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Acknowledgements

Teaching is giving importance to someone. In a sense is an act of love. Without this ingredient there is no Teaching, there is no Learning. During the three years of this beautiful experience I have interrupted my teaching activity at Liceo "Giordano Bruno" in Torino. Still I always had the distinct feeling that all the work carried out during the PhD was well aligned with my past occupation. It has been a sort of extension forward of the road I was walking on.

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Finally, the most important consideration. This journey would have been impossible without Ivan Gnesi. A Physicist with a deep vision of what Physics is and with a rare sensibility for the importance of fostering critical thinking and deep reasoning in every mind and at every level. Ivan has been an ice-breaker and a guide, a sort of "Bud Spencer" always ready to help even at the risk of some punch. The

open discussions with him and all the things he stimulated me to do and to learn are a treasure that I will never forget. My thanks and my appreciation for his priceless efforts toward me go well beyond what I can express with words.

Introduction

When we teach, we do something for students and for us. This fact is well beyond any question. What we do for us in not the focus of this work, what we do for students and how we do it is something we will deeply discuss in the framework of physics teaching. To be more precise this doctoral thesis is the result of three years of Research in Modern Physics Education, the physics related to the two major breakthroughs of the twentieth century: Relativity and Quantum Mechanics. The research was carried out working at the same time at three different levels:

1. Designing teaching approaches to introduce physics concepts. At this level, the main concerns were
   - the definition of the teaching pathway,
   - reflections on the choice of the best mathematical approach,
   - the continuity (when possible) with the approaches used in earlier years introducing classical physics concepts,
2. Research new experimental and practical activities in order to engage students in the learning process and build thought-provoking environments. At this stage, a matter of great concern was
   - the low cost of the experimental apparatus,
   - the intelligibility of the framework,
   - the use of devices and materials already present at school or of easy availability.
3. The test of the educational approaches and methods with groups of students and groups of teachers in order to have a feedback on their effectiveness. The testing stage was performed with courses (both for teachers and students) with a final survey in order to collect statistical data and impressions. A second level test was conducted asking teachers to adopt our approaches or methods in their classrooms. A survey on their experience was then carried out.

Why Physics Education Research in Modern Physics

New Curriculum and the lack of educational good practices

The MIUR (Ministry of University and Research) decree 7 October 2010 n.211 introduces in the Italian scientific high-school physics curriculum new topics [1]. The ordinance adds to the usual curriculum selected aspects of Special Relativity and Quantum Mechanics, placing these subjects in the last year of the secondary school. As a consequence Italian physics teachers, used to deal with the physics up to the Maxwell Theory, found themselves suddenly in front of a huge challenge. The situation was made even worse by the high percentage of physics teachers having a degree in mathematics and not in physics. Two more aspects have to be considered. The first is that both Special Relativity and Quantum Mechanics are very delicate theories, in the sense that their physical interpretation is not easy at all, also for physicists. The second is the lack in teaching and educational good practices suitable for the introduction of modern physics at secondary school level. All these factors have generated and still generate an unpleasant state of anxiety and discomfort in a lot of teachers.

[1] October 7 2010 MIUR, Decree n.211. *Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento*. 2010

Modern Physics: a communication channel between Secondary Schools and Universities

Almost every field of the present-day physics research has a deep connection with Special Relativity and Quantum Mechanics. Since both the theories are in the scientific high-school's curriculum they are good candidates to play a fundamental role in the building of a bridge between schools and the academic environment. One goal the Italian educational system has to achieve is to shape, on the basis of this common ground, a productive exchange of knowledge among the two actors. In later sections we will discuss in more detail about this topic, also proposing a possible method to carry out the task.

How to teach Modern Physics: New Pedagogical Directions in the Italian Secondary School

Teaching Physics in Italy. Considerations.

The common approach to the teaching of Physics in the Italian secondary school is teacher centred, based on the instructor's presentation of ideas. This naturally leads to a general low engagement condition of the students. However, low engagement is not the only drawback of this method: listening to something presented by others do not activate deep reasoning and deep conceptual understanding.

Another issue we have to consider is related to the five years curriculum required for the final exam. A lot of teachers feel a certain stress due to the number of topics they have to cover. The completeness of the curriculum becomes one of the main concerns of the teacher over all the five years, and, as a consequence, the learning process of the classroom becomes a secondary issue. A better equilibrium between development of the curriculum and systematic monitoring of the student reasoning should be reached.

New directions

The keyword to address both the problems above mentioned is "Interaction". More interaction is required between the learner and the subject we teach, more interaction is also desirable among teacher and students. Hereafter the three strategies we suggest to consider.

Interaction Learner-Subject: Inquiry Based Science Education

Basically, to improve the engagement and the quality of the understanding we need to replace some traditional instruction with learner-centred methods [2]. According to the US National Science Education Standards [3], the best way we have is the Inquiry Based Science Education (IBSE)[4]. Essentially with the IBSE approach knowledge is not delivered by the instructor, but is built, under the tutoring of a teacher, by student reasoning, carrying out some active investigation. The interaction between Physics and students becomes the core of the learning scenario; students perceive higher engagement, higher development of reasoning skills and start a building process of autonomous thinking. In order to design a learning environment suitable for this kind of interactions the teacher is required having a certain amount of creativity. He needs to identify simple experiments in order to make the situation easily reproducible by a large percentage of students and to connect the experiment to a well designed tutorial or pedagogical sequence (Fig. 1).

[2] M. Kryjevskaia et al. *Assessing the flexibility of research-based instructional strategies: Implementing Tutorials in Introductory Physics in the lecture environment.* American Journal of Physics, 82, 238-250, 2014

[3] National Research Council. *National Science Education Standards.* The National Academies Press, 1996, Washington DC, https://doi.org/10.17226/4962

[4]

Figure 1: Example of IBSE activity: the entire classroom is working on a science problem; each student has a cheap experimental apparatus in order to perform investigations.

Interaction Teacher-Learner: Continuous Monitoring of the Student Understanding and Reasoning

When teachers work in a classroom they spend a considerable part of their energy in analysing students' facial expression and body posture. In Italy, we are talking about data coming from 20, 25 pupils. Even though this kind of monitoring is fundamental for a lot of reasons, it does not offers a neat representation of the classroom understanding. Nowadays, however, teachers have the possibility to monitor almost in real-time the reasoning of the classroom and to build, during the learning process, check points where to rapidly survey student's opinions and adjusting accordingly their didactic. This can be done in a lot of different ways, ranging from the traditional quick test to the more effective Google Forms method, offering an instantaneous overview of the whole classroom understanding. To act in this way educators are asked to be flexible and to master the topic they are dealing with.

Going Three-Dimensional in Physics Teaching

Usually textbooks give a bi-dimensional view of Physics. As a consequence, the teacher's activity moves over two parameters: the amount of time he will lecture and the amount of exercises he will present to students. Fixing these two parameters he will also define the region in which the understanding of the student will be set. The main concern in this scenario is to create a contact between theory and exercise (Fig. 2).

While this approach offers to the teacher a protected harbour where to move, an unpleasant side effect arises: a self-referential loop is created giving low freedom to the learner thinking. In the mind of the learner a wrong model takes root: Physics as a fixed and closed structure, where everything works as we would wish. This picture of Physics is misleading and, in a sense, boring. Everybody analysing experimental data knows that the core of Physics is the conceptual work to explain what we measure with the theories we have. In order to reach a good understanding of a theory we have to use it as an interpretative gauge for experimental data. This opens the door to a three-dimensional teaching of physics (3DTP), see Figure 3.

Recent studies in Physics Education show the possibility to introduce secondary school students to the analysis of selected data sets coming from the Physics Research environment[5], [6]. An example, connecting Special Relativity to the analysis of data from accelerators experiments will be presented in detail in this thesis. What is important to highlight is that when the theory becomes an interpretative tool of experimental data a deeper understanding of Physics arises, students have the opportunity to see how a scientific result is obtained and also to achieve small discoveries based on their own investigation. 3DTP is thus an innovative tool that could be considered to strengthen the interaction Physics-Students, of course the design of an educational tool of this kind needs a lot of work an thinking

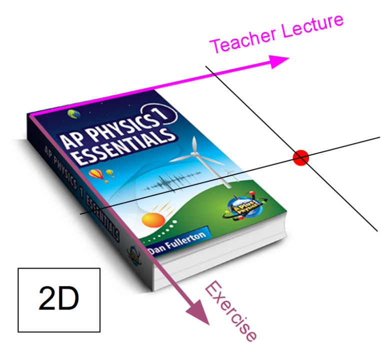


Figure 2: Bi-Dimensional Physics Teaching.

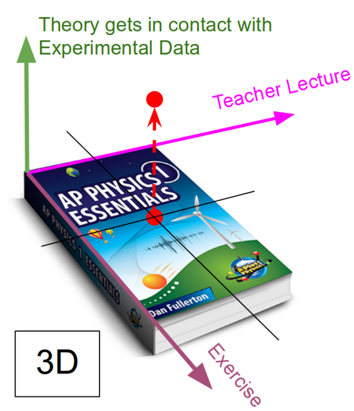


Figure 3: Moving along a new direction, toward the Three-Dimensional Physics Teaching.

[5] Grazzi Stefano and the EEE Collaboration. *EEE Project - Students from all parts of peninsula collaborate to study cosmic rays*. Proceedings of Science, 2017

[6] IPPOG (International Particle Physics Outreach Group). *International Masterclasses*. https://www.physicsmasterclasses.org/

Physics Education Research as the Teachers Continuing Training Starting Point

The European Commission for Education and Training asks for in-service teacher's *continuing professional development*[7]. Fundamentally this requirement is a consequence of the strong correlation between how much time teachers spend in training and the students outcomes[8] and of the focus of the European Council on high quality teaching as key pre-requisite for high quality education and training[9]. The Italian government is basically aligned with the European policy: continuous training is mandatory for all the Italian teachers (Law number 107-July 2015, "La Buona Scuola"). However, continuous training needs to be based on a strong effort in physics education research. A lot of work has to be carried out both in the promotion of modern pedagogy methods and in the research of new approaches to introduce physics concepts. A crucial factor will be the building of a strong connection between physics education researchers and training proposal for Italian high-school teachers. The need for this bridge is evident if we consider the huge distance separating secondary school teachers from journals and international meetings of physics education and from the current physics education research. Since the beginning of my research in physics education at the University of Torino, this was one the main goals of my work. The secondary school teaching expertise, that comes from my background (I'm a mathematics and physics teacher with 18 years of experience) and the opportunity to work in an academic environment have acted as ideal attitudes to establish a contact between the two worlds. The continuous relationship with secondary school teachers, kept alive offering training courses, and the brainstorming with them about new ideas were fundamental tools for the calibration of my "research apparatus". On the other hand, being in contact with other researchers all over the world have opened to my mind new perspectives and horizons.

[7] Francesca Caena. *Quality in Teachers' continuing professional development*. European Commission, 2011

[8] Angrist Lavy. *Does teacher training affects pupils learning? Evidence from matched comparisons in Jerusalem public schools.* Journal of Labor Economics, Vol.19, No.2, 2001; and Kain Rivkin, Hanushek. *Teachers, schools and academic achievement.* Econometrica (The Econometric Society), Vol. 73, No. 2, 2005

[9] Darling-Hammond et al. *Does Teacher Preparation Matter? Evidence about Teacher Certification, Teach for America, and Teacher Effectiveness*. Education Policy Analysis Archives,Vol. 13, No. 42, 2005; Rockoff. *The Impact of Individual Teachers on Student Achievement: Evidence from Panel Data.* The American Economic Review, Vol. 94, No. 2, 2004; and *Council Resolution 2008/C 319/08 of 21.11.08.* Official Journal of the European Union, 2008

Teaching Special Relativity

What is done in the Italian high-school

Usually Special Relativity is introduced in Italian high-schools relying on the derivation of the time dilation and lengths contraction effects[references of three textbooks] and on the consequences related to these facts. Looking through some Italian textbooks we can highlight some common aspect, getting in return a general overview of the teaching method.

- Usually the Lorentz transforms are presented but not derived.
- The constant c appears in the relativistic formulas of almost all text-books; mapping c into 1 seems to be something almost every textbook is scared about.
- the relativistic momentum, when presented, is generally introduced without giving deep explanations about its origin.
- The fundamental role and the physical meaning of invariants, especially of the invariant mass, is barely present when not mentioned at all.

A survey conducted over 35 high-schools teacher spread over the Italian territory reveals interesting features of the "Italian Style" in the teaching of the Einsteinian theory.

The time dedicated to the teaching of this subject in the scientific high-school amounts, on average, to 10 hours with a standard deviation of about 8 hours[10] (Table 1, Figure 4). Special Relativity is planned to be taught in the fifth year, the last year of the high-school. The 88.6 % of the surveyed teachers introduce this subject in the fifth year, only the remaining 11.4 % introduces the theory gradually, beginning from earlier years. This gives a good margin of improvement, since, as will be discussed later, the road towards Special Relativity should be built slowly, finding out approaches that naturally extend classical definitions. In this sense the route needs to be prepared: teachers may plan in advance a set of actions, both in the physics and

[10] Estimates from the gaussian fit of the collected data (Fig. 4).

How many hours do you lecture to introduce Special Relativity to your students (excluding the assessment time)?	
14.3 %	0 to 4 hours
17.1 %	4 to 8 hours
25.7 %	8 to 12 hours
22.9 %	12 to 16 hours
11.4 %	16 to 20 hours
8.6 %	20 to 24 hours

Table 1: Amount of time dedicated to the teaching of Special Relativity in the Italian scientific high-school.

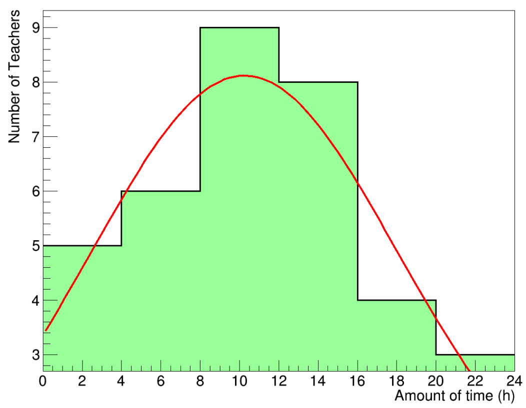


Figure 4: The amount of time dedicated to the teaching of Special Relativity. The histogram is fitted with a Gaussian with a mean value of 10.2 hours and a standard deviation of 7.7 hours.

mathematics curriculum, to be taken since the third year. This attitude gives the opportunity to save a certain amount of time that we will have the opportunity to spend when introducing the Einstein's theory in the fifth year.

The 74.3 % of the teachers introduces the relativistic momentum, but the 79.4 % of the 35 interviewed teachers admits that in textbooks the relativistic momentum is presented with no deep explanation about its derivation and origin. Here we are in presence of an evident educational lack. Roughly speaking, or the relativistic momentum is out of reach of secondary school students or further educational research is needed in order to design new ways to introduce it. In our opinion this fact is of primary importance, since the relativistic relation between momentum, mass and energy is at the core of all the current physics research, being something that students and, more generally, citizens need to understand. All the issues of the presented survey will be addressed in the following sections of this chapter dedicated to the teaching of Special Relativity.

Special Relativity. With $c = 1$.

There are a lot of different approaches to teach Special Relativity at secondary school level. In this section we will present a method which differs from the traditional one adopted in the Italian textbooks.

Beginning with the derivation of the time dilation effect, we will introduce the space-time plane of an observer and we will learn how to draw in it the time and space axes of other observers. Adopting natural units ($c = 1$) we will define the four-position vector $(t, \vec{x})$ from which we will derive the four-velocity vector in a way inspired to the Rob Salgado work. At this point the *invariant* quantity $t^2 - x^2$ is presented and its physical meaning is given, a very important step since in the majority of the Italian textbooks invariants are not introduced nor discussed. Invariants offer the opportunity to discover how distances are measured in the relativistic space-time, therefore the Minkowski distance is introduced. Another important feature of our approach is that the *Lorentz transforms* are derived in a simple and really short way, only based on the time dilation effect. These transforms tell us how space and time coordinates change when we move our point of view from one inertial observer to another one. A very powerful interpretation of the Lorentz transform is given, in analogy with usual rotation in the Euclidean plane they are presented as rotations in the Minkowski space-time, with the only difference that the former rotation leaves the Euclidean distance unchanged, while the latter the Minkowski distance. Looking at the Lorentz transforms as rotations immediately gives us the opportunity to understand how all the four vectors are transformed when we change the observer. If an Euclidean rotation is performed not only a (x, y) vector undergoes the transform, but also the velocities will change components according to the same transform. This also happens in the Minkowski's space-time, thus we automatically learn that also the four-momentum (which is strictly related to four-velocity) transforms according to the same transforms of the space-time vectors. Another new aspect of the teaching method we are proposing is the presence of a part devoted to the *relativistic dynamics*, i.e. to the four momentum, the energy, mass, momentum relativistic relation and the relativistic concept of mass of a system of particles. This topics deserve their space and time in the teaching plan of teacher. They open wide and beautiful horizons toward the current research in nuclear physics and particle physics. It is something we owe students in order to provide them the required tools to look at the world of the scientific research with critical thinking.

A set of interactive applications as well as the theoretical introduction of all the main concepts are offered to the teacher in order to accomplish this task. A set of tutorials fostering the students' autonomous building of knowledge and an enquiry based science education (IBSE) method are proposed as well. Students will have the opportunity to investigate, understand and discover relativistic concepts following a sequence of questions and problems in the tutorials and working in the framework of the interactive applications.

Time dilation

As before mentioned our approach to the teaching of Special Relativity begins more or less as in the traditional way with the time dilation effect. We derive this relativistic phenomenon using the constant speed of light postulate, a statement supported by a lot of experimental evidences:

> The speed of light has the same value for all the inertial observers.

We can consider two observers, O and O': O, at rest with respect to the ground, O', moving rightwards at constant speed v with respect to O. Now we can imagine O holding an apparatus composed by a photon gun directed upwards and an horizontal mirror, L meters above the gun, which reflects photons back to O (Figure ...).

> We have two Observers:
>
> 1. O is an observer at REST with respect to the clock.
> 2. O' is an observer NOT AT REST with respect to the clock.

We focus our attention on the time of flight of the photon, the time to go back and forth from the gun to the gun again. For O, which sees the photon going up and down along the vertical direction and covering a distance $2L$ the time of flight is:

$$t = \frac{2L}{c} \tag{1}$$

Completely different the scenario perceived by O', in his reference system (RS) he will see the clock moving at a speed v leftwards. The back and forth tracks of the photon will be no more vertical, they will appear to him as hypotenuses of right triangles with horizontal legs equal to half the way travelled by the clock during the time of flight t' of the photon. In this case the total distance covered by the photon is:

$$2\sqrt{L^2 + \frac{v^2\, t'^{\,2}}{4}} \tag{2}$$

Since the speed of light is constant for all the observers, this distance has to be equal to ct',

$$2\sqrt{L^2 + \frac{v^2\, t'^{\,2}}{4}} = ct' \tag{3}$$

From this equation we can derive the relation between the two time lapses, t and t', of the two observers:

$$4(L^2 + \frac{v^2 t'^{\,2}}{4}) = c^2 t'^{\,2}, \quad 4L^2 = t'^{\,2}(c^2 - v^2) \tag{4}$$

$$t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot \frac{2L}{c} \tag{5}$$

$$t' = \gamma(v) \cdot t \tag{6}$$

with:

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7}$$

Proper time and time dilation

We may notice that the time interval t has two properties:

- since γ is by definition bigger than 1, t is the shortest time duration of the observed phenomenon;
- is the time measured by the observer AT REST with the clock.

The time of such an observer is called *proper time* and is indicated with the letter τ. Relation [6] becomes:

$$t' = \gamma(v) \cdot \tau \tag{8}$$

> Any observer NOT AT REST with the clock will measure longer time intervals according to the velocity of the clock with respect to him.
>
> Time goes faster for the observer who is NOT AT REST with the clock.

This explains why, if we observe a particle with a certain lifetime moving at nearly the speed of light, its lifetime appears to us so much longer (while for the particle itself nothing changes!).

A maximum velocity?

Since the factor $\gamma(v)$ approaches an infinite value as the speed approaches the speed of light, we may conclude that for an observer not at rest with a so fast moving clock, a finite period of the clock would correspond to a infinite time interval. Now we have two pieces of information, both suggesting that the speed of light is a limit speed. From a mathematical point of view, the first one is given by the domain of the function $\gamma(v)$, defined in the range $]-1, 1[$.

> This means that inside this theory the unit speed can not be reached.

The second comes from the physical meaning of time dilation when the speed of a clock approaches the speed of light: if compared with the time of the clock, the time of all the observers not at rest with it would flow infinitely fast.

Setting c=1

From now on we will assign the value 1 to the speed of light. In this way the β, which is v/c, becomes equal to $v/1 = v$ and even if we will write v we will consider it a dimensionless parameter since is the ratio among two speeds. Furthermore, if we consider 1 as a speed limit, v will range in $[0, 1[$, 1 being the speed of light which is not reachable in this theory (remember the domain of the Lorentz factor).

World Lines

In Special Relativity the line describing how the position of a body varies in time is called *world line* (WL). In figure (5) we have plotted three different WLs in t, x space-time plane.

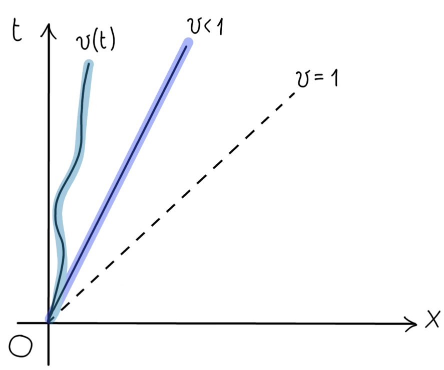


Figure 5: Different WLs.

The dotted line, at 45 degrees, represents the WL of a photon, the other straight line is the WL of a body moving at constant speed lower than the speed of light. The curve line is the WL of a system moving not at constant speed but always being slower that light. We have to accept that time in Special Relativity is associated to the vertical axis, while the position x is represented along the horizontal one. Furthermore we have to accept that a point in space-time is expressed by (t, x), with time in the first position and x in the second.

The time axes of different inertial observers

By definition, every observer is at rest in his own space-time reference system (RS), always being in the origin of his space axis. However, since time flows, his position in space-time will also move along the time axis (Fig. 6).

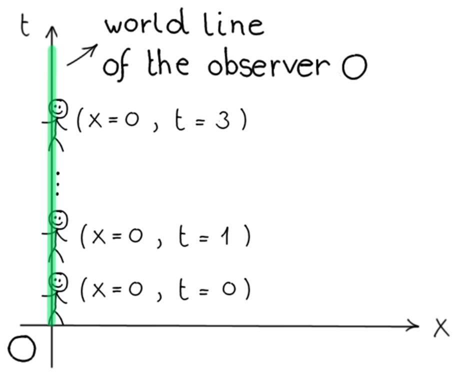


Figure 6: The position in space-time of the "owner" of the Reference System.

