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Ph. D. Thesis

Anisotropic gravitational interactions: variational approach and symbolic dynamics

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To my parents

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"Because the ideas I had about supernatural beings came to me the same way my mathematical ideas did, so I took them seriously."

— John F. Nash

"Quante certezze. Non so se invidiarti o provare una forma di ribrezzo."

— Jep Gambardella, *La grande bellezza*

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Introduction

This work gathers the results contained in the research papers [5] and [6] as a compendium of my recent research, accomplished during my PhD program at the University of Turin. The contents of the above papers are very related and logically interconnected, so that it is worth to give a joint account of the main motivations that have led to the study of the questions answered therein. A more technical and rigorous introduction will precede every one of the two chapters, in order to equip the reader with the notations and settings needed in the two treated problems.

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The N-centre problem of Celestial Mechanics

A deep view and comprehension on how the N-centre problem is treated and understood in the contemporary literature can not take place without a glimpse on the most classical and famous problem of Celestial Mechanics, from which it actually came to life: the N-body problem. The really challenging, and actually simple in its formulation, question that many mathematicians have attempted to answer through the centuries is the following: how does a finite number of heavy bodies move in the Euclidean space under their mutual gravitational attraction? To fix the ideas, we can define the positions of the bodies as N functions $x_k\colon I \subseteq \mathbb{R} \to \mathbb{R}^3$ which evolve with respect to a time variable and N positive numbers $m_k > 0$ that represent their masses. In this way, the Newton's law provides N second order ordinary differential equations that rule the motion of the system:

$$m_k \ddot{x}_k(t) = -\sum_{j \neq k} \frac{m_k m_j (x_k(t) - x_j(t))}{|x_k(t) - x_j(t)|^3}, \quad \text{for every } k = 1, \dots, N.$$

The 2-body problem has been solved and clearly understood, thanks to the works of Newton and Kepler. On the other hand, when $N \geq 3$, the situation changes dramatically and the problem, if considered in a completely general setting, is really far from being solved. In particular, the dynamical system results to be not analytically integrable and the presence of a huge singular set, which is represented by every possible collision between the bodies, is responsible for a very complex dynamics. Indeed, if we introduce the gravitational potential

$$V(x) = V(x_1, \dots, x_N) \doteq \sum_{j \neq k} \frac{m_j m_k}{|x_j - x_k|}$$

the system joins a Hamiltonian structure, with respect to the Hamiltonian function

$$H(x,v) \doteq K(v) - V(x) = \frac{1}{2}\sum_{i=1}^{N}\frac{|v_i|^2}{m_i} - \sum_{j\neq k}\frac{m_j m_k}{|x_j - x_k|}$$

where the singularity of the potential V is determined by the collision set, i.e., whenever $x_k = x_j$ for some $k \neq j$. These singularities of the system determine the non-completeness of the associated flow (see [31]), raising up the complexity of the problem and inducing a chaotic behaviour on the orbits (see [29]). A lot of results have been obtained in the study of such singularities, mainly analysing the behaviour of a collision trajectory producing asymptotic estimates (see [64, 69, 63, 57, 58, 7]), but also employing the powerful tool of the McGehee coordinates introduced in [52], in which the singularity is blown-up and the flow is extended through collisions glueing a collision manifold in the phase space.

Classically, there is a great interest in periodic solutions of Hamiltonian systems since, according to the Poincaré conjecture, a complex and possibly chaotic dynamics for the trajectories is strictly connected with the existence of a dense set of periodic solutions. Some of the hurdles that arise in looking for periodic trajectories of the N-body problem can be partially tackled if one considers some particular situations, taking into account a less general setting. A first approach to find families of periodic solutions consists in imposing symmetries on the motion of the bodies. In this context variational techniques reveal to be very efficient and have induced a plethora of results to enrich the set of periodic solutions (see [13, 35, 22, 21, 65, 10, 8]).

Another simplified, but still far from being trivial, version of the N-body problem can be introduced as follows: consider $(N+1)$ heavy bodies and assume that one of them is moving much faster than the others. In this way, N of the bodies can be assumed motionless, while the one remaining is moving under their attraction (we assume its mass to be equal to 1). This is how the N-centre problem of Celestial Mechanics is usually stated, assuming as an approximation that N bodies are fixed and thus they represent a finite number of *centres* of mass. In $\mathbb{R}^3$, assuming that the Coriolis' and centrifugal forces are neglected, if we denote by $c_1, \ldots, c_N \in \mathbb{R}^3$ the position of the centres, by $m_1, \ldots, m_N > 0$ their masses and we let $x\colon I \subseteq \mathbb{R} \to \mathbb{R}^3$ be the motion law of the moving particle, the equation of motion is the following

$$\ddot{x}(t) = -\sum_{j=1}^{N}\frac{m_j(x(t)-c_j)}{|x(t)-c_j|^3}.$$

Again, the previous equation has Hamiltonian structure, with Hamiltonian

$$H(x,v) = K(v) - V(x) = \frac{1}{2}|v|^2 - \sum_{j=1}^{N}\frac{m_j}{|x-c_j|}$$

and, in this setting, a singularity for V occurs whenever $x = c_j$ for some j. The function $H(x,v)$ represents exactly the total energy of the system and thus a conservation of the energy characterizes every solution of the equation of motion in this way

$$\frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h \in \mathbb{R}, \quad \text{for every } t \in I.$$

As for the N-body problem, the number of the centres plays an essential role with respect to the integrability of the system. If $N = 1$ we end up with the Kepler problem, which is integrable and whose solutions are conic sections. A very interesting situation appears when we consider two centres of mass: note that in this case the equation of motion approximates the one of the restricted three body problem, where by *restricted* we mean that one of the three bodies has a negligible mass with respect to the others. Euler started to study the 2-centre problem in 1760 and he showed that it is integrable ([34]), while explicit solutions have been provided by Jacobi in his famous book *Vorlesungen über Dynamik* (1866) (see also [4] for a more recent explanation of these results). Concerning the applications, the 2-centre problem revealed to be very useful in the determination of the orbit of a satellite, assuming that the Earth is placed in one of the two centres (see [67]); moreover, the 2-centres can be also considered as atomic nuclei and the particle can play the role of an electron in the model of a diatomic molecule (see [68] for further details). As expected, also in the N-centre problem the integrability of the system fails when we consider more than 2 centres. A first step in this direction has been made by Bolotin in [14], where the author showed that the planar N-centre problem is not analytically integrable when $N \geq 3$ if we restrict the dynamical system to energy shells $H^{-1}(h)$, with $h > 0$. Concerning the 3-dimensional case, the non-integrability of the system has been discussed separately in [15] by Bolotin and Negrini and in [48] by Knauf and Taimanov: in the first paper, the authors showed the presence of positive topological entropy for non-negative energies $h \geq 0$, while in the second one the authors showed that over a threshold $h > \tilde{h} \gg 1$, no real-analytic integral exists for the spatial N-centre problem, $N \geq 3$. To conclude this brief digression on integrability, we also observe that in [47] the authors proved that, again over a high energy threshold, the N-centre problem is completely integrable through $\mathcal{C}^\infty$-integrals both in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Many research papers have also provided a qualitative description of this dynamical system. A fundamental contribution to the planar problem at positive energy is contained in [44]: therein the authors used global analysis methods and Riemannian geometry in order to give a rigorous description of the scattering for the N-centre problem. Moreover, the existence of a symbolic dynamics in positive energy shells has been established in the same paper. A generalization and some extensions of this result have been given in [45] for the 3-dimensional setting; the author used perturbation techniques at high positive energies in order to analyse the orbits of the system, providing again a scattering theory for the spatial N-centre problem over a certain energy threshold. In [18] the authors used min-max methods in order to find unbounded trajectories with prescribed ingoing and outgoing directions in the space, assuming the energy to be 0.

When one considers negative energy shells it is useful to introduce the so-called *Hill's region*

$$\mathcal{R}_h \doteq \left\{x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} : V(x) + h \geq 0\right\}$$

since, in contrast with the case $h \geq 0$, this set is strictly contained in the punctured plane. In this situation it is not possible to involve global arguments based on the fundamental group of $\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$ as in [44], since the non-boundedness of the Hill's region introduces a degeneration on the Jacobi metric. Indeed, a portion of the boundary $\partial\mathcal{R}_h$ has null Jacobi measure and thus it would be a minimizer for the Jacobi functional, but it would not solve the N-centre problem with fixed ends. This is why in [61] the authors used a finite dimensional reduction in order to obtain closed periodic solutions for the planar N-centre problem at negative energies. They separated the proof inside and outside a ball of radius $R > 0$, finding solution arcs with two different techniques and then glueing them together on the circle ∂B_R. Other results, with much stricter assumptions, have been obtained for the negative energy case in this way:

- in [16] the authors considered 3 centres in the plane, one of them placed very far from the others, and used perturbation techniques in order to show the dynamical system is chaotic through the Poincaré-Melnikov theory;
- in [32] perturbation methods for the 3-centre problem are used as well, assuming this time that the third centre is less attracting than the others and that the energy is very small in absolute value, obtaining invariant sets of chaotic quasi-colliding solutions.

We conclude this discussion on the N-centre problem observing that it is also possible to find periodic solutions prescribing the period $T > 0$, without any information on the energy of the system. Two interesting contributions in this direction can be found in [20, 70].

The anisotropic Kepler problem

In the research paper [37], Gutzwiller firstly introduced the anisotropic Kepler problem as a classical mechanical system that approximates a quantum system. A natural situation in which this system arises is when one analyses the ground states of an electron near the donor impurity of a semi-conductor. In the perspective of Gutzwiller, the anisotropy resides in the kinetic energy of the planar system through an anisotropic mass tensor, while the potential is induced by a Coulumbic force field. Motivated by these physical applications, firstly on the basis of numerical computations ([37]) and after with analytical methods ([38]), he provided a qualitative description of periodic solutions in negative energy shells (see also [39]). In particular, he showed that a continuous and *one-to-one* map can be constructed between the initial conditions and the binary sequences composed by two real numbers, actually proving that the Poincaré map is topologically conjugate to a Bernoulli shift (cf. Definition 2.1.5).

Deepening a more geometrical point of view, another series of remarkable contributions in this direction have been given by Devaney during the same years. In the research paper [25], he introduced the one-parameter family of Hamiltonian systems driven by

$$H_\mu(x,v) = K_\mu(v) - V(x) = \frac{1}{2}(\mu v_1^2 + v_2^2) - \frac{1}{|x|},$$

with $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\},\ v = (v_1, v_2) \in \mathbb{R}^2$ and $\mu \geq 1$. When $\mu = 1$, the system reduces to the classical Kepler problem and the energy and angular momentum integrals provide the integrability; on the other hand, when μ grows beyond 1, the spherical symmetry of the system disappears and the angular momentum is no longer conserved along a solution. This fact completely changes the phase portrait of the system and the regular structure of the orbits. Indeed, as the parameter μ increases, the trajectories start to draw extensive oscillations about the x_2 axis due to the anisotropic kinetic energy K_μ. In particular, when $\mu > 9/8$, these oscillations become highly random: for instance, one can find orbits that oscillate an arbitrarily large number of times about the x_2-axis before they cross the x_1-axis. As a result, the integrability of the system ceases to exist when anisotropy is introduced and this suggests to investigate the existence of a symbolic dynamics. This whole analysis has been carried out in [25], where the author in particular showed that, for an open and dense subset of parameters μ contained in $(9/8, +\infty)$, the anisotropic Kepler problem displays symbolic dynamics. Notice that when $0 < \mu \leq 1$ the situation symmetrically reflects, then producing oscillations about the x_1-axis. The key investigation objects are the *bi-collision* trajectories, which have been studied using the McGehee's change of coordinates ([52, 53]). This technique has been introduced in the context of the collinear three-body problem and consists in a *blow-up* of the singularity in the phase space, which is replaced by an invariant torus, the so-called *collision manifold*. In this way, the flow is extended beyond the singularity and bi-collision trajectories reveal to be heteroclinic solutions of the new dynamical system (for other perspectives on the qualitative analysis of this system we refer also to [29, 27, 19, 40]).

Another considerable difficulty in this context is represented by the non regularizability of collision trajectories. When the total energy is negative, the Kepler problem admits a cylinder of trajectories which start and collapse again in the origin after a certain time, the so-called *homothetic trajectories*. These solutions are not defined for all the times, but it is a classical result that the *Levi-Civita transform* permits to extend them through collisions, so that they display a *bounce* at the collision instant and then result analytically regularized (see [64]). This can be done also employing a topological surgery technique, using the so-called *isolating blocks* ([33, 23]), but also in a variational fashion ([55, 56]). The regularization then reveals to be a powerful tool in applications in order to avoid collisions and to build periodic solutions (see for instance [61, 35]), but Devaney discovered that this process does not agree with anisotropic context. Indeed, in [26], he used the approach introduced by Easton in [33] to show that there exists an open and dense set of parameters μ in $(1, +\infty)$ such that the corresponding anisotropic Kepler problem is not regularizable by surgery.

In [29] Devaney remarked that an equivalent formulation of this problem, via an easy change of variables, can be considered and thus one can take into account the

Hamiltonian

$$\tilde{H}_\mu(x,v) = K(v) - V_\mu(x) = \frac{1}{2}|v|^2 - \frac{1}{\sqrt{\mu x_1^2 + x_2^2}},$$

where the kinetic energy K is standard, while the anisotropy is now contained inside the potential V_μ. From this point of view, further generalisations of V_μ could be considered, as has been done in the recent papers [11, 12]. The authors introduced a wider class of anisotropic problems, considering a family of singular homogeneous potentials

$$\begin{cases} V \in \mathcal{C}^2(\mathbb{R}^d \setminus \{0\}) \\ V(x) = |x|^{-\alpha} V(x/|x|), \end{cases}$$

with $\alpha \in (0,2)$ and $d \geq 2$ and studying the zero-energy dynamical system

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 = V(x(t)) \end{cases},$$

whose solutions are usually referred as *parabolic trajectories*. In this context, it is extremely useful to introduce the *central configurations* of V as all the unitary vectors that are critical points for the restriction $V|_{\mathbb{S}^{d-1}}$. Indeed, a zero-energy solution $x(t)$ enjoys asymptotic properties: its norm $|x(t)|$ blows up when $t \to \pm\infty$ and, if we assume that the set of central configurations is discrete, we have that

$$\frac{x(t)}{|x(t)|} \to \xi^\pm, \quad \text{as } t \to \pm\infty,$$

where $\xi^\pm$ are central configurations for V. In [11, 12], the existence of entire parabolic trajectories with prescribed asymptotic directions at infinity is provided using a variational approach, in which a zero-energy solution is characterized as a Morse minimizer for the Jacobi metric. Following the approach of McGehee in [52], parabolic solutions are shown to correspond to heteroclinic connections between two saddles in the collision manifold and their existence is strictly related to the choice of the homogeneity degree. In particular, in [11] the authors deepen the study in the planar case; when $d = 2$, one can introduce polar coordinates $x = (r\cos\vartheta, r\sin\vartheta)$ and write

$$V(x) = V(r\cos\vartheta, r\sin\vartheta) = r^{-\alpha} U(\vartheta),$$

where U is exactly the restriction of V to the sphere $\mathbb{S}^1$. With this notations, a central configuration for V is actually an angle $\vartheta \in \mathbb{S}^1$ and this allows to analyse more precisely the qualitative behaviour of the trajectories. If we fix $\vartheta^+, \vartheta^- \in \mathbb{S}^1$ minimal non-degenerate central configurations, then there exists at most one homogeneity degree $\bar{\alpha} = \bar{\alpha}(U, \vartheta^+, \vartheta^-) \in (0,2)$ for which a parabolic Morse minimizer exists. In particular, this threshold of existence is also related to the absence of collisions for fixed-ends Bolza problems (see also [24]), but also to the existence of non-collision periodic orbits having a prescribed winding number; we want to remark that these results are very close to the minimizing properties of Keplerian ellipses studied by Gordon in [36]. Parabolic trajectories for systems driven by homogeneous potentials have also been analysed from an index theory point of view in [42].

The study of this wide class of anisotropic potentials reveals to be very useful in the applications of some particular situations arising from the N-body problem. As enlightened by Devaney in [28, 30], after a combination of Jacobi and McGehee coordinates, the planar isosceles three-body problem can be reduced to a dynamical system driven by an anisotropic potential of the form

$$U(\vartheta) = \frac{1}{\sqrt{2}\cos\vartheta} + \frac{4\varepsilon^{3/2}}{\sqrt{2\varepsilon + 4\sin^2\vartheta}},$$

where the two symmetric masses $m_1 = m_2 = 1$ and the third mass $m_3 = \varepsilon > 0$. Analogous situations can appear when 4 bodies are considered and particular symmetry conditions are required. For instance, in [60, 49], two degenerate situations of the four body problem with only two degrees of freedom are considered: the symmetric collinear four body problem and the rectangular four body problem. In these settings, chosen changes of coordinates allow to reduce the dynamical system to a unique ordinary differential equation, which involves an anisotropic potential. In this way, the total collapse and ejection-collision trajectories are characterized exploiting an analysis of the flow on the collision manifold. Analogous anisotropic potentials come out when one considers the rhomboidal four body problem as in [50], but also in a charged setting (see [1]). To conclude this brief discussion on the applications, we also cite the recent research paper [2], in which the authors study ejection-collision orbits for the symmetric collinear four body problem, dealing with an anisotropic potential arising in this context.

Main results

In this last paragraph we collect the main results contained in the two following chapters.

Minimal collision arcs asymptotic to central configurations ([5])

In Chapter 1 we introduce a class of singular anisotropic homogeneous potentials $V(x) = |x|^{-\alpha}V(x/|x|)$, with $\alpha \in (0,2)$ and $x \in \mathbb{R}^d$ ($d \geq 2$), and we consider the conservative Newtonian system

$$\ddot{x}(t) = \nabla V(x(t)),$$

or the same system driven by lower order perturbations of V. Our objects of interest are those trajectories which collide with the origin in finite time (*collision solutions*). Assuming that the set of central configurations of V is discrete, the normalized configuration $x_{coll}(t)/|x_{coll}(t)|$ of a collision solution $x_{coll}(t)$ converges to a central configuration $s^* \in \mathbb{S}^{d-1}$ for V. Because of the occurrence of such singular trajectories, the flow associated to the dynamical system is not complete. Using McGehee coordinates, the flow can be extended to the collision manifold having central configurations as stationary points, endowed with their stable and unstable manifolds. We focus on the case in which the asymptotic central configuration s^* is a global minimizer of the restriction of V to the sphere $\mathbb{S}^{d-1}$ and we take into account the local stable manifold $\mathcal{W}^S_{loc}(s^*)$ associated with s^*. If we introduce the Jacobi-length functional

$$\mathcal{L}_h(x) = \int_0^T |\dot{x}(t)|\sqrt{h + V(x(t))}\, dt$$

and we define the space of H^1-collision paths starting from a position $q \in \mathbb{R}^d \setminus \{0\}$ as

$$H^q_{coll} = \{x \in H^1([0,T];\mathbb{R}^d) : \, x(0) = q, \; x(T) = 0, \; |x(t)| < |q|, \; \forall\, t \in (0,T)\}$$

with some restrictions on $|q|$, since $\alpha \in (0,2)$, we can find at least a minimizer for $\mathcal{L}_h$ in the above space. Our main goal then is to show that the local stable manifold $\mathcal{W}^S_{loc}(s^*)$ coincides with that of the initial data of minimal collision arcs for $\mathcal{L}_h$. This characterisation may be extremely useful in building complex trajectories with a broken geodesic method. The proof takes advantage of a generalised version of the Sundman's monotonicity formula.

Symbolic dynamics for the anisotropic N-centre problem ([6])

In Chapter 2 we consider an anisotropic version of the planar N-centre problem in which every centre is associated with a different homogeneous potential. In particular, given $c_1, \dots, c_N \in \mathbb{R}^2$ the positions of the centres, we consider $V_1, \dots, V_N \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\})$ homogeneous functions and we introduce the singular potential

$$V(x) = \sum_{j=1}^{N} V_j(x - c_j);$$

if we denote by $x(t)$ the position of the moving particle in the plane, we study the conservative Newtonian system

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = -h. \end{cases}$$

We assume that the energy $-h$ is negative, with $h << 1$ and we require the following hypotheses on the potential V

$$(1) \qquad \begin{cases} V_j \text{ is } -\alpha_j\text{-homogeneous, } \forall\, j = 1, \dots, N \\ 0 < \alpha_1 \le \alpha_2 \le \ldots \le \alpha_N \\ \alpha_1 < 2 \\ \alpha_j > \bar{\alpha}_j, \; \forall\, j = 1, \dots, N \\ V_j \text{ admits at least a stricly minimal central configuration, } \forall\, j = 1, \dots, N, \end{cases}$$

where $\bar{\alpha}_j \in (0, 2)$ is a threshold homogeneity degree which guarantees the absence of collisions with the centre c_j and will be rigorously specified later (see (V), Remark 2.1.1 at page 32 and Theorem 2.4.23 at page 78). Our main result then is to prove that, if V satisfies assumptions (1), then there exist infinitely many periodic solutions for the anisotropic N-centre problem in negative energy shells, which are collision-less and free of self-intersections.

The idea of the proof is to build such solutions as a juxtaposition of different pieces of trajectories, following the approach of [61], which is based on the broken geodesics method introduced by Seifert in [59] in a completely different context. Since the energy is negative, the Hill's region is a proper subset of $\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$ and an arc contained in its boundary has null length with respect to the Jacobi metric. Because of the presence of this degeneration, global arguments (see for instance [44]) do not apply in this context and we need to separate the proof close/far from the singularity set. In particular, when the particle is close to the centres, the solution arcs are provided through the minimization of the Maupertuis' functional under a suitable topological constraint. On the other hand, when the particle travels far from the singularity set, a perturbation technique is employed, including all the centres in a small ball and thus reducing the system to a perturbed anisotropic Kepler problem driven by one of the potentials associated to the centres. As a final step, we alternate an outer and an inner arc and then we *glue* them together, in order to obtain a closed periodic solution. Even if every single trajectory obtained in the two previous steps is smooth, we need to show that such smoothness is preserved through the junctions. This will be made by minimizing the Jacobi-length functional with respect to the endpoints, so that a minimizer will match the derivatives in the contact points. We point out that requiring the energy to be small in absolute value is a fundamental assumption. Indeed, if h becomes very large, then the Hill's region $\{x \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\} : V(x) \geq h\}$ could be disconnected and this would represent a pathologic situation in this strategy of proof.

The presence of infinitely many periodic solutions allows us to characterize this dynamical system with a symbolic dynamics. Using pairs composed by a partition of the centres and a minimal non-degenerate central configuration of the potential with the smallest homogeneity degree, we determine a finite alphabet $\mathcal{Q}$. Then, we consider the metric space of bi-infinite sequences of symbols $\mathcal{Q}^{\mathbb{Z}}$ and we introduce the discrete dynamical system induced by the right Bernoulli shift map. In this way, we determine an invariant subset of the original dynamical system and we show that the first return map is topologically semi-conjugate to the Bernoulli shift. Assuming (1), the symbolic dynamics is collision-less and for m minimal non-degenerate central configurations the number of symbols is $m(2^{N-1} - 1)$, assuming furthermore that $N \geq 2$ and $m \geq 1$, with one of the inequalities holding strictly. As a final remark, if the threshold on the homogeneity degrees $\alpha_j > \bar{\alpha}_j$ in (1) is violated for some j, collisions with the centres may arise; nonetheless, within certain limits, the existence of a symbolic dynamics persists, using a smaller set of symbols and taking into account the collisions.

Contents

1. Minimal collision arcs asymptotic to central configurations

1.1. Introduction and main result

Many papers of the recent literature are focused on the variational properties of expanding (parabolic or hyperbolic) or collapsing trajectories for N-body and N-centre type problems (see e.g. [11, 12, 18, 17, 9, 51]), framing them in a Morse-theoretical perspective. Indeed, in addition to answering natural questions about the nature of these motions, the variational approach is a fruitful tool when building complex trajectories exploiting gluing techniques (cf. Chapter 2, but also [6, 61]). These applications are the original motivations for this work and show a natural link with the next chapter, although we believe that the obtained result is interesting in itself. In order to state it in detail, we need some preliminaries on the motion near collision for a class of singular anisotropic homogeneous potentials of degree $-\alpha$, with $\alpha \in (0, 2)$ (and their lower order perturbations).

In this chapter we consider the Newtonian system of ordinary differential equations

$$\ddot{x}(t) = \nabla V(x(t)), \tag{1.1}$$

whose solutions satisfy the energy relation

$$\frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h, \tag{1.2}$$

with $h \in \mathbb{R}$. It is possible to reword equations (1.1)-(1.2) using the Hamiltonian formalism, choosing, as usual, the total energy to be the Hamiltonian function. Taking into account the singularity of V, we introduce the *configuration space* $\mathbb{R}^d \setminus \{0\}$ and *the phase space* $(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$ of the system. Since we will study fixed-energy trajectories, it makes sense to restrict our discussion to the $(2d-1)$-dimensional *energy shell*

$$\mathcal{H}_h = \left\{ (q,p) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d : \ \frac{1}{2}|p|^2 - V(q) = h \right\},$$

and thus, every solution of (1.1)-(1.2) can be seen as an evolving pair $(q,p) \in \mathcal{H}_h$ which solves

$$\begin{cases} \dot{q} = p \\ \dot{p} = \nabla V(q). \end{cases} \tag{1.3}$$

Our potential V is a *not too singular* perturbation of a $-\alpha$-homogeneous potential S. To be precise, for $d \geq 2$, let us introduce a function $U \in \mathcal{C}^2(\mathbb{S}^{d-1})$ such that

$$(s^*0) \qquad \begin{cases} \exists\, s^* \in \mathbb{S}^{d-1} \text{ s.t. } U(s) \geq U(s^*) > 0, \ \forall s \in \mathbb{S}^{d-1}; \\ \exists\, \delta, \rho > 0 \ : \ \forall s \in \mathbb{S}^{d-1} \text{ s.t. } |s - s^*| < \delta \implies U(s) - U(s^*) \geq \rho |s - s^*|^2, \end{cases}$$

and then consider a potential $V \in \mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$ such that

$$(V0) \qquad \begin{cases} V = S + W; \\ S \in \mathcal{C}^2(\mathbb{R}^d \setminus \{0\}) \text{ and } S(x) = |x|^{-\alpha} U(x/|x|), \text{ for some } \alpha \in (0,2); \\ \lim\limits_{|x| \to 0} |x|^{\alpha'} (W(x) + |x| \cdot |\nabla W(x)|) = 0, \text{ for some } \alpha' < \alpha. \end{cases}$$

Here, S has a singularity in the origin and it represents a generalization of the anisotropic Kepler potential introduced by Gutzwiller ([37, 38, 39]). On the other hand, the perturbation term W becomes negligible with respect to S as $|x| \to 0$. In particular, recalling that a *central configuration* for S is a unitary vector which is a critical point of the restriction of S to the sphere, the assumptions (s^*0) on U state that s^* is a globally minimal non-degenerate central configuration for S. As an example, for $d = 2$ and $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$, one can consider the function

$$V(x) = S(x) + W(x) = \frac{1}{\sqrt{\mu x_1^2 + x_2^2}} + \frac{1}{\sqrt[4]{x_1^2 + x_2^2}},$$

where $\mu > 1$. Notice that the third requirement in $(V0)$ is fulfilled for every $\alpha' \in (1/2, 1)$. As another interesting application, we can consider the N-centre problem driven by N anisotropic potentials satisfying hypotheses (s^*0)-$(V0)$. In this situation, when the particle is significantly close to one of the centres, the remaining $N-1$ centres generate a potential with the same properties of the perturbation function W. We are concerned with the behaviour of those trajectories which collide with the attraction centre in finite time (*collision solutions*). It is well known that, as $|q(t)| \to 0$, the normalized configuration $q(t)/|q(t)|$ has infinitesimal distance from the set of central configurations of S. In particular, if this set is discrete, any collision trajectory admits a limiting central configuration $\hat{s} \in \mathbb{S}^{d-1}$ (see for instance [7, 35, 12]), that is

$$\lim_{t \to T} \frac{q(t)}{|q(t)|} = \hat{s}, \tag{1.4}$$

for some $T > 0$. Given a central configuration $\hat{s} \in \mathbb{S}^{d-1}$ for S, we define the set of initial conditions for (1.3) in $\mathcal{H}_h$ which evolve to collision with limiting configuration $\hat{s}$

$$\mathcal{S}_h(\hat{s}) = \{(q,p) \in \mathcal{H}_h : \text{ the solution of (1.3), with } q(0) = q, \ p(0) = p, \text{ satisfies (1.4)}\}.$$

The corresponding motion is termed $\hat{s}$-asymptotic trajectory and we want to remark that the above set is non-empty, since the $\hat{s}$-homothetic trajectory with energy h is entirely contained in it.

Following McGehee ([52, 53]), it is possible to prove the next result (see Sections 1.2-1.3 for a step-by-step proof in the planar unperturbed case), in order to give a dynamical interpretation of the set $\mathcal{S}_h(s^*)$ when $s^* \in \mathbb{S}^{d-1}$ satisfies (s^*0). From now on, we will consider the tangent bundle $T\mathbb{S}^{d-1}$, which is nothing but the union of all the pairs $\{(s,u) : u \in T_s\mathbb{S}^{d-1}\}$, where $s \in \mathbb{S}^{d-1}$ and $T_s\mathbb{S}^{d-1}$ is the tangent space of the sphere $\mathbb{S}^{d-1}$ in s. Let us observe that both $\mathbb{S}^{d-1}$ and $T_s\mathbb{S}^{d-1}$ are $(d-1)$-dimensional manifolds. Moreover, let us recall that an equilibrium point for a non-linear dynamical system is *hyperbolic* if all the eigenvalues of the Jacobian matrix of the associated vector field have real part different from zero.

Lemma 1.1.1. *Given $h \in \mathbb{R}$, consider a potential $V \in \mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$ and $s^* \in \mathbb{S}^{d-1}$ satisfying respectively $(V0)$ and (s^*0). Then, there exists a diffeomorphism*

$$\begin{aligned} \phi\colon \mathcal{H}_h &\to [0,+\infty) \times T\mathbb{S}^{d-1} \\ (q,p) &\mapsto \phi(q,p) = (r,s,u) \end{aligned}$$

such that, for some $\mathcal{C}^2$-vector field $F\colon [0,+\infty) \times T\mathbb{S}^{d-1} \to [0,+\infty) \times T\mathbb{S}^{d-1}$ and a certain time rescaling $\tau = \tau(t)$, considering the dynamical system (where "$'$" stands for the derivative with respect to τ)

$$(r', s', u') = F(r,s,u), \tag{1.5}$$

we have:

(i) to a solution $(q,p) = (q(t),p(t))_{t\in[0,T)} \subseteq \mathcal{H}_h$ of (1.3) *there corresponds a solution $(r,s,u) = (r(\tau),s(\tau),u(\tau))_{\tau\geq 0} \subseteq [0,+\infty) \times T\mathbb{S}^{d-1}$ of* (1.5);

(ii) $(0,s^*,0)$ is a hyperbolic equilibrium point for (1.5);

(iii) there exists a d-dimensional stable manifold $\mathcal{W}^S$ for $(0,s^*,0)$, which is locally the graph of a $\mathcal{C}^2$-function $\Psi\colon \mathcal{U} \to T_s\mathbb{S}^{d-1}$, where $\mathcal{U} \subseteq [0,+\infty) \times \mathbb{S}^{d-1}$ is a sufficiently small neighbourhood of $(0,s^*)$ and $\Psi(0,s^*) = 0$.

In other words, defining

$$\mathcal{W}^S_{loc} = \mathcal{W}^S \cap (\mathcal{U} \times \Psi(\mathcal{U})),$$

it turns out that

(iv) in a neighbourhood of the origin, $\mathcal{S}_h(s^*)$ corresponds to $\mathcal{W}^S_{loc}$ through the diffeomorphism ϕ, so that a s^*-asymptotic collision trajectory will be represented by an orbit contained in $\mathcal{W}^S_{loc}$.

To have an idea of the geometrical description of the coordinates (r,s,u) introduces in the previous lemma, we point out that (r,s) is the expression in polar coordinates of the position coordinate q, while u is a rescaled version of the velocity p.

Our goal is to establish a link between orbits contained in $\mathcal{W}^S_{loc}$ and collision trajectories which minimize the geometric functional naturally associated with the Hamiltonian system. For this reason, let us introduce the Jacobi-length functional

$$\mathcal{L}_h(y) = \int_0^T |\dot{y}|\sqrt{h+V(y)},$$

for $y \in H^1([0,T];\mathbb{R}^d)$ such that $|\dot{y}| > 0$ and $h + V(y) > 0$. It is well known that a critical point y of $\mathcal{L}_h$ corresponds to a classical solution on $(0,T)$ of (1.3) in $\mathcal{H}_h$ for a certain $T > 0$, if $|y(t)| \neq 0$ for every $t \in (0,T)$ (see for instance [4, 46, 54]).

In particular, for a properly chosen $\bar{r} = \bar{r}(h) > 0$ and for $q \in B_{\bar{r}} = B_{\bar{r}}(0)$, introducing the set of collision paths

$$H^q_{coll} = \{y \in H^1([0,T];\mathbb{R}^d) :\ y(0) = q,\ y(T) = 0,\ |y(t)| < |q|,\ t \in (0,T)\},$$

contrary to the case $\alpha \geq 2$, in which $\mathcal{L}_h$ is never finite on collisions, when $\alpha \in (0,2)$, we are able to find at least a minimizer for the Jacobi length in the above space. Such a minimizer is not necessarily unique; indeed, any of these minimal paths is associated with the starting velocity $\dot{y}(0)$ of the trajectory. This leads to the construction of the multivalued map

$$\begin{aligned} \mathcal{F}_h \colon B_{\bar{r}} &\to \mathcal{P}(T_q\mathbb{R}^d) \\ q &\mapsto \mathcal{F}_h(q) = \left\{ \dot{y}(0) :\ y = \arg\min_{H^q_{coll}} \mathcal{L}_h \right\} \end{aligned}$$

in which $\mathcal{F}_h(q)$ represents the set of all the initial velocities for which a minimal collision arc exists.

Now, in the fashion of Lemma 1.1.1, without loss of generality, we can assume that $\mathcal{U} = [0,\bar{r}) \times B_{\bar{\delta}}(s^*)$ for some $\bar{\delta} > 0$, so that

$$\mathcal{W}^S_{loc} = \mathcal{W}^S_{loc}(\bar{r}, \bar{\delta}).$$

In this way, our main result consists in showing that, if $\bar{r}$ and $\bar{\delta}$ are sufficiently small, a collision minimizer starting at $q \in B_{\bar{r}}$, with $q/|q| \in B_{\bar{\delta}}(s^*)$, is actually unique and its ϕ-corresponding orbit is entirely contained in $\mathcal{W}^S_{loc}$. This means that, for such starting points, the set $\mathcal{F}_h(q)$ is not only a singleton, but it verifies $\phi(q, \mathcal{F}_h(q)) \in \mathcal{W}^S_{loc}$. For this reason, it makes sense to introduce another *local set* in the phase space, which is *spanned* by all the unique minimizers above mentioned

$$\mathfrak{M}_h(\bar{r}, \bar{\delta}) = \{\phi(q, \mathcal{F}_h(q)) :\ q \in B_{\bar{r}},\ q/|q| \in B_{\bar{\delta}}(s^*)\},$$

and to state our core result in this way:

Theorem 1.1.2. *Given $h \in \mathbb{R}$, consider a potential $V \in \mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$ and $s^* \in \mathbb{S}^{d-1}$ verifying respectively* $(V0)$ *and* (s^*0)*. Then, there exist $\bar{r} = \bar{r}(h) > 0$ and $\bar{\delta} = \bar{\delta}(s^*) > 0$ such that*

$$\mathcal{W}^S_{loc}(\bar{r}, \bar{\delta}) = \mathfrak{M}_h(\bar{r}, \bar{\delta}).$$

Remark 1.1.3. *The assumption* (s^*0) *that the minimal central configuration is non-degenerate, though stringent, holds generically. It can be easily lifted in some particular situations, for example in the case of the* $-\alpha$*-homogeneous* N*-body problem, that is when*

$$V(q_1, \cdots, q_N) = \sum_{i \neq j} \frac{m_i m_j}{|q_i - q_j|^\alpha} \ .$$

In this case the potential is invariant under common rotations of all the bodies and obviously no central configuration can be non-degenerate. However, our main result still holds true under the assumption of non-degeneration of the $SO(d)$*-orbit of the minimal central configuration under examination. Indeed, using again McGehee change of coordinates and extending the flow on the collision manifold, Lemma 1.1.1 can be rephrased in terms of a normally invariant manifold of stationary points endowed with their stable and unstable (local) manifolds. Given this alteration, the statement and proof easily follows.*

For the sake of a better comprehension and visualization of the proofs, we will carry out our work in a simplified case, which can be easily generalized to the setting introduced above. In particular, from now on we will take into account a planar anisotropic Kepler problem as proposed in [11] and we will work in negative energy shells, i.e., we will assume

- $d = 2$;
- $W \equiv 0$;
- $h < 0$.

Useful complementary material needed for the proof in the more general setting will be provided in Section 1.6. The paper is organised as follows: Section 1.2 introduces the collision manifold for the planar case and recalls the main features of the extended flow, whereas Section 1.3 is devoted to the analysis of the extended flow near its critical points. The object of Section 1.4 are Bolza minimizing arcs and their properties, while the Main Theorem will be eventually proved in Section 1.5 in the unperturbed and planar case, whereas in Section 1.6 we will discuss the modifications needed to cover the perturbed d-dimensional case.

1.2. The collision manifold for the planar problem

As aforementioned, we will develop this and the following sections working on the plane and with an unperturbed potential V. The following construction, which is the two-dimensional version of Lemma 1.1.1, exploits a technique firstly introduced by R. McGehee in the study of the collinear 3-body problem ([52, 53]) and furthermore employed by Devaney and others for the anisotropic Kepler problem ([25, 29, 30, 43]). Exploiting a space-time change of coordinates, this method consists in attaching a *collision manifold* to the phase space, where the flow can be extended in a suitable way, having central configurations as stationary points, endowed with their stable and unstable manifolds. In particular, a very similar approach with possibly different time parameterization can be found in [11, 12, 42]. We shall follow here the Devaney's approach ([25]). For our purposes, for a point $x \in \mathbb{R}^2 \setminus \{0\}$ it makes sense to introduce polar coordinates $x = (q_1, q_2) = (r\cos\vartheta, r\sin\vartheta)$, where

$$r = \sqrt{q_1^2 + q_2^2}, \qquad \vartheta = \begin{cases} \arctan(q_2/q_1) & \text{if } q_1 > 0,\ q_2 \geq 0 \\ \pi/2 & \text{if } q_1 = 0,\ q_2 > 0 \\ \arctan(q_2/q_1) + \pi & \text{if } q_1 < 0 \\ 3\pi/2 & \text{if } q_1 = 0,\ q_2 < 0 \\ \arctan(q_2/q_1) + 2\pi & \text{if } q_1 > 0,\ q_2 < 0 \end{cases},$$

with $r > 0$ and $\vartheta \in [0, 2\pi)$. In this way, any $-\alpha$-homogeneous potential $V \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\})$ can be written as

$$V(x) = r^{-\alpha} U(\vartheta),$$

where $U \in \mathcal{C}^2(\mathbb{S}^1)$, $U > 0$ and

$$U(\vartheta) = V(\cos\vartheta, \sin\vartheta).$$

Hypotheses on V: In this setting, the original assumptions $(V0)$-(s^*0) reduce respectively to:

$$(V1) \qquad \begin{cases} V \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\}); \\ V(x) = |x|^{-\alpha} U(x/|x|), \text{ with } \alpha \in (0,2) \text{ and } U \in \mathcal{C}^2(\mathbb{S}^1), \end{cases}$$

and

$$(U1) \qquad \exists\, \vartheta^* \in \mathbb{S}^1 \text{ s.t. } U(\vartheta) \geq U(\vartheta^*) > 0\ \forall\, \vartheta \in \mathbb{S}^1 \text{ and } U''(\vartheta^*) > 0.$$

With these notations, we study the motion and energy equations in the plane

$$(1.6) \qquad \begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h, \end{cases}$$

with $h < 0$. As usual, the conservation of energy forces every solution of (1.6) to be included into the *Hill's region*

$$\mathcal{R}_h = \left\{ x \in \mathbb{R}^2 \setminus \{0\} :\ V(x) + h \geq 0 \right\}.$$

Now, since

$$\nabla r = r^{-1}(q_1, q_2), \qquad \nabla\vartheta = r^{-2}(-q_2, q_1),$$

we can compute

$$\nabla V(x) = r^{-\alpha-2}\left[-\alpha U(\vartheta)(q_1, q_2) + U'(\vartheta)(-q_2, q_1)\right].$$

In this way, introducing the momentum vector $(p_1, p_2) = (\dot{q}_1, \dot{q}_2)$, we can rewrite equations (1.6) as

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2 \\ \dot{p}_1 = r^{-\alpha-2}\left[-U'(\vartheta)q_2 - \alpha U(\vartheta)q_1\right] \\ \dot{p}_2 = r^{-\alpha-2}\left[U'(\vartheta)q_1 - \alpha U(\vartheta)q_2\right], \end{cases} \tag{1.7}$$

and

$$\frac{1}{2}\left(p_1^2 + p_2^2\right) - r^{-\alpha}U(\vartheta) = h.$$

If we are not on the boundary of $\mathcal{R}_h$, we have that $|p| \neq 0$ and so, for every solution of (1.7), we can find smooth functions $z > 0$ and $\varphi \in [0, 2\pi)$ in such a way that $p_1 = r^{-\alpha/2}z\cos\varphi$, $p_2 = r^{-\alpha/2}z\sin\varphi$, choosing

$$z = \sqrt{2U(\vartheta) + 2hr^{\alpha}}. \tag{1.8}$$

By standard calculations, equations (1.7) become

$$\begin{cases} \dot{r} = r^{-\alpha/2}z\cos(\varphi - \vartheta) \\ \dot{\vartheta} = r^{-1-\alpha/2}z\sin(\varphi - \vartheta) \\ \dot{z} = r^{-1-\alpha/2}\left[U'(\vartheta)\sin(\varphi - \vartheta) + \alpha hr^{\alpha}\cos(\varphi - \vartheta)\right] \\ \dot{\varphi} = \frac{1}{z}r^{-1-\alpha/2}\left[U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta)\right] \end{cases} \tag{1.9}$$

and this system has a singularity when $r = 0$, which indeed corresponds to the collision set $\{0\} \subseteq \mathbb{R}^2$ of problem (1.6). Introducing a new time variable τ which verifies

$$\frac{dt}{d\tau} = zr^{1+\alpha/2},$$

the singularity of (1.9) can be *removed* in order to extend the vector field to the singular boundary $\{r = 0\}$. The effect of this rescaling is to *blow-up* the instant of an eventual collision, so that the particle will *virtually never reach* the singularity. In this way, we can rewrite (1.9) as (here "$'$" denotes the derivative with respect to τ)

$$\begin{cases} r' = rz^2\cos(\varphi - \vartheta) \\ \vartheta' = z^2\sin(\varphi - \vartheta) \\ z' = z\left[U'(\vartheta)\sin(\varphi - \vartheta) + \alpha hr^{\alpha}\cos(\varphi - \vartheta)\right] \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta). \end{cases}$$

Moreover, the conservation of energy, together with definition (1.8), allows us to eliminate the variable z from the system, and thus to consider the 3-dimensional system

$$\begin{cases} r' = 2r(U(\vartheta) + hr^{\alpha})\cos(\varphi - \vartheta) \\ \vartheta' = 2(U(\vartheta) + hr^{\alpha})\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta) \end{cases} \tag{1.10}$$

which we shortly denote by $(r', \vartheta', \varphi') = F(r, \vartheta, \varphi)$, with $F\colon [0, +\infty) \times T\mathbb{S}^1 \to [0, +\infty) \times T\mathbb{S}^1$. Since $r' = 0$ when $r = 0$, the boundary $\{r = 0\}$ is an invariant set for the above system. In other words, we can restrict the vector field F to $\{r = 0\}$ and study the independent dynamical system

$$\begin{cases} \vartheta' = U(\vartheta)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta) \end{cases} \tag{1.11}$$

which defines a 2-dimensional *collision manifold* and whose stationary points are

$$(\hat{\vartheta}, \hat{\vartheta} + k\pi), \text{ with } k \in \mathbb{Z} \text{ and } U'(\hat{\vartheta}) = 0.$$

If we denote by $JF|_{\{r=0\}}$ the Jacobian matrix of the vector field associated to (1.11) and we evaluate it at $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$, we obtain

$$JF|_{\{r=0\}}(\hat{\vartheta}, \hat{\vartheta} + k\pi) = \cos(k\pi)U(\hat{\vartheta}) \begin{pmatrix} -2 & 2 \\ \frac{U''(\hat{\vartheta})}{U(\hat{\vartheta})} - \alpha & \alpha \end{pmatrix},$$

whose eigenvalues are

$$\mu^{\pm} = \frac{1}{2}\cos(k\pi)U(\hat{\vartheta}) \left\{ \alpha - 2 \pm \left[(\alpha - 2)^2 + 8\frac{U''(\hat{\vartheta})}{U(\hat{\vartheta})} \right]^{\frac{1}{2}} \right\}.$$

For the sake of completeness, we present here a full characterization of the equilibrium points of (1.11), depending on the value of $U''(\hat{\vartheta})$.

For an equilibrium point $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *of* (1.11) *we can have:*

- *if* $U''(\hat{\vartheta}) < -\frac{(\alpha-2)^2}{8}U(\hat{\vartheta})$, *then* $\mu^{\pm} \in \mathbb{C} \setminus \mathbb{R}$ *and thus*
  - *if* k *is even then* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is a* sink;
  - *if* k *is odd then* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is a* source;
- *if* $U''(\hat{\vartheta}) = -\frac{(\alpha-2)^2}{8}U(\hat{\vartheta})$, *then* $\mu^- = \mu^+ \in \mathbb{R}$ *and thus*
  - *if* k *is even then* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is* asymptotically stable;
  - *if* k *is odd then* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is* unstable;
- *if* $U''(\hat{\vartheta}) > -\frac{(\alpha-2)^2}{8}U(\hat{\vartheta})$, *then*
  - *if* $U''(\hat{\vartheta}) > 0$, *then* $\mu^- \cdot \mu^+ < 0$ *and thus* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is a* saddle;
  - *if* $U''(\hat{\vartheta}) = 0$, *then one of the eigenvalues is zero and thus*
    - *if* k *is even then* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is a* stable degenerate node;
    - *if* k *is odd then* $(\hat{\vartheta}, \hat{\vartheta} + k\pi)$ *is a* unstable degenerate node;
  - *if* $0 > U''(\hat{\vartheta}) > -\frac{(\alpha-2)^2}{8}U(\hat{\vartheta})$, *then sign*$(\mu^-)$ = *sign*(μ^+) *and thus*

* *if k is even then $(\hat\vartheta, \hat\vartheta + k\pi)$ is a* stable tangent-node;
* *if k is odd then $(\hat\vartheta, \hat\vartheta + k\pi)$ is a* unstable tangent-node.

Now, we assume again $(V1)$-$(U1)$ and so, in particular, $(\vartheta^*, \vartheta^* + k\pi)$ is a saddle equilibrium point for (1.11). Coming back to the 3-dimensional system (1.10), the Jacobian of F evaluated in the stationary points $(0, \vartheta^*, \vartheta^* + k\pi)$ is

$$JF(0, \vartheta^*, \vartheta^* + k\pi) = U(\vartheta^*)\cos(k\pi) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & \frac{U''(\vartheta^*)}{U(\vartheta^*)} - \alpha & \alpha \end{pmatrix}.$$

In this way, we note that the r-eigenvalue is always non-zero, i.e.

- if k is odd the orbit *enters* in the collision manifold;
- if k is even the orbit *leaves* the collision manifold.

Since we are interested in studying the behaviour of a trajectory which approaches the singularity, we focus our attention on the case in which k is odd. With the choice of $k = 1$, the Jacobian becomes

$$JF(0, \vartheta^*, \vartheta^* + \pi) = \begin{pmatrix} -2U(\vartheta^*) & 0 & 0 \\ 0 & 2U(\vartheta^*) & -2U(\vartheta^*) \\ 0 & \alpha U(\vartheta^*) - U''(\vartheta^*) & -\alpha U(\vartheta^*) \end{pmatrix},$$

the eigenvalues are

$$\begin{aligned} \lambda_r &= -2U(\vartheta^*) < 0 \\ \lambda^\pm &= \frac{2-\alpha}{2} U(\vartheta^*) \pm \frac{1}{2}\sqrt{(2-\alpha)^2 U(\vartheta^*)^2 + 8U(\vartheta^*)U''(\vartheta^*)} \gtrless 0 \end{aligned}$$

and the relative eigendirections are

$$\begin{aligned} v_r &= (1, 0, 0) \\ v^\pm &= \left(0, 1, \frac{1}{2} + \frac{\alpha}{4} \pm \frac{1}{4}\sqrt{(2-\alpha)^2 + 8\frac{U''(\vartheta^*)}{U(\vartheta^*)}}\right). \end{aligned}$$

Remark 1.2.1. *The orbits of the stable manifold associated to the equilibrium point $(0, \vartheta^*, \vartheta^* + \pi)$ will enter in the collision manifold being tangent to v_r or v^-, depending on the sign of the quantity*

$$\frac{U''(\vartheta^*)}{U(\vartheta^*)} - (4 - \alpha)$$

(for instance, if negative, the tangency will be with respect to v^-). In particular, for the classical Kepler problem in which $\alpha = 1$, $U(\vartheta^*) = 1$ and $U''(\vartheta^*) = 0$, the above quantity is always negative.

1.3. The stable manifold

In the previous section we have shown the existence of an invariant set for (1.10), the *collision manifold* $\{r = 0\}$. Moreover, from the linearisation of the vector field, it follows that the only way for a point in the phase space to evolve *entering* in $\{r = 0\}$ is to belong to the stable set of an equilibrium point $(0, \hat{\vartheta}, \hat{\vartheta} + k\pi)$, with $U'(\hat{\vartheta}) = 0$ and k odd. For this reason, in this section we focus our attention on the study of the *stable manifold* of such equilibrium points, with the not restrictive choice of $k = 1$. We start our analysis with the case $h = 0$, which presents a simplified and clearer structure.

1.3.1. Collision orbits for $h = 0$

In this setting, system (1.10) reads

$$\begin{cases} r' = 2rU(\vartheta)\cos(\varphi - \vartheta) \\ \vartheta' = 2U(\vartheta)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta) \end{cases} \tag{1.12}$$

and the set

$$\{(r, \hat{\vartheta}, \hat{\vartheta} + \pi) : \, r \geq 0, \; U'(\hat{\vartheta}) = 0\}$$

is invariant for the system and it gathers all the collision $\hat{\vartheta}$-homothetic trajectories. Once $\hat{\vartheta} \in \mathbb{S}^1$ critical point for U is fixed, such a set reduces to a ray which enters the collision manifold in the equilibrium point $(0, \hat{\vartheta}, \hat{\vartheta} + \pi)$.

Lemma 1.3.1. *Assume* $(V1)$-$(U1)$. *Then, there exist* $\delta > 0$, *a stable manifold* W^s *for the equilibrium point* $(\vartheta^*, \vartheta^* + \pi)$ *of* (1.11) *and a* $\mathcal{C}^2$*-function* $\Psi \colon (\vartheta^* - \delta, \vartheta^* + \delta) \to \mathbb{S}^1$ *such that* $\Psi(\vartheta^*) = \vartheta^* + \pi$ *and for every* $\vartheta \in (\vartheta^* - \delta, \vartheta^* + \delta)$

- $(\vartheta, \varphi) \in W^s$ *if and only if* $\varphi = \Psi(\vartheta)$;
- $\varphi < \vartheta + \pi$ *if* $\vartheta \in (\vartheta^* - \delta, \vartheta^*)$ *[resp.* $\varphi > \vartheta + \pi$ *if* $\vartheta \in (\vartheta^*, \vartheta^* + \delta)$*].*

Moreover, $\mathbb{R}_{\geq 0} \times W^s$ *is the stable manifold of the equilibrium point* $(0, \vartheta^*, \vartheta^* + \pi)$ *for system* (1.12)*.*

Proof. We firstly analyse the 2-dimensional system (1.11), keeping in mind the eigendirections v^+ and v^- computed in the previous section. From the *Stable Manifold Theorem* (see for instance [41],[66]) we have that there exist W^u, W^s $\mathcal{C}^2$-curves on the collision manifold $\{r = 0\}$ such that

- $(\vartheta^*, \vartheta^* + \pi) \in W^u, W^s$;
- W^s is tangent to v^- and W^u is tangent to v^+ in $(\vartheta^*, \vartheta^* + \pi)$;
- for every $(\vartheta^+, \varphi^+) \in W^u$ and $(\vartheta^-, \varphi^-) \in W^s$ we have

$$\lim_{\tau \to \pm\infty} (\vartheta^{\pm}(\tau), \varphi^{\pm}(\tau)) = (\vartheta^*, \vartheta^* + \pi).$$

In particular, since $(\vartheta^*, \vartheta^*+\pi)$ is a hyperbolic equilibrium point, W^s is locally the graph of a $\mathcal{C}^2$-curve $\varphi = \varphi(\vartheta)$, i.e.

$$W^s = \left\{(\vartheta, \varphi(\vartheta)) :\ \vartheta \in (\vartheta^* - \delta, \vartheta^* + \delta) \text{ with } \delta > 0,\ \varphi \in \mathcal{C}^2,\ \varphi(\vartheta^*) = \vartheta^* + \pi\right\},$$

which is tangent to v^- in $(\vartheta^*, \vartheta^*+\pi)$. This is a consequence of the fact that the *local stable manifold* has the same dimension of the *stable eigenspace* (*Hartman-Grobman Theorem*, see for instance [66]).

