Algebraic numbers close to 1: results and methods

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Francesco Amoroso

Dipartimento di Matematica
Via Carlo Alberto, 10
10123 Torino - Italy
e-mail: amoroso@dm.unipi.it

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§1 Introduction.

Let $\alpha \neq 1$ be an algebraic number and denote by $F(x) = a(x-\alpha_1)\cdots(x-\alpha_d)$ ($a > 0$) its minimal polynomial over $\mathbb{Z}$. We are interested in lower bounds for $|\alpha - 1|$ depending on the degree d of α and on its Mahler measure

$$M(\alpha) = M(F) = a\prod_{h=1}^{d}\max(|\alpha_h|, 1).$$

Liouville's inequality gives

$$\log|\alpha - 1| \geq -(d-1)\log 2 - \log M(\alpha). \tag{1.1}$$

This bound is sharp if $M(\alpha)$ is large, say $\log M(\alpha) \geq$ (constant) $\times d$. Otherwise, better results are known, obtained using two different methods. In 1979 M. Mignotte [M1] gave the lower bound:

$$\log|\alpha - 1| \geq -4\sqrt{d}\log(4d) \tag{1.2}$$

provided that $M(\alpha) \leq 2$. In this range, (1.2) improves (1.1) by a factor $\sqrt{d}(\log d)^{-1}$. To prove (1.2), Mignotte applied the Gel'fond method. He used Siegel's Lemma to construct a polynomial P with integer coefficients and low height vanishing at 1 with relatively high multiplicity. Then (1.2) follows by applying Liouville's inequality.

Recently M. Mignotte and M. Waldschmidt [MW] improved (1.2) by finding for any $\mu > \log M(\alpha)$ the lower bound

$$\log|\alpha - 1| \geq -\frac{3}{2}\sqrt{d\mu\log^+(d/\mu)} - 2\mu - \log^+(d/\mu), \tag{1.3}$$

where $\log^+ x = \max(\log x, 0)$ ($x > 0$). The approach of M. Mignotte and M. Waldschmidt differs from Mignotte's paper essentially by two arguments: firstly, they use the Schneider method; secondly they employ an interpolation determinant, which avoids the use of Siegel's Lemma.

The inequality (1.3) was slightly improved upon by Y. Bugeaud, M. Mignotte and F. Normandin [BMN] and by A. Dubickas [Du], who found better values for the constants involved, also giving simpler proofs.

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The aim of this paper is to briefly describe the crucial steps in the proofs of these lower bounds. In section 2 we first describe Mignotte's approach (Gel'fond's method and Siegel's Lemma: Theorem 2.1). Then we study the relation between the behaviour of the height of a product of cyclotomic polynomials with prescribed vanishing at 1 and the lower bounds for $|\alpha - 1|$ (Theorem 2.2). This allows us to describe the limit of Gel'fond's method. We also show that the determinant which appears in Mignotte and Waldschmidt's approach (Schneider's method and interpolation determinant) is a product of cyclotomic polynomials with high vanishing at 1 and low height, hence a "good" auxiliary function for Gel'fond's method. This remark gives, by using Theorem 2.2, an alternative proof of the main result of [MW] (Theorem 2.3). In section 3 we describe the main result of [A2], an explicit construction of algebraic numbers close to 1, which shows that the inequality (1.3) is almost sharp. Finally, in section 4 we discuss a generalization in several variables of these results.

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§2 1-dimensional results.

In this section we shall prove bounds of the shape $|\alpha-1| \geq f\big(\deg\alpha, M(\alpha)\big)$. In order to prove such a bound, we assume $|\alpha| \leq 1$, since otherwise $\beta = 1/\alpha$ satisfies $|\beta| < 1$, $|\beta - 1| \leq |\alpha - 1|$, deg β = deg α and $M(\beta) = M(\alpha)$. Obviously, we also assume $\alpha \neq 0$.

We start with Mignotte's approach, which uses Gel'fond 's method and Siegel's Lemma.

First step: construction of the auxiliary function.

