 = c_{15}^6 = c_{13}^4 = c_{14}^4 = c_{13}^3 = c_{23}^3 = c_{24}^3 = 0,$$

which imply $b_1(\mathfrak{g}) \geq 4$, a contradiction. Then, we have to impose $c_{34}^5 = 0$. Let us assume $c_{12}^6 \neq 0$. Again, a straightforward computation shows that $d^2 f^6 = 0$ implies

$$c_{25}^3 = c_{25}^4 = c_{15}^4 = 0, \quad c_{13}^3 = -c_{14}^4, \quad c_{23}^3 = -c_{13}^4 - c_{15}^6.$$

Now, let us look at the mean convex condition. Using (2.4), we obtain that the matrix $(\beta_{m\overline{n}})$ associated with $d\hat{\rho}$, with respect to the basis $(\xi^1 = f^1 + if^2, \xi^2 = f^3 + if^4, \xi^3 = f^5 + if^6)$, is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{15}^6 - i(c_{24}^3 + c_{14}^4) \\ 0 & c_{15}^6 + i(c_{24}^3 + c_{14}^4) & -c_{14}^5 - c_{13}^6 + c_{24}^6 - c_{23}^5 \end{pmatrix}.$$

Therefore, $d\hat{\rho}$ is semipositive if and only if $c_{15}^6 = 0$, $c_{24}^3 = -c_{14}^4$ and $-c_{14}^5 - c_{13}^6 + c_{24}^6 - c_{23}^5 > 0$. In particular, $c_{15}^6 = 0$ and $c_{24}^3 = -c_{14}^4$ imply that the Jacobi identities hold if and only if $c_{13}^4 = c_{14}^4 = 0$. However, this also implies $df^3 = df^4 = 0$, so that $b_1(\mathfrak{g}) \geq 4$, meaning that we have to discard this case as well. Therefore, $c_{34}^5 = c_{12}^6 = 0$ and, as a consequence,

$$\begin{aligned}
df^3 &= -c_{13}^3 f^{13} - (c_{13}^4 + c_{15}^6) f^{14} - c_{25}^4 f^{15} - c_{23}^3 f^{23} - c_{24}^3 f^{24} - c_{25}^3 f^{25}, \\
df^4 &= -c_{13}^4 f^{13} - c_{14}^4 f^{14} - c_{15}^4 f^{15} - c_{13}^3 f^{23} - (c_{13}^4 + c_{15}^6) f^{24} - c_{25}^4 f^{25}, \\
df^5 &= -(c_{14}^6 + c_{23}^6 + c_{24}^5) f^{13} - c_{14}^5 f^{14} + (c_{14}^4 + c_{13}^3) f^{15} - c_{23}^5 f^{23} - c_{24}^5 f^{24} \\
&\quad + (c_{23}^3 + c_{13}^4 + c_{15}^6) f^{25}, \\
df^6 &= -c_{13}^6 f^{13} - c_{14}^6 f^{14} - c_{15}^6 f^{15} - c_{23}^6 f^{23} - c_{24}^6 f^{24} - (c_{24}^3 - c_{13}^3) f^{25}.
\end{aligned} \tag{2.6}$$

In particular, f^{12} is a non-exact 2-form belonging to $\Lambda^2(\ker d)$ such that $f^{12} \wedge d\mathfrak{g}^* = 0$. On the other hand, a simple computation shows that for every Lie algebra $\mathfrak{g}_i$, for $i = 11, 14, 21$, a 2-form $\alpha \in \Lambda^2(\ker d)$ such that $\alpha \wedge d\mathfrak{g}_i^* = 0$ is necessarily exact, leading to a contradiction. This concludes the non-existence part of the proof in the case $b_1 = 3$.

Now, we consider the remaining case $b_1(\mathfrak{g}) \geq 4$. An explicit example of mean convex half-flat structure on $\mathfrak{g}_i$, $i = 25, 28, 29, 30, 31, 32, 33$, is given in Table 6.3. Then, we only need to prove the non-existence of mean convex half-flat structures on $\mathfrak{g}_{27}$.

Let (ω, ρ) be a mean convex half-flat structure on $\mathfrak{g}_{27}$. We claim that there exists an adapted basis (f_i) with dual basis (f^i) such that $df^1 = df^2 = df^3 = 0$ and $f_6 \in \mathfrak{z}(\mathfrak{g}_{27})$. By Lemma 2.12, we can assume $U_2 = U_3$, with $\dim_{\mathbb{R}} U_3 = 2$. We recall that U_4 has dimension two or four. Let us suppose $\dim_{\mathbb{R}} U_4 = 4$, first. We note that, in this case, the existence of an adapted basis (f_i) for (ω, ρ) such that $f_6 \in \mathfrak{z}(\mathfrak{g}_{27})$ and $V_4 = U_4 = \langle f^1, f^2, f^3, f^4 \rangle$ follows from the previous discussion on $\mathfrak{g}_{21}$, where we only used $\dim_{\mathbb{R}} U_2 = 2$ and $\dim_{\mathbb{R}} U_4 = 4$. In particular, since $V_4 = \ker d$ on $\mathfrak{g}_{27}$, in this case we also have $df^1 = df^2 = df^3 = df^4 = 0$. When $\dim_{\mathbb{R}} U_4 = 2$ instead, since $U_2 = U_3 = U_4$, the discussion is the same as for $\mathfrak{g}_{11}$ and $\mathfrak{g}_{14}$, where we only used $U_3 = U_4$ to find an adapted basis such that $df^1 = df^2 = 0$ and f_6 lying in the center. In particular, since by construction $f^1, f^2, f^3 \in V_4$, on $\mathfrak{g}_{27}$ we also have $df^3 = 0$, since $V_4 = \ker d$. This proves our claim on $\mathfrak{g}_{27}$. Now, using this claim we shall show that $\mathfrak{g}_{27}$ does not admit any mean convex half-flat structures. Like in the previous cases, by contradiction, let us suppose there exists a nilpotent Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{g}_{27}$ admitting a mean convex half-flat structure (ω, ρ). Then, we may assume that there exists an adapted basis (f_i) for (ω, ρ) such that $df^1 = df^2 = df^3 = 0$ and $V_5 = \langle f^1, f^2, f^3, f^4, f^5 \rangle$, so that $f_6 \in \mathfrak{z}(\mathfrak{g})$. Therefore,

$$df^k = -\sum_{\substack{i<j \\ i,j=1}}^{5} c_{ij}^k f^{ij}, \quad k = 4, 5, 6.$$

By imposing the unimodularity of $\mathfrak{g}$ and the half-flat condition for (ω, ρ), we get

$$\begin{aligned}
df^4 =& c_{15}^6 f^{13} - c_{14}^4 f^{14} - c_{15}^4 f^{15}, \\
df^5 =& c_{34}^5 f^{12} - (c_{24}^5 + c_{14}^6 + c_{23}^6) f^{13} - c_{14}^5 f^{14} + c_{14}^4 f^{15} - c_{23}^5 f^{23} \\
& - c_{24}^5 f^{24} - c_{34}^5 f^{34}, \\
df^6 =& - c_{12}^6 f^{12} - c_{13}^6 f^{13} - c_{14}^6 f^{14} - c_{15}^6 f^{15} - c_{23}^6 f^{23} - c_{24}^6 f^{24} + c_{12}^6 f^{34}.
\end{aligned} \tag{2.7}$$

Since $b_1(\mathfrak{g}) = 4$, there should exist a closed 1-form linearly independent from f^1, f^2 and f^3. Moreover, since $\ker d = V_4 \subset V_5 = \langle f^1, f^2, f^3, f^4, f^5 \rangle$, the matrix C associated with

$$d\colon \langle f^4, f^5 \rangle \to \Lambda^2 V_5 = \Lambda^2 \langle f^1, f^2, f^3, f^4, f^5 \rangle$$

must have rank equal to 1. This is equivalent to requiring that C is not the zero matrix and all the 2×2 minors of C vanish. After eliminating all the zero rows, we have

$$C = \begin{pmatrix} 0 & c_{34}^5 \\ c_{15}^6 & -c_{24}^5 - c_{14}^6 - c_{23}^6 \\ -c_{14}^4 & -c_{14}^5 \\ -c_{15}^4 & c_{14}^4 \\ 0 & -c_{23}^5 \\ 0 & -c_{24}^5 \\ 0 & -c_{34}^5 \end{pmatrix}.$$

By using that (f_i) is an adapted basis and (2.4), we get

$$(\beta_{m\bar{n}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{15}^4 & c_{15}^6 - ic_{14}^4 \\ 0 & c_{15}^6 + ic_{14}^4 & -c_{14}^5 - c_{13}^6 + c_{24}^6 - c_{23}^5 \end{pmatrix}.$$

Let us suppose $c_{15}^4 = 0$. Then, $(\beta_{m\bar{n}})$ being positive semi-definite implies $c_{14}^4 = c_{15}^6 = 0$, from which it follows that $\mathfrak{g}$ is 2-step nilpotent, so that we can discard this case since $\mathfrak{g}_{27}$ is 3-step nilpotent. Thus, we have to impose $c_{15}^4 \neq 0$. As a consequence, $d^2 f^i = 0$, $i = 4, 5, 6$, if and only if $c_{24}^5 = c_{34}^5 = c_{24}^6 = c_{23}^5 = c_{12}^6 = 0$, from which it follows that $b_1(\mathfrak{g}) = 4$ holds if and only if

$$c_{14}^5 = -\frac{c_{14}^4}{c_{15}^4}, \quad c_{14}^6 = \frac{c_{14}^4 c_{15}^6 - c_{15}^4 c_{23}^6}{c_{15}^4}.$$

Then, $\mathfrak{g}$ must have structure equations

$$\begin{aligned} df^1 &= df^2 = df^3 = 0, \\ df^4 &= c_{15}^6 f^{13} - c_{14}^4 f^{14} - c_{15}^4 f^{15}, \\ df^5 &= -\frac{c_{14}^4 c_{15}^6}{c_{15}^4} f^{13} + \frac{(c_{14}^4)^2}{c_{15}^4} f^{14} + c_{14}^4 f^{15}, \\ df^6 &= -c_{13}^6 f^{13} - \frac{c_{14}^4 c_{15}^6 - c_{15}^4 c_{23}^6}{c_{15}^4} f^{14} - c_{15}^6 f^{15} - c_{23}^6 f^{23}. \end{aligned} \tag{2.8}$$

By (2.8), $\mathfrak{g}$ has the same central and derived series as $\mathfrak{g}_{27}$. Note that, if $c_{23}^6 = 0$, then $\mathfrak{g}$ is almost abelian, so it cannot be isomorphic to $\mathfrak{g}_{27}$. Thus, we can suppose $c_{23}^6 \neq 0$. By [26], a 6-dimensional 3-step nilpotent Lie algebra having $b_1 = 4$ and admitting a half-flat structure must be isomorphic to either $\mathfrak{g}_{25}$ or $\mathfrak{g}_{27}$. In addition, $b_2(\mathfrak{g}_{25}) = 6$, while $b_2(\mathfrak{g}_{27}) = 7$. We show that we cannot have $b_2(\mathfrak{g}) = 7$, which is a contradiction. To this aim, we need to compute the space $\mathcal{Z}^2\mathfrak{g}$ of closed 2-forms of $\mathfrak{g}$. By a direct computation using (2.8) and $c_{23}^6 \neq 0$, it follows that $\dim \mathcal{Z}^2\mathfrak{g} = \dim \Lambda^2 V_4 + 2 = 8$. Therefore, in order to get $b_2(\mathfrak{g}) = 7$, we have to require that the space $\mathcal{B}^2$ of exact 2-forms of $\mathfrak{g}$ is 1-dimensional. This is equivalent to asking that the linear map

$$d|_{\langle f^4, f^5, f^6 \rangle} \colon \langle f^4, f^5, f^6 \rangle \to \Lambda^2 \mathfrak{g}^*$$

has rank equal to 1. Let E denote the matrix associated with $d|_{\langle f^4, f^5, f^6 \rangle}$ in the induced basis (f^{ij}) of $\Lambda^2\mathfrak{g}^*$. Eliminating all the zero rows, one has

$$E = \begin{pmatrix} c_{15}^6 & -\frac{c_{14}^4 c_{15}^6}{c_{15}^4} & -c_{13}^6 \\ -c_{14}^4 & \frac{(c_{14}^4)^2}{c_{15}^4} & -\frac{c_{14}^4 c_{15}^6 - c_{15}^4 c_{23}^6}{c_{15}^4} \\ -c_{15}^4 & c_{14}^4 & -c_{15}^6 \\ 0 & 0 & -c_{23}^6 \end{pmatrix}.$$

Then, E has rank 1 if and only if E is not the zero matrix and all the 2×2 minors of E vanish. Notice that the minor $c_{23}^6 c_{15}^4$ is different from zero, since we have already excluded both cases $c_{23}^6 = 0$ and $c_{15}^4 = 0$. Then, $\mathfrak{g}$ cannot be isomorphic to $\mathfrak{g}_{27}$ and we obtain a contradiction. This concludes the case $b_1 \geq 4$ and the proof of the theorem. $\square$

Remark 2.13. A 6-dimensional nilpotent Lie algebra $\mathfrak{g}$ with $b_1(\mathfrak{g}) = 2$ admitting mean convex half-flat structures also admits double SU(3)-structures. See Table 6.3 for an explicit example. This is not true for different values of the first Betti number.

Under the hypothesis of exactness, we can prove the following.

Theorem 2.14. *Let $\mathfrak{g}$ be a 6-dimensional nilpotent Lie algebra admitting an exact mean convex* $\mathrm{SL}(3,\mathbb{C})$*-structure. Then, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{18}$ or $\mathfrak{g}_{28}$. Moreover, up to a change of sign, every exact definite 3-form ρ on $\mathfrak{g}_{18}$ and $\mathfrak{g}_{28}$ is mean convex and $\mathfrak{g}_{28}$ is the only nilpotent Lie algebra admitting mean convex coupled structures, up to isomorphism.*

Proof. Among the 6-dimensional nilpotent Lie algebras admitting half-flat structures, as shown in the proof of [53, Theorem 4.1], the only Lie algebras that can admit exact $\mathrm{SL}(3,\mathbb{C})$-structures are isomorphic to $\mathfrak{g}_4$, $\mathfrak{g}_9$ or $\mathfrak{g}_{28}$. Therefore, by Theorem 2.10, $\mathfrak{g}_{28}$ is the only nilpotent Lie algebra among them which can admit a mean convex structure. In particular, a coupled mean convex structure on $\mathfrak{g}_{28}$ is given in Table 6.3. This example was first found in [53], up to a change of sign of the definite 3-form. For the remaining nilpotent Lie algebras $\mathfrak{g}_i$, for $i = 3, 5, 17, 18, 19, 20, 23, 26$, which can admit mean convex $\mathrm{SL}(3,\mathbb{C})$-structures by Theorem 2.10, we prove that $\mathfrak{g}_{18}$ is the only one that admits exact definite 3-forms. To see this, let (e^j) be the basis of $\mathfrak{g}_i^*$ as listed in Table 6.1. Then, the generic exact 3-form ρ on $\mathfrak{g}_i$ is given by $d\eta$, where

$$\eta = \sum_{i<j} p_{ij} e^{ij}, \quad p_{ij} \in \mathbb{R}. \tag{2.9}$$

By an explicit computation, one can show that, on $\mathfrak{g}_i$, for $i = 3, 17, 19, 23, 26$, $\lambda(\rho) = 0$, while, on $\mathfrak{g}_5$ and $\mathfrak{g}_{20}$, $\lambda(\rho) = p_{56}^4 > 0$. Finally, on $\mathfrak{g}_{18}$, $\lambda(\rho) = -4p_{56}^4$. Then, if $p_{56} \neq 0$, $\rho = d\eta$ is a definite 3-form on $\mathfrak{g}_{18}$. Moreover, $(e^1 - iJ_\rho e^1, e^3 - iJ_\rho e^3, e^5 - iJ_\rho e^5)$ is a basis for $\Lambda^{1,1}\mathfrak{g}_{18}^*$ and, with respect to this basis, the matrix $(\beta_{m\overline{n}})$ associated with the $(2,2)$-form $d\hat{\rho}$ is $\mathrm{diag}(0, 0, -4p_{56})$. Then, when $p_{56} < 0$, ρ is mean convex, otherwise $-\rho$ is. By a direct computation one can check that the same conclusions hold also for $\mathfrak{g}_{28}$. In particular, the generic exact 3-form $\rho = d\eta$, with η as in (2.9), is definite as long as $p_{56} \neq 0$. Moreover, $(e^1 - iJ_\rho e^1, e^3 - iJ_\rho e^3, e^5 - iJ_\rho e^5)$ is a basis of $\Lambda^{1,1}\mathfrak{g}_{28}^*$, for every exact definite ρ and, with respect to this basis, the matrix $(\beta_{m\overline{n}})$ associated with the $(2,2)$-form $d\hat{\rho}$ is $\mathrm{diag}(0, 0, -4p_{56})$. □

2.4 The Hitchin flow

In this section we study the mean convex property in relation to the Hitchin flow equations (2). We recall that the solution $(\omega(t), \rho(t))$ of (2) starting from a half-flat structure remains half-flat as long as it exists. However, the same does not happen in general for special classes of half-flat structures. Then, a natural question is whether the Hitchin flow equations preserve the mean convexity of the initial data $(\omega(0), \rho(0))$. A first example of solution preserving the mean convex condition of the initial data, up to change of sign of $\rho(0)$, was found in [54, Proposition 5.4]. In this case the initial structure is coupled. More generally, when the Hitchin flow solution $(\omega(t), \rho(t))$ preserves the coupled condition of the initial data, then $\rho(t) = f(t)\rho(0)$, where $f\colon I \to \mathbb{R}$ is a non-zero smooth function with $f(0) = 1$ (for more details see [54, Proposition 5.2]). Then, a coupled solution preserves the mean convexity of the initial data as long as it exists. Some further remarks can be made in other special cases. If $(\omega(t), \rho(t))$ is a solution to (2) starting from a strictly mean convex half-flat structure (ω, ρ), the solution remains mean convex, at least for small times, by continuity. This occurs, for instance, for double structures. In particular cases, the mean convex property of the double initial data is preserved for all times:

Proposition 2.15. *Let M be a connected 6-manifold endowed with a double structure (ω, ρ). If $(\omega(t), \rho(t))$ is a double solution to (2) defined on some $I \subseteq \mathbb{R}$, $0 \in I$, i.e., $d\hat{\rho}(t) = \nu_0(t)\omega^2(t)$ for each $t \in I$, for some smooth nowhere-vanishing function $\nu_0 \colon I \to \mathbb{R}$, then there exists a nowhere-vanishing smooth function $f \colon I \to \mathbb{R}$ such that $\omega(t) = f(t)\omega(0)$. Conversely, if $(\omega(t), \rho(t))$ is a solution to (2) with $\omega(t) = f(t)\omega(0)$, then it is a double solution.*

Proof. Let $(\omega(t), \rho(t))$ be a solution to (2) with $\omega(t) = f(t)\omega(0)$. From (2), one gets

$$d\hat{\rho}(t) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\omega(t)^2\right) = -\frac{1}{2}\frac{\partial}{\partial t}\left(f^2(t)\omega(0) \wedge \omega(0)\right) = -f(t)f'(t)\omega(0)^2.$$

Then, $\omega(t) = f(t)\omega(0)$ is a double solution with $\nu_0(t) = -\frac{d}{dt}\ln f(t)$. Conversely, if $d\hat{\rho}(t) = \nu_0(t)\omega(t)^2$, then

$$\frac{\partial}{\partial t}\omega(t) \wedge \omega(t) = -d\hat{\rho}(t) = -\nu_0(t)\omega(t)^2.$$

Since the wedge product with $\omega(t)$ is injective on 2-forms, this is equivalent to $\frac{\partial}{\partial t}\omega(t) = -\nu_0(t)\omega(t)$, whose unique solution is $\omega(t) = f(t)\omega(0)$, with $f(t) = e^{-\int_0^t \nu_0(s)ds}$. ☐

We now provide an explicit example of double solution to (2) and show that a double solution with double initial data may not exist.

Example 2.16. Consider the double SU(3)-structure (ω, ρ) on $\mathfrak{g}_{24}$ given in Table 6.3. The solution to the Hitchin flow equations with initial data (ω, ρ) is double and it is explicitly given by

$$\begin{aligned}
\omega(t) &= \left(1 - \frac{5}{2}t\right)^{\frac{1}{5}} \omega, \\
\rho(t) &= -\left(1 - \frac{5}{2}t\right)^{\frac{6}{5}} e^{123} + e^{145} + e^{246} + e^{356}.
\end{aligned}$$

In particular $d\hat{\rho}(t) = \nu_0(t)\omega^2(t)$ with $\nu_0(t) = (2-5t)^{-1} > 0$, for each t in the maximal interval of definition $I = (-\infty, \frac{2}{5})$. Consider now the double SU(3)-structure (ω, ρ) on $\mathfrak{g}_6$ given in Table 6.3. The solution to the Hitchin flow equation with initial data (ω, ρ) is given by

$$\begin{aligned}
\omega(t) &= f_1(t)\left(e^{15} - e^{24}\right) - f_2(t)e^{36}, \\
\rho(t) &= h_1(t)e^{123} + \left(h_2(t) - 1\right)e^{134} - e^{146} - e^{235} + e^{256} - e^{345} + h_2(t)e^{126},
\end{aligned}$$

where $f_1(t), f_2(t), h_1(t), h_2(t)$ satisfy the following autonomous ODE system:

$$\begin{cases} f_1' = \dfrac{1}{2f_1^3 f_2}\left(2h_2 - 1\right), \\ f_2' = -\dfrac{1}{2f_1^4 f_2}\left(2f_1 + f_2\left(2h_2 - 1\right)\right), \\ h_1' = -2f_1, \\ h_2' = -f_2, \end{cases}$$

with initial conditions $f_1(0) = f_2(0) = h_1(0) = 1$, $h_2(0) = 0$, which, by known theorems, admits a unique solution with given initial data. In particular, this solution is not a double solution. A direct computation shows that the eigenvalues $\lambda_i(t)$ of the matrix $(\beta_{m\overline{n}}(t))$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = \lambda_2 = \sqrt{-h_2^2 + h_1 + h_2}, \quad \lambda_3 = (1 - 2h_2)\sqrt{-h_2^2 + h_1 + h_2}.$$

In particular, the mean convex property is preserved for small times as expected.

To our knowledge, the question of whether the Hitchin flow preserves the mean convexity of the initial data when the $(2,2)$-form is not positive but just semi-positive is still open. Nonetheless, some easy considerations can be made in order to obtain a better understanding of the problem. Let M be a compact real analytic 6-dimensional manifold endowed with a half-flat mean convex SU(3)-structure (ω, ρ). Since the unique solution to (2) starting from (ω, ρ) is a one-parameter family of half-flat structures $(\omega(t), \rho(t))$, we can write

$$d\hat{\rho}(t) = (\nu_0(t)\omega(t) - \nu_2(t)) \wedge \omega(t),$$

where $\nu_0(t) \in C^\infty(M)$ and $\nu_2(t) \in \Lambda^2_8 M$ with respect to $J_{\rho(t)}$ for every $t \in I$, where I is the maximal interval of definition of the flow. Then, $d\hat{\rho}(t) \wedge \omega(t) = \nu_0(t)\omega(t)^3$ and, since $\nu_0(0) > 0$ by the mean convexity of the initial data, by continuity we have $\nu_0(t) > 0$ at least for small times. By (2), as long as $\nu_0(t) > 0$, the volume form $\omega(t)^3$ is pointwise decreasing:

$$\frac{\partial}{\partial t}(\omega(t)^3) = \frac{\partial}{\partial t}(\omega(t)^2) \wedge \omega(t) + \frac{\partial}{\partial t}\omega(t) \wedge \omega(t)^2 = -3d\hat{\rho}(t) \wedge \omega(t) = -3\nu_0(t)\omega(t)^3.$$

Moreover, $\omega(t)^2$ is a positive $(2,2)$-form with respect to $J_{\rho(t)}$ for all $t \in I$ and, from the second equation in (2), we know that $-\partial_t(\omega^2(t))$ remains a $(2,2)$-form with respect to $J_{\rho(t)}$ for each $t \in I$ such that $-\partial_t(\omega^2(t))\big|_{t=0} = 2d\hat{\rho}(0)$ is semi-positive. Then, the Hitchin flow solution preserves the mean convexity of the initial data if and only if $-\partial_t(\omega^2(t)) = 2d\hat{\rho}(t)$ remains semi-positive. The essential difficulty in this problem lies in the fact that the link between the positivity of $\omega^2(t)$ and the mean convexity of the initial data is not sufficient to ensure the mean convexity of the solution, since also the almost complex structure evolves in a non-linear way under the equation $\partial_t(\rho(t)) = d\omega(t)$.

Let us look at the behaviour of (2) on a specific mean convex example.

Example 2.17. Consider the mean convex half-flat structure (ω, ρ) on $\mathfrak{g}_{25}$ given in Table 6.3 and consider the family of solutions to the second equation in (2), starting from (ω, ρ):

$$\begin{aligned} \omega(t) &= -a_1(t)e^{13} + \frac{1}{a_2(t)}e^{45} + a_2(t)e^{26}, \\ \rho(t) &= e^{156} + b_1(t)e^{124} - e^{235} - e^{346} + b_2(t)(e^{125} - e^{234}), \end{aligned}$$

where $a_1(t), a_2(t), b_1(t), b_2(t)$ satisfy the ODE system

$$\begin{cases} a_1' = -\dfrac{1}{2a_1a_2}\left(2a_2^2b_2+1\right), \\ a_2' = \dfrac{1}{2a_1^2}\left(2a_2^2b_2-1\right) \end{cases} \tag{2.10}$$

and the normalization condition $\sqrt{b_1-b_2^2}=a_1$, with initial data $a_1(0)=a_2(0)=b_1(0)=1$, $b_2(0)=0$. This system defines a family of solutions to $\frac{1}{2}\partial_t(\omega(t)^2)=-d\hat{\rho}(t)$ depending on $b_2(t)$. Then, if $b_2(t)=a_1(t)-1$, for instance, $d\hat{\rho}(t)$ is not semi-positive, at least for small times $t>0$. Anyway, the unique solution to (2) starting from (ω,ρ), given by (2.10), together with

$$\begin{cases} b_1' = -\dfrac{1}{a_2}, \\ b_2' = a_2, \end{cases}$$

preserves the mean convexity of the initial data.

By a direct computation, one may show that the mean convexity of (ω,ρ) is never strict for all the examples proposed in Table 6.3 on $\mathfrak{g}_{10}$, $\mathfrak{g}_{13}$, $\mathfrak{g}_{16}$, $\mathfrak{g}_{22}$, $\mathfrak{g}_{29}$, $\mathfrak{g}_{30}$, $\mathfrak{g}_{31}$, $\mathfrak{g}_{32}$, $\mathfrak{g}_{33}$, since there is always a vanishing eigenvalue λ_i of the matrix associated with $d\hat{\rho}$. This follows also by Proposition 2.9, since these Lie algebras do not admit double structures (see [25]). This means that, a priori, a solution to the Hitchin flow equations starting from one of these pair might not preserve the mean convexity of the initial data. Anyway, we show that this happens in our examples, at least for small times. Therefore, it would be interesting to determine if this is always the case.

Example 2.18. Let (ω,ρ) be the mean convex half-flat structure on $\mathfrak{g}_{10}$ given in Table 6.3. The unique solution to the Hitchin flow equations starting from (ω,ρ) is given by

$$\begin{aligned} \omega(t) &= -\frac{a_1(t)}{2}e^{13}+a_2(t)e^{46}-\frac{1}{a_2(t)}e^{25}, \\ \rho(t) &= -\frac{1}{2}e^{236}-\left(\frac{1}{2}+b_1(t)\right)e^{234}+\frac{1}{2}e^{345}+e^{156} \\ &\quad +(b_1(t)-1)e^{145}+b_2(t)e^{124}+b_1(t)e^{126}, \end{aligned}$$

where $a_1(t), a_2(t), b_1(t), b_2(t)$ satisfy the following ODE system:

$$\begin{cases} a_1' = -\dfrac{a_2^2(6b_1+1)+1}{2a_1a_2}, \\ a_2' = \dfrac{a_2^2(6b_1+1)-1}{2a_1^2}, \\ b_1' = a_2, \\ b_2' = -\dfrac{1}{a_2}, \end{cases} \tag{2.11}$$

with initial conditions $a_1(0)=a_2(0)=b_2(0)=1$, $b_1(0)=0$. The mean convex property is preserved for small times since the eigenvalues $\lambda_i(t)$ of the matrix $(\beta_{m\overline{n}}(t))$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1=0, \quad \lambda_2=a_1, \quad \lambda_3=(6b_1+1)a_1.$$

Example 2.19. Consider the mean convex half-flat structure (ω, ρ) on $\mathfrak{g}_{13}$ given in Table 6.3. The solution to (2) starting from (ω, ρ) is thus given by

$$\begin{aligned}\omega(t) &= a_1(t)e^{13} + a_2(t)e^{46} + \frac{1}{a_2(t)}e^{25},\\ \rho(t) &= -e^{236} + e^{234} - e^{345} + e^{156} + (1 + b_1(t))e^{145} - b_2(t)e^{124} + b_1(t)e^{126},\end{aligned}$$

where $a_1(t), a_2(t), b_1(t), b_2(t)$ satisfy the following ODE system:

$$\begin{cases} a_1' = -\dfrac{1 + a_2^2(2b_1 + 1)}{2a_1a_2}, \\ a_2' = \dfrac{a_2^2(2b_1 + 1) - 1}{2a_1^2}, \\ b_1' = a_2, \\ b_2' = -\dfrac{1}{a_2}, \end{cases} \tag{2.12}$$

with initial conditions $a_1(0) = a_2(0) = b_2(0) = 1$, $b_1(0) = 0$. The mean convex property is preserved for small times since the eigenvalues $\lambda_i(t)$ of the matrix $(\beta_{m\overline{n}}(t))$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = 0, \quad \lambda_2 = a_1, \quad \lambda_3 = (2b_1 + 1)a_1.$$

Example 2.20. Let (ω, ρ) be the mean convex half-flat structure on $\mathfrak{g}_{16}$ given in Table 6.3. The solution to the Hitchin flow equations with initial data (ω, ρ) is given by

$$\begin{aligned}\omega(t) &= a_1(t)e^{13} + a_2(t)e^{26} - \frac{1}{a_2(t)}e^{45},\\ \rho(t) &= (2 + b_1(t))e^{124} - \frac{\sqrt{2}}{2}e^{156} + (b_1(t) - 1)e^{235} + \frac{\sqrt{2}}{2}e^{346} + b_2(t)(e^{125 - e^{234}}),\end{aligned}$$

with coefficients satisfying

$$\begin{cases} a_1' = \dfrac{2b_2 - a_2^2(1 + 2b_1)}{4a_1a_2}, \\ a_2' = \dfrac{2b_2 + a_2^2(1 + 2b_1)}{4a_1^2}, \\ b_1' = a_2, \\ b_2' = -\dfrac{1}{a_2} \end{cases}$$

and having initial conditions $a_1(0) = a_2(0) = 1$, $b_1(0) = b_2(0) = 0$. The eigenvalues associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = 0, \quad \lambda_2 = \frac{a_1(2b_1 + 1)}{(b_1 - 1)^2}, \quad \lambda_3 = \frac{2\sqrt{2}a_1b_2}{b_1 - 1}.$$

Therefore, the mean convexity of the initial data is preserved for small times.

Example 2.21. Let (ω, ρ) be the mean convex half-flat structure on $\mathfrak{g}_{22}$ given in Table 6.3. The solution to the Hitchin flow equations with this initial data is given by

$$\begin{aligned}\omega(t) &= a_1(t)e^{16} + a_2(t)e^{23} + \frac{1}{a_1(t)}e^{45},\\ \rho(t) &= b_1(t)e^{124} - e^{135} - e^{256} - e^{346} + b_2(t)(e^{125} - e^{134}),\end{aligned}$$

with coefficients satisfying

$$\begin{cases} a_1' = \dfrac{a_1^2 - 2b_2}{2a_2^2}, \\ a_2' = -\dfrac{a_1^2 + 2b_2}{2a_1a_2}, \\ b_1' = -a_1, \\ b_2' = \dfrac{1}{a_1} \end{cases}$$

and having initial conditions $a_1(0) = a_2(0) = b_1(0) = 1$, $b_2(0) = 0$. The eigenvalues $\lambda_i(t)$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = a_2, \quad \lambda_2 = 0, \quad \lambda_3 = 2a_2b_2.$$

Therefore, the mean convexity of the initial data is preserved for small times.