> In space-time the time axis represents the WL of the observer.

Now we consider another observer, O', moving at speed v with respect to O. His time will flow differently if compared to the time of O, therefore his time axis will be denoted by a different letter: t'. Again, his WL in his RS will be represented by his time axis (red line in figure 7).

The WL of O', the red line, if represented in the RS of O becomes inclined (Fig. 8): we have discovered how to represent the time axis of O' in the space-time of O.

This is an important result that gives us the opportunity to draw the time axes of different observers in the same RS.

The space axes of different inertial observers

Once chosen a RS, we can now draw the time axis of any observer, this will help us to draw the corresponding space axis. According to the light speed invariance postulate, the speed of light is the same for all the inertial observers. Since the speed of light is always 1, in

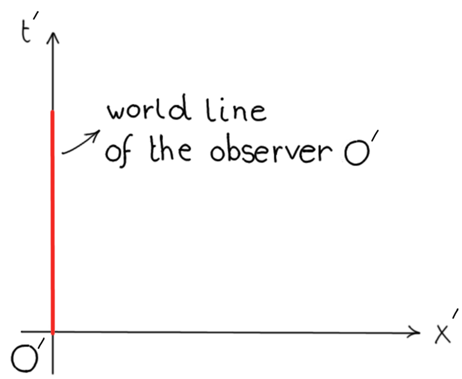


Figure 7: The World line of the observer O′ in his own Reference System.

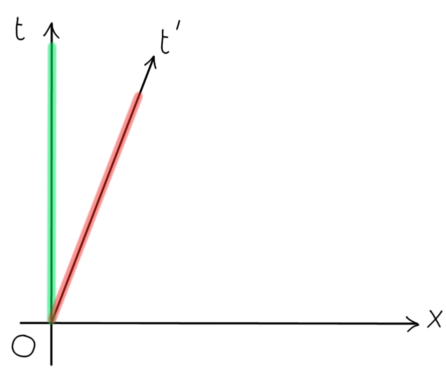


Figure 8: The WL of O′ represented in the Reference System of O.

every RS the space interval travelled by a photon must be equal to the required time interval so that their ratio always gives 1. In other words the WL of a photon has to be the line of symmetry of the time and space axes. Hence, once drawn the time axis of an observer, the space axis will be symmetrically disposed with respect to the photon WL (Fig. 9 and 10). Now we know how to draw both the time axis and the position axis of any observer in a certain RS.

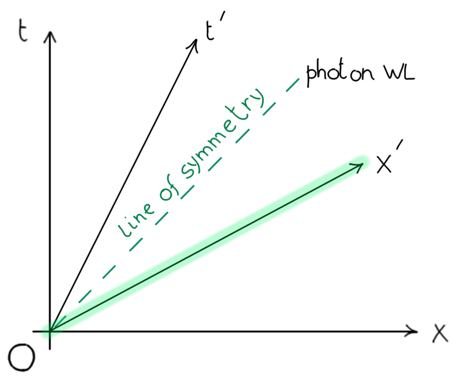


Figure 9: The space axis is symmetrically disposed with respect to the WL of a photon.

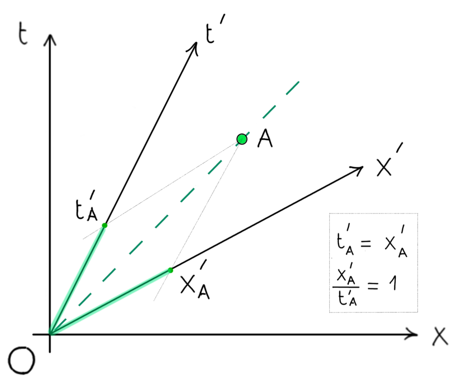


Figure 10: The symmetry guarantees that the speed of light is 1 in any RS.

Four vectors notation

Although we will mostly work in two dimensions, t and x, space-time has four: a time dimension plus three spatial coordinates, $\vec{x}$. Therefore, in order to define a point in space-time we will use this notation:

$$(t, \vec{x}) \tag{9}$$

A point in space-time is an *event*. In order to define it 4 coordinates are needed, therefore the event (9) can also be thought of as a vector with 4 components. In Special Relativity an event corresponds to a *4-vector*, more specifically it corresponds to the so called *4-position* vector which we label with the symbol $\overset{4}{x}$.

Here we have to be clear about units.

The components of the *4-position* are not homogeneous form this point of view, the first has time units, the other three have spatial units. In order to avoid this problem in Special Relativity the temporal component is multiplied by c, the speed of light. Therefore, actually, the *4-position* is:

$$\overset{4}{x} = (ct, \vec{x})\,, \tag{10}$$

however, since $c = 1$, we will always write

$$\overset{4}{x} = (t, \vec{x})\,. \tag{11}$$

Also the space-time plane has to adhere to this choice. The time axis will become ct, which is the distance travelled by light in t seconds. But, again, being c equal to 1, it will not appear. The first component is the *temporal* component and will be denoted by x_0, the remaining three are the *spatial* components and will be denoted by x_1, x_2, x_3.

For educational purposes, we will maintain the $\overset{4}{x}$ notation even if working in two dimensions (one temporal and one spatial component).

From now on the speed of light c will be hidden in our formulas and in our space-time planes, we will learn how to manage this situation and in return this will greatly simplify our mathematical expressions. Side notes all along this text will help the reader to become familiar with this practice.

The theory of Special Relativity revolves around these 4 four vectors, as we will discuss in detail later, there are deep reasons to consider them as fundamental quantities. Here we give some anticipation: they are strictly connected to physical quantities, called *invariants*, that do not depend on the inertial observer. Furthermore they are very useful to introduce the relativistic concepts of Energy and Momentum.

Space-time explorers: the 4-velocity

But what more often torments me is the suspicion that this border does not exist, that this kingdom stretches limitlessly, and that despite the extent of my advances, I shall never be able to reach the end.
— Dino Buzzati, The Seven Messengers

Now we are going to explore the space-time from two different points of view[11]. We will consider a space-time plane (t, x) at rest with an observer and we will recruit a lot of messengers asking them to move along this plane with different speeds and to stop when their clock reads 1 second[12]. The "owner" of the space-time plane, the one at rest with it, will move along the vertical time axis, the others will move along world lines with different slopes according to their speed.

[11] This section is devoted to the derivation of the relativistic 4-velocity and is strongly inspired by the Rob Salgado work

[12] R B Salgado. *Spacetime Trigonometry and Analytic Geometry I: The Trilogy of the Surveyors.* AAPT Topical Conference: Teaching General Relativity to Undergraduates, Syracuse, NY, 2006

When they stop, they are asked to measure their temporal and spatial coordinate in the (t, x) RS.

> This is a crucial point: we are investigating the (t, x) space-time plane, thus we are interested in measuring coordinates in this RS. Messengers have to report the coordinates of their stopping point with respect to the t and x axes, not in with respect to their personal RS!

We will use *Galilean* and *Relativistic* explorers to perform the same task and we start with the *Galilean* surveyors. In Galileo Physics all the observers measure the same time, so all of them come to a stop at the same temporal coordinate in the (t, x) plane: $t = 1s$. Concerning the spatial coordinate, they report different positions according to their speed: $x = v \cdot 1s$. Therefore the set of stopping points will form the horizontal line $t = 1$ (Fig. 11) and, in general, the final four-position of the messengers will be:

$$\overset{4}{x}_{Gal.} = (1s, 1s \cdot v) \,. \tag{12}$$

We can go further and imagine how would the Galilean messengers act in order to derive a 4-velocity from the "4-position" vector (12).

They know that velocity is displacement divided by time. So they divide the four-position by the time they planned to travel: 1 s. In

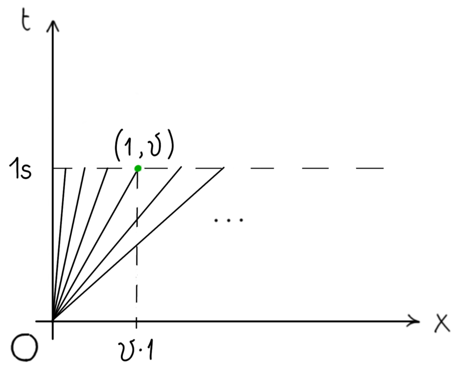


Figure 11: The dot represents the stopping point of a generic observer moving at a speed v.

this way they get a quite good 4-velocity vector:

$$\overset{4}{v}_{Gal.} = (1, v)\,. \tag{13}$$

The first component is dimensionless, the second is the velocity of the messenger. The 4-vector is tangent to the space-time diagram (the WL) of the surveyor, which is a crucial requirement for a velocity vector!

It is time for the *Relativistic* explorers to come into play. They are asked to accomplish exactly the same task of the *Galileans*. However now things change, they will experience time dilation effects. The messengers are like the particle of the example we made in the time dilation section: they are moving clocks in the (t, x) RS, so if their clock reads 1s, a longer time has elapsed for the owner of the $(t,\ x)$ RS who sees them as clocks in motion. The 1 second interval of the surveyors will result in a time $t = \gamma(v) \cdot 1s$. The spatial coordinate of the stopping point will be the product between the travel time and the speed of the messenger : $x = \gamma \cdot 1s \cdot v$. In general, the stopping 4-position will be (Fig. 12):

$$\overset{4}{x} = (\gamma \cdot 1, \gamma \cdot 1 \cdot v). \tag{14}$$

Remember that the speed of light 1 is present but not written in the first component, so that the units of this component are position units.

Now the *Relativistic* explorers have the chance to learn from history and derive the 4-velocity in the same way their ancestors, the *Galilean* surveyors, did before. They divide the 4-position by the time elapsed for every messenger, the time shared by all of them: 1 s. What they get is this vector:

$$\overset{4}{v} = (\gamma, \gamma v) \tag{15}$$

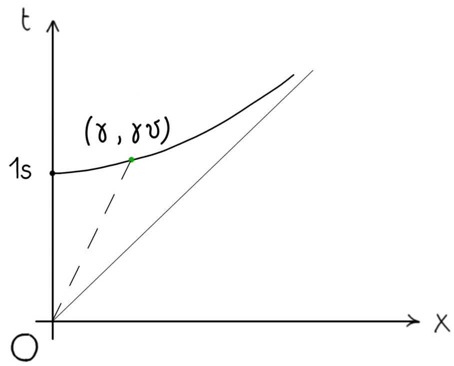


Figure 12: The Stopping position of a generic relativistic traveller lays on the line shown in this plot.

Indeed this is a velocity 4-vector, both components have velocity units (actually the first one is γ multiplied by the speed of light 1)[13]. The ratio between its components gives the speed of the messenger:

$$\frac{\gamma v}{\gamma} = v \tag{16}$$

[13] Dismissing natural units the 4-velocity would be $(\gamma c, \gamma v)$.

thus, again, the vector is tangent to the WL of the surveyor. This 4-vector is the *relativistic 4-velocity*. The observer that sees a body (a messenger in our example) moving in his RS will ascribe to it this 4-velocity. As we will soon discover, the result we have obtained represents a sort of gate that connects us with the relativistic dynamics, opening the doors of the surprising relativistic relation between energy, mass and momentum.

The first relativistic invariant

In the previous section we had two different points of view. An observer A was the owner of the RS in which we were representing the travel of the explorers. He was the observer seeing others moving. Observers of type B were the explorers moving with different speeds v with respect to A. We have seen that 1 second of the observers B corresponds to $\gamma(v) \cdot 1$ seconds for A. Now we want to make explicit what is happening before our eyes.

We have a well defined event E:

> E = the space-time position where the explorer B, travelling at speed v, stops after 1s of his own clock.

However different observers, A and B, ascribe to the same event different space-time coordinates. For B the event has coordinates $(1, 0)$ (his clock reads 1 second and in his frame he is always in the origin of his x-axis). For A the coordinates are $(\gamma(v), \gamma(v) \cdot v)$

(the clock of A reads a longer time and the distance travelled is the product of elapsed time and speed).

Observer B coordinates: $E = (1, 0)$
Observer A coordinates: $E = (\gamma, \gamma \cdot v)$

This is a fact: in Special Relativity space and time coordinates have no absolute physical meaning, they depend on the observer. This is why the word *Relativity* is part of the name of this theory. But we cannot stop here, we must take one step further. Even if the coordinates differ from one observer to another there is something all the observer agree on: the difference among the square of time coordinate and the square of the space coordinate of an event!

The quantity $t^2 - x^2$ has the same value for all the inertial observers.

This is something we can easily prove. For A we have:

$$t_A{}^2 - x_A{}^2 = \gamma^2 - \gamma^2 v^2 = \gamma^2(1 - v^2) = 1\,, \tag{17}$$

for B:

$$t_B{}^2 - x_B{}^2 = t_B^2 = 1\,. \tag{18}$$

We have found an *invariant* quantity, something which is the same for all the inertial observers. In addition this number has a precise physical meaning: it is the time squared of the clock which is at rest with the physical phenomenon we are considering, is the square *proper time* τ^2:

$$t^2 - x^2 = \tau^2\,. \tag{19}$$

In our example we were considering a moving observer who stops after one second of his clock and the event was his stopping point in space-time. All the infinite inertial observers will ascribe to that event different space-time coordinates but all of them will agree on the value of $t^2 - x^2$ which will give them the square time read by the clock of the observer whose motion we are studying[14].

[14] Should we decide not to work in natural units $c = 1$, the invariant would become $(ct)^2 - x^2 = c^2\tau^2$.

If we are considering a physical system moving in our space-time RS and this system according to us has coordinates (t, x), any other inertial observer will ascribe to it different coordinates. However all the observers will agree on the value of the *invariant* $t^2 - x^2$ which is related to the *proper time* of the physical system under study.

This is a key point in the theory of Special Relativity, something we cannot neglect if we aim to give a reasonable introduction to the modern conception of space and time. Einstein was not saying that everything is relative, on the contrary he has proved the existence of something which has absolute physical meaning. The philosophical implications of his theory have to consider this crucial aspect of the theory. A huge number of students (age from 17 to 18) study both physics and philosophy, we hope these considerations to become part of a deeper discussion among students and teachers belonging to different branches of knowledge.

Distances in space-time

So far (Fig. 13) we have seen that the WL of the moving observer B represents his time axis t_B and we know he ascribes a time 1 to the event E. Thus the segment OE possesses a double meaning,

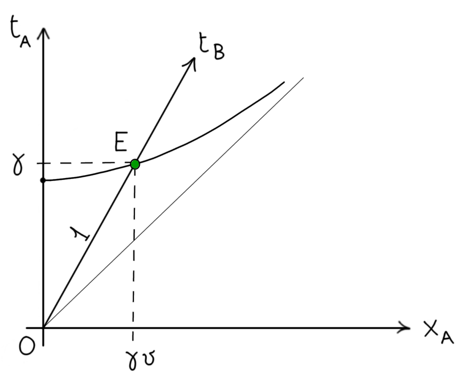


Figure 13: The segment OE represents the time the observer B ascribes to the event E.

- represents the elapsed time in the RS of B, which is 1.
- is the distance of the point E from the origin of the space-time plane of A.

Here comes the surprise! Evaluating the length OE with the coordinates $(\gamma, \gamma v)$ and the Pythagoras's Theorem we would not reach the value 1. In order to consistently close the loop we must accept that distances in the plane of A are measured according to a modified rule, a Pythagoras's theorem with a minus instead of a plus:

$$OE^2 = t^2 - x^2 \tag{20}$$

which we know (17) gives exactly 1!

We have reached another important point, in Special Relativity distances in space-time are measured with a sort of modified

Pythagoras' Theorem: the sum of squares is substituted by the difference. The physical meaning of this length is a time squared. The time required to the cover the segment OE of the WL, measured in the RS of the observer moving along the segment. The new distance[15] among points in space-time is called the Minkowski's distance[16].

Minkowski Distance from the Origin $= t^2 - x^2$

The curve $t^2 - x^2 = 1$ is an hyperbola and represents all the points whose Minkowski's distance from the origin is equal to 1. Being at equal distance from a point, in the Euclidean geometry means circumference. In the Minkowski's geometry the same concept leads to an hyperbola. The length of the segment connecting each point of the hyperbola with the origin represents 1 second for the observer who is moving along the segment.

[15] Also if $t^2 - x^2$ is the squared distance, for brevity we will call it the distance.

[16] We have considered the distance of a point E from the origin of the space-time plane. But our conclusions are valid also for distances among two generic points, the only difference being using Δx and Δt instead of x and t.

Length contraction

The length of a ruler is measured by an observer O who sees the ruler passing in front of him with speed v (Fig. 14).

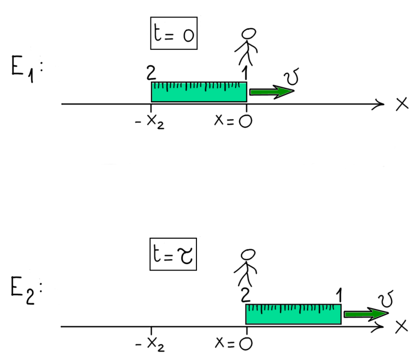


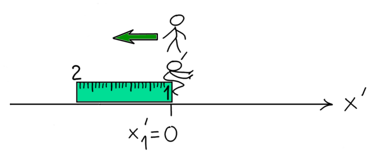
Figure 14: Length of a ruler passing in front of an observer.

O waits for the two ends of the ruler to pass in front of him and measures the time interval between these two events (Fig. 14). From this time interval he will compute the length of the ruler simply multiplying it by the speed of the ruler. The end number 1 is in front of him in $t = 0$, so the event E_1 is $(t = 0, x = 0)$. According to his clock the end number 2 passes after τ seconds, so E_2 is $(\tau, 0)$. The two events take place exactly where the observer O is, in other words they lay on his WL, so the time separation he measures is a *proper time* (this justifies the choice of the symbol τ). For him the length of the ruler is:

$$L = \tau \cdot v\,. \tag{21}$$

e Now we imagine a second observer sitting on the ruler, for example on the end number 1. He will measure with his clock the time separation between the same two events E_1 and E_2.

Figure 15: The observer sitting on the ruler.



However his point of view is different, he will see O moving at speed $-v$ towards him and he will measure the time taken by the observer O to "run" along the ruler. As we have learned by time dilation he is not at rest with the clock of O so his time will flow faster. The dilated interval will be $\gamma\tau$, therefore for him the length L' of the ruler will be bigger than L:

$$L' = \gamma\tau \cdot v\,. \tag{22}$$

In figure 16 are represented the WLs of the two ends of the ruler in the RS of the observer O. The WL of the end number 1 is also the WL of the observer sitting on the ruler, hence it represents its time axis t'. The two events are on the time axis of O, so his time interval is a proper time interval τ. The corresponding time measured by O' is marked with the red dot on his time axis.

We may conclude that

> an observer who sees a moving ruler will ascribe to it a smaller length, L, with respect to the length L' ascribed by an observer at rest with the ruler itself.
>
> $$L = \frac{L'}{\gamma} \tag{23}$$

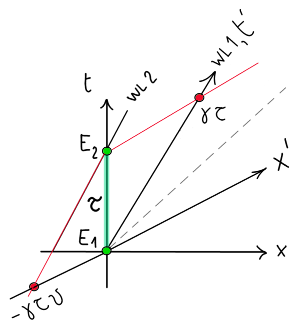


Figure 16: World Lines of the ends of the ruler in the RS of O.

Who is afraid of the Lorentz transforms?

In this section we present a fast way to derive the Lorentz transforms basing our reasoning on what we learned so far. We will start having a look to the Galilean transforms between two different inertial observers, these transforms will inspire us in defining a general form for the relativistic one. Then, just using what we know about time dilation, we will evaluate the parameters of the general relativistic transforms we are looking for. At the end we will test our equations to see if they respect the light speed invariance postulate. If yes, we will accept them.

Here we have Galilean equations connecting space-time coordinates of two inertial observes (O and O' which is moving at speed v with respect to O):

$$\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned} \tag{24}$$

Is a set of linear equations involving both time and space coordinates. Now we assume that the relativistic transforms are also linear. One reason to adopt this assumption is that we would like the relativistic transforms to be a more general case of the Galilean one for some values of the parameters. Therefore we are looking for transforms like these:

$$\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned} \tag{25}$$

In order to find the correct values of the four parameters A, B, C, D, we will consider two physical situations that, to some extent, we already know how to handle .

1. In the RS of O we consider a led in $x = 0$ blinking every τ seconds. We focus on the blink event $(\tau, 0)$ (Fig. 17, top plot). Now we substitute the event coordinates in the generic transforms and see

what are the corresponding coordinates for the observer O'.

$$\begin{aligned} x' &= A \cdot 0 + B\tau \ \rightarrow \quad x' = B\tau \\ t' &= C \cdot 0 + D\tau \ \rightarrow \quad t' = D\tau \end{aligned} \tag{26}$$

From the second equation and our knowledge about time dilation we immediately see that $\boxed{D = \gamma}$. Dividing both sides of the first equation by t' and considering that x'/t' is $-v$, the speed of the led from the O' point of view, we get

$$\begin{aligned} \tfrac{x'}{t'} &= B\tfrac{\tau}{t'} \rightarrow \\ -v &= B\gamma^{-1} \end{aligned} \tag{27}$$

Thus $\boxed{B = -\gamma v}$.

2. Now we move in the RS of O' and consider the same led in $x' = 0$, blinking every τ seconds according to his clock. Again we deal with the event $(\tau, 0)$ and see what are the coordinates t, x for the other observer (Fig. 17, bottom plot). Same technique: we substitute the coordinates t', x' of the event in our equations.

$$\begin{aligned} 0 &= Ax - \gamma vt \\ \tau &= Cx + \gamma t \end{aligned} \tag{28}$$

From the first equation we discover that $\boxed{A = \gamma}$. Dividing the second one by t we have

$$\frac{1}{\gamma} = Cv + \gamma \tag{29}$$

Since $1/\gamma - \gamma = -\gamma v^2$, $\boxed{C = -\gamma v}$.

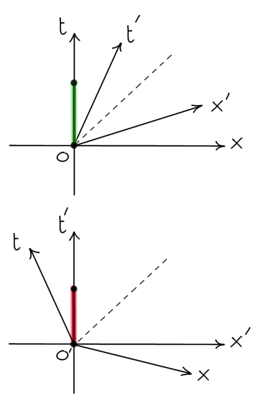


Figure 17: The two situations considered to derive the Lorentz Transforms.

We have the relativistic equations to transform the space-time coordinates of one inertial observer to the coordinates of any other inertial observer. In their beauty:

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx) \end{aligned} \tag{30}$$

Considering that in natural units $\beta = v$ we can also write them in the following way:

$$\begin{aligned} x' &= \gamma(x - \beta t) \\ t' &= \gamma(t - \beta x) \end{aligned} \tag{31}$$

We can almost hear your protests for the physical units that seem not to be correct. Do not worry about that, they are correct. In a following chapter we will face this issue once and for all.