Let $m < N$ be two positive integers; the Bombieri and Vaaler version of Siegel's Lemma (see [BV], Theorem 1) gives a non-zero polynomial $P \in \mathbb{Z}[x]$ of degree $< N$, vanishing at 1 with multiplicity $m(P) \geq m$ and such that its height $H(P)$ (i.e. the maximum modulus of its coefficients) satisfies the inequality [(1)]

$$\log H(P) \leq \frac{m^2}{2(N-m)} \log\frac{cN}{m}, \qquad c = \frac{1}{4}\exp\frac{3}{2}.$$

Since the maximum modulus $|P|$ of P on the unit circle is bounded by $N \cdot H(P)$, we deduce the inequality

$$\log|P| \leq \frac{m^2}{2(N-m)} \log\frac{cN}{m} + \log N. \tag{2.1}$$

(1) By using the box-principle we easily obtain the weaker estimate

$$\log H(P) \leq \frac{m(m+1)}{2(N-m)} \log N.$$

The bound of [BV] allows us to save a factor $\sqrt{2}$ in Theorem 2.1.

Second step: multiplicity estimate.

To ensure that $P(\alpha) \neq 0$ we must have $N \leq m + d$. It is convenient to choose N as large as possible: therefore we fix $N = m + d$.

Third step: maximum principle.

Let P be any polynomial of degree $\leq N$, vanishing at 1 with multiplicity $\geq m$ such that $P(\alpha) \neq 0$. For any embedding $\sigma\colon \mathbb{Q}(\alpha) \to \mathbb{C}$ we have

$$|\sigma P(\alpha)| \leq \max\{1, |\sigma\alpha|\}^N |P|. \tag{2.2}$$

Moreover, by the maximum principle on the disk $|z| = N/(N-m)$,

$$|P(\alpha)| \leq |\alpha - 1|^m \frac{N^N}{m^m (N-m)^{N-m}} |P| \leq |\alpha - 1|^m \left(\frac{eN}{m}\right)^m |P|. \tag{2.3}$$

Since $P(\alpha) \neq 0$, the inequalities (2.2) and (2.3) together give

$$1 \leq a^N \prod_\sigma |\sigma P(\alpha)| \leq |\alpha - 1|^m \left(\frac{eN}{m}\right)^m M(\alpha)^N |P|^d,$$

whence we obtain

$$-\log|\alpha - 1| \leq 1 + \log\frac{N}{m} + \frac{N}{m}\log M(\alpha) + \frac{d}{m}\log|P|. \tag{2.4}$$

Last step: choice of the parameter m.

Substituting (2.1) and the contraint $N = m + d$ into (2.4), we obtain

$$-\log|\alpha - 1| \leq \log(2eM(\alpha)) + \frac{d}{m}\log(2dM(\alpha)) + \frac{m}{2}\log\frac{2cd}{m} \tag{2.5}$$

for any positive integer $m \leq d$. We choose

$$m = 1 + \left[2\sqrt{\frac{d\log(2dM(\alpha))}{\log(2d)}}\right].$$

Assume first $2\sqrt{d\log(2dM(\alpha))} < d\sqrt{\log(2d)}$. Then $m \leq d$ and (2.5) yields, using the inequality $2cd/m \leq c\sqrt{d} < \sqrt{2d}$,

$$-\log|\alpha - 1| < \log(2eM(\alpha)) + \frac{1}{4}\log(2d) + \sqrt{d\log(2d)\log(2dM(\alpha))}. \tag{2.6}$$

Otherwise, if $2\sqrt{d\log(2dM(\alpha))} \geq d\sqrt{\log(2d)}$, the right hand side of (2.6) is

$$\geq \log M(\alpha) + \frac{1}{2}d\log(2d) > \log M(\alpha) + (d-1)\log 2$$

and Liouville's inequality (1.1) implies (2.6). Therefore, (2.6) holds in any case. We have proved:

Theorem 2.1.

Let $\alpha \neq 1$ be an algebraic number of degree d. Then

$$|\alpha - 1| \geq 2^{-5/4} e^{-1} d^{-1/4} M(\alpha)^{-1} \exp\left\{-\sqrt{d \log(2d) \log(2dM(\alpha))}\right\}.$$

It can be shown that any polynomial with integer coefficients having small degree and high vanishing at 1 must also vanish at several roots of unity (see [BV], section 5, and [A1]).