Example 2.22. Consider the mean convex half-flat structure (ω, ρ) on $\mathfrak{g}_{29}$ given in Table 6.3. The solution to the Hitchin flow equations with initial data (ω, ρ) is given by

$$\begin{aligned}\omega(t) &= a_1(t)(e^{13} + e^{24}) - \frac{1}{a_1(t)}e^{56},\\ \rho(t) &= b_1(t)(e^{126} - e^{145}) + b_2(t)e^{235} - e^{346},\end{aligned}$$

with coefficients satisfying

$$\begin{cases} a_1' = \dfrac{b_1 - 2b_2}{2b_1b_2}, \\ b_1' = -\dfrac{1}{a_1}, \\ b_2' = \dfrac{1}{a_1} \end{cases}$$

and having initial conditions $a_1(0) = b_1(0) = b_2(0) = 1$. The eigenvalues of $d\hat{\rho}(t)$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{b_1^2(2b_2 - b_1)}{a_1}.$$

In particular, the mean convexity of (ω, ρ) is preserved for small times.

Example 2.23. Let (ω, ρ) be the mean convex half-flat structure on $\mathfrak{g}_{30}$ given in Table 6.3. The solution to the Hitchin flow equations starting from (ω, ρ) is given by

$$\begin{aligned}\omega(t) &= a_1(t)(e^{13} - e^{24}) + \frac{1}{a_1(t)}e^{56},\\ \rho(t) &= (b_1(t) - 1)e^{126} + e^{236} + e^{145} + (1 - b_1(t))e^{345} + e^{146} + e^{125},\end{aligned}$$

with coefficients satisfying

$$\begin{cases} a_1' = -\dfrac{b_1^3 - 3b_1^2 + 2b_1 + 1}{2a_1^2}, \\ b_1' = \dfrac{1}{a_1} \end{cases}$$

and having initial conditions $a_1(0) = 1$, $b_1(0) = 0$. The eigenvalues $\lambda_i(t)$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{a_1(b_1^3 - 3b_1^2 + 2b_1 + 1)}{(b_1 - 1)^2}.$$

Therefore, the mean convexity of the initial data is preserved for small times.

Example 2.24. Consider the mean convex half-flat structure (ω, ρ) on $\mathfrak{g}_{31}$ given in Table 6.3. The solution to the Hitchin flow equations with initial data (ω, ρ) is explicitly given by

$$\begin{aligned} \omega(t) &= -\sqrt{1-2t}\, e^{14} - e^{35} + e^{26}, \\ \rho(t) &= -e^{346} - e^{245} + e^{156} - (1-2t)\, e^{123}. \end{aligned}$$

The eigenvalues associated with $d\hat{\rho}(t)$ are

$$\lambda_1(t) = 0, \quad \lambda_2(t) = \frac{1}{\sqrt{1-2t}}, \quad \lambda_3(t) = \sqrt{1-2t}.$$

Therefore, the mean convexity of (ω, ρ) is preserved by the solution to (2), as long as it exists.

Example 2.25. Let (ω, ρ) be the mean convex half-flat structure on $\mathfrak{g}_{32}$ given in Table 6.3. The solution to the Hitchin flow equations starting from (ω, ρ) is given by

$$\begin{aligned} \omega(t) &= -a_1(t)(e^{24} + \sqrt{2}\, e^{13}) - \frac{1}{a_1(t)} e^{56}, \\ \rho(t) &= (b_1(t) - 1)e^{125} + e^{146} - e^{236} + (2 + b_1(t))e^{345}, \end{aligned}$$

with coefficients satisfying

$$\begin{cases} a_1' = -\dfrac{2b_1 + 1}{4a_1^2} \\ b_1' = \dfrac{1}{a_1} \end{cases}$$

and having initial conditions $a_1(0) = 1$, $b_1(0) = 0$. The eigenvalues $\lambda_i(t)$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{-2b_1^2 + b_1 + 1}{\sqrt{2}a_1(b_1 + 2)}.$$

Therefore, the mean convexity of the initial data is preserved for small times.

Example 2.26. Let (ω, ρ) be the mean convex half-flat structure on $\mathfrak{g}_{33}$ given in Table 6.3. The solution to the Hitchin flow equations starting from (ω, ρ) is given by

$$\begin{aligned} \omega(t) &= -a_1(t)(e^{13} + e^{24}) - a_2(t)e^{56}, \\ \rho(t) &= -b_1(t)e^{125} + e^{146} - e^{236} + e^{345}, \end{aligned}$$

with coefficients satisfying

$$\begin{cases} a_1' = -\dfrac{1}{2a_1\sqrt{b_1}} \\ a_2' = \dfrac{a_2}{2a_1^2\sqrt{b_1}} \\ b_1' = -a_2 \end{cases}$$

and having initial conditions $a_1(0) = a_2(0) = b_1(0) = 1$. The eigenvalues $\lambda_i(t)$ associated with $d\hat{\rho}(t)$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{b_1}.$$

Therefore, the mean convexity of the initial data is preserved for small times.

2.5 Tamed closed $\mathrm{SL}(3, \mathbb{C})$-structures on solvable Lie algebras

Let M be a 6-dimensional smooth manifold endowed with a closed $\mathrm{SL}(3, \mathbb{C})$-structure $\rho \in \Lambda^3(M)$ inducing the almost complex structure $J_\rho \in \mathrm{End}(TM)$.

Definition 2.27. The 3-form ρ is called *tamed* if there exists a symplectic form $\Omega \in \Lambda^2(M)$ taming J_ρ.

As already observed in [35], a compact 6-manifold M cannot admit a mean convex $\mathrm{SL}(3, \mathbb{C})$-structure ρ tamed by a symplectic form Ω, since

$$\int_M d\hat{\rho} \wedge \Omega = \int_M \hat{\rho} \wedge d\Omega = 0.$$

Notice that, when the normalization condition $\rho \wedge \hat{\rho} = \frac{2}{3}\Omega^3$ is satisfied and $d\Omega = 0$, the pair (Ω, ρ) defines a symplectic half-flat structure.

Since we consider invariant tamed closed $\mathrm{SL}(3, \mathbb{C})$-structures on solvmanifolds, we can work in the same way as in the previous sections at the level of solvable unimodular Lie algebras. We then prove the following result.

Theorem 2.28. *Let $\mathfrak{g}$ be a 6-dimensional unimodular solvable (non-abelian) Lie algebra. Then, $\mathfrak{g}$ admits tamed closed $\mathrm{SL}(3, \mathbb{C})$-structures if and only if it has symplectic half-flat structures. If $\mathfrak{g}$ is nilpotent, then it is isomorphic to $\mathfrak{g}_{24}$ or $\mathfrak{g}_{31}$ as listed in Table 6.1. If $\mathfrak{g}$ is solvable, then it is isomorphic to one among $\mathfrak{g}_{6,38}^0$, $\mathfrak{g}_{6,54}^{0,-1}$, $\mathfrak{g}_{6,118}^{0,-1,-1}$, $\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1)$, $A_{5,7}^{-1,\beta,-\beta}\oplus\mathbb{R}$, $A_{5,17}^{0,0,1}\oplus\mathbb{R}$ and $A_{5,17}^{\alpha,-\alpha,1}\oplus\mathbb{R}$, as listed in Table 6.2. Moreover, all nine Lie algebras admit closed $\mathrm{SL}(3, \mathbb{C})$-structures tamed by a symplectic form Ω such that $d\Omega^{1,1} \neq 0$.*

Proof. First we prove the theorem in the nilpotent case. 6-dimensional symplectic nilpotent Lie algebras were classified in [73] (see also [121]) and their structure equations are listed in Table 6.1. For each of these Lie algebras, we consider a pair $(\rho, \Omega) \in \Lambda^3\mathfrak{g}_i^* \times \Lambda^2\mathfrak{g}_i^*$ explicitly given by

$$\rho = \sum_{i<j<k} p_{ijk}\, e^{ijk}, \quad \Omega = \sum_{r<s} h_{rs}\, e^{rs},$$

with $p_{ijk}, h_{rs} \in \mathbb{R}$, and impose the two conditions $d\rho = 0$ and $d\Omega = 0$, which are both linear in the coefficients p_{ijk}, h_{rs}. Then, Ω is a symplectic form provided that it is non-degenerate,

Table 2.1: Tamed closed SL$(3,\mathbb{C})$-structure on 6-dimensional nilpotent Lie algebras

$\mathfrak{g}$	Structure constants	Tamed closed SL$(3,\mathbb{C})$-structure
$\mathfrak{g}_{24}$	$(0,0,0,e^{12},e^{13},e^{23})$	$\rho=-e^{125}-e^{146}-e^{156}-e^{236}-e^{245}-e^{345}-e^{356}$ $\Omega=e^{13}+\frac{1}{2}e^{14}-\frac{1}{2}e^{24}+e^{26}+e^{35}+e^{36}$
$\mathfrak{g}_{31}$	$(0,0,0,0,e^{12},e^{13})$	$\rho=e^{123}+2e^{145}+e^{156}+e^{235}+e^{246}+e^{345}$ $\Omega=e^{16}-e^{25}-e^{34}+e^{36}$

i.e., $\Omega^3\neq 0$. By [38, Lemma 3.1], a real Lie algebra $\mathfrak{g}$ endowed with an almost complex structure J such that $J\mathfrak{z}(\mathfrak{g})\cap[\mathfrak{g},\mathfrak{g}]\neq\{0\}$ cannot admit a symplectic form Ω taming J. If we assume $\lambda(\rho)<0$, we may then apply this result on each $\mathfrak{g}_i$ by considering the almost complex structure J_ρ induced by ρ. We notice that, for every $\mathfrak{g}_i$ listed in Table 6.1, $e_6\in\mathfrak{z}(\mathfrak{g}_i)$. A direct computation on each $\mathfrak{g}_i$ for $i=3,4,5,6,7,8,9,10,13,18,19,20,28,29,30$, shows that $J_\rho e_6\in[\mathfrak{g}_i,\mathfrak{g}_i]$, for every J_ρ induced by a closed 3-form ρ. On $\mathfrak{g}_i$, for $i=23,26,33$, the same obstruction holds since an explicit computation shows that the map

$$\pi\circ J_\rho\colon \mathfrak{z}(\mathfrak{g}_i)\to\mathfrak{g}_i/[\mathfrak{g}_i,\mathfrak{g}_i]$$

has non-trivial kernel, where π denotes the projection onto $\mathfrak{g}_i/[\mathfrak{g}_i,\mathfrak{g}_i]$. This means that, for each ρ, one can find a non-zero element in the center of $\mathfrak{g}_i$ whose image under J_ρ lies entirely in $[\mathfrak{g}_i,\mathfrak{g}_i]$. For all the other cases, let $\Omega=\Omega^{1,1}+\Omega^{2,0}+\Omega^{0,2}$ be the decomposition of Ω in types with respect to J_ρ and denote by ω the $(1,1)$-form $\Omega^{1,1}:=\frac{1}{2}(\Omega+J_\rho\Omega)$. Then, in order to have a closed SL$(3,\mathbb{C})$-structure tamed by Ω, we have to require that ω is positive, i.e., that the symmetric 2-tensor $g:=\omega(\cdot,J_\rho\cdot)$ is positive definite. Denote by $g_{ij}:=g(e_i,e_j)$ the coefficients of g with respect the dual basis $(e_1,\dots,e_6)$ of $\mathfrak{g}$. Then, a direct computation on $\mathfrak{g}_i$, for $i=11,12,21,22,27$, shows that g_{66} always vanishes, so we may discard these cases as well. We may then restrict our attention to the remaining Lie algebras $\mathfrak{g}_{24}$ and $\mathfrak{g}_{31}$. As shown in [28, Theorem 2.4], these are the only 6-dimensional non-abelian nilpotent Lie algebras carrying a symplectic half-flat structure, up to isomorphism. Explicit examples of closed SL$(3,\mathbb{C})$-structures tamed by a symplectic form Ω such that $d\Omega^{1,1}\neq 0$ are given in Table 2.1. This proves the first part of the theorem.

Using the classification results in [103, Theorem 2] for 6-dimensional symplectic unimodular (non-nilpotent) solvable Lie algebras, for each Lie algebra one can compute the metric coefficients g_{ij} of g with respect to the basis $(e_1,\dots,e_6)$ for $\mathfrak{g}$ as listed in Table 6.2. It turns out that, if $\mathfrak{g}$ is one among $\mathfrak{g}_{6,3}^{0,-1}$, $\mathfrak{g}_{6,10}^{0,0}$, $\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$, $\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}$, $\mathfrak{g}_{6,21}^{0}$, $\mathfrak{g}_{6,36}^{0,0}$, $\mathfrak{g}_{6,78}$, $A_{5,8}^{-1}\oplus\mathbb{R}$, $A_{5,13}^{-1,0,\gamma}$, $A_{5,14}^{0}\oplus\mathbb{R}$, $A_{5,15}^{-1}\oplus\mathbb{R}$, $A_{5,17}^{0,0,\gamma}\oplus\mathbb{R}$, $A_{5,18}^{0}\oplus\mathbb{R}$ and $A_{5,19}^{-1,2}\oplus\mathbb{R}$, each closed definite 3-form ρ induces a J_ρ such that $g_{11}=0$. In a similar way, if $\mathfrak{g}$ is $\mathfrak{g}_{6,15}^{-1}$ or $\mathfrak{g}_{6,18}^{-1,-1}$, then $g_{44}=0$, while when $\mathfrak{g}$ is $\mathfrak{n}_{6,84}^{\pm1}$, $\mathfrak{e}(2)\oplus\mathbb{R}^3$ or $\mathfrak{e}(1,1)\oplus\mathbb{R}^3$, we have $g_{33}=0$. Finally, when $\mathfrak{g}=\mathfrak{e}(1,1)\oplus\mathfrak{h}$, then $g_{66}=0$. In some other cases, g cannot ever be positive definite since, for each closed ρ inducing an almost complex structure J_ρ, one has $g_{rr}=-g_{kk}$ for some $r\neq k$. In particular, when $\mathfrak{g}=\mathfrak{g}_{6,70}^{0,0}$, then $g_{11}=-g_{22}$, when $\mathfrak{g}=\mathfrak{e}(2)\oplus\mathfrak{e}(2)$, then $g_{55}=-g_{66}$ and, when $\mathfrak{g}$ is $\mathfrak{e}(2)\oplus\mathfrak{e}(1,1)$ or $\mathfrak{e}(2)\oplus\mathfrak{h}$, then $g_{22}=-g_{33}$. As shown in [47, Propositions 3.1, 4.1 and 4.3], for the remaining Lie algebras $\mathfrak{g}_{6,38}^{0}$, $\mathfrak{g}_{6,54}^{0,-1}$, $\mathfrak{g}_{6,118}^{0,-1,-1}$, $\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1)$, $A_{5,7}^{-1,\beta,-\beta}\oplus\mathbb{R}$, $A_{5,17}^{0,0,1}\oplus\mathbb{R}$, $A_{5,17}^{\alpha,-\alpha,1}\oplus\mathbb{R}$, as listed in Table 6.2, a symplectic half-flat structure always exists. Explicit

Table 2.2: Tamed closed SL(3, ℂ)-structure on unimodular symplectic non-nilpotent solvable Lie algebras

$\mathfrak{g}$	Structure constants	Tamed closed SL(3, ℂ)-structure
$\mathfrak{g}_{6,38}^{0}$	$(e^{23}, -e^{36}, e^{26}, e^{26}-e^{56}, e^{36}+e^{46}, 0)$	$\rho = -e^{124} - e^{135} + e^{236} - e^{456}$ $\Omega = -2e^{16} + e^{23} - e^{25} + e^{34}$
$\mathfrak{g}_{6,54}^{0,-1}$	$(e^{16}+e^{35}, -e^{26}+e^{45}, e^{36}, -e^{46}, 0, 0)$	$\rho = e^{125} - e^{136} + e^{246} + e^{345}$ $\Omega = e^{14} + e^{23} + e^{34} + \frac{4}{3}e^{56}$
$\mathfrak{g}_{6,118}^{0,-1,-1}$	$(-e^{16}+e^{25}, -e^{15}-e^{26}, e^{36}-e^{45}, e^{35}+e^{46}, 0, 0)$	$\rho = e^{126} + e^{135} + e^{145} - e^{245} + e^{346}$ $\Omega = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$	$(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45})$	$\rho = -e^{125} - e^{126} + e^{135} - e^{145} - e^{246} + e^{345} + e^{346}$ $\Omega = -e^{14} + e^{23} - 2e^{56}$
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$	$(e^{15}, -e^{25}, \beta e^{35}, -\beta e^{45}, 0, 0), \quad -1 \leq \beta < 0$	$\rho = -e^{126} - e^{145} - e^{235} - e^{346}$ $\Omega = -e^{13} + e^{15} + e^{24} + e^{56}$
$A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$	$(\alpha e^{15}+e^{25}, -e^{15}+\alpha e^{25}, -\alpha e^{35}+e^{45}, -e^{35}-\alpha e^{45}, 0, 0), \quad \alpha > 0$	$\rho = e^{125} + e^{136} + e^{145} + e^{246} - e^{345}$ $\Omega = -e^{14} + e^{23} - e^{56}$
$A_{5,17}^{0,0,1} \oplus \mathbb{R}$	$(e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0)$	$\rho = e^{145} - e^{136} + e^{246} + e^{235} - e^{346} - e^{456}$ $\Omega = e^{12} - e^{13} + e^{24} - e^{56}$

examples of closed SL(3, ℂ)-structures ρ tamed by a symplectic form Ω such that $d\Omega^{1,1} \neq 0$ are given in Table 2.2. □

Remark 2.29. 1. By [47, Remarks 3.2 and 4.4], the solvable Lie groups corresponding to each solvable Lie algebra admitting closed tamed SL(3, ℂ)-structures admit compact quotients by lattices (for further details see [15, 44, 126, 131]).

2. Let $\mathfrak{g}$ be a real 6-dimensional Lie algebra endowed with a closed SL(3, ℂ)-structure ρ tamed by a symplectic 2-form Ω. Then, the 3-form

$$\varphi = \rho + \Omega \wedge dt$$

defines a closed G_2-structure on $\mathfrak{g} \oplus \mathbb{R}$. Therefore, as an application of Theorem 2.28, we classify decomposable solvable Lie algebras of the form $\mathfrak{g} \oplus \mathbb{R}$ admitting a closed G_2-structure. In particular, in the nilpotent case, this result was already obtained in [27].

Chapter 3

Closed G_2-structures

In this chapter, we characterize the structure of a 7-dimensional Lie algebra with non-trivial center endowed with a closed G_2-structure. Using this result, we classify all unimodular Lie algebras with non-trivial center admitting closed G_2-structures, up to isomorphism, and we show that six of them arise as the contactization of a symplectic Lie algebra. The main contents and results of this chapter were published in [58].

3.1 Central extensions and contactizations

In this section, we review the general construction of central extensions and contactizations of a given Lie algebra. Let $\mathfrak{h}$ be a real Lie algebra of dimension $n \geq 2$ and denote by $[\cdot,\cdot]_{\mathfrak{h}}$ its Lie bracket. Consider a 2-form $\omega_0 \in \Lambda^2\mathfrak{h}^*$ that is closed with respect to the Chevalley-Eilenberg differential $d_{\mathfrak{h}}$ of $\mathfrak{h}$.

Definition 3.1. The *central extension* of $(\mathfrak{h}, \omega_0)$ is the $n+1$-dimensional real Lie algebra

$$\mathfrak{g} := \mathfrak{h} \oplus \mathbb{R}z,$$

endowed with the Lie bracket

$$[z, \mathfrak{h}] = 0, \qquad [x, y] = -\omega_0(x, y)z + [x, y]_{\mathfrak{h}},\ x, y \in \mathfrak{h}. \tag{3.1}$$

It is clear from this definition that the vector z belongs to the center of $\mathfrak{g}$. More precisely, the center of $\mathfrak{g}$ is given by

$$\mathfrak{z}(\mathfrak{g}) = (\mathfrak{z}(\mathfrak{h}) \cap \mathrm{Rad}(\omega_0)) \oplus \mathbb{R}z,$$

where $\mathrm{Rad}(\omega_0) := \{x \in \mathfrak{h} \mid \omega_0(x, y) = 0,\ y \in \mathfrak{h}\}$.

Remark 3.2. The central extension of $(\mathfrak{h}, \omega_0)$ only depends on the cohomology class $[\omega_0] \in \mathcal{H}^2(\mathfrak{h})$. Indeed, different representatives of $[\omega_0]$ give rise to isomorphic central extensions.

In the following, we shall denote by $\theta \in \mathfrak{g}^*$ the dual of z with respect to the complement $\mathfrak{h}$, namely $\theta(z) = 1$, $\theta|_{\mathfrak{h}} = 0$. Let d denote the Chevalley-Eilenberg differential of $\mathfrak{g}$. Then, $d\theta$ defines an exact 2-form on $\mathfrak{g}$ that coincides with ω_0 on $\mathfrak{h}$ and satisfies $\iota_z d\theta = 0$. Thus, we can write $d\theta = \omega_0$ on $\mathfrak{g}$ by extending ω_0 to $\mathfrak{g}$ via the condition $\iota_z\omega_0 = 0$. When ω_0 is zero, the Lie algebra $\mathfrak{g}$ is simply the direct sum of $\mathfrak{h}$ and the abelian Lie algebra $\mathbb{R}$. When $(\mathfrak{h}, \omega_0)$ is a symplectic Lie algebra of dimension $2n$, the previous construction gives rise to a contact Lie algebra $(\mathfrak{g}, \theta)$ of dimension $2n+1$ with center $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}z$. Indeed, $\theta \in \mathfrak{g}^*$ satisfies the condition $(d\theta)^n \wedge \theta \neq 0$ and so it is a contact form on $\mathfrak{g}$ (see [1]). Moreover, $\mathrm{Rad}(\omega_0) = \{0\}$, whence $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}z$. Notice that z is the *Reeb vector* of the contact structure θ, as $\theta(z) = 1$. This motivates the following.

Definition 3.3. The contact Lie algebra $(\mathfrak{g}, \theta)$ obtained from the symplectic Lie algebra $(\mathfrak{h}, \omega_0)$ via the construction described above is called the *contactization* of $(\mathfrak{h}, \omega_0)$.

It is easy to characterize contact Lie algebras arising as the contactization of a symplectic Lie algebra, as the next result shows (see also [34]).

Proposition 3.4. *A contact Lie algebra $(\mathfrak{g}, \theta)$ is the contactization of a symplectic Lie algebra $(\mathfrak{h}, \omega_0)$ if and only if the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is not trivial.*

Proof. If $(\mathfrak{g}, \theta)$ is the contactization of a symplectic Lie algebra $(\mathfrak{h}, \omega_0)$, then the assertion is true. Conversely, let us assume that $(\mathfrak{g}, \theta)$ is a contact Lie algebra of dimension $2n+1$ with non-trivial center. Then, $\mathfrak{z}(\mathfrak{g})$ is 1-dimensional and it is spanned by the Reeb vector z of the contact structure θ (cf. [6, Proposition 1]). Consequently, we can consider the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$, where the $2n$-dimensional subspace $\mathfrak{h} := \ker\theta$ is a Lie algebra with respect to the bracket

$$[x, y]_{\mathfrak{h}} := [x, y] - \theta\left([x, y]\right) z, \quad x, y \in \mathfrak{h}.$$

Let ω_0 be the 2-form on $\mathfrak{h}$ defined as $\omega_0(x, y) = d\theta(x, y)$, for all $x, y \in \mathfrak{h}$. A direct computation shows that ω_0 is closed with respect to the Chevalley-Eilenberg differential $d_{\mathfrak{h}}$ of $\mathfrak{h}$. Moreover, ω_0 is non-degenerate. Indeed, $(d\theta)^n \wedge \theta$ is a volume form on $\mathfrak{g}$ and contracting it with z gives $(d\theta)^n \neq 0$, as $\theta(z) = 1$ and $\iota_z d\theta = -\theta([z, \cdot]) = 0$. Therefore, $(\mathfrak{g}, \theta)$ is the contactization of the symplectic Lie algebra $(\mathfrak{h}, \omega_0)$. $\square$

3.2 Closed G_2-structures on central extensions and contactizations

In this section, we investigate the structure of a 7-dimensional Lie algebra $\mathfrak{g}$ with non-trivial center endowed with a closed G_2-structure. From the previous section, $\mathfrak{g}$ is the central extension of a 6-dimensional Lie algebra $\mathfrak{h}$ endowed with a closed 2-form ω_0. As a special case of this, when $\omega_0^3 \neq 0$, $\mathfrak{g}$ admits both a closed G_2-structure φ and a contact structure θ.

More generally, every 7-manifold admitting G_2-structures is spin, therefore it also admits almost contact structures. The interplay between these structures and the existence of contact structures on 7-manifolds endowed with special types of G_2-structures have been recently investigated in [8, 32]. Moreover, it is possible to construct examples of compact 7-manifolds with both a G_2-structure and a contact structure as follows. In [17], Boothby and Wang showed that an even-dimensional compact manifold M endowed with a symplectic form ω_0 with integral periods is the base of a principal $\mathbb{S}^1$-bundle $\pi\colon N \to M$ having a connection 1-form θ that defines a contact structure on N and satisfies the structure equation $d\theta = \pi^*\omega_0$. If M is 6-dimensional and it also admits a definite 3-form ρ and a non-degenerate 2-form

$\widetilde{\omega}$ which is taming for the almost complex structure J induced by ρ and one of the two orientations of M, then the total space N has a natural G_2-structure defined by the 3-form $\varphi = \pi^*\widetilde{\omega} \wedge \theta + \pi^*\rho$. A special case of this construction occurs when the pair $(\widetilde{\omega}, \rho)$ defines an SU(3)-structure on M. On the other hand, a G_2-structure which is invariant under a free $\mathbb{S}^1$-action on a 7-manifold N induces an SU(3)-structure on the orbit space $N/\mathbb{S}^1$ (see [7]). Furthermore, the G_2-structure $\varphi = \pi^*\widetilde{\omega} \wedge \theta + \pi^*\rho$ on the total space of the $\mathbb{S}^1$-bundle $\pi\colon N \to M$ is closed whenever the 2-form $\widetilde{\omega}$ on M is symplectic and ρ satisfies the condition $d\rho = -\omega_0 \wedge \widetilde{\omega}$.

Going back to the Lie algebra setting, we can state the next results, which are reminiscent of the construction described above. We begin with the following.

Proposition 3.5. *Let $\mathfrak{h}$ be a 6-dimensional Lie algebra and let ω_0 be a closed 2-form on it. Assume that $\mathfrak{h}$ admits a definite 3-form ρ and a symplectic form $\widetilde{\omega}$ such that*

a) $\widetilde{\omega}$ is a taming form for the almost complex structure J_ρ on $\mathfrak{h}$ induced by ρ and one of the two orientations of $\mathfrak{h}$,

b) $d\rho = -\widetilde{\omega} \wedge \omega_0$.

Then, the 7-dimensional Lie algebra $\mathfrak{g} := \mathfrak{h} \oplus \mathbb{R}z$ obtained as the central extension of $(\mathfrak{h}, \omega_0)$ is endowed with a closed G_2-structure defined by the 3-form

$$\varphi = \widetilde{\omega} \wedge \theta + \rho.$$

Proof. The hypothesis on ρ and $\widetilde{\omega}$ guarantee that the 3-form $\varphi = \widetilde{\omega} \wedge \theta + \rho$ defines a G_2-structure on $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$. Moreover, since $\widetilde{\omega}$ is closed and $\omega_0 = d\theta$, we have

$$d\varphi = d\widetilde{\omega} \wedge \theta + \widetilde{\omega} \wedge d\theta + d\rho = \widetilde{\omega} \wedge \omega_0 + d\rho = 0. \qquad \square$$

Corollary 3.6. *Let $\mathfrak{h}$ be a 6-dimensional Lie algebra. Assume that $\mathfrak{h}$ admits a definite 3-form ρ and a symplectic form $\widetilde{\omega}$ such that*

a) $\widetilde{\omega}$ is a taming form for the almost complex structure J_ρ on $\mathfrak{h}$ induced by ρ and one of the two orientations of $\mathfrak{h}$,

b) $d\rho = 0$.

Then, the Lie algebra direct sum $\mathfrak{g} := \mathfrak{h} \oplus \mathbb{R}$ is endowed with a closed G_2-structure defined by the 3-form

$$\varphi = \widetilde{\omega} \wedge \theta + \rho.$$

Proof. It follows from Proposition 3.5 with $\omega_0 = 0$. □

The next result is a converse of Proposition 3.5.

Proposition 3.7. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra endowed with a closed G_2-structure φ. Assume that the center of $\mathfrak{g}$ is not trivial, consider a non-zero vector $z \in \mathfrak{z}(\mathfrak{g})$ and let $\theta \in \mathfrak{g}^*$ be such that $\theta(z) = 1$. Then, $\mathfrak{g}$ is the central extension of a 6-dimensional Lie algebra $(\mathfrak{h}, \omega_0)$, and the closed G_2-structure can be written as $\varphi = \widetilde{\omega} \wedge \theta + \rho$, where ρ is a definite 3-form on $\mathfrak{h}$, $\widetilde{\omega}$ is a taming symplectic form for J_ρ and $d\rho = -\widetilde{\omega} \wedge \omega_0$.*

Proof. Consider the 6-dimensional subspace $\mathfrak{h} := \ker\theta$ of $\mathfrak{g}$. Then, $\mathfrak{h}$ is a Lie algebra with respect to the bracket

$$[x,y]_{\mathfrak{h}} := [x,y] - \theta([x,y])\,z, \quad x,y \in \mathfrak{h}, \tag{3.2}$$

and the 2-form $\omega_0 := d\theta|_{\mathfrak{h}\times\mathfrak{h}}$ on $\mathfrak{h}$ is closed with respect to the Chevalley-Eilenberg differential of $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$. In particular, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ is the central extension of $(\mathfrak{h}, \omega_0)$. Since $\Lambda^3\mathfrak{g}^* = (\Lambda^2\mathfrak{h}^* \otimes \mathbb{R}\theta) \oplus \Lambda^3\mathfrak{h}^*$, there exist some $\widetilde{\omega} \in \Lambda^2\mathfrak{h}^*$, $\rho \in \Lambda^3\mathfrak{h}^*$ such that $\varphi = \widetilde{\omega} \wedge \theta + \rho$. Clearly, ρ is a definite 3-form on $\mathfrak{h}$ and $\widetilde{\omega} = \iota_z\varphi$ is a non-degenerate 2-form taming J_ρ. Moreover, $\widetilde{\omega}$ is symplectic. Indeed,

$$0 = \mathcal{L}_z\varphi = d(\iota_z\varphi) = d\widetilde{\omega},$$

as $z \in \mathfrak{z}(\mathfrak{g})$. Finally, we have

$$0 = d\varphi = \widetilde{\omega} \wedge d\theta + d\rho = \widetilde{\omega} \wedge \omega_0 + d\rho,$$

where the last identity follows from $\iota_z d\theta = 0$. □

Corollary 3.8. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra endowed with a closed G_2-structure φ. If $\mathfrak{g} := \mathfrak{h} \oplus \mathbb{R}$ as Lie algebras direct sum, then the G_2-structure can be written as $\varphi = \widetilde{\omega} \wedge \theta + \rho$, where $\widetilde{\omega} \in \Lambda^2\mathfrak{h}^*$ is a symplectic 2-form and $\rho \in \Lambda^3\mathfrak{h}^*$ is a closed $\mathrm{SL}(3,\mathbb{C})$-structure tamed by $\widetilde{\omega}$.*

Proof. It follows from Proposition 3.7 with $\omega_0 = 0$. □

Remark 3.9. 1) It follows from [55] that every 7-dimensional unimodular Lie algebra with non-trivial center admitting closed G_2-structures is necessarily solvable. On the other hand, there exist unimodular solvable centerless Lie algebras admitting closed G_2-structures, see e.g. [97, Example 4.7].

2) Every vector $z \in \mathfrak{z}(\mathfrak{g})$ satisfies $\mathcal{L}_z\varphi = 0$. More generally, if $x \in \mathfrak{g}$ satisfies $\mathcal{L}_x\varphi = 0$, then $\mathcal{L}_x g_\varphi = 0$, whence it follows that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$ is skew-symmetric. Consequently, if the Lie algebra $\mathfrak{g}$ is completely solvable, namely the spectrum of ad_v is real for all $v \in \mathfrak{g}$, then every vector x satisfying $\mathcal{L}_x\varphi = 0$ must belong to the center of $\mathfrak{g}$.

The following result is a consequence of Proposition 3.7.