Lorentz Transforms: units considerations

Both the Lorentz transforms we have written (30) and (31) are correct from the point of view of the units. Everything is fine thanks to the hidden speed of light, which is 1, and, for this reason, does not appear. Performing the same calculation we did before writing ct instead of t, and v^2/c^2 inside each γ factor we would detect where the speed of light has to be. However repeating the evaluation with c and c^2 flying from one side to another is not particularly uplifting. So we can also just apply some dimensional common sense to discover that equations (30) if not expressed in natural units would become:

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - \tfrac{v}{c^2}x) \end{aligned} \tag{32}$$

while equations (31) would be:

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ t' &= \gamma(ct - \beta x) \end{aligned} \tag{33}$$

We checked for you, believe us. Save energies for the physical meaning of this beautiful theory.

Testing the Lorentz Transforms

We want to be sure that the transforms wee have derived are a good set of equations for Special Relativity. We want they to respect the light speed invariance postulate. Doing this test is quite easy. We just consider the equation of the WL of a photon in the RS of an observer O. The WL is represented by $x = t$, now we see how it transforms in the RS of another inertial observer O' moving with any speed v with respect to O. Therefore, in the transforms (31) we substitute x with t. This is what we get:

$$\begin{aligned} x' &= \gamma(t - \beta t) \\ t' &= \gamma(t - \beta t) \end{aligned} \tag{34}$$

The ratio between the two equations gives

$$\frac{x'}{t'} = 1 \tag{35}$$

So we can conclude that the equation of the WL of the same photon in the new RS is $x' = t'$, meaning that the photon still moves with the same speed 1. For us is enough, these equation represent the fundamental equations of the Special Relativity. They are the *Lorentz Transforms*.

We apply the *Lorentz Transforms* every time we want to move from one inertial observer to another. The fundamental quantity to perform this change of coordinates is the speed v the second observer is moving with respect to the first.

Lorentz Transforms and Rotations in the Minkowski's space-time

Lorentz Transforms switch our point of view from one inertial observer to another one moving at speed v with respect to the first. We already know that when this action is performed the space-time coordinates of the events change, while the distance $t^2 - x^2$ remains the same for the two observers. In this sense Lorentz transforms act in the same way rotations act in the Euclidean plane (x, y)!

Rotations in the Euclidean space (x, y):
Coordinates (x, y) change
Distances $D^2 = x^2 + y^2$ are invariants

Lorentz transforms in the Minkowski (t, x)
Coordinates (t, x) change
Distances $D^2 = t^2 - x^2$ are invariants

A point that is 1 from the origin in the Euclidean plane will remain 1 from the origin after a rotation: it will change coordinates while remaining on the unit circumference $x^2 + y^2 = 1$. The same happens for the Lorentz transforms in the Minkowski space-time: an event 1 from the origin will change coordinates while remaining on the hyperbola $t^2 - x^2 = 1$.

The coordinates of a point on the Euclidean unit circumference are $(cos\,\theta, sin\,\theta)$ and rotations are performed by means of coefficients related to these coordinates. To rotate a point (x, y) counterclockwise of an angle θ we have to apply these transforms:

$$\begin{aligned} x' &= x \cdot cos\,\theta - y \cdot sin\,\theta \\ y' &= x \cdot sin\,\theta + y \cdot cos\,\theta \end{aligned} \tag{36}$$

The coordinates of a point on the unit hyperbola in the Minkowski plane are $(cosh\,\Phi, sinh\,\Phi)$ [17]. Therefore we may expect that something similar should apply to the Lorentz transforms. This is a sort of prediction, we are driven to think that the coefficients of the Lorentz Transforms (31) (i.e., γ and $-\gamma\beta$) should be expressed by means of the hyperbolic trigonometric functions $cosh\,\Phi$, $sinh\,\Phi$. This is exactly what happens! In fact it is not difficult to prove that

$$(\gamma)^2 - (\gamma\beta)^2 = 1 \tag{37}$$

exactly as $(cosh\,\Phi)^2 - (sinh\,\Phi)^2 = 1$. As expected, we can establish a connection between the coefficients of the Lorentz Transform and the hyperbolic functions[18]:

[17] Remember the fundamental relation for the hyperbolic functions: $cosh^2\Phi - sinh^2\Phi = 1$.

[18] It is important to have in mind the relation

$$cosh\,\Phi = \frac{1}{\sqrt{1 - tanh^2\Phi}} \tag{38}$$

which helps in noticing the connection between γ and the hyperbolic cosine.

$$\begin{aligned} \gamma &= cosh\,\phi \\ \gamma\beta &= sinh\,\phi \\ \beta &= tanh\,\phi \end{aligned} \tag{39}$$

These considerations do not only show an intrinsic mathematical elegance and beauty of the theory, but also play a key role in shaping our way of thinking. We try to clarify the new point of view in the following scheme.

A *Lorentz transform* involves a change from one inertial observer to another one.

A *rotation* changes the coordinates of any point, or any vector, without modifying its distance from the origin, or its length.

A *Lorentz transform* is a *rotation* in the Minkowski's space-time.

It is important to point out that a rotation changes the components of any vector according to the same coefficients. In other words any vector is rotated applying the set of equations (36). This is a very trivial observation if we are considering the Euclidean plane, but, based on our experience, becomes somehow not straightforward in the Minkowski's space. So we are going to explicitly state that the same is true for the Lorentz transforms. They act on any vector in the same way, not only on the four position $\overset{4}{x}$. The four-velocity $\overset{4}{v}$, for example, transforms in the same way the four-position does. If we call the first component of the four velocity v_0 and the second v_1 these components transform in the usual Lorentz way:

$$\begin{aligned} v_0' &= \gamma(v_0 - \beta v_1) \\ v_1' &= \gamma(v_1 - \beta v_0). \end{aligned} \tag{40}$$

Another crucial aspect concerns the modulus of the four-vectors. In the Euclidean space given a generic vector $\vec{a} = (a_x, a_y)$ its square modulus is obtained with the Pythagoras theorem:

$$a^2 = a_x^2 + a_y^2 \quad (41)$$

In the Minkowski's space-time a generic four-vector $\overset{4}{a} = (a_t, a_x)$ has a first component which we call the temporal component, a second component which we call the spatial component and its square modulus is given by the modified Pythagoras theorem:

$$a^2 = a_t^2 - a_x^2\,. \quad (42)$$

The components of a four-vector have no absolute physical meaning, they Lorentz transform when we change the inertial observer. The only quantities with an absolute physical meaning are the Minkowski's lengths of the vectors!

We already know what the length of the four-position represents, in the section dedicated to the relativistic dynamics we will learn something unexpected about the modulus of a vector very close to the four-velocity: the *four-momentum*.

Relativistic Dynamics

Survey about the teaching of the four-momentum in the Italian upper secondary school

Time dilation and lengths contraction are fascinating and fundamental aspects of Special Relativity, however it is our opinion that there are other important parts of the theory. In order to give an overall view of Special Relativity two more key concepts have to be discussed in some detail: *Invariant quantities* and the physics behind the *four-momentum*. Interestingly, a lot of Italian high-school textbooks focus all their efforts in the first part of the theory (time dilation, length contraction, speed addition formula) coming with little energy and few ideas to the second one. In this section we will present the results of a survey about the teaching of the four-momentum in the Italian secondary school conducted on 35 teachers spread over the Italian territory.

Teaching Special Relativity is mandatory in the last year (student's age 18-19) of the Italian scientific Lyceum (upper secondary school specialised in science studies). From the survey it emerges that on average teachers dedicate 10 hours to the teaching of Special

Relativity (Fig. 18) with a standard deviation of about 8 hours (Fig. 4).

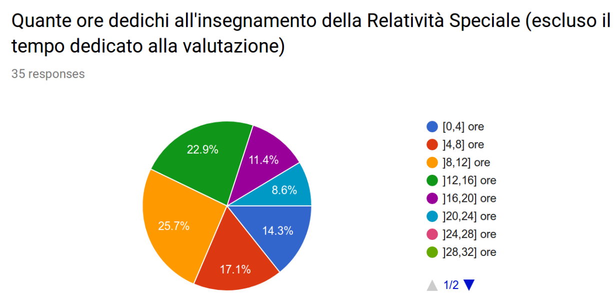


Figure 18: The question: How many hours do you spend to teach Special Relativity? (excluding the time for the assessment).

It is also clear that that nearly 89% of the teachers do not distribute the teaching of the modern concept of space-time over the last three years of high-school. They teach Special Relativity in "*one shot*, 10 hours, nearly one month of school (Fig, 19).

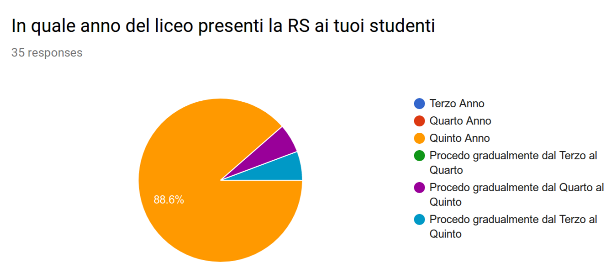


Figure 19: The question: In which school year do you introduce Special Relativity?. Possible answers: 3rd year | 4th year | 5th year (last year of high-school) | gradually from the 3rd to the 4th | gradually from the 4th to the 5th | gradually from the 3rd to the 5th.

When asked specifically about the teaching of the relativistic momentum, 74.3% of the teachers answer that they introduce it to their students (Fig. 20).

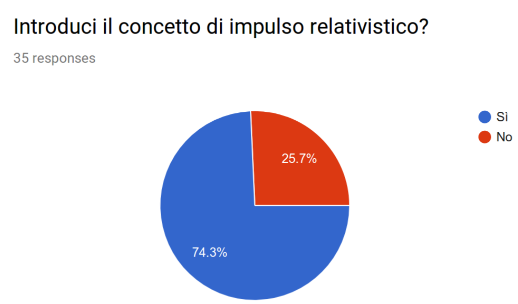


Figure 20: The question: Do you introduce to your students the relativistic momentum? Answers: Blue-yes, Red-No.

However the 79.4% admits that there is no clear explanation about the relativistic four-momentum, which is a given quantity whose origins are somewhat obscure. In the previous section dedicated to the *4-velocity* we have taken a decisive step towards the definition of the *4-momentum*. In the following one we will introduce it and start to analyse its physical meaning. As we will clearly understand going

Figure 21: The question: Analysing some textbooks we noticed that the mathematical expression of the relativistic momentum is not derived. Do you agree with this? Answers: Blue-yes, Red-No.

on with the reading of this chapter, the *4-momentum* is a fundamental quantity to grasp a lot of fundamental ideas behind the current research in physics. Here we provide a short list of key concepts that cannot be well understood without having the meaning of the *4-momentum* well in mind.

- How is it possible to produce heavy particles from collisions of light particles.
- Modern interpretation of interactions (force carriers or messenger particles).
- Why some interaction channels among particles become possible only when a certain energy threshold is exceeded.
- The concept of resonance in particle physics and the understanding of the beautiful plot from CMS (Compact Muon Solenoid experiment, CERN) (Fig. 22).

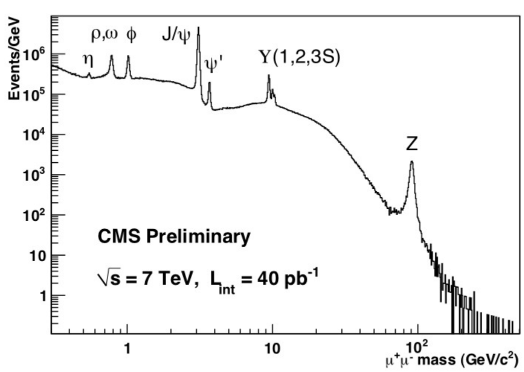


Figure 22: Di-muon mass from proton-proton collisions in LHC (CMS Experiment).

All the items of this list will be addressed and deeply discussed in the following sections and chapters. Furthermore on line resources, materials and interactive application will be presented and given to the teacher as a support for the teaching.

The 4-momentum

Once we have defined a 4-velocity in space-time the way to the 4-momentum is straight. Again we can act according to the analogy with the Galilean physics: simply multiply the 4-velocity by the mass of a particle. Special Relativity is an extension of Galilean physics, same attitudes and behaviours: in the Galilean space-time we multiply $\overset{4}{v}_{Gal.}$ to obtain the 4-momentum, the same we do in the relativistic world. Therefore the relativistic 4-momentum is[19]:

$$\overset{4}{p} = (m\gamma, m\gamma v)\,. \tag{43}$$

The meaning of the time component p_0 and of the spatial component p_1 of this 4-vector is still not clear, however we can start considering that again they depend on the observer. Since the γ factor affects both of them, they depend on the speed v the particle is moving with respect to the observer. Now we can list a number of general aspects we already know about this new 4-vector.

1. We know how it transforms under a change of inertial observer, i.e. under a Lorentz transform. We know we are performing a rotation and that all the 4-vectors in the Minkowski space-time will rotate in the same way:

$$\begin{aligned} p'_0 &= \gamma(p_0 - \beta p_1) \\ p'_1 &= \gamma(p_1 - \beta p_0). \end{aligned} \tag{44}$$

2. We know how to evaluate its length and that its square modulus has an absolute physical meaning:

$$\overset{4}{p}{}^2 = p_0^2 - p_1^2 \tag{45}$$

Thus, first of all, we are going to evaluate the square modulus of the 4-momentum.

$$\begin{aligned} &\overset{4}{p}{}^2 = m^2 \cdot (\gamma^2 - (\gamma v)^2) \\ &\text{and for the relation (37)} \\ &\overset{4}{p}{}^2 = m^2\,. \end{aligned} \tag{46}$$

[20,21]This is remarkable:

> The square modulus of the 4-momentum is the square mass of the physical system:
>
> $$\overset{4}{p}{}^2 = m^2\,. \tag{47}$$
>
> This means that in Special Relativity the mass m of a system is an *invariant* ! It does not depend on the observer and, since the velocity of the system depends on the observer, it does not depend on the velocity of the system!

[19] $\overset{4}{p} = (m\gamma c, m\gamma v)$.

[20] Actually there is another way to reach the same result taking full advantage of the invariance of the modulus. We can evaluate $\overset{4}{p}{}^2$ in whatever reference system we choose, the best is the one at rest with respect to the particle, in this way $v = 0$, $\gamma = 1$ and as a consequence the result is m^2.

[21] In usual units: $\overset{4}{p}{}^2 = m^2c^2$.

In other words, do not think that the mass is a function of the speed. We will come back to this in a later section.

$$m \quad \text{IS NOT} \quad m(v) \quad . \tag{48}$$

Another important aspect is that the relation (47) connects a kinetic quantity like the momentum to the mass.

The time component of the 4-momentum

In order to understand the physical meaning of the time component $p_0 = \frac{m}{\sqrt{1-v^2}}$ we have to do some mathematics. We have an uncomfortable $(1-v^2)^{-1/2}$ and we want to transform it in some more friendly mathematical expression. While this operation is quite easy using derivatives, we will follow a way involving mathematical skills as low as possible.

Math Box

We begin noticing that

$$\begin{aligned} &(1-x)^2 = \mathbf{1} - \mathbf{2x} + x^2 \\ &(1-x)^3 = \mathbf{1} - \mathbf{3x} + 3x^2 + x^3 \\ &\ldots \end{aligned} \tag{49}$$

This sequence of identities can be generalised in this way:

$$\begin{aligned} (1-x)^n = \mathbf{1} - \mathbf{nx} \quad + \\ + \text{(higher order terms in x)} \end{aligned} \tag{50}$$

Our $\gamma = (1-v^2)^{-1/2}$ factor belongs to the category of the functions $f(x) = (1-x)^n$, we just have to declare the following correspondences: $x \to v^2$ and $n \to -1/2$. Therefore we apply the rule (50):

$$p_0 = m\left(1 + \frac{1}{2}v^2 + (\text{high}\, v^2)\right) \tag{51}$$

and we write p_0 as follow

$$p_0 = m + \left(\frac{1}{2}mv^2 + m \cdot \text{high}(v^2)\right). \tag{52}$$

We have to say that each term on the right hand side has momentum units, thus being an energy divided by a speed. In fact if we point out the presence of the speed of light, the terms on the right hand side would be $mc + c \cdot (\frac{1}{2}m\frac{v^2}{c^2} + \ldots)$. From the (52) we can reach three conclusions:

1. The terms in parentheses have the units of an energy divided by c and depend on v, thus they represent the *relativistic kinetic energy* T (of course divided by c).
2. Also the first term must be an energy divided by c, a form of energy coming from the mass!
3. p_0 is a sum of energies divided by c, thus from now on we will represent it with this symbol: E/c.

However, as physicists do, we work in natural units, so instead of E/c, T/c or mc we will write E, T, m thus writing the (52) in a very simple form:

$$E = m + T\,. \tag{53}$$

Summarising[22]:

[22] $p_0 = \frac{E}{c}$
$\frac{E}{c} = mc + \frac{T}{c}$ or
$E = mc^2 + T$.

> The time component of the 4-momentum is
>
> $$p_0 = E \tag{54}$$
>
> E is the *relativistic energy* of a system.
>
> $$E = m + T \tag{55}$$
>
> T is the *relativistic kinetic energy*. For low speeds T becomes the classical kinetic energy.

The "restyled" version of the 4-momentum and the Energy Relation

We are quite close to the "restyled" version of the 4-momentum. We know that the first component is E; the second component $m\gamma v$ is called the *relativistic momentum* and denoted with the symbol p. Therefore we have:

$$\text{Old fashion:} \quad \overset{4}{p} = (m\gamma, m\gamma v)$$

> $$\text{Restyled version:} \quad \overset{4}{\mathbf{p}} = (\mathbf{E}, \mathbf{p}). \tag{56}$$

Furthermore, the invariant square modulus of $\overset{4}{p}$ is m^2 and we have the fundamental *energy relation* in Relativity:

> $$m^2 = E^2 - p^2$$
>
> or better,
>
> $$\mathbf{E^2} = \mathbf{m^2} + \mathbf{p^2} \tag{57}$$

Mass and momentum in Relativity can be considered as different form of energy, the sum of their squares adds up to the total square energy of the system! Part of the energy comes from matter, part from a kinetic quantity: the momentum.

Both the components of the 4-momentum (E, p) represent energy forms which depend on the observer. In this sense energy and momentum have no absolute physical value.

> The only form of energy with absolute connotations a system has at its disposal is the mass m.

There is the question of units. If energy is expressed in eV, the momentum has to be expressed in eV/c and the mass in eV/c^2. But both nuclear and particle physics physicists work in natural units ($c = 1$) thus energies, masses and momenta are measured in eV (Fig. 23) and the energy relation is the (57) instead of $E^2 = m^2c^4 + p^2c^2$.

Figure 23: From *The Review of Particle Physics* of the Particle Data Group.

The geometrical interpretation of the Energy Relation

The relativistic energy relation lends itself to an immediate geometrical and visual interpretation which may be a useful tool for teacher and can work in parallel with the usual algebraic approach (and in some circumstances can even substitute it).

The equation (55) gives us the chance to build a segment with length E and divide it in two parts: one with length T and the other with length m. (Fig. 24 left).

We can now build two squares, one over the segment E, whose area represents the term E^2 of equation (57) and one over the segment m, whose area represents the term m^2 (Fig. 24 right). With a compass we draw a semicircle with diameter equal to E and a circle

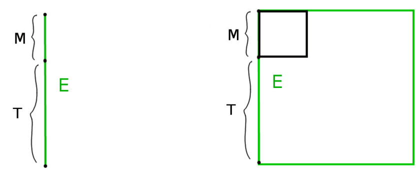


Figure 24: First stage of the geometrical model of the energy relation.

with radius m (Fig. 25). From their point of intersection we trace a chord of the semicircle with length m (Fig. 26).

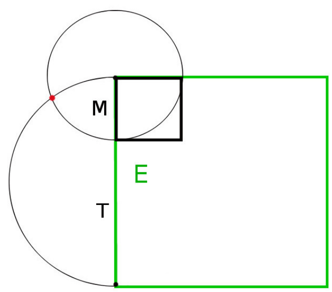


Figure 25: Drawing the two circles.

The inscribed triangle is a right triangle and for it the Pythagoras' theorem $E^2 = m^2 + p^2$ will surely hold! Therefore the purple segment in figure 26 represents the modulus of the momentum p.

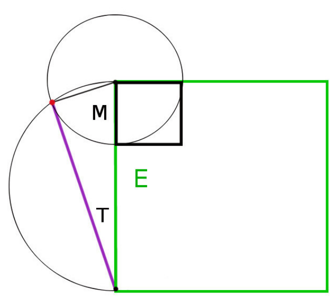


Figure 26: The Energy Square.

The on-line Application for the Energy Relation, and how it works.

We will call the geometrical approach to the energy relation the Energy Square method (ES). This method was implemented in a Geogebra[23] application (ESa) and published on web site. The site name is **PhE**[24] (which stands for Physics Education), it was built to test some of the educational tools developed during our research. Figure 27 shows how the ESa looks like.

[23] Markus Hohenwarter et al. *Geogebra*. https://www.geogebra.org/

[24] Lorenzo Galante. *PhE*. https://sites.google.com/view/physedu/

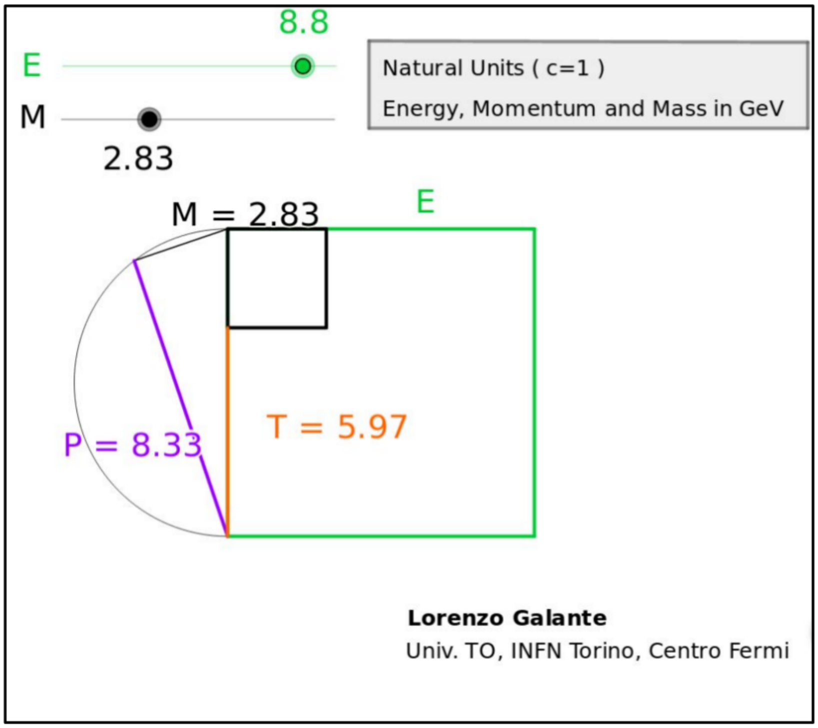


Figure 27: The Energy Square application (ESa).