Let $\alpha = \exp(2j\pi i/k)$ be a primitive k-th root of unity ($k > 1$). Then $d = \deg\,\alpha = \phi(k)$ and $|\alpha - 1| = 2|\sin(j\pi/k)| \geq 2\sin(\pi/k)$. Since $\sin x \sim x$ for $x \to 0$ and

$$\liminf_{k \to +\infty} \frac{\phi(k) \log\log k}{k} = e^{-\gamma}$$

where γ is Euler's constant (see [HW] , Theorem 328), we have

$$|\alpha - 1| \geq \big(1 + o_d(1)\big) \frac{2\pi}{e^\gamma} (d \log\log d)^{-1} \tag{2.7}$$

where $o_d(1)$ is a function of d satisfying $\lim\limits_{d \to +\infty} o_d(1) = 0$. Hence, in order to find lower bounds for $|\alpha - 1|$, we can assume that α is not a root of unity. (2)

Denote by $\Phi_k(x)$ the k-th cyclotomic polynomial and let

$$\Gamma = \left\{x^{e_0} \Phi_1^{e_1} \cdots \Phi_k^{e_k}, \text{ such that } k \in \mathbb{N} \text{ and } e_0, \ldots, e_k \in \mathbb{Z}\right\}.$$

Also let, for $t \geq 0$, $\Gamma(t)$ be the set of non-constant polynomials $P \in \Gamma$ satisfying the inequality $\log |P| \leq t \cdot \deg\, P$. Notice that $\Gamma(t) \neq \emptyset$, since $x \in \Gamma(t)$ for any $t \geq 0$. We define a function $r\colon [0, +\infty) \to [0, 1]$ by setting

$$r(t) = \sup_{P \in \Gamma(t)} \frac{m(P)}{\deg P},$$

where $m(P)$ denotes the multiplicity of P at $x = 1$. Let α be an algebraic number of degree d which is not a root of unity and let $h = d^{-1} \log M(\alpha) > 0$ be its logarithmic height. Let also $t > 0$; from the definition of $r(t)$ it follows that for any $\varepsilon > 0$ there exists a polynomial $P \in \Gamma$ of degree N, vanishing at 1 with multiplicity m such that $\log |P| \leq t \cdot N$ and $m/N \geq (1 + \varepsilon)^{-1} r(t)$. Then (2.4) implies

$$-\log |\alpha - 1| \leq 1 + \log \frac{1 + \varepsilon}{r(t)} + (1 + \varepsilon) \frac{d(h + t)}{r(t)}.$$

Since this inequality holds for any $\varepsilon > 0$, we have:

(2) The lower bound (2.7) is stronger than the result available in the general case. This suggests that also for algebraic numbers of very small height (say $M(\alpha)$ bounded by an absolute constant) strong results can be proved.

Theorem 2.2.

For any algebraic number α of degree d and logarithmic height $h > 0$ and for any $t > 0$ we have

$$\log|\alpha - 1| \geq -1 - \log\frac{1}{r(t)} - \frac{d(h+t)}{r(t)}.$$

For small values of h a good choice of the parameter t is $t = h$:

Corollary 2.1.

For any algebraic number α of degree d and logarithmic height $h > 0$ we have

$$\log|\alpha - 1| \geq -1 - \log\frac{1}{r(h)} - \frac{2dh}{r(h)}.$$

It can be shown that $r(t) \leq c\sqrt{t}$ for some absolute constant $c > 0$ (see [M1], [A1] and [BV], section 5). Hence the limit of Gel'fond's method seems to be a lower bound of the shape

$$\log|\alpha - 1| \geq -1 - \log\frac{1}{c} - \frac{1}{2}\log\frac{d}{\log M(\alpha)} - \frac{2}{c}\sqrt{d\log M(\alpha)}.$$