Corollary 3.10. *Let $\mathfrak{g}$ be a 7-dimensional nilpotent Lie algebra endowed with a closed G_2-structure φ. Then $\mathfrak{g}$ is the central extension of a 6-dimensional nilpotent Lie algebra $\mathfrak{h}$ admitting symplectic structures. Moreover, $\mathfrak{g}$ is the contactization of a 6-dimensional symplectic nilpotent Lie algebra $(\mathfrak{h}, \omega_0)$ if and only if $\mathfrak{g}$ is isomorphic to one of the following:*

$$\begin{aligned}
\mathfrak{n}_9 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25}),\\
\mathfrak{n}_{10} &= (0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}),\\
\mathfrak{n}_{11} &= (0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{15}+e^{16}-3e^{26}),\\
\mathfrak{n}_{12} &= (0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-2e^{16}+2e^{25}).
\end{aligned}$$

Proof. Since a nilpotent Lie algebra has non-trivial center, the first assertion immediately follows from Proposition 3.7. By the classification result of [27], a 7-dimensional nilpotent Lie algebra admitting closed G_2-structures is isomorphic to one of the following:

$$\begin{aligned}
\mathfrak{n}_1 &= (0,0,0,0,0,0,0),\\
\mathfrak{n}_2 &= (0,0,0,0,e^{12},e^{13},0),\\
\mathfrak{n}_3 &= (0,0,0,e^{12},e^{13},e^{23},0),\\
\mathfrak{n}_4 &= (0,0,e^{12},0,0,e^{13}+e^{24},e^{15}),\\
\mathfrak{n}_5 &= (0,0,e^{12},0,0,e^{13},e^{14}+e^{25}),\\
\mathfrak{n}_6 &= (0,0,0,e^{12},e^{13},e^{14},e^{15}),\\
\mathfrak{n}_7 &= (0,0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}),\\
\mathfrak{n}_8 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}),\\
\mathfrak{n}_9 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25}),\\
\mathfrak{n}_{10} &= (0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}),\\
\mathfrak{n}_{11} &= (0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{15}+e^{16}-3e^{26}),\\
\mathfrak{n}_{12} &= (0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-2e^{16}+2e^{25}).
\end{aligned}$$

By [92, Theorem 4.2], a decomposable nilpotent Lie algebra cannot admit any contact structures. Consequently, the Lie algebras $\mathfrak{n}_1$, $\mathfrak{n}_2$ and $\mathfrak{n}_3$ cannot be the contactization of any symplectic Lie algebra. 7-dimensional indecomposable nilpotent Lie algebras admitting contact structures have been classified in [92]. Comparing this classification with the one above, we see that $\mathfrak{g}$ must be isomorphic to one among $\mathfrak{n}_9$, $\mathfrak{n}_{10}$, $\mathfrak{n}_{11}$, $\mathfrak{n}_{12}$. For each of these Lie algebras, $\mathfrak{z}(\mathfrak{n}_i) = \mathbb{R}e_7$ and the 2-form de^7 induces a symplectic form on the 6-dimensional nilpotent Lie algebra $\mathfrak{h}_i := \ker e^7$ with Lie bracket defined as in (3.2). □

Let us now consider a 7-dimensional Lie algebra $\mathfrak{g}$ with non-trivial center endowed with a closed G_2-structure φ. Then, from the previous discussion we can assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ is the central extension of a 6-dimensional Lie algebra $(\mathfrak{h}, \omega_0 := d\theta|_{\mathfrak{h}\times\mathfrak{h}})$, and that $\varphi = \widetilde{\omega}\wedge\theta + \rho$, with $d\rho = -\widetilde{\omega}\wedge\omega_0$ and $d\widetilde{\omega} = 0$. From Section 1.5, we also know that $\mathfrak{h}$ admits an SU(3)-structure (ω, ψ_+) such that $\varphi = u\,\omega\wedge\eta + \psi_+$, with $u := |z|_\varphi$ and $\eta := u^{-2}z^\flat = \eta_\mathfrak{h} + \theta$, for some $\eta_\mathfrak{h} \in \mathfrak{h}^*$. In particular, $\mathfrak{h}$ is the g_φ-orthogonal complement of $\mathbb{R}z$ in $\mathfrak{g}$ if and only if $\eta_\mathfrak{h} = 0$. It is worth observing that this setting generalizes the one considered in [56, Section 6.1], which corresponds to the case where both $\eta_\mathfrak{h}$ and ω_0 vanish, i.e., to the direct sum of Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ endowed with a closed G_2-structure inducing the product metric.

We now investigate the properties of the SU(3)-structure (ω, ψ_+) on $\mathfrak{h}$. Since $u\,\omega = \widetilde{\omega}$, we immediately see that $d\omega = 0$ holds. Consequently, by (1.12), we have

$$\begin{aligned}
d\psi_+ &= w_2^+ \wedge \omega + w_1 \wedge \psi_+,\\
d\psi_- &= w_2^- \wedge \omega + Jw_1 \wedge \psi_+,
\end{aligned} \tag{3.3}$$

for some unique $w_1 \in \mathfrak{h}^*$ and $w_2^\pm \in \Lambda^2_8\mathfrak{h}^*$.

Lemma 3.11. *The 2-form $d\eta \in \Lambda^2\mathfrak{h}^*$ has no component in $\Lambda^2_1\mathfrak{h}^* = \mathbb{R}\omega$, that is, $d\eta\wedge\omega^2 = 0$. Moreover, the intrinsic torsion forms w_2^- and w_1 are related to the components $(d\eta)_k$ of $d\eta$ in $\Lambda^2_k\mathfrak{h}^*$, $k = 6, 8$, as follows:*

$$u\,(d\eta)_6 = -*_g(w_1\wedge\psi_+),$$
$$u\,(d\eta)_8 = -w_2^+.$$

In particular, $w_1 = \frac{u}{2} *_g (\psi_+\wedge d\eta)$.

Proof. The condition $d\varphi = 0$ is equivalent to $d\psi_+ = -u\,\omega\wedge d\eta$. Since ω is symplectic and we have $\omega\wedge\psi_+ = 0$, we get $d\eta\wedge\omega^2 = 0$. Now, according to the SU(3)-irreducible decomposition $\Lambda^2\mathfrak{h}^* = \Lambda^2_1\mathfrak{h}^*\oplus\Lambda^2_6\mathfrak{h}^*\oplus\Lambda^2_8\mathfrak{h}^*$, this implies $(d\eta)_1 = 0$ and we thus have $d\eta = (d\eta)_6 + (d\eta)_8$, with $(d\eta)_6\wedge\omega = *_g(d\eta)_6$ and $(d\eta)_8\wedge\omega = -*_g(d\eta)_8$, see (1.12). Therefore,

$$d\psi_+ = -u\,\omega\wedge d\eta = -u *_g (d\eta)_6 + u *_g (d\eta)_8.$$

Comparing this expression with the one in (3.3), we obtain the identities relating $(d\eta)_6$ and $(d\eta)_8$ with w_1 and w_2^+, respectively. Finally, to obtain the expression of w_1, it is sufficient to notice that

$$u\,d\eta\wedge\psi_+ = u\,(d\eta)_6\wedge\psi_+ = -*_g(w_1\wedge\psi_+)\wedge\psi_+ = -2 *_g w_1,$$

where the last identity follows from Lemma 1.54. □

The 2-form $d\eta$ belongs to $\Lambda^2_6\mathfrak{h}^*\oplus\Lambda^2_8\mathfrak{h}^*$, hence it satisfies the following condition (see [10, Remark 2.7]):

$$d\eta\wedge\omega = -J *_g d\eta. \tag{3.4}$$

In the next lemma, we describe the intrinsic torsion form τ of the closed G_2-structure $\varphi = u\,\omega\wedge\eta + \psi_+$ on $\mathfrak{g}$ in terms of the intrinsic torsion forms of the SU(3)-structure (ω, ψ_+) on $\mathfrak{h}$.

Lemma 3.12. *The intrinsic torsion form $\tau\in\Lambda^2_{14}\mathfrak{g}^*$ of the closed G_2-structure $\varphi = u\,\omega\wedge\eta + \psi_+$ is given by*

$$\tau = w_2^- - *_g(Jw_1\wedge\psi_+) - 2u\,Jw_1\wedge\eta,$$

while its Hodge dual is

$$*_\varphi\tau = u *_g w_2^-\wedge\eta - u\,Jw_1\wedge\psi_+\wedge\eta + 2 *_g Jw_1.$$

Consequently, $|\tau|^2_\varphi = |w_2^-|^2_g + 6|w_1|^2_g$.

Proof. Recall $\tau = -*_\varphi d *_\varphi \varphi$, with $*_\varphi\varphi = \frac{1}{2}\omega^2 + u\,\psi_-\wedge\eta$. We first compute

$$\begin{aligned} d *_\varphi \varphi &= u\,d\psi_-\wedge\eta - u\,\psi_-\wedge d\eta = u\,d\psi_-\wedge\eta - u\,\psi_-\wedge(d\eta)_6 \\ &= u\,d\psi_-\wedge\eta - 2 *_g Jw_1, \end{aligned}$$

where we used $(d\eta)_8\wedge\psi_- = 0$, Lemma 3.11 and the identity $*_g(w_1\wedge\psi_+)\wedge\psi_- = -2 *_g Jw_1$.

Using now the relation between the Hodge operators $*_\varphi$ and $*_g$ together with the identity $d\psi_- = -*_g w_2^- + Jw_1\wedge\psi_+$, we obtain

$$\tau = -*_\varphi d *_\varphi \varphi = -*_g d\psi_- - 2\,u\,Jw_1\wedge\eta = w_2^- - *_g(Jw_1\wedge\psi_+) - 2u\,Jw_1\wedge\eta.$$

From this expression and the relation between $*_\varphi$ and $*_g$, one can easily compute $*_\varphi\tau$. To obtain the norm of τ, it is then sufficient to use the identity $\tau \wedge *_\varphi\tau = |\tau|^2_\varphi \mathrm{Vol}_{g_\varphi}$ and the identity (4) of Lemma 1.54. $\square$

We now examine an example of closed G_2-structure on the nilpotent Lie algebra $\mathfrak{n}_9$ in the light of Propositions 3.5 and 3.7.

Example 3.13. Consider the 6-dimensional nilpotent Lie algebra $\mathfrak{h}$ with structure equations

$$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}).$$

The following 2-forms are symplectic forms on $\mathfrak{h}$:

$$\omega_0 = e^{16} + e^{25} + e^{34}, \quad \widetilde{\omega} = -e^{12} - e^{14} - e^{35} + e^{26}.$$

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}e_7$ be the contactization of $(\mathfrak{h}, \omega_0)$ with contact form $\theta = e^7$ and Reeb vector $z = e_7$. Then, $\mathfrak{g}$ has structure equations

$$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{25} + e^{34})$$

and so it coincides with the Lie algebra $\mathfrak{n}_9$ described in the proof of Corollary 3.10. It is easy to check that $\widetilde{\omega} \wedge \omega_0 = -d\rho$ holds, where

$$\rho = e^{124} - e^{125} - e^{136} - e^{234} - e^{345} + e^{456}$$

is a definite 3-form on $\mathfrak{h}$. The almost complex structure J induced by (ρ, e^{123456}) on $\mathfrak{h}$ is given by

$$Je_1 = -e_4 - e_5,\ Je_2 = e_6,\ Je_3 = e_2 - e_5,\ Je_4 = e_1 - e_3 - e_6,\ Je_5 = e_3 + e_6,\ Je_6 = -e_2.$$

The 2-form $\widetilde{\omega}$ is a taming form for J, as for every non-zero vector $\xi = \xi^k e_k \in \mathfrak{g}$ we have

$$\widetilde{\omega}(\xi, J\xi) = \sum_{k=1}^{6} (\xi^k)^2 + \xi^1\xi^6 + \xi^2\xi^5 - \xi^3\xi^6 - \xi^4\xi^5 > 0.$$

Therefore, $\varphi = \widetilde{\omega} \wedge e^7 + \rho$ defines a closed G_2-structure on $\mathfrak{g}$. The metric induced by φ has the following expression:

$$g_\varphi = 2^{\frac{1}{3}} \left[\sum_{i=1}^{7} e^i \odot e^i + (e^1 \odot e^6 + e^2 \odot e^5 - e^3 \odot e^6 - e^4 \odot e^5) + (e^2 - e^4 + e^5) \odot e^7 \right],$$

where $e^i \odot e^k := \frac{1}{2}\left(e^i \otimes e^k + e^k \otimes e^i\right)$.

On the other hand, we can start with the Lie algebra $\mathfrak{g}$ endowed with the closed G_2-structure φ and consider the SU(3)-structure induced by it on the g_φ-orthogonal complement $\mathfrak{u}$ of the 1-dimensional subspace generated by $e_7 \in \mathfrak{z}(\mathfrak{g})$. We have $u = |e_7|_\varphi = 2^{\frac{1}{6}}$ and

$$\eta = u^{-2}\,(e_7)^\flat = \frac{1}{2}\left(e^2 - e^4 + e^5\right) + e^7.$$

The closed G$_2$-structure φ can be written as $\varphi = u\,\omega\wedge\eta + \psi_+$, where the pair

$$\omega = u^{-1}\,\widetilde{\omega} = 2^{-\frac{1}{6}}\left(-e^{12} - e^{14} - e^{35} + e^{26}\right),$$
$$\psi_+ = -\frac{1}{2}\,e^{125} - e^{136} + \frac{1}{2}\,e^{145} - e^{234} - \frac{1}{2}\,e^{246} + \frac{1}{2}\,e^{256} - \frac{1}{2}\,e^{345} + \frac{1}{2}\,e^{235} + e^{456}$$

defines an SU(3)-structure on the vector subspace $\mathfrak{u} \subset \mathfrak{g}$. Notice also that $\mathfrak{h} := \ker\theta$ is a Lie algebra with respect to the bracket (3.2) and that it is endowed with an SU(3)-structure (ω, ψ_+) whose expression with respect to the basis $(e^1, \dots, e^6)$ of $\mathfrak{h}^*$ is the same as the one appearing above (cf. Remark 1.53). The metric induced by (ω, ψ_+) on $\mathfrak{h}$ is

$$\begin{aligned} g = {} & 2^{\frac{1}{3}}\left(e^1\odot e^1 + e^3\odot e^3 + e^6\odot e^6\right) + \frac{3}{4}\,2^{\frac{1}{3}}\left(e^2\odot e^2 + e^4\odot e^4 + e^5\odot e^5\right) \\ & + 2^{\frac{1}{3}}\left[e^1\odot e^6 - e^3\odot e^6 + \frac{1}{2}\left(e^2\odot e^4 + e^2\odot e^5 - e^4\odot e^5\right)\right] \end{aligned}$$

and we have $g_\varphi = g + u^2\,\eta\otimes\eta$.

The results of Proposition 3.5 are also useful to produce examples of 7-dimensional solvable non-nilpotent Lie algebras admitting closed G$_2$-structures, as the next example shows.

Example 3.14. On the 6-dimensional unimodular solvable non-nilpotent Lie algebra $\mathfrak{g}_{6,70}^{0,0}$ with structure equations

$$\left(-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0\right),$$

consider the closed 2-forms

$$\omega_0 = 2e^{34}, \quad \widetilde{\omega} = -e^{13} - e^{24} - e^{56}$$

and the definite 3-form

$$\rho = e^{125} - e^{146} + e^{236} - e^{345}.$$

Then, we have $d\rho = -\widetilde{\omega}\wedge\omega_0$ and that the almost complex structure J induced by the pair (ρ, e^{123456}) is given by

$$Je_1 = -e_3,\ Je_2 = -e_4,\ Je_3 = e_1,\ Je_4 = e_2,\ Je_5 = -e_6,\ Je_6 = e_5.$$

In particular, $\widetilde{\omega}$ is a taming form for J, as for every non-zero vector $\xi = \xi^k e_k \in \mathfrak{g}_{6,70}^{0,0}$ we have

$$\widetilde{\omega}(\xi, J\xi) = \sum_{k=1}^{6}(\xi^k)^2 > 0.$$

The pair $(\widetilde{\omega}, \rho)$ defines an SU(3)-structure on $\mathfrak{g}_{6,70}^{0,0}$, since $\widetilde{\omega}\wedge\rho = 0$ and $3\rho\wedge J_\rho\rho = 2\widetilde{\omega}^3$. The central extension of $(\mathfrak{g}_{6,70}^{0,0}, \omega_0)$ is given by

$$\mathfrak{g} = (-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0, 2e^{34})$$

and it is isomorphic to the Lie algebra $\mathfrak{s}_9$ of Theorem 3.15 below. By Proposition 3.5, we know that the 3-form $\varphi = \widetilde{\omega}\wedge e^7 + \rho$ defines a closed G$_2$-structure on $\mathfrak{g}$. Notice that the 1-form e^7 does not define a contact structure on $\mathfrak{g}$, as the closed 2-form ω_0 is degenerate.

More generally, one can consider the list of all 6-dimensional unimodular solvable non-nilpotent Lie algebras admitting symplectic structures (see [103]) and determine which of them admit a structure $(\omega_0, \widetilde{\omega}, \rho)$ satisfying the hypothesis of Proposition 3.5. This allows one to obtain further examples of solvable non-nilpotent Lie algebras admitting closed G_2-structures.

3.3 A classification result

In this section, we classify all 7-dimensional unimodular Lie algebras with non-trivial center admitting closed G_2-structures, up to isomorphism. Every such Lie algebra must be solvable by the results of [55]. If it is nilpotent, then it is isomorphic to one of the Lie algebras $\mathfrak{n}_1, \dots, \mathfrak{n}_{12}$, by [27]. To complete the classification, we have to determine which unimodular solvable non-nilpotent Lie algebras with non-trivial center admit closed G_2-structures. We can state the following result.

Theorem 3.15. *Let $\mathfrak{g}$ be a 7-dimensional unimodular solvable non-nilpotent Lie algebra with non-trivial center. Then, $\mathfrak{g}$ admits closed G_2-structures if and only if it is isomorphic to one of the following:*

$$\begin{aligned}
\mathfrak{s}_1 &= (e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 0),\\
\mathfrak{s}_2 &= (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, 0),\\
\mathfrak{s}_3 &= (-e^{16} + e^{25}, -e^{15} - e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0, 0),\\
\mathfrak{s}_4 &= (0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}, 0),\\
\mathfrak{s}_5 &= (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0, 0),\\
\mathfrak{s}_6 &= (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0, 0),\ \ \alpha > 0,\\
\mathfrak{s}_7 &= (e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0, 0),\\
\mathfrak{s}_8 &= (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, e^{34}),\\
\mathfrak{s}_9 &= (-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0, e^{34}),\\
\mathfrak{s}_{10} &= \left(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 2\,e^{16} + e^{25} - e^{34} + \sqrt{3}\,e^{24} + \sqrt{3}\,e^{35}\right),\\
\mathfrak{s}_{11} &= \left(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 2\,e^{16} + e^{25} - e^{34} - \sqrt{3}\,e^{24} - \sqrt{3}\,e^{35}\right).
\end{aligned}$$

In particular, $\mathfrak{g}$ is the contactization of a symplectic Lie algebra if and only if it is isomorphic either to $\mathfrak{s}_{10}$ or to $\mathfrak{s}_{11}$.

Proof. Since the central extension of a nilpotent Lie algebra is nilpotent, by Proposition 3.7 we can assume that $\mathfrak{g}$ is the central extension of a 6-dimensional unimodular solvable non-nilpotent Lie algebra $(\mathfrak{h}, \omega_0)$ admitting symplectic structures. Recall that $\mathfrak{g}$ is determined by any representative in the cohomology class $[\omega_0] \in \mathcal{H}^2(\mathfrak{h})$, up to isomorphism. Moreover, $\mathfrak{h}$ is isomorphic to one of the Lie algebras listed in Table 6.2 (cf. [47, 103]).

If $\omega_0 = 0$, then $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and the abelian Lie algebra $\mathbb{R}$. As a consequence of Theorem 2.28, $\mathfrak{g}$ admits closed G_2-structures if and only if $\mathfrak{h}$ admits symplectic half-flat SU(3)-structures. Therefore, by [47, Theorem 1.1], $\mathfrak{g}$ must be isomorphic to one of the Lie algebras $\mathfrak{s}_1 \cong \mathfrak{g}_{6,38}^0 \oplus \mathbb{R}$, $\mathfrak{s}_2 \cong \mathfrak{g}_{6,54}^{0,-1} \oplus \mathbb{R}$, $\mathfrak{s}_3 \cong \mathfrak{g}_{6,118}^{0,-1,-1} \oplus \mathbb{R}$, $\mathfrak{s}_4 \cong \mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1) \oplus \mathbb{R}$, $\mathfrak{s}_5 \cong A_{5,7}^{-1,-1,1} \oplus \mathbb{R}^2$, $\mathfrak{s}_6 \cong A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}^2$, $\mathfrak{s}_7 \cong A_{5,17}^{0,0,1} \oplus \mathbb{R}^2$.

We can then focus on the case where $\omega_0 \neq 0$ and $\mathfrak{h}$ is one of the Lie algebras appearing in Table 6.2. To determine those having a central extension admitting closed G$_2$-structure, we proceed as follows. First, we compute a basis of the second cohomology group $\mathcal{H}^2(\mathfrak{h})$ using the structure equations given in Table 6.2. Then, we consider a non-zero representative ω_0 of the generic element in $\mathcal{H}^2(\mathfrak{h})$, and we look for closed non-degenerate 2-forms $\widetilde{\omega} \in \Lambda^2\mathfrak{h}^*$ such that $\widetilde{\omega} \wedge \omega_0$ is exact (cf. Proposition 3.7). A computation shows that there are no exact 4-forms of this type when $\mathfrak{h}$ is a decomposable Lie algebra not isomorphic to $A_{5,15}^{-1} \oplus \mathbb{R}$ or to $A_{5,18}^{0} \oplus \mathbb{R}$. Let us prove this claim, for instance, for the first decomposable Lie algebra appearing in Table 6.2, namely $\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$. A basis for its second cohomology group is given by

$$\left([e^{13}], [e^{24}], [e^{56}]\right),$$

and we can consider the non-zero representative

$$\omega_0 = f_1\, e^{13} + f_2\, e^{24} + f_3\, e^{56},$$

where $f_1, f_2, f_3 \in \mathbb{R}$ satisfy $f_1^2 + f_2^2 + f_3^2 \neq 0$. The generic closed non-degenerate 2-form $\widetilde{\omega}$ on $\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$ has the following expression:

$$\widetilde{\omega} = h_1\, e^{13} + h_2 \left(e^{23} - \frac{1}{2}\, e^{16}\right) + h_3\, e^{24} + h_4\, e^{26} + h_5\, e^{36} + h_6\, e^{46} + h_7\, e^{56},$$

for some $h_i \in \mathbb{R}$ such that $h_1\, h_3\, h_7 \neq 0$. Now, we compute

$$\begin{aligned}\widetilde{\omega} \wedge \omega_0 = &-(f_1 h_3 + f_2 h_1)\, e^{1234} - f_1 h_4\, e^{1236} + f_1 h_6\, e^{1346} + (f_1 h_7 + f_3 h_1)\, e^{1356}\\ &- \frac{1}{2} f_2 h_2\, e^{1246} - f_2 h_5\, e^{2346} + (f_2 h_7 + f_3 h_3)\, e^{2456} + f_3 h_2\, e^{2356},\end{aligned}$$

and we see that this 4-form is exact only if the coefficients of e^{1234}, e^{1356} and e^{2456} vanish, namely

$$\begin{cases} f_1\, h_3 + f_2\, h_1 = 0,\\ f_1\, h_7 + f_3\, h_1 = 0,\\ f_2\, h_7 + f_3\, h_3 = 0.\end{cases}$$

This is a homogeneous linear system in the variables f_i's whose unique solution under the constraint $h_1\, h_3\, h_7 \neq 0$ is $f_1 = f_2 = f_3 = 0$. Thus, $\widetilde{\omega} \wedge \omega_0$ cannot be exact if $\omega_0 \neq 0$. A similar discussion leads us to ruling out all of the decomposable Lie algebras listed in Table 6.2 with the exception of $A_{5,15}^{-1} \oplus \mathbb{R}$ and $A_{5,18}^{0} \oplus \mathbb{R}$. In the remaining two cases, $\mathfrak{h}$ is the direct sum of a 5-dimensional ideal $\mathfrak{k}$ and $\mathbb{R}$. A computation shows that there exist pairs $(\widetilde{\omega}, \omega_0)$ satisfying the required conditions only when $\omega_0 \in \Lambda^2\mathfrak{k}^*$. In detail, if $\mathfrak{h} \cong A_{5,15}^{-1} \oplus \mathbb{R}$, then the possible 2-forms are given by

$$\widetilde{\omega} = h_1\,(e^{14} - e^{23}) + h_2\, e^{15} + h_3\, e^{24} + h_4\, e^{25} + h_5\, e^{35} + h_6\, e^{45} + h_7\, e^{56}, \quad \omega_0 = a\, e^{24},$$

where $a, h_i \in \mathbb{R}$ and $a\, h_1\, h_7 \neq 0$. If $\mathfrak{h} \cong A^0_{5,18} \oplus \mathbb{R}$, then the possible 2-forms are given by

$$\widetilde{\omega} = k_1\,(e^{13} + e^{24}) + k_2\, e^{15} + k_3\, e^{25} + k_4\, e^{34} + k_5\, e^{35} + k_6\, e^{45} + k_7\, e^{56}, \quad \omega_0 = b\, e^{34},$$

where $b, k_i \in \mathbb{R}$ and $b\, k_1\, k_7 \neq 0$. Since, in both cases, $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}$ and $\omega_0 \in \Lambda^2\mathfrak{k}^*$, all possible central extensions of $(\mathfrak{h}, \omega_0)$ split as the Lie algebra direct sum of a 6-dimensional ideal and $\mathbb{R}$. If such an extension admits closed G_2-structures, then it must be isomorphic to one among $\mathfrak{s}_1, \ldots, \mathfrak{s}_7$.

We are then left with the indecomposable Lie algebras appearing in Table 6.2. Also in this case, with analogous computations as before, one can check that there are no pairs $(\widetilde{\omega}, \omega_0)$ satisfying the required conditions when $\mathfrak{h}$ is an indecomposable Lie algebra not isomorphic to one among $\mathfrak{g}^0_{6,38}$, $\mathfrak{g}^{0,-1}_{6,54}$, $\mathfrak{g}^{0,0}_{6,70}$. In the remaining three cases, we claim that $\mathfrak{h}$ has a central extension admitting closed G_2-structures.

If $\mathfrak{h} \cong \mathfrak{g}^0_{6,38}$, then there exist closed non-degenerate 2-forms $\widetilde{\omega}$ such that $\widetilde{\omega} \wedge \omega_0$ is exact if and only if either $\omega_0 = a\left(2\, e^{16} + e^{25} - e^{34} + \sqrt{3}\, e^{24} + \sqrt{3}\, e^{35}\right)$, for some $a \neq 0$, or $\omega_0 = b\left(2\, e^{16} + e^{25} - e^{34} - \sqrt{3}\, e^{24} - \sqrt{3}\, e^{35}\right)$, for some $b \neq 0$. These forms are not cohomologous, so they give rise to non-isomorphic central extensions of $\mathfrak{h}$. In the former case, the central extension of $(\mathfrak{h}, \omega_0)$ is isomorphic to $\mathfrak{s}_{10}$ and admits closed G_2-structures. An example is given by

$$\varphi = e^{123} - 4\, e^{145} + 2\, e^{167} - \sqrt{3}\, e^{247} + e^{256} + e^{257} - e^{346} - e^{347} - \sqrt{3}\, e^{357}.$$

In the latter case, the central extension of $(\mathfrak{h}, \omega_0)$ is isomorphic to $\mathfrak{s}_{11}$ and admits closed G_2-structures. An example is given by

$$\varphi = e^{123} - 4\, e^{145} + 2\, e^{167} + \sqrt{3}\, e^{247} - e^{256} + e^{257} + e^{346} - e^{347} + \sqrt{3}\, e^{357}.$$

Both $\mathfrak{s}_{10}$ and $\mathfrak{s}_{11}$ are contactizations, since the 2-form ω_0 is non-degenerate.

If $\mathfrak{h} \cong \mathfrak{g}^{0,-1}_{6,54}$, then $\omega_0 = a\, e^{34}$, for some $a \neq 0$. The central extension of $(\mathfrak{h}, \omega_0)$ is isomorphic to $\mathfrak{s}_8$ and it admits closed G_2-structures. An example is given by

$$\varphi = e^{147} + e^{237} + e^{567} + e^{125} - e^{136} + \frac{1}{2}(e^{146} - e^{236}) + \frac{5}{4} e^{246} + e^{345}.$$

If $\mathfrak{h} \cong \mathfrak{g}^{0,0}_{6,70}$, then $\omega_0 = a\, e^{34}$, for some $a \neq 0$. The central extension of $(\mathfrak{h}, \omega_0)$ is isomorphic to $\mathfrak{s}_9$ and admits closed G_2-structures. An example is given by

$$\varphi = e^{137} + e^{247} + 2\, e^{567} - e^{125} + e^{146} - e^{236} + e^{345},$$

see also Example 3.14.

To conclude the proof, we first observe that the Lie algebras $\mathfrak{s}_8$ and $\mathfrak{s}_9$ cannot be the contactization of a symplectic Lie algebra. Indeed, in both cases ω_0 is a closed degenerate 2-form on the unimodular Lie algebra $\mathfrak{h}$, thus every representative of $[\omega_0] \in \mathcal{H}^2(\mathfrak{h})$ is a degenerate 2-form. Finally, a direct computation shows that the remaining Lie algebras $\mathfrak{s}_1, \ldots, \mathfrak{s}_7$ do not admit any contact structures. $\square$

Remark 3.16. Notice that there are some misprints in [103] that have been corrected in Table 6.2, see also the appendix in [47].

Corollary 3.17. *A 7-dimensional Lie algebra with non-trivial center admitting torsion-free G_2-structures is either abelian or isomorphic to $\mathfrak{s}_7$.*

Proof. Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with non-trivial center endowed with a torsion-free G_2-structure φ. Then, the metric g_φ induced by φ is Ricci-flat and thus flat by [2]. Consequently, the results of [107] imply that either $\mathfrak{g}$ is abelian, or $\mathfrak{g}$ splits as a g_φ-orthogonal direct sum $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{a}$, where $\mathfrak{b}$ is an abelian subalgebra, $\mathfrak{a}$ is an abelian ideal and the endomorphism ad_x is skew-adjoint for every $x \in \mathfrak{b}$. In the latter case, $\mathfrak{g}$ is a unimodular 2-step solvable Lie algebra and the eigenvalues of ad_x are purely imaginary for every $x \in \mathfrak{g}$ (cf. [72, Section 2.8]). Among the Lie algebras obtained in Theorem 3.15, the 2-step solvable ones are $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, $\mathfrak{s}_5$, $\mathfrak{s}_6$ and $\mathfrak{s}_7$. The first four Lie algebras in this list do not admit flat metrics, as the following endomorphisms have real spectrum: $\mathrm{ad}_{e_6} \in \mathrm{Der}(\mathfrak{s}_2)$, $\mathrm{ad}_{e_6} \in \mathrm{Der}(\mathfrak{s}_3)$, $\mathrm{ad}_{e_1} \in \mathrm{Der}(\mathfrak{s}_4)$, $\mathrm{ad}_{e_5} \in \mathrm{Der}(\mathfrak{s}_5)$. Also the Lie algebra $\mathfrak{s}_6$ can be ruled out, since ad_{e_5} has complex eigenvalues that are not purely imaginary. Finally, the Lie algebra $\mathfrak{s}_7$ admits torsion-free G_2-structures. An example is given by the G_2-structure

$$\varphi = e^{137} + e^{247} + e^{567} + e^{125} - e^{146} + e^{236} - e^{345},$$

which induces the metric $g_\varphi = \sum_{i=1}^{7} e^i \odot e^i$. □

Remark 3.18. The simply connected solvable Lie groups whose Lie algebra is one among $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, $\mathfrak{s}_5$, $\mathfrak{s}_7$ admit lattices, and this is the case also for the family of Lie algebras $\mathfrak{s}_6 \cong A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}^2$, for certain values of the parameter $\alpha > 0$ (see e.g. [47] and the references therein). We now show that the simply connected Lie groups with Lie algebra $\mathfrak{s}_8$ or $\mathfrak{s}_9$ admit lattices, too. Indeed, since they are both almost nilpotent, it is possible to construct a lattice using the following criterion.

Proposition 3.19. [15] *Let $G = \mathbb{R} \ltimes_\mu H$ be an almost nilpotent Lie group with nilradical H and let $\mathfrak{g} = \mathbb{R} \ltimes_D \mathfrak{h}$ be its Lie algebra, where $\mathfrak{h} := \mathrm{Lie}(H)$ and $D \in \mathrm{Der}(\mathfrak{h})$ is such that $d\mu(t)|_{1_\mathrm{G}} = \exp(tD)$. If there exists $t_0 \in \mathbb{R} - \{0\}$ and a rational basis $(x_1, \dots, x_n)$ of $\mathfrak{h}$ such that the coordinate matrix of $\exp(t_0 D)$ in such a basis is integral, then $\Gamma := t_0\mathbb{Z} \ltimes_\mu \exp(\mathbb{Z}\langle x_1, \dots, x_n\rangle)$ is a lattice in G.*

Let us consider the Lie algebra $\mathfrak{s}_9$ with the basis $(e_1, \dots, e_7)$ as in Theorem 3.15. We can write $\mathfrak{s}_9 = \mathbb{R} \ltimes_D \mathfrak{h}$, where $\mathfrak{h} = \langle e_1, \dots, e_5, e_7\rangle$ is a nilpotent Lie algebra with structure equations

$$(e^{35}, e^{45}, 0, 0, 0, e^{34}), \tag{3.5}$$

and

$$D = \mathrm{ad}_{e_6}|_{\mathfrak{h}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $t_0 = 2\pi$, this basis satisfies Proposition 3.19 guaranteeing the existence of a lattice in the simply connected Lie group corresponding to $\mathfrak{s}_9$.