Now, since the second and third equations of system (1.12) are uncoupled for every $r \geq 0$, if we consider the set

$$\mathbb{R}_{\geq 0} \times W^s = \{(r, \vartheta, \varphi) :\ r \geq 0,\ (\vartheta, \varphi) \in W^s\},$$

defining the flow associated to (1.12) as $\Phi^\tau = \Phi^\tau(r, \vartheta, \varphi)$, we have that for every $(r, \vartheta, \varphi) \in \mathbb{R}_{\geq 0} \times W^s$

$$\lim_{\tau \to +\infty} \Phi^\tau(r, \vartheta, \varphi) = (0, \vartheta^*, \vartheta^* + \pi).$$

Indeed, $\Phi^\tau(r, \vartheta, \varphi) \in \mathbb{R}_{\geq 0} \times W^s$ for every $\tau > 0$, since the projection of the vector field F_0 on the manifold $\{r = 0\}$, $\pi_{r=0} F_0(\Phi^\tau(r, \vartheta, \varphi))$, is tangent to W^s for every $\tau > 0$, where F_0 represents the vector field associated to (1.12). □

1.3.2. Collision orbits for $h < 0$

When $h < 0$, we come back again to system (1.10), which we recall here for the reader's convenience

$$\begin{cases} r' = 2r(U(\vartheta) + hr^\alpha)\cos(\varphi - \vartheta) \\ \vartheta' = 2(U(\vartheta) + hr^\alpha)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta). \end{cases}$$

The collision manifold $\{r = 0\}$ is exactly the same of the zero-energy system and we still have the invariance of the collision homothetic trajectories set

$$\{(r, \hat{\vartheta}, \hat{\vartheta} + \pi) :\ r \geq 0,\ U'(\hat{\vartheta}) = 0\}.$$

However, we point out that for $h < 0$ the set $\mathbb{R}_0^+ \times W^s$ is not the stable manifold for $(0, \hat{\vartheta}, \hat{\vartheta} + \pi)$. Beside that, assuming $(V1)$-$(U1)$, the hyperbolicity of the fixed point $(0, \vartheta^*, \vartheta^*+\pi)$ still guarantees the existence of a 2-dimensional stable manifold $\mathcal{W}^s$, which contains the homothetic trajectories and the 1-dimensional stable manifold W^s.

Now, the dynamical systems (1.12) and (1.10) share the same linearisation with respect to the equilibrium point $(0, \vartheta^*, \vartheta^* + \pi)$. These means that, below some $r^* > 0$, they are topologically equivalent. In particular, we can imagine $\mathcal{W}^s \cap \{r < r^*\}$ as a $\mathcal{C}^2$ h-deformation of $[0, r^*) \times W^s$, in which the 1-dimensional components $\{0\} \times W^s$ and $(0, r^*) \times \{\vartheta^*\} \times \{\vartheta^* + \pi\}$ stay always fixed.

As a consequence of these discussions, we deduce the following analytic and geometric description of $\mathcal{W}^s$ in a neighbourhood of the equilibrium point, which locally generalizes Lemma 1.3.1.

Lemma 1.3.2. *Assume* $(V1)$-$(U1)$. *Given* $h < 0$, *there exist* $r_{loc} > 0$, $\delta_{loc} > 0$ *and a* $\mathcal{C}^2$-*function* $\Psi\colon [0, r_{loc}) \times (\vartheta^* - \delta_{loc}, \vartheta^* + \delta_{loc}) \to \mathbb{S}^1$ *such that* $\Psi(0, \vartheta^*) = \vartheta^* + \pi$ *and for every* $(r, \vartheta) \in (0, r_{loc}) \times (\vartheta^* - \delta_{loc}, \vartheta^* + \delta_{loc})$

- $(r, \vartheta, \varphi) \in \mathcal{W}^s$ *if and only if* $\varphi = \Psi(r, \vartheta)$;
- $\varphi < \vartheta + \pi$ *if* $\vartheta \in (\vartheta^* - \delta_{loc}, \vartheta^*)$ [*resp.* $\varphi > \vartheta + \pi$ *if* $\vartheta \in (\vartheta^*, \vartheta^* + \delta_{loc})$].

In other words, we have just characterized locally $\mathcal{W}^s$ *as the graph of a function* Ψ

$$\mathcal{W}^s_{loc} \doteq \{(r, \vartheta, \Psi(r, \vartheta)) : \, r \in [0, r_{loc}), \; \vartheta \in (\vartheta^* - \delta_{loc}, \vartheta^* + \delta_{loc})\} \subseteq \mathcal{W}^s.$$

Remark 1.3.3. *To better understand the meaning of the previous lemma, we can refer to the configurations space the behaviour of a point evolving in* $\mathcal{W}^s_{loc}$ *and eventually entering in the collision manifold. In particular, Lemma 1.3.2 guarantees the existence of a cone*

$$\mathcal{C} = \{q = (q_1, q_2) \in \mathbb{R}^2 : \; |q| \leq r_{loc}, \; \arctan(q_2/q_1) \in (\vartheta^* - \delta_{loc}, \vartheta^* + \delta_{loc})\}$$

such that, for every trajectory which starts in $\mathcal{C}$ *and reaches the collision being tangent to* ϑ^*, *it never leaves* $\mathcal{C}$ *(see Figure 1.1). We would stress that this confinement result is strictly addressed to those orbits which, in the phase space, are contained in* $\mathcal{W}^s_{loc}$. *Indeed, every collision orbit that is not tangent to* ϑ^* *in the origin is not necessarily contained in the cone* $\mathcal{C}$.

1.4. Collision orbits as Bolza minimizers

The first task of this work is to highlight the relationship which stands between the dynamical nature of this problem and our variational approach. Therefore, we present here the minimality argument which leads to the existence of a solution for (1.1)-(1.2) and provide further properties of this underlying variational structure of the problem. The *Maupertuis' Principle* states that every critical point of a suitable functional, which could be either the Lagrange-action, the Jacobi-length or the Maupertuis' functional, if properly manipulated is a classical solution of (1.1)-(1.2) (see [4],[3]). In the first part of this section, we state and prove a similar result which guarantees the existence of a trajectory which reaches the origin in finite time, once a critical point is provided. From now on, we will always consider $h < 0$ fixed and assume $(U1)$-$(V1)$, unless differently specified.

1.4.1. The Maupertuis' Principle for collision trajectories

Given $q \in \mathcal{R}_h \setminus \{0\}$, consider the space of all the collision H^1-paths starting from q and reaching the origin in finite time $T > 0$

$$\widehat{H}^q_{coll} = \{u \in H^1([0, T]; \mathbb{R}^2) : \; u(0) = q, \; u(T) = 0, \; u(t) \in \mathcal{R}_h \text{ for every } t \in (0, T)\}.$$

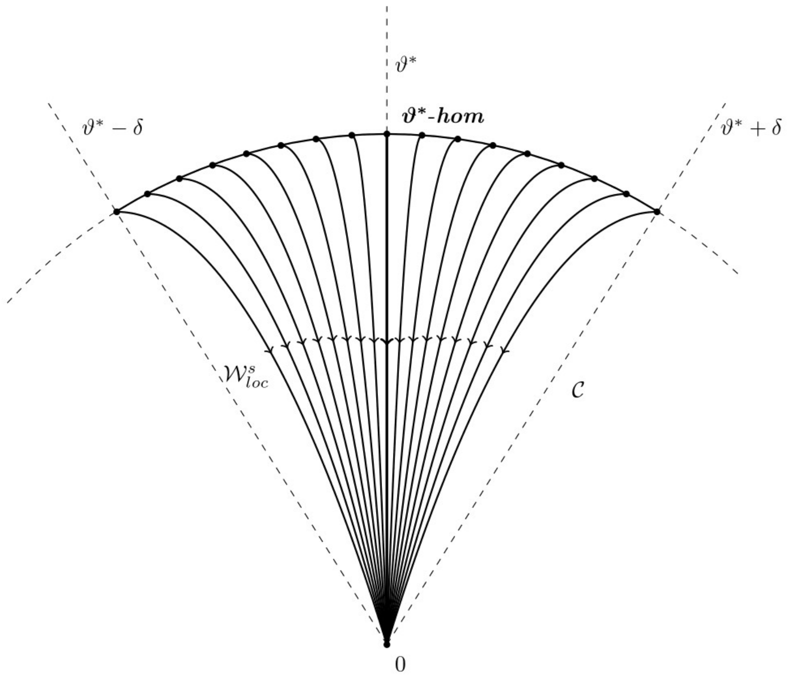

Figure 1.1.: The local stable manifold characterized in Remark 1.3.3.

Moreover, let us introduce the Maupertuis' functional $\mathcal{M}_h \colon \widehat{H}^q_{coll} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\mathcal{M}_h(u) = \frac{1}{2} \int_0^T |\dot{u}(s)|^2 \, ds \int_0^T (h + V(u(s))) \, ds$$

which is differentiable over $\widehat{H}^q_{coll}$ and, if $\mathcal{M}_h(u) > 0$, it makes sense to define the quantity

$$\omega = \left(\frac{\int_0^T (h + V(u))}{\frac{1}{2} \int_0^T |\dot{u}|^2} \right)^{\frac{1}{2}} > 0. \tag{1.13}$$

It is well-known that critical points of the Maupertuis' functional in a suitable space, if reparametrized, solves classically the motion and energy equations (1.1)-(2.3) (see for instance [3] and Appendix C for further details). As a starting point, we show that a minimizer of $\mathcal{M}_h$ in $\widehat{H}^q_{coll}$ at a positive level, if exists, cannot collide in the interior of its interval of definition.

Lemma 1.4.1. *Let* $u \in \widehat{H}^q_{coll}$ *be a minimizer of* $\mathcal{M}_h$*, with* $\mathcal{M}_h(u) > 0$*. Then,* $u(t) \neq 0$ *for every* $t \in (0, T)$*.*

Proof. Assume by contradiction that there exists $\tau \in (0, T)$ such that $u(\tau) = 0$. Observe that the path $v(t) = u(t \cdot \tau / T)$ defined for $t \in [0, T]$ belongs to $\widehat{H}^q_{coll}$. Since the Maupertuis' functional is invariant through time rescalings, with a standard change of variable we obtain

$$\begin{aligned}\mathcal{M}_h(u) &= \left(\frac{1}{2}\int_0^\tau |\dot u|^2 + \frac{1}{2}\int_\tau^T |\dot u|^2\right)\left(\int_0^\tau (h+V(u)) + \int_\tau^T (h+V(u))\right)\\ &= \frac{1}{2}\int_0^\tau |\dot u|^2 \int_0^\tau (h+V(u)) + \frac{1}{2}\int_0^\tau |\dot u|^2 \int_\tau^T (h+V(u))\\ &\quad + \frac{1}{2}\int_\tau^T |\dot u|^2 \int_0^\tau (h+V(u)) + \frac{1}{2}\int_\tau^T |\dot u|^2 \int_\tau^T (h+V(u))\\ &= \mathcal{M}_h(v) + [\text{positive terms}].\end{aligned}$$

The reason why the remaining terms are positive is that a path in $\widehat{H}^q_{coll}$ cannot leave the Hill's region $\mathcal{R}_h$. Indeed, if $V(u)+h=0$ in one of the two intervals, the path would lie constantly on the boundary of the Hill's region, so that it will never reach the origin. On the other hand, if $|\dot u|^2 = 0$, then there would be no motion and this would lead to $q=0$ which is impossible, since $q \in \mathcal{R}_h \setminus \{0\}$. In this way, we reach a contradiction for the minimality of u in $\widehat{H}^q_{coll}$. □

In the next theorem we show that a minimizer of the Maupertuis' functional at a positive level actually solves equations (1.1)-(2.3) except for collision and its normalized configuration converges to a minimal non degenerate central configuration of V as it approaches the origin. We can consider this result as a collision counterpart of the well-known *Maupertuis' principle*.

Theorem 1.4.2. *Let $u \in \widehat{H}^q_{coll}$ be a minimizer for $\mathcal{M}_h$ such that $\mathcal{M}_h(u) > 0$. Then, for ω given by* (1.13), *$x(t) = u(\omega t)$ is a classical solution of* (1.1)-(1.2) *in $[0, T/\omega)$ such that*

- *$x(0) = q$, $x(T/\omega) = 0$;*
- *$x(t)/|x(t)| \to \vartheta^*$ as $t \to (T/\omega)^-$, with $\vartheta^* \in \mathbb{S}^1$ central configuration for V;*
- *for some positive constant K, we have $|x(t)| \sim K(T/\omega - t)^{2/(2+\alpha)}$ as $t \to (T/\omega)^-$.*

Proof. Since $\mathcal{M}_h'(u) = 0$ we have

$$\mathcal{M}_h'(u)[v] = \int_0^T \dot u \cdot \dot v \int_0^T (h+V(u)) + \frac{1}{2}\int_0^T |\dot u|^2 \int_0^T \nabla V(u)\cdot v = 0,$$

for every $v \in H^1_0([0,T];\mathbb{R}^2)$ and so, since $\mathcal{M}_h(u) > 0$

$$\omega^2 \int_0^T \dot u \cdot \dot v + \int_0^T \nabla V(u) \cdot v = 0,$$

for every $v \in H^1_0([0,T];\mathbb{R}^2)$. In other words, u is a weak solution of the equation

$$\omega^2 \ddot u = \nabla V(u), \tag{1.14}$$

but also, by standard regularity arguments and by Lemma 1.4.1, a classical solution of the same equation in $[0,T)$. Now, it is readily checked that $x(t)=u(\omega t)$ solves (1.1)-(1.2) in $[0,T/\omega)$ and that the required boundary conditions are satisfied. The limiting behaviours as $t\to(T/\omega)^-$ follow from the well-known asymptotic estimates (see [7, 12, 11]). Moreover, from equation (1.14), we deduce that there exists $k\in\mathbb{R}$ such that

$$\frac{\omega^2}{2}|\dot{u}(t)|^2-V(u(t))=k,$$

for every $t\in[0,T)$. Integrating the above equation over $[0,T)$, we necessarily get $k=h$ and the energy equation (1.2) for x holds as well. □

1.4.2. Existence through direct methods

In what follows, we show the existence of minimizers for the Maupertuis' functional, which correspond to collision trajectories through Theorem 1.4.2. However, it will be clear that such motions cannot start too far from the singularity. The initial distance $r=|q|$ of the particle is indeed linked to the well-known *Lagrange-Jacobi* inequality (see for instance [69]), which we prove below in our setting.

Lemma 1.4.3 (Lagrange-Jacobi inequality). *Define* $U_{min}=\min_{\vartheta\in\mathbb{S}^1}U(\vartheta)$ *and*

$$r_{LJ}=\left[\frac{(2-\alpha)U_{min}}{-2h}\right]^{\frac{1}{\alpha}}>0.$$

For every solution x of (1.1)-(1.2) *such that $|x|<r_{LJ}$, we have that the moment of inertia $I(x(t))=\frac{1}{2}|x(t)|^2$ is strictly convex with respect to t. In particular, for a solution $x(t)=r(t)e^{i\vartheta(t)}$ which collides with the origin after a time $T>0$, we have $r'(t)<0$ in $(0,T)$.*

Proof. By standard calculations and using (1.1)-(1.2) we obtain

$$\begin{aligned}\frac{d^2}{dt^2}I(x(t))&=\langle\nabla V(x(t)),x(t)\rangle+2(V(x(t))+h)\\&=|x(t)|^{-\alpha}(2-\alpha)U(\vartheta(t))+2h\\&>(2-\alpha)r_{LJ}^{-\alpha}U_{min}+2h=0.\end{aligned}$$

□

The previous result suggests to consider a smaller minimization set than $\widehat{H}^q_{coll}$ and to require the *natural constraint* for a path not to leave the ball where it started from. Indeed, the Lagrange-Jacobi inequality assures that for a solution of (1.1)-(1.2) starting from the interior of $B_{r_{LJ}}$, its radial component is strictly decreasing and so it cannot leave the ball $B_{r_{LJ}}$. To be precise, given $r>0$ and $q\in\partial B_r$, we introduce the set of all the H^1-paths which start in q and collapse in the origin in finite time, without leaving the ball B_r

$$H^q_{coll}\doteq\{u\in H^1([0,1];\mathbb{R}^2):\ u(0)=q,\ u(1)=0,\ |u(s)|\leq r \text{ for every } s\in[0,1]\}.$$

Here and later, in order to simplify the notation, we have set $T = 1$.

We now present two lemmata which allow us to apply the direct method of the calculus of variations; to this aim, we will often make use of the Poincaré inequality, which clearly holds in the space H^q_{coll}.

Lemma 1.4.4. *For every $q \in B_{r_{LJ}}$, there exists a positive constant C such that*

$$0 < C \leq \inf_{u \in H^q_{coll}} \mathcal{M}_h(u) < +\infty.$$

Proof. Fix $r \in (0, r_{LJ})$ and $q \in \partial B_r$. From the definition of r_{LJ} given in Lemma 1.4.3, we have that, for every $u \in H^q_{coll}$

$$\int_0^1 (h + V(u))\, ds \geq \int_0^1 (h + r^{-\alpha} U_{min})\, ds > h + r_{LJ}^{-\alpha} U_{min} = -\frac{\alpha h}{2 - \alpha} > 0. \tag{1.15}$$

Moreover, for $u \in H^q_{coll}$, we can write

$$r = |u(1) - u(0)| \leq \int_0^1 |\dot{u}|\, ds \leq \left(\int_0^1 |\dot{u}|^2\, ds \right)^{1/2}$$

and so, together with (1.15), we obtain

$$\mathcal{M}_h(u) = \int_0^1 |\dot{u}|^2\, ds \int_0^1 (h + V(u))\, ds \geq -\frac{\alpha h r^2}{2 - \alpha} = C > 0, \quad \text{for every } u \in H^q_{coll}.$$

Moreover, since $u \in H^q_{coll}$, then $\dot{u} \in L^2([0,1]; \mathbb{R}^2)$ and $V(u) \in L^1([0,1]; \mathbb{R}^2)$, by means of the limiting behaviour provided in Theorem 1.4.2. This proves the upper bound and concludes the proof. □

Lemma 1.4.5. *For every $q \in B_{r_{LJ}}$, $\mathcal{M}_h$ is coercive on H^q_{coll}.*

Proof. Fix $r \in (0, r_{LJ})$ and $q \in \partial B_r$ and consider $(u_n)_n \subseteq H^q_{coll}$, such that $\|u_n\|_{H^1} \to +\infty$ as $n \to +\infty$. Since $|u_n(s)| \leq r$ for every $s \in [0,1]$ and for every $n \in \mathbb{N}$, we obtain that necessarily

$$\lim_{n \to +\infty} \int_0^1 |\dot{u}_n|^2\, ds = +\infty$$

and so, together with (1.15), we conclude that $\mathcal{M}_h(u_n) \to +\infty$. □

We are about to prove that a minimizer of $\mathcal{M}_h$ exists and thus, invoking Theorem 1.4.2, a collision trajectory $x(t)$ satisfying (1.1)-(1.2) can be provided.

Theorem 1.4.6. *Given $h < 0$ and $r_{LJ} > 0$ as in Lemma 1.4.3, the Maupertuis' functional*

$$\mathcal{M}_h(u) = \int_0^1 |\dot{u}|^2\, ds \int_0^1 (h + V(u))\, ds$$

admits at least a minimizer $u \in H^q_{coll}$ at a positive level, for every $q \in B_{r_{LJ}}$.

Proof. Let us fix $r \in (0, r_{LJ})$ and $q \in \partial B_r$. Since the weak convergence implies the uniform one in H^1, we first observe that H^q_{coll} is weakly closed in H^1 (equivalently, the weak closure follow from the fact that H^q_{coll} is convex and strongly closed).

Now, let us consider a sequence $(u_n)_n \subseteq H^q_{coll}$ such that

$$\mathcal{M}_h(u_n) \to \inf_{u \in H^q_{coll}} \mathcal{M}_h(u),$$

as $n \to +\infty$. From 1.4.4 and 1.4.5 we have that $(u_n)_n$ is bounded in H^1 and so $u_n \rightharpoonup u$ in H^1 up to a subsequence. In particular, since H^q_{coll} is weakly closed, $u \in H^q_{coll}$. Moreover, from the weakly lower semi-continuity of the L^2 norm, we deduce that

$$\int_0^1 |\dot{u}|^2 \, ds \leq \liminf_{n \to \infty} \int_0^1 |\dot{u}_n|^2 \, ds \tag{1.16}$$

and, from Lemma 1.4.4, for every $n \in \mathbb{N}$ we obtain that

$$0 < C \leq \mathcal{M}_h(u_n) < \infty$$

and thus $V(u_n) \in L^1(0,1)$, for every $n \in \mathbb{N}$. This implies that the set $\{t \in [0,1] : u_n(t) = 0\}$ has null measure and hence, since u_n converges to u uniformly, we have that $V(u_n) \to V(u)$ almost everywhere. We can now use Fatou's Lemma to deduce that $V(u) \in L^1(0,1)$ and that

$$\int_0^1 (h + V(u)) \, ds \leq \liminf_{n \to \infty} \int_0^1 (h + V(u_n)) \, ds.$$

This, together with (1.16), proves that

$$\mathcal{M}_h(u) \leq \liminf_{n \to +\infty} \mathcal{M}_h(u_n) = \inf_{u \in H^q_{coll}} \mathcal{M}_h(u).$$ □

1.4.3. A compactness lemma

Theorem 1.4.6 shows that once $h < 0$ and $q \in B_{r_{LJ}}$ are fixed, we can always find at least a minimizer of the Maupertuis' functional in the space H^q_{coll}. In this way, if we fix $r \in (0, r_{LJ})$, we can define a function $\psi_h \colon \partial B_r \to \mathbb{R}^+$ such that

$$\psi_h(q) \doteq \min_{u \in H^q_{coll}} \mathcal{M}_h(u) \quad \text{for } q \in \partial B_r.$$

Remark 1.4.7. *In the next proposition we are going to make use of the Jacobi-length functional*

$$\mathcal{L}_h(u) = \int_0^1 |\dot{u}| \sqrt{h + V(u)} \, ds,$$

which, for a path $u \in H^1([0,1]; \mathbb{R}^2)$, is well-defined whenever $h + V(u) \geq 0$. Therefore, it makes sense to consider paths which live far from the boundary of the Hill's region $\partial \mathcal{R}_h$. Nonetheless, *with our choice of r_{LJ} provided in Lemma 1.4.3, this condition is already satisfied. Indeed, taking $q \in B_{r_{LJ}}$ and $u \in H^q_{coll}$, we can write*

$$h + V(u) \geq h + r_{LJ}^{-\alpha} U_{min} = -\frac{\alpha h}{2-\alpha} = C > 0.$$

Moreover, from the energy equation, we clearly have $|\dot{u}| > 0$.

Proposition 1.4.8. *For $h < 0$ and $r \in (0, r_{LJ})$, the function ψ_h is Lipschitz continuous on ∂B_r. In other words, there exists $L = L(r_{LJ}) > 0$ such that*

$$|\psi_h(q_2) - \psi_h(q_1)| \leq L|\vartheta_2 - \vartheta_1|, \qquad \text{for every } q_1 = re^{i\vartheta_1}, q_2 = re^{i\vartheta_2} \in \partial B_r.$$

Proof. Fix $h < 0$ and $r \in (0, r_{LJ})$. Given $q \in \partial B_r$, for a path $u \in H^q_{coll}$ we can define the Jacobi-length functional

$$\mathcal{L}_h(u) = \int_0^1 |\dot{u}| \sqrt{h + V(u)}\, dt,$$

which is linked to $\mathcal{M}_h$ in this way:

$$2 \min_{H^q_{coll}} \mathcal{M}_h = (\min_{H^q_{coll}} \mathcal{L}_h)^2.$$

Therefore, if we define the function $\omega_h(q) \doteq \min_{H^q_{coll}} \mathcal{L}_h$ for $q \in \partial B_r$ and we show that it is Lipschitz continuous we are done.

Fix $q_1 = re^{i\vartheta_1}, q_2 = re^{i\vartheta_2} \in \partial B_r$ and consider the circular path

$$u_{arc}(t) = re^{i((1-t)\vartheta_1 + t\vartheta_2)}, \quad \text{for } t \in [0,1].$$

We have

$$\begin{aligned} \mathcal{L}_h(u_{arc}) &= r|\vartheta_2 - \vartheta_1| \int_0^1 \sqrt{h + r^{-\alpha} U(\vartheta(t))}\, dt \\ &< L|\vartheta_2 - \vartheta_1|, \quad \text{where } L = L(r_{LJ}) = r_{LJ}^{1-\alpha/2} \sqrt{U_{max}}. \end{aligned}$$

Now, since $\mathcal{L}_h$ is a length, it is invariant under time rescaling and so we can write

$$\min_{H^{q_1}_{coll}} \mathcal{L}_h \leq \mathcal{L}_h(u_{arc}) + \min_{H^{q_2}_{coll}} \mathcal{L}_h.$$

Finally, from the definition of ω_h, we obtain

$$\omega_h(q_1) \leq \omega_h(q_2) + L|\vartheta_2 - \vartheta_1|$$

and, with the same argument

$$\omega_h(q_2) \leq \omega_h(q_1) + L|\vartheta_2 - \vartheta_1|. \qquad \square$$

Corollary 1.4.9. *Given $h < 0$, $r^* \in (0, r_{LJ})$, $h^* \in (h, 0)$ and $\vartheta^* \in \mathbb{S}^1$, consider three sequences $(h_k)_k \subseteq \mathbb{R}$, $(r_k)_k \subseteq \mathbb{R}^+$ and $(\vartheta_k)_k \subseteq \mathbb{S}^1$ such that*

- $h_k \in (h, 0)$ *for every* $k \in \mathbb{N}$ *and* $h_k \to h^*$ *as* $k \to +\infty$*;*
- $0 < r_k < r_{LJ}$ *for every* $k \in \mathbb{N}$ *and* $r_k \to r^*$ *as* $k \to +\infty$*;*
- $q_k = r_k e^{i\vartheta_k} \to q^* = r^* e^{i\vartheta^*}$ *as* $k \to +\infty$*.*

Then

$$\min_{H_{coll}^{q_k}} \mathcal{M}_{h_k} \leq \min_{H_{coll}^{q^*}} \mathcal{M}_{h^*} + O(|\vartheta^* - \vartheta_k|) + O(|h^* - h_k|) + O(|r^* - r_k|),$$

as $k \to +\infty$*.*

Proof. Fix $h < 0$, $r^* \in (0, r_{LJ})$, $h^* \in (h, 0)$ and $\vartheta^* \in \mathbb{S}^1$. Consider the three sequences as in the statement and fix $k \in \mathbb{N}$. We can write

$$\psi_{h_k}(q_k) = \min_{H_{coll}^{q_k}} \mathcal{M}_{h_k}, \quad \psi_{h^*}(q^*) = \min_{H_{coll}^{q^*}} \mathcal{M}_{h^*},$$

so that

$$\psi_{h_k}(q_k) - \psi_{h^*}(q^*) = \psi_{h_k}(q_k) - \psi_{h_k}(q^*) + \psi_{h_k}(q^*) - \psi_{h^*}(q^*).$$

Let us start by the estimate of the term $\psi_{h_k}(q^*) - \psi_{h^*}(q^*)$ on the right-hand side. Consider $u^* \in H_{coll}^{q^*}$ such that

$$\psi_h(q^*) = \min_{H_{coll}^{q^*}} \mathcal{M}_{h^*} = \mathcal{M}_{h^*}(u^*).$$

For the minimality of u^*, we obtain

$$\begin{aligned}\psi_{h_k}(q^*) - \psi_{h^*}(q^*) &\leq \mathcal{M}_{h_k}(u^*) - \mathcal{M}_{h^*}(u^*) \\ &\leq |h^* - h_k| \int_0^1 |\dot{u}^*|^2 \, ds \leq C_1 |h^* - h_k|,\end{aligned}$$

with $C_1 > 0$. Now, concerning the term $\psi_{h_k}(q_k) - \psi_{h_k}(q^*)$, if we consider the point

$$q_k' = \frac{r^*}{r_k} q_k \in \partial B_{r^*}$$

we can write

$$\psi_{h_k}(q_k) - \psi_{h_k}(q^*) = \psi_{h_k}(q_k) - \psi_{h_k}(q_k') + \psi_{h_k}(q_k') - \psi_{h_k}(q^*).$$

Take $v_k \in H_{coll}^{q_k'}$ such that

$$\psi_{h_k}(q_k') = \min_{H_{coll}^{q_k'}} \mathcal{M}_{h_k} = \mathcal{M}_{h_k}(v_k)$$

and define the path

$$\tilde{v}_k = \frac{r_k}{r^*} v_k \in H_{coll}^{q_k}.$$

We can write

$$
\begin{aligned}
\psi_{h_k}(q_k) - &\psi_{h_k}(q_k') \\
&\leq \mathcal{M}_{h_k}(\tilde{v}_k) - \mathcal{M}_{h_k}(v_k) \\
&= \int_0^1 |\dot{\tilde{v}}_k|^2\, ds \int_0^1 (h_k + V(\tilde{v}_k))\, ds - \int_0^1 |\dot{v}_k|^2\, ds \int_0^1 (h_k + V(v_k))\, ds \\
&= \int_0^1 |\dot{v}_k|^2\, ds \int_0^1 \left[h_k \left(\frac{r_k}{r^*}\right)^2 - h_k + \left(\frac{r_k}{r^*}\right)^{2-\alpha} V(v_k) - V(v_k) \right] ds \\
&= \int_0^1 |\dot{v}_k|^2\, ds \int_0^1 \left[h_k \frac{r_k + r^*}{(r^*)^2}(r_k - r^*) + V(v_k) \frac{r_k^{2-\alpha} - (r^*)^{2-\alpha}}{(r^*)^{2-\alpha}} \right] ds \\
&\leq C_2(|r^* - r_k|),
\end{aligned}
$$

with $C_2 > 0$. Finally, since $q_k', q^* \in \partial B_{r^*}$, we can apply Proposition 1.4.8 to obtain

$$
\psi_{h_k}(q_k') - \psi_{h_k}(q^*) \leq L|\vartheta^* - \vartheta_k|.
$$

□

Now we prove the following compactness lemma on sequences of minimizers of the Maupertuis' functional.

Lemma 1.4.10. *Given $h < 0$, $r^* \in (0, r_{LJ})$, $h^* \in (h, 0)$ and $\vartheta^* \in \mathbb{S}^1$, consider three sequences $(h_k)_k \subseteq \mathbb{R}^+$, $(r_k)_k \subseteq \mathbb{R}$ and $(\vartheta_k)_k \subseteq \mathbb{S}^1$ such that*

- $h_k \in (h, 0)$ *for every* $k \in \mathbb{N}$ *and* $h_k \to h^*$ *as* $k \to +\infty$;
- $0 < r_k < r_{LJ}$ *for every* $k \in \mathbb{N}$ *and* $r_k \to r^*$ *as* $k \to +\infty$;
- $q_k = r_k e^{i\vartheta_k} \to q^* = r^* e^{i\vartheta^*}$ *as* $k \to +\infty$.

Define the classes

$$
H_{coll}^{q_k} = \{u \in H^1([0,1];\mathbb{R}^2) :\ u(0) = q_k,\ u(1) = 0,\ |u(s)| \leq r_k \text{ for every } s \in [0,1]\}
$$

and

$$
H_{coll}^{q^*} = \{u \in H^1([0,1];\mathbb{R}^2) :\ u(0) = q^*,\ u(1) = 0,\ |u(s)| \leq r^* \text{ for every } s \in [0,1]\}.
$$

If u_k is a minimizer of $\mathcal{M}_{h_k}$ in $H_{coll}^{q_k}$ for every $k \in \mathbb{N}$, then

(i) $u_k \to u^*$ *in* $H^1([0,1];\mathbb{R}^2)$;

(ii) $u_k \to u^*$ *in* $C^2([0,b];\mathbb{R}^2)$, *for every* $b < 1$.

In particular, u^* is a minimizer of $\mathcal{M}_{h^*}$ in the class of paths $H_{coll}^{q^*}$.

Proof. Fix $h < 0$, $r^* \in (0, r_{LJ})$, $h^* \in (h, 0)$ and $\vartheta^* \in \mathbb{S}^1$ and consider the sequences $(h_k)_k$, $(r_k)_k$ and $(\vartheta_k)_k$ as in the statement. For every $k \in \mathbb{N}$, consider a minimizer u_k of $\mathcal{M}_{h_k}$ in $H^{q_k}_{coll}$. For $k \in \mathbb{N}$ and for every $s \in [0,1]$, following Remark 1.4.7, we have

$$h_k + V(u_k(s)) > h + r_k^{-\alpha} U_{min} > h + r_{LJ}^{-\alpha} U_{min} = C > 0.$$

In this way, we can write

$$\inf_{H^{q_k}_{coll}} \mathcal{M}_{h_k} = \mathcal{M}_{h_k}(u_k) = \int_0^1 |\dot{u}_k|^2 \, ds \int_0^1 (h_k + V(u_k)) \, ds > C \int_0^1 |\dot{u}_k|^2 \, ds$$

and so, by Lemma 1.4.4 and the Poincaré inequality, the sequence $(u_k)_k$ is bounded in H^1 and hence $u_k \rightharpoonup u^*$ in H^1 and uniformly. So, from Fatou's lemma, we have

$$\mathcal{M}_{h^*}(u^*) \leq \liminf_{k \to \infty} \mathcal{M}_{h_k}(u_k).$$

Now, suppose by contradiction that there exists a minimizer u_{min} of $\mathcal{M}_{h^*}$ in $H^{q^*}_{coll}$ such that

$$\mathcal{M}_{h^*}(u_{min}) < \mathcal{M}_{h^*}(u^*).$$

From Corollary 1.4.9, we actually obtain that, as $k \to +\infty$

$$\mathcal{M}_{h_k}(u_k) \leq \mathcal{M}_{h^*}(u_{min}) + O(|\vartheta^* - \vartheta_k|) + O(|h^* - h_k|) + O(|r^* - r_k|)$$

and so

$$\liminf_{k \to \infty} \mathcal{M}_{h_k}(u_k) \leq \mathcal{M}_{h^*}(u_{min}),$$

which leads to a contradiction. Therefore, u^* is a minimizer. Moreover, the same argument leads to a strong convergence in H^1 assuming by contradiction that

$$\int_0^1 |\dot{u}^*|^2 \, ds < \limsup_{k \to \infty} \int_0^1 |\dot{u}_k|^2 \, ds.$$

Finally, since $r_k \to r^* > 0$, we have $\inf_k r_k > 0$ and so we can consider $B_{\tilde{r}}$ with $\tilde{r} = \frac{1}{2} \inf_k r_k$. From Lemma 1.4.3, there exists a sequence $(b_k)_k$ such that $|u_k(b_k)| = \tilde{r}$ and $0 < b_k < 1$ for every $k \in \mathbb{N}$. Defining $b = \inf_k b_k$, again from Lemma 1.4.3 we deduce that $0 < b < 1$. In this way, we obtain

$$\omega^2 \ddot{u}_k(t) = \nabla V(u_k(t)) \quad \text{for } t \in [0,b], \text{ for every } k \in \mathbb{N}$$

and thus $\nabla V(u_k)$ converges uniformly to $\nabla V(u^*)$ on $[0,b]$. This proves that u_k converges in $C^2([0,b])$. □

1.5. Proof of Theorem 1.1.2

In general, the minimizer provided by Theorem 1.4.6 is not unique. However, for a particular class of collision trajectories, we have the following result.

Lemma 1.5.1. *Assume* $(U1)$*-*$(V1)$*. Given* $h < 0$, $r^* \in (0, [-U(\vartheta^*)/h]^{1/\alpha}]$ *and taking* $q^* = r^* e^{i\vartheta^*}$*, there exists a unique minimizer for the Maupertuis' functional* $\mathcal{M}_h$ *in* $H^{q^*}_{coll}$*. This arc is nothing but the monotone* ϑ^**-homothetic collision trajectory.*

Proof. Let us define the homothetic trajectory as $u_{hom}(s) = r_{hom}(s)e^{i\vartheta^*}$, with $r_{hom}(0) = r^*$, $r_{hom}(1) = 0$ and $\dot{r}_{hom}(s) < 0$ for every $s \in (0,1)$. For every $u \in H^{q^*}_{coll}$ we can write $u(s) = r(s)e^{i\vartheta(s)}$, so that

$$\begin{aligned}\mathcal{M}_h(u) &= \int_0^1 |\dot{r}e^{i\vartheta} + ir\dot{\vartheta}e^{i\vartheta}|^2\, ds \int_0^1 (h + r^{-\alpha}U(\vartheta))\, ds \\ &\geq \int_0^1 \dot{r}^2\, ds \int_0^1 (h + r^{-\alpha}U_{min})\, ds \geq \mathcal{M}_h(u_{hom}).\end{aligned}$$

Indeed, the last inequality is strict if $r(s)$ is not monotone in $(0,1)$, otherwise we end up with an equality, since the Maupertuis' functional is invariant under time rescaling (see the proof of Theorem 1.4.2). □

In order to enlarge the set of those minimizers which are also unique, we are going to exploit the dynamical features of our problem. Therefore, we come back to our study started in Sections 1.2-1.3. From Lemma 1.3.2 and Remark 1.3.3 we have a precise characterization of the local stable manifold $\mathcal{W}^s_{loc}$ of the fixed point $(0, \vartheta^*, \vartheta^* + \pi)$. Given a starting point q, the Maupertuis' functional $\mathcal{M}_h$ does not necessarily admit a unique minimizer on the class of collision paths H^q_{coll} but, actually, this set of minimizers is in 1-1 correspondence with the set of their starting velocities. This fact is a consequence of the uniqueness of the solutions of the relative Cauchy problems and it suggests to establish a link between every minimizer and its initial velocity. In this way, we can introduce the set of the reparametrizations through McGehee's coordinates of every minimizer with starting position q as follows

$$\mathfrak{m}_h(q) = \{\gamma_\varphi :\ \text{rep}(\gamma_\varphi) \text{ minimizes } \mathcal{M}_h \text{ in } H^q_{coll},\ \text{with } \varphi \in \mathbb{S}^1\}, \tag{1.17}$$

with $\gamma_\varphi = \gamma_\varphi(\tau)$ for $\tau > 0$ and where rep(γ_φ) represents a suitable reparametrization of γ_φ. With this notation, the angle φ is nothing but the direction of the starting velocity, since its module is already fixed for the conservation of energy.

Following this preliminary discussion, we state and prove below the main result of this chapter, which is nothing but the planar unperturbed version of Theorem 1.1.2 presented in the Introduction of this paper, in a negative energy shell.

Theorem 1.5.2. *Assume* $(U1)$*-*$(V1)$*. Given* $h < 0$*, there exist* $\bar{r} > 0$ *and* $\bar{\delta} > 0$ *such that, defining*

$$Q_0 = \{q_0 = re^{i\vartheta_0} :\ r \in (0, \bar{r}) \text{ and } \vartheta_0 \in (\vartheta^* - \bar{\delta}, \vartheta^* + \bar{\delta})\},$$

and $\mathfrak{m}_h$ *as in* (1.17)*, then*

$$\mathcal{W}^s_{loc}(\bar{r},\bar{\delta}) = \bigcup_{q_0\in Q_0} \mathfrak{m}_h(q_0). \tag{1.18}$$

Moreover, for every $q_0 \in Q_0$*, the Maupertuis' functional* $\mathcal{M}_h$ *admits a unique minimizer in* $H^{q_0}_{coll}$.

Proof. Fix $h < 0$ and, adopting the notations of Lemma 1.3.2 and Lemma 1.4.3, choose

$$\bar{r} = \min\{r_{loc}, r_{LJ}\}.$$

We start with showing that the uniqueness of a minimizer follows from inclusion ($\supseteq$) in (1.18). Indeed, Lemma 1.3.2 guarantees that $\mathcal{W}^s_{loc}$ is the graph of a $\mathcal{C}^2$-function, which associates to every initial position $q_0 \in Q_0$ a unique initial velocity. Therefore, once q_0 is fixed, there necessarily exists a unique minimal arc solving the correspondent fixed-end collision problem.

Inclusion ($\supseteq$): Take $q_0 = re^{i\vartheta_0}$, with $r \in (0,\bar{r})$ and $\vartheta_0 \in (\vartheta^* - \delta_{loc}, \vartheta^* + \delta_{loc})$ (as in Lemma 1.3.2). Therefore, by Theorem 1.4.6, there exists $\gamma = \gamma_{\varphi_0} \in \mathfrak{m}_h(q_0)$ for some $\varphi_0 \in \mathbb{S}^1$. Our goal is to show that, up to make δ_{loc} smaller, γ is entirely contained in the local stable manifold $\mathcal{W}^s_{loc}$. The orbit $\gamma(\tau) = (r(\tau), \vartheta(\tau), \varphi(\tau))$ solves (1.10) in $(0,+\infty)$ so that, by the definition of stable manifold, we have

$$\gamma \in \mathcal{W}^s \iff \begin{cases} r(\tau) \to 0 & \text{as } \tau \to +\infty \\ \vartheta(\tau) \to \vartheta^* & \text{as } \tau \to +\infty \\ \varphi(\tau) \to \varphi^* = \vartheta^* + \pi & \text{as } \tau \to +\infty, \end{cases} \tag{1.19}$$

but also, from Lemma 1.3.2

$$\gamma \in \mathcal{W}^s_{loc} \iff \text{for every } \tau > 0,\ \varphi(\tau) = \Psi(r(\tau), \vartheta(\tau)).$$

Assume by contradiction that there exists $(\vartheta_k(0))_k \subseteq \mathbb{S}^1$ such that

$$q_k = re^{i\vartheta_k(0)} \to q^* = re^{i\vartheta^*} \quad \text{as } k \to +\infty,$$

but there exists a sequence of reparametrized minimizers $(\gamma_k)_k \subseteq (\mathfrak{m}_h(q_k))_k$ such that

$$\gamma_k = \{\gamma_k(\tau) = (r_k(\tau), \vartheta_k(\tau), \varphi_k(\tau)) :\ \tau \geq 0\} \not\subseteq \mathcal{W}^s_{loc}, \quad \text{for every } k \in \mathbb{N}.$$

Notice that, from Lemma 1.4.10 and Lemma 1.5.1, the sequence $(\gamma_k)_k$ converges in $H^1([0,1];\mathbb{R}^2)$, and thus uniformly in $[0,1]$, to the ϑ^*-homothetic motion. For this reason, we can split our proof in two situations, whose discriminant is the uniform convergence of the sequence $(\vartheta_k)_k = (\vartheta_k(\tau))_k$. Indeed, despite the convergence of $(\gamma_k)_k$, for instance it could happen that the sequence of angle functions $(\vartheta_k)_k$ starts to oscillate dramatically as k goes to $+\infty$.

Case 1: Assume that $\gamma_k \not\subseteq \mathcal{W}^s_{loc}$ for infinitely many k and that

$$\lim_{k\to+\infty} \sup_{\tau\geq 0} |\vartheta_k(\tau) - \vartheta^*| = 0.$$

By (1.19), we necessarily have that there exists $\bar{\varepsilon} > 0$ such that

$$\limsup_{k\to+\infty} \sup_{\tau\geq 0} |\varphi_k(\tau) - \varphi^*| = \bar{\varepsilon} > 0.$$

In this way, up to subsequences, from the definition of *sup* and *limsup* we can find a sequence $(\tau_k)_k \subseteq [0, +\infty)$ such that

$$|\varphi_k(\tau_k) - \varphi^*| \geq \frac{\bar{\varepsilon}}{2}, \quad \text{for every } k \in \mathbb{N}. \tag{1.20}$$

Now, we perform the following change of variables and time shifting

$$\begin{cases} \tilde{r}_k(\tau) = \frac{r}{r_k(\tau_k)} r_k(\tau + \tau_k) \\ \tilde{\vartheta}_k(\tau) = \vartheta_k(\tau + \tau_k) \\ \tilde{\varphi}_k(\tau) = \varphi_k(\tau + \tau_k) \end{cases}$$

and, if we define $\lambda_k = r_k(\tau_k)/r \leq 1$, we have that the orbit $\tilde{\gamma}_k = (\tilde{r}_k(\tau), \tilde{\vartheta}_k(\tau), \tilde{\varphi}_k(\tau))$ for $\tau \geq 0$ solves the system

$$\begin{cases} \tilde{r}_k' = 2\tilde{r}_k(U(\tilde{\vartheta}_k) + h_k \tilde{r}_k^\alpha) \cos(\tilde{\varphi}_k - \tilde{\vartheta}_k) \\ \tilde{\vartheta}_k' = 2(U(\tilde{\vartheta}_k) + h_k \tilde{r}_k^\alpha) \sin(\tilde{\varphi}_k - \tilde{\vartheta}_k) \\ \tilde{\varphi}_k' = U'(\tilde{\vartheta}_k) \cos(\tilde{\varphi}_k - \tilde{\vartheta}_k) + \alpha U(\tilde{\vartheta}_k) \sin(\tilde{\varphi}_k - \tilde{\vartheta}_k), \end{cases}$$

where $h_k = \lambda_k^\alpha h$ and so $h_k \in [h, 0)$. Now, denoting by x_k the reparametrization of the trajectory γ_k in time t in the configurations space, x_k solves the problem

$$\begin{cases} \ddot{x}_k = \nabla V(x_k) \\ \frac{1}{2}|\dot{x}_k|^2 - V(x_k) = h, \end{cases}$$

for every $k \in \mathbb{N}$. Therefore, the function

$$\tilde{x}_k(t) = \frac{x_k(t_k + \lambda_k^{1+\alpha/2} t)}{\lambda_k},$$

with t_k such that $\tilde{x}_k(0) = x_k(0)$, will solve the problem

$$\begin{cases} \frac{d^2}{dt^2}\tilde{x}_k(t) = \nabla V(\tilde{x}_k(t)) \\ \frac{1}{2}|\frac{d}{dt}\tilde{x}_k(t)|^2 - V(\tilde{x}_k(t)) = h_k, \end{cases}$$

for every $k \in \mathbb{N}$. Hence, under a suitable change of scales, there exists a sequence $(\tilde{u}_k)_k$ of minimizers of the Maupertuis' functional, with starting point respectively in $(\tilde{q}_k)_k$ such that

$$\tilde{q}_k = \tilde{r}_k(0)e^{i\tilde{\vartheta}_k(0)} = re^{i\vartheta_k(\tau_k)} \to q^* = re^{i\vartheta^*} \quad \text{as } k \to +\infty.$$

For Lemma 1.4.10, we have that such a sequence of minimizers converges in H^1 and thus uniformly to a minimal arc connecting q^* to the origin. On the other hand, from (1.20) we have that $\tilde{\varphi}_k(0) = \varphi_k(\tau_k) \not\to \varphi^* = \vartheta^* + \pi$ as $k \to +\infty$. This means that the limit arc cannot be the ϑ^*-homothetic trajectory, which is impossible for Proposition 1.5.1.

Case 2: Assume that $\gamma_k \notin \mathcal{W}^s$ for infinitely many k and that there exists $\bar{\varepsilon} > 0$ such that

$$\limsup_{k\to+\infty} \sup_{\tau\geq 0} |\vartheta_k(\tau) - \vartheta^*| = \bar{\varepsilon}. \tag{1.21}$$

Hence, up to subsequences, there exists a sequence $(\tau_k)_k \subseteq [0,\infty)$ such that

$$|\vartheta_k(\tau_k) - \vartheta^*| \geq \frac{\bar{\varepsilon}}{2}, \quad \text{for every } k \in \mathbb{N}. \tag{1.22}$$

Moreover, since from Lemma 1.4.10 and Proposition 1.5.1 the sequence $(\gamma_k)_k$ converges C^2 to the homothetic motion on every bounded interval, we necessarily deduce that $\tau_k \to +\infty$ as $k \to +\infty$. In particular, since every γ_k is a collision arc, again from Lemma 1.4.10 we have that

$$r_k(\tau_k) \to 0, \quad \text{as } k \to +\infty. \tag{1.23}$$

Now, for every $k \in \mathbb{N}$ define the orbit $\tilde{\gamma}_k = (\tilde{r}_k, \tilde{\vartheta}_k, \tilde{\varphi}_k)$ such that

$$\begin{cases} \tilde{r}_k(\tau) = r_k(\tau + \tau_k) \\ \tilde{\vartheta}_k(\tau) = \vartheta_k(\tau + \tau_k) \\ \tilde{\varphi}_k(\tau) = \varphi_k(\tau + \tau_k), \end{cases}$$

for $\tau \in [-\tau_k, +\infty)$. We have that, for every $k \in \mathbb{N}$, $\tilde{\gamma}_k$ verifies equations

$$\begin{cases} \tilde{r}_k' = 2\tilde{r}_k(U(\tilde{\vartheta}_k) + h\tilde{r}_k^\alpha)\cos(\tilde{\varphi}_k - \tilde{\vartheta}_k) \\ \tilde{\vartheta}_k' = 2(U(\tilde{\vartheta}_k) + h\tilde{r}_k^\alpha)\sin(\tilde{\varphi}_k - \tilde{\vartheta}_k) \\ \tilde{\varphi}_k' = U'(\tilde{\vartheta}_k)\cos(\tilde{\varphi}_k - \tilde{\vartheta}_k) + \alpha U(\tilde{\vartheta}_k)\sin(\tilde{\varphi}_k - \tilde{\vartheta}_k). \end{cases}$$

Since $\tau_k \to +\infty$, we have that, for every $T > 0$, there exists $\bar{k} \in \mathbb{N}$ such that, for every $k \geq \bar{k}$

$$\tau_k > T.$$

Let us fix $T > 0$. For $\tau \in [-T, T]$ and for every $k \geq \bar{k}$ we have

$$\tau + \tau_k \in [0, +\infty)$$

and so

$$\tilde{r}_k(\tau) = r_k(\tau + \tau_k) \leq r_k(0) < \bar{r}.$$

Moreover,

$$\tilde{\vartheta}_k(\tau), \tilde{\varphi}_k(\tau) \in \mathbb{S}^1$$

and

$$|\tilde{r}_k'(\tau)| \leq 2\bar{r}(U_{max} - h\bar{r}^{\alpha}) = C < +\infty$$

and, with analogous calculations, the same holds for $\tilde{\vartheta}_k'(\tau)$ and $\tilde{\varphi}_k'(\tau)$. From the Ascoli-Arzelà theorem, we have that

$$(\tilde{r}_k, \tilde{\vartheta}_k, \tilde{\varphi}_k) \to (\tilde{r}, \tilde{\vartheta}, \tilde{\varphi}) \quad \text{as } k \to +\infty$$

uniformly on $[-T, T]$. Moreover, from (1.23) we deduce that

$$-\tilde{r}_k^{\alpha}(\tau)h \leq -\tilde{r}_k^{\alpha}(0)h = -r_k^{\alpha}(\tau_k)h \to 0 \quad \text{as } k \to +\infty.$$

This, together with the uniform convergence, implies that $(\tilde{r}, \tilde{\vartheta}, \tilde{\varphi})$ satisfy the equations

$$\begin{cases} \tilde{r}' = 2\tilde{r}U(\tilde{\vartheta})\cos(\tilde{\varphi} - \tilde{\vartheta}) \\ \tilde{\vartheta}' = 2U(\tilde{\vartheta})\sin(\tilde{\varphi} - \tilde{\vartheta}) \\ \tilde{\varphi}' = U'(\tilde{\vartheta})\cos(\tilde{\varphi} - \tilde{\vartheta}) + \alpha U(\tilde{\vartheta})\sin(\tilde{\varphi} - \tilde{\vartheta}), \end{cases} \tag{1.24}$$

on $[-T, T]$. Repeating the same argument for every $T > 0$, we have that the sequences converges uniformly on every compact of $\mathbb{R}$, with limit defined and verifying (1.24) on the whole $\mathbb{R}$. Moreover, let us notice that

$$\tilde{r}_k(0) = r_k(\tau_k) \to 0 = \tilde{r}(0) \quad \text{as } k \to +\infty$$

and so, for the uniqueness of the solution of a Cauchy problem, we actually deduce that $\tilde{r}(\tau) \equiv 0$. This means that the solution of (1.24) is actually a motion on the collision manifold $\{r = 0\}$.

Let us now investigate the asymptotic behaviour of $\tilde{\vartheta}$. From the non-degeneracy of ϑ^*, it is not restrictive to assume that $\bar{\varepsilon} = d/2$ in (1.21), where

$$d \doteq \min\{|\vartheta^* - \hat{\vartheta}| : \hat{\vartheta} \in \mathbb{S}^1, \ U'(\hat{\vartheta}) = 0, \ \hat{\vartheta} \neq \vartheta^*\}.$$

In this way, from (1.22), we can deduce that for every $k \in \mathbb{N}$

$$|\vartheta_k(\tau) - \vartheta^*| < \frac{d}{2}, \quad \text{for every } \tau \in [0, \tau_k)$$

and so, in other words, for every $k \in \mathbb{N}$

$$|\tilde{\vartheta}_k(\tau) - \vartheta^*| < \frac{d}{2}, \quad \text{for every } \tau \in [-\tau_k, 0).$$

From the convergence of $\tilde{\vartheta}_k$ to $\tilde{\vartheta}$ we can easily deduce that

$$|\tilde{\vartheta}(\tau) - \vartheta^*| \leq \frac{d}{2}, \quad \text{for every } \tau \in (-\infty, 0). \tag{1.25}$$

Now, it is known (see [29, 11]) that

$$\lim_{\tau \to \pm\infty} \tilde{\vartheta}(\tau) = \vartheta^{\pm}, \quad \lim_{\tau \to \pm\infty} \tilde{\varphi}(\tau) = \vartheta^{\pm} + \pi,$$

with $\vartheta^{\pm}$ central configuration for U. Assume by contradiction that $\vartheta^- \neq \vartheta^*$, i.e. that for every ε there exists $\tau_\varepsilon \in \mathbb{R}$ such that

$$\tau < \tau_\varepsilon \implies |\tilde{\vartheta}(\tau) - \vartheta^-| < \varepsilon.$$

Choosing $\varepsilon = d/4$ and using (1.25) we head to a contradiction and so necessarily $\vartheta^- = \vartheta^*$.