Moving the two sliders students can change the mass and the energy of the system they want to simulate. The geometric construction follows in real time the choices of the user, showing the areas representing m^2, E^2. The difference between this two areas is easy to visualise and represents p^2. In the next section we will give some ideas about how to use this educational tool in a classroom.

How to use the ESa in a classroom

The ESa may be used as a learning environment where to make students become familiar with the new relativistic vision of energy. But, and this is probably the most interesting feature, it can also be a useful tool to activate inquiry processes in the learning scenario. With the ESa students can explore the relativistic energy relation, discover hidden meanings and reach interesting results by their own . Here we give some examples, a list of activities and explorations that can be proposed to a classroom.

TUTORIAL on the Energy Relation

- **Becoming familiar with E, T, m, p.**

  1. With the ESa simulate a proton ($m \sim 0.9\, GeV$) with energy

$E = 9\,GeV$. In the geometric representation you get identify the area corresponding to E^2, m^2 and p^2. Assess the significance of the area m^2 and p^2 with respect to E^2.

2. You have a monatomic Helium gas at room temperature, evaluate the kinetic energy of one He nucleus and convert it in eV. Now try to show that the momentum of the nucleus is negligible if compared to the momentum of a pion colliding with $p = 217\,MeV$[25].

[25] Momentum units should be MeV/c, in natural units we can write MeV.

- **The ratio p/E.**

1. *The domain of the ratio p/E.* Working with the ESa define the domain of the ratio p/E.
2. *Increasing mass.* Starting from the configuration of a proton with $E = 9\,GeV$ move the mass slider up to the maximum allowed value. Observe what happens while you move the slider and reflect on what is going on. What happens when the mass reaches the maximum allowed value? The particle has changed? What is its speed?
3. *Decreasing mass.* Starting with the same proton of the previous activity, move the mass slider towards zero. Observe what happens while you move the slider. Which particle do you have when you reach zero? What is the speed in natural units of the particle you got? What is the value of the ratio P/E for a massless particle?
4. *The physical meaning of the ratio p/E.* On the basis of the previous considerations could you give a physical meaning to the ratio p/E

- **From the geometric model to the break-up with classical physics.**

1. In your opinion is there a substantial difference among a situation in which the mass approaches zero and one in which the mass is fixed and finite while the energy E approaches infinity?
2. On the basis of your answer to the previous question, what is the speed of a particle with finite mass and infinite energy?
3. Would you expect such a result from the classical physics laws?

We would emphasise that in the activity about the ratio p/E students can discover its physical meaning (which is β or v in natural units) also if they are not aware of the mathematical expression of the 4-momentum $(m\gamma, m\gamma v)$. This means that there is the possibility to introduce the relativistic dynamics starting from the energy relation (without deriving it) and working with the ESa. While this educational approach falls short for the lack of completeness and full derivation of all the quantities, it is extremely agile and can lead students to understand the basic of the relativistic dynamics and also to analyse particle interaction events. In this sense it could be a good strategy for a teacher who wants to introduce some particle physics concepts while having to face strong time limits. The agile approach was used and tested during the 30 hours course *Painuc in the Classroom* designed to introduce students, aged from 16 to 18, to the analysis of nuclear collisions of pions against Helium nuclei. The focus of the course was not the theory of Special Relativity, even if some relativistic concept was necessary in order to have the minimum "knowledge kit" to conduct the analysis.

The mass of a system of two particles

Suppose we have two particles. We have seen that a 4-vector $(E, \vec{p})$ may be associated to each one of them. So the particle 1 has energy E_1 and momentum $\vec{p}_1$ and the particle 2 energy E_2 and momentum $\vec{p}_2$. They may be considered as a system with a total energy given by the sum of the energies and a total momentum given by the vectorial sum of the two momenta:

$$\begin{aligned} E_{tot} &= E_1 + E_2 \\ \vec{p}_{tot} &= \vec{p}_1 + \vec{p}_2 \end{aligned} \tag{58}$$

Therefore just like a single particle may be described by a 4-momentum a system is defined by

$$(E_{tot},\, \vec{p}_{tot})\,. \tag{59}$$

As always, both the energy component and the momentum will depend on the observer, but the difference among the square of the components (E^2 - $\vec{p}^2$) will be an *invariant*. As we know this invariant represents the energy of the system that does not depend on the observer. We will call it the *mass* of the system: the energy of the system having an absolute connotation.

$$m^2 = E^{\,2}_{tot} - \vec{p}^{\,2}_{tot} \tag{60}$$

In the following sections this *mass* will be discussed in some detail with two different educational approaches. A more usual method that proceeds through the mathematical analysis of this quantity and a method based on an environment in which student, guided by a tutorial, can explore the concept and reach important results with their own reasoning. We will start presenting the second method, which is the more interesting from the educational point of view.

Investigating the mass of a system of particles with the application ESa2

In order to build an environment where to explore the mass of a system of two particles an interactive Geogebra application called ESa2 was designed (Fig. 28).

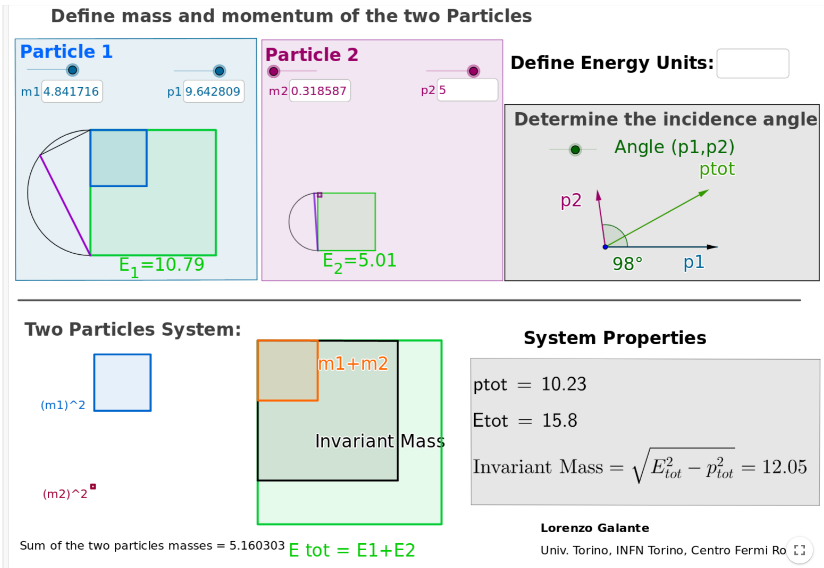


Figure 28: The ESa2 application.

ESa2 is available on the site **PhE**[26]. The table below shows what the ESa2 application can do.

[26] Lorenzo Galante. *PhE*. https://sites.google.com/view/physedu/

POSITION on the DISPLAY	FUNCTION
Top Left (blue box)	Defines m_1 and p_1 of the 1^{st} particle
	Shows the Particle 1 Energy Squares
Top Centre (purple box)	Defines m_2 and p_2 of the 2^{nd} particle
	Shows the Particle 2 Energy Squares
Top Right (unit box)	Defines Energy Units
Top Right (grey box)	Defines the angle between $\vec{p}_1$ and $\vec{p}_2$
	Shows the vectors $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_{tot}$
Bottom Left	Shows the single particle mass square
Bottom Centre	Shows the mass $m_1 + m_2$ (orange square)
Bottom Centre	Shows the system mass (black square)
Bottom Centre	Shows the total energy (green square)
Bottom Right (grey box)	Shows p_{tot}, E_{tot} and system mass

In this section we present a tutorial, partly based on the ESa2, proposed to a classroom of 20 students. It will show the educational path we have asked our students to follow. In the next section we will describe in some details how the teacher could use the tutorial in order to give to entire classroom a common and correct knowledge. The aims of the activities were:

- Present the Esa2 application.
- Become familiar with the mass of a system of particles.
- Learn how to evaluate the mass of a system of particles.
- Understand that for a system of particles energy coming from the momentum is transferred to the mass of the system.
- Become familiar with the *centre of mass* reference system as the RS in which $\vec{p}_{tot} = 0$.

Students were given a tutorial and were asked to work in small groups (2 ore 3 students). The tutorial guided them through a series of situations pushing them to explore fundamental topics related to the mass of a system of two particles. A certain amount of time was assigned for every step of the tutorial, then a plenary session was supervised by the instructor in order to collect the main results coming from the groups and to define a common and verified knowledge.

TUTORIAL on the Mass of a System of 2 particles

1. **The mass of two electrons.**

   We start considering a system composed by two electrons ($m \sim 0.5\,MeV$) having a momentum of $5\,MeV$ and $1\,MeV$ respectively. For now the angle between the two momenta is not important, we fix it to 180 degrees.

   (a) *Become familiar with the ESa2 application*. Open the Esa2 application and define your energy units in the unit box. With the keyboard or using the sliders define masses and momenta of the two particles, then set the angle between the momenta. Esa2 evaluates the total energy E_{tot} of the system (adds the energies of the two electrons) and the total momentum $\vec{p}_{tot}$ (shown as a green vector, vectorial sum of the two momenta). Then it evaluates the mass of the system according to Special Relativity rules: $\sqrt{E_{tot}^2 - \vec{p}_{tot}^{\,2}}$. In the bottom part of the application you will see a green square whose edge is $E_{tot} = E_1 + E_2$. Inside the E_{tot} square there is a black square, whose edge is the two-particle system mass, and an orange one representing the simple sum $m_1 + m_2$ of the two masses.

   (b) Think about the relativistic mass of the two electrons and compare it with the sum of the two electron masses.

(c) Try to evaluate by your self the system mass of the two electrons. (Hint: you are working in natural units, so you can express m, p and E in MeV). Compare your result with the value given by ESa2.

2. **Where does the mass come from?**

   The previous step made us think about the value of the system mass which is bigger if compared to the sum of the masses of the particles composing the system itself. We may remember that the system mass represents the system energy that does not depend on the observer. The question is: where does the energy that increases the system mass come from?

3. **How can we use this aspect of Special Relativity?**

   We may ask how we can take advantage of this beautiful aspect of nature. We can create systems with a mass bigger than the sum of the masses of the parts "stealing" momentum from the particles of the system itself. How would you act with the two electrons to create something 4.7 times more massive than them?

4. **The Higgs and LHC**

   One of the main tasks of LHC was the detection of the Higgs boson. Search in the Internet for the mass of the Higgs boson. Simulate with ESa2 the collision between two LHC protons: consider the momentum of both equal to $1 TeV$, the direction is opposite. (mass of the proton $\sim 1\,GeV$). (Hint: use TeV units and insert the proton mass with the keyboard: $0.001\,TeV$). There is a chance to create particles like the Higgs in the p-p collisions of LHC?

5. **At the core of the energy of a system**

   The geometric situation of the two electrons of the task number 1 is shown in figure 29. We have already stated that E_{tot} is not a quantity that plays a role at the core of the energetic issue: it not an invariant. On the contrary the energetic core is represented by the mass of the system which is the same for every observer. Also the term p_{tot}, represented in figure 29 by the 'L' shaped green region depends on the observer. In your opinion does an observer for whom the term p_{tot} is zero exist?

6. **The Centre of Mass in particle interactions**

   Consider an interaction among two particles, 1 and 2, which gives as output two particles, 3 and 4. Is the Centre of Mass of the two bodies 1 and 2 (before the interaction) the same as the Centre of Mass of the bodies 3 and 4?

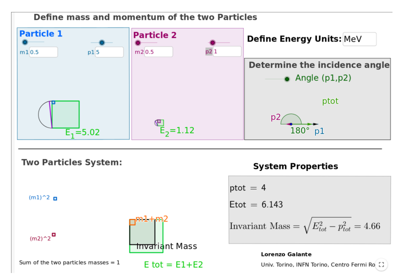


Figure 29:

7. **Mass of a system in particle interactions**

   Consider the same interaction described before, $(1, 2) \rightarrow (3, 4)$. Is the mass of the system (1, 2) the same as the mass of (3, 4)?

8. **Photons mass**

   With the application Esa2 verify that the mass of a system composed by two photons is not zero! We can have mass from massless particles.

9. **The importance of the angle**

   With the application ESa2 explore the role played by the angle between the momenta of the two particles in defining the mass of the system. Can your conclusions help in understanding why physicists build colliders designed to generate "head-on" collisions?

Comments on the system's mass tutorial

1. **The mass of two electrons.**

   In this activity a reflection on the mass of a two electron system is proposed. As shown in figure 29 students learn that the mass of the system can be bigger than the sum of the masses of the two particles. The teacher has the opportunity to underline the importance of the mass which represents the only invariant expression of the energy of the system. At this point there is the possibility to introduce the term energetic ' *core*':

   > We can call the mass of a system the energetic '*core*' of the system. It is the only energetic term with an absolute physical meaning. The energetic '*core*' of a system may be bigger than the sum of the '*cores*' of the single particles!

2. **Where does the mass come from?**

   The answer is: from the momenta p_1 and p_2 of the particles composing the system. If we put two particles together two bodies there is the possibility to 'transfer' energy from the momentum component to the mass. No doubts about this: energy is given by mass and momentum, if the system has more mass then expected the extra amount comes from the momentum drawer!

3. **How can we use this aspect of Special Relativity?**

   To bring out mass from the two electron system we need to make them interact. For example, with a collision. Something similar was done for a long time at CERN with the LEP (Large Electron

Positron collider), but in this case collision was among electrons and positrons. In this way the weak bosons W and Z, with masses around 80, 90 *GeV*, were detected. The starting point was two particles with a 0.5 MeV mass.

WHY ACCELERATORS - 1

> One reason why physicists build accelerators is the creation of higher mass products.

Making high momentum particles collide we can get collision products with higher mass extracting energy from the momentum. How much higher? At most equal to te mass of the system: $m = \sqrt{E_{tot}^2 - p_{tot}^2}$, which, as we have seen, is bigger than $m_1 + m_2$.
We have to say that this is not the only goal. Studying structures at smaller spatial scales is another fundamental task of the accelerators based experiments. However to understand this we need to wait for a later section and to put together Special Relativity and Quantum Mechanics. The topic will be addressed in a later section.

4. **The Higgs and LHC**

The discovery of the Higgs boson was one the major task of LHC. In fact the 125 *GeV* of the Higgs' mass are well below the 2 *TeV* of the mass of the system composed by the two LHC protons (each one with a momentum of 1 *TeV*). However we have to consider that in p-p collision a lot of fast particles are produced and each of them carries away momentum and mass. That is why to have a chance to produce an Higgs we need to exceed the mass of the boson of nearly one order of magnitude. Since one possible decay channel for the Higgs particle is the two γs decay, one way to discover the Higgs is looking for events with photon pairs whose mass is the Higgs mass (30)

A beautiful plot that may be considered the manifesto for the production of heavier particles is shown in figure 22. The plot comes from the CMS experiment (Compact Muon Solenoid), one of the main LHC experiments. In the preliminary test stage CMS has detected the main particles detected in decades of research. The plot is the result of p-p collision with a mass of 7 *TeV*, CMS made a selection of all the event with the production of muons and anti-muons and measured their momenta. The mass of the system muon, anti-muon was then evaluated for all the possible

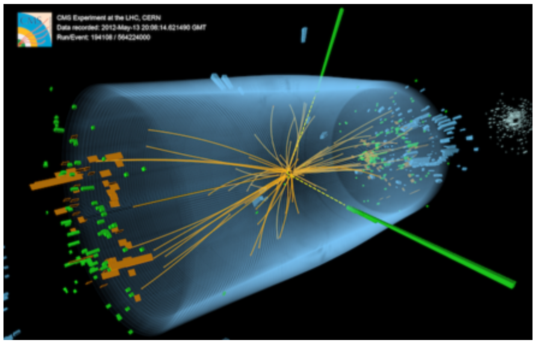


Figure 30: The two photons system (two dotted and straight yellow tracks) whose mass amounts to the Higgs mass. This is a candidate Higgs boson event from collisions between protons in the LHC. [CERN for the ATLAS and CMS Collaborations]

combinations and histogrammed in this plot. The peaks come from the muon anti-muon pairs produced in the decay of a neutral particle. From collision p-p all the neutral particles up to the Z boson were produced. Everything was made with two particles each one having a mass of 1 GeV. This plot is a wonderful proof of the validity of the Special Relativity and in particular of the mass of a system.

5. **At the core of the energy of a system**

   The *Centre of Mass* reference system is defined as the RS in which the total momentum p_{tot} is zero. This RS is very important in Special Relativity because in this reference frame the total energy E_{tot} is the mass of the system thus representing the energetic *'core'* of the system itself. The laboratory system of a collider, as LHC for example, is the Centre of Mass reference system.

6. **The Centre of Mass in particle interactions**

   Before or after an interaction the Centre of Mass reference system does not change. It is a direct consequence of the momentum conservation law: if $\vec{p}_{tot} = 0$ before the collision, it will also be zero after. The momenta $\vec{p}_3$ and $\vec{p}_4$ are different from those of the particles 1 and 2, but they still add up to zero.

   > If we are in the RS of the Centre of Mass before a collision we still are in the RS of the Centre of Mass after the collisions.

7. **Mass of a system in particle interactions**

   The mass of a system of is conserved in particle interactions. We know that the total momentum and the total energy are conserved, as a consequence also the mass (which is derived from both these quantities) will be conserved.

The Mass of a system is conserved in collisions.

The mass of a system from a theoretical point of view

As we have already seen the 4-momentum of a system composed by many bodies is:

$$\overset{4}{p} = (E_{tot} = E_1 + E_2 + \ldots \quad , \quad \vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 + \ldots) \tag{61}$$

Now we want to investigate from a mathematical point of view the meaning of its invariant modulus, which, as we already know, is the mass of the system. We can imagine a collision of two particles (named 1 and 2) that produces two other particles (3 and 4) in the final state. Since the value of 61 does not depend on the RS choice, it is convenient to choose the Centre of Mass frame where $\vec{p}_{tot} = 0$:

$$|\overset{4}{p}| = \sqrt{E_{tot}^2 - 0} = \sqrt{m_1^2 + p_{1cm}^2} + \sqrt{m_2^2 + p_{2cm}^2} \tag{62}$$

From the conservation laws of energy and momentum we have that also the mass of the system has to be conserved so the quantity (62) has to be equal to

$$|\overset{4}{p}| = \sqrt{m_3^2 + p_{3cm}^2} + \sqrt{m_4^2 + p_{4cm}^2} \tag{63}$$

KEY POINT!

$|\overset{4}{p}|$ is a conserved quantity: its value before the collision is equal to its value after the collision.

From these two equations we can recap and understand the meaning of the absolute quantity $|\overset{4}{p}|$:

1. It is an energy
2. It is an *invariant*, independent from the reference frame.
3. It is the mass of the system.
4. If the particles in the final state are produced at rest then $|\overset{4}{p}| = m_3 + m_4$. Thus $|\overset{4}{p}|$ represents the maximum energy of the system that can be completely converted in masses of the final products.
5. We have used the expression 'maximum energy' because if the products are not at rest then part of this energy goes in kinetic energy (momenta)

6. Thanks to the two momenta p_1, p_2 the value of $|\overset{4}{p}|$, the system mass, is bigger than the sum of the masses of the single parts of the system (62). This means that the momentum of the particles can be converted in mass of the reaction product. That is why colliders exist.

When a particle decays, it is at rest in its CM frame, so the invariant mass of the products is the mass of the decaying particle itself. This can help in understanding the beauty of the CMS plot (Fig. 22) commented in one of the previous sections.

A terminological issue

We have often used the term *Mass of the System*, that represents the *Relativistic* concept of mass (63), eg the highest mass of a particle that could be produced by the system in case it is produced at rest. From now on we will simply use the term *Mass*. This choice represents and requires from the learner an higher level of understanding, a sort of mutation in the way the concept is conceived.

The Mass and the GZK cut-off

Relativistic Dynamics is not only applied to particle physics. Here we give an example of how to use relativistic concepts to explain a fundamental aspect of Cosmic Ray physics. Cosmic Rays (CR) are particles, almost protons, originated by astrophysical sources like stars, supernovae, active galaxy nuclei, that impact our atmosphere with a wide spectrum of energies. The flux of CRs as a function of energy is shown in figure 31. In this section we are interested in

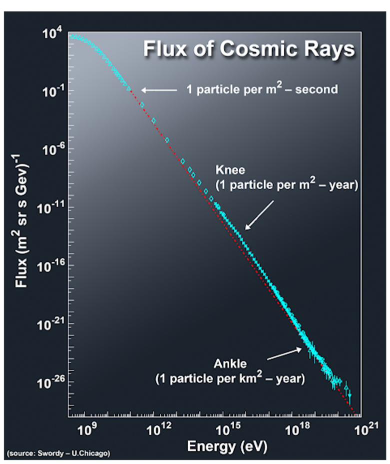


Figure 31: Cosmic ray Flux. (credit: HAP / A. Chantelauze)

a specific feature of this plot: the so called *GZK cutoff*, that is the sudden drop in the flux at energies exceeding $10^{20}eV$. The most accepted explanation for this experimental evidence comes from three scientists, Greisen, Zatepin and Kuzmin, whose initials (GZK) give the name to physical phenomenon. Their idea is based on the interaction of protons (which constitute the majority of the CRs) with the photons of the Cosmic Background Radiation (CBR):

$$p + \gamma_{CBR} \quad \rightarrow ... \tag{64}$$

The energy of the photon is fixed by the CBR temperature ($T \sim 2.7\,K$) and can be calculated considering the value of the Boltzmann constant (K= $8.617\,eV\,K^{-1}$). The energy of the proton is a variable. When the energy of the proton exceed a certain threshold, the mass of the system (p, γ_{CBR}) reaches the value of the mass of an excited state of the proton: the so called Δ *resonance (1232)* which immediately decays in a neutral pion and a proton. In other words when the energy of the proton exceeds a certain value this channel of reaction activates:

$$\begin{array}{rcl} p + \gamma_{CBR} & \rightarrow & \Delta(1232) \\ \Delta(1232) & \rightarrow & \pi_0 + p \end{array} \tag{65}$$

This possibility breaks down the flux of protons at energies exceeding the activation threshold, in fact those protons will have the chance to interact with the CBR photons thus splitting their energy among the two final products: the π_0 and the p.

A Geogebra[27] interactive application called GZKa was designed in order to introduce students to this phenomenon. Figure 32 shows the GZKa. With this application students have the opportunity to get in contact with the GZK cutoff also testing their understanding of the new concept of mass. The application works on the basis of numerical values given by the user. The Table further on summarises the actions and the numerical values required from the user in order to carry out the exploration: they can discover by themselves the value of the threshold.

[27] Markus Hohenwarter et al. *Geogebra*. https://www.geogebra.org/

The application, accordingly to the position of the proton's energy slider, returns in the green box the value of the mass of the (p, γ_{CBR}) system. A red vertical bar moves along the horizontal axis of the plot in the lower part of the application, representing the mass of the system. As the mass of the system increases the bar moves toward the right along the energy axis of the plot. In this way the user can directly discover when the GZK reaction is activated.