Let us now focus on the Lie algebra $\mathfrak{s}_8$ with the basis $(e_1, \ldots, e_7)$ as in Theorem 3.15. We note that $\mathfrak{s}_8 = \mathbb{R} \ltimes_D \mathfrak{h}$, where the structure equations of the nilpotent Lie algebra $\mathfrak{h} = \langle e_1, \ldots, e_5, e_7 \rangle$ are those given in (3.5) and $D = \mathrm{ad}_{e_6}|_{\mathfrak{h}} = \mathrm{diag}(1, -1, 1, -1, 0, 0)$. Let $t_0 = \ln\left(\frac{3+\sqrt{5}}{2}\right)$. We note that $\exp(t_0 D) = E^{-1}BE$, where

$$B = \begin{pmatrix} 3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} \frac{2}{3-\sqrt{5}} & \frac{2}{3+\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3-\sqrt{5}} & \frac{2}{3+\sqrt{5}} & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{5} \end{pmatrix}.$$

Thus, the integer matrix B is the matrix associated with $\exp(t_0 D)$ with respect to a suitable basis $(f_1, \ldots, f_5, f_7)$ of $\mathfrak{h}$. Moreover, $\mathfrak{h}$ has rational structure equations

$$(f^{25}, 0, f^{45}, 0, 0, f^{24})$$

in such a basis. The existence of a lattice in the simply connected solvable Lie group with Lie algebra $\mathfrak{s}_8$ then follows.

Chapter 4

Laplacian solitons

We study the existence of self-similar solutions of the G_2-Laplacian flow on the Lie algebras from our previous classification. In particular, we prove that every semi-algebraic Laplacian soliton on a unimodular solvable Lie algebra with 1-dimensional center must be expanding and we determine all unimodular Lie algebras with center of dimension at least two that admit semi-algebraic Laplacian solitons, up to isomorphism. The main results of this chapter were published in [58].

4.1 The G_2-Laplacian flow

In [19], Bryant introduced a geometric flow for closed G_2-structures. Let N be a 7-manifold endowed with a closed G_2-structure φ_0.

Definition 4.1. The G_2-*Laplacian flow* starting from φ_0 is the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi_0, \end{cases} \tag{4.1}$$

where $\Delta_\varphi = d \circ d^*_\varphi + d^*_\varphi \circ d$ is the Hodge Laplacian with respect to the metric g_φ and $d^*_\varphi = - *_\varphi \circ d \circ *_\varphi$ is the codifferential of d induced by φ.

The solution to (4.1) is a one-parameter family of G_2-structures $\varphi(t)$ which preserves the closure of the initial data, i.e.,

$$d\varphi(t) = 0, \quad t \in I,$$

where I is the maximal definition interval of the solution. Assume N is compact. Since $\varphi(t)$ evolves in the same cohomology class of the initial data φ_0, we can write

$$\varphi(t) = \varphi_0 + d\eta(t),$$

for some time-dependent 2-form $\eta(t)$. Using this, Hitchin showed that the G_2-Laplacian flow is the gradient flow of a volume functional [82]. Let $[\varphi_0]_+$ be the open subset of G_2-structures in the cohomology class $[\varphi_0] = \{\varphi_0 + d\beta \mid \beta \in \Lambda^2(N)\}$ of φ_0. The volume functional

$\mathcal{H}\colon [\varphi_0]_+ \to \mathbb{R}_{>0}$ is defined by

$$\mathcal{H}(\varphi) = \frac{1}{7}\int_N \varphi \wedge *_\varphi \varphi = \int_N *_\varphi 1.$$

Consider its restriction

$$\begin{aligned} \tilde{\mathcal{H}}\colon \{\eta \in \Lambda^2(N) \,|\, \varphi_0 + d\eta \in [\varphi_0]_+\} &\to \mathbb{R}_{>0} \\ \eta &\mapsto \mathcal{H}(\varphi_0 + d\eta). \end{aligned}$$

Its gradient $\nabla\tilde{\mathcal{H}}$ is a vector field defined on an open subset of $\Lambda^2(N)$ satisfying

$$\frac{\partial}{\partial t}\tilde{\mathcal{H}}(\eta(t)) = d\tilde{\mathcal{H}}(\eta'(t)) = \left\langle \eta'(t), \nabla\tilde{\mathcal{H}} \right\rangle.$$

On the other hand, one can show that

$$\frac{\partial}{\partial t}\tilde{\mathcal{H}}(\eta(t)) = \left\langle \eta'(t), d^*_{\varphi(t)}\varphi(t) \right\rangle.$$

Therefore,

$$\nabla\tilde{\mathcal{H}} = d^*_{\varphi(t)}\varphi(t). \tag{4.2}$$

If we consider the gradient flow of $\tilde{\mathcal{H}}$,

$$\frac{\partial}{\partial t}\eta(t) = \nabla\tilde{\mathcal{H}},$$

from the previous discussion we have

$$\begin{aligned} \frac{\partial}{\partial t}\eta(t) = \nabla\tilde{\mathcal{H}} &= d^*_{\varphi(t)}\varphi(t) \\ &= d^*_{\varphi(t)}(\varphi_0 + d\eta(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t) &= \frac{\partial}{\partial t}(\varphi_0 + d\eta(t)) = d\eta'(t) \\ &= dd^*_{\varphi(t)}(\varphi_0 + d\eta(t)) = dd^*_{\varphi(t)}\varphi(t) = \Delta_{\varphi(t)}\varphi(t), \end{aligned}$$

which is exactly the G_2-Laplacian flow. Thus, along the flow, the volume will increase unless $\varphi(t)$ is torsion-free. Moreover, by (4.2), $\varphi \in [\varphi_0]_+$ is a critical point for the volume functional if and only if $d^*_\varphi\varphi = 0$, i.e., if φ is torsion-free.

Short-time existence of the solutions in the compact case was proved in [22]; in particular, since the G_2-Laplacian flow appears to have the wrong sign for the parabolicity, using DeTurck's trick, Bryant and Xu modified the Laplacian flow by a gauge fixing $\mathcal{L}_{V(t)}\varphi(t) = d\iota_{V(t)}\varphi(t)$ for some vector field $V(t)$, so that the new flow

$$\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t) + d\iota_{V(t)}\varphi(t) \tag{4.3}$$

was strictly parabolic in the direction of closed forms. Therefore, short-time existence was proved by applying the Nash Moser inverse function theorem to (4.3).

Theorem 4.2. [22] *Assume that N is compact and $\varphi(0)$ is a closed* G_2*-structure on N. Then, the Laplacian flow has a unique solution for a short time $t \in [0,\varepsilon)$ with ε depending on $\varphi(0)$.*

The stationary points of the Laplacian flow are the harmonic φ, which on compact manifolds are exactly the torsion-free G_2-structures, since

$$\nabla^{g_\varphi}\varphi = 0 \iff \begin{cases} d\varphi = 0, \\ d *_\varphi \varphi = 0, \end{cases}$$

where ∇^{g_φ} is the Levi-Civita connection of the metric g_φ induced by φ. This is true also in the non-compact setting, as shown in [101]. We shall review this result in the next section. Therefore, the Laplacian flow provides an important tool for studying the existence of torsion-free G_2-structures on 7-manifolds admitting closed G_2-structures. Anyway, the Laplacian flow does not always converge to a torsion-free G_2-structure even if it has long-time existence. Indeed, examples of long-time existence on 7-manifolds not admitting torsion-free G_2-structures are known in literature (see, for instance, [19]). Examples of solutions to the Laplacian flow which exist for all times and converge to a torsion-free G_2-structure can be found in [45, 56, 86]. In [102], Lotay and Wei proved that torsion-free G_2-structures are (weakly) dynamically stable along the Laplacian flow: this means that, if the initial data φ_0 is sufficiently close to a given torsion-free G_2-structure $\bar{\varphi}$, then the solution to the flow exists for all times and converges to a torsion-free G_2-structure in the same orbit of $\bar{\varphi}$.

In the next section, we shall focus on *self-similiar* solutions of the G_2-Laplacian flow, i.e., solutions which differ from the initial data φ_0 just by time-dependent scalings and pullbacks by diffeomorphisms.

4.2 Laplacian solitons

Let N be a smooth 7-manifold.

Definition 4.3. A closed G_2-structure φ on N is called a *Laplacian soliton* if

$$\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X\varphi, \tag{4.4}$$

where $\lambda \in \mathbb{R}$ is a constant and X is a complete vector field of N.

Laplacian solitons are initial data to self-similar solutions of the Laplacian flow. More precisely, consider

$$\begin{aligned} \alpha(t) &\coloneqq \left(1 + \frac{2}{3}\lambda t\right)^{\frac{3}{2}}, \\ X(t) &\coloneqq \alpha(t)^{-\frac{2}{3}} X, \end{aligned}$$

and let $\Phi(t)$ be the family of diffeomorphisms generated by the vector field $X(t)$ such that $\Phi(0) = \mathrm{Id}$. One obtains that

$$\varphi(t) \coloneqq \alpha(t)\,\Phi(t)^*\varphi(0)$$

is a solution to the Laplacian flow which differs from the initial data only by a time-dependent scaling factor $\alpha(t)$ and a pullback by $\Phi(t)$. Depending on the sign of λ, we say that a Laplacian soliton φ is *expanding* if $\lambda > 0$, *steady* if $\lambda = 0$, or *shrinking* if $\lambda < 0$.

In recent years, there has been a lot of interest in Laplacian solitons. In [99], Lin proved that on compact 7-manifolds there are no shrinking Laplacian solitons and that the only steady ones are given by torsion-free G_2-structures. In [101], Lotay and Wei looked at the stronger eigenform condition

$$\Delta_\varphi \varphi = \lambda \varphi, \tag{4.5}$$

proving that a closed G_2-structure φ satisfying (4.5) has to be expanding ($\lambda > 0$) or torsion-free ($\lambda = 0$). Therefore, stationary points of the G_2-Laplacian flow are exactly torsion-free G_2-structures, even if N is not compact. Combining the results obtained in [99] and in [101], one has that every non trivial Laplacian soliton on a compact manifold N has to be expanding and the vector field X in (4.4) has to be different from zero. This result is very interesting if we think of the analogous for the Ricci flow. The *Ricci flow*

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)), \\ g(0) = g_0, \end{cases} \tag{4.6}$$

is a geometric flow evolving a Riemannian metric g_0 in the direction of its Ricci tensor (for further details, see [79]). Initial data of self-similar solutions of (4.6) are the so-called *Ricci solitons*, i.e., Riemannian metrics g satisfying

$$\mathrm{Ric}(g) = \lambda\, g + \mathcal{L}_X\, g, \tag{4.7}$$

for some complete vector field X and some constant $\lambda \in \mathbb{R}$. When $X = 0$, a Ricci soliton g is called *Einstein* and we refer to g simply as an *Einstein metric*. Unlike Laplacian solitons, there are many examples of compact manifolds admitting Einstein metrics for different values of $\lambda \in \mathbb{R}$.

The first example of an expanding Laplacian soliton which is not an eigenform for its Hodge Laplacian was found by Lauret on a nilpotent Lie group [95]. The shrinking condition appears to be rarer and all the known examples of Lie groups admitting these Laplacian solitons have trivial center. See, for instance, [97, 112] for some explicit examples. Other interesting examples of Laplacian solitons have been found in [63, 86]. In particular, in [63], the author provided a first example of shrinking Laplacian soliton of gradient type, i.e., one where the vector field X in (4.4) is the gradient of a smooth function. In [86], the authors provided a first example of cohomogeneity one solution to the G_2-Laplacian flow existing for all times and converging to a torsion-free G_2-structure. It is interesting to note that, up to now, there are no known examples of compact manifolds admitting Laplacian solitons.

In the next section, we shall focus on the existence of Laplacian solitons on unimodular solvable Lie groups with non trivial center. As shown by Lauret in [96], these Laplacian solitons correspond to *semi-algebraic Laplacian solitons* on the corresponding Lie algebras. Regarding the existence of compact examples, we recall that, if a Lie group G admits a compact quotient via a lattice Γ, a Laplacian soliton on G may not descend to a Laplacian soliton on the compact quotient Γ/G, since the vector field X in (4.4) may not be invariant with respect to the action of G restricted to Γ. We remark that, in [57], the authors found the first example of a left-invariant closed G_2-structure on a (non-unimodular) solvable Lie group satisfying (4.4) for $\lambda = 0$ and a left-invariant vector field X.

4.3 Semi-algebraic Laplacian solitons on the central extension of a Lie algebra

Let G be a 7-dimensional simply connected Lie group with Lie algebra $\mathfrak{g}$. Consider a derivation D of $\mathfrak{g}$ and denote by X_D the vector field on G induced by the one-parameter group of automorphisms $F_t \in \mathrm{Aut(G)}$ with derivative $dF_t|_e = \exp(tD) \in \mathrm{GL}(\mathfrak{g})$.

Definition 4.4. [96] A left-invariant closed G_2-structure φ on G is said to be a *semi-algebraic Laplacian soliton* if it satisfies the Laplacian soliton equation (4.4) with respect to some vector field X_D corresponding to a derivation $D \in \mathrm{Der}(\mathfrak{g})$. In this case, $\mathcal{L}_{X_D}\varphi = D^*\varphi$, so that equation (4.4) can be rewritten as follows:

$$\Delta_\varphi \varphi = \lambda\varphi + D^*\varphi, \tag{4.8}$$

where

$$A^*\beta(x_1, \dots, x_k) \coloneqq \beta(Ax_1, \dots, x_k) + \cdots + \beta(x_1, \dots, Ax_k),$$

for every $A \in \mathfrak{gl}(\mathfrak{g})$, $x_1, \dots, x_k \in \mathfrak{g}$ and $\beta \in \Lambda^k\mathfrak{g}^*$. When the g_φ-adjoint D^t of D is also a derivation of $\mathfrak{g}$, the G_2-structure φ is called an *algebraic Laplacian soliton.*

We note that $\Delta_\varphi\varphi = dd^*_\varphi\varphi = d\tau$, τ being the intrinsic torsion form of φ. Let us focus on the case where $\mathfrak{g}$ is a unimodular Lie algebra with non-trivial center. By the results of Section 3.2, we can assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ is the central extension of a 6-dimensional unimodular Lie algebra $(\mathfrak{h}, \omega_0)$. Moreover, every closed G_2-structure φ on $\mathfrak{g}$ can be written both as $\varphi = \widetilde{\omega} \wedge \theta + \rho$, with $d\rho = -\widetilde{\omega} \wedge \omega_0$ and $d\widetilde{\omega} = 0$, and as $\varphi = u\,\omega \wedge \eta + \psi_+$, where (ω, ψ_+) is an $\mathrm{SU}(3)$-structure on $\mathfrak{h}$, $u \coloneqq |z|_\varphi$ and $\eta \coloneqq u^{-2}z^\flat = \eta_\mathfrak{h} + \theta$, for some $\eta_\mathfrak{h} \in \mathfrak{h}^*$.

If φ is a semi-algebraic Laplacian soliton, the condition (4.8) is equivalent to a set of equations involving either the forms $(\widetilde{\omega}, \rho)$ or the $\mathrm{SU}(3)$-structure (ω, ψ_+) on $\mathfrak{h}$. In the following, we shall see that it is possible to obtain information on the semi-algebraic Laplacian soliton φ under suitable assumptions. We are interested in the case where z is an eigenvector of D, as this happens whenever $\mathfrak{g}$ is the contactization of a symplectic Lie algebra. Indeed, in that case the center of $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}z$ and it is preserved by all derivations of $\mathfrak{g}$.

Henceforth, we assume that φ is a semi-algebraic Laplacian soliton on the unimodular Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ and that it satisfies the equation (4.8) with respect to a derivation $D \in \mathrm{Der}(\mathfrak{g})$ such that $Dz = cz$, for some $c \in \mathbb{R}$. Then, we have $D^*\theta = \alpha + c\,\theta$, with $\alpha \in \mathfrak{h}^*$. We let $\widetilde{D} \coloneqq \pi_\mathfrak{h} \circ D|_\mathfrak{h} \in \mathfrak{gl}(\mathfrak{h})$, where $\pi_\mathfrak{h} \colon \mathfrak{g} \to \mathfrak{h}$ denotes the projection onto $\mathfrak{h}$.

Using the expression of τ obtained in Lemma 3.12, we see that $\varphi = \widetilde{\omega}\wedge\theta + \rho = u\,\omega\wedge\eta + \psi_+$ solves the equation $d\tau = \Delta_\varphi\varphi = \lambda\varphi + D^*\varphi$ if and only if the following equations hold on $\mathfrak{h}$:

$$\begin{cases} 2\,d(Jw_1) = -\widetilde{D}^*\omega - (c+\lambda)\,\omega, \\ dw_2^- - d*_g(Jw_1 \wedge \psi_+) - 2u\,(d(Jw_1)\wedge\eta_\mathfrak{h} - Jw_1 \wedge d\eta) = u\,\omega\wedge\alpha + \widetilde{D}^*\rho + \lambda\rho, \end{cases} \tag{4.9}$$

where w_1, w_2^- are the intrinsic torsion forms of the SU(3)-structure (ω, ψ_+) and $\rho = u\,\omega\wedge\eta_{\mathfrak{h}} + \psi_+$. Recall that the 2-form $d\eta$ depends on the intrinsic torsion forms w_2^+ and w_1 of (ω, ψ_+) as shown in Lemma 3.11.

The equations (4.9) allow us to relate the constant λ to the eigenvalue c and the norm of the intrinsic torsion form of the semi-algebraic Laplacian soliton. Before stating the result, we show a preliminary lemma.

Lemma 4.5. *Let* (ω, ψ_+) *be an* SU(3)*-structure on a* 6*-dimensional real vector space* V *and let* $A \in \mathfrak{gl}(V)$*. Then,*

$$A^*\psi_+ \wedge \psi_- = A^*\omega \wedge \omega^2 = \frac{1}{3}\,(\mathrm{tr}\,A)\,\omega^3.$$

Proof. Consider a basis $(e_1, \dots, e_6)$ of V which is adapted to the SU(3)-structure (ω, ψ_+). Then,

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \quad \psi_- = e^{136} + e^{145} + e^{235} - e^{246},$$

with respect to the dual basis $(e^1, \dots, e^6)$ of V^*. Now, a direct computation shows that

$$A^*\psi_+ \wedge \psi_- = A^*\omega \wedge \omega^2 = 2\,(\mathrm{tr}A)\,e^{123456} = \frac{1}{3}\,(\mathrm{tr}\,A)\,\omega^3. \qquad \square$$

Proposition 4.6. *The constant* λ *is given by*

$$\lambda = -3\,c - \frac{1}{2}\left(|w_2^-|_g^2 + 6\,|w_1|_g^2\right) = -3\,c - \frac{1}{2}\,|\tau|_\varphi^2.$$

Proof. Wedging the first equation of (4.9) by the closed 4-form ω^2, we obtain

$$\widetilde{D}^*\omega \wedge \omega^2 + (c+\lambda)\omega^3 = -2\,d(Jw_1) \wedge \omega^2 = -2\,d(Jw_1 \wedge \omega^2) = 0,$$

as every 5-form on the unimodular Lie algebra $\mathfrak{h}$ is closed. Then, by Lemma 4.5, we get

$$\mathrm{tr}\,\widetilde{D} = -3(c+\lambda). \tag{4.10}$$

Let us now consider the second equation in (4.9). Wedging both sides by ψ_- and using the compatibility condition $\omega \wedge \psi_- = 0$, we obtain

$$\left(dw_2^- - d*_g(Jw_1 \wedge \psi_+) - 2u\,(d(Jw_1) \wedge \eta_{\mathfrak{h}} - Jw_1 \wedge d\eta)\right) \wedge \psi_- = \left(\widetilde{D}^*\rho + \lambda\rho\right) \wedge \psi_-. \tag{4.11}$$

Since $\rho = u\,\omega \wedge \eta_{\mathfrak{h}} + \psi_+$ and $\omega \wedge \psi_- = 0$, the right-hand side of (4.11) can be rewritten as follows

$$\begin{aligned}\left(\widetilde{D}^*\rho + \lambda\rho\right) \wedge \psi_- &= u\,\widetilde{D}^*\omega \wedge \eta_{\mathfrak{h}} \wedge \psi_- + \widetilde{D}^*\psi_+ \wedge \psi_- + \lambda\psi_+ \wedge \psi_- \\ &= u\,(-2\,d(Jw_1) - (c+\lambda)\,\omega) \wedge \eta_{\mathfrak{h}} \wedge \psi_- + \frac{1}{3}\,(\mathrm{tr}\,\widetilde{D})\,\omega^3 + \frac{2}{3}\,\lambda\,\omega^3 \\ &= -2u\,d(Jw_1) \wedge \eta_{\mathfrak{h}} \wedge \psi_- + \frac{1}{3}\,(\mathrm{tr}\,\widetilde{D})\,\omega^3 + \frac{2}{3}\,\lambda\,\omega^3,\end{aligned}$$

where the second equality follows from the first equation of (4.9), Lemma 4.5 and the normalization condition $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$. The summands appearing in the left-hand side of (4.11) can be rewritten as follows. Since $w_2^- \in \Lambda^2_8\mathfrak{h}^*$, we have

$$dw_2^- \wedge \psi_- = d(w_2^- \wedge \psi_-) - w_2^- \wedge d\psi_- = -w_2^- \wedge (- *_g w_2^- + Jw_1 \wedge \psi_+) = |w_2^-|_g^2 \,\mathrm{Vol}_g.$$

Since every 5-form on $\mathfrak{h}$ is closed, we get

$$\begin{aligned} -d *_g (Jw_1 \wedge \psi_+) \wedge \psi_- &= *_g(Jw_1 \wedge \psi_+) \wedge d\psi_- = *_g (Jw_1 \wedge \psi_+) \wedge (- *_g w_2^- + Jw_1 \wedge \psi_+) \\ &= Jw_1 \wedge 2 *_g (Jw_1) = 2\,|Jw_1|_g^2 \,\mathrm{Vol}_g = 2\,|w_1|_g^2 \,\mathrm{Vol}_g, \end{aligned}$$

where we used the identity (4) of Lemma 1.54. Finally, by Lemma 3.11 and the identity (3) of Lemma 1.54, we have

$$2u\, Jw_1 \wedge d\eta \wedge \psi_- = -2\, Jw_1 \wedge *_g(w_1 \wedge \psi_+) \wedge \psi_- = 4\, Jw_1 \wedge *_g(Jw_1) = 4\,|w_1|_g^2 \,\mathrm{Vol}_g.$$

Hence, equation (4.11) becomes

$$\left(|w_2^-|_g^2 + 6\,|w_1|_g^2\right) \mathrm{Vol}_g = \frac{1}{3}\,(\mathrm{tr}\,\widetilde{D})\,\omega^3 + \frac{2}{3}\,\lambda\,\omega^3.$$

Recalling that $\mathrm{Vol}_g = \frac{1}{6}\,\omega^3$, we have

$$\frac{1}{2}\,\left(|w_2^-|_g^2 + 6\,|w_1|_g^2\right) = \mathrm{tr}\,\widetilde{D} + 2\,\lambda.$$

Now, the claim follows by combining this identity with $\mathrm{tr}\,\widetilde{D} = -3(c+\lambda)$ and recalling that $|w_2^-|_g^2 + 6\,|w_1|_g^2 = |\tau|_\varphi^2$ (cf. Lemma 3.12). □

As a consequence of Proposition 4.6, we have the following.

Corollary 4.7. *Let $(\mathfrak{g}, \theta)$ be the contactization of a symplectic unimodular Lie algebra $(\mathfrak{h}, \omega_0)$ and let φ be a semi-algebraic Laplacian soliton on $\mathfrak{g}$ such that $\Delta_\varphi\varphi = \lambda\varphi + D^*\varphi$, for some $D \in \mathrm{Der}(\mathfrak{g})$. Then,*

$$\lambda = |w_2^-|_g^2 + 6\,|w_1|_g^2 = |\tau|_\varphi^2$$

and φ is expanding.

Proof. Since $\mathfrak{g}$ is the contactization of a symplectic Lie algebra $(\mathfrak{h}, \omega_0)$, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ and $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}z$. In particular, $Dz = cz$, for some $c \in \mathbb{R}$. Therefore, by Proposition 4.6, the constant λ is given by

$$\lambda = -3\,c - \frac{1}{2}\left(|w_2^-|_g^2 + 6\,|w_1|_g^2\right) = -3\,c - \frac{1}{2}\,|\tau|_\varphi^2.$$

Recall that $\omega_0 = d\theta$ on $\mathfrak{g}$. Since $D \in \mathrm{Der}(\mathfrak{g})$, we see that

$$D^*\omega_0 = D^*(d\theta) = d(D^*\theta) = d(\alpha + c\,\theta) = d\alpha + c\,\omega_0.$$

On the other hand, ω_0 is a non-degenerate 2-form on the unimodular Lie algebra $\mathfrak{h}$. Consequently,

$$\frac{1}{3}\,(\operatorname{tr}\widetilde{D})\,\omega_0^3 = \widetilde{D}^*\omega_0 \wedge \omega_0^2 = D^*\omega_0 \wedge \omega_0^2 = (d\alpha + c\,\omega_0) \wedge \omega_0^2 = c\,\omega_0^3,$$

as every 5-form on $\mathfrak{h}$ is closed. Now, from the proof of Proposition 4.6, we know that $3c = \operatorname{tr}\widetilde{D} = -3c - 3\lambda$, whence $-2c = \lambda$. Therefore, we have $\lambda = |w_2^-|_g^2 + 6\,|w_1|_g^2 = |\tau|_\varphi^2$.

To conclude the proof, we observe that $\lambda = 0$ holds if and only if φ is torsion-free. By Corollary 3.17, torsion-free G_2-structures do not occur on the contactization of any symplectic unimodular Lie algebra. Thus, $\lambda > 0$ and φ is expanding. $\square$

The previous result applies, for instance, to the nilpotent Lie algebra $\mathfrak{n}_{12}$ endowed with the closed G_2-structure considered in [45, Theorem 3.6].

Example 4.8. Consider the nilpotent Lie algebra $\mathfrak{n}_{12}$ and let $(e^1, \dots, e^7)$ be the basis of $\mathfrak{n}_{12}^*$ for which the structure equations are the following:

$$\left(0,0,0,\frac{\sqrt{3}}{6}e^{12},\frac{\sqrt{3}}{12}e^{13}-\frac{1}{4}e^{23},-\frac{\sqrt{3}}{12}e^{23}-\frac{1}{4}e^{13},\frac{\sqrt{3}}{12}e^{16}-\frac{1}{4}e^{15}+\frac{\sqrt{3}}{12}e^{25}+\frac{1}{4}e^{26}-\frac{\sqrt{3}}{6}e^{34}\right).$$

Recall that $\mathfrak{n}_{12}$ is the contactization of a 6-dimensional symplectic nilpotent Lie algebra (cf. Corollary 3.10). The 3-form

$$\varphi = e^{167} + e^{257} + e^{347} + e^{135} - e^{124} - e^{236} - e^{456}$$

defines a closed G_2-structure on $\mathfrak{n}_{12}$ inducing the metric $g_\varphi = \sum_{i=1}^7 e^i \odot e^i$. The corresponding intrinsic torsion form is $\tau = \frac{1}{2}\left(e^{56} - e^{37}\right)$. A computation shows that φ is an expanding algebraic Laplacian soliton solving the equation $\Delta_\varphi\varphi = \lambda\varphi + D^*\varphi$ with $\lambda = \frac{1}{2} = |\tau|_\varphi^2$ and

$$D = -\frac{1}{8}\operatorname{diag}(1,1,0,2,1,1,2) \in \operatorname{Der}(\mathfrak{n}_{12}).$$

In addition to $\mathfrak{n}_{12}$, also the non-abelian nilpotent Lie algebras $\mathfrak{n}_2, \dots, \mathfrak{n}_7$ admit (semi-)algebraic Laplacian solitons (see [111]). However, it is currently not known whether semi-algebraic Laplacian solitons occur on the nilpotent Lie algebras $\mathfrak{n}_8$, $\mathfrak{n}_9$, $\mathfrak{n}_{10}$ and $\mathfrak{n}_{11}$.

Using Corollary 4.7 and Proposition 3.7, we can show that semi-algebraic Laplacian solitons do not exist on $\mathfrak{n}_9$.

Proposition 4.9. *The nilpotent Lie algebra*

$$\mathfrak{n}_9 = (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25})$$

does not admit any semi-algebraic Laplacian solitons.

Proof. As we observed in Corollary 3.10, the Lie algebra $\mathfrak{n}_9$ is the contactization of the nilpotent Lie algebra $\mathfrak{h} = (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24})$ endowed with the symplectic form $\omega_0 = e^{16} + e^{34} + e^{25}$. In particular, $\mathfrak{z}(\mathfrak{n}_9) = \mathbb{R}e_7$.

By Proposition 3.7, every closed G_2-structure on $\mathfrak{n}_9$ can be written as $\varphi = \widetilde{\omega} \wedge e^7 + \rho$, where ρ is a definite 3-form on $\mathfrak{h}$, $\widetilde{\omega}$ is a symplectic form taming J_ρ and $d\rho = -\widetilde{\omega} \wedge \omega_0$ (see

Example 3.13 for an explicit case). This G_2-structure is a semi-algebraic Laplacian soliton solving $\Delta_\varphi\varphi = \lambda\varphi + D^*\varphi$, for some $\lambda \in \mathbb{R}$ and some $D \in \mathrm{Der}(\mathfrak{n}_9)$, if and only if the system (4.9) is satisfied. By Corollary 4.7, we must have $De_7 = ce_7$ and $\lambda = -2c = |\tau|^2_\varphi > 0$. Moreover, the first equation in (4.9) forces the 2-form on $\mathfrak{h}$

$$\beta := \widetilde{D}^*\widetilde{\omega} + (c+\lambda)\,\widetilde{\omega} = \widetilde{D}^*\widetilde{\omega} - c\,\widetilde{\omega}$$

to be exact. We shall show that, if this last condition holds for every derivation D and every symplectic form $\widetilde{\omega}$ on $\mathfrak{n}_9$, then $\widetilde{\omega} \wedge \omega_0$ cannot be exact.

The matrix associated with the generic derivation $D \in \mathrm{Der}(\mathfrak{n}_9)$ with respect to the basis $(e_1, \ldots, e_7)$ is given by

$$D = \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2h_1 & 0 & 0 & 0 & 0 & 0 \\ h_2 & h_3 & 3h_1 & 0 & 0 & 0 & 0 \\ h_4 & h_5 & h_3 & 4h_1 & 0 & 0 & 0 \\ h_6 & h_7 & -h_2 & 0 & 5h_1 & 0 & 0 \\ h_8 & h_9 & h_7 - h_4 & -h_2 & h_3 & 6h_1 & 0 \\ h_{10} & h_{11} & h_9 - h_6 & h_7 - 2h_4 & -h_5 - h_2 & h_3 & 7h_1 \end{pmatrix},$$

with $h_j \in \mathbb{R}$. In particular, $c = 7h_1$ and we can assume $h_1 < 0$.

The generic closed 2-form $\widetilde{\omega}$ on $\mathfrak{h}$ is

$$\widetilde{\omega} = f_1 e^{12} + f_2 e^{13} + f_3 e^{14} + f_4\left(e^{15} + e^{24}\right) + f_5\left(e^{16} + e^{34}\right) + f_6 e^{23} + f_7 e^{25} + f_8\left(e^{26} - e^{35}\right),$$

which is non-degenerate if and only if $f_5^2 f_7 - f_3 f_8^2 \neq 0$. Now, we have

$$\begin{aligned} \beta &= \widetilde{D}^*\widetilde{\omega} - 7h_1\,\widetilde{\omega} \\ &= (h_3 f_2 - 4\,h_1 f_1 + h_5 f_3 - h_4 f_4 + h_7 f_4 + h_9 f_5 - h_2 f_6 - h_6 f_7 - h_8 f_8)\,e^{12} \\ &\quad + (h_3 f_3 - 3\,h_1 f_2 - h_2 f_4 - 2\,h_4 f_5 + f_5 h_7 + h_6 f_8)\,e^{13} \\ &\quad + (h_3 f_5 - h_1 f_4 - h_2 f_8)\left(e^{15} + e^{24}\right) \\ &\quad + (h_3 f_4 - h_5 f_5 - 2\,h_1 f_6 - h_2 f_7 - f_8 h_4 + 2\,h_7 f_8)\,e^{23} \\ &\quad - 2h_1 f_3\,e^{14} + h_1 f_8\,e^{26} - h_1 f_8\,e^{35}. \end{aligned}$$

Since the space of exact 2-forms on $\mathfrak{h}$ is generated by $e^{12}, e^{13}, e^{15} + e^{24}, e^{23}$ and, since $h_1 < 0$, we see that β is exact if and only if $f_3 = f_8 = 0$. Consequently, $\widetilde{\omega}$ is non-degenerate if and only if $f_5 f_7 \neq 0$. This last constraint implies that $\widetilde{\omega} \wedge \omega_0$ cannot be exact, since the space of exact 4-forms is spanned by

$$e^{1234}, e^{1235}, e^{1245}, e^{1236}, e^{1246} + e^{1345}, e^{1256} - e^{2345}, e^{1356} - e^{2346}. \qquad \square$$

By similar arguments, the same non-existence result holds more generally on some central extensions of solvable non-nilpotent Lie algebras.