To conclude the proof, consider the function

$$v(\tau) = \sqrt{U(\tilde{\vartheta}(\tau))}\cos(\tilde{\varphi}(\tau) - \tilde{\vartheta}(\tau)),$$

which is non-decreasing and non-constant on every solution of (1.24) (see Theorem 4, [11]), so that

$$-\sqrt{U_{min}} = -\sqrt{U(\vartheta^*)} = \lim_{\tau \to -\infty} v(\tau) < \lim_{\tau \to +\infty} v(\tau) = -\sqrt{U(\vartheta^+)}.$$

This is clearly a contradiction since ϑ^* is a global minimal central configuration for U.

Inclusion ($\subseteq$): From any $(r, \vartheta_0, \varphi_0) \in \mathcal{W}^s_{loc}$ it starts a unique orbit that ends in the equilibrium point $(0, \vartheta^*, \vartheta^* + \pi)$, which connects the point $q_0 = re^{i\vartheta_0}$ to the origin in the configuration space. This is nothing but a reparametrization of a minimal fixed-end arc: indeed, a minimizer from q_0 to the origin exists by means of Theorem 1.4.6 and it is unique, as we have already shown. □

1.6. General setting

This final section collects some useful remarks for the adaptation of the previous proof in the general setting presented in the introduction. This material is mainly thought to ease the reader's comprehension, for it will be clear that the argument used in the proof is exactly the same. We made the choice to split the generalization in two sub-cases. In the first one, we take into account a perturbed potential and a conservative system with possibly non-negative energy. The second one is focused on the higher dimensional case.

1.6.1. $d = 2$, $W \not\equiv 0$, $h \in \mathbb{R}$

As a first step, we set equation (1.1) again in the plane ($d = 2$), but we perturb our potential V exactly as stated in ($V0$). Moreover, we wish to work also in non-negative energy shells, so that equation (1.2) will be given with $h \in \mathbb{R}$. Using polar coordinates (r, ϑ) and adopting the same argument as in Section 1.2, we find an analogous of system (1.10) in our actual setting, i.e., the dynamical system

$$\begin{cases} r' = 2r(U(\vartheta) + r^\alpha W(r,\vartheta) + hr^\alpha)\cos(\varphi - \vartheta) \\ \vartheta' = 2(U(\vartheta) + r^\alpha W(r,\vartheta) + hr^\alpha)\sin(\varphi - \vartheta) \\ \varphi' = (U'(\vartheta) - r^\alpha W_\vartheta)\cos(\varphi - \vartheta) + (\alpha U(\vartheta) + r^{\alpha+1} W_r)\sin(\varphi - \vartheta) \end{cases}, \tag{1.26}$$

where we have set
$$W_r = \frac{\partial W}{\partial r}(r,\vartheta), \quad W_\vartheta = \frac{\partial W}{\partial \vartheta}(r,\vartheta).$$
It is easy to notice that the collision manifold $\{r = 0\}$ induced by (1.26) is nothing but the one described by (1.11) in Section 1.2, regardless of the sign of h. This fact also implies that the dynamical systems (1.26) and (1.10) share not only the same equilibrium points $(0, \vartheta^*, \vartheta^*+k\pi)$, but also the same linearisation. As a consequence, with the help of minor changes, the dynamical characterization provided in Section 1.3 naturally extends to the setting considered above and it is possible to reformulate Lemma 1.3.2.

On second thought, basically all the proofs contained in Section 1.4 strongly depend on the *Lagrange-Jacobi inequality* (Lemma 1.4.3) and its consequences. In particular, when $W \equiv 0$, the $-\alpha$-homogeneity of V and the convexity of the inertial moment allow us to provide all the useful (upper or lower) estimates on the term $h + V$. Again, this can be reset in our new framework, since the hypotheses $(V0)$ on the perturbation W tell us that we can recover a $-\alpha$-homogeneity on V when r is sufficiently small. Indeed, the term $r^\alpha W + r^{\alpha+1}|\nabla W| \to 0$ as $r \to 0$, so that, eventually choosing a smaller r_{LJ} in Lemma 1.4.3, we can carry out again the entire argument.

Finally, we want to remark that a complementary choice of $h \geq 0$ is not dramatic in this setting. In particular, if $W \equiv 0$, this will induce the choice $r_{LJ} = +\infty$ and thus the presence of an infinite Hill's region, as expected in a parabolic or hyperbolic problem. On the other hand, if $W \not\equiv 0$, we could still have a bound on r_{LJ}, depending on the sign of W close to the singularity.

1.6.2. $d > 2$, $W \not\equiv 0$, $h \in \mathbb{R}$

In this higher dimensional setting, the construction presented in Sections 1.2-1.3 needs to be properly modified, in order to take into account the more abstract nature of this case. We want to make once more clear that the variational approach of Section 1.4-1.5 is not affected by taking into account higher dimensions. Moreover, since the discussion of the previous paragraph on the lower order perturbations does not change for $d > 2$, we will assume $W \equiv 0$. In order to face the dynamical complications, we will basically adopt the technique introduced by R. McGehee in [52] (see also [25, 29, 12]) in order to sketch a proof for Lemma 1.1.1. As a starting point, for $x = x(t) \in \mathbb{R}^d$, introduce the new variables
$$\begin{cases} r(t) = |x(t)| \\ s(t) = r(t)^{-1}x(t) \\ v(t) = r(t)^{\alpha/2}\langle \dot{x}(t), s(t)\rangle \\ u(t) = r(t)^{\alpha/2}\pi_{T_s\mathbb{S}^{d-1}}\dot{x}(t), \end{cases}$$
where $\pi_{T_s\mathbb{S}^{d-1}}$ represents the orthogonal projection on the tangent space of $\mathbb{S}^{d-1}$, i.e.,
$$\pi_{T_s\mathbb{S}^{d-1}} z = z - \langle z, s\rangle s, \quad \text{for every } z \in \mathbb{R}^d.$$
To have an intuitive description the coordinates (r, s, v, u), we point out that (r, s) is the expression in polar coordinates of the position x, while (v, u) is the decomposition of

the velocity $\dot{x}$ in the tangent bundle $T\mathbb{S}^{d-1}$ (cf. Lemma 1.1.1). In this way, slowing down the time with the time-rescaling

$$dt = r^{1+\alpha/2} d\tau,$$

solutions of (1.1) will be equivalent to solutions of

$$\begin{cases} r' = rv \\ v' = \frac{\alpha}{2} v^2 + |u|^2 - \alpha V(s) \\ s' = u \\ u' = -\frac{2-\alpha}{\alpha} vu - |u|^2 s + \nabla_T V(s), \end{cases} \tag{1.27}$$

where the homogeneity gives $V(x) = r^{-\alpha} V(s)$ and

$$\nabla_T V(s) = \nabla V(s) - \langle \nabla V(s), s \rangle s = \pi_{T_s \mathbb{S}^{d-1}} \nabla V(s)$$

is commonly known as the tangential gradient of V. The conservation of energy law (1.2) in this variables translates to

$$\frac{1}{2} \left(|u|^2 + v^2 \right) - V(s) = r^\alpha h,$$

and defines the energy shell

$$\mathcal{H}_h = \left\{ (r, v, s, u) \in (0, +\infty) \times \mathbb{R} \times \mathbb{S}^{d-1} \times T_s \mathbb{S}^{d-1} : \frac{1}{2} (|u|^2 + v^2) - V(s) = r^\alpha h \right\}.$$

In $\mathcal{H}_h$ the variable v reads

$$v^\pm(r, s, u) = \pm \sqrt{2(V(s) + r^\alpha h) - |u|^2}.$$

The choice of v^-/v^+ corresponds to the choice of studying in/outgoing trajectories to/from the singularity $r = 0$. Indeed, equations (1.27) can be reduced to a (r, s, u)-system, admitting $\{r = 0\}$ as an invariant set. We denote by Λ such a set, which is commonly known as *collision manifold*, which actually is a smooth manifold of dimension $2d - 2$ (see [29, Proposition 1, pag.234]). Since we are interested in collision trajectories, we will take into account v^-, so that in Λ system (1.27) reads

$$\begin{cases} s' = u \\ u' = \frac{2-\alpha}{\alpha} u \sqrt{2V(s) - |u|^2} - |u|^2 s + \nabla_T V(s). \end{cases} \tag{1.28}$$

From the linearisation of (1.28), it is possible to deduce the hyperbolicity of the equilibrium points of the (r, s, u)-system

$$p^* = (0, s^*, 0) \quad \text{such that } \nabla_T V(s^*) = 0,$$

as long as V is a Morse function (see [29, Proposition 4, pag. 237]). This leads to the existence of stable and unstable manifolds $\mathcal{W}^S$ and $\mathcal{W}^U$ for p^*, with $\dim \mathcal{W}^S + \dim \mathcal{W}^U = 2d - 1$. Again for our purpose of studying ingoing collision orbits, we naturally choose the r-eigenvalue to be negative so that (still following [29], Lemma 5, pag.238) we infer that $\dim \mathcal{W}^S = d$ while $\dim \mathcal{W}^U = d - 1$.

The hyperbolicity of p^* gives rise to a local description of the manifold $\mathcal{W}^S$ as the graph of a $\mathcal{C}^2$-function in the variables (r, s) (see [66], Theorem 7.3).

2. Symbolic dynamics for the anisotropic N-centre problem

2.1. Introduction and main results

We consider the planar N-centre problem of Celestial Mechanics, in which we associate to every centre a non-radial anisotropic potential. In order to do that, we name the position of the $N \geq 2$ centres by $c_1, \ldots, c_N \in \mathbb{R}^2$ and we introduce a finite family of non-negative singular functions $V_1, \ldots, V_N \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\})$. Moreover, we require every function V_j to be $-\alpha_j$-homogeneous so that we will consider the most general anisotropic behaviour for our system, assuming so far that $\alpha_j \in (0, 2)$ (stricter, though fundamental assumptions will be added later). In this way, for every $x \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$, our model will be driven by the total potential

$$V(x) = \sum_{j=1}^{N} V_j(x - c_j) = \sum_{j=1}^{N} |x - c_j|^{-\alpha_j} V_j \left(\frac{x - c_j}{|x - c_j|} \right),$$

recalling that any $-\alpha_j$-homogeneous function V_j easily verifies

$$V_j(y) = V_j \left(|y| \frac{y}{|y|} \right) = |y|^{-\alpha_j} V_j \left(\frac{y}{|y|} \right),$$

whenever $y \neq 0$. Clearly, $V \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\})$ and the equation of motion is the following

$$\ddot{x}(t) = \nabla V(x(t)), \tag{2.1}$$

where $x = x(t)$ represents the position of the moving particle at time $t \in \mathbb{R}$. Without loss of generality we can assume

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_N,$$

thus admitting that more than one centre might have the same homogeneity degree. As it will be clear in the following, the smallest degree of homogeneity α_1 plays an important role in our treatment. For instance, if we assume that $\alpha_1 = \alpha_2 = \ldots = \alpha_k$ for some $1 \leq k < N$, it is useful to put $\alpha \doteq \alpha_1$ and to gather all the $-\alpha$-homogeneous potential in this way

$$W(x) \doteq \sum_{i=1}^{k} V_i(x - c_i) = \sum_{i=1}^{k} |x - c_i|^{-\alpha} V_i \left(\frac{x - c_i}{|x - c_i|} \right),$$

so that $W \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{c_1, \dots, c_k\})$ and

$$V(x) = W(x) + \sum_{j=k+1}^{N} |x - c_j|^{-\alpha_j} V_j\left(\frac{x - c_j}{|x - c_j|}\right). \tag{2.2}$$

Note that, if $k = N$, the problem becomes much simpler because we would have a unique homogeneity degree. In particular, when $N = 1$ we end up with an anisotropic Kepler problem driven by a $-\alpha$-homogeneous potential. This is exactly the situation studied in [11] in which the authors provided non-collision Bolza minimizers with a blow-up analysis.

At this point, let us present the rigorous hypotheses that we will require on the potentials that characterize this problem. For every $j = 1, \dots, N$, we introduce the restriction to the 1-dimensional sphere of every potential V_j, defining $U_j \doteq V_j|_{\mathbb{S}^1}$ and let also

$$U(\vartheta) \doteq \sum_{i=1}^{k} U_i(\vartheta), \quad \text{for } \vartheta \in \mathbb{S}^1.$$

Clearly $U_1, \dots, U_N, U \in \mathcal{C}^2(\mathbb{S}^1)$, so that we can finally state our precise hypotheses on V in this way:

$$(V) \qquad \begin{cases} \alpha < 2; \\ \exists\, (\vartheta_l^*)_{l=0}^{m-1} \subseteq \mathbb{S}^1 \ : \ \forall\, l = 0, \dots, m-1,\ m > 0,\ U''(\vartheta_l^*) > 0,\ U(\vartheta) \geq U(\vartheta_l^*) > 0,\ \forall\, \vartheta \in \mathbb{S}^1; \\ \forall\, j = 1, \dots, N\ \exists\, \vartheta_j \in \mathbb{S}^1 : U_j(\vartheta) \geq U_j(\vartheta_j) > 0,\ \forall\, \vartheta \in \mathbb{S}^1,\ U_j''(\vartheta_j) > 0; \\ \forall\, j = 1, \dots, N\ \alpha_j > \bar{\alpha}_j(U_j, \vartheta_j). \end{cases}$$

Remark 2.1.1. *The previous requirements on V are referred to the strength of the homogeneities α_j and to the extremality with respect to U_j of some particular directions. To be precise, the assumptions $(V)_1$ and $(V)_4$ require some thresholds on the homogeneity degrees, which will play a fundamental role in the dynamics both close and far from the centres. Moreover, recalling that a central configuration for a potential is a critical point of its restriction to the unit sphere, $(V)_2$ and $(V)_3$ guarantee the existence of a finite number of strictly minimal central configurations for every potential U_j, but also for the sum of the $-\alpha$-homogeneous potentials $U_1, \dots, U_k$.*

A *non-collision solution* of (2.1) is a function $x \colon J \subseteq \mathbb{R} \to \mathbb{R}^2$ such that $x(t) \neq c_j$ for every $t \in J$ and for every $j = 1, \dots, N$ and that solves (2.1) in the classical sense. Given $h > 0$, we are interested in those non-collision solutions of equation (2.1) which are confined in the 3-dimensional negative energy shell

$$\mathcal{E}_h = \left\{ (x, v) \in (\mathbb{R}^2 \setminus \{c_1, \dots, c_N\}) \times \mathbb{R}^2 : \ \frac{1}{2}|v|^2 - V(x) = -h \right\}$$

and thus, every solution of (2.1) should verify the energy conservation law

$$\frac{1}{2}|\dot{x}|^2 - V(x) = -h. \tag{2.3}$$

Moreover, equation (2.1) has a Hamiltonian structure in $(\mathbb{R}^2 \setminus \{c_1, \dots, c_N\}) \times \mathbb{R}^2$, driven by the Hamiltonian function

$$H(x, v) = \frac{1}{2}|v|^2 - V(x),$$

which coincides with the total energy once (2.1) is written as the first order autonomous system $\dot{z} = F(z)$, where

$$z = \begin{pmatrix} x \\ v \end{pmatrix} \quad F(z) = \begin{pmatrix} v \\ \nabla V(x) \end{pmatrix}.$$

The conservation of the energy also implies that such solutions will be confined to the *Hill's region*

$$\mathcal{R}_h = \{x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} : V(x) \geq h\}. \tag{2.4}$$

Remark 2.1.2. *From now on, without loss of generality, we will assume that* $\max_j |c_j| \leq 1$ *and we define*

$$\mathfrak{m} = \min_{j=1,\dots,N} \min_{\mathbb{S}^1} U_j.$$

Then, for $x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\}$ *we have*

$$V(x) \geq \mathfrak{m} \sum_{j=1}^{N} |x - c_j|^{-\alpha_j} \geq \frac{\mathfrak{m}}{(|x| + 1)^\alpha}.$$

Then, if we fix $h > 0$*, those* x *such that* $|x| \leq (\mathfrak{m}/h)^{1/\alpha} - 1$ *necessarily belong to the Hill's region* $\mathcal{R}_h$ *associated with such* h*. We can argue in this way to put a bound on* h *in order to rule out those* h *for which* $\mathcal{R}_h = \emptyset$*. To have a well-posed problem, we need to require that every centre lies inside* $\mathcal{R}_h$ *and so that the moving particle can reach at least every region (except for collisions) of the ball* B_1*. This is surely the case if* $(\mathfrak{m}/h)^{1/\alpha} - 1 > 1$ *(see Figure 2.1); this means that our problem makes sense at least for those energies* h *such that*

$$0 < h < \frac{\mathfrak{m}}{2^\alpha} \doteq \tilde{h}.$$

For this reason, from now on we will always assume $h \in (0, \tilde{h})$*. This is actually really natural in our approach to this problem, since later in this paper we will make use of perturbation methods, which work fine only when the particle is very far from the centres. Indeed, as for the classical Kepler problem, small negative energies allow the particle to reach regions farther from the singular set.*

In order to state our main result, we need to introduce some further notation. Following [61], we consider all the possible partitions of the N centres in two disjoint non-empty and non-ordered sets, which are exactly

$$\frac{1}{2}\left(\binom{N}{1} + \dots + \binom{N}{N-1}\right) = \frac{1}{2}\left(\sum_{k=0}^{N}\binom{N}{k} - 2\right) = 2^{N-1} - 1,$$

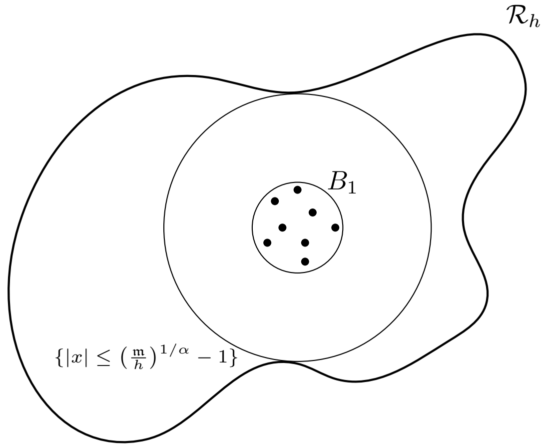

Figure 2.1.: An example of Hill's region for the anisotropic N-centre problem that includes a ball of radius greater than 1 (see Remark 2.1.2).

and we denote the set of these partitions as

$$\mathcal{P} = \{P_j : \ j = 0, \dots, 2^{N-1} - 2\}.$$

Due to the non-radial nature of the potential which describes our system, we need to take into account the non-degenerate minimal central configurations of the $-\alpha$-homogeneous component of V. Indeed, they will play a fundamental role in the construction of a periodic solution for (2.1) as a peculiarity of anisotropic problems (see [12, 11, 5]). We define the finite set (see (V))

$$\Xi \doteq \{\vartheta^* \in \mathbb{S}^1 : \ U'(\vartheta^*) = 0 \text{ and } U''(\vartheta^*) > 0\} = \{\vartheta_0^*, \dots, \vartheta_{m-1}^*\}$$

and we associate to every central configuration in Ξ a partition of $\mathcal{P}$. In this way, we collect together all the possible choices in the set

$$\mathcal{Q} = \{Q_j : \ j = 0, \dots, m(2^{N-1} - 1) - 1\}.$$

Moreover, define the following subsets

$$\begin{aligned}
\mathcal{P}_0 &= \{Q_0, \dots, Q_{m-1}\} = \{Q_{0\cdot m+0}, Q_{0\cdot m+1}, \dots, Q_{0\cdot m+(m-1)}\} \\
\mathcal{P}_1 &= \{Q_m, \dots, Q_{2m-1}\} = \{Q_{1\cdot m+0}, Q_{1\cdot m+1}, \dots, Q_{1\cdot m+(m-1)}\} \\
&\ \vdots \\
\mathcal{P}_l &= \{Q_{lm}, \dots, Q_{(l+1)m-1}\} = \{Q_{l\cdot m+0}, Q_{l\cdot m+1}, \dots, Q_{l\cdot m+(m-1)}\} \\
&\ \vdots \\
\mathcal{P}_{2^{N-1}-2} &= \{Q_{(2^{N-1}-2)m}, \dots, Q_{(2^{N-1}-1)m-1}\} \\
&= \{Q_{(2^{N-1}-2)\cdot m+0}, Q_{(2^{N-1}-2)\cdot m+1}, \dots, Q_{(2^{N-1}-2)\cdot m+(m-1)}\}
\end{aligned}$$

where $\mathcal{P}_l$ represents all the possible combinations of the partition P_l of the centres and a central configuration chosen in Ξ and, of course, $\mathcal{Q} = \mathcal{P}_0 \cup \ldots \cup \mathcal{P}_{2^{N-1}-2}$. In this way, it is easy to verify that, for $j \in \{0, \ldots, m(2^{N-1}-1)-1\}$, if $j \equiv r \mod m$ for $r \in \{0, \ldots, m-1\}$ it means that $Q_j \in \mathcal{P}_l$, with $l = (j-r)/m$. In particular $Q_j = Q_{lm+r}$, so that, in other words, Q_j *represents* the pair (P_l, ϑ_r^*).

Remark 2.1.3. *We will show that our system has a symbolic dynamics and that the alphabet of symbols will be exactly the set $\mathcal{Q}$. For this reason, in order to have a non-trivial symbolic dynamics, we need to have at least 2 elements in $\mathcal{Q}$, so that we will assume*

$$N \geq 3, \ m \geq 1 \quad \textit{or} \quad N \geq 2, \ m \geq 2.$$

Moreover, for $n \in \mathbb{N}$ and $(Q_{j_1}, \ldots, Q_{j_n}) \in \mathcal{Q}^n$, consider the element Q_{j_k} for some $k \in \{1, \ldots, n\}$. It is useful to introduce the quotient and the remainder of the division of j_k by m in this way

$$j_k = l_k m + r_k, \tag{2.5}$$

so that the element Q_{j_k} will refer to the partition P_{l_k} and the central configuration $\vartheta_{r_k}^*$. Note that the symbol Q_{j_k} will reflect the geometrical behaviour of solution arc: according to (2.5)*, the corresponding arc will divide the centres realizing the partition P_{l_k} and then, when it travels far from the centres, it will pass close to the central configuration ϑ_{r_k}.*

Our main result is to prove the existence of periodic solutions of (2.1) in negative energy shells (see Figure 2.2).

Theorem 2.1.4. *Assume that $N \geq 3$ and $m \geq 1$ or, equivalently, $N \geq 2$ and $m \geq 2$. Consider a function V satisfying* (V)*. There exists $\bar{h} > 0$ such that, for every $h \in (0, \bar{h})$, $n \in \mathbb{N}_{\geq 1}$ and $(Q_{j_0}, \ldots, Q_{j_{n-1}}) \in \mathcal{Q}^n$, there exists a periodic non-collision and self-intersection-free solution $x = x(Q_{j_0}, \ldots, Q_{j_{n-1}}; h)$ of* (2.1) *satisfying* (2.3)*, which depends on $(Q_{j_0}, \ldots, Q_{j_{n-1}})$ in this way: there exists $\bar{R} = \bar{R}(h) > 0$ such that the solution x crosses $2n$ times the circle $\partial B_{\bar{R}}$ in one period, at times $(t_k)_{k=0,\ldots,2n-1}$, in such a way that, according to* (2.5)*:*

- *in the interval (t_{2k}, t_{2k+1}) the solution stays outside $B_{\bar{R}}$ and there exists a neighbourhood $\mathcal{U}_{r_k} = \mathcal{U}(\bar{R}e^{i\vartheta_{r_k}^*})$ such that*
$$x(t_{2k}), x(t_{2k+1}) \in \mathcal{U}_{r_k};$$
- *in the interval (t_{2k+1}, t_{2k+2}) the solution stays inside $B_{\bar{R}}$ and separates the centres according to the partition P_{l_k}.*

As a consequence of the above theorem, we will prove that the dynamical system considered admits a symbolic dynamics. In order to state this result, we need to introduce some general notations on symbolic dynamics. Consider a finite set S with at least two elements and introduce therein the discrete metric $\rho(s_j, s_k) = \delta_{jk}$, where δ_{jk} is the Kronecker delta and $s_j, s_k \in S$. Consider the set of bi-infinite sequences of elements of S

$$S^{\mathbb{Z}} \doteq \{(s_m)_{m\in\mathbb{Z}} : \ s_m \in S, \ \text{for all } m \in \mathbb{Z}\}$$

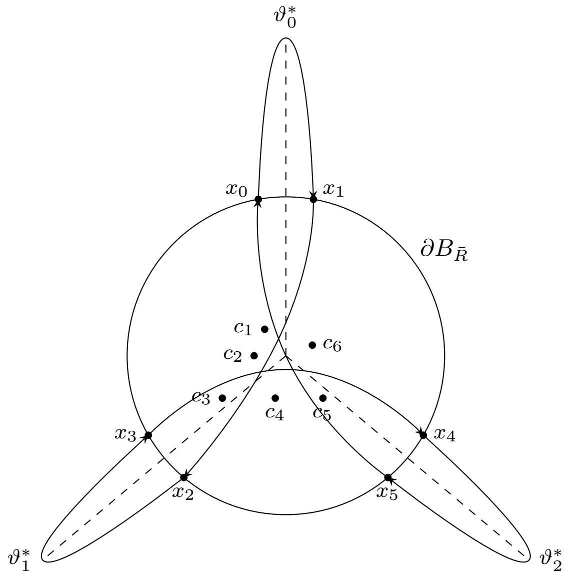

Figure 2.2.: An example of classical periodic solution provided in Theorem 2.1.4

and make it a metric space with the distance

$$d((s_m),(t_m)) \doteq \sum_{m\in\mathbb{Z}} \frac{\rho(s_m,t_m)}{2^{|m|}},$$

defined for every $(s_m),(t_m) \in S^{\mathbb{Z}}$. Introduce also the Bernoulli right shift as the map

$$\begin{aligned} T_r \colon S^{\mathbb{Z}} &\to S^{\mathbb{Z}} \\ (s_m) &\mapsto T_r((s_m)) \doteq (s_{m+1}), \end{aligned}$$

which actually determines the discrete dynamical system $(S^{\mathbb{Z}}, T_r)$. Then, we have the following definition.

Definition 2.1.5. *Let S be a finite set, $\mathcal{E}$ be a metric space and $\mathfrak{R}\colon \mathcal{E} \to \mathcal{E}$ be a continuous map. Then, we say that the dynamical system $(\mathcal{E},\mathfrak{R})$ has a symbolic dynamics with set of symbols S if there exist a subset $\Pi \subseteq \mathcal{E}$ which is invariant through $\mathfrak{R}$ and a continuous and surjective map $\pi\colon \Pi \to S^{\mathbb{Z}}$ such that the diagram*

$$\begin{array}{ccc} \Pi & \xrightarrow{\mathfrak{R}} & \Pi \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S^{\mathbb{Z}} & \xrightarrow{T_r} & S^{\mathbb{Z}} \end{array}$$

commutes. In other words, we are saying that the map $\mathfrak{R}|_\Pi$ is topologically semi-conjugate to the Bernoulli right shift T_r in the metric space $(S^{\mathbb{Z}}, d)$.

Remark 2.1.6. *Many properties of the discrete dynamical system* $(S^{\mathbb{Z}}, T_r)$ *are known, which point toward the concept of chaos. Indeed, it is possible to prove that (see for instance [66]):*

- $(S^{\mathbb{Z}}, T_r)$ *has a dense countable set of periodic points, since all the periodic sequences are periodic points for the Bernoulli shift;*
- $(S^{\mathbb{Z}}, T_r)$ *displays high sensitivity with respect initial data, i.e., if we define as* T_r^k *the* k*-th iteration of the Bernoulli shift, we have that for any* $\varrho > 0$ *there exist two arbitrarily close sequences* $(s_m), (t_m) \in S^{\mathbb{Z}}$ *such that*

$$\sup_{k\in\mathbb{Z}} d(T_r^k((s_m)), T_r^k((t_m))) \geq \varrho;$$

- *the previous property actually holds for several initial data, to be precise, the dynamical system* $(S^{\mathbb{Z}}, T_r)$ *has positive topological entropy.*

For this reason, the existence of a symbolic dynamics for a dynamical system $(\mathcal{E}, \mathfrak{R})$ *reflects a very complex behaviour of its trajectories. Indeed, the semi-conjugacy through the map* π *relates the dynamical properties of the Bernoulli shift with the ones of the first return map* $\mathfrak{R}$*. We point out that this in general is not enough to show that the dynamical system* $(\mathcal{E}, \mathfrak{R})$ *is chaotic, since the projection map* π *is generally not invertible. However, it is clear that the existence of a symbolic dynamics is a necessary condition for the density of periodic orbits and the presence of chaos, so that one usually proves it as an intermediate step in this direction.*

Theorem 2.1.7. *In the same setting of Theorem 2.1.4, take* $h \in (0, \bar{h})$*, with* $\bar{h} > 0$ *therein defined. Then, there exists a subset* Π_h *of the energy shell* $\mathcal{E}_h$*, a first return map* $\mathfrak{R}\colon \Pi_h \to \Pi_h$ *and a continuous and surjective map* $\pi\colon \Pi_h \to \mathcal{Q}^{\mathbb{Z}}$*, such that the diagram*

$$\begin{array}{ccc} \Pi_h & \xrightarrow{\ \mathfrak{R}\ } & \Pi_h \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{Q}^{\mathbb{Z}} & \xrightarrow{\ T_r\ } & \mathcal{Q}^{\mathbb{Z}} \end{array}$$

commutes. In other words, for any h *sufficiently small, the anisotropic* N*-centre problem at energy* $-h$ *admits a symbolic dynamics with sets of symbols* $\mathcal{Q}$*.*

2.1.1. Outline of the proof

The key idea is to consider a different N-centre problem starting from the dynamical system (2.1) and the energy equation (2.3). Defining a suitable rescaled version of potential V, we end up with the problem

$$\begin{cases} \ddot{y}(t) = \nabla V^{\varepsilon}(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V^{\varepsilon}(y(t)) = -1, \end{cases} \tag{2.6}$$

where $\varepsilon = h^{1/\alpha} > 0$ and V^{ε} takes into account the rescaled centres $c_j' = \varepsilon c_j$. In this way, all the new centres are confined in the ball $B_{\varepsilon}(0)$ and collapse to the origin as the energy

h of the original problem becomes very small, since $\varepsilon \to 0^+$ as $h \to 0^+$. It turns out that it is equivalent to look for periodic solutions of (2.1)-(2.3) and periodic solutions of (2.6). Moreover, as ε becomes very small, outside a ball of radius $R \gg \varepsilon$ and centred in the origin, the potential V_ε is a small perturbation of a suitable anisotropic Kepler-like potential. This fact, which, together with the previous discussion, is the content of Section 2.2, allows us to split the proof of the main result outside and inside the ball $B_R(0)$ and to carry out a *broken geodesics argument*.

In Section 2.3 we prove the existence of pieces of solutions of (2.6), starting in $\partial B_R(0)$ and lying outside $B_R(0)$.

In Section 2.4 we show how to build solution arcs which start in ∂B_R and go through the centres without collisions.

In Section 2.5 we glue together the pieces of solutions obtained in the previous sections, in order to obtain periodic solutions of (2.6) and thus of (2.1)-(2.3).

In Section 2.6 we show that this dynamical system admits a symbolic dynamics with respect to a chosen set of symbols.

2.2. A useful rescaling

Given $\varepsilon > 0$ and $y \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$, let us introduce the rescaled potential

$$V^\varepsilon(y) \doteq W^\varepsilon(y) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j - \alpha} V_j(y - \varepsilon c_j), \tag{2.7}$$

where

$$W^\varepsilon(y) \doteq \sum_{i=1}^{k} V_i(y - \varepsilon c_i).$$

Notice that, with this notations and recalling that we have assumed that $\max |c_j| \leq 1$ (see Remark 2.1.2), the new centres εc_j will be included inside the ball B_ε.

Proposition 2.2.1. *Let $V \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\})$ be defined as in* (2.2) *and $x \in \mathcal{C}^2((a,b); \mathbb{R}^2)$ be a classical solution of*

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = -h, \quad h > 0. \end{cases} \tag{2.8}$$

Then, in the interval $(h^{\frac{\alpha+2}{2\alpha}} a, h^{\frac{\alpha+2}{2\alpha}} b)$, the function

$$y(t) \doteq h^{1/\alpha} x(h^{-\frac{\alpha+2}{2\alpha}} t)$$

solves the problem

$$\begin{cases} \ddot{y}(t) = \nabla V^\varepsilon(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V^\varepsilon(y(t)) = -1, \end{cases} \tag{2.9}$$

where $\varepsilon = h^{1/\alpha}$ and V^ε is defined as in (2.7).

Conversely, if $y \in \mathcal{C}^2((c,d);\mathbb{R}^2)$ is a solution of (2.9) *then, taking $h = \varepsilon^\alpha$, the function*

$$x(t) \doteq h^{-1/\alpha} y(h^{\frac{\alpha+2}{2\alpha}} t)$$

is a solution of (2.8) *in the interval $(h^{-\frac{\alpha+2}{2\alpha}} c, h^{-\frac{\alpha+2}{2\alpha}} d)$.*

Proof. Suppose that $x \in \mathcal{C}^2((a,b);\mathbb{R}^2)$ is a solution of (2.8), set $\varepsilon = h^{1/\alpha} > 0$ and define

$$y(t) \doteq h^{1/\alpha} x(h^{-\frac{\alpha+2}{2\alpha}} t) = \varepsilon x(\varepsilon^{-\frac{\alpha+2}{2}} t).$$

Let us start by checking that $y(t)$ satisfies the energy relation in (2.9). A straightforward computation leads to

$$\frac{1}{2}|\dot{y}(t)|^2 = \frac{\varepsilon^{-\alpha}}{2}|\dot{x}(\varepsilon^{-\frac{\alpha+2}{2}} t)|^2 = \varepsilon^{-\alpha} V\left(x(\varepsilon^{-\frac{\alpha+2}{\alpha}} t)\right) - 1;$$

moreover, for every $x \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$, we have that

$$\begin{aligned} V(x) &= W(x) + \sum_{j=k+1}^{N} V_j(x - c_j) \\ &= \varepsilon^\alpha \sum_{i=1}^{k} V_i(\varepsilon x - \varepsilon c_i) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j} V_j(\varepsilon x - \varepsilon c_j) \\ &= \varepsilon^\alpha \left(\sum_{i=1}^{k} V_i(\varepsilon x - \varepsilon c_i) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j - \alpha} V_j(\varepsilon x - \varepsilon c_j) \right). \end{aligned}$$

Therefore, for $t \in (\varepsilon^{\frac{\alpha+2}{2}} a, \varepsilon^{\frac{\alpha+2}{2}} b)$, we get

$$\begin{aligned} \frac{1}{2}|\dot{y}(t)|^2 &= \sum_{i=1}^{k} V_i\left(\varepsilon x\left(\varepsilon^{-\frac{\alpha+2}{2}} t\right) - \varepsilon c_i\right) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j - \alpha} V_j\left(\varepsilon x\left(\varepsilon^{-\frac{\alpha+2}{2}} t\right) - \varepsilon c_j\right) - 1 \\ &= V^\varepsilon(y(t)) - 1. \end{aligned} \tag{2.10}$$

Again, by calculation, we obtain

$$\ddot{y}(t) = \varepsilon^{-\alpha-1} \ddot{x}(\varepsilon^{-\frac{\alpha+2}{2}} t) = \varepsilon^{-\alpha-1} \nabla V(x(\varepsilon^{-\frac{\alpha+2}{2}} t))$$

and, for every $x \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$

$$\begin{aligned} \nabla V(x) &= \varepsilon^{\alpha+1} \sum_{i=1}^{k} \nabla V_i(\varepsilon x - \varepsilon c_i) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j+1} \nabla V_j(\varepsilon x - \varepsilon c_j) \\ &= \varepsilon^{\alpha+1} \left(\sum_{i=1}^{k} \nabla V_i(\varepsilon x - \varepsilon c_i) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j - \alpha} \nabla V_j(\varepsilon x - \varepsilon c_j) \right). \end{aligned}$$

Finally, for every $t \in (\varepsilon^{\alpha+2}a, \varepsilon^{\alpha+2}b)$, we have

$$\ddot{y}(t) = \sum_{i=1}^{k} \nabla V_i(y(t) - \varepsilon c_i) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j - \alpha} \nabla V_j(y(t) - c_j) = \nabla V^\varepsilon(y(t)).$$

This, together with (2.10), concludes the proof, since the converse follows in an analogous way. □

In the rest of this section we show that, outside a ball of radius $R > \varepsilon > 0$, if ε is sufficiently small problem (2.9) can be seen as a perturbation of a Kepler problem, driven by a sum of $-\alpha$-homogeneous potentials. We start by showing a limiting behaviour for V^ε as $\varepsilon \to 0^+$.

Proposition 2.2.2. *Let $\delta > 0$ and V^ε be defined as in* (2.7)*. Then, for every $y \in \mathbb{R}^2 \setminus B_\delta$*

$$V^\varepsilon(y) = W^0(y) + O(\varepsilon^\gamma) \quad \text{as } \varepsilon \to 0^+,$$

where $\gamma \doteq \min\{1, \alpha_{k+1} - \alpha\} > 0$ and, according to (2.7)

$$W^0(y) = \sum_{i=1}^{k} V_i(y) = |y|^{-\alpha} \sum_{i=1}^{k} V_i\left(\frac{y}{|y|}\right). \tag{2.11}$$

Moreover, the potential V^ε is smooth with respect to ε and $V^\varepsilon \to W^0$ uniformly as $\varepsilon \to 0^+$ on every compact subset of $\mathbb{R}^2 \setminus \{0\}$.

Proof. As a starting point, since $\varepsilon \to 0^+$, we can assume $\delta > \varepsilon$; moreover, if we fix $j \in \{1, \dots, N\}$ and $|y| > \delta$, for every $\sigma \in \mathbb{R}$ we have

$$\begin{aligned}
|y - \varepsilon c_j|^{-\sigma} &= \left[|y|^2 - 2\varepsilon\langle y, c_j\rangle + \varepsilon^2 |c_j|^2\right]^{-\sigma/2} \\
&= |y|^{-\sigma}\left[1 - 2\varepsilon\frac{\langle y, c_j\rangle}{|y|^2} + \varepsilon^2\frac{|c_j|^2}{|y|^2}\right]^{-\sigma/2} \\
&= |y|^{-\sigma} + \varepsilon\sigma\frac{\langle y, c_j\rangle}{|y|^{\sigma+2}} + o(\varepsilon) = |y|^{-\sigma} + O(\varepsilon).
\end{aligned}$$

In this way, for every $j \in \{1, \dots, N\}$, as $\varepsilon \to 0^+$ we can write

$$V_j\left(\frac{y - \varepsilon c_j}{|y - \varepsilon c_j|}\right) = V_j\left((y - \varepsilon c_j)\left(\frac{1}{|y|} + O(\varepsilon)\right)\right) = V_j\left(\frac{y}{|y|} + O(\varepsilon)\right)$$

and so

$$V_j\left(\frac{y - \varepsilon c_j}{|y - \varepsilon c_j|}\right) = V_j\left(\frac{y}{|y|}\right) + \left\langle \nabla V_j\left(\frac{y}{|y|}\right), O(\varepsilon)\right\rangle + o(\varepsilon) = V_j\left(\frac{y}{|y|}\right) + O(\varepsilon).$$

To conclude, we write

$$\begin{aligned}V^\varepsilon(y) &= \sum_{i=1}^{k} |y-\varepsilon c_i|^{-\alpha} V_i\left(\frac{y-\varepsilon c_i}{|y-\varepsilon c_i|}\right) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j-\alpha}|y-\varepsilon c_j|^{-\alpha_j} V_j\left(\frac{y-\varepsilon c_j}{|y-\varepsilon c_j|}\right)\\ &= |y|^{-\alpha}\sum_{i=1}^{k} V_i\left(\frac{y}{|y|}\right) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j-\alpha}|y|^{-\alpha_j} V_j\left(\frac{y}{|y|}\right) + O(\varepsilon)\\ &= W^0(y) + O(\varepsilon^\gamma),\end{aligned}$$

where $\gamma = \min\{1, \alpha_{k+1} - \alpha\} > 0$.

To conclude, the uniform convergence on compact subsets of $\mathbb{R} \setminus \{0\}$ is an easy consequence of the fact that the singularity set of V^ε is proportional to ε. □

Remark 2.2.3. *Observe that the potential W^0 defined in* (2.11) *is singular in the origin, while the potential W^ε has multiple poles at $\varepsilon c_1, \ldots, \varepsilon c_k$. Thus, it turns out that assumption $(V)_2$ requires that W^0 admits m strictly minimal central configurations, for some $m > 0$ (see also Remark 2.1.1).*

To conclude this section, we notice that the energy bound found in Remark 2.1.2 for problem (2.8) corresponds to the following bound on the parameter ε for problem (2.9)

$$\varepsilon \in (0, \tilde\varepsilon), \quad \text{where } \tilde\varepsilon = \tilde h^{1/\alpha} = \frac{\mathfrak{m}^{1/\alpha}}{2}, \tag{2.12}$$

where we recall that $\mathfrak{m} = \min_{j=1,\ldots,N} \min_{\mathbb{S}^1} V_j|_{\mathbb{S}^1}$. Naturally, this bound guarantees that the ball B_ε containing the rescaled centres is completely included in the Hill's region of problem (2.9)

$$\mathcal{R}_\varepsilon \doteq \{y \in \mathbb{R}^2 : \, V^\varepsilon(y) \geq 1\}.$$

Indeed, following the same computations of Remark 2.1.2, if $\varepsilon \in (0, \tilde\varepsilon)$ and $|y| \leq \varepsilon$, then

$$V^\varepsilon(y) \geq \mathfrak{m}|y - \varepsilon c_1|^{-\alpha} \geq 1.$$

2.3. Outer dynamics

At this point, the idea is to exploit a perturbation argument suggested by Proposition 2.2.2 and to build pieces of periodic solutions for (2.9), which lie far from the centres and that will be denoted as *outer arcs*. Note that, if $y\colon J \to \mathbb{R}^2$, with $J \subseteq \mathbb{R}$ is a solution of (2.9), then

$$V^\varepsilon(y(t)) \geq 1, \quad \text{for every } t \in J;$$

for this reason, we need to show that there exists an $R > 0$ such that, for every $\varepsilon \in (0, \tilde\varepsilon)$,

$$B_\varepsilon \subset B_R \subset \{y \in \mathbb{R}^2 : \, V^\varepsilon(y) \geq 1\} = \mathcal{R}_\varepsilon. \tag{2.13}$$

Following the same approach as in the end of the previous section, we have that, for any $\varepsilon \in (0, \tilde{\varepsilon})$,

$$B_{\mathfrak{m}^{1/\alpha}-\varepsilon} \subset \{y \in \mathbb{R}^2 : \, V^\varepsilon(y) \geq 1\}.$$

Hence, choosing

$$R \in (\tilde{\varepsilon}, \mathfrak{m}^{1/\alpha} - \tilde{\varepsilon}) \tag{2.14}$$

the inclusions (2.13) hold for any $\varepsilon \in (0, \tilde{\varepsilon})$.

In Section 2.2 we have shown that it is possible to obtain a useful rescaling of the potential V, putting all the centres inside a ball of radius $\varepsilon > 0$ and thus considering the rescaled potential

$$V^\varepsilon(y) = \sum_{i=1}^{k} V_i(y - \varepsilon c_i) + \sum_{j=k+1}^{N} \varepsilon^{\alpha_j - \alpha} V_j(y - \varepsilon c_j),$$

so that a periodic solution of our initial problem is equivalent to a solution of a N centre problem driven by V^ε and with energy -1 (see Proposition 2.2.1). Moreover, if ε is sufficiently small, by Proposition 2.2.2 we know that outside a ball of radius $R > \varepsilon > 0$ the motion follows the dynamics of a perturbed $-\alpha$-homogeneous anisotropic Kepler problem. Inspired by this, we are going to look for solutions of the ε-problem (2.9) which start in $\partial B_R(0)$ and travel in $\mathbb{R}^2 \setminus B_R(0)$; note that, in this setting, R will satisfy (2.14). These solution arcs will be found as perturbed solutions of an anisotropic Kepler problem driven by W^0; given $p_0, p_1 \in \partial B_R(0)$, we are going to look for solutions of the following problem

$$\begin{cases} \ddot{y}(t) = \nabla V^\varepsilon(y(t)) & t \in [0, T] \\ \dfrac{1}{2}|\dot{y}(t)|^2 - V^\varepsilon(y(t)) = -1 & t \in [0, T] \\ |y(t)| > R & t \in (0, T) \\ y(0) = p_0, \quad y(T) = p_1, \end{cases} \tag{2.15}$$

for some $T > 0$ possibly depending on ε.

2.3.1. Homothetic solutions for the anisotropic Kepler problem

The core of our perturbation argument consists in focusing on some special trajectories of an anisotropic Kepler problem driven by W^0, in order to study the behaviour of the close-by orbits. For this reason, we take $\varepsilon = 0$ and we consider the problem

$$\begin{cases} \ddot{x} = \nabla W^0(x), \quad x \in \mathbb{R}^2 \setminus \{0\} \\ \frac{1}{2}|\dot{x}|^2 - W^0(x) = -1, \end{cases} \tag{2.16}$$

recalling that $W^0 \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\})$ is a $-\alpha$-homogeneous anisotropic potential (see Proposition 2.2.2). Note that, if we introduce polar coordinates $x = (r \cos\vartheta, r \sin\vartheta)$ with $r > 0$

and $\vartheta \in [0, 2\pi)$, the potential W^0 can be written as

$$W^0(x) = r^{-\alpha} \sum_{i=1}^{k} V_i(\cos\vartheta, \sin\vartheta) = r^{-\alpha} \sum_{i=1}^{k} U_i(\vartheta) = r^{-\alpha} U(\vartheta),$$

where $U_i \doteq V_i|_{\mathbb{S}^1}$ and $U \doteq \sum_{i=1}^{k} U_i$ (see (V)). From the energy equation in (2.16), the boundary of the Hill's region for this problem is the curve parametrized by polar coordinates in this way

$$\partial\mathcal{R}_0 \doteq \{U(\vartheta)^{1/\alpha}(\cos\vartheta, \sin\vartheta) : \vartheta \in [0, 2\pi)\}.$$

From this, the following definition makes sense:

Definition 2.3.1. *For any $\vartheta \in [0, 2\pi)$, for every $R > 0$ which satisfies*

$$0 < R < U(\vartheta)^{\frac{1}{\alpha}}, \tag{2.17}$$

define $\xi = Re^{i\vartheta}$. An out-in homothetic solution *for* (2.16) *which starts in ξ is a function $\hat{x}_\xi$ which solves* (2.20) *and such that*

$$\hat{x}_\xi(t) \doteq \lambda(t)\xi,$$

where $\lambda \colon [0, T_\xi] \to \mathbb{R}^+$ and

$$\begin{cases} \lambda(t) > 1 \text{ for every } t \in (0, T_\xi), \\ \lambda(0) = 1 = \lambda(T_\xi), \end{cases} \tag{2.18}$$

for some $T_\xi > 0$.

We aim to understand which conditions are satisfied a posteriori on ξ and λ once a homothetic solution for (2.20) is provided. If we plug $\hat{x}_\xi \doteq \lambda\xi$ into the motion equation $\ddot{x} = \nabla W^0(x)$, we obtain

$$\ddot{\lambda}(t)\xi = \lambda(t)^{-\alpha-1}\nabla W^0(\xi) \tag{2.19}$$

and thus the Euler's theorem for homogeneous functions gives

$$R^2\ddot{\lambda}(t) = -\alpha\lambda(t)^{-\alpha-1}W^0(\xi).$$

In this way, we obtain the equation

$$\ddot{\lambda}(t) = -\mu_\xi\lambda(t)^{-\alpha-1}, \tag{2.20}$$

with $\mu_\xi = \alpha R^{-\alpha-2}U(\vartheta_\xi)$, i.e., λ solves a 1-dimensional $-\alpha$-Kepler problem. Since the homothetic solution $\hat{x}_\xi$ has energy -1 and $\lambda(0) = 1$, we can associate to equation (2.20) the initial conditions

$$\lambda(0) = 1, \quad \dot{\lambda}(0) = \frac{1}{R}\sqrt{2(W^0(\xi) - 1)}$$

obtaining a solution which satisfies (2.18). Moreover, recalling that for $x \in \mathbb{R}^2$ the moment of inertia of x is defined as $I(x) = \frac{1}{2}|x|^2$ and $\nabla I(x) = x$, comparing (2.19) and (2.20), we note that ξ solves the equation

$$\nabla W^0(\xi) + \mu_\xi \nabla I(\xi) = 0. \tag{2.21}$$

Definition 2.3.2. *A central configuration for W^0 is a critical point of W^0 constrained to a level surface of the inertial moment I. In other words, a central configuration is a vector $\xi \in \partial B_R(0)$ that verifies* (2.21)*.*

To sum up, we have found out that a homothetic motion for (2.16) is a function $\hat{x}_\xi = \lambda\xi$ such that $\xi \in \partial B_R(0)$ is a central configuration for W^0, $R > 0$ verifies (2.17) and $\lambda\colon [0, T_\xi] \to \mathbb{R}^+$ is the unique solution of

$$\begin{cases} \ddot{\lambda}(t) = -\mu_\xi \lambda(t)^{-\alpha-1} \\ \lambda(0) = 1, \ \dot{\lambda}(0) = \frac{1}{R}\sqrt{2(W^0(\xi) - 1)}. \end{cases}$$

From now on, when we consider the quantity $R > 0$, we will always assume (2.17) and we will refer to $\xi = (R\cos\vartheta_\xi, R\sin\vartheta_\xi)$ or simply to ϑ_ξ as a central configuration for W^0, meaning that ξ verifies (2.21) or, equivalently, that

$$U'(\vartheta_\xi) = 0.$$

Collecting together all the previous discussions, given a central configuration $\xi \in \partial B_R(0)$, we can consider the following Cauchy problem

$$\begin{cases} \ddot{x}(t) = \nabla W^0(x(t)) \\ x(0) = \xi, \quad \dot{x}(0) = v_\xi = \frac{1}{R}\sqrt{2(W^0(\xi) - 1)}\xi, \end{cases} \tag{2.22}$$

which admits as unique solution the homothetic trajectory $\hat{x}_\xi$, that reaches again the position ξ after a time $T_\xi > 0$, with opposite velocity.

2.3.2. Shadowing homothetic solutions in the anisotropic Kepler problem

In Proposition 2.2.2 we have seen that V^ε reduces to W^0 as $\varepsilon \to 0^+$, together with all the ε-centres collapsing to the origin. For this reason, the aim of this paragraph is to provide an intermediate result, i.e., to prove the existence of trajectories for problem (2.16) which start very close to a given homothetic trajectory $\hat{x}_\xi$. In other words, we investigate the existence of a solution for

$$\begin{cases} \ddot{x}(t) = W^0(x(t)), & t \in [0, \overline{T}] \\ \dfrac{1}{2}|\dot{x}(t)|^2 - W^0(x(t)) = -1, & t \in [0, \overline{T}] \\ |x(t)| > R, & t \in (0, \overline{T}) \\ x(0) = p_0, \ x(\overline{T}) = p_1, \end{cases}$$

where p_0 and p_1 are chosen sufficiently close to a central configuration $\xi \in \partial B_R$ for W^0. This indeed corresponds to show the existence of solutions for problem (2.15) when $\varepsilon = 0$ and we believe that it is a result of independent interest.

In the previous paragraph we have seen that a homothetic solution for an anisotropic Kepler problem driven by W^0 is actually the unique solution of the Cauchy problem (2.22). For our convenience, we need a characterization of the homothetic motion $\hat{x}_\xi$ in Hamiltonian formalism and so we reword (2.22) as

$$\begin{cases} \dot{z} = F(z) \\ z(0) = z_\xi, \end{cases} \tag{2.23}$$

where

$$z = \begin{pmatrix} x \\ v \end{pmatrix}, \; F(z) = \begin{pmatrix} v \\ \nabla W^0(x) \end{pmatrix} \text{ and } z_\xi = \begin{pmatrix} \xi \\ v_\xi \end{pmatrix}.$$

According to this, to satisfy the energy constraint in (2.16) we restrict the domain of the vector field F to the 3-dimensional energy shell

$$\mathcal{E} = \left\{ (x, v) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 : \frac{1}{2}|v|^2 - W^0(x) = -1 \right\}$$

and we term $\hat{z}_\xi$ the homothetic solution in the Hamiltonian formalism, i.e., the unique solution of (2.23). Introducing the flow associated to the differential equation in (2.23)

$$\begin{aligned} \Phi \colon \Omega \subset \mathbb{R} \times \mathcal{E} &\to \mathcal{E} \\ (t, z) &\mapsto \Phi(t, z) = \Phi^t(z) \end{aligned}$$

we notice that

$$\hat{z}_\xi(t) = \Phi^t(z_\xi) \quad \text{and} \quad \hat{x}_\xi(t) = \pi_x(\hat{z}_\xi(t)) = \pi_x \Phi^t(z_\xi),$$

where $\pi_x(z)$ and $\pi_v(z)$ represent the two canonic projections of z. Now, if we introduce the 2-dimensional inertial surface

$$\Sigma = \{(x, v) \in \mathcal{E} : |x| = R\} \subseteq \mathcal{E}$$

it turns out that both the starting and ending point of the homothetic motion lie on Σ, i.e., $\hat{z}_\xi(0), \hat{z}_\xi(T_\xi) \in \Sigma$. Moreover, since the initial conditions ξ and v_ξ are parallel, with a slight abuse of notation on the gradient of I we have that

$$\langle F(\xi, v_\xi), \nabla I(\xi) \rangle = \langle v_\xi, \xi \rangle \neq 0$$

and so the field F is *transversal* to Σ in (ξ, v_ξ).

Inspired by this, it is easy to prove the following proposition and thus to define a first return map on Σ.