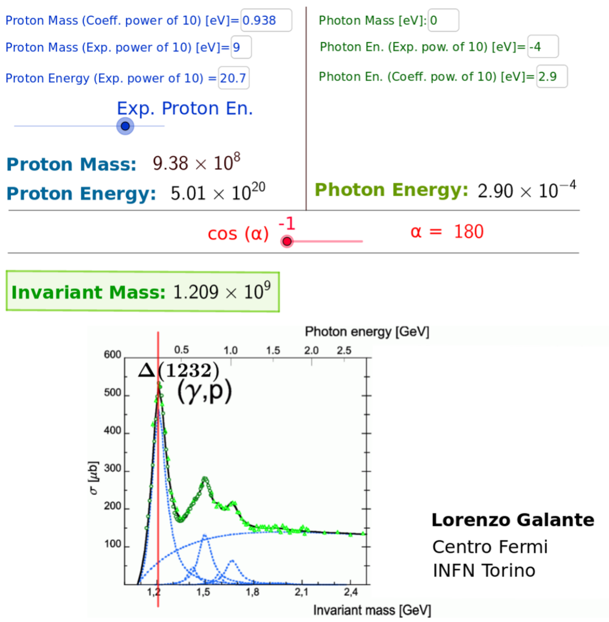


Figure 32: The GZKa application.

Position on the Display	User Action
Top Left (first two blue lines)	proton mass $m_p = a \cdot 10^b$ user have to insert a, b
Top Left (blue slider)	proton energy $E_p = 10^c$ user have to insert c
Top Right (first green line)	mass of the photon user have to insert 0
Top Right (2^{nd} and 3^{rd} green lines)	photon energy $E_\gamma = d * 10^e$ user have to insert d, e
Middle (red slider)	Angle between p and γ user have to choose among three angles (0, 90, 180)

A Learning Environment for the 3D Teaching of Special Relativity

We should have textbooks showing practical applications of the Theory of Special Relativity.
— Anonymous Italian physics teacher, From a Survey on the teaching of SR

It is our believe that the best way to understand a theory is to use it to interpret data coming from an experiment. If we do not try to design educational approaches or environments in which the contact between theory and experiment occurs right in front of the student's eyes, we loose a great opportunity to teach Physics in a proper way. Students are both the citizens of the present and of the future, they deserve a better understanding of science as well as of the scientific methods adopted by the current research. The all society needs young people to acquire cognitive skills and tools in order to give them the chance to build a personal and critical opinion about the choices and the applications deriving from the scientific research. Creating a place where students can analyse experimental data using their theoretical knowledge is what we call three-dimensional Teaching of Physics (3DTP) (pag. 8).

In the previous chapter we presented an innovative approach to the teaching of Special Relativity, in this section we present **ParPLE** (Particle Physics Learning Environment), our efforts to build a learning environment where students can use Special Relativity as a tool to understand how data from particle collisions are handled by physicists. In this way they can directly taste the flavour of the physics research in particle physics. ParPLE is a *mixed* learning environment. With 'mixed' we mean that a list of activities belonging to different categories can be carried out inside ParPLE.

ParPLE gives students and teachers the opportunity to perform the following tasks:

1. Access data from a real particle physics experiment (PAINUC[28]), directly analyse them using a C++ Root template and a small introductory course on Root.

2. Find interactive applications where to explore Physics concepts useful to understand and to analyse particle interactions[29].

3. Analyse collisions through interactive applications on selected pictures of emblematic events.

4. Perform on your desk low cost experiments that will introduce you to the physics of particle collisions.

[28] N. Angelov et al. *Two prong* π *-* ^4He *interactions at 106 MeV*. Eur.Phys. J. A 34, 255269, 2007

[29] The applications designed to better understand relativistic concept were deeply discussed in the previous chapter.

All the activities proposed inside ParPLE are designed to promote active learning processes and to stimulate inquiry skills. According to the available time the teacher will have the opportunity to choose the level of freedom for the students. In ParPLE he will find the possibility to arise questions and problems that student will investigate by their own using the tools available in the environment (student centred approach) as well as the opportunity to directly guide students along different paths to make them understand fundamental concepts of Special Relativity and particle physics (teacher centred approach). All the levels between this two extreme approaches will be available.

The ParPLE environment

The ParPLE environment consists of a series of activities that provide to the teacher educational paths along Special Relativity and Particle Physics. In this chapter we will describe all the activities following a logical scheme. However, as in any self-respecting environment, the teacher will have the opportunity to move in it with complete freedom, changing the ordering according to his preferences, using only parts of what is available and even designing new activities based on ParPLE. In the next chapter, dedicated to the test of these educational methodologies, we will give two practical examples of how ParPLe was used by two Italian teachers as well as examples of how the author of this work used ParPLE in courses for teachers and students.

The Gravitational Accelerator

The first step we have to move is to make collisions in more than one dimension crystal clear to students, in order to achieve this goal we need a collision maker, we need to bring an accelerator into the classroom. This seems to be a "mission impossible" but with a bit of creativity we can do it at the incredible cost of nearly 0.02 €. Of course, we will use the gravitational field instead of the electric one, and two euro cent coins will be our colliding particles. We will call the accelerating apparatus GA (Gravitational Accelerator).

Figure 33: The Gravitational Accelerator.

How to Build a GA.
The equipment consists of a carton box nearly 15 cm high, two A4 sheet of paper and two 1 euro cent coins. Figure 33 shows how to assemble the apparatus. One of the sheets, folded in two along its long edge, is placed on the box and acts as a slide offering to the experimenter always the same potential drop. The second A4 sheet of paper is the surface over which the coin glides after the gravitational fall. The coin falls along the paper slide starting from an at rest condition: the apparatus is a "gravitational accelerator" (GA) providing a mono-chromatic beam of coins. Two masses at the bottom of the slide (two rubbers, for example) secure the slide position so as to maintain the same configuration during the experimental phase.

Collisions with the GA.

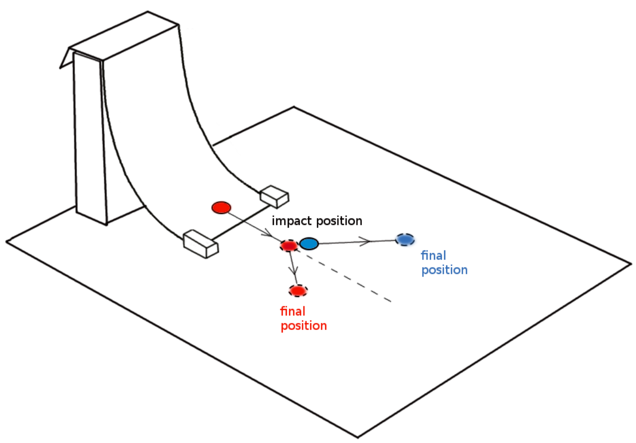


Figure 34: Collisions among coins with the GA.

The GA is a useful tool to carry out experiments on collisions with different values of the impact parameter. Basically what we have to do is to place a second coin on the A4 surface waiting for a collision with the accelerated one (Fig. 34). After the collision the two coins will come to a stop, therefore as we will discuss soon, we will have the opportunity to draw their tracks and, from the length of their trajectories, to determine some kinematic parameter.

Momentum conservation law with the GA. Theoretical considerations.

Analysing collisions generated with the GA we can quantitatively investigate the momentum conservation law in two dimensions. This is a very important step to be moved before entering the realm of particles collisions, since conservation laws hold also in particle physics. Furthermore, as we will see later in the direct analysis of one PAINUC event, the discovery of a missing momentum in an interaction among particles can lead us to the research for an undetected particle.

What we need is a way to measure the velocities of the coins one instant before and one instant after the impact. To find this way we may notice that a constant force F decelerates the coins while they slide over the horizontal A4 surface. F acts until the coins come to a stop, so the following relations hold:

$$FL = \frac{1}{2} \cdot m\, v^2 \tag{66}$$

$$v = \sqrt{\frac{2\,F\,L}{m}} \tag{67}$$

where L is the length of the track of the coin form the point of impact to the stopping point. We thus have a direct proportionality between the velocity v of the coins at the beginning of their ride over the horizontal surface and the square root of the length of the track L.

$$v \propto \sqrt{L} \tag{68}$$

A simple and wonderful result, since from the track of a coin we can extract both the direction and the magnitude of a vector which is proportional to the velocity of the coin.

Now we can go back to the momentum conservation law and see how it applies to the collisions provided by the GA. The sliding coin will start from an at rest condition at the top of the slide. The collision will be among twin coins (1 euro cent coins) so the masses will fade away from the conservation relation (eq. 69):

$$\vec{v}_{1\,in} + \vec{v}_{2\,in} = \vec{v}_{1\,fin} + \vec{v}_{2\,fin} \tag{69}$$

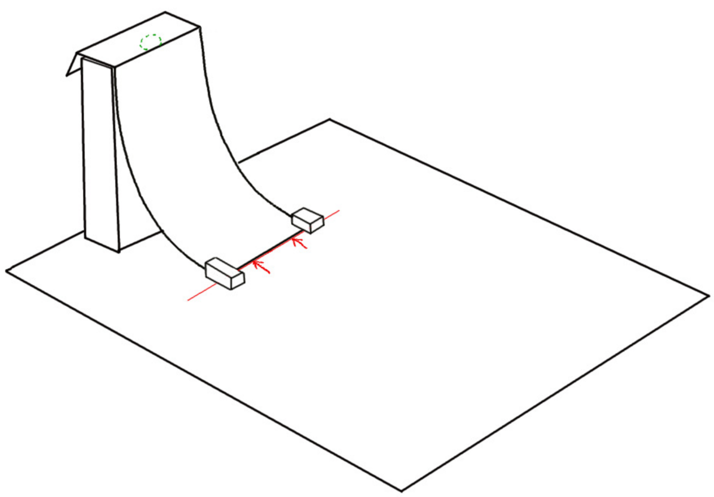
In this scenario we can verify the conservation law simply measuring three velocities (the target coin is initially at rest). Therefore, in order to test the law we have to check the validity of equation (70).

$$\vec{v}_{1\,in} = \vec{v}_{1\,fin} + \vec{v}_{2\,fin} \tag{70}$$

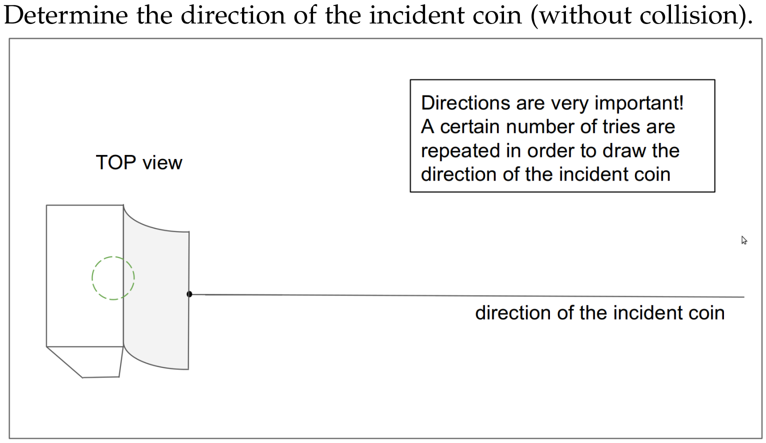
Since from the lengths of the tracks we can derive a vector proportional to the velocity of a coin we have the chance to evaluate the three vectors proportional to the three velocities in equation (70) and we can discover the momentum conservation law measuring collision among coins accelerated with the GA. This is something that should be experienced by students before entering in the world of particles interactions. The experimental procedure is given in the following paragraph.

Momentum conservation law with the GA. Experimental strategy.
In this section we present a step by step procedure in order to carry out the experiment on the momentum conservation law.

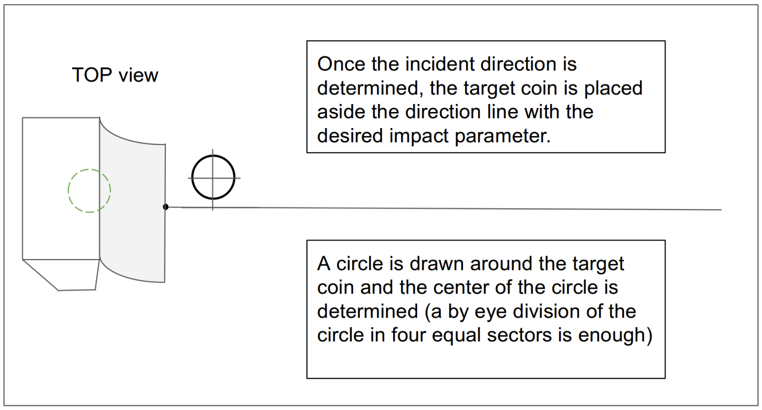
- Fix the starting position of the sliding coin (see dashed circle on top of the slide), draw a line to be sure that the final edge of the slide will always be in the same position. Add small weights on top of the box in order to avoid undesired changes in position of the apparatus.



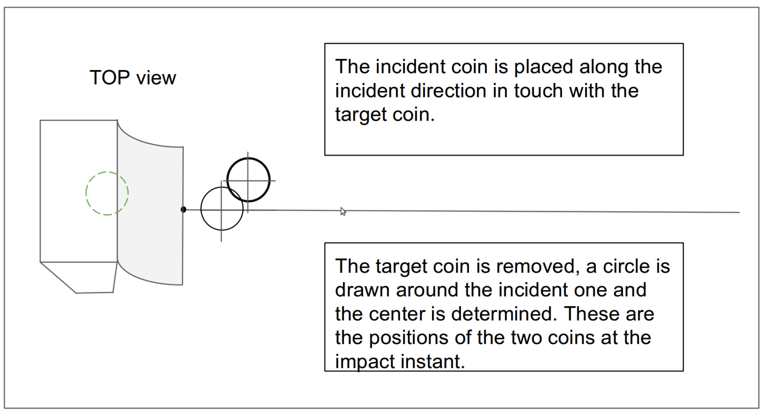
- Determine the direction of the incident coin (without collision).



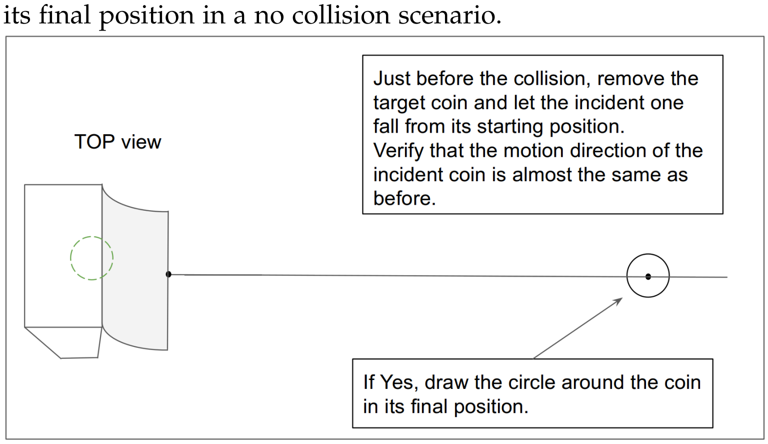
- Choose the position of the target coin.



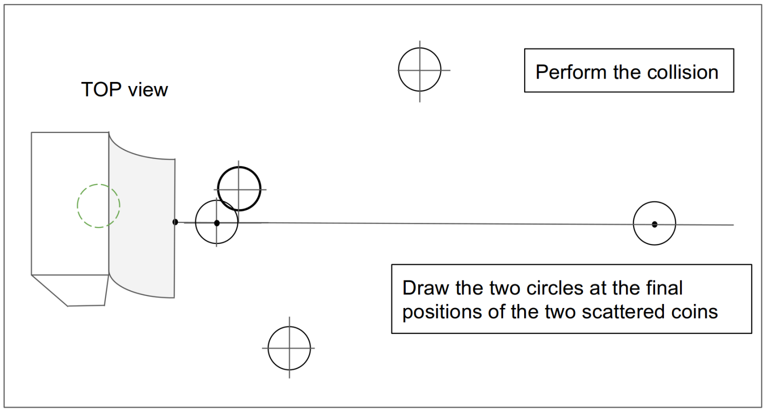
– Determine the position of the incident coin at the instant of impact.



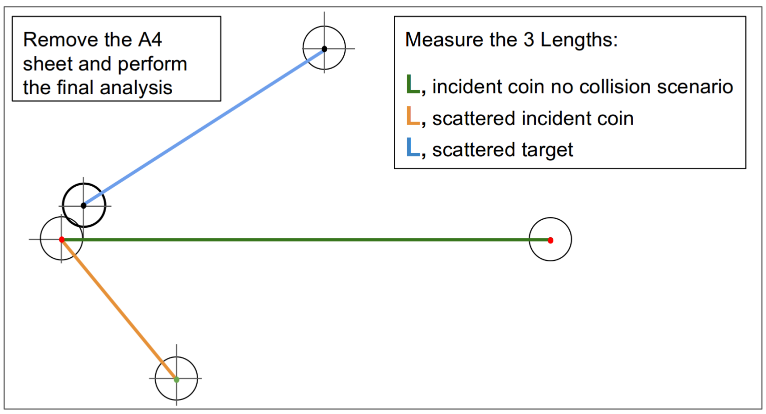
– Confirm the initial direction of the incident coin and determine its final position in a no collision scenario.



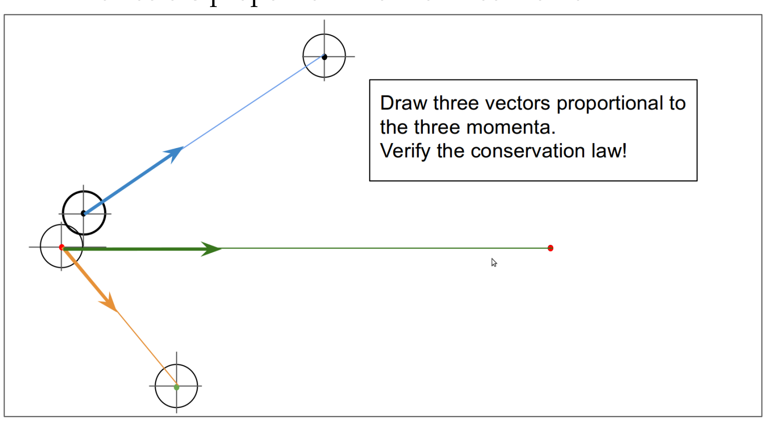
– Perform the collision. Put the target coin in the previously decided impact position and let the incident coin fall from its starting position.



– Remove the A4 sheet and measure the three lengths of the tracks.



– Draw the vectors proportional to the three momenta.



– Verify the conservation law.

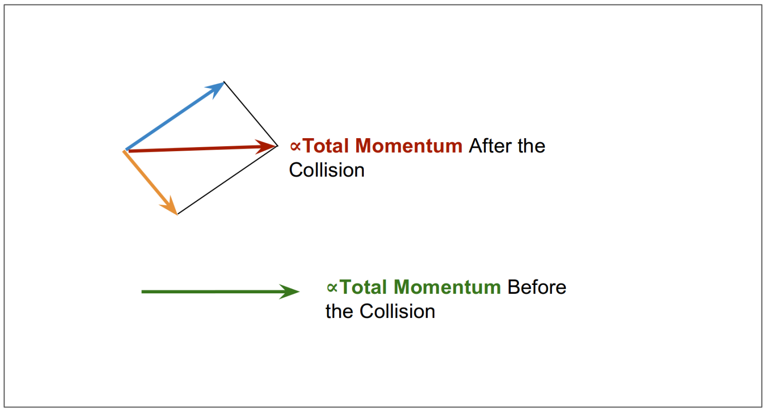


Figure 35 shows a typical experimental result. Carrying out enquires

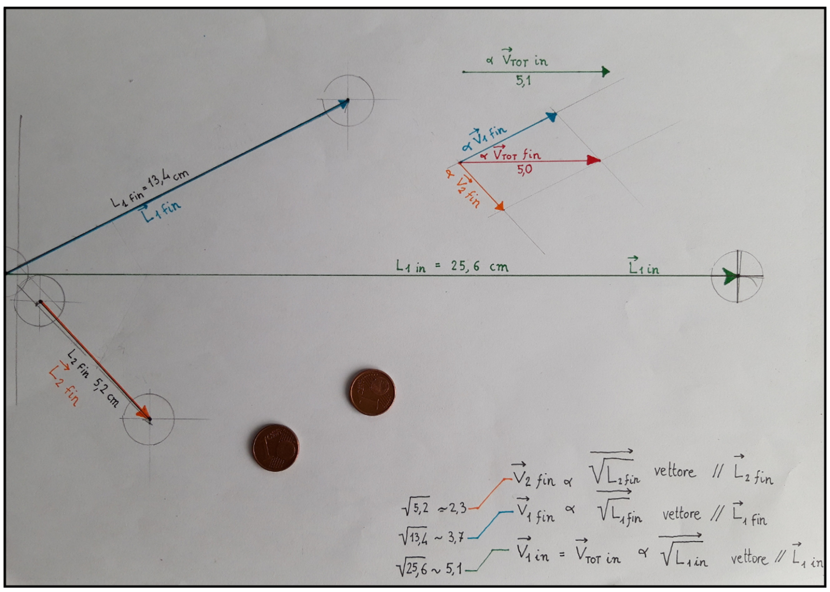


Figure 35: Typical experimental result for the conservation law with the GA.

on collisions performed with the GA students get a clear collision model to refer to. The exchange of momentum between colliding coins, the importance of the impact parameter as well as the momentum conservation law will literally pass under their eyes and through their hands. This a fundamental step before entering the realm of particle interactions.

The PAINUC experiment gives data to Schools

Now we introduce data analysis in ParPLE since, according to the 3DTP approach, we want to create a direct contact between students and data from real particle physics experiments.

Therefore from now we will talk about how students can work on pictures (e.g. Fig. 36) or directly on numerical data from real collisions among pions and Helium nuclei (π, ^4He). Here we briefly present the main aspect of the experiment from which data are gathered, in the following sections the two different approaches to the analysis (with pictures or numerical data) are deeply described.

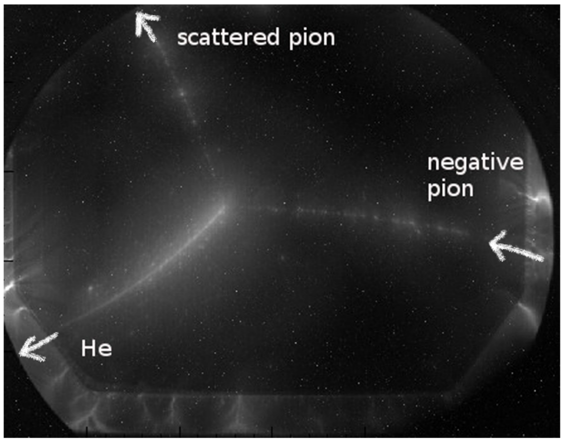


Figure 36: A typical (π, ^4He) collision in the PAINUC experiment.

Data come from the PAINUC experiment which is a collaboration between the Dubna *Joint Institute for Nuclear Research* and the *INFN-Torino*.

The experimental apparatus of PAINUC is designed to investigate nuclear matter[30] through collisions between low energy pions ad a fixed target given by Helium nuclei. The primary experiment components are:

[30] A system of interacting nucleons (protons ad neutrons).