Proposition 4.10. *The 7-dimensional solvable Lie algebras*

$$\begin{aligned} \mathfrak{s}_8 &= (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, e^{34}), \\ \mathfrak{s}_9 &= (-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0, e^{34}) \end{aligned}$$

do not admit semi-algebraic Laplacian solitons.

Proof. We prove the result only for $\mathfrak{s}_8$, since the discussion for $\mathfrak{s}_9$ is analogous. We recall that $\mathfrak{s}_8$ is the central extension of $(\mathfrak{h}, \omega_0)$, where $\mathfrak{h} = \mathfrak{g}_{6,54}^{0,-1}$ and $\omega_0 = e^{34}$. Then, we consider the generic closed 2-form $\widetilde{\omega}$ and the generic 3-form ρ on $\mathfrak{h}$ satisfying the necessary condition $d\rho = -\tilde{\omega} \wedge \omega_0$. By an explicit computation using the structure equations of Table 6.2, one has

$$\widetilde{\omega} = f_1(e^{14} + e^{23}) + f_2 e^{34} + f_3(e^{16} + e^{35}) + f_4 e^{36} + f_5(e^{45} - e^{26}) + f_6 e^{46} + f_7 e^{56}$$

and

$$\begin{aligned}\rho =& p_1 e^{125} - f_3 e^{134} + p_2 e^{136} + p_3 e^{145} + (p_4 + f_7) e^{146} + p_5 e^{156} - f_5 e^{234} \\ &+ p_6 e^{235} + p_4 e^{236} + p_7 e^{246} + p_8 e^{256} + p_9 e^{345} + p_{10} e^{346} + p_{11} e^{356} + p_{12} e^{456},\end{aligned}$$

for some $f_1, \dots, f_7, p_1, \dots, p_{12} \in \mathbb{R}$. In particular $\tilde{\omega}^3 = 6 f_1^2 f_7 e^{123456} \neq 0$ if and only if $f_1 f_7 \neq 0$. Now, by (4.9) and (4.10), in order to have a semi-algebraic Laplacian soliton on $\mathfrak{s}_8$, we have to require that the 2-form

$$\tilde{D}^* \tilde{\omega} - \frac{1}{3}(\operatorname{tr} \tilde{D}) \tilde{\omega}$$

is exact on $\mathfrak{h}$, for some $\tilde{D} \in \operatorname{Der}(\mathfrak{h})$. The matrix of the generic derivation D of $\mathfrak{s}_8$ is given by

$$D = \begin{pmatrix} h_1 & 0 & h_2 & 0 & h_3 & h_4 & 0 \\ 0 & h_5 & 0 & h_6 & h_7 & h_8 & 0 \\ 0 & 0 & h_9 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_9 + h_5 - h_1 & 0 & -h_7 & 0 \\ 0 & 0 & 0 & 0 & -h_9 + h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_7 & h_3 & h_{10} & h_{11} & 2h_9 + h_5 - h_1 \end{pmatrix},$$

for some $h_1, \dots, h_{11} \in \mathbb{R}$. In particular, $\alpha = -h_7 e^3 + h_3 e^4 + h_{10} e^5 + h_{11} e^6$ and $c = 2h_9 + h_5 - h_1$. If we compute $\tilde{D}^* \tilde{\omega} - \frac{1}{3}(\operatorname{tr} \tilde{D}) \tilde{\omega}$ for the generic $\tilde{D} = \pi_{\mathfrak{h}} \circ D|_{\mathfrak{h}}$ on $\mathfrak{h}$, we have that its projection onto e^{14} has to vanish, since e^{14} is not exact on $\mathfrak{h}$. This forces

$$\frac{1}{3} f_1 (h_5 + 2h_9 - h_1) = 0,$$

hence $h_1 = h_5 + 2h_9$. Then, $c = 0$ and $\lambda = -(h_5 + h_9) \leq 0$, by Proposition 4.6. In particular, by (4.10), $\lambda = 0$ holds if and only if $\operatorname{tr} \tilde{D} = 0$ which corresponds to the case where the G_2-structure $\varphi = \rho + \tilde{\omega} \wedge e^7$ is torsion-free. However, $\mathfrak{s}_8$ cannot admit torsion-free G_2-structures by Corollary 3.17. In particular, we may assume $h_5 > -h_9$. Let us look at the second equation in (4.9). We note that the projection of the 3-form $\tilde{D}^* \rho + \lambda \rho + \tilde{\omega} \wedge \alpha$ onto e^{125} has to vanish, since $e^{125} \notin \mathbb{R}\omega_0 \wedge \mathfrak{h}^*$ and it is not exact on $\mathfrak{h}$. This is equivalent to $2p_1(h_5 + h_9) = 0$, which implies $p_1 = 0$, since $h_5 > -h_9$. It is straightforward to check that $p_1 = 0$ implies that the bilinear form b_φ defined on $\mathfrak{s}_8$ is never positive-definite, since $b_\varphi(e_1, e_1) = b_\varphi(e_2, e_2) = 0$. □

By [27] and Theorem 3.15, we know that a 7-dimensional unimodular Lie algebra with 1-dimensional center admitting closed G_2-structures is isomorphic to one among $\mathfrak{n}_8$, $\mathfrak{n}_9$, $\mathfrak{n}_{10}$,

$\mathfrak{n}_{11}$, $\mathfrak{n}_{12}$, $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, $\mathfrak{s}_8$, $\mathfrak{s}_9$, $\mathfrak{s}_{10}$ and $\mathfrak{s}_{11}$. By Proposition 4.9 and Proposition 4.10, the only ones which may admit semi-algebraic Laplacian solitons are $\mathfrak{n}_8$, $\mathfrak{n}_{10}$, $\mathfrak{n}_{11}$, $\mathfrak{n}_{12}$, $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, $\mathfrak{s}_{10}$ and $\mathfrak{s}_{11}$.

Examples of expanding semi-algebraic Laplacian solitons are known on $\mathfrak{n}_{12}$ (see Example 4.8) and on the Lie algebras $\mathfrak{s}_2$ and $\mathfrak{s}_4$ (see [56, Proposition 6.5]). In the remaining cases, it is still not known whether semi-algebraic Laplacian solitons exist. However, if there are any, they must be expanding. This follows from Corollary 4.7 when the Lie algebra is one among $\mathfrak{n}_{10}$, $\mathfrak{n}_{11}$, $\mathfrak{n}_{12}$, $\mathfrak{s}_{10}$ and $\mathfrak{s}_{11}$, while it follows by a direct computation when the Lie algebra is one among $\mathfrak{n}_8$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ and $\mathfrak{s}_4$. We shall see it in detail in the next proposition.

Proposition 4.11. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra isomorphic to either $\mathfrak{n}_8$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ or $\mathfrak{s}_4$. If $\mathfrak{g}$ admits a semi-algebraic Laplacian soliton, then it is expanding.*

Proof. We work case-by-case. Let us start with the Lie algebra $\mathfrak{n}_8$, which is the central extension of $(\mathfrak{h}, \omega_0)$, where $\mathfrak{h}$ is the nilpotent Lie algebra whose structure equations are

$$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24})$$

and

$$\omega_0 = e^{16} + e^{34}.$$

The generic closed 2-form in this coframe is

$$\widetilde{\omega} = f_1 e^{12} + f_2 e^{13} + f_3 e^{14} + f_4(e^{15} + e^{24}) + f_5(e^{16} + e^{34}) + f_6 e^{23} + f_7 e^{24} + f_8 e^{25} + f_9(e^{26} - e^{35}),$$

for some $f_i \in \mathbb{R}$, $i = 1, \ldots, 9$. In particular, $\tilde{\omega}^3 = 6(f_5^2 f_8 - f_3 f_9^2)e^{123456}$. The matrix of the generic derivation D of $\mathfrak{n}_8$ is given by

$$D = \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 & 0 & 0 \\ h_3 & h_4 & h_1 + h_2 & 0 & 0 & 0 & 0 \\ h_5 & h_6 & h_4 & 2h_1 + h_2 & 0 & 0 & 0 \\ h_7 & h_8 & -h_3 & 0 & h_1 + 2h_2 & 0 & 0 \\ h_9 & h_{10} & -h_5 + h_8 & -h_3 & h_4 & 2(h_1 + h_2) & 0 \\ h_{11} & h_{12} & h_{10} & h_8 - 2h_5 & -h_6 & h_4 & 2h_2 + 3h_1 \end{pmatrix},$$

for some $h_1, \ldots, h_{12} \in \mathbb{R}$. In particular, $\alpha = h_{11}e^1 + h_{12}e^2 + h_{10}e^3 + (h_8 - 2h_5)e^4 - h_6 e^5 + h_4 e^6 \in \mathfrak{h}^*$ and $c = 2h_2 + 3h_1$. Since the 2-form

$$\tilde{D}^* \tilde{\omega} - \frac{1}{3}(\operatorname{tr} \tilde{D})\tilde{\omega}$$

has to be exact on $\mathfrak{h}$, its projections onto $e^{14}, e^{16}, e^{25}, e^{26}, e^{34}, e^{35}$ have to vanish. This is equivalent to the following system of equations

$$\begin{cases} f_3(h_1 - 2h_2) = 0, \\ f_5(2h_1 - h_2) = 0, \\ f_8(2h_1 - h_2) = 0, \\ f_9(h_1 - 2h_2) = 0. \end{cases} \tag{4.12}$$

If $h_1 = 2h_2$, then $c = 8h_2$, $\operatorname{tr} D = 29h_2$ and $\lambda = -15h_2$. In particular, by Proposition 4.6, $h_2 \leq 0$, so that $\lambda \geq 0$ holds, with $\lambda = 0$ only if the semi-algebraic Laplacian soliton arises from a torsion-free G_2-structure, which is impossible by Corollary 3.17. If $h_1 \neq 2h_2$, instead, (4.12) implies $f_3 = f_9 = 0$ and, since the vanishing of both f_5 and f_8 implies the degeneracy of $\tilde{\omega}$, (4.12) admits solutions if and only if $h_2 = 2h_1$. In particular, $\lambda > 0$ holds as before.

Now, let us consider the Lie algebra $\mathfrak{s}_2$. We recall that this Lie algebra is obtained as the Lie algebra direct sum $\mathfrak{g}_{6,54}^{0,-1} \oplus \mathbb{R}$, so in this case we have $\omega_0 = de^7 = 0$. The matrix of the generic derivation D of $\mathfrak{s}_2$ is given by

$$D = \begin{pmatrix} h_1 & 0 & h_2 & 0 & h_3 & h_4 & 0 \\ 0 & h_5 & 0 & h_6 & h_7 & h_8 & 0 \\ 0 & 0 & h_9 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_9 + h_5 - h_1 & 0 & -h_7 & 0 \\ 0 & 0 & 0 & 0 & -h_9 + h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{10} & h_{11} & h_{12} \end{pmatrix},$$

$h_1, \ldots, h_{12} \in \mathbb{R}$. In particular, $\alpha = h_{10}e^5 + h_{11}e^6 \in \mathfrak{h}^*$ is closed. We note this is always true when $\omega_0 = 0$, since, by differentiating $D^*e^7 = \alpha + ce^7$, we get $0 = d\alpha$ since $d \circ D = D \circ d$. In particular, (4.9) reduces to asking that the forms

$$\tilde{D}^*\tilde{\omega} - \frac{1}{3}(\operatorname{tr} \tilde{D})\tilde{\omega}, \quad \tilde{D}^*\rho + \lambda\rho + \tilde{\omega} \wedge \alpha$$

are exact on $\mathfrak{g}_{6,54}^{0,-1}$, where $\lambda = -\frac{1}{3}(\operatorname{tr} D + 2c)$. Let us consider a pair $(\tilde{\omega}, \rho) \in \Lambda^2\mathfrak{h} \times \Lambda^3\mathfrak{h}$ of generic forms closed forms on $\mathfrak{h} = \mathfrak{g}_{6,54}^{0,-1}$. Then,

$$\begin{aligned} \widetilde{\omega} =& f_1(e^{14} + e^{23}) + f_2(e^{16} + e^{35}) + f_3(e^{45} - e^{26}) + f_4e^{34} + f_5e^{36} + f_6e^{46} + f_7e^{56}, \\ \rho =& p_1e^{125} + p_2e^{136} + p_3e^{145} + p_4(e^{146} + e^{236}) + p_5e^{156} + p_6e^{235} + p_7e^{246} + p_8e^{256} \\ &+ p_9e^{345} + p_{10}e^{346} + p_{11}e^{356} + p_{12}e^{456}, \end{aligned}$$

for some $f_1, \ldots, f_7, p_1, \ldots, p_{12} \in \mathbb{R}$. We recall that this is equivalent to the closure of the 3-form $\varphi = \rho + \tilde{\omega} \wedge e^7$ on $\mathfrak{s}_2$, since $\omega_0 = 0$. Then, $\tilde{\omega}^3 = 6f_1^2f_7e^{123456}$, so we may assume both f_1 and f_7 different from zero. Now, since e^{14} is not exact on $\mathfrak{g}_{6,54}^{0,-1}$, the projection of $\tilde{D}^*\tilde{\omega} - \frac{1}{3}(\operatorname{tr} \tilde{D})\tilde{\omega}$ onto this direction, given by $\frac{1}{3}f_1(h_5 + 2h_9 - h_1)$, has to vanish. This occurs if and only if $h_1 = h_5 + 2h_9$, since $f_1 \neq 0$. Analogously, the projection of $\tilde{D}^*\rho + \lambda\rho + \tilde{\omega} \wedge \alpha$ onto e^{125}, given by $p_1(2(h_5 + h_9) - h_{12})$, has to vanish, since e^{125} is not exact. Let us suppose $p_1 = 0$, first. A direct computation shows that, if we consider the generic closed 3-form $\varphi = \rho + \tilde{\omega} \wedge e^7$, then the bilinear symmetric tensor b_φ is never positive-definite, since $b_\varphi(1,1) = b_\varphi(2,2) = 0$. If $p_1 \neq 0$ instead, then $h_{12} = 2(h_5 + h_9)$ which implies

$$\operatorname{tr} D = \frac{5}{2}c, \quad \lambda = -\frac{3}{2}c.$$

In particular $\operatorname{tr} D - 7c = -\frac{9}{2}c$. Then, by Proposition 4.6, $\lambda \geq 0$ holds, with $\lambda = 0$ only in the torsion-free case, which cannot occur by Corollary 3.17. This concludes the proof in the case where $\mathfrak{g}$ is isomorphic to $\mathfrak{s}_2$.

By a very similar discussion, the same result holds for $\mathfrak{s}_3$ and $\mathfrak{s}_4$. □

When the center of the Lie algebra is at least 2-dimensional, we have the following classification result.

Theorem 4.12. *Let* $\mathfrak{g}$ *be a* 7*-dimensional unimodular Lie algebra with* $\dim \mathfrak{z}(\mathfrak{g}) \geq 2$ *admitting closed* G_2*-structures. Then,* $\mathfrak{g}$ *admits semi-algebraic Laplacian solitons if and only if it is isomorphic to one among* $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$, $\mathfrak{n}_4$, $\mathfrak{n}_5$, $\mathfrak{n}_6$, $\mathfrak{n}_7$, $\mathfrak{s}_5$, $\mathfrak{s}_6$ *and* $\mathfrak{s}_7$.

Proof. If $\mathfrak{g}$ is nilpotent, then it must be isomorphic to one among $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$, $\mathfrak{n}_4$, $\mathfrak{n}_5$, $\mathfrak{n}_6$ and $\mathfrak{n}_7$, by the classification result in [27]. Every G_2-structure φ on the abelian Lie algebra $\mathfrak{n}_1$ is torsion-free and thus solves the equation $\Delta_\varphi \varphi = \lambda\varphi + D^*\varphi$ with $\lambda = 0$ and $D = 0 \in \mathrm{Der}(\mathfrak{n}_1)$. In the remaining cases, the existence of semi-algebraic Laplacian solitons is known from [111].

We can then focus on the case where $\mathfrak{g}$ is solvable non-nilpotent. By Theorem 3.15, $\mathfrak{g}$ must be isomorphic to one among $\mathfrak{s}_1$, $\mathfrak{s}_5$, $\mathfrak{s}_6$ and $\mathfrak{s}_7$. Examples of semi-algebraic Laplacian solitons on $\mathfrak{s}_5$ and $\mathfrak{s}_6$ were given in [56, Proposition 6.5]. By Corollary 3.17, the Lie algebra $\mathfrak{s}_7$ admits torsion-free G_2-structures, which are semi-algebraic Laplacian solitons with $\lambda = 0$ and $D = 0 \in \mathrm{Der}(\mathfrak{s}_7)$.

To conclude the proof, we must show that the Lie algebra $\mathfrak{s}_1$ does not admit any semi-algebraic Laplacian solitons. Let us assume by contradiction that φ is a semi-algebraic Laplacian soliton on $\mathfrak{s}_1$. Then, as $\mathfrak{s}_1 \cong \mathfrak{g}^0_{6,38} \oplus \mathbb{R}$, we can write $\varphi = \widetilde{\omega} \wedge e^7 + \rho$, where e^7 spans $\mathbb{R}^*$ and $\widetilde{\omega}$ and ρ are closed forms on $\mathfrak{g}^0_{6,38}$. In particular, we have

$$
\begin{aligned}
\widetilde{\omega} &= f_1\left(2\,e^{16} + e^{25} - e^{34}\right) + f_2\,e^{23} + f_3\left(e^{24} + e^{35}\right) + f_4\,e^{26} + f_5\,e^{36} + f_6\,e^{46} + f_7\,e^{56},\\
\rho &= p_1\,e^{123} + p_2\left(e^{124} + e^{135}\right) + p_3\,e^{126} + p_4\,e^{136} + p_5\left(e^{146} - e^{235}\right) + p_6\left(e^{156} + e^{234}\right)\\
&\quad + p_7\,e^{236} + p_8\,e^{246} + p_9\,e^{256} + p_{10}\,e^{346} + p_{11}\,e^{356} + p_{12}\,e^{456},
\end{aligned}
$$

with the $f_1, \ldots, f_7, p_1, \ldots, p_{12} \in \mathbb{R}$. The symmetric bilinear form b_φ induced by φ as in (1.1) satisfies $b_\varphi(e_1, e_1) = -2p_2^2 f_1\,e^{1234567}$ and $b_\varphi(e_4, e_4) = -p_2 f_1 p_{12} e^{1234567}$. Since b_φ is definite, we must have $p_2 p_{12} f_1 \neq 0$.

The generic derivation $D \in \mathrm{Der}(\mathfrak{s}_1)$ has the following expression with respect to the basis $(e_1, \ldots, e_7)$ of $\mathfrak{s}_1$:

$$
D = \begin{pmatrix}
h_1 & h_2 & h_3 & 0 & 0 & h_4 & h_5\\
0 & \frac{1}{2}h_1 & h_6 & 0 & 0 & h_2 & 0\\
0 & -h_6 & \frac{1}{2}h_1 & 0 & 0 & h_3 & 0\\
0 & h_7 & h_8 & \frac{1}{2}h_1 & h_6 & h_9 & 0\\
0 & -h_8 & h_7 & -h_6 & \frac{1}{2}h_1 & h_{10} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & h_{11} & h_{12}
\end{pmatrix},
$$

with $h_i \in \mathbb{R}$. Since φ is a semi-algebraic Laplacian soliton, there is some $\lambda \in \mathbb{R}$ such that the 3-form $D^*\varphi + \lambda\varphi$ on $\mathfrak{s}_1$ is exact. Under the constraint $p_2 p_{12} f_1 \neq 0$, this implies that $\lambda = 0$ and that $\mathfrak{z}(\mathfrak{s}_1) = \langle e_1, e_7 \rangle \subset \ker D$. By Proposition 4.6, we then have $|\tau|_\varphi = 0$, i.e., the G_2-structure φ is torsion-free. However, $\mathfrak{s}_1$ does not carry any torsion-free G_2-structures by Corollary 3.17. $\square$

Chapter 5

Balanced SU(3)-structures of cohomogeneity one

A Hermitian structure (g, J) on a $2n$-dimensional manifold M is called *balanced* if $d\omega^{n-1} = 0$, ω being the fundamental form associated with (g, J). Balanced metrics have been extensively studied in [11, 52, 60, 61, 66, 106, 114]. In this chapter, we consider balanced metrics on complex manifolds with holomorphically trivial canonical bundle, most commonly known as balanced SU(n)-structures. Such structures are of interest for both Hermitian geometry and string theory, especially in dimension $2n = 6$, since they provide the ideal setting for the Hull–Strominger system. We provide a non-existence result for balanced non-Kähler SU(3)-structures which are invariant under a cohomogeneity one action on simply connected 6-manifolds. The discussion and the main results of this chapter were published in [5].

5.1 Cohomogeneity one manifolds

Here, we recall the basic structure of cohomogeneity one manifolds. For further details, see for instance [3, 13, 84, 85, 132].

Definition 5.1. A cohomogeneity one manifold is a connected smooth manifold M with a left action $\alpha\colon G \times M \to M$ of a compact Lie group G having an orbit of codimension one.

From now on, let us assume that M is a simply connected cohomogeneity one manifold and that G is connected. By the compactness of G, the action α is proper and there exists a G-invariant Riemannian metric g on M (cf. Theorem 1.15). This is equivalent to saying that the G-action on (M, g) is isometric (cf. Definition 1.14). Moreover, we assume that the action α is almost effective. As usual, we denote by $\pi\colon M \to M/G$ the canonical projection and we equip M/G with the quotient topology relative to π. By a result of Bérard Bergery [13], the quotient space M/G is homeomorphic to a circle or an interval. As we are assuming that M is simply connected, we have that M/G is homeomorphic to an interval I. The inverse images of the interior points of the orbit space M/G are *principal orbits* in the sense of Definition 1.22, while the inverse images of the boundary points are *singular orbits* in the sense of Definition 1.26. As usual, we denote by M^{princ} the union of all principal orbits, which is an open dense subset of M (cf. Theorem 1.25) and by G_p the isotropy group at $p \in M$.

First, we will suppose M is compact. It follows that M/G is homeomorphic to the closed interval $I = [-1, 1]$. Denote by $\mathcal{O}_1$ and $\mathcal{O}_2$ the two singular orbits $\pi^{-1}(-1)$ and $\pi^{-1}(1)$, respectively, and fix $q_1 \in \mathcal{O}_1$. By the compactness of the G-orbits, there exists a minimizing geodesic $\gamma_{q_1}\colon [-1, 1] \to M$ from q_1 to $\mathcal{O}_2$ which is orthogonal to every principal orbit. We call *normal geodesic* a geodesic orthogonal to every principal orbit. Let $\gamma\colon [-1, 1] \to M$ be a normal geodesic between $\pi^{-1}(-1)$ and $\pi^{-1}(1)$. Up to rescaling, we can always suppose that the orbit space M/G is such that $\pi \circ \gamma = \mathrm{Id}_{[-1,1]}$. Then, by Kleiner's Lemma, there exists a subgroup K of G such that $G_{\gamma(t)} = K$ for all $t \in (-1, 1)$ and K is subgroup of $G_{\gamma(-1)}$ and $G_{\gamma(1)}$ (cf. also Proposition 1.23).

For M non-compact, M/G is homeomorphic either to an open interval or to an interval with a closed end. In the former case, M is a product manifold $M \cong I \times G/K$. In the latter case, there exists exactly one singular orbit and $M/G \cong I$ where $I = [0, L)$ and L is either a positive number or $+\infty$. Analogously to the compact case, there exists a normal geodesic $\gamma\colon [0, L) \to M$ such that $\gamma(0) \in \pi^{-1}(0)$ and we can suppose $\pi \circ \gamma = \mathrm{Id}_{[0,L)}$. In addition, there exists a subgroup K of G such that $G_{\gamma(t)} = K$ for all $t \in (0, L)$ and K is a subgroup of $H \coloneqq G_{\gamma(0)}$. So, we have

- $\pi^{-1}(t) \cong G/K$ for all $t \in \mathring{I}$,
- $M^{\mathrm{princ}} = \bigcup_{t\in\mathring{I}} \pi^{-1}(t) = \bigcup_{t\in\mathring{I}} G \cdot \gamma(t)$,
- for every $p_1, p_2 \in M^{\mathrm{princ}}$, $G \cdot p_1$ and $G \cdot p_2$ are diffeomorphic.

Therefore, up to conjugation along the orbits, when M is compact we have three possible isotropy groups $H_1 \coloneqq G_{\gamma(-1)}$, $H_2 \coloneqq G_{\gamma(1)}$ and $K \coloneqq G_{\gamma(t)}$, $t \in (-1, 1)$. When M is non-compact and has one singular orbit, instead, we have two possible isotropy groups $H \coloneqq G_{\gamma(0)}$ and $K \coloneqq G_{\gamma(t)}$, $t \in (0, L)$. From all of the above, we have

$$M^{\mathrm{princ}} \cong \mathring{I} \times G/K,$$

so that, by fixing a suitable global coordinate system, we can decompose the G-invariant metric g as

$$g_{\gamma(t)} = dt^2 + g_t, \tag{5.1}$$

where dt^2 is the $(0, 2)$-tensor corresponding to the vector field $\xi \coloneqq \gamma'(t)$ evaluated at the point $\gamma(t)$ and g_t is a G-invariant metric on the homogeneous orbit $G\cdot\gamma(t)$ through the point $\gamma(t) \in M$.

Now, assume M is compact. By the density of M^{princ} in M and Theorem 1.21, M is homotopically equivalent to

$$\left(G \times_{H_1} S_{\gamma(-1)}\right) \cup_{G/K} \left(G \times_{H_2} S_{\gamma(1)}\right), \tag{5.2}$$

where the geodesic balls $S_{\gamma(\pm1)} \coloneqq \exp\left(B_{\varepsilon^\pm}(0)\right)$, $B_{\varepsilon^\pm}(0) \subset T_{\gamma(\pm1)}\left(G \cdot \gamma(\pm1)\right)^\perp$, are normal slices to the singular orbits in $\gamma(\pm1)$. Here, $G \times_{H_i} S_{\gamma(\pm1)}$ is the associated fiber bundle to the principal bundle $G \to G/H_i$ with type fiber $S_{\gamma(\pm1)}$. By Theorem 1.18, M is also homotopically equivalent to

$$\left(G \times_{H_1} B_{\varepsilon^-}(0)\right) \cup_{G/K} \left(G \times_{H_2} B_{\varepsilon^+}(0)\right). \tag{5.3}$$

The isotropy groups H_i act on $B_{\varepsilon^\pm}(0)$ via the slice representation and, since the boundary of the tubular neighborhood $\mathrm{Tub}(\mathcal{O}_i) := G \times_{H_i} B_{\varepsilon^\pm}(0)$, $i = 1, 2$, is identified with the principal orbit G/K and the G-action on $\mathrm{Tub}(\mathcal{O}_i)$ is identified with the H_i-action on $B_{\varepsilon^\pm}(0)$, then H_i acts transitively on the sphere $S^{l_i} := \partial B_{\varepsilon^\pm}$, $l_i > 0$ again with isotropy K. The normal spheres S^{l_i} are thus the homogeneous spaces H_i/K, $i = 1, 2$. The H_i-action on S^{l_i}, $i = 1, 2$, may be ineffective, but it is sufficient to quotient H_i by the ineffective kernel to obtain an effective action. Transitive effective actions of compact Lie groups on spheres were classified by Borel and are summarized in Table 5.1.

H	$\mathrm{SO}(n)$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathrm{Sp}(n)\mathrm{U}(1)$	$\mathrm{Sp}(n)$	G_2	$\mathrm{Spin}(7)$	$\mathrm{Spin}(9)$
K	$\mathrm{SO}(n-1)$	$\mathrm{U}(n-1)$	$\mathrm{SU}(n-1)$	$\mathrm{Sp}(n-1)\mathrm{Sp}(1)$	$\mathrm{Sp}(n-1)\mathrm{U}(1)$	$\mathrm{Sp}(n-1)$	$\mathrm{SU}(3)$	G_2	$\mathrm{Spin}(7)$
$S^l = H/K$	S^{n-1}	S^{2n-1}		S^{4n-1}			S^6	S^7	S^{15}

Table 5.1: Transitive effective actions of compact Lie groups on spheres

The collection of G with its isotropy groups $G \supset H_1, H_2 \supset K$ is called the *group diagram* of the cohomogeneity one manifold M. Conversely, let $G \supset H_1, H_2 \supset K$ be compact groups with $H_i/K = S^{l_i}$, $i = 1, 2$. By the classification of transitive actions on spheres, one has that the H_i-action on S^{l_i} is linear, so that it can be extended to an action on $B_{\varepsilon^\pm}(0)$ bounded by S^{l_i}, $i = 1, 2$. Therefore, (5.3) defines a cohomogeneity one manifold M. Analogously, if M is a non-compact cohomogeneity one manifold with one singular orbit, we define the *group diagram* of M to be the collection of G and the isotropy groups $G \supset H \supset K$, where the homogeneous space H/K is a sphere. The converse is also true: the group diagram defines a non-compact cohomogeneity one manifold M. In these cases, M is homotopically equivalent to $G \times_H B_\varepsilon(0)$, where $B_\varepsilon(0) \subseteq T_{\gamma(0)}(G \cdot \gamma(0))^\perp$ as before.

We study cohomogeneity one manifolds up to the equivalence given in Definition 1.19. Moreover, if a cohomogeneity one manifold M has group diagram $G \supset H_1, H_2 \supset K$ or $G \supset H \supset K$, one can show that each of the following operations results in a G-equivariantly diffeomorphic manifold:

1. switching H_1 and H_2,

2. conjugating each group in the diagram by the same element of G,

3. replacing H_i (respectively H) with aH_ia^{-1} (respectively aHa^{-1}), where a is an element of the identity component of the normalizer of K.

5.2 Balanced SU(3)-structures on cohomogeneity one 6-manifolds

We recall that an SU(3)-structure (ω, ψ_+) on a 6-manifold M is called *balanced* if

$$\begin{cases} d\psi_+ = 0, \\ d\psi_- = 0, \\ d\omega^2 = 0. \end{cases}$$

where $\psi_- = J\psi_+$, J being the almost complex structure on M induced by the 3-form ψ_+. We denote by $\Psi = \psi_+ + i\psi_-$ the induced $(3,0)$-form on M. Following [49] and Remark 1.38, a balanced SU(3)-structure can be equivalent defined as a triple (g, J, Ψ) of tensors on M satisfying the following conditions:

- J is integrable, i.e., (M, J) is a complex manifold. We recall that, for SU(3)-structures, the integrability of J is equivalent to the condition $(d\Psi)^{2,2} = 0$,
- Ψ is a nowhere-vanishing holomorphic $(3,0)$-form,
- $d\omega^2 = 0$, ω being the associated fundamental form.

In particular, we are interested in the non-Kählerian case, i.e., $d\omega \neq 0$.

Remark 5.2. We can equivalently say that an SU(3)-structure (g, J, Ψ) on M is balanced if and only if

$$\begin{cases} d\Psi = 0, \\ d\omega^2 = 0, \end{cases}$$

since $d\psi_\pm = 0$ if and only if $d\Psi = 0$. Moreover, $d\Psi = 0$ if and only if $\Psi = \psi_+ + i\psi_-$ is holomorphic and the induced almost complex structure $J = J_{\psi_+}$ is integrable.

Remark 5.3. From the formulas in [10], we have that, if (ω, ψ_+) is a balanced non-Kähler SU(3)-structure on a 6-dimensional smooth manifold M, the scalar curvature of the associated metric g is strictly negative.

Let M be a simply connected cohomogeneity one 6-manifold for the almost effective action of a compact connected Lie group G and let (ω, ψ_+) be a balanced SU(3)-structure on M which is invariant under the cohomogeneity one action. We are thus requiring G to preserve the SU(3)-structure on M. For the convenience of the reader, we recall that

- G preserves the induced metric g if and only if α^h is an isometry for g, for each $h \in G$,
- G preserves the induced complex structure J if and only if J commutes with the differential $d\alpha^h$, for each $h \in G$,
- G preserves the induced 3-form Ψ if and only if $(\alpha^h)^*\Psi = \Psi$, for each $h \in G$.