Proposition 2.3.3. *Given $\xi \in \partial B_R(0)$ central configuration for W^0, there exists a neighbourhood $\mathcal{U} \times \mathcal{V}$ of (ξ, v_ξ) and a function $T \in \mathcal{C}^1(\mathcal{U} \times \mathcal{V}; \mathbb{R}^+)$ such that*

- $T(\xi, v_\xi) = T_\xi$;
- *for every* $(x,v) \in \mathcal{U} \times \mathcal{V}$*, for* $t > 0$ *holds*

$$\Phi^t(x,v) \in \Sigma \text{ if and only if } t = T(x,v).$$

Moreover, if we define $\frac{d}{dz}\Phi^t(z)$ *as the derivative of* Φ *with respect to* $z = (x,v)$ *(see Appendix B), given* $z_0 \in \mathcal{U} \times \mathcal{V}$ *we have that*

$$\langle \nabla T(z_0), \zeta \rangle = -\frac{\left\langle \pi_x \Phi^{T(z_0)}(z_0), \pi_x \frac{d}{dz}\Phi^{T(z_0)}(z)\Big|_{z=z_0} \zeta \right\rangle}{\left\langle \pi_x \Phi^{T(z_0)}(z_0), \pi_v \Phi^{T(z_0)}(z_0) \right\rangle},$$

for every $\zeta \in \mathcal{T}_{z_0}(\mathcal{U} \times \mathcal{V})$*, where* $\mathcal{T}_{z_0}$ *denotes the tangent space at the vector* z_0*.*

Proof. Defining the map $G(x,v) = |x|^2 - R^2$ for every $(x,v) \in \mathcal{E}$, we can consider its composition with the flow Φ, to obtain the $\mathcal{C}^1$ map

$$\begin{aligned} f \colon \mathbb{R} \times \mathcal{E} &\to \mathbb{R} \\ (t,x,v) &\mapsto f(t,x,v) \doteq G(\Phi^t(x,v)). \end{aligned}$$

Since $f(T_\xi, \xi, v_\xi) = G(\xi, -v_\xi) = 0$ and

$$\begin{aligned} \frac{\partial}{\partial t} f(t,x,v)\Big|_{(T_\xi,\xi,v_\xi)} &= \left\langle \nabla G(\Phi^{T_\xi}(\xi, v_\xi)), F(\Phi^{T_\xi}(\xi, v_\xi)) \right\rangle \\ &= \left\langle (2\xi, 0), (-v_\xi, \nabla W^0(\xi)) \right\rangle \neq 0, \end{aligned}$$

the first part of the statement easily follows from the Implicit Function Theorem. Moreover, for every $(x,v) \in \mathcal{U} \times \mathcal{V}$ we have

$$\nabla T(x,v) = -\frac{\nabla_{x,v} f(T(x,v),x,v)}{\frac{\partial}{\partial t} f(T(x,v),x,v)} = -\frac{\nabla G(\Phi^{T(x,v)}(x,v)) J_{x,v} \Phi^{T(x,v)}(x,v)}{\left\langle \nabla G(\Phi^{T(x,v)}(x,v)), F(\Phi^{T(x,v)}(x,v)) \right\rangle}.$$

Therefore, given $z_0 = (x_0, v_0) \in \mathcal{U} \times \mathcal{V}$ we can define the differential

$$\begin{aligned} dT(z_0) \colon \mathcal{T}_{z_0}(\mathcal{U} \times \mathcal{V}) &\to \mathbb{R} \\ \zeta \longmapsto dT(z_0)\zeta \quad &= \langle \nabla T(z_0), \zeta \rangle, \end{aligned}$$

with

$$\begin{aligned} \langle \nabla T(z_0), \zeta \rangle &= -\frac{\left\langle \nabla G(\Phi^{T(z_0)}(z_0)), \frac{d}{dz}\Phi^{T(z_0)}(z)\Big|_{z=z_0} \zeta \right\rangle}{\langle \nabla G(\Phi^{T(z_0)}(z_0)), F(\Phi^{T(z_0)}(z_0)) \rangle} \\ &= -\frac{\left\langle \pi_x \Phi^{T(z_0)}(z_0), \pi_x \frac{d}{dz}\Phi^{T(z_0)}(z)\Big|_{z=z_0} \zeta \right\rangle}{\left\langle \pi_x \Phi^{T(z_0)}(z_0), \pi_v \Phi^{T(z_0)}(z_0) \right\rangle}. \end{aligned}$$

□

Hence, given $\xi \in \partial B_R(0)$ central configuration for W^0 and $\mathcal{U}, \mathcal{V}$ as in Proposition 2.3.3, for every $(x_0, v_0) \in \mathcal{U} \times \mathcal{V}$ there exists a unique instant $T(x_0, v_0) > 0$ such that

$$(x_1, v_1) \doteq \Phi^{T(x_0,v_0)}(x_0, v_0) \in \Sigma.$$

In this way, if we fix $x_0 \in \mathcal{U}$, we can define the arriving point x_1 as a function of $v_0 \in \mathcal{V}$

$$x_1(v_0) \doteq \pi_x(\Phi^{T(x_0,v_0)}(x_0, v_0)). \tag{2.24}$$

Our aim is to prove that the previous map is invertible, so that we would be able to build solution arcs starting in a point $x_0 \in \partial B_R(0)$ and arriving in another point $x_1 \in \partial B_R(0)$, with x_0, x_1 sufficiently close to ξ.

Theorem 2.3.4. *Given $\xi \in \partial B_R$ a central configuration for W^0 such that $U''(\vartheta_\xi) \geq 0$, the map x_1 defined in* (2.24) *is invertible in a neighbourhood of v_ξ.*

The proof of Theorem 2.3.4 is rather technical and relies on a series of lemmata which we state and prove below.

Lemma 2.3.5. *Assume that there exists $\xi \in \partial B_R$ central configuration for W^0. Following the notations of Proposition 2.3.3, define the map*

$$\begin{aligned} g \colon \mathcal{U} \times \mathcal{V} &\to \Sigma \\ (x, v) &\mapsto g(x, v) \doteq \Phi^{T(x,v)}(x, v). \end{aligned}$$

Then, g is $\mathcal{C}^1$-differentiable over $\mathcal{U} \times \mathcal{V}$ and

$$dg(z_\xi)\zeta = \frac{d}{dz}\Phi^{T_\xi}(z)\Big|_{z=z_\xi} \zeta + F(\Phi^{T_\xi}(z_\xi))\langle \nabla T(z_\xi), \zeta\rangle,$$

for every $\zeta \in \mathcal{T}_{z_\xi}(\mathcal{U} \times \mathcal{V})$.

Proof. In order to prove this result, we need to give a characterization of the partial derivative of the flow Φ with respect to the variable v. First of all, observe that for the $\mathcal{C}^1$-dependence on initial data of the flow Φ^t and for Proposition 2.3.3, the map g is well defined and $\mathcal{C}^1$-differentiable over $\mathcal{U} \times \mathcal{V}$. If we call as usual $z = (x, v)$ and $z_\xi = (\xi, v_\xi)$, we can observe that

$$g(z_\xi) = (\xi, -v_\xi)$$

and, following the notation introduced in Appendix B, the differential of g in the point $z_\xi \in \mathcal{U} \times \mathcal{V}$ is the linear map

$$\begin{aligned} dg(z_\xi) \colon \mathcal{T}_{z_\xi}(\mathcal{U} \times \mathcal{V}) &\to \mathcal{T}_{(\xi,-v_\xi)}\Sigma \\ \zeta &\mapsto dg(z_\xi)\zeta = \frac{d}{dz}\left[\Phi^{T(z)}(z)\right]_{z=z_\xi} \zeta. \end{aligned} \tag{2.25}$$

In this way, if we compute the derivative of g with respect to z we get

$$\frac{d}{dz}\left[\Phi^{T(z)}(z)\right]_{z=z_\xi} = \lim_{\|\eta\|\to 0} \frac{\Phi^{T(z_\xi+\eta)}(z_\xi+\eta) - \Phi^{T(z_\xi)}(z_\xi)}{\eta}$$
$$= \lim_{\|\eta\|\to 0} \frac{\Phi^{T(z_\xi+\eta)}(z_\xi+\eta) - \Phi^{T(z_\xi+\eta)}(z_\xi)}{\eta} + \lim_{\|\eta\|\to 0} \frac{\Phi^{T(z_\xi+\eta)}(z_\xi) - \Phi^{T(z_\xi)}(z_\xi)}{\eta}$$
$$= \frac{d}{dz}\Phi^{T(z_\xi)}(z)\Big|_{z=z_\xi} + F(\Phi^{T(z_\xi)}(z_\xi))\nabla T(z_\xi).$$

Finally, recalling that $T(z_\xi) = T_\xi$, we have shown that

$$dg(z_\xi)\zeta = \frac{d}{dz}\Phi^{T_\xi}(z)\Big|_{z=z_\xi}\zeta + F(\Phi^{T_\xi}(z_\xi))\langle \nabla T(z_\xi), \zeta\rangle,$$

for every $\zeta \in \mathcal{T}_{z_\xi}(\mathcal{U}\times\mathcal{V})$. □

Lemma 2.3.6. *In the same setting of Lemma 2.3.5, given $\zeta \in \mathcal{T}_{z_\xi}(\mathcal{U}\times\mathcal{V})$ and $t \in (0, T_\xi)$, define*

$$q(t) \doteq \pi_x \frac{d}{dz}\Phi^t(z)\Big|_{z=z_\xi}\zeta$$

and name also $s_\xi = \xi/|\xi|$ the ξ-direction unit vector and its orthogonal unit vector $s_\tau \doteq s_\xi^\perp$. Then, the projection of q over the direction s_τ

$$q_\tau(t) = \langle q(t), s_\tau\rangle s_\tau$$

solves the problem

$$\begin{cases} \ddot{q}_\tau = \langle \nabla^2 W^0(\hat{x}_\xi(t))s_\tau, s_\tau\rangle q_\tau \\ q_\tau(0) = \langle \pi_x\zeta, s_\tau\rangle s_\tau, \end{cases}$$

recalling that $\hat{x}_\xi$ is the unique (homothetic) solution of (2.22).

Proof. Following Appendix B, we know that the partial derivative of Φ with respect to z satisfies the variational equation along the homothetic solution $\hat{x}_\xi$, which gives us information about how the flow is sensitive under variations made on the initial condition $z(0) = (x(0), \dot{x}(0))$. Since the Jacobian matrix of the vector field F in z reads

$$JF(z) = \begin{pmatrix} 0_2 & I_2 \\ \nabla^2 W^0(x) & 0_2 \end{pmatrix},$$

by Remark B.1, the variational equation along the homothetic solution reads

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{d}{dz}\Phi^t(z)\Big|_{z=z_\xi}\zeta\right) = \begin{pmatrix} 0_2 & I_2 \\ \nabla^2 W^0(\hat{x}_\xi(t)) & 0_2 \end{pmatrix} \dfrac{d}{dz}\Phi^t(z)\Big|_{z=z_\xi}\zeta, \\ \dfrac{d}{dz}\Phi^0(z)\Big|_{z=z_\xi}\zeta = \zeta, \end{cases}$$

for every $\zeta \in \mathcal{T}_{z_\xi}(\mathcal{U} \times \mathcal{V})$. In this way, writing

$$\begin{pmatrix} q(t) \\ w(t) \end{pmatrix} = \frac{d}{dz}\Phi^t(z)\Big|_{z=z_\xi}\zeta,$$

we see that $q(t)$ must satisfy the problem

$$\begin{cases} \ddot{q} = \nabla^2 W^0(\hat{x}_\xi(t))q \\ q(0) = \pi_x \zeta. \end{cases} \tag{2.26}$$

Now, we can decompose q in two orthogonal components

$$q = q_\xi + q_\tau = \langle q, s_\xi \rangle s_\xi + \langle q, s_\tau \rangle s_\tau$$

and so, by the first equation in (2.26), we get

$$\begin{aligned} \ddot{q}_\xi + \ddot{q}_\tau &= \nabla^2 W^0(\hat{x}_\xi(t))q_\xi + \nabla^2 W^0(\hat{x}_\xi(t))q_\tau \\ &= \langle q, s_\xi \rangle \nabla^2 W^0(\hat{x}_\xi(t))s_\xi + \langle q, s_\tau \rangle \nabla^2 W^0(\hat{x}_\xi(t))s_\tau. \end{aligned}$$

From (A.2) (Appendix (A)), the vectors $\nabla^2 W^0(\hat{x}_\xi(t))s_\xi$ and $\nabla^2 W^0(\hat{x}_\xi(t))s_\tau$ are respectively parallel to s_ξ and s_τ. Thus, problem (2.26) can be projected along the tangential direction s_τ to finally obtain

$$\begin{cases} \ddot{q}_\tau = \langle \nabla^2 W^0(\hat{x}_\xi(t))s_\tau, s_\tau \rangle q_\tau \\ q_\tau(0) = \langle \pi_x \zeta, s_\tau \rangle s_\tau. \end{cases}$$

and conclude the proof. □

In the proof of the next lemma we are going to use again the differential of the flow Φ. Therefore, following again Appendix B, it is useful to provide a characterization of the elements of the tangent space of the surface Σ. If we take into account the maps

$$H(x, v) = \frac{1}{2}|v|^2 - W^0(x), \quad G(x, v) = |x|^2 - R^2, \ (x, v) \in \mathcal{E}$$

with gradients

$$\nabla H(x, v) = (-\nabla W^0(x), v), \quad \nabla G(x, v) = 2(x, 0),$$

we note that

$$\mathcal{E} = H^{-1}(-1), \quad \Sigma = H^{-1}(-1) \cap G^{-1}(0).$$

Therefore, for a point $(x, v) \in \Sigma$, we have that

$$(q, w) \in \mathcal{T}_{(x,v)}\Sigma \Leftrightarrow \begin{cases} \langle v, w \rangle - \langle \nabla W^0(x), q \rangle = 0 \\ \langle x, q \rangle = 0, \end{cases}$$

and, if $\xi \in \partial B_R(0)$ is a central configuration for W^0, from (2.21) we get that

$$(q, w) \in \mathcal{T}_{(\xi, -v_\xi)}\Sigma \Leftrightarrow \begin{cases} \langle v_\xi, w \rangle = 0 \\ \langle \xi, q \rangle = 0. \end{cases} \tag{2.27}$$

Lemma 2.3.7. *Given $\xi \in \partial B_R(0)$ central configuration for W^0 such that $U''(\vartheta_\xi) \geq 0$, the jacobian matrix*

$$\pi_x \frac{\partial}{\partial v}[\Phi^{T(x,v)}(x,v)] \tag{2.28}$$

is invertible in (ξ, v_ξ).

Proof. Recall that $(\xi, v_\xi) = z_\xi$ and assume by contradiction that the matrix (2.28) is not invertible in z_ξ. Following again the notations of Proposition 2.3.3, from Lemma 2.3.5 and, in particular, from (2.25) it is clear that

$$\frac{\partial}{\partial v}\left[\Phi^{T(x,v)}(x,v)\right]_{(x,v)=z_\xi} = dg(z_\xi)\Big|_{\{0\}\times\mathcal{T}_{v_\xi}(\mathcal{V})}.$$

This means that our absurd hypothesis can be translated as follows: there exists $\overline{\zeta} = (0, \overline{w}) \in \mathcal{T}_{(\xi,v_\xi)}(\mathcal{U}\times\mathcal{V})$, with $\overline{w} \neq 0$ such that

$$\pi_x dg(z_\xi)\overline{\zeta} = 0.$$

In this way, by Lemma 2.3.5, we have that

$$\pi_x \left(\frac{d}{dz}\Phi^{T_\xi}(z)\Big|_{z=z_\xi}\overline{\zeta} + F(\Phi^{T_\xi}(z_\xi))\langle\nabla T(z_\xi), \overline{\zeta}\rangle\right) = 0$$

and so

$$\pi_x \frac{d}{dz}\Phi^{T_\xi}(z)\Big|_{z=z_\xi}\overline{\zeta} = -\langle\nabla T(z_\xi), \overline{\zeta}\rangle\pi_x F(\Phi^{T_\xi}(z_\xi)) = \langle\nabla T(z_\xi), \overline{\zeta}\rangle v_\xi,$$

recalling that, by (2.25)

$$\pi_x \frac{d}{dz}\Phi^{T_\xi}(z)\Big|_{z=z_\xi}\overline{\zeta} \in \pi_x\mathcal{T}_{(\xi,-v_\xi)}\Sigma. \tag{2.29}$$

At this point, since ξ and v_ξ are parallel, by (2.27) and (2.29) we deduce that necessarily

$$\langle\nabla T(z_\xi), \overline{\zeta}\rangle = 0.$$

This means that, if we take

$$q(t) = \pi_x \frac{d}{dz}\Phi^t(z)\Big|_{z=z_\xi}\overline{\zeta}$$

then

$$q(T_\xi) = 0$$

and thus, clearly

$$q_\tau(T_\xi) = \langle q(T_\xi), s_\tau\rangle s_\tau = 0.$$

Now, from Lemma 2.3.6, we know that the projection of q on the direction s_τ, $q_\tau(t)$, solves the Sturm-Liouville problem

$$\begin{cases} \ddot{q}_\tau + c(t)q_\tau = 0 \\ q_\tau(0) = 0 = q_\tau(T_\xi), \end{cases} \tag{2.30}$$

where by (A.4) (Appendix A)

$$c(t) \doteq -\langle \nabla^2 W^0(\hat{x}_\xi(t))s_\tau, s_\tau\rangle = |\hat{x}_\xi(t)|^{-\alpha-2}(\alpha U(\vartheta_\xi) - U''(\vartheta_\xi)).$$

The function $u(t) \doteq |\hat{x}_\xi(t)|$ is nothing but a normalized version of the 1-dimensional homothetic trajectory already studied in Paragraph 2.3.1. Indeed, $u(t) = |\hat{x}_\xi(t)| = \lambda(t)R$, where $\lambda(t)$ solves the 1-dimensional α-Kepler problem

$$\begin{cases} \ddot{\lambda} + \alpha R^{-\alpha-2}U(\vartheta_\xi)\lambda^{-\alpha-1} = 0 \\ \lambda(0) = 1 = \lambda(T_\xi), \end{cases}$$

and so, by calculation, u solves

$$\begin{cases} \ddot{u} + \alpha u^{-\alpha-2}U(\vartheta_\xi)u = 0 \\ u(0) = R = u(T_\xi). \end{cases} \tag{2.31}$$

Now, since $U''(\vartheta_\xi) \geq 0$, we have that

$$c(t) \leq \alpha u(t)^{-\alpha-2}U(\vartheta_\xi)$$

and therefore, if we apply the Sturm comparison theorem to (2.30) and (2.31) we have that there exists $\overline{T} \in (0, T_\xi)$ such that $u(\overline{T}) = 0$. This is finally a contradiction and concludes the proof, since $|\hat{x}_\xi(t)|$ cannot be null in the interval $[0, T_\xi]$. □

Remark 2.3.8. *Following the notation of the previous proof, since by* (A.3) *(Appendix A) we have that*

$$c_1(t) \doteq -\langle \nabla^2 W^0(\hat{x}_\xi(t))s_\xi, s_\xi\rangle = -u(t)^{-\alpha-2}\alpha(\alpha+1)U(\vartheta_\xi)$$

one could think to study the Sturm-Liouville problem

$$\begin{cases} \ddot{q}_\xi + c_1(t)q_\xi = 0 \\ q_\xi(0) = 0 = q_\xi(T_\xi) \end{cases}$$

instead of problem (2.30)*. Since it is always true that*

$$c_1(t) \leq \alpha u(t)^{-\alpha-2}U(\vartheta_\xi),$$

then we should drop the hypothesis $U''(\vartheta_\xi) \geq 0$. However, this would not lead to a contradiction in our argument, since in this case $q_\xi(t)$ and $u(t)$ are proportional and so we would not deduce from the Sturm theorem the existence of a null point for u in the interval $(0, T_\xi)$.

At this point the proof of Theorem 2.3.4 is very easy to get.

Proof of Theorem 2.3.4. It is enough to observe that

$$\frac{\partial}{\partial v}x_1(v)\big|_{v=v_\xi} = \frac{\partial}{\partial v}\pi_x\left[\Phi^{T(\xi,v)(\xi,v)}\right]_{v=v_\xi} = \pi_x\frac{\partial}{\partial v}\left[\Phi^{T(x,v)}(x,v)\right]_{(x,v)=(\xi,v_\xi)},$$

which is invertible for Lemma 2.3.7. □

Now, we are ready to prove the main result of this section, which concerns the existence of outer arcs for the anisotropic Kepler problem.

Theorem 2.3.9. *Let $\xi = (R\cos\vartheta_\xi, R\sin\vartheta_\xi)$ be a central configuration for W^0. Assume that*

$$U''(\vartheta_\xi) \geq 0.$$

Then, there exists a neighbourhood $\mathcal{U}_\xi$ of ξ on ∂B_R such that, for any $p_0, p_1 \in \mathcal{U}_\xi$ there exist $\overline{T} > 0$ and a solution $x = x(t)$ of

$$\begin{cases} \ddot{x}(t) = \nabla W^0(x(t)), & t \in [0,\overline{T}] \\ \frac{1}{2}|\dot{x}(t)|^2 - W^0(x(t)) = -1, & t \in [0,\overline{T}] \\ |x(t)| > R, & t \in (0,\overline{T}) \\ x(0) = p_0,\ x(\overline{T}) = p_1. & \end{cases}$$

Moreover, x depends on a $\mathcal{C}^1$-manner on the endpoints p_0, p_1.

Proof. Define the shooting map

$$\begin{aligned} \Psi\colon \mathcal{U}\times\mathcal{U}\times\mathcal{V} &\to \mathbb{R}^2 \\ (p_0,p_1,v_0) &\mapsto \Psi(p_0,p_1,v_0) \doteq x(T(p_0,v_0);p_0,v_0) - p_1, \end{aligned}$$

where the sets $\mathcal{U}$ and $\mathcal{V}$ are respectively the neighbourhoods of ξ and v_ξ found in Proposition 2.3.3, $T\colon \mathcal{U}\times\mathcal{V}$ is the $\mathcal{C}^1$ first return map defined in the same proposition and $x(\cdot;p_0,v_0)$ is the unique solution of the Cauchy problem

$$\begin{cases} \ddot{x}(t) = \nabla W^0(x(t)) \\ x(0) = p_0, \quad \dot{x}(0) = v_0 \end{cases} \tag{2.32}$$

in the time interval $[0, T(p_0,v_0)]$. Note that, following the notation of Lemma 2.3.7, we have

$$x(t;p_0,v_0) = \pi_x\Phi^t(p_0,v_0), \quad \text{for every } t\in[0,T(p_0,v_0)].$$

The map Ψ is $\mathcal{C}^1$ in its domain both for the $\mathcal{C}^1$ dependence of the solutions of the Cauchy problem (2.32) on initial data and time and for the differentiability of the first return map T (see Proposition 2.3.3). Moreover, we have that

$$\Psi(\xi,\xi,v_\xi) = x(T(\xi,v_\xi);\xi,v_\xi) - \xi = \pi_x\Phi^{T_\xi}(\xi,v_\xi) - \xi = 0$$

and

$$\frac{\partial \Psi}{\partial v_0}(p_0, p_1, v_0)\Big|_{(\xi,\xi,v_\xi)} = \frac{\partial}{\partial v_0} x(T(p_0, v_0); p_0, v_0)\Big|_{(\xi,v_\xi)} = \pi_x \frac{\partial}{\partial v}\left[\Phi^{T(p,v)}(p, v)\right]_{(\xi,v_\xi)},$$

which is invertible thanks to Lemma 2.3.7. Therefore, by the Implicit Function Theorem, we have that there exist a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of v_ξ, a neighbourhood $\mathcal{U}_\xi \subseteq \mathcal{U}$ of ξ and a unique $\mathcal{C}^1$ function $\eta\colon \mathcal{U}_\xi \times \mathcal{U}_\xi \to \mathcal{V}'$ such that $\eta(\xi, \xi) = v_\xi$ and

$$\Psi(p_0, p_1, \eta(p_0, p_1)) = 0 \quad \text{for every } (p_0, p_1) \in \mathcal{U}_\xi \times \mathcal{U}_\xi.$$

This actually means that, if we fix $(p_0, p_1) \in \mathcal{U}_\xi \times \mathcal{U}_\xi$, we can find a solution x of (2.32), defined in the time interval $[0, \overline{T}]$, with $v_0 = \eta(p_0, p_1)$ and $\overline{T} = T(p_0, \eta(p_0, p_1)) = T(p_0, v_0)$. Furthermore, note that this solution has constant energy -1, since

$$(p_0, \eta(p_0, p_1)) = (p_0, v_0) \in \mathcal{U}_\xi \times \mathcal{V}' \subset \mathcal{U} \times \mathcal{V} \subset \Sigma \subset \mathcal{E}.$$

The $\mathcal{C}^1$-dependence on initial data is a straightforward consequence of the Implicit Function Theorem. □

2.3.3. Outer solution arcs for the N-centre problem

We conclude this section with the proof of the existence of an outer solution arc for the anisotropic N-centre problem driven by V^ε. As a starting point, we recall that, by Proposition 2.2.2, if $|y| > R > 0$, then

$$V^\varepsilon(y) = W^0(y) + O(\varepsilon^\gamma), \quad \text{as } \varepsilon \to 0^+$$

for a suitable $\gamma > 0$. This suggests to repeat the proof of Theorem 2.3.9, this time taking into account the perturbation induced by the presence of the centres. Before we start with the proof, it is useful to recall the set of strictly minimal central configurations of W^0, defined as

$$\Xi = \{\vartheta^* \in \mathbb{S}^1 : \ U'(\vartheta^*) = 0 \text{ and } U''(\vartheta^*) > 0\} = \{\vartheta_0^*, \ldots, \vartheta_{m-1}^*\}.$$

Note that, actually, as it is clear from the assumptions of Theorem 2.3.9, it would be enough to require the (not necessarily strict) minimality of the above central configurations. Beside that, the non-degeneration of such critical points will be a fundamental requirement on Section 2.5 and however we decide to keep it since it is a natural assumption in anisotropic settings (see for instance [12, 11, 5]).

Theorem 2.3.10. *Assume that the assumptions* (V) *on the potentials* $(V_j)_{j=1}^N$ *are satisfied and fix* $R > 0$ *as in* (2.14)*. Then, there exists* $\varepsilon_{ext} > 0$ *such that, for any* $\vartheta^* \in \Xi$ *minimal non-degenerate central configuration for* W^0*, defining* $\xi^* \doteq Re^{i\vartheta^*}$*, there exists a neighbourhood* $\mathcal{U}_{ext}(\xi^*)$ *of* ξ^* *on* ∂B_R *with the following property:*

for every $\varepsilon \in (0, \varepsilon_{ext})$, for any pair of endpoints $p_0, p_1 \in \mathcal{U}_{ext}(\xi^*)$, there exist $T_{ext} = T_{ext}(p_0, p_1; \varepsilon) > 0$ and a unique solution $y_{ext}(t) = y_{ext}(t; p_0, p_1; \varepsilon)$ of the outer problem

$$\begin{cases} \ddot{y}_{ext}(t) = \nabla V^\varepsilon(y_{ext}(t)) & t \in [0, T_{ext}] \\ \frac{1}{2}|\dot{y}_{ext}(t)|^2 - V^\varepsilon(y_{ext}(t)) = -1 & t \in [0, T_{ext}] \\ |y_{ext}(t)| > R & t \in (0, T_{ext}) \\ y_{ext}(0) = p_0, \quad y_{ext}(T_{ext}) = p_1. \end{cases}$$

Moreover, the solution depends on a $\mathcal{C}^1$-manner on its endpoints p_0 and p_1.

Proof. Recalling the definition (2.12) of $\tilde{\varepsilon}$, define the shooting map

$$\begin{aligned} \Psi \colon [0, \tilde{\varepsilon}) \times \mathcal{U} \times \mathcal{U} \times \mathcal{V} &\to \mathbb{R}^2 \\ (\varepsilon, p_0, p_1, v_0) &\mapsto \Psi(\varepsilon, p_0, p_1, v_0) \doteq y(T(p_0, v_0); p_0, v_0; \varepsilon) - p_1, \end{aligned}$$

where the sets $\mathcal{U}$ and $\mathcal{V}$ are respectively the neighbourhoods of ξ and v_ξ found in Proposition 2.3.3, $T \colon \mathcal{U} \times \mathcal{V} \to \mathbb{R}^+$ is the $\mathcal{C}^1$ first return map defined in the same proposition and $y(\cdot; p_0, v_0; \varepsilon)$ is the unique solution of the Cauchy problem

$$\begin{cases} \ddot{y}(t) = \nabla V^\varepsilon(y(t)) \\ y(0) = p_0, \quad \dot{y}(0) = v_0, \end{cases} \tag{2.33}$$

in the time interval $[0, T(p_0, v_0)]$. Note that, following the notation of Lemma 2.3.7, we have

$$y(t; p_0, v_0; 0) = \pi_x \Phi^t(p_0, v_0) \quad \text{for every } t \in [0, T(p_0, v_0)].$$

Moreover, the map Ψ is $\mathcal{C}^1$ in its domain both for the dependence of the solutions of (2.33) on the initial data and time, for the differentiability of V^ε with respect to ε (see Proposition 2.2.2) and for the $\mathcal{C}^1$ differentiability of the map T in $\mathcal{U} \times \mathcal{V}$ (see Proposition 2.3.3).

We furthermore note that

$$\Psi(0, \xi, \xi, v_\xi) = y(T(\xi, v_\xi); \xi, v_\xi; 0) - \xi = \pi_x \Phi^{T_\xi}(\xi, v_\xi) - \xi = 0$$

and

$$\begin{aligned} \frac{\partial \Psi}{\partial v_0}(\varepsilon, p_0, p_1, v_0)\Big|_{(0,\xi,\xi,v_\xi)} &= \frac{\partial}{\partial v_0} y(T(p_0, v_0); p_0, v_0; \varepsilon)\Big|_{(0,\xi,\xi,v_\xi)} \\ &= \pi_x \frac{d}{dv}\left[\Phi^{T(p,v)}(p, v)\right]_{(\xi, v_\xi)}, \end{aligned}$$

which is invertible thanks to Lemma 2.3.7 (see also Figure 2.3). Therefore, by the Implicit Function theorem, we have that there exist a neighbourhood $\mathcal{V}' \subset \mathcal{V}$ of v_ξ, $\varepsilon_{ext} \in (0, \tilde{\varepsilon})$, a neighbourhood $\mathcal{U}_{ext}(\xi^*) \subset \mathcal{U}$ of ξ and a unique $\mathcal{C}^1$ function $\eta \colon [0, \varepsilon_{ext}) \times \mathcal{U}_{ext}(\xi^*) \times \mathcal{U}_{ext}(\xi^*) \to \mathcal{V}'$ such that $\eta(0, \xi, \xi) = v_\xi$ and

$$\Psi(\varepsilon, p_0, p_1, \eta(\varepsilon, p_0, p_1)) = 0 \quad \text{for every } (\varepsilon, p_0, p_1) \in [0, \varepsilon_{ext}) \times \mathcal{U}_{ext}(\xi^*) \times \mathcal{U}_{ext}(\xi^*).$$

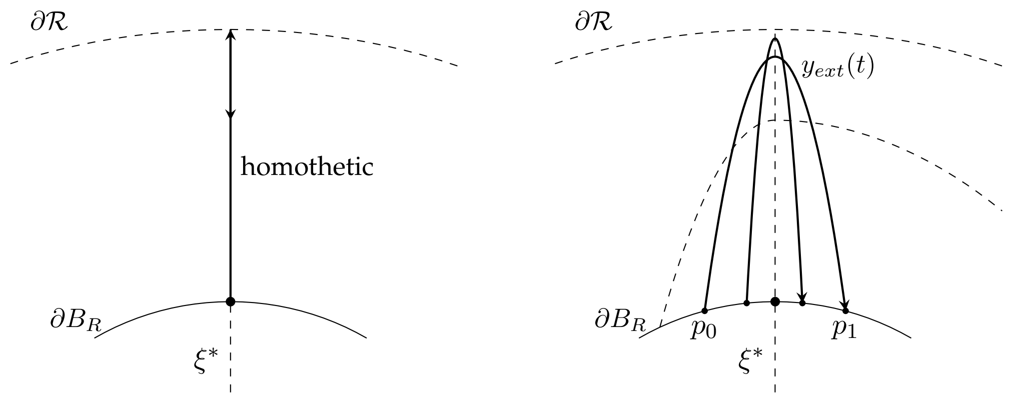

Figure 2.3.: The proof of Theorem 2.3.10: here $\partial\mathcal{R}$ denotes the boundary of the Hill's region for the rescaled N-centre problem driven by V^ε. On the left side we have drawn the homothetic trajectory through ξ^*: it is a 1-dimensional motion that starts in ξ^*, it reaches the boundary $\partial\mathcal{R}$ and then it hits again ∂B_R in ξ^*. On the right we can see that, if we shoot with initial position sufficiently close to ξ^*, there will be a first return on the sphere, guaranteed by the transversality of the flow. On the other hand, the dashed trajectory on the right could never reach again the sphere since its starting point is outside the existence neighbourhood provided in the theorem.

This actually means that, if we fix $\varepsilon \in [0, \varepsilon_{ext})$ and $(p_0, p_1) \in \mathcal{U}_{ext}(\xi^*) \times \mathcal{U}_{ext}(\xi^*)$, we can find a unique solution y_{ext} of (2.15), defined in the time interval $[0, T_{ext}]$, starting with velocity $v_0 = \eta(\varepsilon, p_0, p_1)$ and such that $T_{ext} = T(p_0, \eta(\varepsilon, p_0, p_1))$ in the fashion of Proposition 2.3.3. Finally, note that this solution has constant energy -1, since

$$(p_0, \eta(\varepsilon, p_0, p_1)) \in \mathcal{U}_{ext}(\xi^*) \times \mathcal{V}' \subset \mathcal{U} \times \mathcal{V} \subset \Sigma \subset \mathcal{E}.$$

To conclude, the $\mathcal{C}^1$-dependence on the endpoints is a straightforward consequence on the perturbation technique used in the proof. □

We conclude this section providing upper and lower bounds for the time interval in which an external solution is defined, that will be useful later in this work.

Lemma 2.3.11. *Let $\varepsilon \in (0, \varepsilon_{ext})$, let $\vartheta^* \in \mathbb{S}^1$ be a minimal non-degenerate central configuration for W^0 and $\mathcal{U}_{ext}(\xi^*)$ be its neighbourhood on ∂B_R found in Theorem 2.3.10. Let $p_0, p_1 \in \mathcal{U}_{ext}(\xi^*)$ and let $y_{ext}(\cdot; p_0, p_1; \varepsilon)$ be the unique solution found in Theorem 2.3.10, defined in its time interval $[0, T_{ext}(p_0, p_1; \varepsilon)]$. Then, there exist $c, C > 0$ such that*

$$c \leq T_{ext}(p_0, p_1; \varepsilon) \leq C.$$

Such constants do not depend on the choice of p_0, p_1 inside the neighbourhood.

Proof. The proof is a direct consequence of the continuous dependence of the solution on initial data and of its perturbative nature. □

2.4. Inner dynamics

This section is named *Inner dynamics* since we will look for solution arcs of the rescaled N-centre problem (2.6), which bridge any pair of points of ∂B_R ($R >> \varepsilon > 0$ already chosen in Section 2.3) and lie inside the ball B_R along their motion. It is clear that the main difficulty is given by the possible interactions with the centres; indeed, since we are looking for classical solutions with fixed end-points, we need to avoid every possible collision. Moreover, since we will be working inside B_R, we cannot make use of Proposition 2.2.2 and thus perturbation techniques do not apply in this case. For this reason, following [61], we opt for a variational approach and our inner solution arcs will be (reparametrizations of) minimizers of a suitable geometric functional. In the last two sections of this paper we will build closed periodic orbits for the anisotropic N-centre problem as a juxtaposition of outer and inner arcs using a broken geodesics technique and, as a corollary, we will link this result with the presence of a symbolic dynamics. In the Section 2.1 we have already defined those symbols that will compose the alphabet of our dynamics, which can be roughly thought as all the possible choices of a suitable partition of the centres and of a central configuration for the leading potential W^0. With some intermediate steps, we will define a suitable topological constraint that forces every inner arc to separate the centres according to a prescribed partition. To be clear, the main result of this section is to prove that, for $\varepsilon > 0$ sufficiently small and for any $p_1, p_2 \in \partial B_R$, there exists a solution $y(\cdot; p_1, p_2; \varepsilon)$ of the following problem

$$\begin{cases} \ddot{y}(t) = \nabla V^\varepsilon(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - V^\varepsilon(y(t)) = -1 & t \in [0,T] \\ |y(t)| < R & t \in (0,T) \\ y(0) = p_1, \quad y(T) = p_2, \end{cases} \tag{2.34}$$

for some $T > 0$, possibly depending on ε, and such that the trajectory y separates the centres according to a chosen partition.

2.4.1. Functional setting and variational principles

In order to follow a variational approach, we introduce the set of admissible paths on which we will minimize some suitable geometric functionals. We build our setting referring to the starting equations (2.1)-(2.3) and thus we take into account again the potential V and the energy $-h < 0$ is fixed. However, we notice that a scaling on the centres, and thus on the whole problem (see problem (2.9)), does not affect the following discussion. Recalling the notations of the Section 2.1, we fix p_1, p_2 inside the open Hill's region $\mathring{\mathcal{R}}_h$ (see (2.4)) and we define

$$\hat{H} = \hat{H}_{p_1,p_2}([a,b]) \doteq \left\{ u \in H^1([a,b];\mathbb{R}^2) \;\middle|\; \begin{array}{l} u(a) = p_1, \; u(b) = p_2, \\ u(t) \neq \; c_j \; \forall \, t \in [a,b], \; \forall \, j \end{array} \right\},$$

i.e., all the H^1-paths that join p_1, p_2 and do not collapse on the centres, and also the H^1-collision paths

$$\mathfrak{Coll} = \mathfrak{Coll}_{p_1,p_2}([a,b]) \doteq \left\{ u \in H^1([a,b];\mathbb{R}^2) \,\middle|\, \begin{array}{l} u(a) = p_1,\ u(b) = p_2,\ \exists\, t \in [a,b], \\ \exists\, j \in \{1,\dots,N\} \text{ s.t. } u(t) = c_j \end{array} \right\}.$$

We introduce also the set

$$\begin{aligned} H = H_{p_1,p_2}([a,b]) &\doteq \hat{H}_{p_1,p_2}([a,b]) \cup \mathfrak{Coll}_{p_1,p_2}([a,b]) \\ &= \{u \in H^1([a,b];\mathbb{R}^2) :\ u(a) = p_1,\ u(b) = p_2\} \end{aligned}$$

and it is easy to check that H is the closure of $\hat{H}$ with respect to the weak topology of $H^1([a,b];\mathbb{R}^2)$. Let us define the Maupertuis' functional as

$$\begin{aligned} \mathcal{M}_h(\cdot) \doteq \mathcal{M}_h([a,b];\cdot)\colon H_{p_1,p_2}([a,b]) &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ u &\longmapsto \mathcal{M}_h(u) \doteq \frac{1}{2}\int_a^b |\dot{u}(t)|^2\,dt \int_a^b (-h + V(u(t)))\,dt \end{aligned}$$

which is differentiable over the non-collision paths space $\hat{H}$. The next classical result, known as the *Maupertuis' principle* (see [3]), establishes a link between classical solutions of the equation $\ddot{x} = \nabla V(x)$ at energy $-h$ and critical points at a positive level of $\mathcal{M}_h$ in the space $\hat{H}$. Note that, if $\mathcal{M}_h(u) > 0$ for some $u \in H$, then we can define the positive quantity

$$\omega^2 \doteq \frac{\int_a^b (-h + V(u))}{\frac{1}{2}\int_a^b |\dot{u}|^2}, \tag{2.35}$$

that plays and important role in the next result.

Theorem 2.4.1 (The Maupertuis' principle). *Let $u \in \hat{H}_{p_1,p_2}([a,b])$ be a critical point of $\mathcal{M}_h$ at a positive level and let $\omega > 0$ be defined by* (2.35)*. Then, $x(t) \doteq u(\omega t)$ is a classical solution of the fixed-end problem*

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) & t \in [a/\omega, b/\omega] \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = -h & t \in [a/\omega, b/\omega] \\ x(a/\omega) = p_1,\ x(b/\omega) = p_2 \end{cases}$$

while u itself is a classical solution of

$$\begin{cases} \omega^2\ddot{u}(t) = \nabla V(u(t)) & t \in [a,b] \\ \frac{\omega^2}{2}|\dot{u}(t)|^2 - V(u(t)) = -h & t \in [a,b] \\ u(a) = p_1,\ u(b) = p_2. \end{cases}$$

The converse holds also true, i.e., if x is a classical solution of the fixed-end problem above in a certain interval $[a', b']$, then, setting $\omega = 1/(a' - b')$, $u(t) = x(t/\omega)$ is a critical point of $\mathcal{M}_h([a,b];\cdot)$ at a positive level, for some suitable values a, b.

Proof. Since u is a critical point for $\mathcal{M}_h$, then

$$\int_a^b \langle \dot{u}, \dot{v} \rangle \int_a^b (-h + V(u)) + \frac{1}{2} \int_a^b |\dot{u}|^2 \int_a^b \langle \nabla V(u), v \rangle = 0, \quad \forall\, v \in H_0^1([a,b];\mathbb{R}^2).$$

Since $\mathcal{M}_h(u) > 0$ we thus have

$$\omega^2 \int_a^b \langle \dot{u}, \dot{v} \rangle = - \int_a^b \langle \nabla V(u), v \rangle, \quad \forall\, v \in H_0^1([a,b];\mathbb{R}^2)$$

and so u solves weakly (and thus classically by routine regularity arguments)

$$\omega^2 \ddot{u}(t) = \nabla V(u(t)), \quad \forall\, t \in [a,b].$$

If we define

$$g(t) \doteq \frac{\omega^2}{2} |u(t)|^2 - V(u(t))$$

we immediately get that $g(t)$ is constant in $[a,b]$ and, by (2.35) we necessarily have that $g(t) \equiv -h$ and the energy conservation follows.

Moreover, defining $x(t) \doteq u(\omega t)$ we have

$$\ddot{x}(t) = \nabla V(x(t)), \quad \forall\, t \in [a,b].$$ □

In order to apply direct methods of the Calculus of Variations to $\mathcal{M}_h$ we will work in H, which is weakly closed in H^1. As a first step we show that a (possibly colliding) minimizer of $\mathcal{M}_h$ in H preserves the energy almost everywhere.

Lemma 2.4.2. *If $u \in H$ is a minimizer of $\mathcal{M}_h$ at a positive level, then*

$$\frac{\omega^2}{2} |\dot{u}(t)|^2 - V(u(t)) = -h \quad \textit{for a.e. } t \in [a,b].$$

Proof. It is enough to observe that u is an extremal with respect to time reparametrizations which keep the ends fixed, i.e., if $\varphi \in \mathcal{C}_c^\infty(a,b)$ and we define $u_\lambda(t) \doteq u(t + \lambda\varphi(t))$ for $\lambda \in \mathbb{R}^+$, then

$$\frac{d}{d\lambda} \mathcal{M}_h(u_\lambda) \Big|_{\lambda=0} = 0. \tag{2.36}$$

Let us prove (2.36). If λ is small enough then the function $t \to t + \lambda\varphi(t)$ is increasing in $[a,b]$ and so it is invertible. Through the change of variable $s = t + \lambda\varphi(s)$ we have

$$\mathcal{M}_h(u_\lambda) = \frac{1}{2} \int_a^b |\dot{u}(s)|^2 (1 + \lambda\dot{\varphi}(t(s)))\, ds \int_a^b \frac{V(u(s)) - h}{1 + \lambda\dot{\varphi}(t(s))}\, ds,$$

where $t(s) = s - \lambda\varphi(t)$. The one-parameter family of functions $t_\lambda(s) \doteq t(s)$ uniformly converges to 0 in $[a,b]$ as $\lambda \to s$ since

$$|t_\lambda(s) - s| \le \lambda \|\varphi\|_\infty, \quad \forall\, t \in [a,b].$$

For this reason we have

$$
\begin{aligned}
\frac{d}{d\lambda}\mathcal{M}_h(u_\lambda)\Big|_{\lambda=0} &= \frac{1}{2}\int_a^b |\dot{u}(s)|\dot{\varphi}(s)^2\, ds \int_a^b (-h+V(u(s)))\; ds \\
&\quad - \frac{1}{2}\int_a^b |\dot{u}(s)|^2\, ds \int_a^b (-h+V(s))\,\dot{\varphi}(s)\, ds \\
&= \int_a^b \left[\frac{1}{2}\left(\int_a^b (V(s)-h)\; ds\right)|\dot{u}(s)|^2 - \frac{1}{2}\|\dot{u}\|_2^2\,(-h+V(u(s)))\right]\dot{\varphi}(s)\, ds \\
&= 0
\end{aligned}
$$

for every $\varphi \in \mathcal{C}_0^\infty(a,b)$, and thus (2.36) is proved. In particular, we deduce that

$$
\frac{1}{2}\left(\int_a^b (V(s)-h)\; ds\right)|\dot{u}(s)|^2 - \frac{1}{2}\|\dot{u}\|_2^2\,(-h+V(u(s))) = k \quad \text{a.e. in } [a,b]
$$

for some $k \in \mathbb{R}$. Now, since $\mathcal{M}_h(u) > 0$ we have

$$
\frac{\omega^2}{2}|\dot{u}(s)|^2 = V(u(s)) - h + k, \quad \text{a.e. in } [a,b]
$$

and from (2.35) the proof is complete. □

The lack of additivity of $\mathcal{M}_h$ induces the introduction of the Jacobi-length functional

$$
\mathcal{L}_h(u) \doteq \int_0^1 |\dot{u}(t)|\sqrt{-h+V(u(t))}\, dt
$$

whose domain is the weak H^1-closure of the set

$$
H_h^{p_1,p_2}([a,b]) \doteq \left\{ u \in H^1([a,b];\mathbb{R}^2) \;\middle|\; \begin{array}{l} u(a) = p_1,\ u(b) = p_2, \\ V(u(t)) > h,\ |\dot{u}(t)| > 0,\ \text{for every } t \in [a,b] \end{array} \right\}.
$$

Indeed, Theorem 2.4.1 could be rephrased for $\mathcal{L}_h$ and thus classical solutions will be suitable reparametrizations of critical points of $\mathcal{L}_h$ (see for instance [54] and Appendix C for more precise details on this functional). Finally, we notice that the Maupertuis' functional is not additive, while it is well-known that the Jacobi-length functional is and it is also invariant under reparametrizations, since it is a length. Despite that, exploiting the correspondence which stands between minimizers of $\mathcal{M}_h$ and minimizers of the Jacobi-length functional (see Proposition C.3), an easy proof leads to the following proposition.

Proposition 2.4.3. *Let u be a minimizer of $\mathcal{M}_h([a,b];\cdot)$ in $H_{p_1,p_2}([a,b])$. Then, for any subinterval $[c,d] \subseteq [a,b]$, the restriction $u|_{[c,d]}$ is a minimizer of $\mathcal{M}_h([c,d];\cdot)$ in the space $H_{u(c),u(d)}([c,d])$.*

2.4.2. Minimizing through direct methods

At this point, we go back to the N-centre problem (2.34), introducing the notation $c'_j \doteq \varepsilon c_j$ for the ε-centres included in B_ε. We aim to prove the existence of a minimizer for the Maupertuis' functional, requiring the following topological constraint: an inner arc has to cross the ball B_ε, dividing the centres into two non-trivial subsets. This can be done introducing the winding number with respect to every centre; but since a path in $\hat{H}$ is not necessarily closed, we need to close it artificially. Let us fix $[a,b] \subseteq \mathbb{R}$, $p_1, p_2 \in \partial B_R$ and write

$$p_1 = Re^{i\vartheta_1}, \quad p_2 = Re^{i\vartheta_2}$$

for $\vartheta_1, \vartheta_2 \in [0, 2\pi)$. For $u \in \hat{H}_{p_1,p_2}([a,b])$, if $p_1 \neq p_2$ we close u glueing an arc of ∂B_R in counter-clockwise direction, i.e., we define

$$\Gamma_u(t) \doteq \begin{cases} \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\vartheta_2)} & t \in (b, b+\vartheta_1+2\pi-\vartheta_2) \end{cases} & \text{if } \vartheta_1 < \vartheta_2 \\ u(t) & t \in [a,b] \qquad\qquad\qquad\qquad \text{if } \vartheta_1 = \vartheta_2 \\ \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\vartheta_2)} & t \in (b, b+\vartheta_1-\vartheta_2) \end{cases} & \text{if } \vartheta_1 > \vartheta_2 \end{cases}$$

and so, we can introduce the winding number of u with respect to a centre c'_j as

$$\text{Ind}(u; c'_j) \doteq \frac{1}{2\pi i} \int_{\Gamma_u} \frac{dz}{z - c'_j} \in \mathbb{Z}, \quad \text{for all } j = 1, \dots, N.$$

Since a path u has to separate the centres with respect to a given partition in two non-trivial subsets, we can choose the parity of the winding numbers $\text{Ind}(u; c_j)$ as a dichotomy property. Following this, we introduce the set of *admissible winding vectors*

$$\mathfrak{I}^N \doteq \{l \in \mathbb{Z}_2^N : \exists\, j,k \in \{1,\dots,N\},\ j \neq k, \text{ s.t. } l_j \neq l_k\} \tag{2.37}$$

and, for $l \in \mathfrak{I}^N$ (which we fix from now on), we consider the class of paths

$$\hat{H}_l \doteq \{u \in \hat{H} : \text{Ind}(u; c'_j) \equiv l_j \pmod 2,\ \forall\, j = 1, \dots, N\}.$$

Of course, the above set is not closed with respect to the weak topology of H^1 and so, as before, we include the collision paths in our minimization set. For $j \in \{1, \dots, N\}$ define the set

$$\mathfrak{Coll}_l^j \doteq \{u \in H : \text{Ind}(u; c'_k) \equiv l_k \pmod 2\ \forall\, k \neq j \text{ and } \exists\, t \in [a,b] \text{ s.t. } u(t) = c'_j\}$$

i.e., the collision paths behaving like a path in $\hat{H}_l$ with respect to every centre, except for c'_j in which the particle collides. In the same way, we can include two collision centres c'_{j_1}, c'_{j_2} defining

$$\mathfrak{Coll}_l^{j_1,j_2} \doteq \left\{ u \in H \;\middle|\; \begin{array}{l} \text{Ind}(u; c'_k) \equiv l_k \pmod 2\ \forall\, k \neq j_1, j_2 \text{ and} \\ \exists\, t_1, t_2 \in [a,b] \text{ s.t. } u(t_1) = c'_{j_1}, u(t_2) = c'_{j_2} \end{array} \right\}$$

and so on

$$\begin{aligned}
&\mathfrak{Coll}_l^{j_1,j_2,j_3} \doteq \dots, \\
&\quad \vdots \\
&\mathfrak{Coll}_l^{1,\dots,N} = \mathfrak{Coll}^{1,\dots,N} \doteq \{u \in H :\ u \text{ collides in every centre}\}.
\end{aligned}$$

At this point, we can collect together all the admissible collision paths with respect to a fixed winding vector $l \in \mathfrak{I}^N$ in the set

$$\mathfrak{Coll}_l \doteq \bigcup_{j=1}^{N} \mathfrak{Coll}_l^j \cup \bigcup_{1\leq j_1<j_2\leq N} \mathfrak{Coll}_l^{j_1,j_2} \cup \dots \cup \mathfrak{Coll}_l^{1,\dots,N}$$

and prove the following result.

Proposition 2.4.4. *The set*

$$H_l \doteq \hat{H}_l \cup \mathfrak{Coll}_l$$

is weakly closed in H^1.

Proof. Take $(u_n) \subseteq H_l$ such that $u_n \rightharpoonup u$ in H^1 and so, in particular, u_n uniformly converges to u in $[a,b]$. Then, if u has a collision then $u \in \mathfrak{Coll}_l$. On the other hand, if u is collision-free, then the uniform convergence implies the existence of $n_0 \in \mathbb{N}$ such that

$$u_n \in \hat{H}_l \ \forall\, n \geq n_0 \implies u \in \hat{H}_l. \qquad \square$$

Finally, we look for solution arcs which lie inside B_R along their trajectory, and so it makes sense to add another constraint on them. For this reason we will restrict our investigation to the sets:

$$\begin{aligned}
\hat{K}_l &\doteq \hat{K}_l^{p_1,p_2}([a,b]) \doteq \{u \in \hat{H}_l :\ |u(t)| \leq R,\ \forall\, t \in [a,b]\} \\
K_l &\doteq K_l^{p_1,p_2}([a,b]) \doteq \{u \in H_l :\ |u(t)| \leq R,\ \forall\, t \in [a,b]\},
\end{aligned} \tag{2.38}$$

(see Figure 2.4 for an explanation on the geometrical meaning of the constraint induced by these sets).

The following proposition guarantees that we are in the convenient setting to perform a variational argument.

Proposition 2.4.5. *The set* K_l *is weakly closed in* H^1.

Proof. The proof is trivial since K_l is a subset of H_l which is stable under uniform convergence. $\square$

For any $u \in K_l = K_l^{p_1,p_2}([0,1])$, we take into account the Maupertuis' functional

$$\mathcal{M}(u) = \frac{1}{2}\int_0^1 |\dot{u}(t)|^2\,dt \int_0^1 (-1 + V^{\varepsilon}(u(t)))\,dt$$

and we remark two facts:

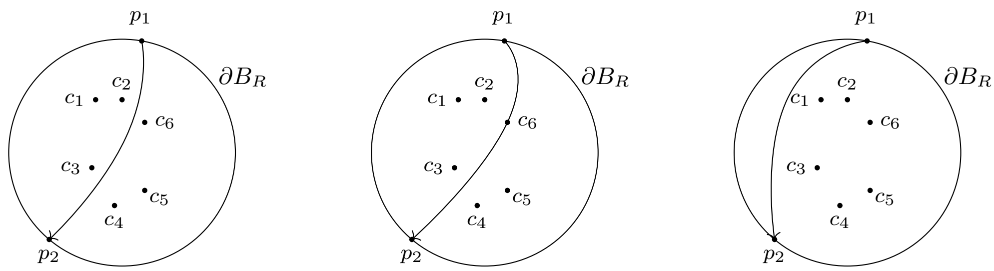

Figure 2.4.: The geometrical interpretation of the topological constraint: in the first picture we have a collision-less path realizing the winding vector $(0,0,0,1,1,1)$; the second arc collides with c_6, thus it belongs to the closed space K_l, with $l = (0,0,0,1,1,0)$ or $l = (0,0,0,1,1,1)$; in the third picture we have a non-admissible path, since it has winding vector $l = (1,1,1,1,1,1) \notin \mathfrak{I}^N$. This also explains why, in order to have the centres geometrically divided by an arc, such arc cannot have indices with the same parity with respect to every centres. Indeed, in this case the path would belong to the space K_l with $l = (1,1,1,1,1,1)$ or $l = (0,0,0,0,0,0)$.