We now consider Schneider's approach. The auxiliary function is a polynomial with coefficients in $\mathbb{Q}(\alpha)$ vanishing at several powers of α. The corresponding interpolation determinant is

$$\Delta = \operatorname{Det}\left(\alpha^{ij}\right)_{\substack{i=0,\ldots,k-1;\\ j=0,\ldots,k-1}} = \prod_{0\leq i<j\leq k-1}(\alpha^j - \alpha^i) \neq 0$$

where $k \in \mathbb{N}$ is a parameter at our disposal. We consider this determinant as a polynomial $\Delta(\alpha) \in \mathbb{Z}[\alpha]$. [3] Then $\Delta \in \Gamma$ and an easy computation shows that Δ has degree $\frac{1}{6}(k-1)k(2k-1)$, vanishes at 0 with multiplicity $\frac{1}{6}(k-1)k(k-2)$ and vanishes at 1 with multiplicity $m := \frac{1}{2}k(k-1)$. Moreover, by Hadamard's inequality, $|\Delta| \leq k^{k/2}$. Consider the polynomial $P(x) = x^{-(k-1)k(k-2)/6}\Delta(x) \in \Gamma$ of degree $N := \frac{1}{6}(k-1)k(k+1)$. We have

$$\frac{\log|P|}{N} \leq (1 + o_k(1))\frac{3\log k}{k^2}, \qquad \frac{m}{N} = (1 + o_k(1))\frac{3}{k},$$

where $o_k(1) \to 0$ for $k \to +\infty$. Let $t > 0$; choosing $k = 1 + \left[\sqrt{\frac{3}{2t}\log\frac{1}{t}}\right]$ we obtain

$$\frac{\log|P|}{N} \leq (1 + o_t(1))t, \qquad \frac{m}{N} = (1 + o_t(1))\sqrt{\frac{6t}{\log 1/t}},$$

where $o_t(1) \to 0$ for $t \to 0$. This proves that

$$r(t) \geq (1 + o_t(1))\sqrt{\frac{6t}{\log 1/t}}.$$

[3] This polynomial was firstly introduced by Dobrowolski (see [D]).

Let α be an algebraic number of degree d and logarithmic height $h > 0$. By applying Corollary 2.1 we find that for any $\varepsilon \in (0,1)$ there exists $C(\varepsilon) > 1$ such that

$$\log|\alpha - 1| \geq -2 - \frac{1}{2}\log\left(\frac{1}{6h}\log\frac{1}{h}\right) - (1+\varepsilon/2)d\sqrt{\frac{2}{3}h\log 1/h}, \tag{2.8}$$

for any algebraic number α of degree d and logarithmic height $h \in \big(0, \log C(\varepsilon)\big)$. Moreover

$$\log M(\alpha) \geq \frac{1}{4}\left(\frac{\log\log d}{\log d}\right)^3, \tag{2.9}$$

by a recent result of [V] concerning the Lehmer problem. A fortiori, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$2 + \frac{1}{2}\log\left(\frac{1}{3h}\log\frac{1}{h}\right) \leq \frac{\varepsilon}{2}d\sqrt{\frac{2}{3}h\log 1/h} \tag{2.10}$$

provided that $d \geq \delta(\varepsilon)$. Collecting (2.8) and (2.10) together we find the following version of the main result of [MW]:

Theorem 2.3.

Let α be an algebraic number of degree d which is not a root of unity. Then for any $\varepsilon > 0$ there exist $C(\varepsilon) > 1$ and $\delta(\varepsilon) > 0$ such that

$$\log|\alpha - 1| \geq -(1+\varepsilon)\sqrt{\frac{2}{3}d\log M(\alpha)\log\frac{d}{\log M(\alpha)}}$$

provided that $M(\alpha) \leq C(\varepsilon)^d$ and $d \geq \delta(\varepsilon)$.

In the original paper [MW] the constant was slightly worse, 1 instead of $\sqrt{2/3}$. The constant $\sqrt{2/3}$ was firstly obtained in [BMN]. A further improvement has been recently proved by A. Dubickas, who replaces $\sqrt{2/3}$ by $\pi/4$ (see [D]). This improvement comes from the use of a more complicated determinant which leads, in our interpretation of the method, to a better asymptotic lower bound for $r(t)$ as $t \to +\infty$.

§3 Explicit construction of algebraic number close to 1.

In this section we recall the main result of [A2]:

Theorem 3.1.