In particular, this implies that the principal isotropy K acts on T_pM preserving $(\omega_p, (\psi_+)_p)$ for every $p \in M$, which means that K can be identified with a subgroup of SU(3). Now, since the J-invariant K-action fixes the subspace $\langle \xi|_p \rangle$ of T_pM, then it fixes $\langle J\xi|_p \rangle$ as well. Let us write T_pM as

$$T_pM = \langle \xi|_p \rangle \oplus \langle J\xi|_p \rangle \oplus V,$$

where V is the 4-dimensional g_p-orthogonal complement of $\langle \xi|_p, J\xi|_p \rangle$ in T_pM. Notice that V is J_p- and K-invariant. To see the K-invariance, let $h \in K$ and $v \in V$ be generic. Then, if $d\alpha_p^h(v) = \lambda(J\xi|_p) + w$, for some $\lambda \in \mathbb{R}$, $w \in V$, we would have $J\left(d\alpha_p^h(v)\right) = d\alpha_p^h(J_pv) = -\lambda\,\xi|_p + J_pw$, which contradicts the fact that the K-action is closed along the G-orbits. Therefore, for each $h \in K$, its action on T_pM is described by a 6×6 block matrix

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline & & \\ \multicolumn{2}{c|}{0} & A \\ & & \end{array}\right)$$

with respect to the decomposition of $T_pM = \langle \xi|_p \rangle \oplus \langle J\xi|_p \rangle \oplus V$. Since the matrix above lies in SU(3), we have $A \in \mathrm{SU}(2)$, hence K can be identified with a subgroup of SU(2). Therefore, $\mathfrak{k} := \mathrm{Lie}(K)$ is $\{0\}$, $\mathbb{R}$, or $\mathfrak{su}(2)$. As observed in [117], all the possible candidate pairs $(\mathfrak{g}, \mathfrak{k})$, with $\mathfrak{g}$ compact, which may admit an SU(3)-structure in cohomogeneity one are:

(a) $\mathfrak{k} = \{0\}$ and

 (1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathbb{R}$,

 (2) $\mathfrak{g} = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{5\text{ times}}$,

(b) $\mathfrak{k} = \mathbb{R}$ and

 (1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

 (2) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$,

 (3) $\mathfrak{g} = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{6\text{ times}}$,

(c) $\mathfrak{k} = \mathfrak{su}(2)$ and

 (1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathbb{R}$,

 (2) $\mathfrak{g} = \mathfrak{su}(2) \oplus \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{5\text{ times}}$,

 (3) $\mathfrak{g} = \mathfrak{su}(3)$.

Under the assumption of simply connectedness of M, we can discard some pairs of this list. If M is compact, by Hoelcher's classification [84, Proposition 3.1], we can readily discard cases (a.2), (b.2), (b.3), (c.1) and (c.2). For the case where M is non-compact and has one singular orbit, we can suitably adapt [84, Proposition 1.8] which deals with the compact case, to obtain:

Proposition 5.4. *Let M be the non-compact cohomogeneity one manifold given by the group diagram $G \supset H \supset K$ with $H/K = S^l$. Then, $\pi_1(M) \cong \pi_1(G/K)/N$ where*

$$N = \ker\{\pi_1(G/K) \to \pi_1(G/H)\} = \mathrm{Im}\{\pi_1(H/K) \to \pi_1(G/K)\}.$$

In particular, M is simply connected if and only if the image of $\pi_1(S^l)$ generates $\pi_1(G/K)$ under the natural inclusions.

We know that $\pi_1(S^l)$ is either $\{0\}$ or $\mathbb{Z}$. Now, we observe that

$\pi_1(G/K) = \mathbb{Z}^2$ in cases (a.1) and (c.1),

$\pi_1(G/K) = \mathbb{Z}^5$ in cases (a.2), (b.3) and (c.2),

$\pi_1(G/K) = \mathbb{Z}^2$ or $\pi_1(G/K) = \mathbb{Z}^3$ in case (b.2).

If M is non-compact and has no singular orbits, $\pi_1(M) = \pi_1(G/K)$. Hence, when M is non-compact, we can discard the pairs (a.1), (a.2), (b.2), (b.3), (c.1) and (c.2), as $\pi_1(M)$ would be infinite. Therefore, the possible pairs which may admit a balanced SU(3)-structure on a simply connected manifold of cohomogeneity one under the almost effective action of a compact connected Lie group G are (a.1) (only when M is compact), (b.1) and (c.3).

Remark 5.5. In case (b.1), we shall need to divide the discussion depending on the embeddings of $\mathfrak{k} = \mathbb{R}$ in $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which, up to isomorphism, are all generated by an element of the form

$$\begin{pmatrix} ip & 0 & 0 & 0 \\ 0 & -ip & 0 & 0 \\ 0 & 0 & iq & 0 \\ 0 & 0 & 0 & -iq \end{pmatrix} \in \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

with fixed $p, q \in \mathbb{N}$. Up to uniform rescalings, which do not change the immersion of $\mathfrak{k}$, we can assume either $(p,q) = (1,0)$ or p, q to be coprime, if neither is zero. Notice that, when $(p,q) = (1,1)$ or $(p,q) = (1,0)$, $\mathfrak{k}$ induces a decomposition of $\mathfrak{g}$ in $\mathrm{Ad}(K)$-modules, some of which are equivalent. In the former case, we shall say that $\mathfrak{k}$ is diagonally embedded in $\mathfrak{g}$, while, in the latter, $\mathfrak{k}$ is said to be trivially embedded in one of the two $\mathfrak{su}(2)$-factors of $\mathfrak{g}$. Instead, when p and q are different and non-zero, the $\mathrm{Ad}(K)$-modules are pairwise inequivalent.

From now on, for each $p \in M^{\mathrm{princ}}$, let $\mathfrak{m}_p =: \mathfrak{m}$ be an $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. For each $p \in M^{\mathrm{princ}}$, we have $T_pM = \langle \xi|_p \rangle \oplus \widehat{\mathfrak{m}}|_p$, where, for every $X \in \mathfrak{g}$, we denoted by $\widehat{X}$ the action field as defined in Definition 1.11. It is known that, since $M^{\mathrm{princ}} \cong \overset{\circ}{I} \times G/K$, every G-invariant structure on M^{princ} can be expressed via a K-invariant structure on $\langle \xi \rangle \oplus \widehat{\mathfrak{m}}$, with $C^\infty(\overset{\circ}{I})$-coefficients. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r$ be the decomposition of $\mathfrak{m}$ into irreducible $\mathrm{Ad}(K)$-modules. Recall that, if the $\mathfrak{m}_i$'s are pairwise inequivalent, then they are orthogonal with respect the metric g_t, for every t (see (5.1)). Otherwise, the expression of the metric strongly depends on the specific equivalence of the modules. In all cases, we recover the whole SU(3)-structure from a pair of G-invariant stable forms (ω, ψ_+) of degree two and three, respectively.

To fix the notations, in what follows, we shall denote by

- $\tilde{\mathcal{B}} = -\mathcal{B}$ the opposite of the Killing Cartan form on $\mathfrak{g}$,
- $(\tilde{e}_i)_{i=1,2,3}$ the standard basis for $\mathfrak{su}(2)$ given by

$$\tilde{e}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tilde{e}_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \tilde{e}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

- $(f_i)_{i=1,\dots,m}$ the generic basis for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{k} = \langle f_1, \dots, f_k \rangle$, $\mathfrak{m} = \langle f_{k+1}, \dots, f_m \rangle$, where $k = \dim \mathfrak{k}$, $m = \dim \mathfrak{g}$,
- $e_1 \coloneqq \xi \cong \frac{\partial}{\partial t}$,
- $e_i \coloneqq \widehat{f_j}$, $j = \dim \mathfrak{k} - 1 + i$, the Killing vector fields on M^{princ} induced by the G-action, for $i = 2, \dots, 6$,
- e^i the dual 1-forms to e_i.

Therefore, in what follows $(e_i)_{i=1,\dots,6}$ will be vectors on M^{princ} which provide a basis for T_pM at each point $p = \gamma(t) \in M^{\text{princ}}$, where $\gamma_p \colon \overset{\circ}{I} \to M$ is a normal geodesic through p. For every principal point p of M, we also denote by γ_p the normal geodesic such that $\gamma_p(0) = p$. Moreover, we recall some basic facts about G-actions which will be useful for our discussion:

- Since $g \cdot \gamma_p = \gamma_{g \cdot p}$ for the uniqueness of the normal geodesic γ starting from the point $g \cdot p$, we have $\Phi_1^{\hat{X}} \circ \Phi_t^{\xi}(p) = \Phi_t^{\xi} \circ \Phi_1^{\hat{X}}(p)$, where Φ_t^v denotes the flow of the vector field v evaluated at time t. This is equivalent to $[\xi, \hat{X}] = 0$, for all $X \in \mathfrak{g}$;
- A k-form α on M^{princ} of the form

$$\alpha = \sum_{i_1 < \dots < i_k = 1}^{6} a_{i_1 \dots i_k} \, e^{i_1 \dots i_k},$$

  with $a_{i_1 \dots i_k} \in C^\infty(\overset{\circ}{I})$ for all $i_1 < \dots < i_k$, is G-invariant if and only if α_p is K-invariant for all $p \in M^{\text{princ}}$.
- If α is a G-invariant k-form on M and $v_1, \dots, v_k$ are G-invariant vector fields on M, then $\alpha\,(v_1, \dots, v_n)\,|_p$ is constant along the G-orbit through p, for all $p \in M$.

5.3 Non-existence result

We first give a local result (Theorem 5.6) for the existence of balanced non-Kähler SU(3)-structures by working on the principal part and then we prove that none of these local solutions can be extended to a global one (Theorem 5.10).

Theorem 5.6. *Let M be a 6-dimensional simply connected cohomogeneity one manifold under the almost effective action of a compact connected Lie group G and let K be the principal isotropy group. Then, the principal part M^{princ} admits a G-invariant balanced non-Kähler* SU(3)*-structure (ω, ψ_+) if and only if M is compact and $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$.*

Proof. From all the above discussion and the previous lemmas, the only possible pairs allowing M^{princ} to support a balanced SU(3)-structure are (a.1) with M compact, (c.3) and (b.1). We investigate these three cases separately.

For each of these cases, we shall consider the generic pair (ω, ψ_+) of G-invariant forms on M^{princ} of degree two and three, respectively, with $C^\infty(\mathring{I})$-coefficients. In order for the pair (ω, ψ_+) to define a G-invariant balanced non-Kähler SU(3)-structure on M^{princ}, we have to impose the following conditions:

(1) the stability conditions:

- $\omega^3 \neq 0$,
- $\lambda := \lambda(\psi_+) < 0$,

(2) the compatibility conditions $\psi_\pm \wedge \omega = 0$,

(3) the normalization conditions:

- $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$,
- $\frac{1}{6}\omega^3 = \pm\sqrt{\det(g)}\, e^{1\dots6}$ where the sign $\pm$ depends on the fixed orientation $\pm e^{1\dots6}$,

(4) $d\psi_\pm = 0$,

(5) the balanced condition $d\omega^2 = 0$,

(6) the non-Kähler condition $d\omega \neq 0$,

(7) the positive-definiteness of the induced symmetric bilinear form $g := \omega(\cdot, J\cdot)$ on M^{princ}.

We start with case (b.1).

5.3.1 Case (b.1): $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R})$

In the notation of Remark 5.5, let us suppose p, q non-zero and coprime with $(p, q) \neq (1, 1)$, first. Consider the $\tilde{\mathcal{B}}$-orthonormal basis of $\mathfrak{g}$ given by

$$
\begin{aligned}
f_1 &= \frac{1}{2\sqrt{2(p^2+q^2)}}\begin{pmatrix} p\tilde{e}_1 & 0 \\ 0 & q\tilde{e}_1 \end{pmatrix}, & f_2 &= \frac{1}{2\sqrt{2(p^2+q^2)}}\begin{pmatrix} q\tilde{e}_1 & 0 \\ 0 & -p\tilde{e}_1 \end{pmatrix}, \\
f_3 &= \frac{1}{2\sqrt{2}}\begin{pmatrix} \tilde{e}_3 & 0 \\ 0 & 0 \end{pmatrix}, & f_4 &= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{e}_3 \end{pmatrix}, \\
f_5 &= \frac{1}{2\sqrt{2}}\begin{pmatrix} \tilde{e}_2 & 0 \\ 0 & 0 \end{pmatrix}, & f_6 &= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{e}_2 \end{pmatrix}
\end{aligned}
\tag{5.4}
$$

and take $\mathfrak{k} = \langle f_1 \rangle$. Notice that, since $\operatorname{rk}(\mathfrak{su}(2)) = 1$, this assumption is not restrictive. The decomposition of $\mathfrak{g}$ into irreducible $\operatorname{Ad}(K)$-modules is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2,$$

with $\mathfrak{a} := \langle f_2 \rangle$ is $\mathrm{Ad}(K)$-fixed, $\mathfrak{b}_1 := \langle f_3, f_5 \rangle$ and $\mathfrak{b}_2 := \langle f_4, f_6 \rangle$, hence $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$. Fix the orientation given by $\Omega = e^{1\dots 6}$ and consider the generic G-invariant 3-form ψ_+ on M^{princ},

$$\psi_+ := p_1\, e^{135} + p_2\, e^{146} + p_3\, e^{235} + p_4\, e^{246},$$

where $p_j \in C^\infty(\overset{\circ}{I})$, $j = 1, \dots, 4$. A simple calculation shows that the stability condition $\lambda(\psi_+) < 0$ never holds, since $\lambda(\psi_+) = (p_1p_4 - p_2p_3)^2 \geq 0$.

Now, let $(p, q) = (1, 0)$ and consider the $\tilde{\mathcal{B}}$-orthogonal basis of $\mathfrak{g}$ given by (5.4) when $(p, q) = (1, 0)$ and assume $\mathfrak{k} = \langle f_1 \rangle$ as before. Then, the decomposition of $\mathfrak{g}$ into irreducible $\mathrm{Ad}(K)$-modules is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3,$$

with $\mathfrak{b}_1 := \langle f_3, f_5 \rangle$, $\mathfrak{a}_1 := \langle f_2 \rangle$, $\mathfrak{a}_2 := \langle f_4 \rangle$ and $\mathfrak{a}_3 := \langle f_6 \rangle$. Observe that the modules $\mathfrak{a}_i$, $i = 1, 2, 3$, are equivalent. Consider the generic G-invariant 3-form ψ_+ on M^{princ}, which is of the form

$$\psi_+ := p_1\, e^{124} + p_2\, e^{126} + p_3\, e^{135} + p_4\, e^{146} + p_5\, e^{235} + p_6\, e^{246} + p_7\, e^{345} + p_8\, e^{356},$$

with $p_j \in C^\infty(\overset{\circ}{I})$, $j = 1, \dots, 8$. It is straightforward to show

$$\lambda(\psi_+) = (p_1p_8 + p_2p_7 - p_3p_6 + p_4p_5)^2 \geq 0.$$

Remark 5.7. By the previous discussion we have that, when $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R})$ with $\mathfrak{k}$ not diagonally embedded in $\mathfrak{g}$, M admits no G-invariant $\mathrm{SL}(3,\mathbb{C})$-structures, i.e., G-invariant stable 3-forms inducing an almost complex structure on M.

Finally, let us consider the case where $\mathfrak{k}$ is diagonally embedded in $\mathfrak{g}$. Without loss of generality, we can assume $(p, q) = (1, 1)$. We consider the $\tilde{\mathcal{B}}$-orthonormal basis of $\mathfrak{g}$ given by (5.4) when $(p, q) = (1, 1)$. The decomposition of $\mathfrak{g}$ into irreducible $\mathrm{Ad}\,(K)$-modules is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2,$$

with $\mathfrak{k} = \langle f_1 \rangle$, $\mathfrak{a} := \langle f_2 \rangle$ being $\mathrm{Ad}(K)$-fixed, $\mathfrak{b}_1 := \langle f_3, f_5 \rangle$ and $\mathfrak{b}_2 := \langle f_4, f_6 \rangle$. Then, $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$. Unlike the case $p \neq q$ both non-zero, here the equivalence of the modules $\mathfrak{b}_1$ and $\mathfrak{b}_2$ implies that the metric g on M^{princ} is not necessarily diagonal but of the form

$$g = dt^2 + f(t)^2 \tilde{\mathcal{B}}|_{\mathfrak{a}\times\mathfrak{a}} + h_1(t)^2 \tilde{\mathcal{B}}|_{\mathfrak{b}_1\times\mathfrak{b}_1} + h_2(t)^2 \tilde{\mathcal{B}}|_{\mathfrak{b}_2\times\mathfrak{b}_2} + \mathcal{Q}|_{\mathfrak{b}_1\times\mathfrak{b}_2},$$

for some $f, h_1, h_2 \in C^\infty(\overset{\circ}{I})$, where $\mathcal{Q}$ denotes a symmetric quadratic form on the isotypic component $\mathfrak{b}_1 \oplus \mathfrak{b}_2$. In particular, the metric coefficients $g_{ij} := g(e_i, e_j)$ must satisfy

$$\begin{aligned}
&g_{1i} = g_{i1} = 0, \quad i = 2, \dots, 6,\\
&g_{2i} = g_{i2} = 0, \quad i = 3, \dots, 6,\\
&g_{33} = g_{55}, \quad g_{35} = g_{53} = 0,\\
&g_{44} = g_{66}, \quad g_{46} = g_{64} = 0.
\end{aligned} \tag{5.5}$$

where e_i, $i = 1, \dots, 6$, are the vector fields defined in the usual way. Fix the orientation given by $\Omega := e^{1\dots 6}$ and consider a pair of G-invariant forms (ω, ψ_+) of degree two and three, given respectively by

$$\begin{aligned}\omega =& h_1\, e^{12} + h_2\, e^{35} + h_3\, e^{46} + h_4(e^{34} + e^{56}) + h_5(e^{36} + e^{45}),\\ \psi_+ =& p_1\, e^{135} + p_2\, e^{146} + p_3(e^{134} + e^{156}) + p_4(e^{136} + e^{145})\\ &+ p_5\, e^{235} + p_6\, e^{246} + p_7(e^{234} + e^{256}) + p_8(e^{236} + e^{245}),\end{aligned}$$

with $h_i, p_j \in C^\infty(\overset{\circ}{I})$, $i = 1, \dots, 5$, $j = 1, \dots, 8$. Moreover, the structure equations are given by

$$de^1 = 0,\ de^2 = \frac{1}{2}\left(e^{35} - e^{46}\right),\ de^3 = -\frac{1}{2}e^{25},\ de^4 = \frac{1}{2}e^{26},\ de^5 = \frac{1}{2}e^{23},\ de^6 = -\frac{1}{2}e^{24}.$$

In order to find a G-invariant balanced non-Kähler SU(3)-structure on M^{princ}, we have to impose the conditions (1) to (7) listed at the beginning of this section, together with (5.5). We shall show that this system of equations is incompatible. This implies there are no G-invariant balanced non-Kähler SU(3)-structures on the corresponding M. In order to see this, we write all conditions in terms of the coefficients h_i, p_j of (ω, ψ_+), for $i = 1, \dots, 5$, $j = 1, \dots, 8$. One has that $d\omega^2 = 0$ holds if and only if

$$\frac{h_1}{2}\left(h_3 - h_2\right) - \left(h_2 h_3 - h_4^2 - h_5^2\right)' = 0$$

and, in particular, $d\omega = 0$ holds if and only if

$$\begin{cases} -\dfrac{h_1}{2} + h_2' = 0,\\ (h_2 + h_3)' = 0,\\ h_4 = h_5 = 0.\end{cases}$$

Similarly, $d\psi_+ = 0$ holds if and only if

$$\begin{cases} p_8' - p_3 = 0,\\ p_7' + p_4 = 0,\\ p_5 = p_6,\\ p_6' = 0.\end{cases} \tag{5.6}$$

Let us suppose that ψ_+ is stable with $\lambda < 0$ and consider the induced almost complex structure J on M^{princ}. Recall that, by G-invariance, $\psi_- = J\psi_+$ needs to be of the same general form of ψ_+, namely

$$\begin{aligned}\psi_- =& q_1 e^{135} + q_2 e^{146} + q_3(e^{134} + e^{156}) + q_4(e^{136} + e^{145})\\ &+ q_5 e^{235} + q_6 e^{246} + q_7(e^{234} + e^{256}) + q_8(e^{236} + e^{245}),\end{aligned}$$

where $q_1, \dots, q_8$ are functions of $p_1, \dots, p_8$. Therefore, one has $d\psi_- = 0$ if and only if

$$\begin{cases} q_8' - q_3 = 0, \\ q_7' + q_4 = 0, \\ q_5 = q_6, \\ q_6' = 0. \end{cases} \tag{5.7}$$

Moreover, (5.5) is equivalent to the system

$$\begin{cases} p_1p_6 + p_2p_6 - 2p_3p_7 - p_4p_8 = 0, \\ h_2(p_3p_8 - p_4p_7) + h_4(p_4p_6 - p_1p_8) + 2h_5(p_1p_7 - p_3p_6) = 0, \\ h_3(p_3p_8 - p_4p_7) + h_4(p_4p_6 - p_2p_8) + 2h_5(p_2p_7 - p_3p_6) = 0, \\ h_5(p_4p_6 - p_1p_8) = 0, \\ h_5(p_2p_8 - p_4p_6) = 0, \\ h_2(p_2p_6 - p_1p_6) + 2h_4(p_1p_7 - p_3p_6) = 0, \\ h_3(p_2p_6 - p_1p_6) + 2h_4(p_3p_6 - p_2p_7) = 0, \end{cases} \tag{5.8}$$

where we have already assumed $p_5 = p_6$ from (5.6). Since $p_6' = 0$ and all the conditions for the G-invariant balanced non-Kähler SU(3)-structure involve only homogeneous polynomials, we can assume either $p_6 = 0$ or $p_6 = 1$, up to scalings. Some possibilities can be excluded using the following lemmas.

Lemma 5.8. *Assume $p_6 = 0$. If $p_1 = 0$, or $p_2 = 0$, or $p_7 = 0$, then conditions* (1)-(7) are incompatible.

Proof. Let us assume $p_1 = 0$. Then, $\lambda(\psi_+) = -2(p_3p_8 - p_4p_7)^2 \leq 0$ and

$$\begin{aligned} q_i &= 0, \quad i = 4, 5, 8, \\ q_3 &= \pm\frac{1}{\sqrt{2}}p_4, \\ q_7 &= \pm\frac{1}{\sqrt{2}}p_8, \end{aligned}$$

where the signs of q_3 and q_7 depend on that of $(p_3p_8 - p_4p_7)$. Then, $d\psi_\pm = 0$ implies $p_3 = p_4 = 0$, from which $\lambda = 0$ follows.

Assume instead $p_2 = 0$. Then, we have $\lambda = -2(p_3p_8 - p_4p_7)^2 \leq 0$, as in the previous case. Moreover, one can easily compute

$$\begin{aligned} q_4 &= q_8 = 0, \\ q_3 &= \pm\frac{1}{\sqrt{2}}\,p_4, \\ q_7 &= \pm\frac{1}{\sqrt{2}}\,p_8, \end{aligned}$$

by which we can draw the same conclusion.

Finally, let us assume $p_7 = 0$. Then, (5.6) implies $p_4 = 0$. In this case, $\lambda(\psi_+) = 2p_8^2(p_1p_2 - p_3^2)$ can be strictly negative and one can compute

$$\begin{aligned} q_i &= 0, \quad i = 3,4,8, \\ q_5 &= \frac{p_1p_8^2}{\sqrt{-\lambda}}, \\ q_6 &= \frac{p_2p_8^2}{\sqrt{-\lambda}}, \\ q_7 &= \frac{p_3p_8^2}{\sqrt{-\lambda}}. \end{aligned}$$

Therefore, assuming $p_8 \neq 0$ to ensure $\lambda(\psi_+) \neq 0$, the requirement $d\psi_\pm = 0$ imposes

$$\begin{cases} p_1 = p_2, \\ q_6' = 0, \\ q_7' = 0, \end{cases}$$

which implies $\lambda \geq 0$. □

Lemma 5.9. *If $h_5 \neq 0$, $p_6 = 1$, $p_8 = 0$, then conditions* (1)-(7) and (5.8) are incompatible.

Proof. From (5.8) and the closure of ψ_+, one has

$$\begin{aligned} p_3 &= 0, \\ p_4 &= 0, \\ p_1 &= -p_2, \end{aligned}$$

from which it follows that $\lambda = -4p_2^2(p_7^2 - 1)$ and

$$q_5 = -\frac{2(p_7^2-1)p_2}{\sqrt{-\lambda}} = -q_6.$$

Thus $q_5 = q_6 = 0$, from (5.7), which would force λ to vanish. □

We can then divide the discussion into the following cases:

1. $h_5 \neq 0$, $p_6 = 0$,
2. $h_5 \neq 0$, $p_6 = 1$,
3. $h_5 = 0$, $p_6 = 0$,
4. $h_5 = 0$, $p_6 = 1$.

We study each case separately.

Case (1). By (5.8), Lemma 5.8 and $d\psi_+ = 0$, it follows that $p_3 = p_8 = h_4 = 0$. Then, $d\psi_- = 0$ implies $p_1 = p_2$. Now, we have $\lambda = 2p_7^2(2p_2^2 - p_4^2)$. The compatibility condition

$\psi_+ \wedge \omega = 0$ holds if and only if $p_2(h_2 + h_3) = 2h_5 p_4$. Then, if $h_2 \neq -h_3$, we can write $p_2 = \frac{2p_4 h_5}{(h_2+h_3)}$. Therefore, (5.8) reduces to

$$\begin{cases} -p_4 p_7 (h_2^2 + h_2 h_3 - 4h_5^2) = 0, \\ -p_4 p_7 (h_3^2 + h_2 h_3 - 4h_5^2) = 0, \end{cases}$$

all of whose solutions imply $\lambda \geq 0$. When $h_2 = -h_3$, the condition $\psi_+ \wedge \omega = 0$ implies $p_4 = 0$, from which $\lambda \geq 0$ follows.

Case (2). By Lemma 5.9, we can assume $p_8 \neq 0$. Then, by (5.8), we have

$$p_1 = p_2,$$
$$p_4 = p_2 p_8.$$

Moreover, since in this case $\lambda = -2(p_8^2 - 2)(p_2 p_7 - p_3)^2$, (5.8) implies $h_4 = 0$ as well. Then, (5.8) implies

$$(p_2 p_7 - p_3)(h_2 p_8 - 2h_5) = 0,$$
$$(p_2 p_7 - p_3)(h_3 p_8 - 2h_5) = 0,$$

from which it follows that $h_2 = h_3 = 2\frac{h_5}{p_8}$, since λ must not vanish. Then, $\psi_+ \wedge \omega = 0$ holds if and only if $p_8^2 - 2 = 0$, which would imply $\lambda = 0$.

Case (3). By (5.8) and Lemma 5.8, we have $h_4 = 0$, which implies $\det(g) = h_1^2 h_2^2 h_3^2$. Then, from (5.8), we also have

$$\begin{aligned} p_3 p_8 &= p_4 p_7, \\ 2p_3 p_7 &= -p_4 p_8. \end{aligned} \tag{5.9}$$

If $p_3, p_8 \neq 0$, then (5.9) implies $p_8^2 + 2p_7^2 = 0$, which contradicts our hypothesis. If $p_8 = 0$, the closure of ψ_+ implies $p_3 = 0$. Then, we only need to discuss the remaining case, $p_3 = 0$. Supposing this is the case, we have that (5.9), together with Lemma 5.8, implies $p_4 = 0$. Under these hypotheses, one can easily compute $\lambda = 2p_1 p_2 (2p_7^2 + p_8^2)$,

$$q_5 = \frac{(2p_7^2 + p_8^2) p_1}{\sqrt{-\lambda}},$$
$$q_6 = \frac{(2p_7^2 + p_8^2) p_2}{\sqrt{-\lambda}},$$

so that $\lambda < 0$ forces $q_5 \neq q_6$, a contradiction.

Case (4). Here, the compatibility condition $\psi_+ \wedge \omega = 0$, which holds if and only if

$$\begin{cases} h_2 = 2h_4 p_7 - h_3, \\ -h_2 p_2 - h_3 p_1 + 2h_4 p_3 = 0, \end{cases} \tag{5.10}$$

together with (5.8), implies that one of the following must hold:

(4.a) $h_4 = 0$,

(4.b) $p_4 = 0$,

(4.c) $2p_7^2 + p_8^2 = 2$.

Let us start with case (4.a). By (5.10), we have $h_2 = -h_3$. In particular, since $\det(g) = h_1^2 h_2^2 h_3^2$, we must have $h_3 \neq 0$. Then, a simple calculation show that $d\omega^2 = 0$ is equivalent to $d\omega = 0$. In case (4.b), by (5.10) and (5.8), we have

$$\begin{aligned} p_1 &= 2p_3p_7 - p_2, \\ h_2 &= 2h_4p_7 - h_3, \end{aligned}$$

from which it follows that $\lambda = -2(2p_7^2 + p_8^2 - 2)(-2p_2p_3p_7 + p_2^2 + p_3^2)$. Moreover, one can show that $q_5 = q_6$ implies $p_2 = p_3p_7$. Now, (5.8) implies $h_4 = 0$, which was already ruled out in the previous case. In case (4.c), again by (5.10) and (5.8), we have

$$\begin{aligned} p_1 &= 2p_3p_7 + p_4p_8 - p_2, \\ h_2 &= 2h_4p_7 - h_3, \end{aligned}$$

which implies $\lambda = 0$. This concludes case (b.1).

5.3.2 Case (c.3): $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{k} = \mathfrak{su}(2)$

Consider the $\tilde{\mathcal{B}}$-orthogonal basis of $\mathfrak{g}$ given by

$$f_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$
$$f_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad f_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad f_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad f_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

Then, $\mathfrak{k} = \langle f_1, f_2, f_3 \rangle$. Let $\mathfrak{a} := \langle f_8 \rangle$ and $\mathfrak{n} := \langle f_4, f_5, f_6, f_7 \rangle$. Hence, $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{n}$. Since the $\mathrm{Ad}(K)$-invariant irreducible modules in the decomposition of $\mathfrak{g}$ are pairwise inequivalent, the metric g on M^{princ} is diagonal. In particular, it is of the form

$$g = dt^2 + h(t)^2 \tilde{\mathcal{B}}|_{\mathfrak{a}\times\mathfrak{a}} + f(t)^2 \tilde{\mathcal{B}}|_{\mathfrak{n}\times\mathfrak{n}},$$

for some positive $h, f \in C^\infty(\mathring{I})$. Moreover, with respect to the frame $(e_i)_{i=1,\ldots,6}$ of M^{princ}, the structure equations are given by

$$\begin{aligned} de^1 &= 0, & de^2 &= -\sqrt{3}e^{36}, & de^3 &= \sqrt{3}e^{26}, \\ de^4 &= -\sqrt{3}e^{56}, & de^5 &= \sqrt{3}e^{46}, & de^6 &= -\sqrt{3}(e^{23} + e^{45}). \end{aligned}$$

Fix the volume form $\Omega = e^{1\ldots6}$. One can easily show that a pair of generic G-invariant forms (ω, ψ_+) on M^{princ} of degree two and three is given by

$$\begin{aligned} \omega =& h_1\, e^{16} + h_2\, (e^{23} + e^{45}) + h_3\, (e^{24} - e^{35}) + h_4 (e^{25} + e^{34}), \\ \psi_+ =& p_1\, (e^{123} + e^{145}) + p_2\, (e^{124} - e^{135}) + p_3 (e^{246} - e^{356}) + p_4 (e^{236} + e^{456}) \\ &+ p_5\, (e^{125} + e^{134}) + p_6\, (e^{256} + e^{346}), \end{aligned}$$

with $h_i, p_j \in C^\infty(\overset{\circ}{I})$, $i = 1, \dots, 4$, $j = 1, \dots, 6$. As we did for case (b.1), we are going to show that the system of equations resulting from imposing conditions (1)-(7) is incompatible. A simple computation shows that $d\psi_+ = 0$ is equivalent to

$$\begin{cases} p_6' - 2\sqrt{3}\, p_2 = 0, \\ p_3' + 2\sqrt{3}\, p_5 = 0, \\ p_4 = p_4' = 0. \end{cases}$$

By the G-invariance,

$$\begin{aligned} \psi_- :=& q_1\,(e^{123} + e^{145}) + q_2\,(e^{124} - e^{135}) + q_3(e^{246} - e^{356}) + q_4(e^{236} + e^{456}) \\ &+ q_5\,(e^{125} + e^{134}) + q_6\,(e^{256} + e^{346}), \end{aligned}$$

where the q_i's are functions of $p_1, \dots, p_6$. Therefore, $d\psi_- = 0$ holds if and only if

$$\begin{cases} q_6' - 2\sqrt{3} q_2 = 0, \\ q_3' + 2\sqrt{3} q_5 = 0, \\ q_4 = q_4' = 0. \end{cases}$$

In particular, it follows from $p_4 = 0$ that we have

$$q_4 = \frac{2(p_3^2 + p_6^2)p_1}{\sqrt{-\lambda}},$$

with $\lambda = -4(p_1^2\,(p_3^2 + p_6^2) + (p_2\,p_6 - p_3\,p_5)^2)$. We suppose that ψ_+ is stable with $\lambda < 0$. Then, $q_4 = 0$ holds if and only if $p_1 = 0$ does. Since p_1 has to be equal to zero, it can be shown that the compatibility condition $\psi_+ \wedge \omega = 0$ is equivalent to the following system of equations:

$$\begin{cases} h_3 p_3 + h_4 p_6 = 0, \\ h_3 p_2 + h_4 p_5 = 0. \end{cases} \tag{5.11}$$

Moreover, the positive-definiteness of g implies $h_1 > 0$. Then, the normalization condition $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$ is equivalent to

$$|p_2 p_6 - p_3 p_5| = h_1(h_2^2 + h_3^2 + h_4^2). \tag{5.12}$$

The balanced condition $d\omega^2 = 0$ is satisfied if and only if

$$2\sqrt{3} h_1 h_2 + (h_2^2 + h_3^2 + h_4^2)' = 0. \tag{5.13}$$

Finally, the Kähler condition $d\omega = 0$ holds if and only if

$$\begin{cases} h_3 = h_4 = 0 \\ \sqrt{3} h_1 + h_2' = 0. \end{cases} \tag{5.14}$$

Multiplying (5.12) by h_4 and using (5.11), we obtain $h_4 h_1(h_2^2 + h_3^2 + h_4^2) = 0$. Since $h_1 > 0$ and $h_2 = h_3 = h_4 = 0$ would imply $\omega^3 = 0$, we necessarily have $h_4 = 0$. Then, (5.11) implies

$$\begin{cases} h_3 p_3 = 0 \\ h_3 p_2 = 0, \end{cases}$$

from which $h_3 = 0$ follows, since $p_2 = p_3 = 0$ would imply $\lambda = 0$. Then, (5.13) reads $h_2(\sqrt{3}h_1 + h_2') = 0$ and, since $h_2 \neq 0$, in order to have $\omega^3 \neq 0$ we have $\sqrt{3}h_1 + h_2' = 0$, forcing $d\omega = 0$. Therefore, every G-invariant balanced SU(3)-structure on the corresponding M is necessarily Kähler. This concludes case (c.3).