- since $\mathcal{M}$ is invariant under time reparametrizations, we have put $a = 0$ and $b = 1$;
- actually, $\mathcal{M} = \mathcal{M}_1^\varepsilon$, but we have omitted this dependence since we will mainly work with both $\varepsilon > 0$ and the energy fixed. When we will *move* such ε or the energy, we will use the more explicit notations.

We now prove three lemmata in order to apply direct methods to the Maupertuis' functional.

Lemma 2.4.6. *There exists $C > 0$ such that*

$$\mathcal{M}(u) \geq C > 0, \quad \textit{for every } u \in K_l.$$

Proof. Since $u \in K_l$ then $|u(t)| \leq R$ for every $t \in [0,1]$ and so

$$|u(t) - c_j'| \leq R + \varepsilon$$

for every $j = 1, \dots, N$ and for every $t \in [0,1]$. Now, recalling that

$$\mathfrak{m} = \min_{j=1,\dots,N} \min_{\mathbb{S}^1} U_j$$

we have that

$$V^\varepsilon(u(t)) \geq |u(t) - c_1'|^{-\alpha}\mathfrak{m} \geq \frac{\mathfrak{m}}{(R+\varepsilon)^\alpha}, \tag{2.39}$$

for every $t \in [0,1]$. Recalling (2.12) and (2.14), we have that $R \in (\varepsilon, \mathfrak{m}^{1/\alpha} - \varepsilon)$ for every ε, hence

$$V^\varepsilon(u(t)) - 1 \geq C > 0.$$

In this way, we have shown that there exists $C > 0$ such that

$$\mathcal{M}(u) \geq C \int_0^1 |\dot{u}(t)|^2 \, dt, \tag{2.40}$$

for every $u \in K_l$. At this point, let us define $t^* \in (0,1)$ as the first instant at which u crosses B_ε. Using the Hölder inequality, we note that

$$0 < C_1 \doteq R - \varepsilon \leq |u(0) - u(t^*)| \leq \int_0^1 |\dot{u}(t)| \, dt \leq \|\dot{u}\|_2 \tag{2.41}$$

and the proof is concluded. □

Lemma 2.4.7. *The Maupertuis' functional $\mathcal{M}$ is coercive in K_l.*

Proof. Take $(u_n) \subseteq K_l$ such that $\|u_n\|_{H^1} \to +\infty$. Since the sequence $(\|u_n\|_2) \subseteq \mathbb{R}$ is bounded we have that necessarily

$$\lim_{n\to+\infty} \|\dot{u}_n\|_2^2 = +\infty$$

and thus the proof is complete for (2.40). □

Lemma 2.4.8. *The Maupertuis' functional $\mathcal{M}$ is weakly lower semi-continuous $(w.l.s.c.)$ in K_l.*

Proof. It is equivalent to show that the set

$$M^C \doteq \{u \in K_l : \mathcal{M}(u) \leq C\}$$

is weakly closed in H^1 for every $C \in \mathbb{R}$. Fix $C \in \mathbb{R}$ and take $(u_n) \subseteq M^C$ such that $u_n \rightharpoonup u \in K_l$ in H^1. Since the H^1-norm is w.l.s.c. we have

$$\|u_n\|_2^2 + \|\dot{u}_n\|_2^2 \leq \liminf_{n\to+\infty} \left(\|u_n\|_2^2 + \|\dot{u}_n\|2^2\right).$$

The weak convergence implies the uniform one and so, in particular, $u_n \to u$ in L^2. For this reason

$$\|\dot{u}_n\|_2^2 \leq \liminf_{n\to+\infty} \|\dot{u}_n\|_2^2.$$

By assumption we have $\mathcal{M}(u_n) \leq C$ for every $n \in \mathbb{N}$ and so, in particular, $V^\varepsilon(u_n) \in L^1(0,1)$ for every $n \in \mathbb{N}$. This means that the set

$$\{t \in [0,1] : u_n(t) = c_j \text{ for some } j\}$$

has null measure and so, again from the uniform convergence of (u_n), we have

$$V^\varepsilon(u_n(t)) \to V^\varepsilon(u(t)) \quad \text{a.e. in } [0,1].$$

Therefore, Fatou Lemma implies that

$$\int_0^1 (-1 + V^\varepsilon(u(t)))\ dt \le \liminf_{n\to+\infty} \int_0^1 (-1 + V^\varepsilon(u_n(t)))\ dt$$

and, in particular, that $V^\varepsilon(u) \in L^1(0,1)$. At this point we can conclude the proof observing that, since

$$\begin{aligned}\mathcal{M}(u) &= \frac{1}{2}\int_0^1 |\dot u(t)|^2\, dt \int_0^1 (-1 + V^\varepsilon(u(t)))\ dt \\ &\le \frac{1}{2}\liminf_{n\to+\infty} \int_0^1 |\dot u_n(t)|^2\, dt \int_0^1 (-1 + V^\varepsilon(u_n(t)))\ dt \\ &\le \limsup_{n\to+\infty} \mathcal{M}(u_n) \\ &\le C,\end{aligned}$$

then $u \in M^C$. □

We are ready to prove the next result which claims the existence of a minimizer for the Maupertuis' functional in the set K_l.

Proposition 2.4.9. *Assume that the assumptions* (V) *on the potentials* $(V_j)_{j=1}^N$ *are satisfied. Fix* $\varepsilon \in (0, \tilde\varepsilon)$ *as in* (2.12)*, fix* $R \in (\tilde\varepsilon, \mathfrak{m}^{1/\alpha} - \tilde\varepsilon)$ *as in* (2.14) *and fix* $l \in \mathfrak{I}^N$*. Then, for any* $p_1, p_2 \in \partial B_R$*, the Maupertuis' functional*

$$\mathcal{M}(u) = \frac{1}{2}\int_0^1 |\dot u(t)|^2\, dt \int_0^1 (-1 + V^\varepsilon(u(t)))\, dt$$

admits a minimizer $u \in K_l^{p_1,p_2}([0,1])$ *at a positive level.*

Proof. Apply the direct method of Calculus of Variations to the Maupertuis' functional, making use of Lemmata 2.4.6, 2.4.7 and 2.4.8. □

Now, if we show that the minimizer $u \in K_l$ verifies:

(CF) u is collision-free,

(R) $|u(t)| < R$ for every $t \in (0,1)$,

we have that

$$\frac{d}{d\lambda}\mathcal{M}(u + \lambda\varphi)\big|_{\lambda=0} = 0 \quad \text{for every } \varphi \in \mathcal{C}_c^\infty(0,1),$$

so that Theorem 2.4.1 applies and we can find a classical solution $y\colon [0,T] \to \mathbb{R}^2$ of the inner problem

$$\begin{cases} \ddot y(t) = \nabla V^\varepsilon(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot y(t)|^2 - V^\varepsilon(y(t)) = -1 & t \in [0,T] \\ |y(t)| < R & t \in (0,T) \\ y(0) = p_1, \quad y(T) = p_2. \end{cases}$$

The next two sections are devoted to show respectively that u joins the properties (CF) and (R), to finally obtain a classical solution arc for the anisotropic N-centre problem inside. As a starting point, we characterize the sets of colliding instants and of the times at which $|u| = R$. In particular, if we define

$$\begin{aligned} T_c(u) &\doteq \{t \in [0,1] : u(t) = c_j' \text{ for some } j \in 1,\dots,N\} \subseteq (0,1) \\ T_R(u) &\doteq \{t \in [0,1] : |u(t)| = R\} \subseteq [0,1], \end{aligned} \tag{2.42}$$

we can easily notice that, since $\mathcal{M}(u) < +\infty$, $T_c(u)$ is a closed set of null measure and its complement $[0,1] \setminus T_c(u)$ is a union of a countable or finite number of open intervals. Moreover, when the minimizer travels along a connected component of $[0,1] \setminus (T_c(u) \cup T_R(u))$, it can be reparametrized to obtain a classical solution of the N-centre problem through Theorem 2.4.1 and the energy is conserved along this path. This is shown in the next lemma.

Lemma 2.4.10. *Given a minimizer $u \in K_l$ of the Maupertuis' functional $\mathcal{M}$:*

(i) u verifies

$$\frac{1}{2}|\dot{u}(t)|^2 - V^\varepsilon(u(t)) = -\frac{1}{\omega^2} \quad \text{a.e. in } [0,1];$$

(ii) if (a,b) is a connected component of $[0,1] \setminus (T_c(u) \cup T_R(u))$ then $u|_{(a,b)} \in \mathcal{C}^2(a,b)$ and

$$\omega^2 \ddot{u}(t) = \nabla V^\varepsilon(u(t)) \quad \text{for every } t \in (a,b).$$

Proof. The proof is a consequence of the minimality of u with respect to compact support variations in $[0,1] \setminus (T_c(u) \cup T_R(u))$ (see the proof of Theorem 2.4.1 and Lemma 2.4.2). □

2.4.3. Qualitative properties of minimizers: absence of collisions and (self-)intersections

In what follows we are going to provide the absence of collisions (CF) for a minimizer u obtained in the previous subsections. In order to do that, we will carry out a local study near-collisions. Since we will be working close to the centres, the radius of the ball B_ε will play no role here. For this reason, without loss of generality we fix ε. Fix an admissible partition of the centres, that corresponds to fix $l \in \mathfrak{I}^N$ and consider a minimizer $u \in K_l$ (see (2.37) and (2.38) for their definitions). To start with, we show that the collisions are isolated. Recalling the definition of $T_c(u)$ in (2.42), this is the content of the next lemma that, moreover, provides a *Lagrange-Jacobi* identity for colliding arcs.

Lemma 2.4.11. *The set $T_c(u)$ is discrete and it has a finite number of elements. In particular, if the minimizer u has a collision with the centre c_j', the function $I(t) \doteq |u(t) - c_j'|^2$ is strictly convex in a neighbourhood of the colliding instant.*

Proof. Without loss of generality assume that u collides with the centre c_1' and assume by contradiction that t_0 is an accumulation point for $T_c(u)$ and $u(t_0) = c_1'$. Since u

is continuous in $[0,1]$, only collision instants with the centre c'_1 can accumulate in t_0. Therefore, there exists a sequence (a_n, b_n) of open intervals such that $(a_n, b_n) \subseteq [0,1]$ for every $n \in \mathbb{N}$, $a_n \to t_0$ and $b_n \to t_0$ as $n \to +\infty$, $u(a_n) = c'_1 = u(b_n)$ for every $n \in \mathbb{N}$ and

$$|u(t) - c'_1| > 0 \quad \text{for all } t \in (a_n, b_n).$$

In particular, again from the continuity of u, we have that, at least for n sufficiently large

$$|u(t) - c'_k| \geq C > 0 \quad \text{and} \quad |u(t)| < R,$$

for all $t \in (a_n, b_n)$, for every $k \neq 1$. Define the inertial moment of u with respect to the centre c'_1 as the function

$$I(t) \doteq |u(t) - c'_1|^2.$$

Now, since (a_n, b_n) is a connected component of $[0,1] \setminus (T_c(u) \cup T_R(u))$ if n is sufficiently large, by Lemma 2.4.10 and using the Euler theorem for homogeneous functions, we can obtain a Lagrange-Jacobi-like identity

$$\begin{aligned} \ddot{I}(t) &= 2\langle \ddot{u}(t), u(t) - c'_1 \rangle + 2|\dot{u}(t)|^2 \\ &= \frac{2}{\omega^2}\langle \nabla V^\varepsilon(u(t)), u(t) - c'_1 \rangle + \frac{4}{\omega^2}\left(V^\varepsilon(u(t)) - 1\right) \\ &= -\frac{4}{\omega^2} + \frac{2}{\omega^2}(2-\alpha)V_1(u(t) - c'_1) + f(u(t)), \end{aligned} \tag{2.43}$$

for every $t \in (a_n, b_n)$ and for some smooth function f. The continuous function $I(t)$

- is positive;
- is zero when $t \in \{a_n, b_n\}$;
- admits a maximizer $\xi_n \in (a_n, b_n)$.

Therefore, we would have $\ddot{I}(\xi_n) \leq 0$ for every $n \in \mathbb{N}$. But, if $n \to +\infty$ the second term in (2.43) blows up, while the others stay bounded. This is clearly a contradiction. For this reason t_0 is an isolated point for $T_c(u)$ and, since $[0,1]$ is compact, in particular we have that $T_c(u)$ is finite. □

In the next two propositions we discuss some important properties of minimizers of $\mathcal{M}$, concerning the (self-)intersections at points which are different from the centres.

Proposition 2.4.12. *Let $u \in K_l^{p_1,p_2}$ be a minimizer of $\mathcal{M}$. Then, u parametrizes a path without self-intersections at points different from the centres.*

Proof. By contradiction, assume that there exist $t_* < t_{**}$ such that $u(t_*) = u(t_{**}) = p \neq c'_j$ for all j and $|p| < R$. Hence, there exists (a,b) connected component of $([0,1] \setminus (T_c(u) \cup T_R(u)))$ such that $t_* \in (a,b)$. From Lemma 2.4.10 we have that $u|_{(a,b)}$ is a classical solution of $\omega^2\ddot{u} = \nabla V(u)$ (and so, in particular, it is $\mathcal{C}^1$ in the same interval) and the conservation of energy implies that

$$\dot{u}(t_*), \dot{u}(t_{**}) \neq 0, \quad |u(t_*)| = |u(t_{**})|.$$

This leads us to consider three possible alternatives

1. $\dot{u}(t_*)$ is transverse to $\dot{u}(t_{**})$;
2. $\dot{u}(t_*)$ is tangent to $\dot{u}(t_{**})$ with opposite direction ($\dot{u}(t_*) = -\dot{u}(t_{**})$);
3. $\dot{u}(t_*)$ is tangent to $\dot{u}(t_{**})$ with same direction ($\dot{u}(t_*) = \dot{u}(t_{**})$).

The alternative (3.) is impossible by means of the uniqueness theorem for Cauchy problems, but also is alternative (2.) since our problem joins time-reversibility.

For (1.) we can produce an explicit variation $v \in K_l$ for which $\mathcal{M}(v) = \mathcal{M}(u)$ but such that $v \notin \mathcal{C}^1$ in a neighbourhood of t_*, which is impossible. To build such a variation is enough to travel along u until t^*, then to change the orientation of the loop between t^* and t^{**}, and then to travel again along u until the end. Notice that the new path v preserves the parity of the winding number with respect to every centre; indeed, for every $j \in \{1, \dots, N\}$ we have

$$\mathrm{Ind}(v; c_j') = \mathrm{Ind}(u; c_j') - 2\,\mathrm{Ind}(u|_{[t^*,t^{**}]}; c_j').$$

$\square$

Remark 2.4.13. *In light of the previous proposition, we can affirm that we could start this minimization process choosing among only those paths with winding index equals to* 0 *or* 1 *with respect to every centre, even if this choice could seem unnatural at the beginning. We also remark that a priori we do not necessarily need the paths to do not self-intersect; nonetheless, Proposition 2.4.12 shows that this is actually an intrinsic property of the minimizers.*

Lemma 2.4.14. *Let $u \in K_l^{p_1,p_2}$ be a minimizer of $\mathcal{M}$, let $q_1 = u(c)$ and $q_2 = u(d)$ for some sub-interval $[c,d] \subseteq [0,1]$. If we define $K^{q_1,q_2}(u)$ as the weak H^1-closure of the space*

$$\hat{K}^{q_1,q_2}(u) \doteq \left\{ v \in H^1([c,d];\mathbb{R}^2) \;\middle|\; \begin{array}{c} v(c) = q_1,\ v(d) = q_2,\ |v| \le R,\ v \text{ is homotopic} \\ \text{to } u|_{[c,d]} \text{ in the punctured ball } B_R \setminus \{c_1', \dots, c_N'\} \end{array} \right\},$$

then

$$\mathcal{M}(u|_{[c,d]}) = \min_{K^{q_1,q_2}(u)} \mathcal{M}.$$

Proof. Assume by contradiction that there exists $w \in K^{q_1,q_2}(u)$ such that

$$\min_{K^{q_1,q_2}(u)} \mathcal{M} = \mathcal{M}(w) < \mathcal{M}(u|_{[c,d]}).$$

The path

$$\tilde{u} \doteq \begin{cases} u|_{[0,c]}(t) & t \in [0,c] \cup [d,1] \\ w & t \in [c,d] \end{cases}$$

belongs to the space $K_l^{p_1,p_2}$ and minimizes $\mathcal{M}$ in that space. This is in contrast with the minimality of u in the same space. $\square$

Proposition 2.4.15. *Let $u \in K_l^{p_1,p_2}$ be a minimizer of $\mathcal{M}$. Let $\tilde{l} \in \mathfrak{I}^N$, $\tilde{p}_1, \tilde{p}_2 \in \partial B_R$ and $v \in K_{\tilde{l}}^{\tilde{p}_1,\tilde{p}_2}$ be a minimizer of $\mathcal{M}$. Then, if u intersects v at least in two distinct points $q_1, q_2 \in B_R \setminus \{c'_1, \ldots, c'_N\}$, the portions of u and v between q_1 and q_2 are not homotopic paths in the punctured ball. As a consequence, if $l = \tilde{l}$, then u cannot intersect v more than once.*

Proof. Since the Maupertuis' functional is invariant under time-reparametrizations, to prove the assertion we can assume that there exist $q_1, q_2 \in B_R \setminus \{c'_1, \ldots, c'_N\}$ such that

$$u(c) = q_1 = v(c)$$
$$u(d) = q_2 = v(d),$$

for some interval $[c,d] \subseteq [0,1]$. Assume by contradiction that the paths $u|_{[c,d]}$ and $v|_{[c,d]}$ are homotopic in the punctured ball $B_R \setminus \{c'_1, \ldots, c'_N\}$; this means in particular that $K^{q_1,q_2}(u) = K^{q_1,q_2}(v)$ (for their definitions see the statement of Lemma 2.4.14). Now, again from Lemma 2.4.14, we deduce that

$$\mathcal{M}(u;[c,d]) = \min_{K^{q_1,q_2}(u)} \mathcal{M} = \min_{K^{q_1,q_2}(v)} \mathcal{M} = \mathcal{M}(v;[c,d]).$$

For this reason, if we define the path (see Figure 2.5)

$$\tilde{u}(t) \doteq \begin{cases} u(t) & \text{if } t \in [0,c) \cup (d,1] \\ v(t) & \text{if } t \in [c,d] \end{cases}$$

we clearly have that $\tilde{u} \in K_l^{p_1,p_2}$ and

$$\mathcal{M}(\tilde{u}) = \mathcal{M}(u) = \min_{K_l^{p_1,p_2}} \mathcal{M}.$$

By Lemma 2.4.11 the instants c and d belong to two connected components of $[0,1] \setminus (T_c(u) \cup T_R(u))$ and therefore Lemma 2.4.10 applies too. This is finally a contradiction since the path $\tilde{u}$ cannot be differentiable in c and d (note that $\dot{u}(c) \neq \dot{v}(c)$ for the uniqueness of solutions of Cauchy problems; the same holds at d for time-reversibility).

For the situation $l = \tilde{l}$ the proof is trivial, once provided Proposition 2.4.12. □

At this point we are ready to start a local analysis in order to rule out the presence of collisions with the centres. Let us now assume that the minimizer u has a collision with the centre c'_j at time t_0. By means of Lemma 2.4.11, we have that there exist $c, d \in [0,1]$ such that

- $c < t_0 < d$ and t_0 is the unique instant of collision of u in $[c,d]$;
- the inertial moment $I(t) = |u(t) - c'_j|^2$ is strictly convex in $[c,d]$.

We define $\bar{p}_1 = u(c)$ and $\bar{p}_2 = u(d)$. Since $u \in \mathcal{C}([c,d];\mathbb{R}^2)$, then there exists $r^* > 0$ such that

(2.44) $$|u(t) - c'_k| \geq r^* > 0 \quad \text{for every } t \in [c,d] \text{ and for every } k \neq j$$

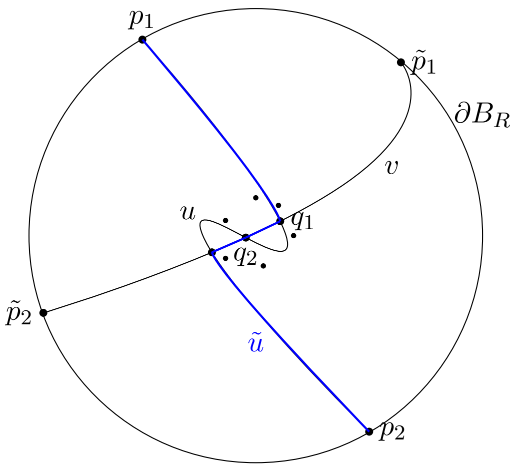

Figure 2.5.: Situation of Proposition 2.4.15

and, without loss of generality, we can assume that $\bar{p}_1, \bar{p}_2 \in \partial B_r(c'_j)$, for some $r < r^*$.

Since we are getting close to the collision, it makes sense to *localize* the potential and to write

$$V^\varepsilon(y) = V_j(y - c'_j) + f^j(y), \quad \text{with } f^j(y) \doteq \sum_{k \neq j} V_k(y - c'_k).$$

Notice that inside the ball $B_r(c'_j)$ the quantity $f^j(y)$ is smooth and bounded.

Let us introduce the space

$$\hat{\mathcal{K}}_l^{\bar{p}_1,\bar{p}_2} \doteq \left\{ v \in H^1([c,d];\mathbb{R}^2) \;\middle|\; \begin{array}{l} v(c) = \bar{p}_1,\ v(d) = \bar{p}_2,\ v(t) \neq c'_j \ \forall\, t \in [c,d],\ \forall\, j \\ \text{the function } G_v(t) \doteq \begin{cases} u(t) & \text{if } t \in [0,c) \cup (d,1] \\ v(t) & \text{if } t \in [c,d] \end{cases} \\ \text{belongs to } \hat{K}_l^{p_1,p_2} \end{array} \right\}$$

and its weak H^1-closure

$$\mathcal{K}_l^{\bar{p}_1,\bar{p}_2} \doteq \hat{\mathcal{K}}_l^{\bar{p}_1,\bar{p}_2} \cup \left\{ v \in H^1([c,d];\mathbb{R}^2) : \ v(c) = \bar{p}_1,\ v(d) = \bar{p}_2,\ G_v \in \mathfrak{Coll}_l \right\}$$

and restrict the Maupertuis' functional to $\mathcal{K}_l^{\bar{p}_1,\bar{p}_2}$ in this way

$$\mathcal{M}_l^{\bar{p}_1,\bar{p}_2} : \mathcal{K}_l^{\bar{p}_1,\bar{p}_2} \to \mathbb{R} \cup \{+\infty\}, \quad \mathcal{M}_l^{\bar{p}_1,\bar{p}_2}(v) = \frac{1}{2}\int_c^d |\dot{v}(t)|^2\, dt \int_c^d \left(-1 + V^\varepsilon(v(t))\right)\, dt.$$

We can repeat the proof of Subsection 2.4.2 and show that $\mathcal{M}_l^{\bar{p}_1,\bar{p}_2}$ admits a minimizer in $\mathcal{K}_l^{\bar{p}_1,\bar{p}_2}$ at a positive level. Moreover, from Lemma 2.4.14, this minimizer is nothing but $v \doteq u|_{[c,d]}$.

For the sake of simplicity assume $c'_j = 0$.

Proposition 2.4.16. *The following behaviour holds:*

$$V^\varepsilon(y) = V_j(y) + C + \mathcal{O}(|y|), \text{ as } y \to 0^+,$$

for some constant $C > 0$. In particular, when $|y|$ is sufficiently small, the problem is a small perturbation of an anisotropic Kepler problem driven by the $-\alpha_j$-homogeneous potential V_j.

Proof. It is enough to observe that, with the same computation of the proof of Proposition 2.2.2, we have that, for every $k \neq j$

$$|y - c_k'|^{-\alpha_k} = |c_k'|^{-\alpha_k} + \mathcal{O}(|y|), \quad \text{as } |y| \to 0^+$$

and

$$V_k\left(\frac{y - c_k'}{|y - c_k'|}\right) = C + \mathcal{O}(|y|), \quad \text{as } |y| \to 0^+.$$

In particular, defining $r^* > 0$ as in (2.44), for $r \in (0, r^*)$ and for $|y| \leq r$ we can write

$$V^\varepsilon(y) = V_j(y) + C + rG_r(|y|),$$

with G_r uniformly bounded with respect to r. □

At this point we need a result from [5] (see also Chapter 1) on the properties of minimal collision orbits for a perturbed anisotropic Kepler problem. In order to take it into account, we need to introduce some further notations. Let r^* be as in (2.44); for $r \in (0, r^*)$ and $q \in \partial B_r$ we define the set of H^1-colliding paths on a generic real interval $[c, d] \subseteq \mathbb{R}$

$$H^q_{coll} \doteq \left\{ w \in H^1([c,d];\mathbb{R}^2) : \ w(c) = q, \ w(d) = 0, \ |w(t)| \leq r, \ \forall\, t \in [c,d] \right\}.$$

Moreover, for a potential $V^\varepsilon \in \mathcal{C}^2(B_r \setminus \{0\})$ which is a perturbation of an anisotropic potential as in Proposition 2.4.16, consider the Maupertuis' functional

$$\mathcal{M}(w) = \frac{1}{2}\int_c^d |\dot{w}(t)|^2\, dt \int_c^d (-1 + V^\varepsilon(w(t)))\, dt$$

for $w \in H^q_{coll}$. Up to choosing a smaller r^*, we proved the following result; in order to ease the notation, we will denote a minimal non-degenerate central configuration for U_j as ϑ^* instead of ϑ_j (see (V)).

Lemma 2.4.17 (Theorem 5.2,[5]; cf Theorem 1.5.2, Remark 1.3.3)**.** *Let $\vartheta^* \in \mathbb{S}^1$ be a minimal non-degenerate central configuration for U_j. There exist $r^* > 0$ and $\delta > 0$ such that, for every $q = re^{i\vartheta}$ with $r < r^*$ and $\vartheta \in (\vartheta^* - \delta, \vartheta^* + \delta)$ there exists a unique minimizer of the Maupertuis' functional in the set of colliding paths H^q_{coll}. In particular, this path cannot leave the cone emanating from the origin and bounded by the arc-neighbourhood $(\vartheta^* - \delta, \vartheta^* + \delta)$.*

The presence of this region foliated by minimal arcs inside a cone spanned by ϑ_j, together with Proposition 2.4.15, suggests to choose one of this paths and to use it as a *barrier*, in order to determine a region of the ball B_r in which a minimizer with endpoints on ∂B_r has to be confined. Indeed, in order to rule out the presence of collisions for a minimizer in $\mathcal{K}_l^{\bar{p}_1,\bar{p}_2}$, we aim to follow the ideas contained in a result from [11], which holds true for minimizers that do not leave a prescribed angular sector. For a $r > 0$, a potential $V^\varepsilon \in \mathcal{C}^2(B_r \setminus \{0\})$ as in Proposition 2.4.16 and $T > 0$, introduce the action functional $\mathcal{A}_T\colon H^1([0,T];\mathbb{R}^2) \to \mathbb{R} \cup \{+\infty\}$ such that

$$\mathcal{A}_T(x) \doteq \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + V^\varepsilon(x(t)) - 1\right) dt.$$

Definition 2.4.18. *We say that $x \in H^1([0,T];\mathbb{R}^2)$ is a* fixed-time Bolza minimizer *associated with the endpoints $x_1 = x(0), x_2 = x(T)$, if, for every $y \in H^1([0,T];\mathbb{R}^2)$ there holds*

$$y(0) = x_1,\ y(T) = x_2 \implies \mathcal{A}_T(x) \le \mathcal{A}_T(y).$$

We recall an important result from [11], which represent our starting point to get the absence of collisions for our minimizers.

Lemma 2.4.19 (Theorem 2, [11])**.** *Consider a perturbed potential $V \in \mathcal{C}^1(\mathbb{R}^2 \setminus \{0\})$ such that, writing in polar coordinates $x = (r\cos\vartheta, r\sin\vartheta)$*

$$V(x) = r^{-\alpha}U(\vartheta) + W(x),$$

where $\alpha \ge 0$ and

$$\lim_{r\to 0^+} r^{\alpha'}\left(W(x) + r|\nabla W(x)|\right) = 0$$

for some $\alpha' < \alpha$. Assume that there exists at least $\vartheta^* \in \mathbb{S}^1$ such that

$$\begin{cases} U(\vartheta) \ge U(\vartheta^*) > 0, & \forall\, \vartheta \in \mathbb{S}^1 \\ U''(\vartheta^*) > 0, \end{cases}$$

i.e., ϑ^* is a minimal non-degenerate central configuration for U, and define

$$\Theta \doteq \{\vartheta \in \mathbb{R} :\ \vartheta = \vartheta^* + 2n\pi,\ \text{for some } n \in \mathbb{Z}\}.$$

Then, for every $\vartheta^- < \vartheta^+ \in \Theta$ there exists $\bar{\alpha}(U,\vartheta^-,\vartheta^+) \in (0,2)$ such that if $\alpha > \bar{\alpha}$ all the fixed-time Bolza minimizers in the angular sector $[\vartheta^-,\vartheta^+]$ are collision-less.

As a first step, we show that the previous lemma can be extended for those H^1-paths with fixed ends which minimize the Maupertuis' functional instead of the action functional. Indeed, with the same proof of Subsection 2.4.2, one can prove the existence of a minimizer for the Maupertuis' functional

$$\mathcal{M}_h(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 \int_0^1 (-h + V(u))$$

in the space of the H^1-paths which join two points within the sector $[\vartheta^-,\vartheta^+]$, for $h \in \mathbb{R}$.

Lemma 2.4.20. *In the same setting of Lemma 2.4.19, if $h \in \mathbb{R}$ and if $\alpha > \bar{\alpha}(U, \vartheta^-, \vartheta^+)$, then all the minimizers of the Maupertuis' functional $\mathcal{M}_h$ within the sector $[\vartheta^-, \vartheta^+]$ are collision-less.*

Proof. Assume that $u \in H^1([0,1];\mathbb{R}^2)$ minimizes the Maupertuis' functional in the set of the H^1-paths which join two points q_1, q_2 within the sector $[\vartheta^-, \vartheta^+]$ and assume also that u has a collision with the origin. If we define $x(t) \doteq u(\omega t)$, with

$$\omega \doteq \left(\frac{\int_0^1 (-h + V(u))}{\frac{1}{2}\int_0^1 |\dot{u}|^2} \right)^{1/2} > 0$$

then, from Theorem 2.4.1, we know that x solves

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) & t \in [0, 1/\omega] \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = -h & t \in [0, 1/\omega] \\ x(0) = q_1, \ x(1/\omega) = q_2. \end{cases}$$

At this point we define $T = 1/\omega$ and we find the fixed-time Bolza minimizer of the action functional associated with the sector $[\vartheta^-, \vartheta^+]$ and we define

$$H_T \doteq \left\{ y \in H^1([0,T];\mathbb{R}^2) : \ y(0) = q_1, \ y(T) = q_2, \ y(t) \in [\vartheta^-, \vartheta^+] \, \forall \, t \in [0,T] \right\}.$$

We call this minimizing path

$$\bar{x} = \arg \min_{y \in H_T} \mathcal{A}_T(y)$$

and, by Lemma 2.4.19, we know that it cannot collide with the origin; this also proves that $x \neq \bar{x}$.

Now, we know that $x(t) = u(t/T)$ for all $t \in [0,T]$, and so we can compute

$$\begin{aligned} \mathcal{A}_T(x) &= \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + V(x(t)) - h \right) dt \\ &= \int_0^1 \left(\frac{1}{2T}|\dot{u}(s)|^2 + TV(u(s)) - Th \right) ds \end{aligned}$$

and, from the conservation of the energy for u and the definition of $\omega = 1/T$, we can find that

$$\mathcal{A}_T(x) = \int_0^1 \frac{1}{T}|\dot{u}(s)|^2 \, ds = \sqrt{2} \left(\int_0^1 |\dot{u}(s)|^2 \, ds \int_0^1 (-h + V(u(s))) \, ds \right)^{1/2} = 2\sqrt{\mathcal{M}_h(u)}.$$

At this point, since the Maupertuis functional is invariant under time-reparametrizations we have

$$2\sqrt{\mathcal{M}_h(u)} = \min_{T>0} \min_{y \in H_T} \mathcal{A}_T(y) \leq \min_{y \in H_T} \mathcal{A}_T(y) = \mathcal{A}_T(\tilde{x}) < \mathcal{A}_T(x) = 2\sqrt{\mathcal{M}_h(u)},$$

which is a contradiction since $\tilde{x}$ is collision-less. □

In the next result we show that it is possible to improve the previous lemma. In particular, a sequence of minimal paths cannot accumulate to a collision path. We will prove this result taking into account the potential V^ε as in Proposition 2.4.16 because it is meaningful to our purposes, but the same result holds in the setting of Lemma 2.4.19.

Lemma 2.4.21. *In the same setting of Lemma 2.4.19, there exists $\bar{r} > 0$ such that, for any $r_0 \in (0, \bar{r})$, for every $k \in \mathbb{N} \setminus \{0\}$, for every sector $[\vartheta^-, \vartheta^+]$ such that $\vartheta^+ - \vartheta^- = 2k\pi$, for every $\alpha > \bar{\alpha}(U_j, \vartheta^-, \vartheta^+)$, there exists $\delta > 0$ such that, for every $q_1, q_2 \in \partial B_{r_0}$ the Bolza minimizer u considered in Lemma 2.4.20 from q_1 to q_2 in the sector $[\vartheta^-, \vartheta^+]$ is such that*

$$\min_{t \in [0,1]} |u(t)| > \delta r_0.$$

Proof. Assume by contradiction that, for every $\bar{r} > 0$, there exists $r_0 \in (0, \bar{r})$, there exists $k \in \mathbb{N}$, $\vartheta^* \in \mathbb{S}^1$ minimal non-degenerate central configuration for U_j and there exists $\alpha > \bar{\alpha}(U_j, \vartheta^*, \vartheta^* + 2k\pi)$ such that, for any $\delta > 0$ there exists $q_1, q_2 \in \partial B_{r_0}$ such that the Maupertuis' minimizer u which connects q_1 and q_2 in the sector $[\vartheta^*, \vartheta^* + 2k\pi]$ is such that

$$\min_{t \in [0,1]} |u(t)| \leq \delta r_0.$$

It is not restrictive to assume instead the following:

- there exists $r_n \to 0^+$ sequence of positive real numbers;
- fix $k \in \mathbb{N}$;
- fix $\vartheta^* \in \mathbb{S}^1$ minimal non-degenerate central configuration for U_j (this is not restrictive since U_j admits just a finite number of them);
- there exists $\alpha_j > \bar{\alpha}(U_j, \vartheta^*, \vartheta^* + 2k\pi)$;
- take $\delta_n \to 0^+$ sequence of positive real numbers;
- take two sequences of points $(q_1^n), (q_2^n) \subseteq (\partial B_{r_n})$;
- consider the sequence of minimizers (u_n) of the Maupertuis' functional

  $$\mathcal{M}(u_n) = \frac{1}{2} \int_0^1 |\dot{u}_n|^2 \int_0^1 (-1 + V^\varepsilon(u_n)),$$

  every one of them respectively in the space

  $$H^n \doteq \{u_n \in H^1([0,1]; \mathbb{R}^2) : \, u_n(0) = q_1^n, \, u_n(1) = q_2^n, \, |u_n| \leq r_n\}$$

  and within the sector $[\vartheta^*, \vartheta^* + 2k\pi]$, such that

$$\min_{t\in[0,1]} |u_n(t)| \leq \delta_n r_n.$$

Define the blow-up sequence

$$v_n(t) \doteq \frac{1}{r_n} u_n(t) \quad \text{for } t \in [0,1], \text{ for every } n \in \mathbb{N}$$

which, for every $n \in \mathbb{N}$, verifies the following:

$$\begin{cases} \bar{q}_1^n \doteq v_n(0),\ \bar{q}_2^n \doteq v_n(1) \in \partial B_1; \\ |v_n(t)| \leq 1 \text{ for every } t \in [0,1]; \\ \min_{t\in[0,1]} |v_n(t)| \leq \delta_n. \end{cases} \tag{2.45}$$

Recalling the behaviour of V^ε (see Proposition 2.4.16), observe that, if we fix $y \in \mathbb{R}^2 \setminus \{0\}$ we can compute

$$V^\varepsilon(r_n y) = r_n^{-\alpha_j} V_j(y) + C + \mathcal{O}(r_n) = r_n^{-\alpha_j} \left(V_j(y) + r_n^{\alpha_j} C + \mathcal{O}(r_n^{\alpha_j+1}) \right)$$

as $n \to +\infty$. In this way we have

$$\begin{aligned} \mathcal{M}(u_n) = \mathcal{M}(r_n v_n) &= \frac{1}{2} \int_0^1 |r_n \dot{v}_n|^2 \int_0^1 (-1 + V^\varepsilon(r_n v_n)) \\ &= r_n^2 \frac{1}{2} \int_0^1 |\dot{v}_n|^2 \int_0^1 r_n^{-\alpha_j} \left(-r_n^{\alpha_j} + V_j(v_n) + r_n^{\alpha} C + \mathcal{O}(r_n^{\alpha_j+1}) \right) \\ &= r_n^{2-\alpha_j} \frac{1}{2} \int_0^1 |\dot{v}_n|^2 \int_0^1 \left(V_j(v_n) + \mathcal{O}(r_n^{\alpha_j}) \right) \end{aligned}$$

and so, if we define

$$\bar{\mathcal{M}}(v_n) \doteq \frac{1}{2} \int_0^1 |\dot{v}_n|^2 \int_0^1 \left(V_j(v_n) + \mathcal{O}(r_n^{\alpha_j}) \right)$$

we have shown that

$$\bar{\mathcal{M}}(v_n) = r_n^{\alpha_j - 2} \mathcal{M}(u_n), \quad \text{for every } n \in \mathbb{N}.$$

Now, since $\bar{\mathcal{M}}(v_n)$ and $\mathcal{M}(u_n)$ are proportional and u_n minimizes $\mathcal{M}$ in H^n, if we define

$$\bar{H}^n \doteq \{ v_n \in H^1([0,1];\mathbb{R}^2) :\ v_n(0) = \bar{q}_1^n,\ v_n(1) = \bar{q}_2^n,\ |v_n| \leq 1 \}$$

we easily deduce that

$$\bar{\mathcal{M}}(v_n) = \min_{\bar{H}^n} \bar{\mathcal{M}}.$$

At this point, we want to show that (v_n) admits a weak limit in the H^1 topology. Since V_j is bounded from below in $\mathbb{S}^1$ we have that there exists $C_1 > 0$ such that

$$\bar{\mathcal{M}}(v_n) \geq C_1 \int_0^1 |\dot{v}_n|^2, \quad \text{for every } n \in \mathbb{N}.$$

On the other hand, since $\mathcal{O}(r_n^{\alpha_j})$ is uniformly bounded as $n \to +\infty$ by a constant $C_2 > 0$, we have that there exists $C_3 > 0$ such that

$$\bar{\mathcal{M}}(v_n) = \min_{\bar{H}^n} \bar{\mathcal{M}} \leq \min_{v \in \bar{H}^n} \frac{1}{2} \int_0^1 |\dot{v}|^2 \int_0^1 V_j(v) + C_2 \leq C_3.$$

Moreover, the sequence (v_n) is uniformly bounded by 1 and so its L^2-norm is too. For this reason, we deduce that there exists $v_0 \in H^1$ such that $v_n \rightharpoonup v_0$ in the H^1-topology and thus uniformly; in particular, from (2.45) and the uniform convergence we have that

$$\begin{cases} \bar{q}_1 \doteq v_0(0), \ \bar{q}_2 \doteq v_0(1) \in \partial B_1; \\ |v_0(t)| \leq 1, \ \text{for every } t \in [0,1]; \\ \min\limits_{t \in [0,1]} |v_0(t)| = 0. \end{cases}$$

In other words, we have shown that the blow-up limit v_0 is a collision path in the space

$$\bar{H} \doteq \{v \in H^1([0,1]; \mathbb{R}^2) : \ v(0) = \bar{q}_1, \ v(1) = \bar{q}_2, \ |v| \leq 1\}$$

in the sector $[\vartheta^*, \vartheta^* + 2\bar{k}\pi]$. For this reason, it is enough to show that v_0 minimizes the Maupertuis' functional

$$\bar{\mathcal{M}}_0(v_0) = \frac{1}{2} \int_0^1 |\dot{v}_0|^2 \int_0^1 V_j(v_0)$$

in the space $\bar{H}$; indeed, we would reach a contradiction thanks to Lemma 2.4.20, since $\alpha_j > \bar{\alpha}(U_j, \vartheta^*, \vartheta^* + 2k\pi)$ and a minimizer cannot have collisions.

From Fatou lemma we have that

$$\bar{\mathcal{M}}_0(v_0) = \frac{1}{2} \int_0^1 |\dot{v}_0|^2 \int_0^1 V_j(v_0) \leq \liminf_{n \to +\infty} \bar{\mathcal{M}}(v_n);$$

on the other hand, since v_n minimizes $\bar{\mathcal{M}}$ in $\bar{H}_n$ for every $n \in \mathbb{N}$, we have that

$$\bar{\mathcal{M}}(v_n) \leq \bar{\mathcal{M}}(v_0) \leq \frac{1}{2} \int_0^1 |\dot{v}_0|^2 \int_0^1 V_j(v_0) + C_4 r_n^{\alpha},$$

for some $C_4 > 0$ and for every $n \in \mathbb{N}$. In this way, we also have that

$$\liminf_{n \to +\infty} \bar{\mathcal{M}}(v_n) \leq \frac{1}{2} \int_0^1 |\dot{v}_0|^2 \int_0^1 V_j(v_0) = \bar{\mathcal{M}}_0(v_0)$$

and so $v_n \to v_0$ strongly in H^1. This shows that v_0 is a minimizer in $\bar{H}$ and concludes the proof. □

At this point, we want to prove something stronger than the previous lemma, which will involve Lemma 2.4.17. Indeed, our idea is to show that it is possible to extend Lemma 2.4.21 to those *sectors* that are determined by two minimal arcs of the *foliation* provided in Lemma 2.4.17. We are interested in those *curved sectors* which have as barriers one minimal arc and its $2k\pi$-copy for some $k \in \mathbb{N}$. Note that in [5], the authors give a particular characterization of the foliation of minimal arcs: it is possible to parametrize every minimal arc with respect to its distance from the origin, thanks to a monotonicity property of the radial variable (see [5, Lemma 4.3]). Recalling that ϑ^* is a minimal non-degenerate central configuration for U_j, we consider the unique minimal arc γ^*, parametrized as the polar curve $\gamma^*(r) = (r, \varphi^*(r))$, such that $\varphi^*(r_0) = \vartheta^*$. For $k \in \mathbb{N}$, we can define

$$\Sigma(\vartheta^*, k) \doteq \{(r, \vartheta(r)) : \varphi^*(r) \leq \vartheta(r) \leq \varphi^*(r) + 2k\pi, \text{ for } 0 \leq r \leq r_0\}$$

and we are able to prove the following result. Again, we will refer the proof to a potential V^ε as in Proposition 2.4.16.

Lemma 2.4.22. *In the same setting of Lemma 2.4.19, there exists $r^* > 0$ such that, for every $r_0 \in (0, r^*)$, for every $k \in \mathbb{N}$, for every $\alpha > \bar{\alpha}(U_j, \vartheta^*, \vartheta^* + 2k\pi)$, there exists $\delta > 0$ such that, for every $q_1, q_2 \in \Sigma(\vartheta^*, k) \cap \partial B_{r_0}$, the Bolza minimizer u which connects q_1 and q_2 is such that:*

(i) *u belongs pointwisely to the sector $\Sigma(\vartheta^*, k)$;*

(ii) *u verifies*

$$\min_{t\in[0,1]} |u(t)| > \delta r_0.$$

Proof. We start with the proof of (ii). Following the same technique used in the proof of Lemma 2.4.21, assume by contradiction that:

- there exists $r_n \to 0^+$ sequence of positive real numbers and, without loss of generality, assume that $r_n \leq r^*$ for n sufficiently large, with $r^* > 0$ as in Lemma 2.4.17;
- fix $k \in \mathbb{N}$;
- fix $\vartheta^* \in \mathbb{S}^1$ minimal non-degenerate central configuration for U_j (this is not restrictive since U_j admits just a finite number of them);
- there exists $\alpha_j > \bar{\alpha}(U_j, \vartheta^*, \vartheta^* + 2k\pi)$;
- take $\delta_n \to 0^+$ sequence of positive real numbers;
- define the sequence of curved sectors

  $$\Sigma_n \doteq \{(r, \vartheta(r)) : \varphi^*(r) \leq \vartheta(r) \leq \vartheta^*(r) + 2k\pi, \text{ for } 0 \leq r \leq r_n\},$$

  where $\gamma^*(r) = (r, \varphi^*(r))$ is the polar curve which parametrizes the unique minimal arc of the foliation provided in Lemma 2.4.17, such that $\varphi^*(r^*) = \vartheta^*$;

- take two sequences of points $(q_1^n), (q_2^n) \subseteq (\Sigma_n \cap \partial B_{r_n})$;
- consider the sequence of minimizers (u_n) of the Maupertuis' functional

$$\mathcal{M}(u_n) = \frac{1}{2} \int_0^1 |\dot{u}_n|^2 \int_0^1 (-1 + V^\varepsilon(u_n)),$$

  every one of them respectively in the space

$$H^n \doteq \{u_n \in H^1([0,1]; \mathbb{R}^2) : \ u_n(0) = q_1^n, \ u_n(1) = q_2^n, \ |u_n| \leq r_n\}$$

  and within the curved sector Σ_n, requiring that every u_n satisfies

$$\min_{t \in [0,1]} |u_n(t)| \leq \delta_n r_n.$$

Define the blow-up sequence

$$v_n(t) \doteq \frac{1}{r_n} u_n(t), \quad \text{for } t \in [0,1], \ \text{for every } n \in \mathbb{N},$$

which, for every $n \in \mathbb{N}$, verifies the following:

$$\begin{cases} \bar{q}_1^n \doteq v_n(0), \ \bar{q}_2^n \doteq v_n(1) \in \partial B_1; \\ |v_n(t)| \leq 1 \text{ for every } t \in [0,1]; \\ \min_{t \in [0,1]} |v_n(t)| \leq \delta_n. \end{cases}$$

With the same proof of Lemma 2.4.21, one can prove that every v_n (at least for n large) minimizes the functional

$$\bar{\mathcal{M}}(v_n) \doteq \frac{1}{2} \int_0^1 |\dot{v}_n|^2 \int_0^1 (V_j(v_n) + \mathcal{O}(r_n^{\alpha_j}))$$

in the space

$$\bar{H}^n \doteq \{v_n \in H^1([0,1]; \mathbb{R}^2) : \ v_n(0) = \bar{q}_1^n, \ v_n(1) = \bar{q}_2^n, \ |v_n| \leq 1\}.$$

Moreover, defining the angular variable $\varphi_n^*(r) \doteq \varphi^*(r_n r)$ and the blow-up sector

$$\bar{\Sigma}_n \doteq \{(r, \vartheta(r)) : \ \varphi_n^*(r) \leq \vartheta(r) \leq \varphi_n^*(r) + 2k\pi, \ \text{for } 0 \leq r \leq 1\},$$

one can easily verify that $v_n \in \bar{\Sigma}_n$ for every $n \in \mathbb{N}$.

At this point, with the same technique of Lemma 2.4.21, one can prove that $v_n \to v_0$ uniformly in $[0,1]$, with v_0 minimizer of the functional

$$\bar{\mathcal{M}}_0(v_0) = \frac{1}{2} \int_0^1 |\dot{v}_0|^2 \int_0^1 V_j(v_0)$$

in the space

$$\bar{H} \doteq \{v \in H^1([0,1];\mathbb{R}^2) : \, v(0) = \bar{q}_1, \; v(1) = \bar{q}_2, \; |v| \le 1\},$$

for some $\bar{q}_1, \bar{q}_2 \in \partial B_1$ and such that

$$\min_{t\in[0,1]} |v_0(t)| = 0. \tag{2.46}$$

Moreover, from Lemma 2.4.17, since $r_n \to 0^+$, we have that the sequence of functions $\varphi_n^* = \varphi_n^*(r)$ uniformly converges to ϑ^* on the r-interval $[0,1]$ and so

$$\bar{\Sigma}_n \to [\vartheta^*, \vartheta^* + 2k\pi] \quad \text{as } n \to +\infty.$$

This means that v_0 minimizes $\bar{\mathcal{M}}$ in $\bar{H}$ within the sector $[\vartheta^*, \vartheta^* + 2k\pi]$ and, thanks to (2.46), has a collision. This is a contradiction for Lemma 2.4.20 and proves (ii).

In order to prove (i) it is enough to observe that a minimizer of the Maupertuis' functional $\mathcal{M}$ with endpoints in the sector $\Sigma(\vartheta^*, k)$ cannot leave this sector. Indeed, $\Sigma(\vartheta^*, k)$ has a minimal collision arc and its $2k\pi$-copy as boundary; these arcs act as a barrier, since Proposition 2.4.15 applies also in this context and a Bolza minimizer cannot intersect another minimal arc more than once. □

We now we extend the previous local study to a global setting,which takes into account all the other centres. In order to do this, we need to show that the local minimization process provides two minimizers which do not collide in c_j' and such that, if juxtaposed, have winding number equal to 1 with respect to c_j'. In this way, if one takes a minimizer $u \in K_l$ and assumes that u collides in c_j', then a contradiction arises. Indeed, the portion of u close enough to c_j' must correspond to one of the two local minimizers above, depending on if $l_j = 0$ or $l_j = 1$.

Theorem 2.4.23. *In the same setting of Lemma 2.4.19, there exists $r^* > 0$ such that, for every $r_0 \in (0, r^*)$, for every $\alpha > \bar{\alpha}(U_j, \vartheta^*, \vartheta^* + 4\pi)$, there exists $\delta > 0$ such that, for every $q_1, q_2 \in \partial B_{r_0}$, there exist two Bolza minimizers u_1 and u_2 which connect q_1 and q_2 such that*

(i) *for every $i = 1, 2$ we have*

$$\min_{t\in[0,1]} |u_i(t)| > \delta r_0;$$

(ii) *the juxtaposition u of u_1 and u_2 is a closed path which has winding number 1 with respect to the origin, up to choose a suitable time-parametrization.*

Proof. Take $q_1 = r_0 e^{i\vartheta_1}, q_2 = r_0 e^{i\vartheta_2} \in \partial B_{r_0}$ and, without loss of generality, assume that $q_1, q_2 \in \Sigma(\vartheta_j, 1)$ so that, in particular $|\vartheta_1 - \vartheta_2| < 2\pi$. Moreover, it is not restrictive to assume that $\vartheta_1 < \vartheta_2$. By Lemma 2.4.22 there exists a Bolza minimizer u_1 which connects q_1 and q_2 and verifies properties (i) and (ii) of such lemma. At this point, define $\tilde{q}_1 \doteq r_0 e^{i(\vartheta_1 + 2\pi)}$ which, of course, coincides with q_1 in the Euclidean space, but not with respect to the curved sectors. Indeed, we have that $\tilde{q}_1 \in \Sigma(\vartheta_j, 2) \setminus \Sigma(\vartheta_j, 1)$ and, of course, also $q_2 \in \Sigma(\vartheta_j, 2)$ (see Figure 2.6). For this reason, again from Lemma 2.4.22,

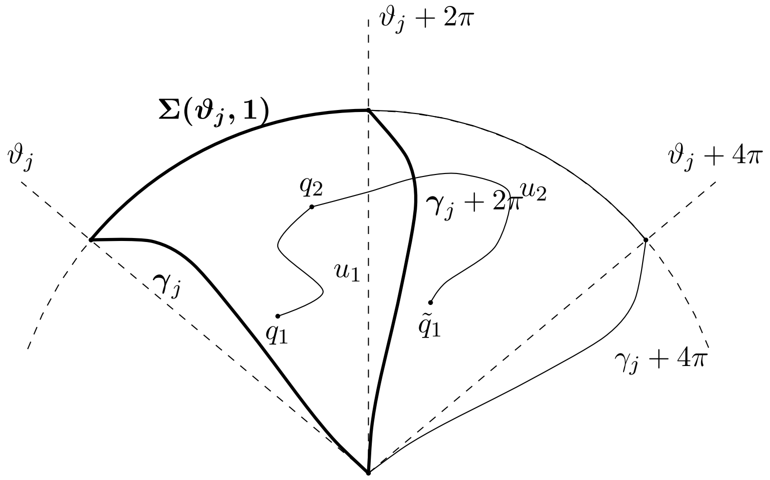

Figure 2.6.: Situation of Theorem 2.4.23. We remark that the picture is not referred to the Euclidean space. Indeed, here we denote by γ_j the unique collision minimizer which starts from $r_0 e^{i\vartheta_j}$ in the fashion of Lemma 2.4.17 and by $\gamma_j + 2\pi$ and $\gamma_j + 4\pi$ its 2π and 4π copies respectively. This minimal arcs determine the curved sectors used in the proof, while the concatenation of u_1 and u_2 is a closed path which winds around the origin.

we deduce the existence of the second minimal arc u_2, which connects q_2 and $\tilde{q}_1$ with the same properties of u_1. Consider the concatenation u of u_1 and u_2, which, of course, is a closed curve from q_1 to itself. Since both u_1 and u_2 are collision-less, the winding number of u with respect to the origin is 1. □

At this point, we are ready to prove that a minimizer $u \in K_l$ for the Maupertuis' functional joins property (CF).