Let r be a positive integer and consider the polynomial

$$G(z) = 1 + (z-1)\prod_{n=1}^{r}(z^{2n-1}+1)$$

of degree $d = 1 + r^2$. Then there exists a root α of G such that $|\alpha - 1| \leq (r^2+1)2^{-r}$. Moreover, the Mahler measure of α is bounded by:

$$\exp\left\{\frac{1}{\pi^2}(\log r)^2 + 3\log r + 7\right\}. \tag{3.1}$$

Notice that the bound (1.3) gives in this case

$$\log|\alpha - 1| \geq -(1+\varepsilon)\frac{\sqrt{2}}{\pi} r(\log r)^{3/2}$$

for any $\varepsilon > 0$ and for any r sufficiently large with respect to ε^{-1}. Hence the term $\sqrt{d}$ in (1.3) cannot be replaced by $d^{1/2-\delta}$ for any $\delta > 0$.

Proof of Theorem 3.1.

Let $\alpha = \alpha_1, \ldots, \alpha_d$ be the roots of G and assume that α is the root closest to 1. From

$$G'(z) = G(z)\sum_{i=1}^{d} \frac{1}{z-\alpha_i}, \qquad z \neq \alpha_1, \ldots, \alpha_d$$

we deduce, taking into account $G(1) = 1$ and $G'(1) = 2^r$,

$$|\alpha - 1| \leq d2^{-r} = (r^2+1)2^{-r}.$$

We now prove (3.1). Let

$$F(z) = \prod_{n=1}^{r} (z^{2n-1} - 1).$$

Since

$$|G(e^{it})| \leq 1 + 2|F(-e^{it})| \leq 3\max\{|F(-e^{it})|, 1\}$$

and since, by Jensen's formula,

$$M(G) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|G(e^{it})| dt\right),$$

we have

$$\log M(G) \leq \log 3 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F(e^{it})|\, dt. \tag{3.2}$$

Using standard Fourier analysis, it can be shown that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F(e^{it})|\, dt \leq \Big(\frac{1}{2}\log r + 2\Big)K_0 + 2\log r + 3$$

where

$$K_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{n=1}^{r} \cos(2n-1)t\right| dt$$

(see [A2], Theorem 1.2). An easy computation shows that $K_0 \leq \frac{2}{\pi^2}(\log r) + 1$. Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F(e^{it})|\, dt \leq \frac{1}{\pi^2}(\log r)^2 + 3\log r + 5$$

and , by (3.2),

$$\log M(G) \leq \frac{1}{\pi^2}(\log r)^2 + 3\log r + 7.$$

□

§4 Generalization in several variables.

The aim of this section is to generalize (1.3) for several algebraic numbers.

Theorem 4.1.
Let $\mathbf{K}$ *be a number field of degree* d *and let* $\alpha^{(1)},\dots,\alpha^{(n)}$ *be multiplicatively independent elements of* $\mathbf{K}$. [4] *Let* $h=h(\alpha^{(1)})\cdots h(\alpha^{(n)})$. *Then*

$$\begin{aligned}\log\max_{j=1,\dots,n}|\alpha^{(j)}-1|\geq &-(n+1)d\left(h\log^{+}\frac{1}{h}\right)^{1/(n+1)}\\ &-2d\sum_{j=1}^{n}h(\alpha^{(j)})-\log\left(3nd^{2}\right).\end{aligned}\tag{4.1}$$

Moreover, for any $\varepsilon>0$ *there exists* $c=c(\varepsilon)>0$ *such that*

$$\begin{aligned}\log\max_{j=1,\dots,n}|\alpha^{(j)}-1|\geq -(1+\varepsilon)\left(\frac{2}{3}(n+1)\right)^{n/(n+1)}d\left(h\log^{+}\frac{1}{h}\right)^{1/(n+1)}\\ -\frac{d}{6}\sum_{j=1}^{n}h(\alpha^{(j)})-\log\left(nd^{2}\right),\end{aligned}$$

provided that $h\leq c^{n}$.

Although a slightly weaker form of this result already appeared in corollary 4 of [MW], but the proof we present here follows a different scheme. We shall see that an explicit auxiliary function is given by the natural generalisation of the determinant Δ of section 2. This auxiliary function is a polynomial in several variables with high vanishing at 1 and low height, hence, by using the method of §2, we easily obtain the proof of Theorem 4.1.