5.3.3 Case (a.1): $\mathfrak{g} = \mathfrak{su}(2) \oplus 2\mathbb{R}$, $\mathfrak{k} = \{0\}$

Since $\mathfrak{k} = \{0\}$, we can write $T_pM \cong \langle e_1|_p \rangle \oplus \hat{\mathfrak{g}}\big|_p$, for each $p \in M^{\mathrm{princ}}$. Moreover, every k-form α on M^{princ} of the form

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq 6} \alpha_{i_1 \ldots i_k} e^{i_1 \ldots i_k},$$

with $\alpha_{i_1\ldots i_k} \in C^\infty(\mathring{I})$, is G-invariant. Let

$$\omega = \sum_{1 \leq i < j \leq 6} h_{ij} e^{ij}, \qquad \psi_+ = \sum_{1 \leq i < j < k \leq 6} p_{ijk} e^{ijk} \tag{5.15}$$

be a pair of generic G-invariant forms on M^{princ} of degree two and three, respectively, with coefficients $h_{ij}, p_{ijk} \in C^\infty(\mathring{I})$. If we choose a $\tilde{\mathcal{B}}$-orthogonal basis of $\mathfrak{su}(2)$ with vectors of constant norm, say

$$f_i = \left(\begin{array}{c|c|c} \tilde{e}_i & & \\ \hline & 0 & \\ \hline & & 0 \end{array}\right), \; i = 1, 2, 3,$$

and extend it to a basis $(f_i)_{i=1,\ldots,5}$ of $\mathfrak{g}$, the structure equations with respect to the frame $(e_i)_{i=1,\ldots,6}$ of M^{princ} are given by

$$de^1 = 0, \quad de^2 = -2e^{34}, \quad de^3 = 2e^{24}, \quad de^4 = -2e^{23}, \quad de^5 = 0, \quad de^6 = 0.$$

Fix the volume form $\Omega := -e^{1\ldots6}$. We consider the forms given in (5.15) and set

$$\begin{aligned}
&p_{134} = p_{234} = 1, \\
&p_{136} = p_{235} = p_{246} = -p_{145} = e^{2t}, \\
&h_{12} = \frac{3}{2}\frac{e^{4t}}{\sqrt{9 + 3e^{6t}}}, \\
&h_{34} = -\frac{1}{3}\left(-3 + \sqrt{9 + 3e^{6t}}\right) e^{-2t}, \\
&h_{35} = h_{36} = h_{46} = -h_{45} = 1, \\
&h_{56} = 2e^{2t},
\end{aligned}$$

for each $t \in (-1, 1)$, with all the other coefficients equal to zero. Then, by performing the change of variable

$$\tilde{t}(t) := \int_0^t a(s)ds, \quad a(s) = \sqrt{\frac{3}{2}}(9 + 3\,e^{6t})^{-\frac{1}{4}} e^{2t},$$

one can easily check that the resulting pair (ω, ψ_+) defines a G-invariant balanced non-Kähler SU(3)-structure on the corresponding M^{princ}. With respect to the t parameter, the metric on M^{princ} is represented by the matrix

$$(g_{ij}) = \begin{pmatrix} \frac{3}{2}\frac{e^{4t}}{\sqrt{9+3e^{6t}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}\frac{e^{4t}}{\sqrt{9+3e^{6t}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{9+3e^{6t}}}{3e^{2t}} & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{3+\sqrt{9+3e^{6t}}}{3e^{2t}} & 1 & 1 \\ 0 & 0 & 1 & 1 & 2e^{2t} & 0 \\ 0 & 0 & -1 & 1 & 0 & 2e^{2t} \end{pmatrix}.$$

However, using the results from [129], we can check that this example cannot be extended to the singular orbits to give a smooth metric on the whole manifold. This concludes the proof of Theorem 5.6. □

We will finally prove our main theorem.

Theorem 5.10. *Let M be a 6-dimensional simply connected cohomogeneity one manifold under the almost effective action of a compact connected Lie group G. Then, M admits no G-invariant balanced non-Kähler* SU(3)*-structures.*

Proof. By Theorem 5.6, we only need to discuss whether there exist balanced non-Kähler SU(3)-structures of cohomogeneity one arising as the compactification of the principal part determined by the pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$.

By [85], a 6-dimensional compact simply connected cohomogeneity one manifold M whose corresponding principal part is determined by the pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$ at the Lie algebra level is G-equivariantly diffeomorphic to the product of two 3-dimensional spheres, i.e., $M \cong S^3 \times S^3$. If we denote by H_i, $i = 1, 2$, the singular isotropy groups for the G-action on M and by $\mathfrak{h}_i = \text{Lie}(H_i)$, $i = 1, 2$, their respective Lie algebras, we have that both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are isomorphic to $\mathbb{R}$ so that both the singular orbits of M are 4-dimensional compact submanifolds of M. Letting b_i be the i-th Betti number of M, then we have $b_4 = 0$. By Michelsohn's obstruction ([106, Corollary 1.7]), if M admitted any 4-dimensional compact complex submanifold S, then M would not admit a balanced metric. Therefore, we can make a few considerations by focusing on one tubular neighborhood of a singular orbit $G \supset H \supset K$ at a time. In particular, we divide the discussion depending on the immersion of $\mathfrak{h} \subset \mathfrak{g}$. Let S be the singular orbit determined by the group diagram $G \supset H \supset K$. We notice that, if S is J-invariant, a complex structure on M would give rise to a complex structure on S, so we can discard all these cases by Michelsohn's obstruction. In particular, we have $T_qM = T_qS \oplus V$, where $V = T_qS^\perp$ is the slice at $q \in S$. Since S is 4-dimensional, V is always a 2-plane. We recall that the H-action on T_qS is given by the adjoint representation, while the H-action on V is given via slice representation. Since V is 2-dimensional, this action is just a rotation on V of a certain speed, say a. Let us start by considering the case where $\mathfrak{h}$ is contained in the center of $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g})$. In this case, the H-action on T_qS is trivial. Therefore, $T_qS \oplus V$ are inequivalent modules of the H-action on T_qM and, since J commutes with the H-action, J preserves T_qS for each $q \in S$, i.e., S is an almost complex submanifold of M and we may apply Michelsohn's obstruction to discard this case. Therefore, we may suppose that $\mathfrak{h}$ has a non-trivial component in the $\mathfrak{su}(2)$-factor of $\mathfrak{g}$. In particular, since $\mathrm{rk}(\mathfrak{su}(2)) = 1$ and the adjoint action ignores components in the center of $\mathfrak{g}$, we may assume $\mathfrak{h} = \langle f_1 \rangle$ without loss of generality and using the notation from Section 5.3.3. Moreover, if we let $\mathfrak{m}$ denote the tangent space to S via the usual identification, the decomposition of $\mathfrak{m}$ in irreducible H-modules is given by

$$\mathfrak{m} = l_0 \oplus l_1,$$

where H acts trivially on l_0 and via rotation of speed d on l_1. Therefore, when the integer a is different from d, the modules l_0, l_1 and V are inequivalent for the H-action and must be preserved by J, as a consequence. In particular, we have $J(T_qS) \subseteq T_qS$ and we may apply Michelsohn's obstruction as before. For the remaining case $a = d$, we have that the two modules l_1 and V are equivalent, hence, $J(l_1 \oplus V) \subseteq l_1 \oplus V$ but not necessarily $J(l_1) \subseteq l_1$. In particular, when this case occurs, the orbit S is not J-invariant and we do not have obstructions to the existence of balanced metrics. Therefore, from now on, we assume this is the case.

Let $\partial/\partial x$ be a vector field such that $(\xi|_q, \partial/\partial x|_q)$ is an orthonormal basis for the slice V and $T_q^*M = \langle e^1|_q, dx|_q, e^3|_q, e^4|_q, e^5|_q, e^6|_q \rangle$. Let $\varphi\colon \mathfrak{h} \to \mathrm{End}(T_qM)$ be the $\mathfrak{h}$-action on T_qM. Then, in order to have l_1 and V $\mathfrak{h}$-equivalent, up to rescaling f_1, $\varphi(f_1)^*$ acts on 1-forms given in the previous basis as

$$\varphi(f_1)^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Fix the volume form $\Omega = e^{1\dots 6}$ and consider the 3-form

$$\begin{aligned} \psi_+ \; &:= p_1 e^{123} + p_2 e^{124} + p_3 e^{125} + p_4 e^{126} + p_5 e^{134} + p_6 e^{135} + p_7 e^{136} + p_8 e^{145} \\ &+ p_9 e^{146} + p_{11} e^{234} + p_{12} e^{235} + p_{13} e^{236} + p_{14} e^{245} + p_{15} e^{246} + p_{16} e^{256} \\ &+ p_{17} e^{345} + p_{18} e^{346} + p_{19} e^{356} + p_{20} e^{456}, \end{aligned}$$

with $p_j \in C^\infty((-1,1))$, $j = 1, \dots, 20$.

The condition $d\psi_+ = 0$ is equivalent to the following ODE system:

$$\begin{cases} p_{11}' = 0, \\ p_{12}' + 2p_8 = 0, \\ p_{13}' + 2p_9 = 0, \\ p_{14}' - 2p_6 = 0, \\ p_{15}' - 2p_7 = 0, \\ p_{17}' + 2p_3 = 0, \\ p_{18}' + 2p_4 = 0, \\ p_{16} = p_{19} = p_{20} = 0. \end{cases} \tag{5.16}$$

From now on, we shall assume $p_{16} = p_{19} = p_{20} = 0$.

Let the slice be $V \cong \mathbb{R}^2$, so that the singular point $q \in \mathcal{O}_1$ is identified with $0 \in \mathbb{R}^2$, and let $r\colon V \to \mathbb{R}$ be the radial distance, such that for $v = (v_1, v_2) \in V$, $r(v) = |v| = \sqrt{v_1^2 + v_2^2}$. Then, $r \notin C^\infty(V)$ and neither are the odd powers of r. Via the exponential map, we can identify $t+1$ with the radial distance r.

Let α be a G-invariant 1-form on M. Then,

$$\alpha(t) = \sum_{i=1}^{6} \alpha_i(t) e^i,$$

for $t \in (-1, 1)$ and some smooth functions α_i, $i = 1, ..., 6$. This expression has to extend smoothly to $t = -1$. In particular, the Taylor expansion of $\alpha_k(t)$ around $t = -1$ for $k \geq 2$ only has even powers of $(t+1)$:

$$\alpha_k(t) \sim \sum_{n>1} a_{k,2n}(t+1)^{2n}.$$

Now, for $2 \leq i < j < k \leq 6$ fixed, the e^{ijk}-coefficients extend smoothly to $t = -1$. Hence,

$$p_{12}(t) \sim \sum_{n>1} a_{2n}(t+1)^{2n},$$

with analogous expressions holding for $p_{13}(t)$, $p_{14}(t)$ and $p_{15}(t)$ around $t = -1$. Therefore, we have $\lim_{t\to-1} p_{12}'(t) = \lim_{t\to-1} p_{13}'(t) = \lim_{t\to-1} p_{14}'(t) = \lim_{t\to-1} p_{15}'(t) = 0$. From (5.16), we obtain $\lim_{t\to-1} p_6(t) = \lim_{t\to-1} p_7(t) = \lim_{t\to-1} p_8(t) = \lim_{t\to-1} p_9(t) = 0$.

The 3-form ψ_+ at $t = 0$ has to be H-invariant, hence it can be written as

$$\begin{aligned}\rho \;&= c_3 e^1 \wedge dx \wedge e^5 + c_4 e^1 \wedge dx \wedge e^6 + c_6 e^{135} + c_7 e^{136} \\ &\quad + c_8 e^{145} + c_9 e^{146} - c_8 dx \wedge e^{35} - c_9 dx \wedge e^{36} \\ &\quad + c_6 dx \wedge e^{45} + c_7 dx \wedge e^{46} + c_{17} e^{345} + c_{18} e^{346},\end{aligned}$$

for some $c_3, c_4, c_6, c_7, c_8, c_9, c_{17}, c_{18} \in \mathbb{R}$. But $c_i = \lim_{t\to-1} p_i(t) = 0$ for $i = 6, 7, 8, 9$. Therefore, one can easily compute

$$\lambda|_{t=-1} = (c_{18}c_3 - c_{17}c_4)^2 \geq 0.$$

This concludes case (a.1).

We note that it is possible to reach a contradiction just by studying the behaviour around one of the singular orbits. However, if we do not use the information coming from Michelsohn's obstruction, the computations get significantly more complicated. The main point is that $d\psi_- = 0$ and the stability condition $\lambda < 0$ imply $p_{10} = 0$. If we assume this too, the 3-form ψ_+ at $t = -1$ can be written as

$$\begin{aligned}\rho \;&= c_1 e^1 \wedge dx \wedge e^3 + c_2 e^1 \wedge dx \wedge e^4 + c_3 e^1 \wedge dx \wedge e^5 + c_4 e^1 \wedge dx \wedge e^6 \\ &\quad + c_5 e^{134} + c_6 e^{135} + c_7 e^{136} + c_8 e^{145} + c_9 e^{146} \\ &\quad + c_{11} dx \wedge e^{34} + c_{12} dx \wedge e^{35} + c_{13} dx \wedge e^{36} + c_{14} dx \wedge e^{45} + c_{15} dx \wedge e^{46} \\ &\quad + c_{17} e^{345} + c_{18} e^{346},\end{aligned}$$

for some $c_i \in \mathbb{R}$, $i = 1, \ldots, 18$, $i \neq 10, 16$. Then, once again, we obtain $\lambda|_{t=-1} = (c_{18}c_3 - c_{17}c_4)^2 \geq 0$ concluding the case. $\square$

Remark 5.11. In case (a.1) and when $\mathfrak{h} = \mathbb{R}$, we also note that we can remove the hypothesis of simply connectedness from the non-compact case and still get a non-existence result. Let M be a 6-dimensional non-compact cohomogeneity one manifold under the almost effective action of a compact connected Lie group G and let K, H be the principal and singular isotropy groups, respectively, with $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \mathbb{R}, \{0\})$. Then, M admits no G-invariant balanced non-Kähler SU(3)-structures.

Remark 5.12. In [66], balanced metrics were constructed on the connected sum of $k \geq 2$ copies of $S^3 \times S^3$. However, it is not known whether $S^3 \times S^3$ admits balanced structures. In [106, Example 1.8], Michelsohn proved that $S^3 \times S^3$ endowed with the Calabi–Eckmann complex structure does not admit any compatible balanced metrics. By [4, Remark 1], in a manifold with six real dimensions, there is no non-Kähler Hermitian metric which is simultaneously balanced and strong Kähler-with-torsion (a.k.a. SKT). In [60], Fino and Vezzoni conjectured that on non-Kähler compact complex manifolds it is never possible to find an SKT metric and also a balanced metric. In [74], an example of an SKT structure on $S^3 \times S^3$ was provided. The key case that needed to be tackled in Theorem 5.10 was precisely $S^3 \times S^3$.

From Theorem 5.10, we get the following corollary.

Corollary 5.13. *There are no non-Kähler balanced* SU(3)*-structures on* $S^3 \times S^3$ *which are invariant under a cohomogeneity one action.*

Remark 5.14. In the non-simply-connected case, by Theorem 5.6, we can discard cases (b.1) and (c.3), as these do not admit local solutions to conditions (1)–(7). Moreover, as observed in [117, Section 3], one can also rule out cases (b.3) and (c.2), as the G-action would not be almost effective, as well as case (c.1) since it would give rise to a 3-dimensional J-invariant subspace, a contradiction.

Chapter 6

Appendix 1

Table 6.1 contains the isomorphism classes of 6-dimensional real nilpotent Lie algebras $\mathfrak{g}_i$, $i = 1, \ldots, 34$, including their first Betti numbers and an indication of whether they admit half-flat structures and symplectic forms. Explicit examples of mean convex closed SU(3)-structures on these Lie algebras, whenever they exist, are given in Table 6.3. We also indicate which ones are half-flat. Table 6.2 contains all 6-dimensional symplectic solvable (non-nilpotent) unimodular Lie algebras, up to isomorphism, specifying which ones admit tamed closed $\mathrm{SL}(3, \mathbb{C})$-structures.

Table 6.1: 6-dimensional real nilpotent Lie algebras, up to isomorphism

$\mathfrak{g}$	Structure constants	$b_1(\mathfrak{g})$	Half-flat	Symplectic
$\mathfrak{g}_1$	$(0,0,e^{12},e^{13},e^{14}+e^{23},e^{34}-e^{25})$	2	–	–
$\mathfrak{g}_2$	$(0,0,e^{12},e^{13},e^{14},e^{34}-e^{25})$	2	–	–
$\mathfrak{g}_3$	$(0,0,e^{12},e^{13},e^{14},e^{15})$	2	–	✓
$\mathfrak{g}_4$	$(0,0,e^{12},e^{13},e^{14}+e^{23},e^{24}+e^{15})$	2	✓	✓
$\mathfrak{g}_5$	$(0,0,e^{12},e^{13},e^{14},e^{23}+e^{15})$	2	–	✓
$\mathfrak{g}_6$	$(0,0,e^{12},e^{13},e^{23},e^{14})$	2	✓	✓
$\mathfrak{g}_7$	$(0,0,e^{12},e^{13},e^{23},e^{14}-e^{25})$	2	✓	✓
$\mathfrak{g}_8$	$(0,0,e^{12},e^{13},e^{23},e^{14}+e^{25})$	2	✓	✓
$\mathfrak{g}_9$	$(0,0,0,e^{12},e^{14}-e^{23},e^{15}+e^{34})$	3	✓	✓
$\mathfrak{g}_{10}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{23})$	3	✓	✓
$\mathfrak{g}_{11}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{23}+e^{24})$	3	✓	✓
$\mathfrak{g}_{12}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{24})$	3	✓	✓
$\mathfrak{g}_{13}$	$(0,0,0,e^{12},e^{14},e^{15})$	3	✓	✓
$\mathfrak{g}_{14}$	$(0,0,0,e^{12},e^{13},e^{14}+e^{35})$	3	✓	–
$\mathfrak{g}_{15}$	$(0,0,0,e^{12},e^{23},e^{14}+e^{35})$	3	✓	–
$\mathfrak{g}_{16}$	$(0,0,0,e^{12},e^{23},e^{14}-e^{35})$	3	✓	–
$\mathfrak{g}_{17}$	$(0,0,0,e^{12},e^{14},e^{24})$	3	–	–
$\mathfrak{g}_{18}$	$(0,0,0,e^{12},e^{13}-e^{24},e^{14}+e^{23})$	3	–	✓
$\mathfrak{g}_{19}$	$(0,0,0,e^{12},e^{14},e^{13}-e^{24})$	3	–	✓
$\mathfrak{g}_{20}$	$(0,0,0,e^{12},e^{13}+e^{14},e^{24})$	3	–	✓
$\mathfrak{g}_{21}$	$(0,0,0,e^{12},e^{13},e^{14}+e^{23})$	3	✓	✓
$\mathfrak{g}_{22}$	$(0,0,0,e^{12},e^{13},e^{24})$	3	✓	✓
$\mathfrak{g}_{23}$	$(0,0,0,e^{12},e^{13},e^{14})$	3	–	✓
$\mathfrak{g}_{24}$	$(0,0,0,e^{12},e^{13},e^{23})$	3	✓	✓
$\mathfrak{g}_{25}$	$(0,0,0,0,e^{12},e^{15}+e^{34})$	4	✓	–
$\mathfrak{g}_{26}$	$(0,0,0,0,e^{12},e^{15})$	4	–	✓
$\mathfrak{g}_{27}$	$(0,0,0,0,e^{12},e^{14}+e^{25})$	4	✓	✓
$\mathfrak{g}_{28}$	$(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23})$	4	✓	✓
$\mathfrak{g}_{29}$	$(0,0,0,0,e^{12},e^{14}+e^{23})$	4	✓	✓
$\mathfrak{g}_{30}$	$(0,0,0,0,e^{12},e^{34})$	4	✓	✓
$\mathfrak{g}_{31}$	$(0,0,0,0,e^{12},e^{13})$	4	✓	✓
$\mathfrak{g}_{32}$	$(0,0,0,0,0,e^{12}+e^{34})$	5	✓	–
$\mathfrak{g}_{33}$	$(0,0,0,0,0,e^{12})$	5	✓	✓
$\mathfrak{g}_{34}$	$(0,0,0,0,0,0)$	6	✓	✓

Table 6.2: 6-dimensional unimodular solvable non-nilpotent Lie algebras admitting symplectic structures, up to isomorphism [47, 103].

$\mathfrak{g}$	Structure equations	decomposable	Tamed closed $\mathrm{SL}(3,\mathbb{C})$-structure
$\mathfrak{g}_{6,3}^{0,-1}$	$(e^{26}, e^{36}, 0, e^{46}, -e^{56}, 0)$		
$\mathfrak{g}_{6,10}^{0,0}$	$(e^{26}, e^{36}, 0, e^{56}, -e^{46}, 0)$		
$\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$	$(-\frac{1}{2}e^{16} + e^{23}, -e^{26}, \frac{1}{2}e^{36}, e^{46}, 0, 0)$	✓	
$\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}$	$(-\frac{1}{2}e^{16} + e^{23}, \frac{1}{2}e^{26}, -e^{36}, e^{46}, 0, 0)$	✓	
$\mathfrak{g}_{6,15}^{-1}$	$(e^{23}, e^{26}, -e^{36}, e^{26} + e^{46}, e^{36} - e^{56}, 0)$		
$\mathfrak{g}_{6,18}^{-1,-1}$	$(e^{23}, -e^{26}, e^{36}, e^{36} + e^{46}, -e^{56}, 0)$		
$\mathfrak{g}_{6,21}^{0}$	$(e^{23}, 0, e^{26}, e^{46}, -e^{56}, 0)$		
$\mathfrak{g}_{6,36}^{0,0}$	$(e^{23}, 0, e^{26}, -e^{56}, e^{46}, 0)$		
$\mathfrak{g}_{6,38}^{0}$	$(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0)$		✓
$\mathfrak{g}_{6,54}^{0,-1}$	$(e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0)$		✓
$\mathfrak{g}_{6,70}^{0,0}$	$(-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0)$		
$\mathfrak{g}_{6,78}$	$(-e^{16} + e^{25}, e^{45}, e^{24} + e^{36} + e^{46}, e^{46}, -e^{56}, 0)$		
$\mathfrak{g}_{6,118}^{0,-1,-1}$	$(-e^{16} + e^{25}, -e^{15} - e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0)$		✓
$\mathfrak{n}_{6,84}^{\pm 1}$	$(-e^{45}, -e^{15} - e^{36}, -e^{14} + e^{26} \mp e^{56}, e^{56}, -e^{46}, 0)$		
$\mathfrak{e}(2) \oplus \mathfrak{e}(2)$	$(0, -e^{13}, e^{12}, 0, -e^{46}, e^{45})$	✓	
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$	$(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45})$	✓	✓
$\mathfrak{e}(2) \oplus \mathbb{R}^3$	$(0, -e^{13}, e^{12}, 0, 0, 0)$	✓	
$\mathfrak{e}(1,1) \oplus \mathbb{R}^3$	$(0, -e^{13}, -e^{12}, 0, 0, 0)$	✓	
$\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$	$(0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45})$	✓	
$\mathfrak{e}(2) \oplus \mathfrak{h}_3$	$(0, -e^{13}, e^{12}, 0, 0, e^{45})$	✓	
$\mathfrak{e}(1,1) \oplus \mathfrak{h}_3$	$(0, -e^{13}, -e^{12}, 0, 0, e^{45})$	✓	
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$	$(e^{15}, -e^{25}, \beta e^{35}, -\beta e^{45}, 0, 0), \quad -1 \leq \beta < 0$	✓	✓
$A_{5,8}^{-1} \oplus \mathbb{R}$	$(e^{25}, 0, e^{35}, -e^{45}, 0, 0)$	✓	
$A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$	$(e^{15}, -e^{25}, \gamma e^{45}, -\gamma e^{35}, 0, 0), \quad \gamma > 0$	✓	
$A_{5,14}^{0} \oplus \mathbb{R}$	$(e^{25}, 0, e^{45}, -e^{35}, 0, 0)$	✓	
$A_{5,15}^{-1} \oplus \mathbb{R}$	$(e^{15} + e^{25}, e^{25}, -e^{35} + e^{45}, -e^{45}, 0, 0)$	✓	
$A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$	$(\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0), \quad \alpha > 0$	✓	✓
$A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$	$(e^{25}, -e^{15}, \gamma e^{45}, -\gamma e^{35}, 0, 0), \quad 0 < \gamma < 1$	✓	
$A_{5,17}^{0,0,1} \oplus \mathbb{R}$	$(e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0)$	✓	✓
$A_{5,18}^{0} \oplus \mathbb{R}$	$(e^{25} + e^{35}, -e^{15} + e^{45}, e^{45}, -e^{35}, 0, 0)$	✓	
$A_{5,19}^{-1,2} \oplus \mathbb{R}$	$(-e^{15} + e^{23}, e^{25}, -2e^{35}, 2e^{45}, 0, 0)$	✓	

Table 6.3: Explicit examples of mean convex closed SU(3)-structures

$\mathfrak{g}$	Mean convex closed SU(3)-structures	Half-flat structures	Half-flat mean convex structures
$\mathfrak{g}_1$	–	–	–
$\mathfrak{g}_2$	–	–	–
$\mathfrak{g}_3$	$\omega = -e^{12} - e^{35} - e^{46}$ $\rho = -\frac{5}{4}e^{136} + \frac{5}{4}e^{145} - e^{156} - e^{234} - e^{236} + e^{245}$	–	–
$\mathfrak{g}_4$	–	✓	–
$\mathfrak{g}_5$	$\omega = -e^{12} - e^{35} - e^{46}$ $\rho = \frac{1}{2}e^{134} - e^{156} - e^{236} + 2e^{245}$	–	–
$\mathfrak{g}_6$	$\omega = e^{15} - e^{24} - e^{36}$ $\rho = e^{123} - e^{134} - e^{146} - e^{235} - e^{256} - e^{345}$	✓	✓
$\mathfrak{g}_7$	$\omega = -\frac{1}{2}e^{15} + \frac{1}{2}e^{24} - \frac{3}{2}e^{36}$ $\rho = -\frac{3}{4}e^{123} + \frac{1}{3}e^{134} - e^{146} + \frac{1}{12}e^{235} - \frac{1}{4}e^{256} + \frac{3}{4}e^{345}$	✓	✓
$\mathfrak{g}_8$	$\omega = e^{15} - e^{24} - \frac{1}{2}e^{36}$ $\rho = e^{123} - e^{134} - \frac{1}{2}e^{146} - e^{235} - \frac{1}{2}e^{256} - e^{345}$	✓	✓
$\mathfrak{g}_9$	–	✓	–
$\mathfrak{g}_{10}$	$\omega = -\frac{1}{2}e^{13} + e^{46} - e^{25}$ $\rho = e^{124} - e^{145} + e^{156} - \frac{1}{2}e^{234} - \frac{1}{2}e^{236} + \frac{1}{2}e^{345}$	✓	✓
$\mathfrak{g}_{11}$	$\omega = \frac{5}{4}e^{13} + \frac{28}{3}e^{24} + e^{25} - \frac{82}{15}e^{26} + \frac{5}{4}e^{34} + e^{35} + e^{45} + \frac{14}{3}e^{46} + e^{56}$ $\rho = 2e^{125} + e^{126} - \frac{5}{4}e^{134} + e^{136} + e^{146} + e^{156} - e^{236} + e^{245} - e^{246}$	✓	–
$\mathfrak{g}_{12}$	–	✓	–
$\mathfrak{g}_{13}$	$\omega = e^{13} + e^{46} + e^{25}$ $\rho = -e^{124} + e^{145} + e^{156} + e^{234} - e^{236} - e^{345}$	✓	✓
$\mathfrak{g}_{14}$	$\omega = e^{13} - e^{26} + e^{45}$ $\rho = -e^{125} - e^{146} + e^{234} + e^{356}$	✓	–
$\mathfrak{g}_{15}$	$\omega = e^{15} + e^{34} - e^{26}$ $\rho = e^{123} + e^{136} - e^{146} + e^{235} - e^{245} + e^{356}$	✓	✓
$\mathfrak{g}_{16}$	$\omega = e^{13} + e^{26} - e^{45}$ $\rho = 2e^{124} - \frac{\sqrt{2}}{2}e^{156} - e^{235} + \frac{\sqrt{2}}{2}e^{346}$	✓	✓
$\mathfrak{g}_{17}$	$\omega = e^{12} + e^{34} + e^{56}$ $\rho = -e^{135} + 2e^{146} + e^{236} + \frac{1}{2}e^{245}$	–	–
$\mathfrak{g}_{18}$	$\omega = e^{12} - e^{34} - e^{56}$ $\rho = e^{135} - \frac{\sqrt{5}}{2}e^{146} + \frac{\sqrt{5}}{2}e^{236} + e^{245} + e^{246}$	–	–
$\mathfrak{g}_{19}$	$\omega = -e^{12} + e^{34} - e^{56}$ $\rho = e^{135} + e^{146} - e^{236} + e^{245}$	–	–
$\mathfrak{g}_{20}$	$\omega = -e^{12} - e^{34} + e^{56}$ $\rho = -e^{135} - e^{146} + e^{235} - e^{236} + e^{245} + e^{246}$	–	–
$\mathfrak{g}_{21}$	$\omega = -e^{12} - e^{34} + e^{56}$ $\rho = -2e^{136} + e^{145} + \frac{1}{2}e^{235} + e^{246}$	✓	–
$\mathfrak{g}_{22}$	$\omega = e^{16} + e^{23} + e^{45}$ $\rho = e^{124} - e^{135} - e^{256} - e^{346}$	✓	✓
$\mathfrak{g}_{23}$	$\omega = e^{12} + e^{34} + e^{56}$ $\rho = 2e^{136} + \frac{1}{2}e^{145} + e^{235} - e^{246}$	–	–
$\mathfrak{g}_{24}$	$\omega = -e^{16} + e^{25} - e^{34}$ $\rho = -e^{123} + e^{145} + e^{246} + e^{356}$	✓	✓
$\mathfrak{g}_{25}$	$\omega = -e^{13} + e^{45} + e^{26}$ $\rho = e^{156} + e^{124} - e^{235} - e^{346}$	✓	✓
$\mathfrak{g}_{26}$	$\omega = e^{16} + e^{23} - e^{36} + e^{45}$ $\rho = -2e^{124} + e^{135} + e^{146} - e^{234} + e^{256}$	–	–
$\mathfrak{g}_{27}$	$\omega = -\frac{\sqrt{3}}{2}e^{12} - e^{45} + e^{36}$ $\rho = e^{135} + e^{146} + e^{234} + e^{235} - e^{256}$	✓	–
$\mathfrak{g}_{28}$	$\omega = -e^{12} - e^{34} + e^{56}$ $\rho = -e^{136} + e^{145} + e^{235} + e^{246}$	✓	✓
$\mathfrak{g}_{29}$	$\omega = e^{13} + e^{24} - e^{56}$ $\rho = e^{126} - e^{145} + e^{235} - e^{346}$	✓	✓
$\mathfrak{g}_{30}$	$\omega = e^{13} - e^{24} + e^{56}$ $\rho = e^{125} - e^{126} + e^{145} + e^{146} + e^{236} + e^{345}$	✓	✓
$\mathfrak{g}_{31}$	$\omega = -e^{14} - e^{35} + e^{26}$ $\rho = -e^{123} + e^{156} - e^{245} - e^{346}$	✓	✓
$\mathfrak{g}_{32}$	$\omega = -\sqrt{2}e^{13} - e^{24} - e^{56}$ $\rho = -e^{125} + e^{146} - e^{236} + 2e^{345}$	✓	✓
$\mathfrak{g}_{33}$	$\omega = -e^{13} - e^{24} - e^{56}$ $\rho = -e^{125} + e^{146} - e^{236} + e^{345}$	✓	✓
$\mathfrak{g}_{34}$	–	✓	–