1. A monochromatic pion beam.

2. A chamber in which tracks leaved by ionising particles are detected (*Streamer Chamber*).

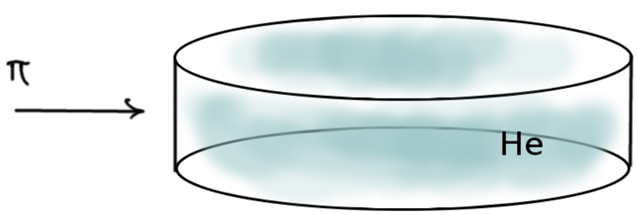


Figure 37: The chamber containing the He gas and the pion beam directed towards the chamber itself.

As shown in figure 37, a charged pion beam is directed towards the Streamer Chamber filled with Helium gas at atmospheric pressure and room temperature. The Helium gas performs a double task:

- *Detection Task*. Allows the detection of the charged particle tracks: a charged particle passing through the gas ionise the Helium thus leaving in the chamber an ionisation track. The ionisation line is then converted in a picture of the particle track.

- *Target Task*. The Helium nuclei also act as the nuclear target (2 protons + 2 neutrons) for the pion beam.

A charged pion travelling in the Helium gas ionise the gas along its trajectory but, sometimes, a collision with a nucleus can also occur. Since the goal of the PAINUC experiment is the study of the nuclear matter, these interactions with the nucleus are of great interest. As we can see in figure 36, a collision will deflect the pion (*scattering process*) also making the nucleus recoil. The ionisation process operated by the pion along its path only serves to take a picture of its trajectory so generating a track. In figure we can recognise the incoming pion track (coming from the right side), the scattered pion and the recoiling Helium tracks. An event like this one is called a *two prong*[31] collision.Figure 38 shows a sketch of the apparatus with the pion beam entering the Streamer Chamber. The beam direction is deflected in the chamber by a uniform magnetic field which is perpendicular to the plane of the figure. The horizontal rods named C6, C7 are scintillators, tiles of plastic material exhibiting emission of photons when struck by a ionising particle. These scintillators are placed on the side of the beam to give an alert in case are crossed by a scattered pion: if this happens a nuclear collision have occurred and the photographic device takes a picture of the event. The signal from the scintillators is called *trigger* signal.

[31] Two tracks in the final state are present.

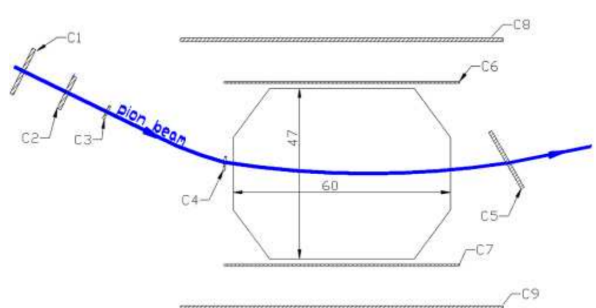


Figure 38: The pion beam enters the Streamer Chamber (octagonal shape) filled with Helium at atmospheric pressure. The beam is deviated by a uniform magnetic field perpendicular to the plane of the image.

Data Analysis form pictures of PAINUC events.

A set of activities to analyse pictures from the PAINUC experiment is presented. Students will learn how to measure the momentum of a particle, how to check the conservation of the momentum in a collision and even how the detection of a missing momentum can lead to the discovery of a reaction channel.

Particle momentum in a uniform magnetic field

A charged particle moving in a constant and uniform magnetic field $\vec{B}$ perpendicular to its velocity follows a curved trajectory. The curvature radius, evaluated applying the Lorentz Force equation, is given by the relation

$$r = \frac{p}{qB} \tag{71}$$

where q is the electric charge and p the momentum. This is evident if we observe a picture of a PAINUC event, like the one in figure 36.

Inside the chamber the particles move nearly at constant speed and the magnetic field, perpendicular to the plane of the image, is approximately uniform so the radius may be considered constant. In the application MFCMa (Magnetic Fields, Charge and Momentum application) (Fig. 39) students can even derive equation (71) changing the values of q, p, B and writing down the value of the radius of the trajectory. Looking at the figure 36 we may notice that the direction

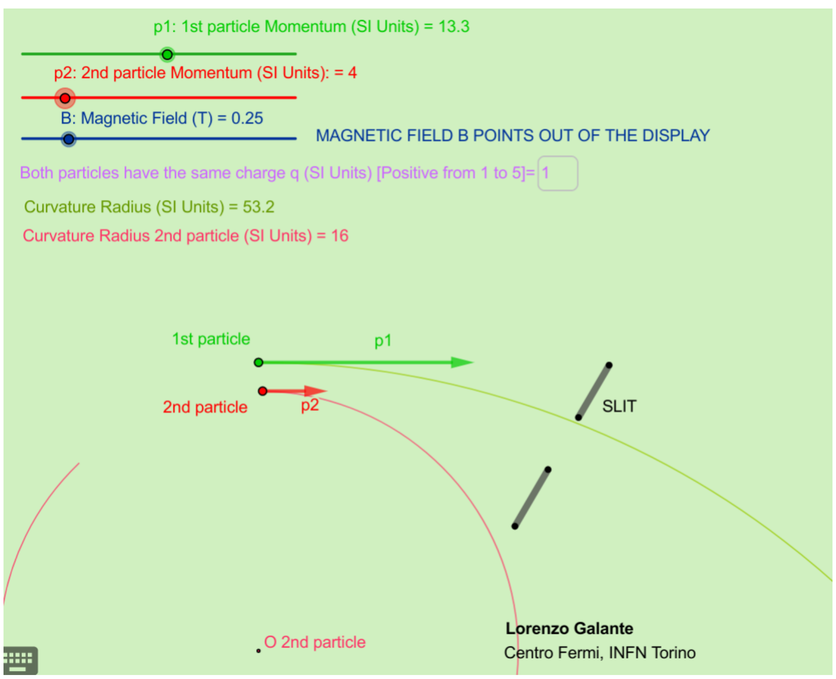


Figure 39: The MFCMa application to explore the relation among the radius of the track, the momentum, the magnetic field and the charge of the particle. Moving the end points of the slit, students can also discover how to build a monochromatic beam.

of rotation depends on the sign of the charge. Negative pions rotate counterclockwise, the positive nucleus follows a clockwise track.

In figure 40 a three prong $(\pi^-, {}^4He)$ collision is shown; analysing the direction of rotation of the tracks students may be challenged in

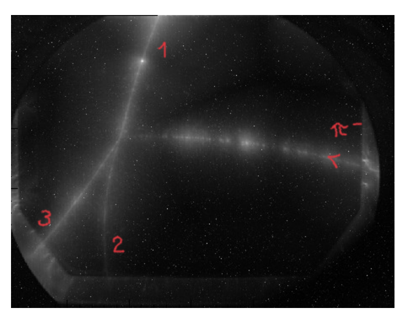


Figure 40: A three prong event in the PAINUC experiment.

discovering the identity of the involved particles.

Measuring the momentum of the particle.

So far students should be familiar with collisions at least in two dimensions, now it is time for them to explore how the momentum of a particle is measured. This is a crucial advance in their knowledge about particle physics, since the momentum represents a key quantity in collisions.

Basically if we have information about the track of the particle moving in a magnetic field we can evaluate its momentum using the relation:

$$p = rqB \tag{72}$$

where r is the radius of the track, q the electric charge of the particle and B the magnetic field[32]. Since The PAINUC experiments provides a beautiful series of pictures of collision events we had the opportunity to build an interactive application allowing students to act like a physicist. The application, again developed in the *Geogebra* framework, is called Measuring Momentum application (MMa), it works on a PAINUC event image and gives the opportunity to fit, with three different circles, the three tracks. In return MMa gives the value of the radii in arbitrary units then converted in meters by the user. For each circle students can drag a point (for example the

[32] Equation (72) comes from the expressions of the magnetic force $F = q\,\vec{v} \wedge \vec{B}$ and of the centripetal force.

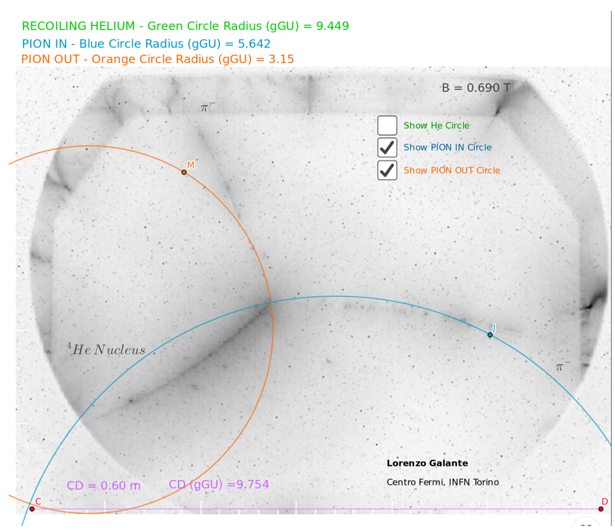


Figure 41: The MMa application.

point M for the orange circle in figure 41) in order to find the best fit of the track. Once all the circles are defined, in the upper part of the application and in arbitrary units (geogebra units, gGU) the radii are given. Since the segment CD shown in the lower part of the image has a known length both in meters and in geogebra units (0.60 m, 9.754 gCU), radii can be converted in meters and from the value of the magnetic field (written in the upper right corner of the image) the momentum p can be evaluated. Some useful tip to get reliable values of the radii is given in the box at the end of this paragraph.

We are using Tesla, meters and Coulomb, so, at first, we express momenta in the International System of Units (SI) then we convert to eV. The conversion process is explained in the following box.

Conversion to eV

In the International System of Units (SI) the units of the momentum $\vec{p} = m\vec{v}$ are:

$$kg \cdot m \cdot s^{-1}. \tag{73}$$

Considering that the kinetic energy, which is expressed in J, is proportional to mv^2 we can also state that

$$kg \cdot m \cdot s^{-1} = \frac{J}{ms^{-1}}. \tag{74}$$

Thus, no surprise if we express the units of momenta as a ratio between energy and velocity.

$$p \rightarrow \frac{Energy}{Velocity} \tag{75}$$

Now we simply have to convert J in eV and ms^{-1} in units of the speed of light c:

$$J \rightarrow eV$$

$$ms^{-1} \rightarrow \text{units of } c\,.$$

Now

$$1\,J = \frac{1}{1.6} \cdot 10^{19}\,eV \tag{76}$$

and, since in the natural units we are adopting $c = 1$,

$$1\,ms^{-1} = \frac{1}{3 \cdot 10^8} \tag{77}$$

Hence

$$\frac{J}{ms^{-1}} = \frac{1}{1.6} \cdot 10^{19}\,eV \cdot 3 \cdot 10^8 = 1.875 \cdot 10^{27}\,eV\,. \tag{78}$$

So, multiplying momenta expressed in SI units by $\boxed{1.875 \cdot 10^{27}}$ we get momenta expressed in eV.

Useful tips to use The **MMa**

1. The radius is really sensitive to small changes in the position of the point. So we suggest to be careful and do your best to pass through the dark grey ionisation spots shown in the picture below (Figure 1).

2. The huge spot in the red circle is not part of the track (Figure 1).

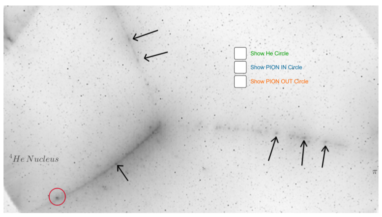


Fig. 1

In order to get a reliable value of the track radius follow this procedure:

1. Place the point M beyond the end point of the track to have full visibility of the track itself (Figure 2).

2. Find two values of the radius and then evaluate the average. Evaluate the first one starting with the point M at one side of the track and stopping as soon as you are satisfied of the fit. Do not exceed the 'satisfactory' position. To get the second radius start with the point M at the other side of the track and again be careful to immediately stop as soon as you think the best fit is met (Figure 2, 3).

3. Evaluate the mean value of the two radii and use it to find the momentum of the particle.

4. The part of the event image in the square box (Figure 2) is the side surface of the chamber, do not consider it during the fitting process.

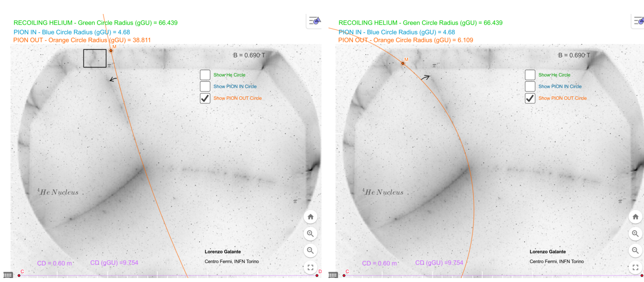


Fig. 2, 3

Momentum distribution of the pion beam

The distribution of the momenta of the incoming pions is given in the plot below. The mean value is 217 MeV and the standard deviation is $\sim$ 19 MeV. Thus, performing your analysis on the proposed events, you should expect an initial pion momentum in that range.

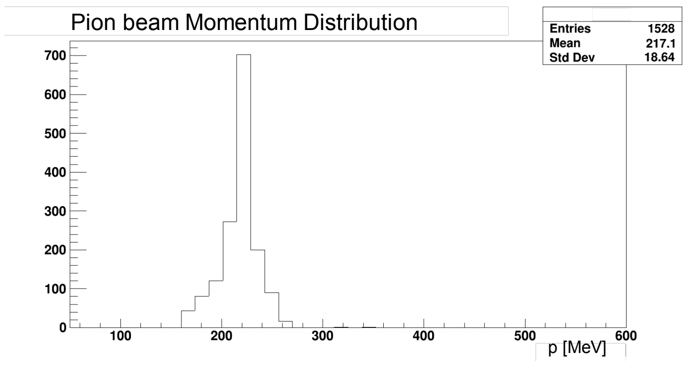


Figure 42: The momentum distribution of the pion beam.

Evaluating the speed of the particles

Once momenta are evaluated student can discover the speed, or the β, of the particles involved in the collision. Two main ways to get the speed:

1. Evaluate the Energy of the particle using the Einstein relation $E^2 = m^2 + p^2$ then evaluate the p/E ratio.

2. Use the $p = m\gamma v$ formula to evaluate the gamma and from the gamma go to the speed.

   Here an example using the first method:

$$p = 220\, MeV \quad , \quad m_\pi = 139\, MeV\,,$$

$$E = \sqrt{m^2 + p^2} = 260 \; MeV\,,$$

$$\frac{p}{E} = 0.85\,.$$

The pion with a 220 MeV momentum is travelling at 85% of the speed of light.

Special Relativity and Quantum Mechanics get together party

In ParPLE we can also discover how joining the two main theories of the 20th century, Special Relativity and Quantum Mechanics, we can introduce one of the main actors of modern physics: the force carrier. As we will learn in the following sections the experimental data available in ParPLE give the opportunity to test some features of the collisions which are strictly related to them.

When two particles interact they exchange momentum. We can consider a GA collision among coins (Fig. 43) to define the transferred momentum and even ask students to measure it.

The transferred momentum $\vec{p}_{\mathrm{Tr}}$ is the difference among the initial and final momentum of one of the bodies involved in the collision:

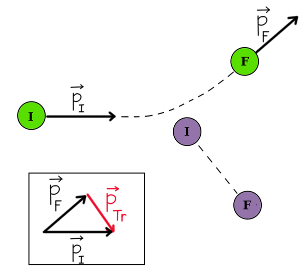


Figure 43: Transferred Momentum in a GA collision among coins.

$$\vec{p}_{\mathrm{Tr}} = \vec{p}_{\mathrm{I}} - \vec{p}_{\mathrm{F}}\,. \tag{79}$$

The same relation holds in particle interactions like those we study in ParPLE (Fig. **??**). Of course, in addition to the momentum two particles also exchange energy, therefore we can say that both energy E_{Tr} and momentum $\vec{p}_{\mathrm{Tr}}$ are transferred from one particle to the other. Now, according to Special Relativity energy and momentum form the 4-momentum vector $(\vec{p}_{\mathrm{Tr}}, E_{\mathrm{Tr}})$ and this 4-vector defines a system (or a particle if we prefer) which is the responsible for the energy and momentum transfer. This particle is the **force carrier**.

We want to show that

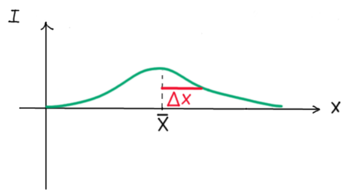
> the **force carrier** plays a key role in determining the spatial scales of an interaction.

With "spatial scales of interaction" we mean the order of magnitude of the spatial region involved in the interaction, for example the whole atom ($10^{-10}m$), the nucleus ($10^{-13}m$ to $10^{-14}m$) or the single nucleon ($10^{-15}m = 1\,fm = 1$ Fermi) might be involved in the collision when a pion interacts with Helium.

To understand the role of the force carrier we need to move two steps: the first one concerns the Heisenberg Uncertainty principle, the second one the distribution in momentum of a system.

1^{st} Step. The Heisenberg Principle

Quantum systems are distributed in space (x) and in momentum (p). In the figure below we have an example of a spatial distribution of a system in the x domain. On the vertical axis the "Intensity" of the system in the considered domain is represented. For a sound it could be the volume, for a body the mass density, etc. Two important parameters are shown: the mean value $\bar{x}$ and the dispersion Δx.



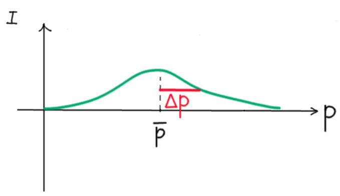
The Heisenberg Principle states that for a quantum system the product of the two dispersions is constant:

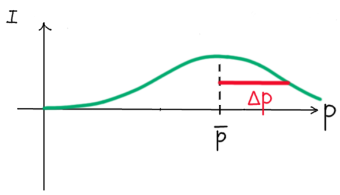
$$\Delta x \cdot \Delta p \sim \hbar \tag{80}$$

This means that the more the system is dispersed in momentum, the less is dispersed in space (and vice versa).

2^{nd} Step. Dispersions and distributions

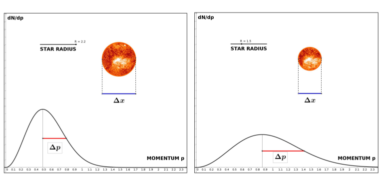
Furthermore we can state that if a system is distributed in the momentum p (see figures below), the bigger is the mean value of the momentum $\overline{p}$ the bigger is the dispersion Δp.



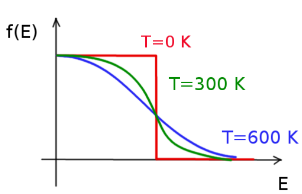


We can give some physical examples in order to convince that this statement is true.

1. *The Maxwell distribution*. We can consider a star and the distribution of the momenta of the proton gas. If the Star collapses the temperature increases as well as the mean value of the momentum and the dispersion of the distribution. The total number of protons involved in the distribution (the area under the plot) is conserved so the distribution has to become broader.



2. *The Fermi distribution*. The distribution of energies of a system consisting of many identical particles that obey the "Pauli exclusion principle" is called the Fermi distribution. If we increase the temperature of the system the mean energy of the fermions increases as well as the dispersion. This is evident considering the plot below



If we join together this two facts we can understand the role of the force carrier in determining the spatial scales of an interaction. If we appeal to the **2nd Step** we might conclude that the bigger is the transferred momentum $\vec{p}_{\text{Tr}}$, which is the momentum of the force carrier, the bigger the dispersion in momentum Δp of the carrier itself. From the **Heisenberg Principle** we know that the bigger the dispersion in momentum of the carrier the smaller its dispersion in space!

Therefore high momentum transfer implies small spatial scales of the interaction.

> The bigger is the transferred momentum, the less the force carrier is dispersed in space. As a consequence, the smaller are the scales of interaction.

The scales of interaction are thus determined by $\vec{p}_{\text{Tr}}$ and we can understand why only by merging together both Special Relativity and Quantum Mechanics.

Math Box

How to evaluate the scales of interaction.

We assume that the order of magnitude of the momentum $\vec{p}_{\text{Tr}}$ and of the dispersion in momentum Δp of the carrier are the same. This seems to be a reasonable assumption (look at the Maxwell Boltzmann distribution, for example). Then we can apply the Heisenberg relation to evaluate the spatial dispersion of the carrier. Here we give an example:

suppose that the transferred momentum is

$p_{\text{Tr}} = 1MeV,$

$\Delta p \sim 1MeV,$

$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{\hbar c}{\Delta p c},$

We know that:

$\hbar c \sim 200MeV\, fm,$

Thus

$\Delta x \sim \frac{200MeV\, fm}{1MeV} \sim 10^{-13}m.$

This is the spatial dispersion of the force carrier which is also the scale of the interaction.

A second reason why physicists build increasingly powerful accelerators can be given exactly at this point.

WHY ACCELERATORS - 2

More powerful accelerators give to the colliding particles higher momentum. Therefore they have a bigger chance to transfer higher momentum and, as a consequence, to interact at smaller scales. In this way scientists have the opportunity to investigate matter at smaller and smaller scales.

Acting like a physicist: analysis of a PAINUC event

Up to this point a lot of tools and knowledge has been given to learners. In ParPLE they had the chance to explore and investigate a lot of concepts, now it is time to take the field, act like a real physicist and perform the analysis of a selected PAINUC event.

The event we are going to face is the one shown in figure 44; as usual a negative pion coming from the left impinges with an Helium nucleus. From the direction of the rotations we can say that the pion in back scattered, while the nucleus recoils forward. The visual framework in which the analysis will be carried out is the Geogebra application EAa (Event Analysis application) shown in figure 45. As the MMa application, EAa offers the opportunity to fit the three trajectories with three different circles and return the radii of the tracks. But now we know how to evaluate the momentum therefore we can go deeper in the analysis. Once the momenta are evaluated by the user they have to be inserted in the white fields

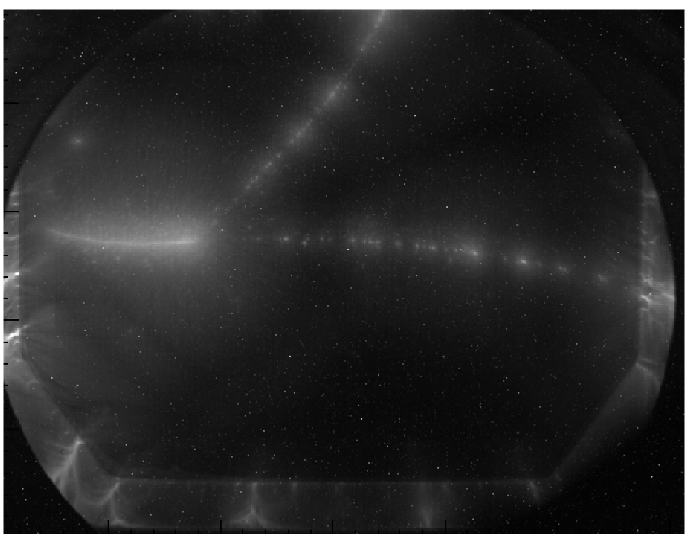


Figure 44: The PAINUC event we propose to analyse.