Proof of Theorem 4.1.
First step: choice of the determinant Δ.

Let $k_1,\dots,k_n$ be positive integers such that $K:=k_1\cdots k_n>1$. We denote by z the vector $(z_1,\dots,z_n)$ and we use the standard convention $z^{\lambda}=z^{\lambda_1}\cdots z^{\lambda_n}$ for a multi-index λ. Let $\Lambda=\{(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n,\ 0\leq\lambda_j\leq k_j-1,\ j=1,\dots,n\}$. We fix an arbitrary total order $<$ on Λ and we consider the Vandermonde determinant

$$\Delta(z)=\operatorname{Det}\left((z^{\lambda})^{j}\right)_{\substack{\lambda\in\Lambda\\ j=0,\dots K-1}}=\prod_{\substack{\lambda,\mu\in\Lambda\\ \mu<\lambda}}(z^{\lambda}-z^{\mu}).$$

Since $\alpha^{(1)},\dots,\alpha^{(n)}$ are multiplicatively independent, $\Delta(\alpha^{(1)},\dots,\alpha^{(n)})\neq 0$.
Second step: upper bound for $|\mathrm{N}_{\mathbb{Q}}^{\mathbf{K}}\Delta(\alpha^{(1)},\dots,\alpha^{(n)})|$ *and main inequality.*

An easy computation shows that Δ has partial degree

$$N_j:=\frac{K(k_j-1)(4Kk_j+K-3k_j)}{12k_j}$$

[4] This assumption is in fact necessary. Otherwise, considering $\alpha^{(j)}=\alpha^{j}$, we could improve the main term $d^{1/2}$ in (1.3), which is impossible according to the result of section 3.

with respect to z_j and vanishes at 1with multiplicity $m = K(K-1)/2$. Moreover, by Hadamard inequality,

$$\max_{|z_j|=1} |\Delta(z_1, \ldots, z_n)| \le K^{K/2}.$$

Hence,

$$|\Delta(\sigma\alpha^{(1)}, \ldots, \sigma\alpha^{(n)})| \le K^{K/2} \prod_{j=1}^{n} \max\{1, |\sigma\alpha|\}^{N_j} \tag{4.2}$$

for any embedding $\sigma\colon \mathbf{K} \to \mathbb{C}$. Assume $|\alpha^{(1)} - 1| \ge |\alpha^{(j)} - 1|$ $(j = 1, \ldots, n)$ and let $N = N_1 + \cdots + N_n$. If $|\alpha^{(1)} - 1| \ge m/(N-m)$, then, by (4.2),

$$|\Delta(\alpha^{(1)}, \ldots, \alpha^{(n)})| \le |\alpha^{(1)} - 1|^m \left(\frac{eN}{m}\right)^m K^{K/2} \prod_{j=1}^{n} \max\{|\alpha_j|, 1\}^{N_j}. \tag{4.3}$$

Assume now $|\alpha^{(1)} - 1| < m/(N-m)$, let $\rho = m/\big(|\alpha^{(1)} - 1|(N-m)\big)$ and consider the polynomial in one variable

$$Q(t) = \Delta\big((\alpha^{(1)} - 1)t + 1, \cdots, (\alpha^{(n)} - 1)t + 1\big).$$

By the maximum principle,

$$\begin{aligned} |\Delta(\alpha^{(1)}, \ldots, \alpha^{(n)})| = |Q(1)| &\le \rho^{-m} \max_{|t|=\rho} |Q(t)| \\ &\le \frac{(|\alpha^{(1)} - 1|\rho + 1)^N}{\rho^m} K^{K/2} \\ &\le |\alpha^{(1)} - 1|^m \left(\frac{eN}{m}\right)^m K^{K/2}. \end{aligned}$$