Chapter 7

Appendix 2

In this appendix, the reader can find the Maple instructions used to derive the main results of Chapter 2.

```
restart;
with(LinearAlgebra); with(difforms);

for i to 6 do defform(e[i] = 1) end do;
for i to 6 do defform(dw[i] = 1) end do;

Sign := proc (p::list) local i, j, n, N; n := nops(p); N := 0;
for i to n-1 do for j from i+1 to n do
if p[j] < p[i] then N := N+1 end if end do end do;
`if`(N::even, 1, -1) end proc:

# Procedure to order wedge products
ord := proc (form) local i, j, k, l, m, n, list1, list2, res, temp; res := 0;
temp := simpform(form);
if wdegree(temp) = 2 then for i to 6 do for j to 6 do
if simpform(&^(e[i], e[j])) <> 0 then list1 := [i, j]; list2 := sort(list1);
res := res+Sign(list1)*coeff(temp, &^(e[i], e[j]))*&^(e[list2[1]], e[list2[2]])
+Sign(list1)*coeff(temp, &^(dw[i], dw[j]))*&^(dw[list2[1]], dw[list2[2]])
end if end do end do
elif wdegree(temp) = 3 then for i to 6 do for j to 6 do for k to 6 do
if simpform(&^(&^(e[i], e[j]), e[k])) <> 0 then list1 := [i, j, k];
list2 := sort(list1); res := res+Sign(list1)*coeff(temp, &^(&^(e[i], e[j]), e[k]))
*&^(&^(e[list2[1]], e[list2[2]]), e[list2[3]])
+Sign(list1)*coeff(temp, &^(&^(dw[i], dw[j]), dw[k]))
*&^(&^(dw[list2[1]], dw[list2[2]]), dw[list2[3]])
end if end do end do end do
elif wdegree(temp) = 4 then
for i to 6 do for j to 6 do for k to 6 do for l to 6 do
if simpform(&^(&^(&^(e[i], e[j]), e[k]), e[l])) <> 0 then
```

```
list1 := [i, j, k, l]; list2 := sort(list1);
res := res+Sign(list1)*coeff(temp, &^(&^(&^(e[i], e[j]), e[k]), e[l]))
*&^(&^(&^(e[list2[1]], e[list2[2]]), e[list2[3]]), e[list2[4]])
+Sign(list1)*coeff(temp, &^(&^(&^(dw[i], dw[j]), dw[k]), dw[l]))
*&^(&^(&^(dw[list2[1]], dw[list2[2]]), dw[list2[3]]), dw[list2[4]])
end if end do end do end do end do
elif wdegree(temp) = 5 then
for i to 6 do for j to 6 do for k to 6 do for l to 6 do for m to 6 do
if simpform(&^(&^(&^(&^(e[i], e[j]), e[k]), e[l]), e[m])) <> 0 then
list1 := [i, j, k, l, m]; list2 := sort(list1);
res := res+Sign(list1)*coeff(temp, &^(&^(&^(&^(e[i], e[j]), e[k]), e[l]), e[m]))
*&^(&^(&^(&^(e[list2[1]], e[list2[2]]), e[list2[3]]), e[list2[4]]), e[list2[5]])
+Sign(list1)*coeff(temp, &^(&^(&^(&^(dw[i], dw[j]), dw[k]), dw[l]), dw[m]))
*&^(&^(&^(&^(dw[list2[1]], dw[list2[2]]), dw[list2[3]]), dw[list2[4]]), dw[list2[5]])
end if end do end do end do end do end do
elif wdegree(temp) = 6 then
for i to 6 do for j to 6 do for k to 6 do for l to 6 do for m to 6 do for n to 6 do
if simpform(&^(&^(&^(&^(&^(e[i], e[j]), e[k]), e[l]), e[m]), e[n])) <> 0 then
list1 := [i, j, k, l, m, n]; list2 := sort(list1);
res := res+Sign(list1)
*coeff(temp, &^(&^(&^(&^(&^(e[i], e[j]), e[k]), e[l]), e[m]), e[n]))
*&^(&^(&^(&^(&^(e[list2[1]], e[list2[2]]), e[list2[3]]), e[list2[4]]),
e[list2[5]]), e[list2[6]])
+Sign(list1)*coeff(temp, &^(&^(&^(&^(&^(dw[i], dw[j]), dw[k]), dw[l]), dw[m]), dw[n]))
*&^(&^(&^(&^(&^(dw[list2[1]], dw[list2[2]]), dw[list2[3]]), dw[list2[4]]),
dw[list2[5]]), dw[list2[6]])
end if end do end do end do end do end do end do end if;
simpform(res) end proc:

# Procedure to differentiate differential forms
dform := proc (form) local i, j, k, temp, res;
res := 0; temp := simpform(form);
if wdegree(temp) = 2 then for i to 6 do for j to 6 do
if simpform(&^(e[i], e[j])) <> 0 then
res := res+coeff(temp, &^(e[i], e[j]))*(&^(de[i], e[j])-&^(e[i], de[j]))
end if end do end do
elif wdegree(temp) = 3 then for i to 6 do for j to 6 do for k to 6 do
if simpform(&^(&^(e[i], e[j]), e[k])) <> 0 then
res := res+coeff(temp, &^(&^(e[i], e[j]), e[k]))
*(&^(&^(de[i], e[j]), e[k])-&^(&^(e[i], de[j]), e[k])+&^(&^(e[i], e[j]), de[k]))
end if end do end do end do end if; ord(res) end proc:

# Chosen orientation
Omega := &^(&^(&^(&^(&^(e[1], e[2]), e[3]), e[4]), e[5]), e[6]):
orient := coeff(Omega, &^(&^(&^(&^(&^(e[1], e[2]), e[3]), e[4]), e[5]), e[6])):
```

```
for i to 20 do defform(p[i] = const) end do;

# Generic 3-form psip:=psi_plus
psip := p[1]*&^(e[1], e[2], e[3])+p[2]*&^(e[1], e[2], e[4])
+p[3]*&^(e[1], e[2], e[5])+p[4]*&^(e[1], e[2], e[6])
+p[5]*&^(e[1], e[3], e[4])+p[6]*&^(e[1], e[3], e[5])
+p[7]*&^(e[1], e[3], e[6])+p[8]*&^(e[1], e[4], e[5])
+p[9]*&^(e[1], e[4], e[6])+p[10]*&^(e[1], e[5], e[6])
+p[11]*&^(e[2], e[3], e[4])+p[12]*&^(e[2], e[3], e[5])
+p[13]*&^(e[2], e[3], e[6])+p[14]*&^(e[2], e[4], e[5])
+p[15]*&^(e[2], e[4], e[6])+p[16]*&^(e[2], e[5], e[6])
+p[17]*&^(e[3], e[4], e[5])+p[18]*&^(e[3], e[4], e[6])
+p[19]*&^(e[3], e[5], e[6])+p[20]*&^(e[4], e[5], e[6]):

# Structure constants
de[1] := [###]:
de[2] := [###]:
de[3] := [###]:
de[4] := [###]:
de[5] := [###]:
de[6] := [###]:

# Compute d(psip) and impose its vanishing
dform(psip)

# Construct the induced almost complex structure
for i to 6 do K1[i] := 0 end do;
for i to 6 do for j to 6 do for k from j+1 to 6 do
if simpform(&^(&^(e[i], e[j]), e[k])) <> 0 then
K1[i] := simpform(K1[i]+coeff(psip, &^(&^(e[i], e[j]), e[k]))*&^(e[j], e[k]));
if simpform(&^(&^(e[j], e[i]), e[k])) <> 0 then
K1[i] := simpform(K1[i]-coeff(psip, &^(&^(e[j], e[i]), e[k]))*&^(e[j], e[k]));
if simpform(&^(&^(e[j], e[k]), e[i])) <> 0 then
K1[i] := simpform(K1[i]+coeff(psip, &^(&^(e[j], e[k]), e[i]))*&^(e[j], e[k]))
end if end if end if end do end do end do;
for i to 6 do K1[i] := ord(&^(K1[i], psip)) end do:

for i to 6 do
K2[i] := orient
*simpform(coeff(K1[i], &^(&^(&^(&^(e[2], e[3]), e[4]), e[5]), e[6]))*e[1]
-coeff(K1[i], &^(&^(&^(&^(e[1], e[3]), e[4]), e[5]), e[6]))*e[2]
+coeff(K1[i], &^(&^(&^(&^(e[1], e[2]), e[4]), e[5]), e[6]))*e[3]
-coeff(K1[i], &^(&^(&^(&^(e[1], e[2]), e[3]), e[5]), e[6]))*e[4]
+coeff(K1[i], &^(&^(&^(&^(e[1], e[2]), e[3]), e[4]), e[6]))*e[5]
```

```
-coeff(K1[i], &^(&^(&^(&^(e[1], e[2]), e[3]), e[4]), e[5]))*e[6]) end do:

K := Matrix(6):
for i to 6 do for j to 6 do K[i, j] := -coeff(K2[j], e[i]) end do end do:

lambda := factor((1/6)*Trace(K.K)):

J := simplify(DiagonalMatrix([1/sqrt(-lambda), 1/sqrt(-lambda), 1/sqrt(-lambda),
1/sqrt(-lambda), 1/sqrt(-lambda), 1/sqrt(-lambda)]).K):


for i to 6 do Je[i] := 0 end do:
for i to 6 do for j to 6 do Je[i] := J[i, j]*e[j]+Je[i] end do end do:

# Construct psim:=J(psip)
psim := 0:
for i to 6 do for j from i+1 to 6 do for k from j+1 to 6 do
psim := ord(psim+coeff(psip, &^(&^(e[i], e[j]), e[k]))*&^(&^(Je[i], Je[j]), Je[k]))
end do end do end do:

# For simplicity, multiply d(psim) by sqrt(-lambda)
dpsim2 := ord(sqrt(-lambda)*dform(psim)):

# Construct a basis of the form {e_i,Je_i,e_j,Je_j,e_k,Je_k}
# To do so, choose three indices {i,j,k}
# and build the change of basis
listJ := [###]:
M := Matrix(6);
M[listJ[1], 1] := 1; M[listJ[2], 3] := 1; M[listJ[3], 5] := 1;
for i to 6 do M[i, 2] := J[i,listJ[1]];
M[i, 4] := J[i,listJ[2]]; M[i, 6] := J[i,listJ[3]] end do:

# Make sure det(M) is non-zero
Determinant(M)


Q := simplify(MatrixInverse(M)):

# Rewrite dpsim2 in complex coordinates, obtaining dpsim3
# dw[i], i=1,2,3, are (1,0)-forms
# dw[i], i=4,5,6, are their conjugate (0,1)-forms
h[1] := 1/2*(dw[1]+dw[4]): h[2] := (I*(1/2))*(dw[1]-dw[4]):
h[3] := 1/2*(dw[2]+dw[5]): h[4] := (I*(1/2))*(dw[2]-dw[5]):
h[5] := 1/2*(dw[3]+dw[6]): h[6] := (I*(1/2))*(dw[3]-dw[6]):
```

```
for i to 6 do g[i] := 0 end do:
for i to 6 do for j to 6 do g[i] := Q[j, i]*h[j]+g[i] end do end do:
dpsim3 := 0:
for i to 6 do for j from i+1 to 6 do
for k from j+1 to 6 do for l from k+1 to 6 do
dpsim3 := ord(dpsim3+coeff(dpsim2, &^(&^(&^(e[i], e[j]), e[k]), e[l]))
*&^(&^(&^(g[i], g[j]), g[k]), g[l])) end do end do end do end do:

# Construct the (1,1)-form beta associated with dpsim3
beta := 0:
for i to 6 do for j from i+1 to 6 do for k from j+1 to 6 do
for l from k+1 to 6 do for r to 6 do for s from r+1 to 6 do
if &^(&^(dw[i], dw[j], dw[k], dw[l]), dw[r], dw[s]) <> 0 then
beta := ord(beta+Sign([i, j, k, l, r, s])
*(2*I)*&^(coeff(dpsim3, &^(dw[i], dw[j], dw[k], dw[l])), dw[r], dw[s]))
end if end do end do end do end do end do end do:

# Construct the 3x3 matrix B associated with beta
B := Matrix(3):
for i to 3 do for j to 3 do
B[i, j] := -(2*I)*coeff(beta, &^(dw[i], dw[j+3])) end do end do:

# Using Sylvester's criterion, check if B is positive semidefinite
# via its principal minors
B[1, 1]

B[2, 2]

B[3, 3]

Determinant(SubMatrix(B, [1, 2], [1, 2]))

Determinant(SubMatrix(B, [1, 3], [1, 3]))

Determinant(SubMatrix(B, [2, 3], [2, 3]))

Determinant(B)
```

Bibliography

[1] D. V. Alekseevsky, *Contact homogeneous spaces*, Funct. Anal. Appl. **24** (1990), 324–325.

[2] D. V. Alekseevsky, B. N. Kimel'fel'd, *Structure of homogeneous Riemannian spaces with zero Ricci curvature*, Funkcional. Anal. i Priložen. **9** (1975), 5–11.

[3] M. M. Alexandrino, R.G. Bettiol, *Lie Groups and Geometric Aspects of Isometric Actions*, Springer, Berlin (2015).

[4] B. Alexandrov, S. Ivanov, *Vanishing theorems on Hermitian manifolds*, Differ. Geom. Appl. **14** (2001), 251–265.

[5] I. Alonso, F. Salvatore, *On the existence of balanced metrics on six-manifolds of cohomogeneity one*, Ann. Glob. Anal. Geom. **61** (2022), 309–331.

[6] A. Andrada, A. Fino, L. Vezzoni, *A class of Sasakian 5-manifolds*, *Transform. Groups* **14** (2009) 493–512.

[7] V. Apostolov, S. Salamon, *Kähler reduction of metrics with holonomy* G_2, Comm. Math. Phys. **246** (2004), 43–61.

[8] M. F. Arikan, H. Cho, S. Salur, *Existence of compatible contact structures on* G_2-*manifolds*, Asian J. Math. **17** (2013), 321–334.

[9] G. Ball, *Quadratic closed* G_2-*structures*, arXiv:2006.14155.

[10] L. Bedulli, L. Vezzoni, *The Ricci tensor of* $\mathrm{SU}(3)$-*manifolds*, J. Geom. Phys. **57** (2007), 1125–1146.

[11] L. Bedulli, L. Vezzoni, *A parabolic flow of balanced metrics*, J. Reine Angew. Math. **723** (2017), 79–99.

[12] F. Belgun, V. Cortés, M. Freibert, O. Goertsches, *On the boundary behavior of left-invariant Hitchin and hypo flows*, J. Lond. Math. Soc. (2) **92** (2015), 41–62.

[13] L. Bérard Bergery, *Sur des nouvelles varietes Riemanniennes d'Einstein*, Publ. de Inst. E. Cartan **6** (1982), 1–60.

[14] M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330.

[15] C. Bock, *On low dimensional solvmanifolds*, Asian J. Math. **20** (2016), 199–262.

[16] E. Bonan, *Sur des variétés riemanniennes à groupe d'holonomie G2 ou Spin(7)*, C. R. Acad. Sci. Paris Sér. A-B, **262** (1966), A127–A129.

[17] W. M. Boothby, H. C. Wang, *On Contact manifolds*, Ann. Math. **68** (1958), 721–734.

[18] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. **126** (1987), 525–576.

[19] R. L. Bryant, *Some remarks on* G_2*-structures*, in: Proceedings of Gökova Geometry-Topology Conference (2006), 75–109.

[20] R. L. Bryant, *Non-embedding and non-extension results in special holonomy*, in: The many facets of geometry, Oxford Univ. Press, Oxford (2010), 346–367.

[21] R. L. Bryant, S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850.

[22] R. Bryant, F. Xu, *Laplacian Flow for Closed* G_2*-Structures: Short Time Behavior*, arXiv:1101.2004.

[23] J.-B. Butruille, *Homogeneous nearly Kähler manifolds*, in: Handbook of pseudo-Riemannian geometry and supersymmetry, vol. 16 of IRMA Lect. Math. Theor. Phys., Eur. Math. Soc., Zürich (2010), 399–423.

[24] S. Chiossi, S. Salamon, *The intrinsic torsion of* $\mathrm{SU}(3)$ *and* G_2 *structures*, Differential geometry, Valencia (2001), 115–133, World Sci. Publishing, River Edge, NJ (2002).

[25] S. Chiossi, A. Swann, G_2*-structures with torsion from half-integrable nilmanifolds*, J. Geom. Phys. **54** (2005), 262–285.

[26] D. Conti, *Half-flat nilmanifolds*, Math. Ann. **350** (2011), 155–168.

[27] D. Conti, M. Fernández, *Nilmanifolds with a calibrated* G_2*-structure*, Differential Geom. Appl. **29** (2011), 493–506.

[28] D. Conti, A. Tomassini, *Special symplectic* 6*-manifolds*, Q. J. Math. **58** (2006), 297–311.

[29] V. Cortés, T. Leistner, L. Schäfer, F. Schulte-Hengesbach, *Half-flat structures and special holonomy*, Proc. Lond. Math. Soc. (3) **102** (2011), 113–158.

[30] A. Corti, M. Haskins, J. Nordström, T. Pacini, G_2*-manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J. **164** (2015), 1971–2092.

[31] D. Crowley, J. Nordström, *New invariants of* G_2*-structures*, Geom. Topol. **19** (2015), 2949-2992.

[32] X. de la Ossa, M. Larfors, M. Magill, *Almost contact structures on manifolds with a* G_2 *structure*, arXiv:2101.12605

[33] J.-P. Demailly, *Complex Analytic and Differential Geometry*, available at `www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

[34] A. Diatta, *Left invariant contact structures on Lie groups*, *Differential Geom. Appl.* **26** (2008), 544–552.

[35] S. Donaldson, *Remarks on* G_2*-manifolds with boundary*, Surv. Differ. Geom. **22** (2017), 103–124.

[36] J. J. Duistermaat, J. A. C. Kolk, *Lie Groups*, Springer, Berlin (2000).

[37] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, AMS, Providence, RI (2002).

[38] N. Enrietti, A. Fino, L. Vezzoni, *Tamed Symplectic forms and Strong Kähler with torsion metrics*, J. Symplectic Geom. **10** (2012), 203–223.

[39] T. Fei, *A construction of non-Kähler Calabi-Yau manifolds and new solutions to the Strominger system*, Adv. Math. **302** (2016), 529–550.

[40] T. Fei, Z. Huang, S. Picard, *A construction of infinitely many solutions to the Strominger system*, J. Differ. Geom. **117** (2021), 23–39.

[41] T. Fei, S.-T. Yau, *Invariant solutions to the Strominger system on complex Lie groups and their quotients*, Commun. Math. Phys. **338** (2015), 1183–1195.

[42] M. Fernández, *An example of a compact calibrated manifold associated with the exceptional Lie group* G_2, J. Differential Geom. **26** (1987), 367–370.

[43] M. Fernández, *A family of compact solvable* G_2*-calibrated manifolds*, Tohoku Math. J. **39** (1987), 287–289.

[44] M. Fernández, M. de León, M. Saralegui, *A six dimensional symplectic solvmanifold without Kähler structures*, Osaka J. Math. **33** (1996), 19–35.

[45] M. Fernández, A. Fino, V. Manero, *Laplacian flow of closed* G_2*-structures inducing nilsolitons*, J. Geom. Anal. **26** (2016), 1808–1837.

[46] M. Fernández, A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Mat. Pura Appl. **132** (1982), 19–45.

[47] M. Fernández, V. Manero, A. Otal, L. Ugarte, *Symplectic half-flat solvmanifolds*, Ann. Glob. Anal. Geom. **43** (2013), 367–383.

[48] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Compact supersymmetric solutions of the heterotic equations of motion in dimensions* 7 *and* 8, Adv. Theor. Math. Phys. **15** (2011), 245–284.

[49] M. Fernández, A. Tomassini, L. Ugarte, R. Villacampa, *Balanced Hermitian metrics from* $\mathrm{SU}(2)$*-structures*, J. Math. Phys. **50** (2009), 033507.

[50] S. Fidanza, R. Minasian, A. Tomasiello, *Mirror symmetric* SU(3)*-structure manifolds with NS fluxes*, Comm. Math. Phys. **254** (2005), 401–423.

[51] J. Fine, C. Yao, *Hypersymplectic* 4*-manifolds, the* G_2*-Laplacian flow, and extension assuming bounded scalar curvature*, Duke Math. J. **167** (2018), 3533–3589.

[52] A. Fino, G. Grantcharov, L. Vezzoni, *Astheno-Kähler and balanced structures on fibrations*, Int. Math. Res. Not. **22** (2019), 7093–7117.

[53] A. Fino, A. Raffero, *Einstein locally conformal calibrated* G_2*-structures*, Math. Z. **280** (2015), 1093–1106.

[54] A. Fino, A. Raffero, *Coupled* SU(3)*-Structures and Supersymmetry*, Symmetry **7** (2015), 625–650.

[55] A. Fino, A. Raffero, *Closed* G_2*-structures on non-solvable Lie groups*, Rev. Mat. Complut. **32** (2019), 837–851.

[56] A. Fino, A. Raffero, *Closed warped* G_2*-structures evolving under the Laplacian flow*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. **20** (2020) 315–348.

[57] A. Fino, A. Raffero, *Remarks on homogeneous solitons of the* G_2*-Laplacian flow*, C. R. Math. Acad. Sci. Paris. **358** (2020), 401–406.

[58] A. Fino, A. Raffero, F. Salvatore, *Closed* G_2*-structures on unimodular Lie algebras with non-trivial center*, Transform. Groups, (2022), doi:10.1007/s00031-021-09683-8.

[59] A. Fino, F. Salvatore, *Closed* $SL(3,\mathbb{C})$*-structures on nilmanifolds*, J. Geom. Phys. **167** (2021), 104289.

[60] A. Fino, L. Vezzoni, *Special Hermitian metrics on compact solvmanifolds*, J. Geom. Phys. **91** (2015), 40–53.

[61] A. Fino, L. Vezzoni, *On the existence of balanced and SKT metrics on nilmanifolds*, Proc. Am. Math. Soc. **144** (2016), 2455–2459.

[62] L. Foscolo, M. Haskins, *New* G_2 *holonomy cones and exotic nearly Kähler structures on the 6-sphere and the product of a pair of 3-spheres*, Annals Math. **185** (2017), 59–130.

[63] U. Fowdar, S^1*-invariant Laplacian flow*, J. Geom. Anal. **32** (2022), doi:10.1007/s12220-021-00784-0.

[64] M. Freibert, *Calibrated and parallel structures on almost Abelian Lie algebras*, arXiv:1307.2542.

[65] M. Freibert, S. Salamon, *Closed* G_2*-eigenforms and exact* G_2*-structures*, Rev. Mat. Iberoam. (2021), doi:10.4171/RMI/1315.

[66] J. Fu, J. Li, S.-T. Yau, *Balanced metrics on non-Kähler Calabi-Yau threefolds*, J. Differ. Geom. **90** (2012), 81–129.

[67] J.-X. Fu, L.-S. Tseng, S.-T. Yau, *Local heterotic torsional models*, Commun. Math. Phys. **289** (2009), 1151–1169.

[68] J.-X. Fu, S.-T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampére equation*, J. Differ. Geom. **78** (2008), 369–428.

[69] M. Garcia-Fernandez, *T-dual solutions of the Hull-Strominger system on non-Kähler threefolds*, J. reine angew. Math. **2020** (2020), 137–150.

[70] G. W. Gibbons, H. Lü, C. N. Pope, K. S. Stelle, *Supersymmetric domain walls from metrics of special holonomy*, Nuclear Phys. B **623** (2002), 3–46.

[71] M-P. Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and $\mathbb{R}$)*, Ph.D. thesis, University of Waterloo (1998).

[72] V. V. Gorbatsevich, A. L. Onishchik, *Lie transformation groups*, in: Lie groups and Lie algebras. I, Encyclopaedia of Mathematical Sciences, vol. 20, Springer-Verlag, Berlin (1993).

[73] M. Goze, Y. Khakimdjanov, *Nilpotent Lie algebras*, Mathematics and its Applications, 361, Kluwer Academic Publishers Group, Dordrecht (1996).

[74] D. Grantcharov, G. Grantcharov, Y.S. Poon: *Calabi-Yau connections with torsion on toric bundles*. J. Differ. Geom. **78** (2008), 13–32.

[75] G. Grantcharov, *Geometry of compact complex homogeneous spaces with vanishing first Chern class*, Adv. Math. **226** (2011), 3136–3159.

[76] A. Gray, *Vector cross products on manifolds*, Trans. Am. Math. Soc. **141** (1969) 465–504.

[77] A. Gray, *Nearly Kähler manifolds*, J. Differential Geom. **4** (1970), 283–309.

[78] A. Gray, *The structure of nearly Kähler manifolds*, Math. Ann. **223** (1976), 223–248.

[79] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

[80] F. R. Harvey, *Spinors and Calibrations*, Perspectives in Mathematics vol. 9, Academic Press, Boston (1990).

[81] R. Harvey, A. Knapp, *Positive (p,p)-forms, Wirtinger's inequality, and currents*, in: Value distribution theory, Part A, M. Dekker (1974), 43–62.

[82] N. Hitchin, *The geometry of three-forms in six dimensions*, J. Differential Geom. **55** (2000), 547–576.

[83] N. Hitchin, *Stable forms and special metrics*, in: Global differential geometry: the mathematical legacy of Alfred Gray, Bilbao (2000), vol. 288 of Contemp. Math., Amer. Math. Soc., Providence, RI (2001), 70–89.

[84] C. A. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, Pac. J. Math. **246** (2010), 129–185.

[85] C. A. Hoelscher, *Diffeomorphism type of six-dimensional cohomogeneity one manifolds*, Ann. Global Anal. Geom. **38** (2010), 1–9.

[86] H. Huang, Y. Wang, C. Yao, *Cohomogeneity-one* G_2*-Laplacian flow on 7-torus*, J. London Math. **98** (2018), 349–368.

[87] C. Hull, *Superstring compactifications with torsion and space-time supersymmetry.* In Turin 1985 Proceedings "Superunification and Extra Dimensions" (1986), 347–375.

[88] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy* G_2, J. Differential Geom. **43** (1996), 291–328.

[89] D. D. Joyce, *Riemannian holonomy groups and calibrated geometry*, volume 12 of Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, (2007).

[90] D. D. Joyce, S. Karigiannis, *A new construction of compact torsion-free* G_2*-manifolds by gluing families of Eguchi-Hanson spaces*, J. Differential Geom. **117** (2021), 255–343.

[91] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.

[92] S. Kutsak, *Invariant contact structures on 7-dimensional nilmanifolds*, Geom. Dedicata **172** (2014) 351–361.

[93] B. Lambert, J. D. Lotay, *Spacelike Mean Curvature Flow*, J. Geom. Anal. **31** (2021), 1291–1359.

[94] M. Larfors, *Revisiting toric* $SU(3)$ *structures*, Fortschr. Phys. **61** (2013), 1031–1055.

[95] J. Lauret, *Geometric flows and their solitons on homogeneous spaces*, Rend. Semin. Mat. Univ. Politec. Torino **74** (2016), 55–93.

[96] J. Lauret, *Laplacian flow of homogeneous* G_2*-structures and its solitons*, Proc. Lond. Math. Soc. **114** (2017), 527–560.

[97] J. Lauret, *Laplacian solitons: questions and homogeneous examples*, Differential Geom. Appl. **54** (2017), 345–360.

[98] J. Lauret, M. Nicolini, *The classification of ERP* G_2*-structures on Lie groups*, Ann. Mat. Pura Appl. **199** (2020), 2489–2510.

[99] C. Lin, *Laplacian solitons and symmetry in* G_2*-geometry*, J. Geom. Phys. **64** (2013), 111–119.

[100] J. D. Lotay, *Geometric Flows of* G_2 *Structures*, in: Lectures and Surveys on G_2-Manifolds and Related Topics, Fields Institute Communications, vol 84, Springer, New York, NY, (2020).

[101] J. D. Lotay, Y. Wei, *Laplacian flow for closed* G_2 *structures: Shi-type estimates, uniqueness and compactness*, Geom. Funct. Anal. **27** (2017), 165–233.

[102] J. D. Lotay, Y. Wei, *Stability of torsion-free* G_2*-structures along the Laplacian flow*, J. Differential Geom. **111** (2019), 495–526.

[103] M. Macrì, *Cohomological properties of unimodular six dimensional solvable Lie algebra*, Differential Geom. Appl. **31** (2013), 112–129.

[104] T. B. Madsen, S. Salamon, *Half-flat structures on* $S^3 \times S^3$, Ann. Glob. Anal. Geom. **44** (2013), 369–390.

[105] L. Magnin, *Sur les algèbres de Lie nilpotentes de dimension* ≤ 7, J. Geom. Phys. **3** (1986), 119–144.

[106] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), 261–295.

[107] J. Milnor, *Curvatures of left invariant metrics on Lie Groups*, Adv. Math. **21** (1976), 293–329.

[108] G. M. Mubarakzyanov, *Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element*, Izv. Vyssh. Uchebn. Zaved. Mat. **1963** (1963), 104–116.

[109] P.-A. Nagy, *Nearly Kähler geometry and Riemannian foliations*, Asian J. Math. **6** (2002), 481–504.

[110] P.-A. Nagy, *On nearly Kähler geometry*, Ann. Global Anal. Geom. **22** (2002), 167–178.

[111] M. Nicolini, *Laplacian solitons on nilpotent Lie groups*, Bull. Belg. Math. Soc. Simon Stevin. **25** (2018), 183–196.

[112] M. Nicolini, *New Examples Of Shrinking Laplacian Solitons*, Q. J. Math. **73** (2021), 239–259.

[113] A. Otal, L. Ugarte, R. Villacampa, *Invariant solutions to the Strominger system and the heterotic equations of motion*, Nucl. Phys. B **920** (2017), 442–474.

[114] D. H. Phong, S. Picard, X. Zhang, *A flow of conformally balanced metrics with Kähler fixed points*, Math. Ann. **374** (2019), 2005–2040.

[115] F. Podestà, A. Raffero, *Homogeneous symplectic half-flat 6-manifolds*, Ann. Glob. Anal. Geom. **55** (2019), 1–15.

[116] F. Podestà, A. Raffero, *On the automorphism group of a symplectic half-flat 6-manifold*, Forum Math. **31** (2019), 265–273.

[117] F. Podestà, A. Spiro, 6*-dimensional nearly Kähler manifolds of cohomogeneity one*, J. Geom. Phys. **60** (2010), 156–164.

[118] M. Pujia, *The Hull-Strominger system and the Anomaly flow on a class of solvmanifolds*, J. Geom. Phys. **170** (2021), 104352.

[119] M. Pujia, L. Ugarte, *The Anomaly flow on nilmanifolds*, Ann. Glob. Anal. Geom. **60** (2021), 501–537.

[120] W. Reichel, *Über die Trilinearen Alternierenden Formen in 6 und 7 Veränder-lichen*, Ph.D. thesis, Greifswald University (1907).

[121] S. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), 311–333.

[122] F. Schulte-Hengesbach, *Half-flat structures on Lie groups*, Ph.D. thesis, University of Hamburg (2010), available at `www.math.uni-hamburg.de/home/schulte-hengesbach/diss.pdf`.

[123] A. Shabanskaya, *Classification of six dimensional solvable indecomposable Lie algebras with a codimension one nilradical over* $\mathbb{R}$, Ph.D. thesis, University of Toledo (2011).

[124] L. Šnobl, P. Winternitz, *Classification and Identification of Lie Algebras*, CRM Monograph Series **33**, American Mathematical Society (2014).

[125] A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (1986), 253–284.

[126] A. Tralle, J. Oprea, *Symplectic manifolds with no Kähler structures*, Lectures Notes in Mathematics, 1661, Springer-Verlag, Berlin (1997).

[127] P. Turkowski, *Solvable Lie algebras of dimension six*, J. Math. Phys. **31**, (1990), 1344–1350.

[128] L. Ugarte, R. Villacampa, *Balanced Hermitian geometry on 6-dimensional nilmanifolds*, Forum Math. **27** (2015), 1025–1070.

[129] L. Verdiani, W. Ziller, *Smoothness Conditions in Cohomogeneity one manifolds*, Transform. Groups **27** (2022), 311–342.

[130] H.-C. Wang, *Complex parallisable manifolds*, Proc. Amer. Math. Soc. **5** (1954), 771–776.

[131] T. Yamada, *A pseudo-Kähler structure on a nontoral compact complex parallelizable solvmanifold*, Geom. Dedicata **112** (2005), 115–122.

[132] W. Ziller, *On the geometry of cohomogeneity one manifolds with positive curvature*, in: Riemannian Topology and Geometric Structures on Manifolds, Progress in Mathematics 271, Birkhäuser, Boston (2009).