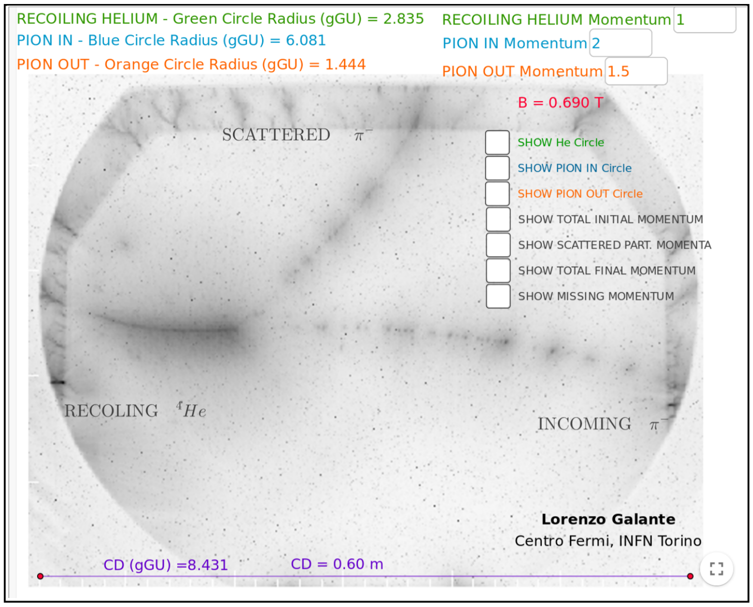


Figure 45: The Event Analysis framework EAa.

on the top-right corner. They have to be expressed in 10^8 eV: so if we get a momentum of 126 MeV we have to input the number 1.26. If asked, by checking the appropriate boxes, the application shows the momentum vectors, which are tangent to the trajectories in the interaction vertex and have a modulus defined by the user. The application draws (Fig. 46):

- The *Total initial momentum*, [Blu vector], which is the the momentum of the incoming pion alone, since the Helium nucleus before the collision may be considered at rest (the order of magnitude of the Helium kinetic energy at room temperature is $KT \sim 10^{-2}\, eV$, $K \sim 9 \cdot 10^{-5} eV\, K^{-1}$ and $T \sim 300\, K$).

- The *Recoiling Helium momentum*, [Green vector].

- The *Scattered pion momentum*, [Orange vector].

- The *Total final momentum*, [Red vector], which is the sum of the Helium and scattered pion momenta.

- The *Missing momentum*, [Black vector], which is the difference between the two total momenta (Initial-Final).

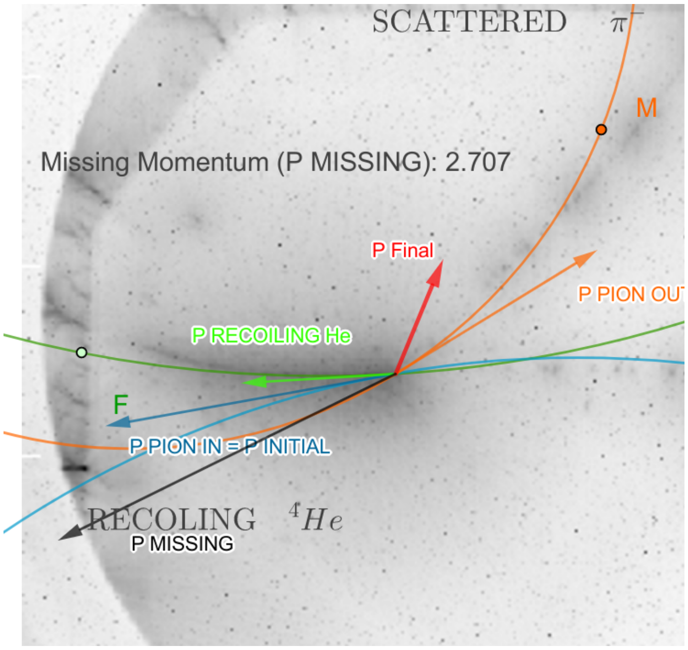


Figure 46: The momentum vectors drawn by EAa. The vectors and the trajectories shown in this image are drawn just to give an overview of the possibilities offered by the application: they are intentionally not reliable.

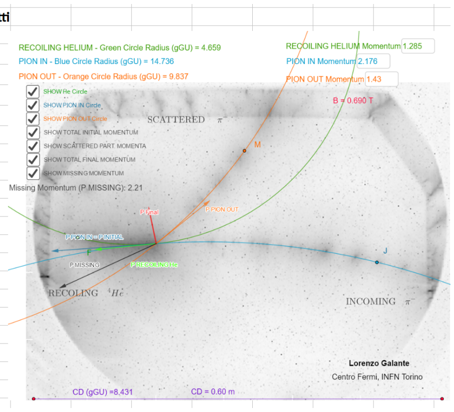
An example of analysis

In this box we show the analysis carried out by a secondary school student during a summer course.

tabella dati analisi evento PAINUC - Andrea Nicoletti

	r [gGU]	r [m]	p [10^8 eV]
^{4}He	4.363	0.31	1.285
π^- IN	14.768	1.051	2.176
π^- OUT	9.708	0.691	1.43

Impulso mancante: 2.21



As pointed out before, students were asked to determine the radii as the average of two measures.

As we can see from the picture in the box, nearly 200 MeV of momentum are missing. We are at a crossroads, we can say that in particle physics the linear momentum is not conserved or we can try to explain the missing quantity in some other way. Probably the best choice is to try finding a way to save the conservation law. Therefore we have to start our enquiry.

The missing momentum can be ascribed to an undetected particle. Undetected means that no track was left in the chamber and, since the track is related to a ionisation, we may guess that the hypothetical particle is neutral. Now we can start the hunt.

Hunting the mysterious particle.

We can start making the list of the neutral particles beginning from the lightest one:

1. π^0, $m \sim 135\,MeV$ (pion)
2. k^0, $m \sim 498\,MeV$ (Kaon)
3. n, $m \sim 940\,MeV$ (Neutron)

The question is whether one of these particles may have been produced in the interaction $(\pi, {}^4He)$. If yes, we have to find which one. In situations like this, a good thing to do is to evaluate the mass of the initial system. We need Special Relativity to perform this calculation, so at this point the Einstein Theory is acting as an analysis tool. We can get the mass with the ESa2, for example inserting these values expressed in GeV (the application needs not-zero momenta, thus for the Helium in the initial state we just have to input a negligible momentum):

π^-, $m = 0.139$, $p = 0.218$;

${}^4He$, $m = 3.727$, $p = 0.001$;

It turns out that the mass of the system before the interaction is: $\boxed{m = 3.99\,\text{GeV}}$. Previously, as a comment to equation (63), we said that the mass of a system is the maximum energy that can be completely converted in masses of the final products. As already pointed out this energy is completely converted in mass only if the products are at rest and, in this case, it reduces to the sum of the final product masses. So we can add up the masses of the detected final products (scattered π and recoiling ${}^4He$) and see what part of the 3.99 GeV is left for the additional neutral particle we are looking for:

$$(3.727\, +\, 0.139)\,\text{GeV} =\, 3.866\,\text{GeV}$$

The detected products almost cover all the energy that could be converted in final mass. We only have 0.124 GeV left, which are not enough even to produce the lightest of the neutral particles (the π_0).

This is the current scenario:

1. We are assuming that a neutral particle has stolen momentum from the interaction scene.
2. Special Relativity tells us that even the lightest of the neutral particles can not be produced.

The mystery thickens! Who is the murderer?

A new reaction channel

Since no neutral particle may be produced in the collision we apparently come to a standstill. However we have a way out. Although we are led to conclude that no neutral particle was created from the initial mass, we may argue that the murderer was already present on the crime scene, even before the interaction occurred.
After all in the Helium nucleus we have two neutral particles: the neutrons! Therefore probably during the collision one neutron of the nucleus was knocked out! We thus have a possible solution to our mystery. We have discovered a new possible reaction channel:

$$\pi^- + {}^4He \rightarrow {}^3He + n + \pi^- \tag{81}$$

This is a *neutron knock out* channel. In particle physics when the products are others than the reagents, the collision is called *inelastic*. So the neutron knock out reaction is an inelastic collision. It turns out that other channels of interactions might be the proton knock out channel, the elastic channel and the radiative channel (in which a photon is emitted):

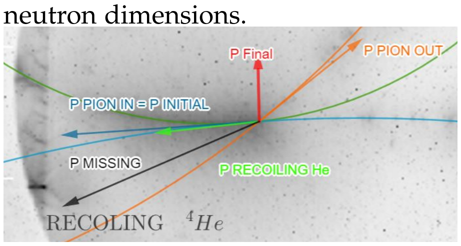
- $\pi^- + {}^4He \rightarrow 3^H + p + \pi^-$ (proton knock out)
- $\pi^- + {}^4He \rightarrow {}^4He + \pi^-$ (elastic)
- $\pi^- + {}^4He \rightarrow {}^4He + \gamma + \pi^-$ (radiative)

But how can we be sure that our thesis is correct?

Testing our thesis

We can perform two nice tests in order to verify the neutron knock out thesis.

1st check : **scales of interaction**.
If we had a neutron knock out it is reasonable to think that the scale of interaction were of the order of magnitude of the neutron dimensions.



looking at the analysis result (see image below) we can easily estimate the order of magnitude of the transferred momentum p_{Tr}, i.e. the momentum of the force carrier. It is evident that the difference between the incoming pion and the scattered pion momenta (Blu and Orange vectors) has the same order of magnitude of the missing momentum (Black vector), thus

$$p_{Tr} \sim 10^2 MeV \,. \tag{82}$$

We can now compute the spatial scales of the interaction occurred in our event:

$$\Delta x \sim \frac{200 MeV\, fm}{100 MeV} \sim 10^{-15} m \tag{83}$$

which are the typical spatial dimensions of a nucleon! The first check confirms our thesis.

2nd check : **Energy conservation**.
We can verify the energy conservation in the reaction (81) assuming that the missing momentum is carried away by the neutron:

[$m_{\pi^-} = 139$ MeV, $m_{4He} = 3727$ MeV, $m_{3He} = 2808$ MeV, $m_n =$ 940 MeV]

$$E_{\text{Tot In}} = E_{\pi\,\text{In}} + E_{4He} = \sqrt{m_\pi^2 + p_\pi^2} + m_{4He} = \boxed{3994 \text{ MeV}}$$

$$E_{\text{Tot Fin}} = E_{\pi\,\text{Fin}} + E_{3He} + E_n = \sqrt{m_\pi^2 + p_\pi^2} + \sqrt{m_{3He}^2 + p_{3He}} + \sqrt{m_n^2 + p_n^2} = \boxed{4022 \text{ MeV}}$$

We have energy conservation within the 0.7% !

Data Analysis from PAINUC numerical data.

In ParPLE teachers and students also have the opportunity to perform a deeper data analysis on a set of nearly 1500 $(\pi, {}^4He)$ events. The analysis is quite different from what was presented in the previous chapter, less visual more abstract, but it allows to move a step further in the understanding of what a physicist has to do in order to extract a scientific result from a dataset.

The analysis framework is ROOT which is the analysis environment developed at CERN and used by particle physicists. The Schools aiming to perform this analysis have to install ROOT on a certain number of computers, then they are given the dataset and are guided through the analysis by a on-line course published on the Internet site **PhE**.

Figure 47: ROOT.

Figure 48: The On-Line Course (in Italian) that guides through the ROOT analysis of a set of 1500 events from the PAINUC experiment.

The dataset concerns nearly 1500 $(\pi, {}^4He)$ collisions with only two particles in the final configuration: the scattered pion and the recoiling Helium nucleus (*two prong* events, see Fig. 48). Student have access to a number of physical variables shown in the list of figure 49. Basically students will use the three components of the momenta of the particles involved in the collision. These variables are labelled *momx*, *momy*, *momz*, while the modulus of the momentum is represented by the variable *mom*. The course is divided in 10 parts and includes video tutorials , pdf tutorials for the student and for the teacher, interactive applications. In order to minimise the coding skills needed to perform the analysis in ROOT (C++ coding

```
/////////////// LOADING TREES FROM ROOT FILE ///////////////////

TTree* TTp2= (TTree*) fTp->Get("Tp2");

// LINKING Tree branches to local variables ////
 TTp2->SetBranchAddress("event_name", event_name);
 TTp2->SetBranchAddress("ev_type", &ev_type);
 TTp2->SetBranchAddress("n_tracks", &n_tracks);
 TTp2->SetBranchAddress("dir_cosx", dir_cosx);
 TTp2->SetBranchAddress("dir_cosy", dir_cosy);
 TTp2->SetBranchAddress("dir_cosz", dir_cosz);
 TTp2->SetBranchAddress("mom", mom);
 TTp2->SetBranchAddress("momx", momx);
 TTp2->SetBranchAddress("momy", momy);
 TTp2->SetBranchAddress("momz", momz);
 TTp2->SetBranchAddress("pid", pid);
 TTp2->SetBranchAddress("vertex", vertex);
 TTp2->SetBranchAddress("dvertexonplane", &dvertexonplane);
 TTp2->SetBranchAddress("dvertexZ", &dvertexZ);
 TTp2->SetBranchAddress("track_lenght_onplane", track_lenght_onplane);
 TTp2->SetBranchAddress("track_lenght", track_lenght);
 TTp2->SetBranchAddress("bright", bright);
 TTp2->SetBranchAddress("rupture", rupture);
 TTp2->SetBranchAddress("rel_loss", rel_loss);
 TTp2->SetBranchAddress("weight", &weight);
 TTp2->SetBranchAddress("charge", &charge);
```

Figure 49: List of the variables stored in the PAINUC dataset.

is required), schools are given a C++ template with blank parts they have to fill in order to carry out the analysis. In this way students only have to deal with commands related to the physics involved and with the initialisation and filling of one or two dimensional histograms. A set of four video tutorials (Fig. 50) introduces to the ROOT framework and to the needed coding skills. In Italy we have

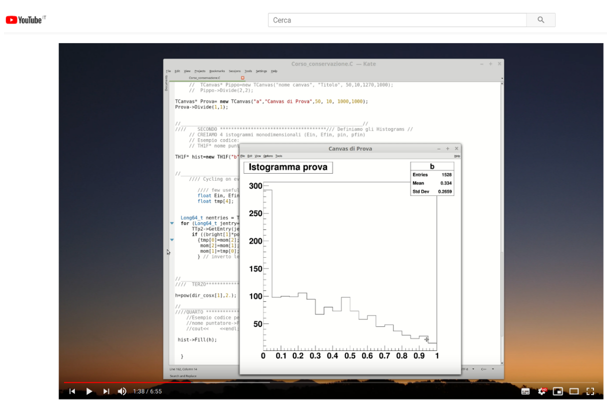


Figure 50: A video tutorial designed to give students the needed coding skills.

three schools which have the data set and ROOT installed on a Linux operative system. The course was used both for the training of in-service teachers and for in-depth courses for students.

In this chapter the 10 sections of the course are individually presented.

Part1 - Mono-chromatic Beams.

In this section the concept of mono-chromatic particle beam is introduced. Students are asked to build the GA (pag. **??**) and to measure the distribution of the track lengths of 20 coins (Fig. 52). In this way they discover that the lengths are distributed even if they expect all

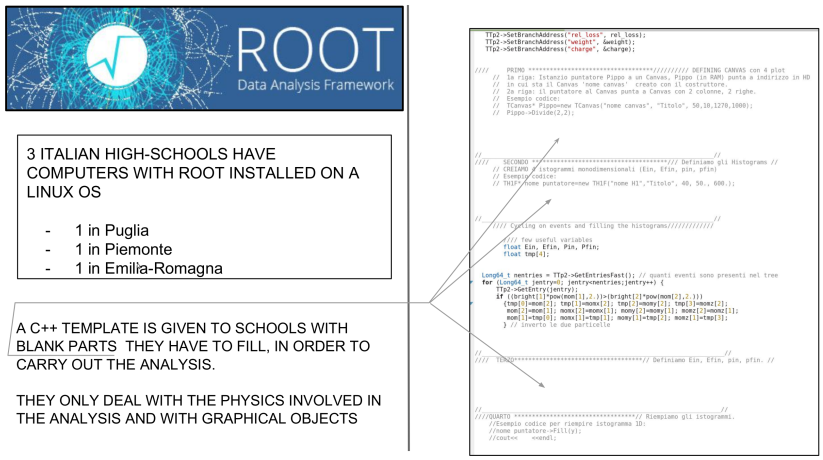


Figure 51: The template to carry out the analysis and the blank parts that have to be filled by students.

the coins to have the same speed at the end of the slide. They are also asked to evaluate the standard deviation of the lengths distribution. The activity is propaedeutic to the next part of the course in which students will perform their first analysis with ROOT plotting the histogram of the momentum distribution of the 1500 incoming pions from the PAINUC dataset. The incoming pions are supposed to form a monochromatic beam of particles since each pion is produced in the same way the others are, nevertheless their momentum is dispersed around a mean value. The GA performs exactly the same: all the coins are accelerated with the same potential drop, starting from the same position and nevertheless the track lengths are distributed. Students learn that in physics we can never assume a variable to have a precise and fixed value.

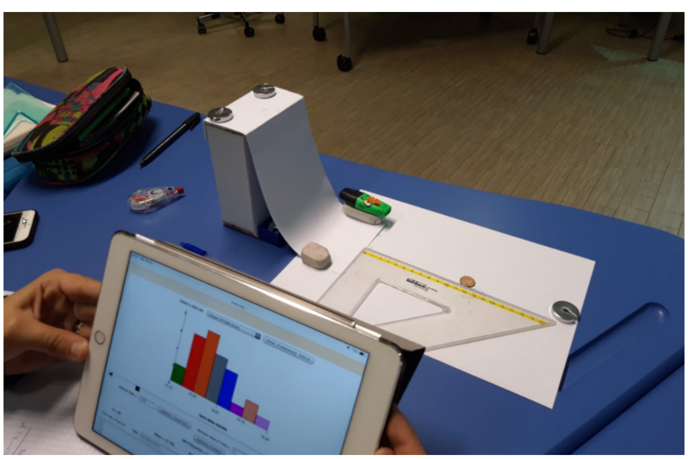


Figure 52: The Length distribution of the mono-chromatic beam of 1 €cent coins according to the measurements carried out by a student.

Part2 - Momentum distribution of the PAINUC pion beam.

Students are asked to carry out their first data analysis with ROOT. The aim is to plot the distribution of the momenta of the incoming pions. A set of short videos provides all the information needed to plot the histogram.

The beam momentum distribution is shown in figure 53, from the analysis we learn that the mean value is $\bar{p} = 217$ MeV and the standard deviation is $\sigma = 19$ MeV.

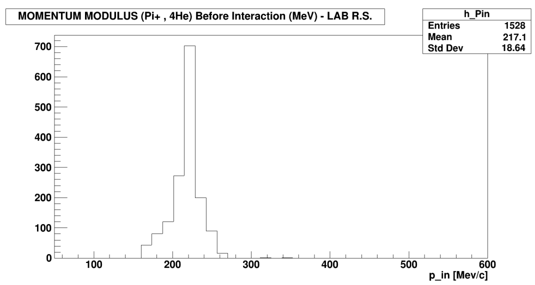


Figure 53: The incoming pion momentum distribution.

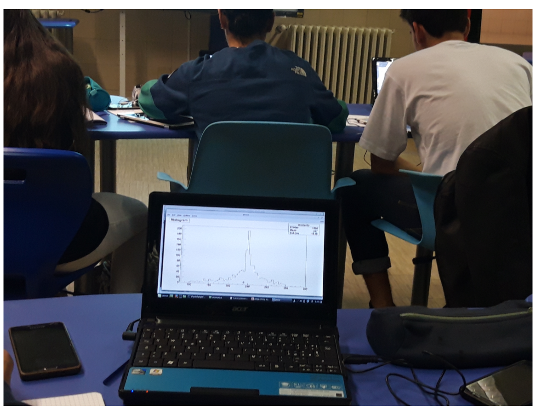


Figure 54: Students at work with their ROOT template aiming to plot the momentum distribution of the pion beam.

Part3 - Familiar with collisions in 2D.

In order to carry out the analysis of the collisions between pions and Helium nuclei, student have to become familiar with collision in more than one dimension. This part is completely devoted to a deep investigation of two-dimensional collision among coins accelerated with the GA. At this stage is very important that student enquire into the conservation of linear momentum being aware of the vectorial nature of the law. This part has been deeply discussed in a previous section of this chapter.

Figure 55: Students enquiring into the momentum conservation law with the GA.

Part4 - Momentum Conservation at sub-nuclear level?

In this part of the course students are asked to test the momentum conservation law in the $(\pi, {}^4He)$ collisions.

> It is probably the first time students have the chance to check if the laws of classical physics hold at nuclear and sub-nuclear level.

Basically they have to move a step further into the analysis of the PAINUC dataset. They have to build a second histogram of the modulus of the total final momentum and compare it with the distribution of the modulus of the total initial momentum. The total final momentum is the vectorial sum of the momenta of the scattered particles (π and recoiling ${}^4He$).

> This point of the activity is a beautiful check point to see if the vectorial nature of the momentum conservation law is well understood.

The total initial momentum is the vectorial sum of the incoming pion and of the Helium nucleus of the gas at room temperature. The order of magnitude of the momentum of the Helium at room temperature may be evaluated from the kinetic energy T:

$$T \sim k \cdot \text{Absolute Temp.} \quad \text{with } K = 8.6\,10^{-5} eV\, K^{-1} \qquad (84)$$

Since the mass of the nucleus is $m_{4He} \sim 3.7 GeV$ and $T \sim 10^{-2} eV$, the relativistic relation $E = T + m$ suggests that the energy of the nucleus is given by the mass with a negligible contribution from the kinetic energy. If we consider the relation $(E^2 = m^2 + p^2)$, we may conclude that $p \sim 0$. Therefore the total initial momentum is given only by the incoming pion momentum.

The result of the analysis is plotted in the following two figures:

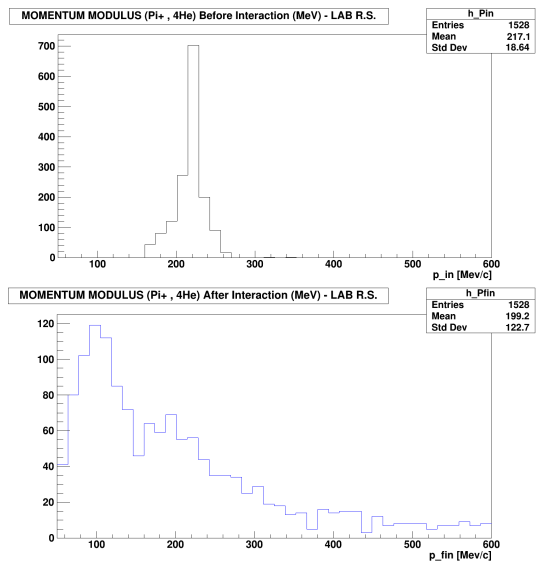


Figure 56: Distribution of the modulus of the total initial momentum (Top). Distribution of the modulus of the total final momentum (Bottom).