Hence (4.3) holds in any case. Let now $a_j > 0$ be the leading coefficient of the minimal equation of α_j over $\mathbb{Z}$, and let d_j its degree. By Lemma 4 of [MW],

$$A = a_1^{N_1 d/d_1} \cdots a_n^{N_n d/d_n} \prod_{l=1}^{d} \sigma_l \Delta(\alpha^{(1)}, \ldots, \alpha^{(n)})$$

is an integer. By using (4.2) and (4.3) we have

$$\begin{aligned} |A| &\le |\alpha^{(1)} - 1|^m \left(\frac{eN}{m}\right)^m K^{dK/2} \prod_{j=1}^{n} \left(a_j^{d/d_j} \prod_{l=1}^{d} \max\{|\sigma_l \alpha^{(j)}|, 1\} \right)^{N_j} \\ &\le |\alpha^{(1)} - 1|^m \left(\frac{eN}{m}\right)^m K^{dK/2} \prod_{j=1}^{n} M(\alpha^{(j)})^{N_j d/d_j} \\ &= |\alpha^{(1)} - 1|^m \left(\frac{eN}{m}\right)^m K^{dK/2} \exp\Big\{ d\big(N_1 h(\alpha^{(1)}) + \cdots + N_n h(\alpha^{(n)})\big)\Big\}. \end{aligned}$$

Since the left hand side of this inequality is ≥ 1, we have

$$\log \frac{1}{|\alpha^{(1)} - 1|} \le 1 + \log \frac{N}{m} + \frac{d}{m}\Big(\frac{1}{2} K \log K + N_1 h(\alpha^{(1)}) + \cdots + N_n h(\alpha^{(n)})\Big). \tag{4.4}$$

An easy computation shows that

$$\frac{N_j}{m} \le \frac{2}{3}\frac{K}{K-1}(k_j-1)+\frac{1}{6}. \tag{4.5}$$

From (4.4) and (4.5) we obtain the main inequality:

$$\begin{aligned}-\log \max_{j=1,\dots,n} |\alpha^{(j)}-1| &\le 1+\log\left(\frac{2}{3}\frac{K}{K-1}\sum_{j=1}^{n}(k_j-1)+\frac{n}{6}\right)\\ &\quad + d\left(\frac{\log K}{K-1}+\frac{2}{3}\frac{K}{K-1}\sum_{j=1}^{n}(k_j-1)h(\alpha^{(j)})+\frac{1}{6}\sum_{j=1}^{n}h(\alpha^{(j)})\right).\end{aligned} \tag{4.6}$$

Last step: choice of the parameter k.

Let $X>1$ and choose

$$k_j = \left[(Xh)^{1/n}h(\alpha^{(j)})^{-1}\right]+1, \qquad j=1,\dots,n.$$

Therefore $K=k_1\cdots k_n \ge X>1$. Since $K\mapsto(\log K)/(K-1)$ and $K\mapsto K/(K-1)$ are both decreasing for $K>1$, we have, by (4.6),

$$\begin{aligned}-\log \max_{j=1,\dots,n} |\alpha^{(j)}-1| &\le d\,\frac{\log X+\frac{2}{3}nX^{(n+1)/n}h^{1/n}}{X-1}+\frac{d}{6}\sum_{j=1}^{n}h(\alpha^{(j)})\\ &\quad +\log\left(\frac{\frac{2}{3}X^{(n+1)/n}h^{1/n}}{X-1}\sum_{j=1}^{n}h(\alpha^{(j)})^{-1}+\frac{n}{6}\right)+1.\end{aligned} \tag{4.7}$$

Let $\Omega=h^{-1/(n+1)}$. To prove (4.1) we distinguish three cases.

• First case: $\Omega<\frac{6}{5}$. The relation between the arithmetic and geometric means shows that

$$\sum_{j=1}^{n}h(\alpha^{(j)}) \ge nh^{1/n}=n\Omega^{-(n+1)/n}\ge\left(\frac{6}{5}\right)^{-2}>\log 2.$$

Hence

$$2\sum_{j=1}^{n}h(\alpha^{(j)})>h(\alpha^{(1)})+\log 2$$

and Liouville's inequality (1.1) implies (4.1).

• Second case: $\frac{6}{5}\le\Omega<4$. An easy computation shows that in this case

$$(n+1)\left(h\log\frac{1}{h}\right)^{1/(n+1)}=(n+1)\Omega^{-1}\big((n+1)\log\Omega\big)^{1/(n+1)}>\log 2.$$

Hence

$$(n+1)\left(h\log\frac{1}{h}\right)^{1/(n+1)}+2\sum_{j=1}^{n}h(\alpha^{(j)})>\log 2+h(\alpha^{(1)})$$

and Liouville's inequality implies again (4.1).