Theorem 2.4.24. *Assume that the assumptions (V) on the potentials $(V_j)_{j=1}^N$ are satisfied and fix $l \in \mathfrak{I}^N$. Fix $\varepsilon \in (0, \bar{\varepsilon})$ as in* (2.12) *and $R \in (\tilde{\varepsilon}, \mathfrak{m}^{1/\alpha} - \tilde{\varepsilon})$ as in* (2.14)*. Then, there exists $\delta > 0$ such that, for every $p_1, p_2 \in \partial B_R$ every minimizer u of the Maupertuis' functional*

$$\mathcal{M}(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 \int_0^1 (-1 + V^\varepsilon(u))$$

in the space $K_l^{p_1,p_2}$ found in Proposition 2.4.9 joins the following properties:

(i) *u is free of self-intersections;*

(ii) *u satisfies*

$$\min_{t\in[0,1]} |u(t) - c_j'| > \delta, \quad \text{for every } j = 1, \dots, N.$$

Therefore, in particular u is collision-less.

Proof. Fix ε and R as in the statement, fix $l \in \mathfrak{I}^N$ and $p_1, p_2 \in \partial B_R$. Assume by contradiction that a minimizer $u \in K_l^{p_1,p_2}$ of the Maupertuis' functional $\mathcal{M}$ has a collision with the centre c_j' for some $j \in \{1, \dots, N\}$; for the sake of simplicity we will assume again $c_j' = 0$. Then, $u \in \mathfrak{Coll}_l^j$, there exists $t_0 \in (0,1)$ such that $u(t_0) = 0$ and in particular

$$\text{Ind}(u; c_k') \equiv l_k \pmod 2, \quad \forall\, k \neq j.$$

Then, localizing the collision as in the beginning if this section, we can find an interval $[c,d] \subseteq [0,1]$ such that:

- $t_0 \in [c,d]$ and the collision is isolated therein;
- $\bar{p}_1 \doteq u(c), \bar{p}_2 \doteq u(d) \in \partial B_r$, with $r < r^*$ and $r^* > 0$ as in Lemma 2.4.17.

Then, by means of Lemma 2.4.14, the restriction $v \doteq u|_{[c,d]}$ is a minimizer of the Maupertuis' functional

$$\mathcal{M}_l^{\bar{p}_1,\bar{p}_2}(v) = \frac{1}{2}\int_c^d |\dot{v}|^2 \int_c^d (-1 + V^\varepsilon(v))$$

in the weak H^1-closure $\mathcal{K}_l^{\bar{p}_1,\bar{p}_2}$ of the H^1 restricted paths

$$\hat{\mathcal{K}}_l^{\bar{p}_1,\bar{p}_2} \doteq \left\{ v \in H^1([c,d];\mathbb{R}^2) : \begin{array}{l} v(c) = \bar{p}_1,\ v(d) = \bar{p}_2,\ v(t) \neq c_j'\ \forall\, t \in [c,d],\ \forall\, j \\ \text{the function } G_v(t) \doteq \begin{cases} u(t) & \text{if } t \in [0,c) \cup (d,1] \\ v(t) & \text{if } t \in [c,d] \end{cases} \\ \text{belongs to } \hat{K}_l^{p_1,p_2} \end{array} \right\}.$$

Since v solves a Bolza problem for the Maupertuis' functional inside B_r, by Theorem 2.4.23, we know that, up to time reparametrizations, v connects $\bar{p}_1$ and $\bar{p}_2$ belonging to $\Sigma(\vartheta_j, 1)$ or to $\Sigma(\vartheta_j, 2) \setminus \Sigma(\vartheta_j, 1)$, depending on the value of the index l_j. Therefore, by claim (i) of Theorem 2.4.23, a contradiction arises both if $l_j = 0$ or $l_j = 1$. Thus u cannot have a collisions and in particular, again from Theorem 2.4.23, (ii) is proved. Claim (i) follows from this property and Proposition 2.4.12. □

2.4.4. Classical solution arcs

In this section we will conclude the proof of the existence of internal arcs, finally showing that the minimizer of the Maupertuis' functional satisfies property (R) introduced at page 64. Indeed, in the previous section we have already showed that the minimizer is collision-less, i.e., we proved (CF). In particular, in the next result we show that, given a minimizer u provided in Proposition 2.4.9, if u has endpoints sufficiently close to minimal non-degenerate central configurations of W^0, then $|u(t)| < R$ whenever $t \in (0,1)$. Recall that W^0 is $-\alpha$-homogeneous (see (2.11)) and it is the *leading* component of the total potential $V^\varepsilon(y)$ as $\varepsilon \to 0^+$ and $|y|$ becomes very large. Indeed, we have already observed along Section 2.3 that, when we are far from the singularity set, the problem reduces to a perturbation of an anisotropic Kepler problem driven by

W^0. This suggests to use compactness properties of sequences of minimizers and their convergence to a minimal collision arc for the anisotropic Kepler problem driven by W^0. In particular, we will consider again Lemma 2.4.17 from [5], in which the authors show that all the collision minimizers starting sufficiently close to minimal non-degenerate central configurations describe a foliation which is strictly contained in a given cone. Recall that, from the second line of assumption (V), as already observed in Remark 2.2.3, the sum potential W^0 admits a finite number of minimal non-degenerate central configurations

$$\Xi \doteq \{\vartheta^* \in \mathbb{S}^1 : \; U'(\vartheta^*) = 0 \text{ and } U''(\vartheta^*) > 0\} = \{\vartheta_0^*, \ldots, \vartheta_{m-1}^*\},$$

where, in polar coordinates $y = (\rho, \vartheta)$

$$W^0(y) = W^0(\rho, \vartheta) = \rho^{-\alpha} U(\vartheta) = \rho^{-\alpha} \sum_{i=1}^{k} U_i(\vartheta).$$

Therefore, it is not restrictive to work with two of this central configurations $\vartheta^*, \vartheta^{**} \in \mathbb{S}^1$, since we are solving a Bolza problem, but it is clear that the result holds choosing any pair (not necessarily distinct) of central configurations.

Theorem 2.4.25. *Assume that the assumptions (V) on the potentials $(V_j)_{j=1}^N$ are satisfied and fix $R > 0$ as in* (2.14). *Then, there exists $\bar{\varepsilon} > 0$ such that, for any $\vartheta^*, \vartheta^{**} \in \Xi$ minimal non degenerate central configurations for W^0, defining $\xi^* \doteq Re^{i\vartheta^*}, \xi^{**} \doteq Re^{i\vartheta^{**}} \in \partial B_R$, there exist two neighbourhoods $\mathcal{U}_{\xi^*}, \mathcal{U}_{\xi^{**}}$ on ∂B_R with the following property:*

$$\forall\, \varepsilon \in (0, \bar{\varepsilon}),\ \forall l \in \mathfrak{I}^N,\ \forall\, p_1 \in \mathcal{U}_{\xi^*},\ \forall\, p_2 \in \mathcal{U}_{\xi^{**}} \text{ there holds } \quad |u(t)| < R, \text{ for all } t \in (0,1).$$

where u is the minimizer of the Maupertuis' functional in the space $K_l^{p_1,p_2}$ provided in Proposition 2.4.9.

Proof. Assume by contradiction that there exist the following sequences:

- $(\varepsilon_n) \subseteq \mathbb{R}^+$, with $\varepsilon_n \to 0^+$,
- $(p_1^n) \subseteq \mathcal{U}_{\xi^*}$ and $(p_2^n) \subseteq \mathcal{U}_{\xi^{**}}$,
- $(t_n) \subseteq (0,1)$,
- a sequence of minimizers $(u_n) \subseteq (K_l^{p_1^n,p_2^n})$ for the sequence of functionals $(\mathcal{M}_n)$ defined by
$$\mathcal{M}_n(u_n) \doteq \frac{1}{2}\int_0^1 |\dot{u}_n|^2 \int_0^1 \left(-1 + V^{\varepsilon_n}(u_n)\right),$$

such that

$$|u_n(t_n)| = R, \quad \text{for all } n \in \mathbb{N}.$$

Up to subsequences and without loss of generality we can assume that

$$\begin{aligned} p_1^n &\to p_1 \in \mathcal{U}_{\xi^*} \\ p_2^n &\to p_2 \in \mathcal{U}_{\xi^{**}} \\ t_n &\to \bar{t} \in [0,1] \end{aligned}$$

as $n \to +\infty$. Indeed, concerning the limits of (p_1^n) and (p_2^n), it is enough to choose smaller neighbourhoods of ξ^* and ξ^{**}. Recalling the limiting behaviour of V^ε as $\varepsilon \to 0^+$ (see Proposition 2.2.2) and thus the definition of the $-\alpha$-homogeneous potential W^0, if we define

$$\mathcal{M}_0(u) \doteq \frac{1}{2}\int_0^1 |\dot{u}|^2 \int_0^1 \left(-1 + W^0(u)\right),$$

from Lemma 2.4.17 we know that there exist a unique $u^* \in H^{p_1}_{coll}$ and a unique $u^{**} \in H^{p_2}_{coll}$ such that

$$\mathcal{M}_0(u^*) = \min_{H^{p_1}_{coll}} \mathcal{M}_0, \quad \mathcal{M}_0(u^{**}) = \min_{H^{p_2}_{coll}} \mathcal{M}_0.$$

In particular, from Proposition 2.4.3, we have that there exists a unique $u_0 \in H^{p_1,p_2}$, where

$$H^{p_1,p_2} \doteq \left\{ u \in H^1([0,1];\mathbb{R}^2) : \ u(0) = p_1, \ u(1) = p_2, \ u(t_0) = 0, \text{ for some } t_0 \in (0,1) \right\},$$

such that

$$\mathcal{M}_0(u_0) = \min_{H^{p_1,p_2}} \mathcal{M}_0,$$

where this path u_0 is nothing but the concatenation of u^* and u^{**}. We claim that

$$\lim_{n\to+\infty} \mathcal{M}_n(u_n) = \mathcal{M}_0(u_0) \tag{2.47}$$

and we start by showing that

$$\liminf_{n\to+\infty} \mathcal{M}_n(u_n) \le \mathcal{M}_0(u_0). \tag{2.48}$$

For every $n \in \mathbb{N}$ let us introduce the Jacobi-length functionals

$$\mathcal{L}_n(u_n) \doteq \int_0^1 |\dot{u}_n| \sqrt{-1 + V^{\varepsilon_n}(u_n)}, \quad \mathcal{L}_0(u_0) \doteq \int_0^1 |\dot{u}_0| \sqrt{-1 + W^0(u_0)}$$

and, since u_n and u_0 are minimizers, we have

$$\mathcal{L}_n(u_n) = \sqrt{2\mathcal{M}_n(u_n)}, \quad \text{for all } n \in \mathbb{N}, \quad \mathcal{L}_0(u_0) = \sqrt{2\mathcal{M}_0(u_0)}.$$

Our idea is to provide an explicit variation $w_n \in K_l^{p_1^n,p_2^n}$ such that, for large n

$$\mathcal{L}_n(u_n) \le \mathcal{L}_n(w_n) \le \mathcal{L}_0(u_0) + \mathcal{O}(\varepsilon_n^\beta), \tag{2.49}$$

for some $\beta > 0$, which would prove (2.48). In order to build such w_n we need to define some points inside B_R:

- define $\hat{p}_1^n \in \varepsilon_n \partial B_R$ as the first intersection between u_0 and the sphere $\varepsilon_n \partial B_R$;
- define $\hat{p}_2^n \in \varepsilon_n \partial B_R$ as the second intersection between u_0 and the sphere $\varepsilon_n \partial B_R$;
- define $q_1^n \doteq \varepsilon_n R \frac{\xi^*}{|\xi^*|} \in \varepsilon_n \partial B_R$ and $q_2^n \doteq \varepsilon_n R \frac{\xi^{**}}{|\xi^{**}|} \in \varepsilon_n \partial B_R$.

Note that, once n is fixed, the points $\hat{p}_1^n, \hat{p}_2^n$ are uniquely determined since both u^* and u^{**} are strictly decreasing with respect to t thanks to the Lagrange-Jacobi inequality (see [5, Lemma 4.3], cf Lemma 1.4.3). Moreover, we define the building blocks of w_n in this way:

- define arc(p_1^n, p_1) as the shorter (in the Euclidean metric) parametrized arc of ∂B_R, connecting p_1^n to p_1 with constant angular velocity;
- define γ_n^* as the portion of u_0 that goes from p_1 to $\hat{p}_1^n$;
- define arc$(\hat{p}_1^n, q_1^n)$ as as the shorter (in the Euclidean metric) parametrized arc of ∂B_R, connecting $\hat{p}_1^n$ to q_1^n with constant angular velocity;
- define φ_n as the minimizer of $\mathcal{L}_n$ in the space $K_l^{q_1^n, q_2^n}$;
- define as above the analogous path composed by the pieces arc$(q_2^n, \hat{p}_2^n)$, γ_n^{**}, arc(p_2, p_2^n), which goes from $\hat{p}_2^n$ to p_2^n.

At this point, we build w_n as the concatenation of the previous pieces with a suitable time parametrization

$$w_n = \begin{cases} \text{arc}(p_1^n, p_1) & \text{from } p_1^n \text{ to } p_1 \\ \gamma_n^* & \text{from } p_1 \text{ to } \hat{p}_1^n \\ \text{arc}(\hat{p}_1^n, q_1^n) & \text{from } \hat{p}_1^n \text{ to } q_1^n \\ \varphi_n & \text{from } q_1^n \text{ to } q_2^n \\ \text{arc}(q_2^n, \hat{p}_2^n) & \text{from } q_2^n \text{ to } \hat{p}_2^n \\ \gamma_n^{**} & \text{from } \hat{p}_2^n \text{ to } p_2 \\ \text{arc}(p_2, p_2^n) & \text{from } p_2 \text{ to } p_2^n \end{cases}$$

(see Figure 2.7). Now, since $\mathcal{L}_n$ is additive, the length of w_n is exactly the sum of the length of every piece and, in particular, since $w_n \in K_l^{p_1^n, p_2^n}$ and u_n is a minimizer of $\mathcal{L}_n$ we have

$$\mathcal{L}_n(u_n) \leq \mathcal{L}_n(w_n).$$

The next estimates on the arch lengths easily follow:

$$\begin{aligned} &p_1^n \to p_1,\ p_2^n \to p_2 && \Longrightarrow \mathcal{L}_n(\text{arc}(p_1^n, p_1)) = \mathcal{O}(1),\ \mathcal{L}_n(\text{arc}(p_2, p_2^n)) = \mathcal{O}(1) \\ &\hat{p}_1^n, \hat{p}_2^n, q_1^n, q_2^n \in \varepsilon_n \partial B_R && \Longrightarrow \mathcal{L}_n(\text{arc}(\hat{p}_1^n, q_1^n)) = \mathcal{O}(\varepsilon_n),\ \mathcal{L}_n(\text{arc}(q_2^n, \hat{p}_2^n)) = \mathcal{O}(\varepsilon_n) \end{aligned}$$

as $n \to +\infty$. From Proposition 2.2.2, we know that, if $y \in \mathbb{R}^2 \setminus B_\delta$ with $\delta > \varepsilon_n$, then

$$V^{\varepsilon_n}(y) = W^0(y) + \mathcal{O}(\varepsilon_n^{\min\{1, \alpha_{k+1} - \alpha\}}), \quad \text{as } n \to +\infty$$

hence

$$\mathcal{L}_n(\gamma_n^*) + \mathcal{L}_n(\gamma_n^{**}) \leq \mathcal{L}_0(u_0) + \mathcal{O}(\varepsilon_n^{\min\{1,\alpha_{k+1}-\alpha\}/2}), \quad \text{as } n \to +\infty.$$

Therefore, to prove (2.49), we need to provide an estimate on $\mathcal{L}_n(\varphi_n)$; to do that, let us define the blow-up sequence

$$\tilde{\varphi}_n(t) \doteq \frac{1}{\varepsilon_n}\varphi_n(t), \quad \text{for } t \in [0,1]$$

and note that

$$\tilde{\varphi}_n(0) = \frac{\xi^*}{|\xi^*|}R \doteq q^* \in \partial B_R, \quad \tilde{\varphi}_n(1) = \frac{\xi^{**}}{|\xi^{**}|}R \doteq q^{**} \in \partial B_R. \tag{2.50}$$

Moreover, recalling the definition (2.7) of V^{ε_n}, for $n \in \mathbb{N}$ and $y \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$ we can compute

$$\begin{aligned}
V^{\varepsilon_n}(\varepsilon_n y) &= \sum_{i=1}^{k} V_i(\varepsilon_n y - \varepsilon_n c_i) + \sum_{j=k+1}^{N} \varepsilon_n^{\alpha_j - \alpha} V_j(\varepsilon_n y - \varepsilon_n c_j) \\
&= \varepsilon_n^{-\alpha} \sum_{i=1}^{k} V_i(y - c_i) + \varepsilon_n^{-\alpha} \sum_{j=k+1}^{N} V_j(y - c_j) \\
&= \varepsilon_n^{-\alpha} V(y)
\end{aligned}$$

and thus we have

$$\begin{aligned}
\mathcal{L}_n(\varphi_n) = \mathcal{L}_n(\varepsilon_n\tilde{\varphi}_n) &= \int_0^1 \varepsilon_n|\dot{\tilde{\varphi}}_n|\sqrt{-1 + V^{\varepsilon_n}(\varepsilon_n\tilde{\varphi}_n)} \\
&= \varepsilon_n \int_0^1 |\dot{\tilde{\varphi}}_n|\sqrt{\varepsilon_n^{-\alpha}(-\varepsilon_n^{\alpha} + V(\tilde{\varphi}_n))} \\
&= \varepsilon_n^{\frac{2-\alpha}{2}} \int_0^1 |\dot{\tilde{\varphi}}_n|\sqrt{-\varepsilon_n^{\alpha} + V(\tilde{\varphi}_n)} \\
&= \varepsilon_n^{\frac{2-\alpha}{2}} \tilde{\mathcal{L}}_n(\tilde{\varphi}_n),
\end{aligned}$$

where we have put

$$\tilde{\mathcal{L}}_n(\tilde{\varphi}_n) \doteq \int_0^1 |\dot{\tilde{\varphi}}_n|\sqrt{-\varepsilon_n^{\alpha} + V(\tilde{\varphi}_n)}, \quad \text{for every } n \in \mathbb{N}.$$

Notice that the function $\tilde{\varphi}_n$ is clearly a minimizer for $\tilde{\mathcal{L}}_n$ in the space

$$S_l^{q^*,q^{**}} \doteq \left\{ \varphi \in H^1([0,1];\mathbb{R}^2) : \begin{array}{l} \varphi(0) = q^*,\ \varphi(1) = q^{**}\ |\varphi| \leq R, \\ \text{Ind}(\varphi; c_j) \equiv l_j \mod 2,\ \forall\, j = 1, \ldots, N \end{array} \right\},$$

where q^* and q^{**} have been defined in (2.50). If we furthermore define the functional

$$\tilde{\mathcal{L}}_0(\varphi) \doteq \int_0^1 |\dot{\varphi}|\sqrt{V(\varphi)}$$

we have that, for every $n \in \mathbb{N}$ and for every test function φ

$$\tilde{\mathcal{L}}_n(\tilde{\varphi}_n) \leq \tilde{\mathcal{L}}_n(\varphi) \leq \tilde{\mathcal{L}}_0(\varphi)$$

and so, in particular

$$\tilde{\mathcal{L}}_n(\tilde{\varphi}_n) \leq \min_{S_l^{q^*,q^{**}}} \tilde{\mathcal{L}}_0 \leq C,$$

for some $C > 0$. This last inequality finally gives the estimates (2.49) and thus (2.48).

At this point, in order to get the claim (2.47), we prove the reverse inequality, i.e.,

$$\mathcal{M}_0(u_0) \leq \liminf_{n\to+\infty} \mathcal{M}_n(u_n). \tag{2.51}$$

Since every component of the potential V^{ε_n} is bounded on $\mathbb{S}^1$ and $|u_n| \leq R$, together with (2.47) we can deduce that there exists $C_1, C_2 > 0$ such that

$$C_1 \geq \mathcal{M}_n(u_n) \geq C_2 \int_0^1 |\dot{u}_n|^2,$$

at least for n large enough. From this, we deduce a uniform bound on the H^1-norm of (u_n) and the existence of a H^1-weak and uniform in $[0,1]$ limit $\bar{u} \in H^{p_1,p_2}$. Fatou lemma, the semi-continuity of the H^1-norm and the a.e. convergence of V^{ε_n} to W^0 in $\mathbb{R}^2$ then give

$$\mathcal{M}_0(\bar{u}) \leq \liminf_{n\to+\infty} \mathcal{M}_n(u_n).$$

At this point, the minimality of u_0 for $\mathcal{M}_0$ in the space H^{p_1,p_2} gives the inequality (2.51) and, together with (2.48), we get the claim (2.47)

$$\lim_{n\to+\infty} \mathcal{M}_n(u_n) = \mathcal{M}_0(u_0)$$

with, in particular

$$u_n \to u_0 \quad \text{uniformly in } [0,1].$$

At this point, a bootstrap technique helped by the conservation of the energy for (u_n) leads to a $\mathcal{C}^1$-convergence outside the collision instant t_0 of u_0; this proves that $|u_0(\bar{t})| = R$. If $\bar{t} \in (0,1)$ this is a contradiction for Lemma 2.4.17, because the minimizer cannot leave the cone therein defined; otherwise, if for instance $t_n \to 0$, we would find that $\dot{u}_0(0)$ is tangent to ∂B_R. This indeed is also a contradiction: up to make $\mathcal{U}_{\xi^*}$ smaller, the unique collision trajectory from $p_1 \in \mathcal{U}_{\xi^*}$ must have initial velocity direction close to the initial velocity of the homothetic motion starting from ξ^*, which is normal to sphere. □

Now, we can finally show that a minimizer of the Maupertuis' functional is actually a reparametrization of a classical solution arc of the inner problem.

Theorem 2.4.26. *Assume that the assumptions* (V) *on the potentials* $(V_j)_{j=1}^N$ *are satisfied and fix* $R > 0$ *as in* (2.14). *Then, there exists* $\varepsilon_{int} > 0$ *such that, for any* $\vartheta^*, \vartheta^{**} \in \Xi$ *minimal*

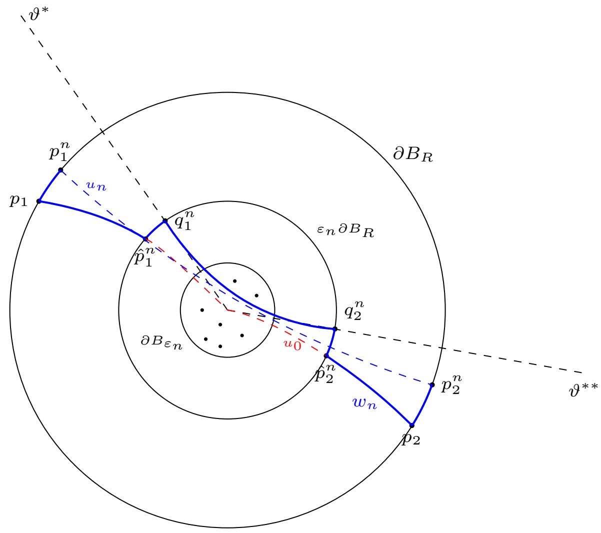

Figure 2.7.: The proof of Theorem 2.4.25: the blue path w_n piecewise built in the proof belongs to the space $K_l^{p_1^n,p_2^n}$, as well as the blue dashed path u_n. This makes it a suitable competitor for u_n and allows to use the minimization argument. In this picture, the red dashed path u_0 represents the limit collision path which belongs to the space H^{p_1,p_2} and that actually connects p_1 and p_2 on ∂B_R.

non degenerate central configurations for W^0, defining $\xi^* \doteq Re^{i\vartheta^*}, \xi^{**} \doteq Re^{i\vartheta^{**}} \in \partial B_R$, there exist two neighbourhoods $\mathcal{U}_{\xi^*}, \mathcal{U}_{\xi^{**}}$ on ∂B_R with the following property:

for any $\varepsilon \in (0, \varepsilon_{int})$, for any $l \in \mathfrak{I}^N$, for any pair of endpoints $p_1 \in \mathcal{U}_{\xi^*}$,$p_2 \in \mathcal{U}_{\xi^{**}}$, there exist $T > 0$ and a classical (collision-less) solution $y \in \hat{K}_l^{p_1,p_2}([0,T])$ of the inner problem

$$\begin{cases} \ddot{y}(t) = \nabla V^\varepsilon(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - V^\varepsilon(y(t)) = -1 & t \in [0,T] \\ |y(t)| < R & t \in (0,T) \\ y(0) = p_1, \quad y(T) = p_2 \end{cases}.$$

In particular, y is a re-parametrization of a minimizer of the Maupertuis' functional in the space $K_l^{p_1,p_2}([0,1])$ and it is free of self-intersections and there exists $\delta > 0$

$$\min_{t\in[0,T]} |y(t) - c_j'| > \delta, \quad \textit{for any } j \in \{1, \dots, N\}.$$

Proof. The proof is a direct consequence of Theorem 2.4.24, Theorem 2.4.25 and Theorem 2.4.1 (the Maupertuis' principle). □

In order to conclude the construction of the interior arcs for the N-centre problem, we need to give a version of Theorem 2.4.26 which takes into account the language of

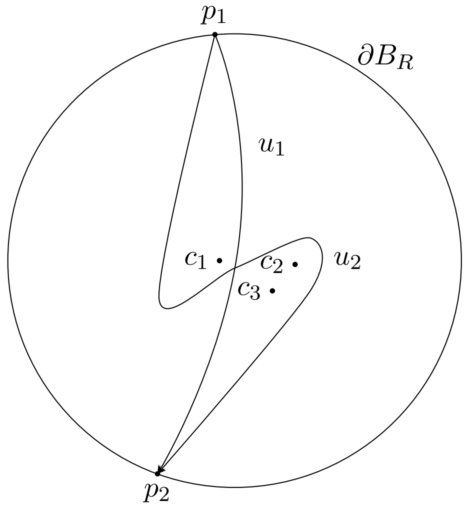

Figure 2.8.: An example of two minimal arcs which realize the same partition of the centres. Indeed, both the paths divide the centres with respect to the partition $\{\{c_1\},\{c_2,c_3\}\}$, but $u_1 \in K^{p_1,p_2}_{(0,1,1)}$ while $u_2 \in K^{p_1,p_2}_{(1,0,0)}$.

partitions. We invite the reader to go back at page 33 and we note that a minimizer $u \in \hat{K}^{p_1,p_2}_l$ which is free of self intersections satisfies the topological constraint of separation of the centres (see Proposition 2.4.12 and Remark 2.4.13). In particular, recalling the definition of the set of all the partitions in two non-trivial subsets of the centres

$$\mathcal{P} = \{P_j : \; j = 0, \dots, 2^{N-1} - 2\}$$

a choice of $l \in \mathfrak{I}^N$ will induce a choice of $P_j \in \mathcal{P}$, for some j and this is not 1-1. Notice that the lack of the 1-1 property is due to the fact that, for instance, if $N = 3$ the winding vectors $(1,0,0)$ and $(0,1,1)$ produce respectively two minimizers that separate the centres with respect to the same partition (see Figure 2.8).

Following the notations introduced in [61], define the map $\mathcal{A}\colon \mathfrak{I}^N \to \mathcal{P}$ which associates to every winding vector

$$l = (l_1, \dots, l_N), \quad \text{with the property} \begin{cases} \text{if } l_k = 0 \text{ then } k \in A_0 \subseteq \{1, \dots, N\} \\ \text{if } l_j = 1 \text{ then } j \in A_1 \subseteq \{1, \dots, N\} \end{cases}$$

the partition

$$\mathcal{A}(l) = \{\{c_k : \; l_k \in A_0\}, \{c_j : \; l_j \in A_1\}\}.$$

As already observed, the map $\mathcal{A}$ is surjective, but not injective, since $\mathcal{A}(l) = \mathcal{A}(\tilde{l})$, for every $l, \tilde{l} \in \mathfrak{I}^N$ such that

$$l_j + \tilde{l}_j = 1, \quad \text{for every } j = 1, \dots, N.$$

At this point, for any $j \in \{0, \dots, 2^{N-1} - 2\}$ and for any $p_1, p_2 \in \partial B_R$, these sets are well-defined

$$
\begin{aligned}
\hat{K}_{P_j} &\doteq \hat{K}^{p_1,p_2}_{P_j}([0,1]) = \{u \in \hat{K}^{p_1,p_2}_{l}([0,1]) : l = \mathcal{A}^{-1}(P_j)\} \\
K_{P_j} &\doteq K^{p_1,p_2}_{P_j}([0,1]) = \{u \in K^{p_1,p_2}_{l}([0,1]) : l = \mathcal{A}^{-1}(P_j)\}.
\end{aligned}
\tag{2.52}
$$

The set K_{P_j} is the weak H^1-closure of $\hat{K}_{P_j}$ and, if $P_j = \mathcal{A}(l) = \mathcal{A}(\tilde{l})$, it turns out that it is exactly the union of two disjoint connected components, i.e.,

$$
K_{P_j} = K_l \cup K_{\tilde{l}}. \tag{2.53}
$$

Remark 2.4.27. *From the previous discussion we deduce that once a partition P_j is fixed, the corresponding minimizer of the Maupertuis' functional is not unique (see Figure 2.8).*

We can now state the main theorem of this section which is readily proven.

Theorem 2.4.28. *Assume that the assumptions* (V) *on the potentials $(V_j)_{j=1}^N$ are satisfied and fix $R > 0$ as in* (2.14)*. Then, there exists $\varepsilon_{int} > 0$ such that, for any $\vartheta^*, \vartheta^{**} \in \mathbb{S}^1$ minimal non degenerate central configurations for W^0, defining $\xi^* \doteq Re^{i\vartheta^*}, \xi^{**} \doteq Re^{i\vartheta^{**}} \in \partial B_R$, there exist two neighbourhoods $\mathcal{U}_{\xi^*}, \mathcal{U}_{\xi^{**}}$ on ∂B_R with the following property:*

for any $\varepsilon \in (0, \varepsilon_{int})$, for any $P_j \in \mathcal{P}$, for any pair of endpoints $p_1 \in \mathcal{U}_{\xi^*}$,$p_2 \in \mathcal{U}_{\xi^{**}}$, there exist $T_1, T_2 > 0$ and two classical (collision-less) solutions $y_1 \in \hat{K}^{p_1,p_2}_{P_j}([0,T_1])$ and $y_2 \in \hat{K}^{p_1,p_2}_{P_j}([0,T_2])$ of the inner problems

$$
\begin{cases}
\ddot{y}(t) = \nabla V^\varepsilon(y(t)) & t \in [0, T_i] \\
\frac{1}{2}|\dot{y}(t)|^2 - V^\varepsilon(y(t)) = -1 & t \in [0, T_i] \\
|y(t)| < R & t \in (0, T_i) \\
y(0) = p_1, \quad y(T) = p_2
\end{cases}
$$

for $i = 1, 2$. In particular, y_1 and y_2 are re-parametrizations of two minimizers of the Maupertuis' functional, every one of them in a different connected component of K_{P_j} (see (2.53)*). Moreover, y_1 and y_2 are free of self-intersections and there exists $\delta > 0$ such that*

$$
\min_{t \in [0,T_i]} |y_i(t) - c_j'| > \delta, \quad \textit{for any } j \in \{1, \dots, N\},
$$

for $i = 1, 2$.

We conclude this section with a property of the inner solution arcs found in Theorem 2.4.28, i.e., we show that there exists a uniform bound on the time intervals of such solutions.

Lemma 2.4.29. *Let $\varepsilon \in (0, \varepsilon_{int})$, let $\vartheta^*, \vartheta^{**} \in \mathbb{S}^1$ be two minimal non-degenerate central configurations for W^0 and $\mathcal{U}^*, \mathcal{U}^{**}$ be their suitable neighbourhoods, let $p_1 \in \mathcal{U}^*$ and $p_2 \in \mathcal{U}^{**}$,* *let $P_j \in \mathcal{P}$ and let $y_{P_j}(\cdot;p_1,p_2;\varepsilon)$ be one of the two classical solutions found in Theorem 2.4.28, defined in its time interval $[0,T_{P_j}(p_1,p_2;\varepsilon)]$. Then, there exist $c,C>0$ such that*

$$c \le T_{P_j}(p_1,p_2;\varepsilon) \le C.$$

Such constants do not depend on the choice of p_1,p_2 and P_j.

Proof. If we denote by $u_{P_j}(\cdot;p_1,p_2;\varepsilon) \in \hat{K}^{p_1,p_2}_{P_j}([0,1])$ the minimizer of the Maupertuis functional that re-parametrizes the solution y_{P_j}, from Theorem 2.4.1 we have that $T_{P_j}(p_1,p_2,\varepsilon) = 1/\omega_{P_j}(p_1,p_2;\varepsilon)$, where

$$\omega_{P_j}(p_1,p_2;\varepsilon) = \frac{\int_0^1(-1+V^\varepsilon(u_{P_j}))}{\frac{1}{2}\int_0^1 |\dot{u}_{P_j}|^2}.$$

Therefore, if we prove equivalent bounds on ω_{P_j} we are done. It is clear that we can fix P_j since $\mathcal{P}$ is finite. Moreover, we will prove the statement for points p_1,p_2 which can belong to all the sphere ∂B_R; this is actually a weaker hypothesis that will give the proof anyway and that will simplify the notations in this context.

Let us fix $q_1,q_2 \in \partial B_R$ and consider a path $\bar{v} \in \hat{K}^{q_1,q_2}_{P_j}([0,1])$ such that there exist $\bar{C},\mu>0$ such that

$$\begin{cases} |\dot{\bar{v}}(t)| = \bar{C}, \text{ for every } t\in[0,1] \\ |\bar{v}(t)-c'_k| \ge \mu, \text{ for evert } t \in [0,1], \text{ for all } k=1,\dots,N. \end{cases}$$

Recalling the definition (2.7) of V^ε and that the restriction of every potential to $\mathbb{S}^1$ is bounded, we have that there exists $C_1>0$ such that

$$\begin{aligned} \mathcal{M}(\bar{v}) &= \frac{1}{2}\int_0^1 |\dot{\bar{v}}(t)|^2\,dt \int_0^1 (-1+V^\varepsilon(\bar{v}(t)))\,dt \\ &\le \frac{\bar{C}^2}{2}\int_0^1 \left(-1+\sum_{i=1}^k \mu^{-\alpha}\max_{\mathbb{S}^1} V_i + \sum_{j=k+1}^N \varepsilon^{\alpha-\alpha_j}\mu^{-\alpha_j}\max_{\mathbb{S}^1} V_j\right) \\ &\le C_1. \end{aligned} \tag{2.54}$$

In general, for every path that connects two points on ∂B_R, i.e., for every

$$v \in \bigcup_{p_1,p_2\in\partial B_R} \hat{K}^{p_1,p_2}_{P_j}([0,1])$$

we have a constant $C_2>$ which does not depends on the endpoints such that

$$\int_0^1 (-1+V^\varepsilon(v)) \ge \frac{\mathfrak{m}}{(R+\varepsilon)^\alpha} - 1 \doteq C_2 > 0 \tag{2.55}$$

(see (2.39) in the proof of Lemma 2.4.6). Moreover, if $\tilde{v} \in \hat{K}^{q_1,q_2}_{P_j}([0,1])$ is a minimizer of $\mathcal{M}$, we have that $\mathcal{M}(\tilde{v}) \le \mathcal{M}(\bar{v})$ and thus, from (2.54) and (2.55) we get

$$\int_0^1 |\dot{\tilde{v}}|^2 \le \frac{2C_1}{C_2}.$$

The previous bound has to be refined because the constant C_1 depends on $\bar{C}$, which is not uniform with respect to q_1 and q_2. To this aim, let us consider $p_1, p_2 \in \partial B_R \setminus \{q_1, q_2\}$ and, up to time re-parametrizations, define the path

$$\hat{v} \doteq \begin{cases} \mathrm{arc}(p_1, q_1) & \text{from } p_1 \text{ to } q_1 \\ \tilde{v} & \text{from } q_1 \text{ to } q_2 \\ \mathrm{arc}(q_2, p_2) & \text{from } q_2 \text{ to } p_2, \end{cases}$$

where $\mathrm{arc}(p_1, q_1)$ (respectively $\mathrm{arc}(q_2, p_2)$) denotes the shortest (in the Euclidean metric) parametrized arc of ∂B_R, connecting p_1 to q_1 (respectively q_2 to p_2) with constant angular velocity. Since the angular velocity is clearly uniformly bounded with respect to the endpoints from above and from the definition of $\hat{v}$ it is easy to see that there exists a constant $C_3 > 0$ which does not depends on p_1 and p_2 such that

$$\mathcal{M}(\hat{v}) \leq C_3.$$

For this reason, this bound is conserved for all the minimizers with endpoints in ∂B_R, i.e.,

$$\mathcal{M}(u_{P_j}(\cdot; p_1, p_2, ; \varepsilon)) \leq C_3, \quad \text{for all } p_1, p_2 \in \partial B_R, \tag{2.56}$$

which, together with (2.55) gives

$$\int_0^1 |\dot{u}_{P_j}(\cdot; p_2, p_2; \varepsilon)|^2 \leq \frac{2C_3}{C_2} \doteq C_4, \quad \text{for all } p_1, p_2 \in \partial B_R. \tag{2.57}$$

Moreover, from (2.41) in the proof of Lemma 2.4.6, we have seen that the constant $C_5 \doteq (R - \varepsilon)^2 > 0$ is a uniform lower bound for the quantity $\|\dot{u}\|_2^2$, for any $u \in \hat{K}_{P_j}^{p_1, p_2}([0,1])$, for any $p_1, p_2 \in \partial B_R$, so that in particular

$$\int_0^1 |\dot{u}_{P_j}(\cdot; p_1, p_2; \varepsilon)|^2 \geq C_5, \quad \text{for all } p_1, p_2 \in \partial B_R. \tag{2.58}$$

This, together with (2.56), proves that

$$\int_0^1 \left(-1 + V^\varepsilon(u_{P_j}(\cdot; p_1, p_2; \varepsilon))\right) \leq \frac{2C_3}{C_5} \doteq C_6 \quad \text{for all } p_1, p_2 \in \partial B_R. \tag{2.59}$$

At this point we have that (2.57) and (2.58) give

$$C_5 \leq \inf_{p_1, p_2 \in \partial B_R} \int_0^1 |\dot{u}_{P_j}(\cdot; p_1, p_2; \varepsilon)|^2 \leq \sup_{p_1, p_2 \in \partial B_R} \int_0^1 |\dot{u}_{P_j}(\cdot; p_1, p_2; \varepsilon)|^2 \leq C_4,$$

while (2.55) and (2.59) lead to

$$\begin{aligned} C_2 \leq \inf_{p_1, p_2 \in \partial B_R} &\int_0^1 \left(-1 + V^\varepsilon(u_{P_j}(\cdot; p_1, p_2; \varepsilon))\right) \\ &\leq \sup_{p_1, p_2 \in \partial B_R} \int_0^1 \left(-1 + V^\varepsilon(u_{P_j}(\cdot; p_1, p_2; \varepsilon))\right) \leq C_6; \end{aligned}$$

the definition of ω_{P_j} then clearly concludes the proof. □

Remark 2.4.30. *In the following, when the partition of the centres will not have a relevance, we will denote one of the internal arcs provided in Theorem 2.4.28 in this way*

$$y_{int}(\cdot; p_1, p_2; \varepsilon),$$

in order to highlight that it is an arc lying inside B_R that connects p_1 and p_2. Moreover, the corresponding neighbourhoods on ∂B_R will be denoted as

$$\mathcal{U}_{int}(\xi^*),\ \mathcal{U}_{int}(\xi^{**}).$$

2.5. Glueing pieces and multiplicity of periodic solutions

Since we have proved the existence of outer and inner fixed-ends solution arcs, respectively in Section 2.3 and Section 2.4, this section is devoted to build periodic trajectories which solve

$$\begin{cases} \ddot{y} = \nabla V^\varepsilon(y) \\ \frac{1}{2}|\dot{y}|^2 - V^\varepsilon(y) = -1, \end{cases}$$

glueing together solution pieces on ∂B_R. The assumptions on R will be again (2.14) as well as the requirements (V) on V^ε. We recall the set of strictly minimal central configurations for the leading potential in the outer dynamics

$$\Xi = \{\vartheta^* \in \mathbb{S}^1 : U'(\vartheta^*) = 0 \text{ and } U''(\vartheta^*) > 0\} = \{\vartheta_0^*, \dots, \vartheta_{m-1}^*\}.$$

Let $\varepsilon \in (0, \min\{\varepsilon_{int}, \varepsilon_{ext}\})$, with $\varepsilon_{int}, \varepsilon_{ext} > 0$ provided in Theorem 2.3.10 and Theorem 2.4.28, and let $n \in \mathbb{N}_{\geq 1}$ be the number of pairs of inner and outer arcs; the idea is to relate a periodic trajectory in the punctured plane with a double sequence of this kind

$$(P_0, \xi_0^*), (P_1, \xi_1^*), \dots, (P_{n-1}, \xi_{n-1}^*) \tag{2.60}$$

where $P_j \in \mathcal{P}$ is a partition of the centres and $\xi_j^* = Re^{i\vartheta_j^*}$, with $\vartheta_j^* \in \Xi$ for every $j = 0, \dots, n-1$. Note that we admit the situation in which two or more elements of the sequence could be equal.

From Theorem 2.3.10 we know that, for every $j = 0, \dots, n-1$ there exists a neighbourhood $\mathcal{U}_{ext}(\xi_j^*) \subseteq \partial B_R$ of ξ_j^* such that, for every $(p_{2j}, p_{2j+1}) \in \mathcal{U}_{ext}(\xi_j^*) \times \mathcal{U}_{ext}(\xi_j^*)$ there exists an outer arc $y_{ext}(\cdot; p_{2j}, p_{2j+1}; \varepsilon)$ which starts in p_{2j} and arrives in p_{2j+1}. This arc actually solves problem (2.15) with boundary conditions in (p_{2j}, p_{2j+1}) and in a suitable time interval $[0, T_{\varepsilon,2j}]$. We have selected $2n$ points on ∂B_R

$$\{p_0, p_1, p_2, \dots, p_{2n-2}, p_{2n-1}\},$$

so that $p_{2j}, p_{2j+1} \in \mathcal{U}_{ext}(\xi_j^*)$ are connected through an outer arc, for every $j = 0, \dots, n-1$.

Now, for any $j = 1, \dots, n-1$, thanks to Theorem 2.4.28 and in view of the notations introduced in Remark 2.4.30, if $p_{2j-1} \in \mathcal{U}_{int}(\xi_{j-1}^*)$ and $p_{2j} \in \mathcal{U}_{int}(\xi_j^*)$, we can connect them through a minimizing inner arc $y_{int}(\cdot; p_{2j-1}, p_{2j}; \varepsilon)$ verifying the partition P_j. Up to time re-parametrizations, the inner arc will be defined in the interval $[0, T_{\varepsilon,2j-1}]$.

At this point, in order to build a closed orbit, for any $j = 0, \dots, n-1$ we introduce a smaller neighbourhood of ξ_j^*

$$\mathcal{U}_j \doteq \mathcal{U}_{ext}(\xi_j^*) \cap \mathcal{U}_{int}(\xi_j^*)$$

and we select an ordered sequence of pairs $(p_{2j}, p_{2j+1}) \in \mathcal{U}_j \times \mathcal{U}_j$. Moreover, to close the orbit, we add a last inner arc, joining p_{2n-1} and $p_{2n} \doteq p_0$ and that realizes the partition P_0. We denote this arc as $y_{int}(\cdot; p_{2n-1}, p_{2n}; \varepsilon)$ and we parametrize it on the interval $[0, T_{\varepsilon,2n-1}]$.

In addition, it is useful to define

$$\mathbf{U} \doteq (\mathcal{U}_0 \times \mathcal{U}_0) \times (\mathcal{U}_1 \times \mathcal{U}_1) \times \dots \times (\mathcal{U}_{n-1} \times \mathcal{U}_{n-1}) \times \mathcal{U}_0 \subseteq (\partial B_R)^{2n+1}$$

and thus to introduce the following closed set

$$\mathcal{S} \doteq \left\{ \mathbf{p} = (p_0, p_1, \dots, p_{2n}) \in \overline{\mathbf{U}} : \, p_0 = p_{2n} \right\} \subseteq (\partial B_R)^{2n+1},$$

which describes all the possible cuts on the sphere ∂B_R which an orbit could do, once sequence (2.60) is fixed. In this way, for every $\mathbf{p} \in \mathcal{S}$, we can define the periodic trajectory $\gamma_{\varepsilon,\mathbf{p}}$ as the alternating concatenation of n outer arcs and n inner arcs, up to time re-parametrizations. This curve will be $\mathbf{T}_\varepsilon$-periodic, where $\mathbf{T}_\varepsilon \doteq \sum_{j=0}^{2n-1} T_{\varepsilon,j}$ and piecewise-differentiable thanks to Theorem 2.3.10 and Theorem 2.4.28. In general, the function $\gamma_{\varepsilon,\mathbf{p}}$ is not $\mathcal{C}^1$ in the junction points and, indeed, the main result of this section is to prove this differentiability through a variational technique.

Remark 2.5.1. *We are going to minimize a geometric functional over $\mathcal{S}$ in order to provide the smoothness of the junctions. Indeed, we have defined $\mathcal{S}$ as a subset of the closure of* $\mathbf{U}$ *to induce compactness. However, we know from Theorem 2.3.10 and Theorem 2.4.28 that the construction of outer and inner arcs works just for the interior points of $\mathcal{U}_j$. For this reason, we might make such neighbourhoods smaller, keeping the same notation. Furthermore, without loss of generality, we assume that the non-degeneracy of every central configuration ϑ_j^* of W^0 is preserved along its corresponding neighbourhood $\mathcal{U}_j$, i.e., we require that the function U is strictly convex on the whole $\mathcal{U}_j$.*

In order to proceed, let us first fix a sequence (2.60) and $\varepsilon \in (0, \min\{\varepsilon_{ext}, \varepsilon_{int}\})$. Define the total Jacobi length function $\mathbf{L} \colon \mathcal{S} \to \mathbb{R}$ as

$$\begin{aligned}
\mathbf{L}(\mathbf{p}) &\doteq \mathcal{L}([0, \mathbf{T}_\varepsilon]; \gamma_{\varepsilon,\mathbf{p}}) \\
&= \sum_{j=0}^{n-1} \mathcal{L}([0, T_{\varepsilon,2j}]; y_{ext}(t; p_{2j}, p_{2j+1}; \varepsilon)) + \sum_{j=1}^{n} \mathcal{L}([0, T_{\varepsilon,2j-1}]; y_{int}(t; p_{2j-1}, p_{2j}; \varepsilon)) \\
&= \sum_{j=0}^{n-1} \int_0^{T_{\varepsilon,2j}} |\dot{y}_{ext}(t; p_{2j}, p_{2j+1}; \varepsilon)| \sqrt{(V^\varepsilon(y_{ext}(t; p_{2j}, p_{2j+1}; \varepsilon)) - 1)} \, dt \\
&\quad + \sum_{j=1}^{n} \int_0^{T_{\varepsilon,2j-1}} |\dot{y}_{int}(t; p_{2j-1}, p_{2j}; \varepsilon)| \sqrt{(V^\varepsilon(y_{int}(t; p_{2j-1}, p_{2j}; \varepsilon)) - 1)} \, dt.
\end{aligned}$$

The compactness of the set $\mathcal{S}$ implies the following result.

Lemma 2.5.2. *There exists* $\bar{\mathbf{p}} \in \mathcal{S}$ *that minimizes* $\mathbf{L}$.

Proof. The proof goes exactly as in Step 1) of Theorem 5.3. of [61]. □

The aim of this section is thus to prove through several steps the following result.

Theorem 2.5.3. *There exists* $\bar{\varepsilon} > 0$ *such that, for every* $\varepsilon \in (0, \bar{\varepsilon})$, *for every* $n \geq 1$ *there exists* $\bar{\mathbf{p}} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{2n}) \in \mathring{\mathcal{S}}$ *such that*

(i) *the following holds*

$$\min_{\mathbf{p} \in \mathcal{S}} \mathbf{L}(\mathbf{p}) = \mathbf{L}(\bar{\mathbf{p}});$$

(ii) *the corresponding function* $\gamma_{\varepsilon, \bar{\mathbf{p}}}$ *is a periodic solution in* $[0, \mathbf{T}_\varepsilon]$ *of the* N*-centre problem*

$$\begin{cases} \ddot{\gamma} = \nabla V^\varepsilon(\gamma) \\ \frac{1}{2}|\dot{\gamma}|^2 - V^\varepsilon(\gamma) = -1. \end{cases} \tag{2.61}$$

The idea is to provide global smoothness of $\gamma_{\varepsilon, \bar{\mathbf{p}}}$ as a consequence of the Euler-Lagrange equation

$$\nabla \mathbf{L}(\bar{\mathbf{p}}) = 0.$$

In order to compute the partial derivatives of $\mathbf{L}$ we need the uniqueness for each of the $2n$ pieces that compose the concatenation $\gamma_{\varepsilon, \bar{\mathbf{p}}}$. The $\mathcal{C}^1$-dependence on initial data guarantees this property for the outer arcs (see Theorem 2.3.10). On the contrary, the Maupertuis' principle that we have used so far to find internal solution arcs, does not provide the uniqueness of such paths (see indeed 2.4.28). To overcome this, it would be necessary to proceed as in [61, 62] and to restrict again the neighbourhoods $\mathcal{U}_j$ in order to work inside a strictly convex neighbourhood. Indeed, it is known that there exists a unique geodesic that connects two points which belong to some neighbourhoods with such property. Since a rigorous treatment in this direction would be very technical and, actually, a repetition of what has been made in the quoted addendum, we will assume that the neighbourhoods $\mathcal{U}_j$ fits this uniqueness properties and thus we will compute directly the partial derivatives and we assume the validity of this lemma without further details.

Lemma 2.5.4. *The function* $\mathbf{L}$ *admits partial derivatives in* $\mathring{\mathcal{S}}$.

2.5.1. Partial derivatives of the Jacobi length with respect to the endpoints

In this paragraph we make the explicit computations of the partial derivatives of $\mathbf{L}$. The only non-trivial contributions involved in the computation of the partial derivative of $\mathbf{L}$ with respect to some p_k are given by the length of a selected pair of outer and inner arcs, i.e., the ones that share the contact point p_k. Therefore, for the sake of simplicity, in the following proofs we are going to consider only the length of the first outer arc

$y_{ext}(t; p_0, p_1; \varepsilon)$ and the last of the inner arcs $y_{int}(t; p_{2n-1}, p_{2n}; \varepsilon) = y_{int}(t; p_{2n-1}, p_0; \varepsilon)$, which connect in p_0.