• Third case: $\Omega \geq 4$. Choose

$$X = \left(\frac{\frac{3}{2}\log(1/h)}{n+1}\right)^{n/(n+1)} h^{-1/(n+1)}.$$

Then

$$\frac{\log X + \frac{2}{3}nX^{(n+1)/n}h^{1/n}}{X-1} = f_n(\Omega)\cdot\left(h\log\frac{1}{h}\right)^{1/(n+1)} \tag{4.8}$$

and

$$\frac{\frac{2}{3}X^{(n+1)/n}h^{1/n}}{X-1} = g_n(\Omega), \tag{4.9}$$

where

$$f_n(t) = \left(\frac{2}{3}(n+1)\right)^{n/(n+1)}\left(1+\frac{n}{(n+1)^2}\frac{\log(\frac{3}{2}\log t)}{\log t}\right)\left(1-\left(\frac{3}{2}\log t\right)^{-n/(n+1)}\frac{1}{t}\right)^{-1}$$

and

$$g_n(t) = \frac{\log t}{t(\frac{3}{2}\log t)^{n/(n+1)}-1}.$$

Moreover, using for instance the inequality (2.9) of P. Voutier,

$$\sum_{j=1}^{n} h(\alpha^{(j)})^{-1} \leq 3nd^2. \tag{4.10}$$

By (4.7), (4.8), (4.9) and (4.10) we have

$$\begin{aligned}-\log\max_{j=1,\dots,n}|\alpha^{(j)}-1| \leq f_n(\Omega)\cdot d\left(h\log\frac{1}{h}\right)^{1/(n+1)} &+ \frac{d}{6}\sum_{j=1}^{n}h(\alpha^{(j)})\\ +\log\left(3d^2ng_n(\Omega)+\frac{n}{6}\right)+1.&\end{aligned} \tag{4.11}$$

A computation shows that $f_n(t)$ and $g_n(t)$ both decrease for $t \geq 4$, $f_n(4) \leq n+1$ and $g_n(4) \leq \frac{3}{10}$. Therefore, by (4.11),

$$\begin{aligned}-\log\max_{j=1,\dots,n}|\alpha^{(j)}-1| &\leq (n+1)d\left(h\log\frac{1}{h}\right)^{1/(n+1)} + \frac{d}{6}\sum_{j=1}^{n}h(\alpha^{(j)})\\ &\quad+\log\left(\frac{9}{10}d^2n+\frac{n}{6}\right)+1\\ &\leq (n+1)d\left(h\log\frac{1}{h}\right)^{1/(n+1)} + \frac{d}{6}\sum_{j=1}^{n}h(\alpha^{(j)})+\log(3d^2n),\end{aligned}$$

which implies (4.1).

To prove the second assertion of Theorem 4.1, we remark that

$$\lim_{t\to+\infty} f_n(t) = \left(\frac{2}{3}(n+1)\right)^{n/(n+1)} \qquad \text{and} \qquad \lim_{t\to+\infty} g_n(t) = 0.$$

Therefore, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) \geq 1$ such that if $\Omega \geq C$ we have $f_n(\Omega) \leq (1+\varepsilon)\left(\frac{2}{3}(n+1)\right)^{n/(n+1)}$ and $3g_n(\Omega) \leq 1/e - 1/6$. Let $c = C^{-2}$. If $h \leq c^n$ then $\Omega \geq C$ and, by (4.11),

$$-\log \max_{j=1,\ldots,n} |\alpha^{(j)} - 1| \leq (1+\varepsilon)\left(\frac{2}{3}(n+1)\right)^{n/(n+1)} d\left(h \log^+ \frac{1}{h}\right)^{1/(n+1)} + \frac{d}{6}\sum_{j=1}^{n} h(\alpha^{(j)}) + 2\log d + \log n.$$

The proof of Theorem 4.1 is now complete.

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Francesco Amoroso,

Dipartimento di Matematica
Via Carlo Alberto, 10
10123 Torino - Italy
e-mail: amoroso@dm.unipi.it