From Theorem 2.3.10 and following the notations introduced at the beginning of this section, given $\vartheta_0^* \in \Xi$ and $\xi_0^* = Re^{i\vartheta_0^*} \in \partial B_R$, we know that, for every $\varepsilon \in (0, \min\{\varepsilon_{ext}, \varepsilon_{int}\})$ and for every $p_0, p_1 \in \mathcal{U}_0$, there exists a unique outer arc $y_{ext}(t) = y_{ext}(t; p_0, p_1; \varepsilon)$ which solves the problem

$$\begin{cases} \ddot{y}_{ext}(t) = \nabla V^\varepsilon(y_{ext}(t)) & t \in [0, T_{\varepsilon,0}] \\ \frac{1}{2}|\dot{y}_{ext}(t)|^2 - V^\varepsilon(y_{ext}(t)) = -1 & t \in [0, T_{\varepsilon,0}] \\ |y_{ext}(t)| > R & t \in (0, T_{\varepsilon,0}) \\ y_{ext}(0) = p_0, \quad y_{ext}(T_{\varepsilon,0}) = p_1. \end{cases} \tag{2.62}$$

It is important to remark that both $T_{\varepsilon,0}$ and y_{ext}, with its first and second derivative, depend on p_0 and p_1, while ε does not depend on p_0 and p_1. In particular, from the proof of Theorem 2.3.10, we have that $T_{\varepsilon,0} = T_{\varepsilon,0}(\varepsilon, p_0, p_1) \doteq T(p_0, \eta(\varepsilon, p_0, p_1))$, where η is the implicit function defined by the shooting map. As usual, we can associate to (2.62) its flow $\Phi^t(p_0, v_0)$ which actually depends on ε too; we omit this dependence to ease the notations. Moreover, keeping in mind the notation of Proposition 2.3.3, we have that

$$y_{ext}(T_{\varepsilon,0}; p_0, p_1; \varepsilon) = \pi_x \Phi^{T_{\varepsilon,0}}(p_0, v_0),$$

where $v_0 = v_0(\varepsilon, p_0, p_1) = \eta(\varepsilon, p_0, p_1) = \dot{y}_{ext}(0; p_0, p_1; \varepsilon)$ and

$$\begin{aligned} \Phi^0(p_0, v_0) &= (p_0, v_0) \in \Sigma \\ \Phi^{T_{\varepsilon,0}}(p_0, v_0) &= (p_1, v_1) \in \Sigma, \end{aligned} \tag{2.63}$$

with $v_1 = v_1(\varepsilon, p_0, p_1) = -\eta(\varepsilon, p_1, p_0)$ and we recall the definition of Σ as the inertial sphere on the phase space

$$\Sigma \doteq \{(x, v) \in \mathcal{E} : |x| = R\}.$$

Finally, we observe that problem (2.62) is time-reversible, since it is not difficult to prove that

$$y_{ext}(t; p_0, p_1; \varepsilon) = y_{ext}(T_{\varepsilon,0} - t; p_1, p_0; \varepsilon) \quad \text{for every } t \in [0, T_{\varepsilon,0}]. \tag{2.64}$$

Consider the length of the external arc $\mathcal{L}_{ext} \colon \mathcal{U}_0 \times \mathcal{U}_0 \to \mathbb{R}_0^+$ such that

$$\mathcal{L}_{ext}(p_0, p_1) \doteq \int_0^{T_{\varepsilon,0}} |\dot{y}_{ext}(t)| \sqrt{(V^\varepsilon(y_{ext}(t)) - 1)}\, dt,$$

which, using the conservation of energy, can be written in the following two equivalent forms

$$\mathcal{L}_{ext}(p_0, p_1) = \frac{1}{\sqrt{2}} \int_0^{T_{\varepsilon,0}} |\dot{y}_{ext}(t)|^2\, dt = \frac{1}{\sqrt{2}} \int_0^{T_{\varepsilon,0}} \left(\frac{1}{2}|\dot{y}_{ext}(t)|^2 + V^\varepsilon(y_{ext}(t)) - 1 \right) dt. \tag{2.65}$$

Lemma 2.5.5. *The function $\mathcal{L}_{ext} \in \mathcal{C}^1(\mathcal{U}_0 \times \mathcal{U}_0)$ and its differential, for every $(p_0, p_1) \in \mathcal{U}_0 \times \mathcal{U}_0$, is $d\mathcal{L}_{ext}(p_0, p_1) \colon \mathcal{T}_{p_0}(\partial B_R) \times \mathcal{T}_{p_1}(\partial B_R) \to \mathbb{R}$ and*

$$d\mathcal{L}_{ext}(p_0,p_1)[\varphi,\psi] = -\frac{1}{\sqrt{2}}\langle v_0(\varepsilon,p_0,p_1),\varphi\rangle + \frac{1}{\sqrt{2}}\langle v_1(\varepsilon,p_0,p_1),\psi\rangle$$
$$= -\frac{1}{\sqrt{2}}\langle \dot{y}_{ext}(0),\varphi\rangle + \frac{1}{\sqrt{2}}\langle \dot{y}_{ext}(T_{\varepsilon,0}),\psi\rangle.$$

Proof. Thanks to the $\mathcal{C}^1$-dependence of problem (2.62) on the initial data and time, the function $\mathcal{L}_{ext}$ is of class $\mathcal{C}^1$ in $\mathcal{U}_0 \times \mathcal{U}_0$. Moreover, from (2.65) we have

(2.66)
$$\begin{aligned}
\frac{\partial}{\partial p_0}\mathcal{L}_{ext}(p_0,p_1) &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial p_0}\left[\int_0^{T_{\varepsilon,0}} |\dot{y}_{ext}(t)|^2\,dt\right]\\
&= \frac{1}{\sqrt{2}}\left[|\dot{y}_{ext}(T_{\varepsilon,0})|^2\frac{\partial T_{\varepsilon,0}}{\partial p_0} + \int_0^{T_{\varepsilon,0}} \frac{\partial}{\partial p_0}\left(\frac{1}{2}|\dot{y}_{ext}(t)|^2 + V^{\varepsilon}(y_{ext}(t)) - 1\right)dt\right]\\
&= \frac{1}{\sqrt{2}}\left[|\dot{y}_{ext}(T_{\varepsilon,0})|^2\frac{\partial T_{\varepsilon,0}}{\partial p_0} + \int_0^{T_{\varepsilon,0}} \left(\dot{y}_{ext}(t)\frac{\partial \dot{y}_{ext}(t)}{\partial p_0} + \nabla V^{\varepsilon}(y_{ext}(t))\frac{\partial y_{ext}(t)}{\partial p_0}\right)dt\right]\\
&= \frac{1}{\sqrt{2}}\Bigg[|\dot{y}_{ext}(T_{\varepsilon,0})|^2\frac{\partial T_{\varepsilon,0}}{\partial p_0} + \left[\dot{y}_{ext}(t)\frac{\partial y_{ext}(t)}{\partial p_0}\right]_0^{T_{\varepsilon,0}}\\
&\quad + \int_0^{T_{\varepsilon,0}} \left(-\ddot{y}_{ext}(t) + \nabla V^{\varepsilon}(y_{ext}(t))\right)\frac{\partial y_{ext}(t)}{\partial p_0}\,dt\Bigg]\\
&= \frac{1}{\sqrt{2}}\left(|\dot{y}_{ext}(T_{\varepsilon,0})|^2\frac{\partial T_{\varepsilon,0}}{\partial p_0} + \left[\dot{y}_{ext}(t)\frac{\partial y_{ext}(t)}{\partial p_0}\right]_0^{T_{\varepsilon,0}}\right).
\end{aligned}$$

Note that the term $\dot{y}_{ext}(t)\frac{\partial \dot{y}_{ext}(t)}{\partial p_0}$ is actually a $(1 \times 2) \cdot (2 \times 2)$ matrix product, so that we would have to transpose the vector $\dot{y}_{ext}(t)$ firstly. Anyway, we are going to omit this and other transpositions in order to ease the notation.

Now, we can also compute the total derivative of the boundary conditions in (2.62) with respect to p_0, obtaining

$$\frac{d}{dp_0}y_{ext}(0) = \frac{\partial y_{ext}(0)}{\partial p_0} = I_2 \tag{2.67}$$

and

$$\frac{d}{dp_0}y_{ext}(T_{\varepsilon,0}) = \dot{y}_{ext}(T_{\varepsilon,0})\frac{\partial T_{\varepsilon,0}}{\partial p_0} + \frac{\partial y_{ext}(T_{\varepsilon,0})}{\partial p_0} = 0_2. \tag{2.68}$$

Moreover, multiplying both sides of (2.68) by $\dot{y}_{ext}(T_{\varepsilon,0})$, we have

$$|\dot{y}_{ext}(T_{\varepsilon,0})|^2\frac{\partial T_{\varepsilon,0}}{\partial p_0} + \dot{y}_{ext}(T_{\varepsilon,0})\frac{\partial y_{ext}(T_{\varepsilon,0})}{\partial p_0} = 0$$

This, together with (2.66) and (2.67), leads to

$$\frac{\partial}{\partial p_0}\mathcal{L}_{ext}(p_0,p_1) = -\frac{1}{\sqrt{2}}\dot{y}_{ext}(0) = -\frac{1}{\sqrt{2}}v_0(\varepsilon,p_0,p_1).$$

In the same way, one could obtain

$$\frac{\partial}{\partial p_1}\mathcal{L}_{ext}(p_0,p_1) = \frac{1}{\sqrt{2}}\left(|\dot{y}_{ext}(T_{\varepsilon,0})|^2\frac{\partial T_{\varepsilon,0}}{\partial p_1} + \left[\dot{y}_{ext}(t)\frac{\partial y_{ext}(t)}{\partial p_1}\right]_0^{T_{\varepsilon,0}}\right)$$

$$\frac{d}{dp_1}y_{ext}(0) = \frac{\partial y_{ext}(0)}{\partial p_1} = 0_2$$

$$\frac{d}{dp_1}y_{ext}(T_{\varepsilon,0}) = \dot{y}_{ext}(T_{\varepsilon,0})\frac{\partial T_{\varepsilon,0}}{\partial p_1} + \frac{\partial y_{ext}(T_{\varepsilon,0})}{\partial p_1} = I_2,$$

and so, multiplying both sides of the last equation by $\dot{y}_{ext}(T_{\varepsilon,0})$, we have that

$$\frac{\partial}{\partial p_1}\mathcal{L}_{ext}(p_0,p_1) = \frac{1}{\sqrt{2}}\dot{y}_{ext}(T_{\varepsilon,0}) = \frac{1}{\sqrt{2}}v_1(\varepsilon,p_0,p_1). \qquad \square$$

As before, for $\varepsilon \in (0, \min\{\varepsilon_{ext}, \varepsilon_{int}\})$, we can consider the length of the inner arc $y_{int}(t) \doteq y_{int}(t; p_{2n-1}, p_0; \varepsilon)$ as the function $\mathcal{L}_{int}\colon \mathcal{U}_{2n-1}\times\mathcal{U}_0 \to \mathbb{R}_0^+$ such that

$$\mathcal{L}_{int}(p_{2n-1},p_0) \doteq \int_0^{T_{\varepsilon,2n-1}} |\dot{y}_{int}(t)|\sqrt{(V^\varepsilon(y_{int}(t))-1)}\,dt$$

and prove the following lemma.

Lemma 2.5.6. *The function $\mathcal{L}_{int} \in \mathcal{C}^1(\mathcal{U}_{2n-1}\times\mathcal{U}_0)$ and its differential, for every $(p_{2n-1},p_0) \in \mathcal{U}_{2n-1}\times\mathcal{U}_0$, is $d\mathcal{L}_{int}(p_{2n-1},p_0)\colon \mathcal{T}_{p_{2n-1}}(\partial B_R)\times\mathcal{T}_{p_0}(\partial B_R) \to \mathbb{R}$ and*

$$d\mathcal{L}_{int}(p_{2n-1},p_0)[\nu,\varphi] = -\frac{1}{\sqrt{2}}\langle \dot{y}_{int}(0),\nu\rangle + \frac{1}{\sqrt{2}}\langle \dot{y}_{int}(T_{\varepsilon,2n-1}),\varphi\rangle.$$

Proof. As we have already remarked at page 93, the differentiability of this length function is a consequence of the results contained in [62], up to restrict the neighbourhoods $\mathcal{U}_{2n-1}$ and $\mathcal{U}_0$. Concerning the computation of the differential, the proof goes exactly as in Lemma 2.5.5. $\square$

2.5.2. The minimizing points of the Jacobi length are not in the boundary

The purpose of this section is to prove the first statement of Theorem 2.5.3. From Lemma 2.5.2 it sufficient to show that the minimizer $\bar{\mathbf{p}}$ does not occur on the boundary of $\mathcal{S}$. As already anticipated, this is made exploiting the minimizing property of $\bar{\mathbf{p}}$. In the next two paragraphs we will study the local behaviour of the external and internal arcs with respect to small variations on the endpoints. In short, what happens is that, if $\bar{\mathbf{p}} \in \partial\mathcal{S}$, then a particular variation on the endpoints gives a contradiction against the minimality of $\bar{\mathbf{p}}$.

An explicit variation on the external path

In order to do this, let us start by defining the matrix function

$$M(t) \doteq \frac{\partial y_{ext}(t)}{\partial p_1} = \pi_x \frac{\partial}{\partial p_1}\Phi^t(p_0, v_0),$$

which, from (2.62), satisfies

$$\ddot{M}(t) = \nabla^2 V^\varepsilon(y_{ext}(t))M(t) \quad \text{for every } t \in [0, T_{\varepsilon,0}].$$

Moreover, from the computations on the boundary conditions (2.63), we have that

$$M(0) = \pi_x \frac{\partial}{\partial p_0}\Phi^0(p_0, v_0) = \pi_x \left(0_2, \frac{\partial v_0}{\partial v_0}\right) = 0_2$$
$$M(T_{\varepsilon,0}) = \pi_x \frac{\partial}{\partial p_1}\Phi^{T_{\varepsilon,0}}(p_0, v_0) = \pi_x \left(I_2, \frac{\partial v_1}{\partial p_1}\right) = I_2,$$

so that $M(t)$ is a solution of the linearised boundary value problem

$$\begin{cases} \ddot{M}(t) = \nabla^2 V^\varepsilon(y_{ext}(t; p_0, p_1; \varepsilon))M(t), & t \in [0, T_{\varepsilon,0}] \\ M(0) = 0_2, \; M(T_{\varepsilon,0}) = I_2. \end{cases} \tag{2.69}$$

Lemma 2.5.7. *Let $M = M(t)$ be a solution of* (2.69)*. Then,*

$$J_{p_0,p_1} v_0(\varepsilon, p_0, p_1) = \left(-\dot{M}(T_{\varepsilon,0}), \dot{M}(0)\right),$$

where $v_0 = \dot{y}_{ext}(0; p_0, p_1; \varepsilon)$ (see the notations at page 94).

Proof. From Lemma 2.5.5 and since problem (2.62) is time-reversible (see (2.64)), we have that

$$\frac{\partial}{\partial p_0} v_0(\varepsilon, p_0, p_1) = \frac{\partial}{\partial p_0}\left(\frac{d}{dt} y_{ext}(t; p_0, p_1; \varepsilon)\Big|_{t=0}\right) = \frac{d}{dt}\left(\frac{\partial y_{ext}(t; p_0, p_1; \varepsilon)}{\partial p_0}\right)_{t=0}$$
$$= \frac{d}{dt}\left(\frac{\partial}{\partial p_0} y_{ext}(T_{\varepsilon,0} - t; p_1, p_0; \varepsilon)\right)_{t=0} = \frac{d}{dt} M(T_{\varepsilon,0} - t)\Big|_{t=0} = -\dot{M}(T_{\varepsilon,0}).$$

$$\frac{\partial}{\partial p_1} v_0(\varepsilon, p_0, p_1) = \frac{\partial}{\partial p_1}\left(\frac{d}{dt} y_{ext}(t; p_0, p_1; \varepsilon)\Big|_{t=0}\right) = \frac{d}{dt}\left(\frac{\partial y_{ext}(t; p_0, p_1; \varepsilon)}{\partial p_1}\right)_{t=0} = \dot{M}(0).$$

□

Let us focus on some details for a moment. The function $M(t)$ is, actually, the solution of the variational equation (see Remark B.1 in Appendix B) around an external arc y_{ext}, which gives information on how the flow associated to such solution changes under infinitesimal variations on the boundary conditions. Moreover, we know that y_{ext} (and thus, M) depends on ε in a $\mathcal{C}^1$ manner, since we have shown that the anisotropic N-centre problem is a perturbation of a Kepler problem driven by W^0 (see Proposition 2.2.2 in

Section 2.2). Therefore, it makes sense to simplify the proof and to consider a particular linearised problem, i.e., the one around the homothetic solution emanating from ξ_0^* and to put $\varepsilon = 0$. Indeed, from Appendix A, we have an explicit characterization of the hessian of a $-\alpha$-homogeneous potential like W^0 when it is valued on an homothetic motion which will be very useful in this context. Finally, the non-degeneracy of W^0, which is fundamental in the forthcoming proof, is guaranteed also in a neighbourhood of ξ_0^*, so that the argument can be easily extended near ξ_0^* (see Remark 2.5.1). Therefore, let us consider the homothetic trajectory $\hat{y}_{\xi_0^*}(t) = y_{ext}(t; \xi_0^*, \xi_0^*; 0)$. Around this solution, problem (2.69) becomes

$$\begin{cases} \ddot{M}(t) = \nabla^2 W^0(\hat{y}_{\xi_0^*}(t))M(t), \quad t \in [0, T_0] \\ M(0) = 0_2, \ M(T_0) = I_2, \end{cases} \tag{2.70}$$

where clearly $T_0 = T_{0,0}$ and it is the first return time of the homothetic motion.

Lemma 2.5.8. *Let $M = M(t)$ be a solution of* (2.70) *and let us define $s_\xi \doteq \xi_0^*/|\xi_0^*|$ and it unitary orthogonal vector $s_\tau = s_\xi^\perp$. Moreover, define the 1-dimensional functions*

$$\begin{aligned} w(t) &\doteq \langle M(t)s_\tau, s_\tau\rangle, \quad v(t) \doteq \langle M(t)s_\tau, s_\xi\rangle \\ c(t) &\doteq -\langle \nabla^2 W^0(\hat{y}_{\xi_0^*}(t))s_\tau, s_\tau\rangle = |\hat{y}_{\xi_0^*}(t)|^{-\alpha-2}\left(\alpha U(\vartheta_0^*) - U''(\vartheta_0^*)\right), \end{aligned}$$

where the last equality has been proven in (A.4) *in Appendix A. Then w solves*

$$\begin{cases} \ddot{w} + c(t)w = 0 \\ w(0) = 0, \ w(T_0) = 1 \end{cases} \tag{2.71}$$

and

$$v \equiv 0 \textit{ in } [0, T_0].$$

Proof. From the definition of w and since $M(t)$ solves (2.70) it is clear that

$$\ddot{w}(t) - \langle \nabla^2 W^0(\hat{y}_{\xi_0^*}(t))M(t)s_\tau, s_\tau\rangle = 0 \quad \text{for every } t \in [0, T_0].$$

Now, since $\nabla^2 W^0(\hat{y}_{\xi_0^*}(t))$ is symmetric and admits s_τ as eigenvector (see (A.2) in Appendix A) with for some eigenvalue $\lambda_\tau(t)$, we can write

$$\begin{aligned} \langle \nabla^2 W^0(\hat{y}_{\xi_0^*}(t))M(t)s_\tau, s_\tau\rangle &= \langle M(t)s_\tau, \nabla^2 W^0(\hat{y}_{\xi_0^*}(t))s_\tau\rangle \\ &= \lambda_\tau(t)w(t) \\ &= \langle \nabla^2 W^0(\hat{y}_{\xi_0^*}(t))s_\tau, s_\tau\rangle w(t). \end{aligned}$$

Hence, considering the boundary conditions satisfied by $M(t)$ in (2.70) we conclude that w solves (2.71).

Concerning v, using the same argument, we deduce that it solves

$$\begin{cases} \ddot{v} = \lambda_\xi(t)v \\ v(0) = 0 = v(T_0), \end{cases}$$

with $\lambda_\xi(t) = |\hat{y}_{\xi_0^*}(t)|^{-\alpha-2}\alpha(\alpha+1)U(\vartheta_0^*)$. Since $\lambda_\xi(t) > 0$ then $v \equiv 0$ in $[0, T_0]$. □

As a consequence of Lemma 2.5.7 and Lemma 2.5.8, we can directly deduce this useful result.

Corollary 2.5.9. *Let M, w and v as in Lemma 2.5.8 and recall the notations of Lemma 2.5.7. Then*

$$J_{p_0,p_1}v_0(0,\xi_0^*,\xi_0^*) = \Big(-\dot M(T_0),\dot M(0)\Big)$$

and, for every $k\in[-1,1]$,

$$\left\langle J_{p_0,p_1}v_0(0,\xi_0^*,\xi_0^*)\begin{bmatrix} s_\tau\\ ks_\tau\end{bmatrix}, s_\tau\right\rangle = -\langle \dot M(T_0)s_\tau,s_\tau\rangle + k\langle \dot M(0)s_\tau,s_\tau\rangle = -\dot w(T_0)+k\dot w(0),$$

and

$$\left\langle J_{p_0,p_1}v_0(0,\xi_0^*,\xi_0^*)\begin{bmatrix} s_\tau\\ ks_\tau\end{bmatrix}, s_\xi\right\rangle = -\dot v(T_0)+k\dot v(0) = 0. \tag{2.72}$$

At this point, our aim is to prove a result which actually gives an estimate of the tangential component of the velocity $v_0 = v_0(0,p_0,p_1)$, with respect to oscillations of p_0 and p_1 around the central configuration ξ_0^*. In order to do this, it is useful to introduce polar coordinates to provide an explicit dependence of p_0 and p_1 on an angular variation. Indeed, we characterize every point $p\in\mathcal U_0$ as a function of the counter-clockwise angle $\phi\in(-\pi,\pi)$ joining p and ξ_0^*, so that

$$p(0)=\xi_0^* = Rs_\xi, \quad \text{and} \quad p(\phi) = R\cos\phi\, s_\xi + R\sin\phi\, s_\tau. \tag{2.73}$$

We furthermore remark that when we write the orthogonal of a vector we mean a counter-clockwise rotation of $\pi/2$ of such vector.

Lemma 2.5.10. *There exists $\delta=\delta(\vartheta_0^*)$, $C=C(\vartheta_0^*)>0$ such that, for any $\phi_0,\phi_1\in\mathbb R$ verifying $0<|\phi_0|<\delta$ and $|\phi_1|\le|\phi_0|$, the following holds*

$$\frac{-\langle v_0(0,p_0(\phi_0),p_1(\phi_1)),p_0(\phi_0)^\perp\rangle}{\phi_0}\ge C.$$

Proof. For the sake of simplicity we introduce these notations

$$v_0(p_0,p_1)\doteq v_0(0,p_0,p_1),$$
$$v_{\xi_0^*}\doteq v_0(p_0(0),p_1(0)).$$

Furthermore, we prove the statement for $\phi_0>0$; if ϕ_0 is negative, the proof is the same up to minor changes. Since $v_{\xi_0^*}$ is the initial velocity of the homothetic motion along a central configuration, it is orthogonal to s_τ, then

$$\langle v_{\xi_0^*},p_0(0)^\perp\rangle = 0$$

and so we can write

$$\begin{aligned}&\langle v_0(p_0(\phi_0),p_1(\phi_1)),p_0(\phi_0)^\perp\rangle\\ &\qquad = \langle v_0(p_0(\phi_0),p_1(\phi_1))-v_{\xi_0^*},p_0(\phi_0)^\perp\rangle + \langle v_{\xi_0^*},p_0(\phi_0)^\perp - p_0(0)^\perp\rangle.\end{aligned}$$

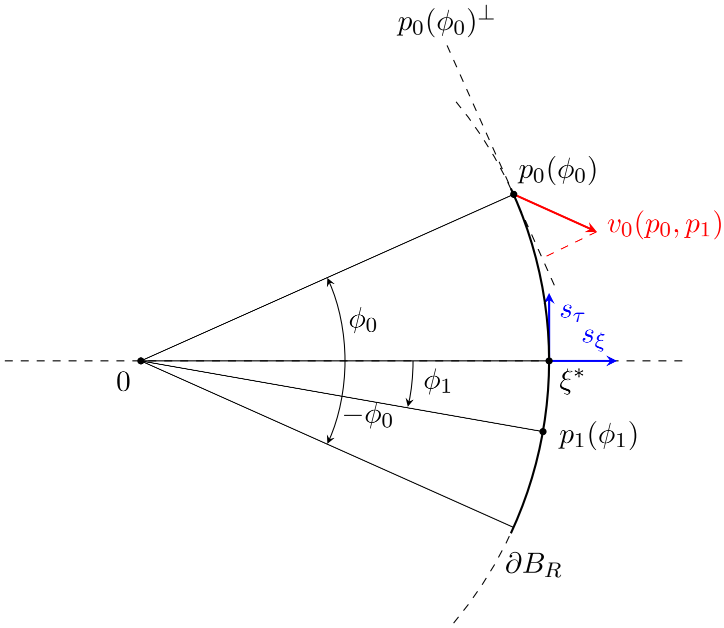

Figure 2.9.: The notations of Lemma 2.5.10 and the behaviour of the initial velocity with respect to variations on the endpoints p_0 and p_1.

From the notations in (2.73), if $\phi \to 0^+$, than we have

$$p(\phi) - p(0) = R\phi s_\tau + o(\phi) \quad \text{and} \quad p(\phi)^\perp - p(0)^\perp = -R\phi s_\xi + o(\phi),$$

hence, as $\phi_0 \to 0^+$,

$$\langle v_{\xi_0^*}, p_0(\phi_0)^\perp - p_0(0)^\perp \rangle = -R|v_{\xi_0^*}|\phi_0 + o(\phi_0).$$

Furthermore, again as $\phi_0 \to 0^+$ (and thus, as $\phi_1 \to 0^+$)

$$\begin{aligned} v_0(p_0(\phi_0), p_1(\phi_1)) - v_{\xi_0^*} &= J_{p_0,p_1} v_0(\xi_0^*, \xi_0^*) \begin{bmatrix} p_0(\phi_0) - p_0(0) \\ p_1(\phi_1) - p_1(0) \end{bmatrix} + o(\phi_0) \\ &= R\, J_{p_0,p_1} v_0(\xi_0^*, \xi_0^*) \begin{bmatrix} \phi_0 s_\tau \\ \phi_1 s_\tau \end{bmatrix} + o(\phi_0). \end{aligned}$$

Assuming now $\phi_1 = k\phi_0$ for some $k \in [-1,1]$, and using (2.72) in Corollary 2.5.9 and the fact that $p_0(0)^\perp = Rs_\tau$, we have

$$\left\langle v_0(p_0(\phi_0), p_1(\phi_1)) - v_{\xi_0^*}, p_0(\phi_0)^\perp \right\rangle = R^2\phi_0 \left\langle J_{p_0,p_1} v_0(\xi_0^*, \xi_0^*) \begin{bmatrix} s_\tau \\ ks_\tau \end{bmatrix}, s_\tau \right\rangle + o(\phi_0).$$

so that

$$\langle v_0(p_0(\phi_0), p_1(\phi_1)), p_0(\phi_0)^\perp \rangle = R\phi_0 \left(R \left\langle J_{p_0,p_1} v_0(\xi_0^*, \xi_0^*) \begin{bmatrix} s_\tau \\ ks_\tau \end{bmatrix}, s_\tau \right\rangle - |v_{\xi_0^*}| \right) + o(\phi_0).$$

In order to conclude we need to prove the existence of a positive constant C, depending on ϕ_0 and k, such that

$$-\left(R\left\langle J_{p_0,p_1}v_0(\xi_0^*,\xi_0^*)\begin{bmatrix}s_\tau\\ ks_\tau\end{bmatrix},s_\tau\right\rangle-|v_{\xi_0^*}|\right)\geq C>0.$$

By means of Corollary 2.5.9, this is equivalent to prove that

$$R\dot{w}(T_0)-kR\dot{w}(0)+|v_{\xi_0^*}|\geq C>0,$$

where $w(t)$ solves (2.71).

Let now $u(t)=|\hat{y}_{\xi_0^*}(t)|$; then u solves the 1-dimensional problem

$$\begin{cases}\ddot{u}+\dfrac{\alpha U(\vartheta_0^*)}{u^{\alpha+2}}u=0\\ u(0)=R=u(T_0).\end{cases} \tag{2.74}$$

Since by assumption $U''(\vartheta_0^*)>0$ we have

$$d(t)\doteq\frac{\alpha U(\vartheta_0^*)}{u(t)^{\alpha+2}}>\frac{\alpha U(\vartheta_0^*)-U''(\vartheta_0^*)}{u(t)^{\alpha+2}}\doteq c(t)\quad\text{in }[0,T_0], \tag{2.75}$$

recalling that the function $c(t)$ has been introduced in Lemma 2.5.8.
Since $u(t)\neq 0$ for any $t\in[0,T_0]$, we can define $f(t)=\frac{w(t)}{u(t)}$ whose derivatives are

$$\dot{f}=\frac{\dot{w}}{u}-\frac{w\dot{u}}{u^2}$$
$$\ddot{f}=\frac{\ddot{w}}{u}-2\frac{\dot{w}\dot{u}}{u^2}-\frac{w\ddot{u}}{u^2}+2\frac{w\dot{u}^2}{u^3}=-2\frac{\dot{u}}{u}\left(\frac{\dot{w}}{u}-\frac{w\dot{u}}{u^2}\right)+\frac{w}{u}\left(\frac{\ddot{w}}{w}-\frac{\ddot{u}}{u}\right).$$

Multiplying both sides by u^2 we deduce that f solves the problem

$$\begin{cases}\dfrac{d}{dt}\left(u^2\dot{f}\right)=(d(t)-c(t))u^2f\\ f(0)=0,\ f(T_0)=1/Rf.\end{cases} \tag{2.76}$$

We want to prove that f is strictly positive in $(0,T_0]$; hence, suppose by contradiction that there exists $t^*\in(0,T_0)$ such that $f(t^*)=0$. Then, it is clear that there exists $t_m\in(0,t^*]$ such that

$$f(t_m)\leq 0,\ \dot{f}(t_m)=0,\ \ddot{f}(t_m)>0.$$

In this way, considering the equation in (2.76) and the inequality (2.75), we get

$$0<u^2(t_m)\ddot{f}(t_m)=(d(t_m)-c(t_m))u^2(t_m)f(t_m)\leq 0,$$

which is a contradiction. Therefore, f is strictly positive in the interval $(0,T_0]$.

Now, integrating the equation in (2.76) in $[0,T_0]$ we get

$$u^2(T_0)\dot{f}(T_0)-u^2(0)\dot{f}(0)>0$$

and, using the explicit expression of $\dot{f}$, we have

$$R\dot{w}(T_0) - R\dot{w}(0) > \dot{u}(T_0) = \frac{1}{R}\langle \hat{y}_{\xi_0^*}(T_0), \dot{\hat{y}}_{\xi_0^*}(T_0)\rangle = -|v_{\xi_0^*}|. \tag{2.77}$$

Moreover, since f cannot vanish in $(0, T_0]$, we have that necessarily $\dot{f}(0) \geq 0$ and thus

$$\dot{w}(0) \geq \frac{w(0)\dot{u}(0)}{u^2(0)} = 0. \tag{2.78}$$

At this point, let us consider $\gamma \in (0,1)$ and define the function

$$\begin{aligned} u_\gamma \colon [0,T_0] &\to \mathbb{R}^+ \\ t &\longmapsto u_\gamma(t) \doteq u(t)^\gamma, \end{aligned}$$

which verifies

$$\ddot{u}_\gamma = \gamma(\gamma-1)u^{\gamma-2}\dot{u}^2 + \gamma u^{\gamma-1}\ddot{u}.$$

In this way, by (2.74) we have

$$-\frac{\ddot{u}_\gamma}{u_\gamma} = \gamma\left[-\frac{\ddot{u}}{u} + (1-\gamma)\frac{\dot{u}^2}{u^2}\right] = \gamma\left[\frac{\alpha U(\vartheta_0^*)}{u^{\alpha+2}} + (1-\gamma)\frac{\dot{u}^2}{u^2}\right];$$

in other words, u_γ solves the problem

$$\begin{cases} \ddot{u}_\gamma + d_\gamma(t)u_\gamma = 0 \\ u_\gamma(0) = R^\gamma = u_\gamma(T_0), \end{cases}$$

where

$$d_\gamma(t) = \gamma\left[\frac{\alpha U(\vartheta_0^*)}{u(t)^{\alpha+2}} + (1-\gamma)\frac{\dot{u}(t)^2}{u(t)^2}\right]. \tag{2.79}$$

Moreover

$$\frac{\dot{u}_\gamma}{u_\gamma} = \frac{\gamma u^{\gamma-1}\dot{u}}{u^\gamma} = \gamma\frac{\dot{u}}{u};$$

therefore, if we show that there exists $\gamma \in (0,1)$ such that $d_\gamma(t) \geq c(t)$ in $[0, T_0]$, we can repeat the previous argument and, as in (2.77), show that for such γ

$$R\dot{w}(T_0) - R\dot{w}(0) + \gamma|v_{\xi_0^*}| \geq 0. \tag{2.80}$$

By (2.79), such inequality is satisfied if there exists $\gamma \in (0,1)$ such that, for every $t \in [0, T_0]$

$$d_\gamma(t) = \gamma\left[\frac{\alpha U(\vartheta_0^*)}{u(t)^{\alpha+2}} + (1-\gamma)\frac{\dot{u}(t)^2}{u(t)^2}\right] \geq \frac{\alpha U(\vartheta_0^*)}{u(t)^{\alpha+2}} - \frac{U''(\vartheta_0^*)}{u(t)^{\alpha+2}} = c(t)$$

or, equivalently, if

$$\gamma(1-\gamma)\frac{\dot{u}(t)^2}{u(t)^2} \geq (1-\gamma)\frac{\alpha U(\vartheta_0^*)}{u^{\alpha+2}} - \frac{U''(\vartheta_0^*)}{u^{\alpha+2}}.$$

Since the left-hand side is always non-negative, it is enough to find a $\gamma \in (0,1)$ such that

$$\frac{U''(\vartheta_0^*)}{\alpha U(\vartheta_0^*)} > 1 - \gamma,$$

but such γ clearly exists since $U''(\vartheta_0^*) > 0$ and $U(\vartheta_0^*) > 0$ and, moreover, does not depends on t. Since now (2.80) is proved, using (2.78) we easily deduce that

$$\dot{w}(T_0) - k\dot{w}(0) + \gamma |v_{\xi_0^*}| \geq 0 \quad \text{for every } k \in [-1,1]$$

and, choosing $C = (1-\gamma)|v_{\xi_0^*}| > 0$, the lemma is finally proved. □

Remark 2.5.11. *One could think that the proof of Lemma 2.5.10 could be concluded with the inequality* (2.77) *since, for every $k \in [-1,1]$*

$$\dot{w}(T_0) - k\dot{w}(0) + |v_{\xi_0^*}| \geq \dot{w}(T_0) - \dot{w}(0) + |v_{\xi_0^*}| = C > 0.$$

However, this estimate would not be enough for the purposes of this work, since we need a uniform estimate which does not depend on ε. On the other hand, the constant $(1-\gamma)|v_{\xi_0^*}|$ provided at the end of the lemma depends only on ϑ_0^* and so it joins this uniformity and allows us to extend this argument also for the N-centre problem driven by V^ε when ε is sufficiently small.

As a consequence of Lemma 2.5.10, of Remark 2.5.11 and of the uniform behaviour of the dynamical system when ε is small and p_0 and p_1 are not far from ξ_0^* (see the discussion at page 97), we can obtain the same result for the N-centre problem. Note that here we will refer to the notations of Lemma 2.5.5 and thus we will consider only the Jacobi length from p_0 to p_1; of course, an equivalent result holds for any pair (p_{2j}, p_{2j+1}), for $j = 1, \ldots, n-1$, since, concerning the external arc, the derivative of $\mathbf{L}$ with respect to p_{2j} involves only the contribute $\mathcal{L}_{ext}(p_{2j}, p_{2j+1})$.

Theorem 2.5.12. *There exists $\bar{\varepsilon}_{ext} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_{ext})$, if*

$$\bar{\mathbf{p}} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{2n}) \in \mathcal{S}$$

is a minimizer of $\mathbf{L}$ provided in Lemma 2.5.2, then there exists $\psi \in \mathcal{T}_{p_0}(\partial B_R)$ and there exists $C_{ext} > 0$ such that

$$\frac{\partial \mathcal{L}_{ext}}{\partial p_0}(\bar{p}_0, \bar{p}_1)[\psi] = -\frac{1}{\sqrt{2}} \langle \dot{y}_{ext}(0), \psi \rangle \leq -C_{ext} < 0.$$

An explicit variation on the internal path

To conclude this section and to finally prove that a minimizer of $\mathbf{L}$ is actually an inner point of $\mathcal{S}$, we need another preliminary result. Indeed, it is necessary to give an estimate of the final velocity of the inner arc $y_{int} = y_{int}(p_{2n-1}, p_0)$, with respect to the tangent space spanned by $p_0^\perp$. As for the external arc, we are going to provide a result for $\varepsilon = 0$ and then we will extend it for ε sufficiently small by uniformity. In order to do this, consider again the notation introduced before the Lemma 2.5.10. We prove the following result.

Lemma 2.5.13. *There exist $\delta = \delta(\vartheta_0^*), C = C(\vartheta_0^*) > 0$ such that, for any $\phi \in \mathbb{R}$ verifying $0 < |\phi| < \delta$, the following holds*

$$\frac{\langle \dot{y}_{int}(T_{0,2n-1}), p_0(\phi)^{\perp} \rangle}{\phi} \geq C,$$

where $p_0(\phi)$ follows the notations in (2.73).

Proof. We have put $\varepsilon = 0$, therefore we are now studying an anisotropic Kepler problem driven by W^0 (see Proposition 2.2.1) and, in particular, y_{int} in this setting is exactly one of the collision trajectories studied in [5] (cf Chapter 1). Actually, y_{int} is a trajectory which emanates from collision; therefore, thanks to the time reversibility, we can consider $w(t) \doteq y_{int}(T_{0,2n-1} - t)$ which is defined again in $[0, T_{2n-1}]$, starts from ∂B_R and finishes in collision with he origin (see also Figure 2.10). For a vector $y \in \mathbb{R}^2$ we will denote by $\widehat{y}$ its angle with the horizontal axis with respect to the canonical basis of $\mathbb{R}^2$. As a starting point without loss of generality we assume $\phi > 0$ (as in the proof of Lemma 2.5.10)and we note that

$$\begin{aligned}
\langle \dot{y}_{int}(T_{0,2n-1}), p_0(\phi)^{\perp} \rangle &= \langle -\dot{w}(0), p_0(\phi)^{\perp} \rangle \\
&= \langle \dot{w}(0), -p_0(\phi)^{\perp} \rangle \\
&= |\dot{w}(0)||p_0(\phi)^{\perp}| \cos\left(\widehat{\dot{w}(0)} - \left(\vartheta_0^* + \phi - \frac{\pi}{2}\right)\right) \\
&= |\dot{w}(0)||p_0(\phi)^{\perp}| \sin\left(\widehat{\dot{w}(0)} - (\vartheta_0^* + \phi) + \pi\right) \\
&= |\dot{w}(0)||p_0(\phi)^{\perp}| \sin\left(\widehat{\dot{w}(0)} - (\vartheta_0^* + \phi) - \pi\right)
\end{aligned} \tag{2.81}$$

and so, since $|\dot{w}(0)|$ and $|-p_0(\phi)^{\perp}|$ are far from 0 as $\phi \to 0^+$, our proof reduces to study the behaviour of the angles (cf Figure 2.10). Actually, it is clear that the angle $\widehat{\dot{w}(0)}$ depends on ϕ and, in particular

$$\widehat{\dot{w}(0)}(\phi) \to \vartheta_0^* + \pi \quad \text{as } |\phi| \to 0^+,$$

since the inner arc tends to the collision homothetic motion as $|\phi| \to 0^+$. This suggests to follow the approach of [5] (cf Chapter 1) and to take into account the McGehee coordinates. The change of variables, which of course depends on ϕ, with respect to the trajectory w then reads

$$\begin{cases} r(t) = |w(t)| \\ \vartheta(t) = \widehat{w(t)} \\ \varphi(t) = \widehat{\dot{w}(t)} \end{cases} \quad \text{with} \quad \begin{cases} r(0) = |w(0)| = R \\ \vartheta(0) = \widehat{w(0)} = \vartheta_0^* + \phi \\ \varphi(0) = \widehat{\dot{w}(0)} \end{cases}.$$

On the other hand, since w is a collision solution at energy -1 of the anisotropic Kepler problem driven by $W^0(w) = r^{-\alpha}U(\vartheta)$, following Section 2 of [5] (cf Section 1.2) (r, ϑ, φ), after a time rescaling, solves

$$\text{(2.82)}\qquad \begin{cases} r' = 2r(U(\vartheta) - r^\alpha)\cos(\varphi - \vartheta) \\ \vartheta' = 2(U(\vartheta) - r^\alpha)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta). \end{cases}$$

Following again Section 2 of [5] (cf Section 1.2), since ϑ_0^* is such that $U'(\vartheta_0^*) = 0$ and $U''(\vartheta_0^*)$, then the triplet $(0, \vartheta_0^*, \vartheta_0^* + \pi)$ is a hyperbolic equilibrium point for (2.82) such that:

- its stable manifold is two-dimensional;
- the two eigendirections that span its stable manifold are

$$\begin{aligned} v_r &= (1, 0, 0) \\ v^- &= \left(0, 1, \frac{1}{2} + \frac{\alpha}{4} + \frac{1}{4}\sqrt{(2-\alpha)^2 + 8\frac{U''(\vartheta_0^*)}{U(\vartheta_0^*)}}\right) \end{aligned}$$

  and the corresponding eigenvalues are

$$\begin{aligned} \lambda_r &= -2U(\vartheta_0^*) < 0 \\ \lambda^- &= \frac{2-\alpha}{2}U(\vartheta_0^*) - \frac{1}{2}\sqrt{(2-\alpha)^2 U(\vartheta_0^*)^2 + 8U(\vartheta_0^*)U''(\vartheta_0^*)} < 0. \end{aligned}$$

Note that the third component of v^- is greater than 1. Moreover, by the main result (Theorem 5.2) in [5] (cf Theorem 1.5.2), we have that $w = (r, \vartheta, \varphi)$ belongs to the stable manifold of $(0, \vartheta_0^*, \vartheta_0^* + \pi)$ and in particular $\varphi(0)$ is a $\mathcal{C}^2$ function of $\vartheta(0) = \vartheta_0^* + \phi$. This is one of consequences of Lemma 3.2 in [5] (cf Lemma 1.3.2), together with the fact that, when $\vartheta(0) \to \vartheta_0^*$ in some way, the growth ratio of $\varphi = \varphi(0)$ as a function of $\vartheta = \vartheta(0)$ tends to the slope of v^- projected on $(\vartheta, \varphi(\vartheta))$. In other words, recalling that $\varphi(\vartheta_0^*) = \vartheta_0^* + \pi$, we have that

$$\frac{\varphi(\vartheta) - (\vartheta_0^* + \pi)}{\vartheta - \vartheta_0^*} \to 1 - \frac{\lambda^-}{2U(\vartheta_0^*)}, \quad \text{as } \vartheta \to \vartheta_0^*$$

and thus

$$\frac{\varphi(\vartheta) - (\vartheta + \pi)}{\vartheta - \vartheta_0^*} \to -\frac{\lambda^-}{2U(\vartheta_0^*)} > 0, \quad \text{as } \vartheta \to \vartheta_0^*.$$

Now, from (2.81) and the above limiting behaviour, since $\vartheta \to \vartheta_0^*$ as $\phi \to 0^+$, we have that

$$\begin{aligned} \frac{\langle \dot{y}_{int}(T_{0,2n-1}), p_0(\phi)^\perp \rangle}{\phi} &= |\dot{w}(0)||p_0(\phi)^\perp| \frac{\sin\left(\widehat{\dot{w}(0)} - (\phi + \vartheta_0^*) - \pi\right)}{\phi} \\ &= R|\dot{w}(0)| \frac{\sin\left(\varphi(\vartheta) - (\vartheta + \pi)\right)}{\vartheta - \vartheta_0^*} \\ &\sim R|\dot{w}(0)| \frac{\varphi(\vartheta) - (\vartheta + \pi)}{\vartheta - \vartheta_0^*}. \end{aligned}$$

To conclude the proof, note that $\dot{w}(0)$ is uniformly bounded in ϕ by a constant $c > 0$ which depends on the initial velocity of the homothetic collision motion. Therefore, the proof is concluded choosing the constant

$$C = C(\vartheta_0^*) \doteq -\frac{c\lambda^-}{2U(\vartheta_0^*)} > 0.$$

□

As we have said, it is possible to extend the previous result for ε sufficiently small. Indeed, from Proposition 2.2.1 we know that the potential V^ε converges uniformly to W^0 on every compact set of $\mathbb{R}^2 \setminus \{0\}$; moreover, we have already seen in Lemma 2.4.21 and Theorem 2.4.25 that, as $\varepsilon \to 0^+$, a sequence of minimizers of the Maupertuis' functional $\mathcal{M}$ converges uniformly to a minimizer of the Maupertuis' functional

$$\mathcal{M}_0(u) \doteq \frac{1}{2}\int_0^1 |\dot{u}|^2 \int_0^1 \left(-1 + W^0(u)\right).$$

As for Theorem 2.5.12, this is enough for the proof of the next result.

Theorem 2.5.14. *There exists $\bar{\varepsilon}_{int} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_{int})$, if*

$$\bar{\mathbf{p}} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{2n-1}) \in \mathcal{S}$$

is a minimizer of $\mathbf{L}$ provided in Lemma 2.5.2, then there exists $\psi \in \mathcal{T}_{p_0}(\partial B_R)$ and there exists $C_{int} > 0$ such that

$$\frac{\partial \mathcal{L}_{int}}{\partial p_0}(\bar{p}_{2n-1}, \bar{p}_0)[\psi] = \frac{1}{\sqrt{2}}\langle \dot{y}_{int}(T_{\varepsilon,2n-1}), \psi\rangle \leq -C_{int} < 0.$$

Proof of (i) of Theorem 2.5.3

Define $\bar{\varepsilon} \doteq \min\{\bar{\varepsilon}_{ext}, \bar{\varepsilon}_{int}\}$, with $\bar{\varepsilon}_{ext}$ and $\bar{\varepsilon}_{int}$ introduced respectively in Theorem 2.5.12 and Theorem 2.5.14, and take $\varepsilon \in (0, \bar{\varepsilon})$. Assume by contradiction that the minimizer $\bar{\mathbf{p}} = (\bar{p}_0\, \bar{p}_1, \ldots, \bar{p}_{2n})$ of $\mathbf{L}$ provided in Lemma 2.5.2 belongs to the boundary ∂S. To accomplish this absurd hypothesis it is not restrictive to assume that $\bar{p}_0 \in \partial\mathcal{U}_0$ and thus to produce a variation on $\bar{p}_0$ such that the total length $\mathbf{L}$ decreases along this variation. This would lead to a contradiction and would conclude the proof.

As a consequence of Theorem 2.5.12 and Theorem 2.5.14 there exist a variation $\psi \in \mathcal{T}_{p_0}(\partial B_R)$ and a constant $C > 0$ such that

$$\begin{aligned}\frac{\partial \mathbf{L}}{\partial p_0}(\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{2n-1})[\psi] &= \frac{\partial \mathcal{L}_{ext}}{\partial p_0}(\bar{p}_0, \bar{p}_1)[\psi] + \frac{\partial \mathcal{L}_{int}}{\partial p_0}(\bar{p}_{2n-1}, \bar{p}_0)[\psi] \\ &= -\frac{1}{\sqrt{2}}\langle \dot{y}_{ext}(0), \psi\rangle + \frac{1}{\sqrt{2}}\langle \dot{y}_{int}(T_{\varepsilon,2n-1}), \psi\rangle \\ &\leq -2C < 0.\end{aligned}$$

Therefore, the minimality of $\bar{\mathbf{p}}$ is in contradiction with the above inequality and thus $\bar{\mathbf{p}}$ is necessarily an inner point of $\mathcal{S}$.

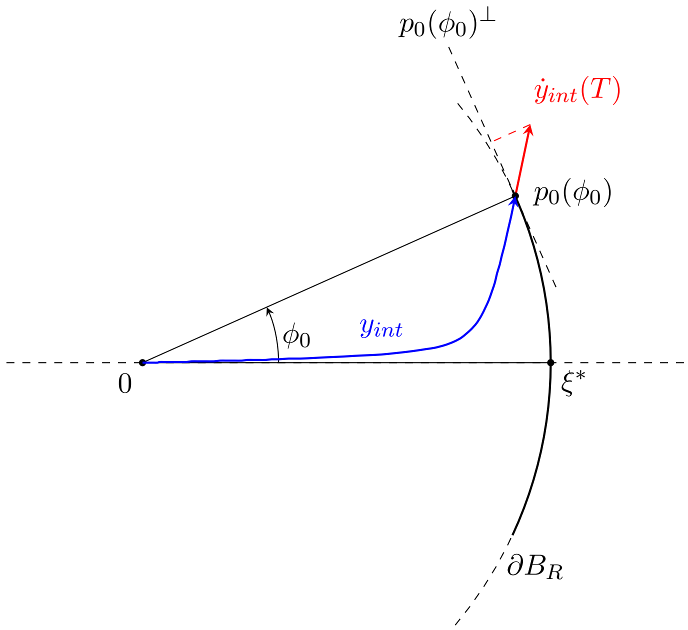

Figure 2.10.: Situation of Lemma 2.5.13: the behaviour of the final velocity of an internal arc.

2.5.3. Smoothness of the minimizers and existence of the corresponding periodic solutions: proof of (ii) of Theorem 2.5.3 and of Theorem 2.1.4

We conclude this section with the proof of the smoothness of the trajectory $\gamma_{\varepsilon,\bar{\mathbf{p}}}$ and thus we provide the existence of a periodic solution of the anisotropic N-centre problem driven by V^{ε}.

Proof of (ii) *of Theorem 2.5.3.* Since point (i) of Theorem 2.5.3 has been proved in the previous paragraph, for $\varepsilon \in (0,\bar{\varepsilon})$ we can consider a minimizer $\bar{\mathbf{p}} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{2n}) \in \mathring{\mathcal{S}}$ for $\mathbf{L}$ and now we know that

$$\frac{\partial \mathbf{L}}{\partial p_k}(\bar{\mathbf{p}}) = 0, \quad \text{for every } k = 0, \dots, 2n.$$

Again, we can assume $k = 0$ and thus, Lemma 2.5.5 and Lemma 2.5.6 give that, for every $\psi \in \mathcal{T}_{p_0}(\partial B_R)$ we have

$$\frac{\partial \mathbf{L}}{\partial p_0}(\bar{\mathbf{p}})[\psi] = \frac{\partial \mathcal{L}_{int}}{\partial p_0}(\bar{p}_{2n-1}, \bar{p}_0)[\psi] + \frac{\partial \mathcal{L}_{ext}}{\partial p_0}(\bar{p}_0, \bar{p}_1)[\psi] = \frac{1}{\sqrt{2}}\langle \dot{y}_{int}(T_{2n-1}) - \dot{y}_{ext}(0), \psi\rangle = 0.$$

The tangent space $\mathcal{T}_{p_0}(\partial B_R)$ is one-dimensional and it is spanned by a unit vector ν which is orthogonal to p_0. Therefore, if we denote by $(\widehat{y,\nu})$ the angle included between the two vectors, we can deduce that

$$|\dot{y}_{int}(T_{\varepsilon,2n-1})| \cos(\widehat{\dot{y}_{int}(T_{\varepsilon,2n-1}),\nu}) = |\dot{y}_{ext}(0)| \cos(\widehat{\dot{y}_{ext}(0),\nu})$$

and the conservation of the energy both for y_{int} and y_{ext} at the point $\bar{p}_0$ leads to the equality

$$|\dot{y}_{int}(T_{\varepsilon,2n-1})| = |\dot{y}_{ext}(0)|. \tag{2.83}$$

This implies that

$$\cos(\widehat{\dot{y}_{int}(T_{\varepsilon,2n-1}),\nu}) = \cos(\widehat{\dot{y}_{ext}(0),\nu})$$

and thus, since $\dot{y}_{int}(T_{\varepsilon,2n-1})$ and $\dot{y}_{ext}(0)$ point outside B_R, we obtain that

$$(\widehat{\dot{y}_{int}(T_{\varepsilon,2n-1}),\nu}) = (\widehat{\dot{y}_{ext}(0),\nu}).$$

This, together with (2.83), shows that $\dot{y}_{int}(T_{\varepsilon,2n-1}) = \dot{y}_{ext}(0)$ and thus $\gamma_{\varepsilon,\bar{\mathbf{p}}}$ is $\mathcal{C}^1([0,\mathbf{T}_\varepsilon])$.

At this point, we have shown that if $\bar{\mathbf{p}} \in \mathring{\mathcal{S}}$ is a minimizer of $\mathbf{L}$, then the corresponding periodic trajectory $\gamma_{\varepsilon,\bar{\mathbf{p}}}(t)$ is a classical solution of the N-centre problem (2.61) at energy -1 when $t \in [0,\mathbf{T}_\varepsilon] \setminus \{0, T_{\varepsilon,0}, \ldots, T_{\varepsilon,2n-1}\}$ and it is a $\mathcal{C}^1$ function in $[0,\mathbf{T}_\varepsilon]$. Since the junctions on ∂B_R are smooth, we can extend the trajectory $\gamma_{\varepsilon,\bar{\mathbf{p}}}$ by $\mathbf{T}_\varepsilon$-periodicity to all $\mathbb{R}$. To conclude the proof of Theorem 2.5.3 we need to show that $\gamma_{\varepsilon,\bar{\mathbf{p}}}$ is $\mathcal{C}^2(\mathbb{R})$. In order to do that, let us consider again the solution arcs y_{int} and y_{ext} which glue on the point $\bar{p}_0$ (the same argument applies for all the other building blocks) and let us compute

$$\begin{aligned}
\lim_{t\to T^-_{\varepsilon,2n-1}} \ddot{\gamma}_{\varepsilon,\bar{\mathbf{p}}}(t) &= \lim_{t\to T^-_{\varepsilon,2n-1}} \ddot{y}_{int}(t) = \lim_{t\to T^-_{\varepsilon,2n-1}} \nabla V^\varepsilon(y_{int}(t)) \\
&= \lim_{t\to 0^+} \nabla V^\varepsilon(y_{ext}(t)) = \lim_{t\to 0^+} \ddot{y}_{ext}(t) \\
&= \lim_{t\to 0} \ddot{\gamma}_{\varepsilon,\bar{\mathbf{p}}}(t).
\end{aligned}$$

This shows that $\gamma_{\varepsilon,\bar{\mathbf{p}}} \in \mathcal{C}^2(\mathbb{R})$ and concludes the proof of Theorem 2.5.3. □

At this point, it remains to show that the existence of multiple periodic solutions holds also for the original N-centre problem, i.e., the problem

$$\begin{cases} \ddot{x} = \nabla V(x) \\ \frac{1}{2}|\dot{x}|^2 - V(x) = -h, \end{cases} \tag{2.84}$$

where V is the potential referred to the original centres $c_1, \ldots, c_N$ (see (2.2)) and h has to be chosen small enough. We recall that the multiplicity of periodic solutions for problem (2.84) is determined both by a choice of a partition of the centres and by a minimal non-degenerate central configuration for W^0. As we have already discussed at page 33, we can describe all the possible behaviours of a periodic solution choosing a finite sequence of elements in the set

$$\mathcal{Q} = \{Q_j : j = 0, \ldots, m(2^{N-1}-1)-1\}.$$

We need to link a sequence of n elements in $\mathcal{Q}$ with a double sequence, composed by n partitions and n minimal non-degenerate central configurations of W^0. This can be done using Remark 2.1.3, which yields the following correspondence

$$(P_{l_0}, \ldots, P_{l_{n-1}}),\ (\vartheta^*_{l_0}, \ldots, \vartheta^*_{l_{n-1}}) \longleftrightarrow (Q_{j_0}, \ldots Q_{j_{n-1}})$$

with $j_k = l_k m + r_k$, for $l_k \in \{0, \ldots, 2^{N-1} - 2\}$, $r_k \in \{0, \ldots, m-1\}$, and thus $j_k \in \{0, \ldots, m(2^{N-1} - 1) - 1\}$. Finally, it is useful to characterize a solution provided in Theorem 2.5.3 with respect to its dependence on P_{l_k} and ϑ_{r_k}. Once $n \geq 1$ and $\varepsilon \in (0, \bar{\varepsilon})$ are fixed, we have a periodic solution $\gamma_{\varepsilon, \bar{\mathbf{p}}}$, with $\bar{\mathbf{p}} = (\bar{p}_0, \ldots, \bar{p}_{2n}) \in \mathring{\mathcal{S}}$. Actually, it is clear from the discussion at page 91 that this solution depends on a choice of n partitions and n minimal non degenerate central configurations, i.e.,

$$\gamma_{\varepsilon;\bar{\mathbf{p}}} = \gamma(\varepsilon; P_{l_0}, \ldots, P_{l_{n-1}}; \vartheta^*_{r_0}, \ldots, \vartheta^*_{r_{n-1}}).$$

Proof of Theorem 2.1.4. First of all, from Proposition 2.2.1, in order to obtain a solution of (2.84) as a rescaling of a solution of the problem driven by V^ε at energy -1, the energy h has to be in $(0, \bar{h})$, with $\bar{h} \doteq \bar{\varepsilon}^\alpha$ and $\bar{\varepsilon} > 0$ is the one defined in Theorem 2.5.3. Moreover, when such h is fixed, a unique $\varepsilon = h^{1/\alpha}$ is determined such that B_ε contains all the scaled centres, as well as a ball B_R which is included in the Hill's region of V^ε and that allows to build periodic solutions for the ε-problem. In particular, from (2.14) and Theorem 2.5.3, we have that R has to verify

$$\bar{\varepsilon} < R < \mathfrak{m}^{1/\alpha} - \bar{\varepsilon},$$

where the constant $\mathfrak{m}$ has been defined in Remark 2.1.2. To such R, via Proposition 2.2.1, we can associate a radius $\bar{R} \doteq h^{-1/\alpha} R > 0$ which plays the same for problem (2.84). Therefore, again by Proposition 2.2.1 and Remark 2.1.3, when $n \geq 1$ and $(Q_{j_0}, \ldots, Q_{j_{n-1}}) \in \mathcal{Q}^n$ are fixed, we can define the function $x = x(Q_{j_0}, \ldots, Q_{j_{n-1}}; h)$ as the rescaling via h of the solution $\gamma(\varepsilon; P_{l_0}, \ldots, P_{l_{n-1}}; \vartheta^*_{r_0}, \ldots, \vartheta^*_{r_{n-1}})$, with the rule

$$j_k = l_k m + r_k, \text{ for every } k = 0, \ldots, n-1.$$

This x will be clearly a classical and periodic solution of problem (2.84) that crosses $2n$-times the circle $\partial B_{\bar{R}}$ in chosen neighbourhoods of the points $\bar{R} e^{i\vartheta^*_{r_k}}$. □

2.6. Existence of a symbolic dynamics

In Theorem 2.1.4 we proved that, whenever the energy h and a sequence of labels $(Q_{j_0}, \ldots, Q_{j_{n-1}})$ of arbitrary length $n \in \mathbb{N}_{\geq 1}$ are fixed, a periodic solution of the anisotropic N-centre problem at energy h that satisfies the geometrical features represented by the above labels exists. This shows the existence of infinitely many periodic solutions in negative energy shells and suggests to investigate the presence of a symbolic dynamics for the dynamical system considered in this work.

To start with, we recall that N is the number of the centres, while m represents the number of minimal non-degenerate central configurations for the leading potential W^0 far from the singularity. We assume that $N \geq 3$ and $m \geq 1$ or, equivalently, that $N \geq 2$ and $m \geq 2$, and we require again hypotheses (V) on the potential V (see Remark 2.1.3). Moreover, we fix $h \in (0, \bar{h})$, where the threshold $\bar{h}$ has been determined in the previous section. By means of Theorem 2.1.4 we have that, for any $n \geq 1$ and for any finite sequence $(Q_{j_0}, \dots, Q_{j_{n-1}}) \subseteq \mathcal{Q}^n$, there exists a classical periodic solution $x = x(Q_{j_0}, \dots, Q_{j_{n-1}}; h)$ of the equation $\ddot{x} = \nabla V(x)$ at energy $-h$ and there exists $\bar{R} = \bar{R}(h) > 0$ such that the solution x crosses the circle $\partial B_{\bar{R}}$ $2n$-times in one period at the instants $(t_k)_{k=0}^{2n-1}$. In particular, for any $k = 0, \dots, n-1$ there exists a neighbourhood $\mathcal{U}_{r_k} \doteq \mathcal{U}(\bar{R}e^{i\vartheta^*_{r_k}})$ on $\partial B_{\bar{R}}$ such that, if we define $x_k \doteq x(t_k)$, we have that

- when $t \in (t_{2k}, t_{2k+1})$ the solution stays outside $B_{\bar{R}}$ and
$$x_{2k},\ x_{2k+1} \in \mathcal{U}_{r_k}$$
- in the interval (t_{2k+1}, t_{2k+2}) (we clearly set $t_{2n} \doteq t_0$ to close the trajectory) the solution stays inside $B_{\bar{R}}$, it separates the centres according to the partition P_{l_k} and
$$x_{2k+2} \in \mathcal{U}_{r_{k+1}},$$

keeping in mind the correspondence

$$Q_{j_k} \longleftrightarrow (P_{l_k}, \vartheta^*_{r_k}), \text{ with } j_k = l_k m + r_k.$$

We recall that this piecewise solution has been determined with several steps in the previous sections, working with a normalized version of the N-centre problem, driven by V^ε. In the same way, thanks to Theorem 2.3.10, Theorem 2.4.28 and Proposition 2.2.1, we can distinguish between the solution arcs outside and inside $B_{\bar{R}}$ in this way:

- we denote by $x_{ext}(\cdot; x_{2k}, x_{2k+1}; h)$ the piece of outer solution which connects x_{2k} and x_{2k+1}, defined on its re-parametrized interval $[0, T_{ext}(x_{2k}, x_{2k+1}; h)]$;
- we denote by $x_{P_{j_k}}(\cdot; x_{2k+1}, x_{2k+2}; h)$ the piece of inner solution which connects x_{2k+1} and x_{2k+2} and separates the centres with respect to the partition P_{l_k}, defined on its re-parametrized interval $[0, T_{P_{j_k}}(x_{2k+1}, x_{2k+2}; h)]$.

We recall that the inner arc for the ε-problem has been determined as a reparametrization of a minimizer of the Maupertuis' functional in Section 2.4. On the other hand, we know that the Maupertuis' principle (Theorem 2.4.1) joins also a vice-versa, i.e., if we set $\omega(x_{2k+1}, x_{2k+2}; h) \doteq 1/T_{P_{j_k}}(x_{2k+1}, x_{2k+2}; h)$, the function

$$v_{P_{j_k}}(t; x_{2k+1}, x_{2k+2}; h) \doteq x_{P_{j_k}}(t/\omega(x_{2k+1}, x_{2k+2}; h); x_{2k+2}, x_{2k+2}; h)$$

will be a critical point of the Maupertuis' functional $\mathcal{M}_h$ at a positive level, in a suitable space. In particular, we can introduce the set of H^1-paths

$$\hat{\mathcal{H}}_{x_{2k+1}, x_{2k+2}}([0,1]) \doteq \left\{ v \in H^1([0,1]; \mathbb{R}^2) \;\middle|\; \begin{array}{l} v(0) = x_{2k+1},\ v(1) = x_{2k+2}, \\ v(t) \neq c_j\ \forall\, t \in [0,1] \forall\, j \end{array} \right\}$$

and its H^1-closure

$$\mathcal{H}_{x_{2k+1}, x_{2k+2}}([0,1]) \doteq \{ v \in H^1([0,1]; \mathbb{R}^2) :\ v(0) = x_{2k+1},\ v(1) = x_{2k+2} \}.$$

Now, recalling that in the ε-problem we have studied the existence of inner solutions inside B_R, with $R = h^{1/\alpha}\bar{R}$, for any $p_{2k+1}, p_{2k+2} \in \partial B_R$ we can consider the points $x_{2k+1} = h^{-1/\alpha}p_{2k+1}$, $x_{2k+2} = h^{-1/\alpha}p_{2k+2} \in \partial B_{\bar{R}}$. Moreover, we recall the analogue of the space defined above in the ε-context, i.e., the spaces $\hat{H}_{p_{2k+1},p_{2k+2}}([0,1])$ and $H_{p_{2k+1},p_{2k+2}}([0,1])$ introduced in Section 2.4 and we consider the bijective map

$$J\colon H_{p_{2k+1},p_{2k+2}}([0,1]) \to \mathcal{H}_{x_{2k+1},x_{2k+2}}([0,1])$$

such that $J(u) = h^{1/\alpha}u$, for any $u \in H_{p_{2k+1},p_{2k+2}}([0,1])$. It is clear that the topological behaviour of an arc in $H_{p_{2k+1},p_{2k+2}}([0,1])$ with respect to the centres c_j' naturally translates on the same behaviour for its image through J with respect to the centres c_j. In light of this, for any $P_j \in \mathcal{P}$, we recall the definition (2.52) of the minimization space $\hat{K}_{P_j}^{p_1,p_2}$ and its H^1-closure $K_{P_j}^{p_1,p_2}$, and we set

$$\begin{aligned}
\hat{\mathcal{K}}_{P_j}^{x_{2k+1},x_{2k+2}}([0,1]) &= J\left(\hat{K}_{P_j}^{p_{2k+1},p_{2k+2}}([0,1])\right),\\
\mathcal{K}_{P_j}^{x_{2k+1},x_{2k+2}}([0,1]) &= J\left(K_{P_j}^{p_{2k+1},p_{2k+2}}([0,1])\right).
\end{aligned}$$

Now, since the inner arc $y_{int}(\cdot;p_{2k+1},p_{2k+2};\varepsilon)$ with respect to the partition P_j provided in Theorem 2.4.28 re-parametrizes a minimizer of the Maupertuis functional in $K_{P_j}^{p_{2k+1},p_{2k+2}}([0,1])$, we can immediately conclude that $v_{P_j}(\cdot;x_{2k+1},x_{2k+2};h)$ will be a minimizer of the Maupertuis' functional $\mathcal{M}_h$ in $\mathcal{K}_{P_j}^{x_{2k+1},x_{2k+2}}([0,1])$.

The rest of this section is devoted to the proof of Theorem 2.1.7 as a consequence of Theorem 2.1.4, i.e., to prove the existence of a symbolic dynamics with set of symbols $\mathcal{Q}$. For this reason, we start with the definition of a suitable subset Π_h of the energy shell

$$\mathcal{E}_h = \left\{(x,v) \in (\mathbb{R}^2 \setminus \{c_1,\dots,c_N\}) \times \mathbb{R}^2 :\ \frac{1}{2}|v|^2 - V(x) = -h\right\}$$

which is a 3-dimensional submanifold of $\mathbb{R}^2 \setminus \{c_1,\dots,c_N\}) \times \mathbb{R}^2$ and it is invariant for the flow Φ^t induced by the vector field

$$\begin{aligned}
F\colon \mathbb{R}^2 \setminus \{c_1,\dots,c_N\}) \times \mathbb{R}^2 &\to \mathbb{R}^2 \times \mathbb{R}^2\\
(x,v) &\mapsto F(x,v) = (v, \nabla V(x)).
\end{aligned}$$

As a starting point, for a neighbourhood $\mathcal{U}_r = \mathcal{U}(\bar{R}e^{i\vartheta_r^*})$ provided in Theorem 2.1.4 ($r = 0,\dots,m-1$), we define the sets of pairs (x,v) such that $x \in \mathcal{U}_r$ and v is not tangent to the same circle, i.e.,

$$\mathcal{E}_{h,\bar{R},r}^{\pm} \doteq \{(x,v) \in \mathcal{E}_h :\ x \in \mathcal{U}_r,\ \langle x,v\rangle \gtrless 0\}.$$

We note that for a pair in $\mathcal{E}_{h,\bar{R},r}^{+}$ the velocity points towards the outer of $B_{\bar{R}}$ (the converse holds for $\mathcal{E}_{h,\bar{R},r}^{-}$) and that both sets are included in the 2-dimensional inertial surface

$$\Sigma_h = \{(x,v) \in \mathcal{E}_h :\ |x| = \bar{R}\}.$$

Therefore, it is clear that, if we consider the restriction $F_h \doteq F|_{\mathcal{E}_h}$ of the vector field, it is transverse to $\mathcal{E}^+_{h,\bar{R},r}$ (for more details we refer to Section 2.3).

For every $(x, v) \in \mathcal{E}^+_{h,\bar{R},r}$ we introduce the sets

$$\mathbb{T}^{\pm}(x, v) \doteq \left\{t \in (0, +\infty) : \Phi^t(x, v) \in \mathcal{E}^{\pm}_{h,\bar{R},s}, \text{ for some } s \in \{0, \dots, m-1\}\right\}$$

and the sets

$$\left(\mathcal{E}^+_{h,\bar{R},r}\right)^{\pm} \doteq \left\{(x, v) \in \mathcal{E}^+_{h,\bar{R},r} : \mathbb{T}^{\pm}(x, v) \neq \emptyset\right\}.$$

Note that in general the sets $\mathbb{T}^{\pm}(x, v)$ could be empty, since the piece of trajectory which starts in a neighbourhood $\mathcal{U}_r$ and points towards the outer of $B_{\bar{R}}$ needs to have an initial velocity v which satisfies a behaviour well described in Lemma 2.5.10. Besides that, note that Theorem 2.1.4 ensures that the sets $\left(\mathcal{E}^+_{h,\bar{R},r}\right)^{\pm}$ are non-empty, since the theorem provides periodic solutions of the equation $\ddot{x} = \nabla V(x)$ that cross the circle $\partial B_{\bar{R}}$ an infinite number of times, exactly inside the neighbourhoods $\mathcal{U}_r$, in which the transversality condition $\langle \dot{x}, x \rangle \gtrless 0$ is clearly satisfied. Moreover, the continuous dependence on initial data, together with the transversality of $\mathcal{E}^+_{h,\bar{R},r}$ with respect to the vector field F guarantee that the set $\left(\mathcal{E}^+_{h,\bar{R},r}\right)^+$ is open. At this point, for $(x, v) \in \left(\mathcal{E}^+_{h,\bar{R},r}\right)^+$ we define

$$T^+_{\text{min}}(x, v) \doteq \inf \mathbb{T}^+(x, v),$$

while for $x \in \left(\mathcal{E}^+_{h,\bar{R},r}\right)^-$ we set

$$T^-_{\text{min}}(x, v) \doteq \inf \mathbb{T}^-(x, v).$$

If we take $(x, v) \in \left(\mathcal{E}^+_{h,\bar{R},r}\right)^+ \cap \left(\mathcal{E}^+_{h,\bar{R},r}\right)^-$ such that $T^-_{\text{min}}(x, v) < T^+_{\text{min}}(x, v)$, we can consider the piece of the orbit emanating from (x, v) between the first two instants in which it crosses again $\partial B_{\bar{R}}$ in two of the admissible neighbourhoods, which is exactly the following restriction of the flow

$$\{\Phi^t(x, v) : t \in [T^-_{\text{min}}, T^+_{\text{min}}]\},$$

where we have omitted the dependence on (x, v) to ease the notations (see Figure 2.11).

Recalling the set of symbols

$$\mathcal{Q} = \{Q_j : j = 0, \dots, m(2^{N-1} - 1) - 1\},$$

and recalling that $\pi_x \Phi^t(x, v)$ denotes the projection on the first component of the flow,

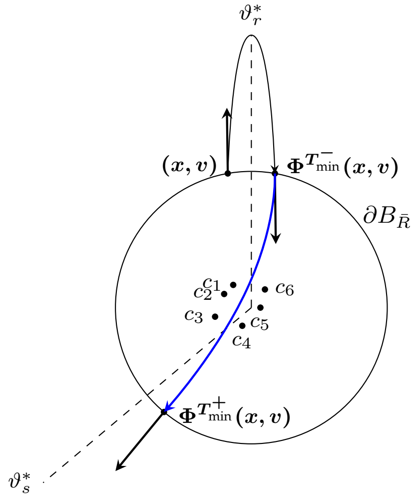

Figure 2.11.: In this picture we can become familiar with the notations introduced above. Indeed, we have drawn in bold the velocity vectors associated to every position in the configuration space, so that it is possible to visualize the flow associated to an initial data $(\boldsymbol{x}, \boldsymbol{v}) \in \left(\mathcal{E}^+_{h,\bar{R},r}\right)^+ \cap \left(\mathcal{E}^+_{h,\bar{R},r}\right)^-$. In particular, the blue arc represents the projection on the configuration space of the restriction of the flow between T^-_{min} and T^+_{min}.

let us now define the set

$$\mathcal{E}^{\mathcal{Q}}_{h,\bar{R}} \doteq \left\{ (x,v) \in \left(\mathcal{E}^+_{h,\bar{R},r}\right)^+ \cap \left(\mathcal{E}^+_{h,\bar{R},r}\right)^- \;\middle|\; \begin{array}{l} r \in \{0,\dots,m-1\},\ T^-_{\text{min}} < T^+_{\text{min}} \\ \{\pi_x \Phi^t(x,v)\}_{t\in[T^-_{\text{min}},T^+_{\text{min}}]} \text{is the re-param.} \\ \text{of a minimizer of } \mathcal{M}_h \text{ in the space} \\ \hat{\mathcal{K}}_{P_l}^{\pi_x\Phi^{T^-_{\text{min}}}(x,v),\pi_x\Phi^{T^+_{\text{min}}}(x,v)}([T^-_{\text{min}}, T^+_{\text{min}}]), \\ \text{for some } l \in \{0,\dots,2^{N-1}-2\}, \\ \text{with } Q_j = Q_{lm+r} \in \mathcal{Q} \end{array} \right\}.$$

The above set is non-empty since Theorem 2.1.4 proves the existence of periodic solutions for the N-centre problem, which then identify an infinite number of points that belong to the set $\mathcal{E}^{\mathcal{Q}}_{h,\bar{R}}$. Indeed, the x-components of these points are nothing but the crosses that the periodic trajectories make on the circle $\partial B_{\bar{R}}$ when they start their motion outside the ball. We can then define a first return map on $\mathcal{E}^{\mathcal{Q}}_{h,\bar{R}}$ in this way

$$\begin{aligned} \mathcal{R}\colon \mathcal{E}^{\mathcal{Q}}_{h,\bar{R}} &\to \mathcal{E}^{\mathcal{Q}}_{h,\bar{R}} \\ (x,v) &\mapsto \mathcal{R}(x,v) \doteq \Phi^{T^+_{\text{min}}}(x,v), \end{aligned}$$

which is continuous as a consequence of our construction. Moreover, we can also define another map $\chi\colon \mathcal{E}^{\mathcal{Q}}_{h,\bar{R}} \to \mathcal{Q}$ such that

$$\chi(x,v) = Q_j, \quad \text{if} \begin{cases} (x,v) \in \left(\mathcal{E}^{+}_{h,\bar{R},r}\right)^{+} \\ \{\pi_x \Phi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]} \in \hat{\mathcal{K}}_{P_l}^{\pi_x\Phi^{T^-_{\min}}(x,v),\pi_x\Phi^{T^+_{\min}}(x,v)}([T^-_{\min},T^+_{\min}]) \\ j = lm + r \end{cases}.$$

We can finally define the set Π_h in this way

$$\Pi_h \doteq \bigcap_{j\in\mathbb{Z}} \mathcal{R}^j(\mathcal{E}^{\mathcal{Q}}_{h,\bar{R}}),$$

which is exactly the set of all the possible initial data which generate solutions that cross the circle $\partial B_{\bar{R}}$ an infinite number of time having velocity directed toward the exterior of $B_{\bar{R}}$; moreover, each of these solutions, every time that travels inside $B_{\bar{R}}$, draws a partition P_l of the centres for some l and minimizing the Maupertuis' functional in the corresponding space $\hat{\mathcal{K}}_{P_l}$. To conclude this preliminary discussion we define also the application π which maps every one of this initial data to its corresponding bi-infinite sequence of symbols, i.e.,

$$\begin{aligned} \pi\colon \Pi_h &\to \mathcal{Q}^{\mathbb{Z}} \\ (x,v) &\mapsto \pi(x,v) \doteq (Q_{j_k})_{k\in\mathbb{Z}}, \text{ with } Q_{j_k} \doteq \chi(\mathcal{R}^k(x,v)); \end{aligned}$$

we also introduce the restriction of the first return map to the invariant submanifold Π_h as $\mathfrak{R} \doteq \mathcal{R}|_{\Pi_h}$. At this point, we proceed with the proof of Theorem 2.1.7, i.e., we need to prove that the map π that we have just defined is surjective and continuous. In order to do that, we need to prove some preliminary property on the pieces of solutions. The first one consists in showing that their intervals of definition are uniformly bounded from above and below.

Lemma 2.6.1. *There exist two constants $c, C > 0$ such that, for any $x_0, x_1 \in \partial B_{\bar{R}}$ for which $x_{ext}(\cdot; x_0, x_1; h)$ exists, for any $x_2, x_3 \in \partial B_{\bar{R}}$ and for any $P_j \in \mathcal{P}$ for which $x_{P_j}(\cdot; x_2, x_3; h)$ exists, we have*

$$\begin{aligned} c \le T_{ext}(x_0, x_1; h) \le C \\ c \le T_{P_j}(x_2, x_3; h) \le C. \end{aligned}$$

Proof. Lemma 2.3.11 and Lemma 2.4.29 provide such uniform bounds for the ε-problem; the conclusion is then a direct consequence of Proposition 2.2.1. □

We also need a compactness lemma on sequences of minimizers of $\mathcal{M}_h$ which separate the centres with respect to the same partition. In particular, we want to prove that if the endpoints of the minimizers converge to a limit pair $(\bar{x}_2, \bar{x}_3)$ then the limit path is itself a minimizer of $\mathcal{M}_h$ in the space $\hat{\mathcal{K}}_{P_j}^{\bar{x}_2,\bar{x}_3}([0,1])$, for a fixed partition P_j.

Lemma 2.6.2. *Let* $(x_2^n) \subseteq \mathcal{U}_2$ *and* $(x_3^n) \subseteq \mathcal{U}_3$ *such that* $(x_2^n, x_3^n) \to (\bar{x}_2, \bar{x}_3)$, *with* $\bar{x}_2 \in \mathcal{U}_3$ *and* $\bar{x}_3 \in \mathcal{U}_3$, *where* $\mathcal{U}_2$ *and* $\mathcal{U}_3$ *are the neighbourhoods of two minimal non-degenerate central configurations of* W^0 *on* $\partial B_{\bar{R}}$ *which guarantee the existence of the internal arcs. Fix also a partition* $P_j \in \mathcal{P}$ *and let* v_n *be a minimizer of* $\mathcal{M}_h$ *in the space* $\hat{\mathcal{K}}_{P_j}^{x_2^n, x_3^n}([0,1])$. *Then, there exist a subsequence* (v_{n_k}) *of* (v_n) *and a minimizer* $\bar{v}$ *of* $\mathcal{M}_h$ *in the space* $\hat{\mathcal{K}}_{P_j}^{\bar{x}_2, \bar{x}_3}([0,1])$ *such that*

$$v_{n_k} \rightharpoonup \bar{v} \text{ in } H^1.$$

Proof. An analogous compactness property has been proved for sequences of minimizers of $\mathcal{M}_{-1}$ in the ε-problem in Lemma 2.4.21 and Theorem 2.4.25. Again, Proposition 2.2.1 gives the proof. □

We are now ready to give the proof of Theorem 2.1.7.

Proof of Theorem 2.1.7. **Surjectivity of π** : Consider a bi-infinite sequence $(Q_{j_n})_{n\in\mathbb{Z}} \subseteq \mathcal{Q}^{\mathbb{Z}}$ and the sequence of finite sequences

$$(Q_{j_0}),\ (Q_{j_{-1}}, Q_{j_0}, Q_{j_1}),\ \ldots\ (Q_{j_{-n}}, Q_{j_{-n+1}}, \ldots, Q_{j_{-1}}, Q_{j_0}, Q_{j_1}, \ldots, Q_{j_{n-1}} Q_{j_n}),\ \ldots$$

If we fix $h \in (0, \bar{h})$, through Theorem 2.1.4 we can associate to each of these sequence a corresponding periodic solution; this will be made using this notation that takes into account the length of the finite sequence

$$(Q_{j_{-n}}, \ldots, Q_{j_{-1}}, Q_{j_0}, Q_{j_1}, \ldots, Q_{j_n}) \longleftrightarrow x^n(\cdot).$$

Without loss of generality, we can define $(x^n(0), \dot{x}^n(0)) \in \Pi_h$ as the initial data such that the first symbol determined by x^n is Q_{j_0}, for every $n \in \mathbb{N}$. In particular, we know that $j_0 = l_0 m + r_0$ and thus this symbol will refer to a first piece of solution, composed by an outer arc with endpoints in the neighbourhood $\mathcal{U}_{r_0}$ and inner arc that agrees with the partition P_{l_0} and that arrives in the neighbourhood $\mathcal{U}_{r_1}$. In this way, for every n we can find a sequence of points $(x_k^n)_{k\in\mathbb{Z}} \subseteq \partial B_{\bar{R}}$ which correspond to the crosses of the periodic trajectory x^n with the circle $\partial B_{\bar{R}}$. Note that since the trajectory is periodic, the sequence of points will be periodic too. We can now take into account the sequence of sequences

$$\{(x_k^n)_{k\in\mathbb{Z}}\}_{n\in\mathbb{N}}$$

in order to start a diagonal process that will imply a convergence on these sequences. If we fix $k = 0$, since $\partial B_{\bar{R}}$ is compact we can extract a subsequence $(x_0^{n_0})_{n_0\in\mathbb{N}}$ such that $x_0^{n_0} \to \bar{x}_0$ as $n_0 \to +\infty$. In the same way, we can fix $k = 1$ and consider the subsequence $(x_1^{n_0})_{n_0\in\mathbb{N}} \subseteq \partial B_{\bar{R}}$ and extract a sub-subsequence $(x_1^{n_1})_{n_1\in\mathbb{N}}$ such that $x_1^{n_1} \to \bar{x}_1$ as $n_1 \to +\infty$. This can be made for every $k \in \mathbb{Z}$, in order to find a subsequence $(x_k^{n_k})_{n_k\in\mathbb{N}}$ such that $x_k \to \bar{x}_k$ as $n_k \to +\infty$. At this point we can consider the diagonal sequence $(x_k^{n_n})_{n\in\mathbb{N}}$ and relabel it as $(x_k^n)_{n\in\mathbb{N}}$, so that

$$\lim_{n\to+\infty} x_k^n = \bar{x}_k, \text{ for all } k \in \mathbb{Z}.$$

Note that all these limit points belong to $\partial B_{\bar{R}}$ so that we can connect two of them with an inner or outer arc; this would actually require that the points are inside the neighbourhoods $\mathcal{U}_{r_k}$ found in Theorem 2.1.4, but up to restrict these neighbourhoods we can repeat the previous argument so that the limits would still be inside a neighbourhood in which the existence is guaranteed. Once this is clear, we can connect the points $\bar{x}_{2k}, \bar{x}_{2k+1} \in \mathcal{U}_{r_k}$ with a unique outer arc using the technique illustrated in Theorem 2.3.10; we can also connect the point $\bar{x}_{2k+1} \in \mathcal{U}_{r_k}$ and the point $\bar{x}_{2k+2} \in \mathcal{U}_{r_{k+1}}$ following the procedure of Theorem 2.4.28 so that the induced path would separate the centres according to the partition P_{l_k}. Repeating this procedure for every $k \in \mathbb{Z}$ we can glue together all these pieces to obtain a continuous function $\bar{x}\colon \mathbb{R} \to\colon \mathbb{R}^2$, using the same technique provided in Section 2.5. We point out that $\bar{x}$ is not unique, since the inner pieces, coming from Maupertuis' minimizers, are not unique. In the following, we are going to show that $\bar{x}$ is a classical periodic solution of the equation $\ddot{x} = \nabla V(x)$ and verifies

$$\begin{cases} (\bar{x}(0), \dot{\bar{x}}(0)) \in \Pi_h \\ \pi((\bar{x}(0), \dot{\bar{x}}(0))) = (Q_{j_k})_{k\in\mathbb{Z}}. \end{cases}$$

If we introduce the set of collision instants of $\bar{x}$ as

$$T_c(\bar{x}) \doteq \{t \in \mathbb{R} : \bar{x}(t) = c_j, \text{ for some } j \in \{1, \dots, N\}\},$$

due to the nature of the sequence (x^n) it is enough to show that $x^n \to \bar{x}$ in a $\mathcal{C}^2$ manner on every compact subset of $\mathbb{R} \setminus T_c(\bar{x})$. To start with, note that if we take $[a, b] \subseteq \mathbb{R}$ such that $\bar{x}(a) = \bar{x}_{2k}$ and $\bar{x}(b) = \bar{x}_{2k+1}$, then the outer arc connecting these two points depends on a continuous manner on the endpoints (see Theorem 2.3.10) and so $x^n \to \bar{x}$ uniformly on $[a, b]$. Moreover, if we take $[c, d] \subseteq \mathbb{R}$ such that $\bar{x}(c) = \bar{x}_{2k+1}$ and $\bar{x}(d) = \bar{x}_{2k+2}$, then the uniform convergence on $[c, d]$ is a straightforward consequence of Lemma 2.6.2. This convergence also determines a unique choice for the inner solution that connects $\bar{x}_{2k+1}$ and $\bar{x}_{2k+2}$, so that now the function $\bar{x}$ is uniquely determined. Moreover, since the internal arcs provided in Theorem 2.4.28 have a uniform distance δ from the centres, this actually proves that the uniform convergence of x^n to $\bar{x}$ takes place in $\mathbb{R} \setminus T_c(\bar{x})$, i.e., $\bar{x}$ has no collision with the centres.

At this point, the function $\nabla V(\bar{x}(\cdot))$ is continuous on the whole $\mathbb{R}$ and, since x^n is a $\mathcal{C}^2$ solution of the equation $\ddot{x} = \nabla V(x)$ by the uniform convergence of x^n to $\bar{x}$ on $[a, b]$ we have that

$$\lim_{n\to+\infty} \ddot{x}^n(t) = \lim_{n\to+\infty} \nabla V(x^n(t)) = \nabla V(\bar{x}(t)),$$

for every $t \in [a, b]$. This means that the sequence $(\dot{x}^n(t))$ is equi-continuous in $[a, b]$; moreover, the energy equation implies that

$$|\dot{x}^n(t)| = \sqrt{V(x^n(t)) - h} \leq C$$

for every $t \in [a, b]$ and for every $n \in \mathbb{N}$, i.e., the sequence $(\dot{x}^n(t))$ is also equi-bounded in $[a, b]$. This fact, together with the uniform convergence, finally shows that the sequence $(x^n(t))$ $\mathcal{C}^2$-converges to $\bar{x}(t)$ in $[a, b]$, for every compact set $[a, b] \subseteq \mathbb{R}$. As a consequence,

$\bar{x}$ is a $\mathcal{C}^2$ solution of the equation $\ddot{x} = \nabla V(x)$ at energy $-h$ on every compact set of $\mathbb{R}$. As a final remark, note that the uniform convergence also implies the conservation of the topological constraint, i.e., the piece of $\bar{x}$ between the points x_{2k+1} and x_{2k+2} will separate the centres with respect to P_{l_k}. This finally proves that $\pi((\bar{x}(0), \dot{\bar{x}}(0))) = (Q_{j_k})_{k\in\mathbb{Z}}$, where $j_k = l_k m + r_k$.

Continuity of $\boldsymbol{\pi}$: We recall that we can endow the set of bi-infinite sequences $\mathcal{Q}^{\mathbb{Z}}$ with the distance

$$d((Q_m), (\tilde{Q}_m)) \doteq \sum_{m\in\mathbb{Z}} \frac{\rho(Q_m, \tilde{Q}_m)}{2^{|m|}}, \ \forall\, (Q_m), (\tilde{Q}_m) \in \mathcal{Q}^{\mathbb{Z}},$$

where ρ is the discrete metric defined through the Kronecker delta. Moreover, for every $m \in \mathbb{Z}$ we define the map

$$\begin{aligned} \pi_m \colon \Pi_h &\to \mathcal{Q} \\ (x, v) &\mapsto \pi_m(x, v) \doteq \chi(\mathfrak{R}^m(x, v)), \end{aligned}$$

i.e., it associates to (x, v) the symbol corresponding to the m-th piece (composed by an outer arc and an inner arc) of the solution with initial data (x, v). Given this, if we fix $(x_0, v_0) \in \Pi_h$, we need to show that for $\lambda > 0$ there exists $\delta > 0$ such that

$$\forall\, (x, v) \in \Pi_h \text{ s.t. } \|(x, v) - (x_0, v_0)\| < \delta \implies \sum_{m\in\mathbb{Z}} \frac{\rho(\pi_m(x, v), \pi_m(x_0, v_0))}{2^{|m|}} < \lambda. \tag{2.85}$$

It is clear that we can find $m_0 \in \mathbb{N}$ such that

$$\sum_{|m|>m_0, m\in\mathbb{Z}} \frac{1}{2^{|m|}} < \lambda.$$

For this reason and for the definition of the metric d in the space $(\mathcal{Q}, d)$, in order to prove (2.85) it is enough to show that two initial data sufficiently close are mapped through π_m to the same symbol Q_m, for any $m \in \{-m_0, \dots, m_0\}$. Therefore to (2.85) it is equivalent to prove that, for any $m_0 \in \mathbb{N}$ there exists $\eta > 0$ such that

$$\forall\, (x, v) \in \Pi_h \text{ s.t. } \|(x, v) - (x_0, v_0)\| < \eta \implies \pi_m(x, v) = \pi_m(x_0, v_0), \ \forall\, |m| \le m_0.$$

If we take $m_0 \in \mathbb{N}$, by means of Lemma 2.6.1 there exists a time interval $[-a, a]$ such that every solution with initial data in Π_h detects at least $2m_0 + 1$ symbols in $[-a, a]$, i.e., it determines at least $4m_0 + 2$ crosses on $\partial B_{\bar{R}}$. Moreover, the solution which emanates from the initial data (x_0, v_0) is collision-free and its projection on the x-component has a uniform distance $\delta > 0$ from the centres (see Theorem 2.4.28), i.e.,

$$|\pi_x \Phi^t(x_0, v_0) - c_j| \ge \delta, \ \forall\, t \in [-a, a], \ \forall\, j \in \{1, \dots, N\},$$

recalling that π_x denotes the projection on the x-component of the flow. At this point, if (x, v) is sufficiently close to (x_0, v_0), the continuous dependence on initial data implies that

$$\left|\pi_x \Phi^t(x, v) - \pi_x \Phi^t(x_0, v_0)\right| < \frac{\delta}{2},$$

for every $t \in [-a, a]$. This fact ensures that the flow associated to (x, v) determines the same $2m_0 + 1$ symbols of the flow associated to (x_0, v_0) so that, in particular

$$\pi_m(x, v) = \pi_m(x_0, v_0), \; \forall\, m \in \{-m_0, \ldots, m_0\}.$$

The proof is then concluded. □

A. On the hessian matrix of homogeneous potentials

We prefer to gather in this appendix some technical computations which concern the differentiation of homogeneous potentials and show to be useful for the purposes of Chapter 2.

Consider a non-negative $-\alpha$-homogeneous function $W \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\})$, with $\alpha \in (0, 2)$. We adopt the notation $w(x) = W(x_1, x_2)$, we introduce polar coordinates in $\mathbb{R}^2$

$$(x_1, x_2) = (r \cos \vartheta, r \sin \vartheta)$$

with $r = \sqrt{x_1^2 + x_2^2} > 0$ and $\vartheta = \arctan(x_2/x_1) \in [0, 2\pi)$, and we define the gradient with respect to x_1 and x_2 as

$$\nabla(\cdot) = (\partial_{x_1}(\cdot), \partial_{x_2}(\cdot));$$

in particular, we note that

$$\nabla r = r^{-1}(x_1, x_2), \quad \nabla\vartheta = r^{-2}(-x_2, x_1).$$

Moreover, we can write

$$W(x_1, x_2) = W(r \sin \vartheta, r \cos \vartheta) = r^{-\alpha} W(\cos \vartheta, \sin \vartheta),$$

and thus, if we let $U(\vartheta) \doteq W(\cos \vartheta, \sin \vartheta)$, we have

$$W(x_1, x_2) = r^{-\alpha} U(\vartheta).$$

In agreement with this notation, we have

$$U'(\vartheta) = \frac{d}{d\vartheta} W(\cos \vartheta, \sin \vartheta) = \langle \nabla W(\cos \vartheta, \sin \vartheta), (-\sin \vartheta, \cos \vartheta) \rangle .$$

In this way, we can compute the gradient of W

$$\begin{aligned} \nabla W(x_1, x_2) &= \nabla(r^{-\alpha} U(\vartheta)) \\ &= \nabla(r^{-\alpha}) U(\vartheta) + r^{-\alpha} \nabla U(\vartheta) \\ &= -\alpha r^{-\alpha-1} U(\vartheta) \nabla r + r^{-\alpha} U'(\vartheta) \nabla \vartheta \\ &= -\alpha r^{-\alpha-2} U(\vartheta)(x_1, x_2) + r^{-\alpha-2} U'(\vartheta)(-x_2, x_1), \end{aligned}$$

and so

$$\begin{aligned} \partial_{x_1} W(x_1, x_2) &= -\alpha r^{-\alpha-2} U(\vartheta) x_1 - r^{-\alpha-2} U'(\vartheta) x_2 \\ \partial_{x_2} W(x_1, x_2) &= -\alpha r^{-\alpha-2} U(\vartheta) x_2 + r^{-\alpha-2} U'(\vartheta) x_1. \end{aligned}$$

In the same way, the second derivatives read

$$\begin{aligned}
\partial_{x_1x_1}W(x_1,x_2) &= \alpha(\alpha+2)r^{-\alpha-4}U(\vartheta)x_1^2 + \alpha r^{-\alpha-4}U'(\vartheta)x_1x_2 - \alpha r^{-\alpha-2}U(\vartheta)\\
&\quad + (\alpha+2)r^{-\alpha-4}U'(\vartheta)x_1x_2 + r^{-\alpha-4}U''(\vartheta)x_2^2\\
\partial_{x_1x_2}W(x_1,x_2) &= \alpha(\alpha+2)r^{-\alpha-4}U(\vartheta)x_1x_2 - \alpha r^{-\alpha-4}U'(\vartheta)x_1^2 + (\alpha+2)r^{-\alpha-4}U'(\vartheta)x_2^2\\
&\quad - r^{-\alpha-4}U''(\vartheta)x_1x_2 - r^{-\alpha-2}U'(\vartheta)\\
\partial_{x_2x_1}W(x_1,x_2) &= \alpha(\alpha+2)r^{-\alpha-4}U(\vartheta)x_1x_2 + \alpha r^{-\alpha-4}U'(\vartheta)x_2^2 - (\alpha+2)r^{-\alpha-4}U'(\vartheta)x_1^2\\
&\quad - r^{-\alpha-4}U''(\vartheta)x_1x_2 + r^{-\alpha-2}U'(\vartheta)\\
\partial_{x_2x_2}W(x_1,x_2) &= \alpha(\alpha+2)r^{-\alpha-4}U(\vartheta)x_2^2 - \alpha r^{-\alpha-4}U'(\vartheta)x_1x_2 - \alpha r^{-\alpha-2}U(\vartheta)\\
&\quad - (\alpha+2)r^{-\alpha-4}U'(\vartheta)x_1x_2 + r^{-\alpha-4}U''(\vartheta)x_1^2.
\end{aligned}$$

At this point, we introduce the tensor (dyadic) product between two vectors and we make use of the following notation

$$(x_1,x_2)\otimes(x_1,x_2) = \begin{pmatrix} x_1^2 & x_1x_2\\ x_1x_2 & x_2^2\end{pmatrix} \qquad (x_1,x_2)\otimes(-x_2,x_1) = \begin{pmatrix} -x_1x_2 & x_1^2\\ -x_2^2 & x_1x_2\end{pmatrix}$$

$$(-x_2,x_1)\otimes(x_1,x_2) = \begin{pmatrix} -x_1x_2 & -x_2^2\\ x_1^2 & x_1x_2\end{pmatrix} \qquad (-x_2,x_1)\otimes(-x_2,x_1) = \begin{pmatrix} x_2^2 & -x_1x_2\\ -x_1x_2 & x_1^2\end{pmatrix}$$

and thus, if we let $x=(x_1,x_2)$ and $x^\perp=(-x_2,x_1)$, we get

$$\begin{aligned}
\nabla^2W(x) &= \alpha(\alpha+2)r^{-\alpha-4}U(\vartheta)x\otimes x - \alpha r^{-\alpha-4}U'(\vartheta)x\otimes x^\perp\\
&\quad - \alpha r^{-\alpha-2}U(\vartheta)I_2 - (\alpha+2)r^{-\alpha-4}U'(\vartheta)x^\perp\otimes x\\
&\quad + r^{-\alpha-4}U''(\vartheta)x^\perp\otimes x^\perp + r^{-\alpha-2}U'(\vartheta)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.
\end{aligned} \tag{A.1}$$

Now, take $\xi=(R\cos\vartheta^*, R\sin\vartheta^*)\in\mathbb{R}^2\setminus\{0\}$ such that

$$U'(\vartheta^*)=0,$$

i.e., following the notations of the previous sections, we assume that ξ is a *central configuration* for W. In this way, from (A.1), we have

$$\nabla^2W(\xi) = \alpha(\alpha+2)R^{-\alpha-4}U(\vartheta^*)\xi\otimes\xi - \alpha R^{-\alpha-2}U(\vartheta^*)I_2 + R^{-\alpha-4}U''(\vartheta^*)\xi^\perp\otimes\xi^\perp.$$

Furthermore, we recall that $(a\otimes b)c=\langle b,c\rangle a$ for every $a,b,c\in\mathbb{R}^2$ and, if we define $s_\xi \doteq R^{-1}\xi$, $s_\xi^\perp \doteq R^{-1}\xi^\perp\in\mathbb{S}^1$, we get this useful characterization of s_ξ and $s_\xi^\perp$ as eigenvectors of the matrix $\nabla^2W(\xi)$

$$\begin{aligned}
\nabla^2W(\xi)s_\xi &= \alpha(\alpha+2)R^{-\alpha-2}U(\vartheta^*)s_\xi - \alpha R^{-\alpha-2}U(\vartheta^*)s_\xi\\
&= \alpha(\alpha+1)R^{-\alpha-2}U(\vartheta^*)s_\xi\\
\nabla^2W(\xi)s_\xi^\perp &= -\alpha R^{-\alpha-2}U(\vartheta^*)s_\xi^\perp + R^{-\alpha-2}U''(\vartheta^*)s_\xi^\perp\\
&= R^{-\alpha-2}\left(-\alpha U(\vartheta^*)+U''(\vartheta^*)\right)s_\xi^\perp,
\end{aligned} \tag{A.2}$$

which correspond to the eigenvalues

(A.3) $$\lambda_\xi = \langle \nabla^2 W(\xi) s_\xi, s_\xi \rangle = \alpha(\alpha+1) R^{-\alpha-2} U(\vartheta^*)$$

(A.4) $$\lambda_{\xi^\perp} = \langle \nabla^2 W(\xi) s_\xi^\perp, s_\xi^\perp \rangle = R^{-\alpha-2} \left(-\alpha U(\vartheta^*) + U''(\vartheta^*) \right).$$

B. On the differential of the flow

This appendix is devoted to some properties of the flow associated to a dynamical system, with a special focus on the so-called *Variational Equation*.

Let $E \subseteq \mathbb{R}^n$ be an open set and let $F : E \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field. Given $z_0 \in E$, consider the following problem

$$\begin{cases} \dot{z} = F(z) \\ z(t_0) = z_0, \end{cases} \tag{B.1}$$

which admits a unique solution $\gamma_{z_0} \colon I(z_0) \to \mathbb{R}^n$, defined on its maximal interval $I(z_0) \subset \mathbb{R}$. Introducing the open set

$$\Omega = \{(t, z) \in \mathbb{R} \times E : \ t \in I(z)\}$$

we can define the flow associated to the differential equation in (B.1) as the map

$$\begin{aligned} \Phi \colon \Omega &\longrightarrow E \\ (t, z) &\to \Phi(t, z) \doteq \gamma_z(t). \end{aligned}$$

We will use often the notation $\Phi^t(z) = \Phi(t, z)$, in order to highlight the dependence on one of the two variables. If we fix $z_0 \in E$ we can thus consider the map

$$\begin{aligned} \Phi(\cdot, z_0) \colon I(z_0) \subset \mathbb{R} &\to E \\ t &\longmapsto \Phi^t(z_0) = \gamma_{z_0}(t), \end{aligned}$$

whose first derivative is nothing but the partial derivative, with respect to t, of the flow Φ. Moreover, since γ_{z_0} solves (B.1), the following chain of equalities holds

$$\frac{\partial}{\partial t}\Phi(t, z_0)\Big|_{t=t_0} = \frac{d}{dt}\Phi^t(z_0)\Big|_{t=t_0} = \dot{\gamma}_{z_0}(t) = F(\gamma_{z_0}(t)) = F(\Phi^t(z_0)).$$

In the same way, if we fix $t_0 \in \mathbb{R}$ in such a way that the set $\Omega_{t_0} = \{z \in E : \ t_0 \in I(z)\}$ is not empty, we can consider the map

$$\begin{aligned} \Phi(t_0, \cdot) \colon \Omega_{t_0} \subset E &\to E \\ z &\longmapsto \Phi^{t_0}(z) = \gamma_z(t_0), \end{aligned}$$

whose jacobian matrix is (with a slight abuse of notation)

$$\frac{\partial}{\partial z}\Phi(t_0, z)\Big|_{z=z_0} = \frac{d}{dz}\Phi^{t_0}(z)\Big|_{z=z_0}.$$

Therefore, given $z_0 \in \Omega_{t_0}$, the differential in z_0 is also well defined as the linear map

$$\begin{aligned} d_z\Phi^{t_0}(z_0)\colon \mathcal{T}_{z_0}\Omega_{t_0} &\to \mathcal{T}_{\Phi^{t_0}(z_0)}E \\ \zeta \mapsto d_z\Phi^{t_0}(z_0)[\zeta] &= \frac{d}{dz}\Phi^{t_0}(z)\Big|_{z=z_0}\zeta, \end{aligned}$$

where $\mathcal{T}_{z_0}$ denotes the tangent space at the vector z_0.

Remark B.1. *It is easy to notice that the flow $\Phi(t,z)$ is $\mathcal{C}^1$ in the variable t. Indeed, if we fix z_0, $\Phi(t,z_0)$ is exactly the unique solution curve $\gamma_{z_0}(t)$ of problem* (B.1) *and therefore it is differentiable. On the other hand, we can't argue in the same way for the differentiability with respect to the variable z, since in general it is not possible to give an explicit expression of the jacobian matrix of Φ with respect to z. In spite of this, it is well known (see for instance [41, 66]) that the jacobian matrix*

$$\frac{d}{dz}\Phi^t(z)\Big|_{z=z_0}$$

satisfies the so called Variational Equation, i.e., it is a solution of the following linearized problem (along the solution curve $\gamma_{z_0}(t) = \Phi^t(z_0)$)

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{d}{dz}\Phi^t(z)\Big|_{z=z_0}\right) = JF(\gamma_{z_0}(t))\dfrac{d}{dz}\Phi^t(z)\Big|_{z=z_0} \\ \dfrac{d}{dz}\Phi^{t_0}(z)\Big|_{z=z_0} = I_n. \end{cases}$$

This fact allows one to extrapolate several useful properties of the spatial differential of Φ and, among them, its continuity.

Finally, we can also define the jacobian matrix of the flow $\Phi = \Phi(t,z)$ in a point $(t_0, z_0) \in \Omega$ as

$$J\Phi(t_0,z_0) = \left(\frac{\partial}{\partial t}\Phi(t,z_0)\Big|_{t=t_0}, \frac{\partial}{\partial z}\Phi(t_0,z)\Big|_{z=z_0}\right) \in \mathbb{R}^{n\times(n+1)}$$

and thus, the differential of Φ in (t_0, z_0) will be the linear map

$$\begin{aligned} d\Phi(t_0,z_0)\colon \mathcal{T}_{(t_0,z_0)}\Omega &\to \mathcal{T}_{\Phi(t_0,z_0)}\Phi(\Omega) \\ (\tau,\zeta) \longmapsto d\Phi(t_0,z_0)[\tau,\zeta] \quad &= J\Phi(t_0,z_0)\begin{pmatrix}\tau \\ \zeta\end{pmatrix} \\ &= \frac{d}{dt}\Phi^t(z_0)\Big|_{t=t_0}\tau + \frac{d}{dz}\Phi^{t_0}(z)\Big|_{z=z_0}\zeta. \end{aligned}$$

C. Variational principles

This final appendix collects some general variational results which involve the functionals used along this work. In particular, we want to establish some important relations between the Maupertuis' functional, the Lagrange-action functional and the Jacobi-length functional.

Consider an open set $\Omega \subseteq \mathbb{R}^2$, a potential $V \in \mathcal{C}^2(\Omega)$ and introduce the second order system

$$\ddot{x} = \nabla V(x). \tag{C.1}$$

Note that (C.1) has Hamiltonian structure and thus we can look for those solutions x which preserve a fixed energy $h \in \mathbb{R}$ along their motion, i.e., such that

$$\frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h \tag{C.2}$$

and, in particular, such solutions will be confined inside the open Hill's region

$$\mathring{\mathcal{R}}_h \doteq \{x \in \Omega : \, V(x) + h > 0\}.$$

For $T > 0$ and $x \in H^1([0,T];\mathbb{R}^2)$ we define the Lagrange-action functional as

$$\mathcal{A}_T(x) \doteq \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + V(x(t)\right) \, dt$$

and it is well known that the *Least Action Principle* affirms that a solution $x\colon [0,T] \to \Omega$ of (C.1) corresponds to a critical point of $\mathcal{A}_T$.

In this work we have mainly used the Maupertuis' functional, which in this setting reads

$$\mathcal{M}_h(u) \doteq \frac{1}{2}\int_0^1 |\dot{u}|^2 \int_0^1 (h + V(u))$$

and it is differentiable in the space

$$H_h \doteq \{u \in H^1([0,1];\Omega) : \, V(u) + h > 0\}.$$

An equivalent result of the Least Action Principle can be stated for $\mathcal{M}_h$, the so-called *Maupertuis' Principle* (see [3], but also Theorem 2.4.1 for a version concerning fixed-ends problems), which also provides a first relation between the critical points of the two functionals.

Theorem C.1. *Let $u \in H_h$ be a critical point of $\mathcal{M}_h$ at a positive level. Define $\omega > 0$ such that*

$$\omega^2 \doteq \frac{\int_0^1 (h + V(u))}{\frac{1}{2}\int_0^1 |\dot{u}|^2}.$$

Then, the function $x(t) \doteq u(\omega t)$ solves (C.1)-(C.2) *in the interval $[0, T]$, with $T \doteq 1/\omega$.*

As a consequence, the function x is also a critical point of $\mathcal{A}_{1/\omega}$ in the space $H^1([0, 1/\omega]; \Omega)$.

The next result refines the correspondence between critical points of $\mathcal{M}_h$ and $\mathcal{A}_T$, showing that in particular a critical point of the Maupertuis' functional minimizes the action for every time $T > 0$.

Proposition C.2. *Let $u \in H_h$ be a critical point of $\mathcal{M}_h$ at a positive level. If, for every $T > 0$, we define*

$$x_T(t) \doteq u\left(\frac{t}{T}\right), \quad \text{for } t \in [0, T],$$

then

$$2\sqrt{\mathcal{M}_h(u)} = \mathcal{A}_{1/\omega}(x_{1/\omega}) + \frac{h}{\omega} = \min_{T>0} \left(\mathcal{A}_T(x_T) + Th\right).$$

Proof. For every $T > 0$, we can compute

$$\begin{aligned} \mathcal{A}_T(x_T) + Th &= \int_0^T \left(\frac{1}{2}|\dot{x}_T(t)|^2 + V(x_T(t)) + Th\right) dt \\ &= \int_0^T \left(\frac{1}{2T^2}|\dot{u}(t/T)|^2 + V(u(t/T)) + h\right) dt \\ &= \int_0^1 \left(\frac{1}{2T}|\dot{u}(s)|^2 + TV(u(s)) + Th\right) ds. \end{aligned}$$

Since u is fixed, the previous quantity depends only on T and it is easy to check that it attains its minimum at

$$T = \left(\frac{\int_0^1 |\dot{u}|^2}{2\int_0^1 (h + V(u))}\right)^{1/2} = \frac{1}{\omega}.$$

□

The previous result also shows a well-known property of the Maupertuis' functional, i.e., that this functional is invariant under time reparameterizations. However, $\mathcal{M}_h$ is not additive, and this suggests the introduction of another geometric functional. The Jacobi-length functional is defined as

$$\mathcal{L}_h(u) = \int_0^1 |\dot{u}(t)|\sqrt{h + V(u(t))}\, dt,$$

for every $u \in H_h$. Note that Theorem C.1 could be rephrased for $\mathcal{L}_h$ and thus classical solutions will be suitable reparameterizations of critical points of $\mathcal{L}_h$ (see for instance

[54]). Moreover, the Jacobi-length functional is also a fundamental tool in differential geometry since $\mathcal{L}_h(u)$ is exactly the Riemannian length of the curve parametrized by u with respect to the Jacobi metric

$$g_{ij}(x) \doteq (-h + V(x))\delta_{ij},$$

δ_{ij} being the Kronecker delta. As the Maupertuis' functional, $\mathcal{L}_h$ is invariant under time reparameterizations and, being a length, it is also additive.

Note that, if $u \in H_h$, the Cauchy-Schwarz inequality easily gives

$$\mathcal{L}_h(u) \leq \sqrt{2\mathcal{M}_h(u)},$$

with the occurrence of the equality if and only if the quotient

$$\frac{|\dot{u}(t)|^2}{V(u(t)) + h}$$

is constant for a.e. $t \in [0, 1]$. This shows that $\mathcal{M}_h$ and $\mathcal{L}_h$ share the same critical points u such that $\mathcal{M}_h(u) > 0$. This is sufficient to give the (easy) proof of the next result.

Proposition C.3. *Let $u \in H_h$ be a critical point of $\mathcal{M}_h$ at a positive level, let ω be defined as in Theorem C.1 and let x_T be defined for every $T > 0$ as in Proposition C.2. Then, the following chain of equalities holds*

$$\mathcal{A}_{1/\omega}(x_{1/\omega}) + \frac{h}{\omega} = 2\sqrt{\mathcal{M}_h(u)} = \frac{2}{\sqrt{2}}\mathcal{L}_h(u).$$

We can say that, up to constant factors and time re-parametrizations, the three functionals coincide on the non-constant critical points of $\mathcal{M}_h$.

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