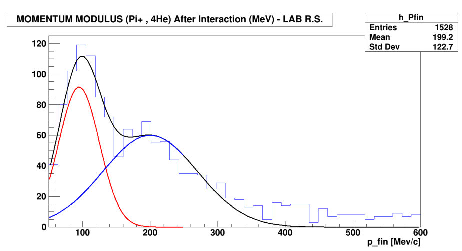


Figure 57: The double gaussian fit of the distribution of the modulus of the final momentum shows very well the two population of events.

We are just comparing the moduli of the total momenta, but this is enough to discover that something is going wrong. Looking at the plot of the total final momentum we can notice the presence of two population of events, one almost centred around the initial total momentum ($217\,MeV$) and one, centred at $\sim 100\,MeV$, showing less momentum in the final state! The final distribution may be fitted with a double gaussian function, the result (Fig. 57) clarifies very well the presence of the two distributions: the red one showing a missing momentum and the blue one in which conservation seems to be respected.

In a huge set of events the momentum conservation law seems not to be verified, we may conclude that the law does not hold at sub-nuclear level or we may decide to look for an undetected particle carrying away the missing momentum. In order to carry out this research student need a basic knowledge about relativistic dynamics. This essential kit is given in the part 5 of the course.

Part5 - Essential Relativistic Dynamics.

Before going deeper into the analysis of the PAINUC data, students need to become familiar with some essential concepts in relativistic dynamics. In this section of the course they are introduced to the following list of relativistic relations.

The four-momentum ($(E, \vec{p})$.

Energy, mass, momentum relation ($E^2 = m^2 + p^2$).

The relativistic kinetic energy T ($E = T + m$).

The meaning of the ratio $\frac{p}{E} = v$.

The meaning of mass in relativity (single particle).

An educational approach to teach these topics is given in the previous chapter about the teaching of Special Relativity, in the section devoted to the relativistic dynamics.

Part6 - Energy losses in the Helium gas.

An undetected particle in the PAINUC experiment is a particle that leaves no ionisation track in the chamber. Therefore some basic information about the ionisation processes in particle physics are given.

- The ionisation process is an electromagnetic effect. In order to ionise matter particles need an electromagnetic interaction with the medium.
- Neutral particles are non ionising particles, therefore they will not leave a track in the PAINUC detector.
- Energy losses due to a ionisation process are often measured and expressed in term of the quantity $\frac{dE}{dx}$, the amount of energy lost per unit length travelled in the medium.

- The *mass depth* η represents the mass per unit area of the number of nucleons the particle will interact with travelling a certain distance in the medium. It is expressed in $g \cdot cm^{-2}$. We can imagine a cylinder of some material, travelled along its length by a particle: η will be proportional to the mass density ρ and to the length L of the cylinder.

$$\eta = \rho L \tag{85}$$

- The energy loss per unit length of a pion in different materials is shown in figure 58. The PAINUC detector is filled with Helium

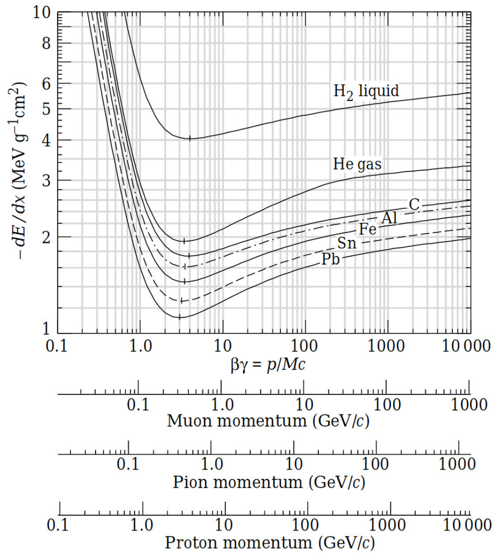


Figure 58: Energy loss per unit length of a pion in different materials.

gas at atmospheric pressure, from the plot we learn that a pion with a momentum of 200 MeV looses 2 MeV per g/cm^2. Therefore we can evaluate how much energy an incoming PAINUC pion will loose in the chamber which is $\sim 1\, m$ long (remember that the incoming pions have a mean momentum of 217 MeV). The first thing we have to do is to evaluate the mass depth of the chamber, assuming that the density of the gas will be 0.2 g/m^3:

$$\eta \sim 0.2 \cdot 10^3 \frac{g}{m^3} \cdot 1\, m \sim 10^{-2} \frac{g}{cm^2} \tag{86}$$

For the incoming pion the PAINUC chamber is equivalent to $10^{-2} g/cm^2$, thus, on average, pions will loose 0.02 MeV of momentum. This amount is negligible if compared to the average momentum of the beam ($\sim 217\, MeV$).

- Each curve shown in figure 58 has a minimum; the region around that minimum is called *minimum ionising* region.

- Experimental plots of the energy loss by ionisation of some charged particle as a function of the momentum is shown in figure 59. It is important to compare the energy loss of a proton and of a pion, which is much higher for the proton. This is due to the dependence of dE/dx on the square of the mass (the mass of a proton is nearly 10 times bigger). In the *minimum ionising* region the ionisation curves may be approximated with this relation:

$$\frac{dE}{dx} \propto \frac{Z^2 m^2}{p^2} \,. \tag{87}$$

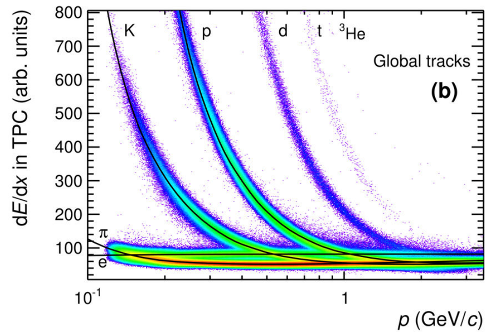


Figure 59: Experimental plots of the energy loss by ionisation of different charged particles as a function of the momentum.

Where Z is the atomic number, i.e the charge of the particle, m the mass and p the momentum.

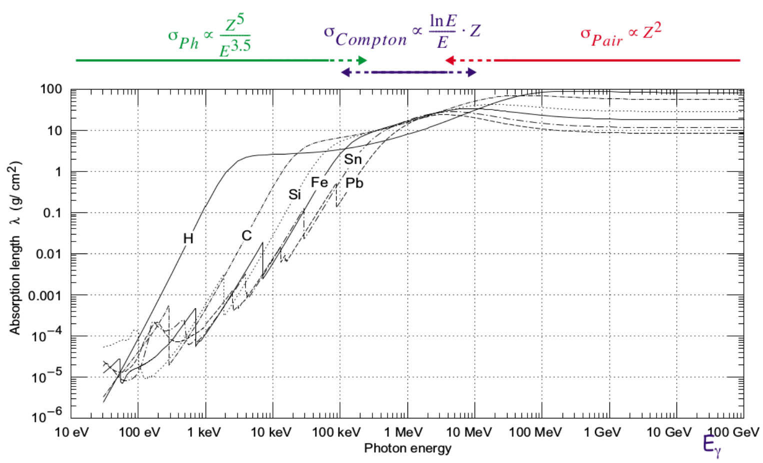


Figure 60: Absorption length of a photon in various materials.

- The absorption length of a photon travelling in various materials is shown in figure 60. To have a rough idea of the absorption length in Helium we can assume that the plot for the Helium gas should be placed somewhere between the plot of the Hydrogen and Carbon. Therefore for a photon with and energy bigger than 10 MeV the order of magnitude of the absorption length will be $\sim\ 10\,g/cm^2$. Considering that the mass depth of the PAINUC chamber is $10^{-2}\,g/cm^2$, we would need 10^3 chambers in order to have one absorption length, i.e to reduce an hypothetical photon beam of a factor $1/e$ ($\sim$ 37%). As a conclusion we may assume that even such photons would go through the detector without leaving a track.

Part7 - Discovery of the first unexpected channel of reaction

Previous similar experiments at the same energies were blind to new reaction channels [33] the only resulting channel was the elastic one:

[33] Ivan Gnesi. *Pion Induced Reactions in the Delta Resonance Energy Region*. PhD Thesis University of Torino, 2010

$$\pi + {}^4He \rightarrow \pi + {}^4He\,, \tag{88}$$

a two prong reaction with only a scattered pion and a recoiling nucleus in the final state. This means that the resolution in measuring the momentum of the particles was not enough to allow the detection of a missing momentum. However we have detected a missing momentum and it is our duty to discover the responsible missing particle.

From the considerations on the energy losses of particles travelling in a medium (see part 6) we know that there are two possible candidates: a **neutral particle** or a **photon**. We can begin looking for emitted photons. In part 5 students learned that Special Relativity is a useful tool to identify a particle, now it is up to them to try. The classroom may be divided in groups and may be asked to find a strategy to discover the undetected particle. For the students the challenge is to use concepts previously learned to solve the problem themselves. Here we present a possible strategy. From Special Relativity we know that the ratio p/E is a great tool to identify photons, if the ratio is 1 the particle is travelling at the speed of light hence being a photon[34]. If we evaluate the missing momentum $\vec{p}_{mis.}$ and the missing energy $E_{mis.}$ we can check if the missing particle is a photon by computing the ratio between the modulus of the missing momentum and the missing energy.

[34] Or a massive particle with a $p >> m$. However, at the energies of the PAINUC experiment, pion beam with $p \sim 217 MeV$, we are not in this situation!

$$\begin{aligned}\vec{p}_{mis.} &= \vec{p}_{\pi\,IN} - (\vec{p}_{\pi\,OUT} + \vec{p}_{^4He}) \\ E_{mis.} &= E_{\pi\,IN} - (E_{\pi\,OUT} + E_{^4He}).\end{aligned} \tag{89}$$

The result of this analysis is shown in figure 61. The plot shows values of the ratio p/E bigger than 1[35] and even negative values. This fact will be surprising for students however is something usual for particle physicists. It is a common error propagation effect. All the plotted quantities are evaluated on the basis of the curvature of the tracks, of the magnetic field in the chamber and so on; errors in each one of this quantities determine further errors in what we are computing. This is a very important lesson student have to learn. How to escape to this problem will be addressed later, now we have to accept that

[35] Meaning that the missing particle is faster than light.

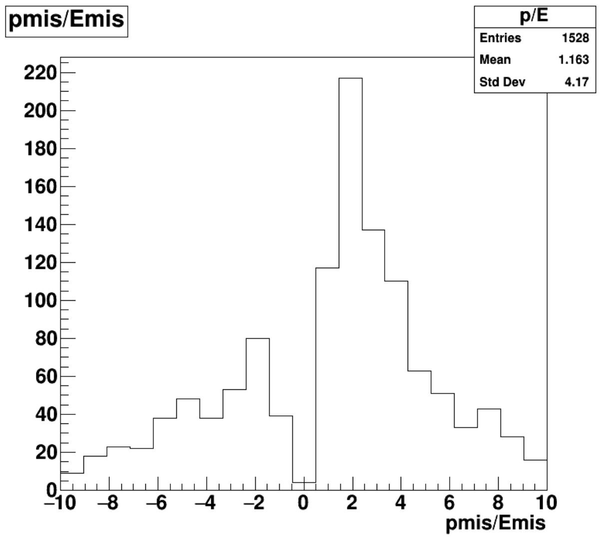


Figure 61: The distribution of the p/E missing for all the analysed events.

In every particle physics experiment the computed quantities are affected by an error propagation effect. Thus one important and challenging task physicist have to face in every experiment is to understand how to interpret what they plot taking into account the error propagation.

Once accepted this fact, we move on in our analysis and we notice that there is a region of our histogram in which we have a number of events with values of the ratio p/E is distributed around 1. These events are candidates for a new reaction channel in which an undetected photon ($p/E = 1$) is emitted:

Radiative reaction channel

$$\pi + {}^4He \rightarrow \pi + {}^4He + \gamma \,. \tag{90}$$

This is a huge result that can be achieved by students. However, in order to proof our thesis and to show without any doubt that we have discovered a channel with the emission of a photon a lot of further actions have to be taken. Actions that, at the moment, we believe to be well beyond the aim of this work. Nevertheless we can make clear what was done in the PAINUC experiment or what is usually done in every particle physics experiment. The keyword to face this issue is *Monte Carlo*. In the following part of this section we will give some explanation about it. The plot shown in figure 62 is a superposition of three analysis. The black triangles show the distribution of $p_{mis.}/E_{mis.}$ from the experimental data, the same build by students and presented in figure 62. The blue line is the

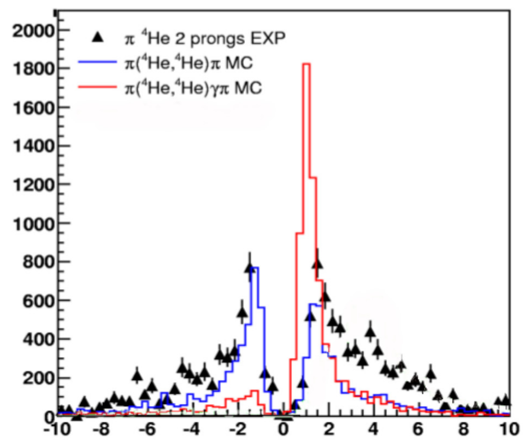


Figure 62: The Monte Carlo analysis of the PAINUC collaboration. On the horizontal axis is represented the $p_{mis.}/E_{mis.}$.

$p_{mis.}/E_{mis.}$ obtained from a computer simulation of elastic events, i.e. events without any other particle produced in the final state ($\pi +^4 He \rightarrow \pi +^4 H$). The red line is the $p_{mis.}/E_{mis.}$ obtained from a computer simulation of events in which a photon is emitted in addition ($\pi +^4 He \rightarrow \pi +^4 H + \gamma$). The red and the blue plot are Monte Carlo simulation: data obtained by a computer program. The program was written to generate momenta of the particles involved in the selected reaction (for example the one with a photon emission) conserving momentum and energy, but also introducing a random error in each momentum simulating exactly the random error introduced by the PAINUC experimental apparatus. If the Monte Carlo simulation is well designed physicists can use this plots to see what values of $p_{mis.}/E_{mis.}$ they have to expect from the PAINUC apparatus for each simulated channel. Therefore the blue plot has to be interpreted in this way:

> According to the Monte Carlo simulation, that also simulate the error generated by the PAINUC apparatus, we have to expect for the elastic processes a distribution of the $p_{mis.}/E_{mis.}$ like the one drawn in blue.

And the red one in this:

> According to the Monte Carlo simulation, that also simulate the error generated by the PAINUC apparatus, we have to expect for the radiative processes a distribution of the $p_{mis.}/E_{mis.}$ like the one drawn in red.

Given these results we are now confident that in the highlighted region of the experimental $p_{mis.}/E_{mis.}$ distribution we have both elastic

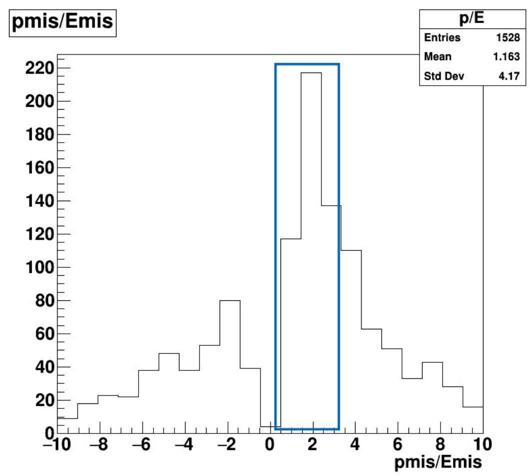


Figure 63: The region of the histogram in which we have both elastic and radiative processes.

and radiative processes (Fig. 63). The last thing to do is to find a way to separate them. One possibility is related to the coplanarity of the momenta. If no photon is emitted the momentum of the incoming pion ($p_{\pi i}$ in Fig. 64) and the two momenta of the scattered pion and recoiling nucleus have to be coplanar. In a radiative reaction this property is no more required (Fig. 65). The ROOT template allows an easy check for the coplanarity of the three momenta: in case of coplanarity the quantity $\vec{p}_{\pi i} \cdot (\vec{p}_{\pi f} \wedge \vec{p}_{Hef})$ is zero.

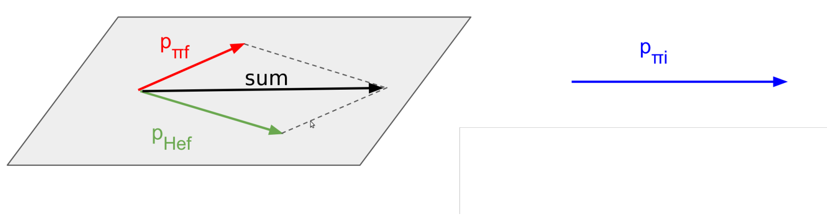


Figure 64: If no photon is emitted the incoming pion momentum has to lie in the plane of the momenta of the scattered pion and of the nucleus.

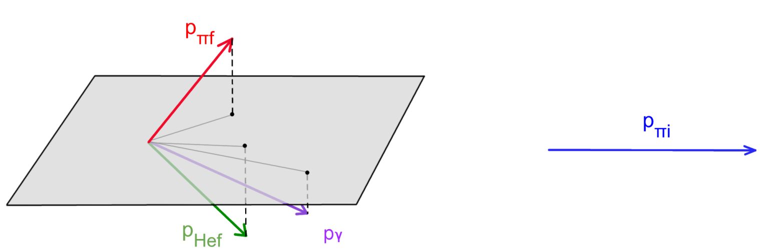


Figure 65: If a photon is emitted the incoming pion momentum generally does not lie in the plane of the momenta of the scattered pion and of the nucleus.

Part8 - Relativistic mass of a system of particles

This part of the course is dedicated to the relativistic mass of a system of particles. This topic is essential to carry out the last analysis which will lead to the discovery of another unexpected channel of reaction. Again, in the previous chapter dedicated to the teaching of Special Relativity how to introduce this topic is addressed.

Part 9 - A second unexpected channel of reaction

If we plot the distribution of the modulus of the missing momentum (Fig. 66) we discover a population of events in which the missing momentum reaches high values of 200 - 300 MeV (Fig. 67). We can

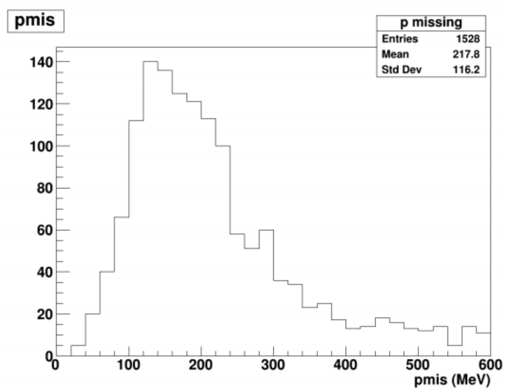


Figure 66: The distribution of the modulus of the missing momentum (89)

ask what kind of particle can take away such a high momentum. We have just discovered the *radiative* channel which involves an undetected γ, thus the first question we can try to answer to is: "Can the photon be the responsible for a 200-300 MeV of missing momentum?". We can answer this question working on the energy

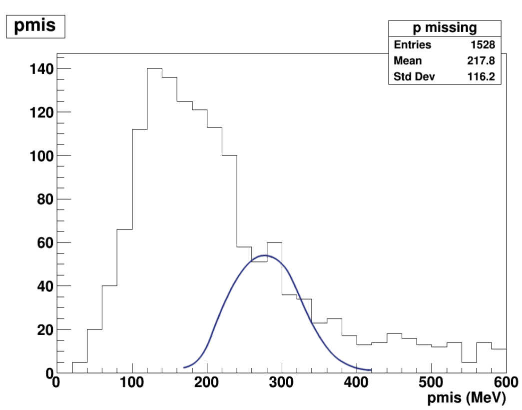


Figure 67: The population of events with a missing momentum distributed around 200 - 300 MeV.

budget of the system: the mass. With the application ESa2 (see page

77) or even by hand we can evaluate the mass of the initial system: the Helium nucleus almost at rest plus the incoming pion. It turns out to be:

$$m = 3990\,MeV\,. \tag{91}$$

We have learned that the mass of a system is an invariant quantity that does not depend on the choice of the Reference System and is best evaluated in the Center of Mass reference system (CM) (see page 50) where it simply becomes the total energy E_{tot}. Therefore the mass of the final system, in the CM, is:

$$m_{fin} = p_\gamma + \sqrt{m_{pi}^2 + \vec{p}\,^2_{\pi\,cm}} + \sqrt{m_{4He}^2 + \vec{p}\,^2_{4He\,cm}}\,. \tag{92}$$

We also know that the mass of a system is conserved, it is the same before and after the interaction, therefore for the radiative process to be possible m_{fin} should not exceed the initial value of 3990 MeV. But we are assuming that the photon is carrying away 200-300 MeV hence

$$m_{fin} = (200-300)\,MeV\; + \;(\geq 139)\,MeV\; + \;(\geq 3727)\,MeV \tag{93}$$

which gives a lower limit for the final mass of 4066 MeV. We have thus proved that

> No radiative process can explain the set of events with the missing momentum distribution shown in blue in figure 67.

In the part dedicated to the energy losses of particles travelling in a medium (part-6 of the course) we learned that both photons with certain energies and neutral particles do not to leave tracks in the PAINUC chamber. We have excluded the possibility of a photon emission, thus we have to consider the production of a neutral particle in the final state. The search for the neutral particle which is carrying away momentum in these events was deeply discussed in a previous section (pages 77-79) therefore will not be repeated here.

We just say that the final result will be the discovery of a second unexpected channel of reaction in which a neutron of the nucleus itself is knocked out:

$$\pi\; + \;^4He \rightarrow \;^3He + n + \pi \tag{94}$$

a so called *neutron knock out* channel.

Test of the methodologies for the teaching of Special Relativity

> I find very useful to share ideas with physics education researchers with an experience in secondary school teaching. Only if you have this kind of experience you can be aware of certain aspects of the school life.
>
> — Anonymous Italian physics teacher

In order to test the tools and the methodologies developed in ParPLE for the teaching of Special Relativity and Particle Physics a number of courses were taught both for students and teachers. In some of these occasions feedback data were collected. This chapter reports data collected in this stage of the research, the list of the courses is given in the following table (Tab. 2).

Italian Teacher Programme	CERN
PAINUC in the classroom	Liceo Majorana
Summer Physics Campus	Bardonecchia
Gamma Factor of the PAINUC particles	Andria
Momentum Conservation in particle Physics	Piacenza
17^{th} Ippog Meeting	GSI Helmholtz Center

Table 2: List of the ParPLE based courses.

CERN Italian Teacher Programme

A set of methodologies to introduce secondary school students to particle physics concepts was presented during the Italian Teacher Programme (ITP) in March 2019. All the methodologies, presented to

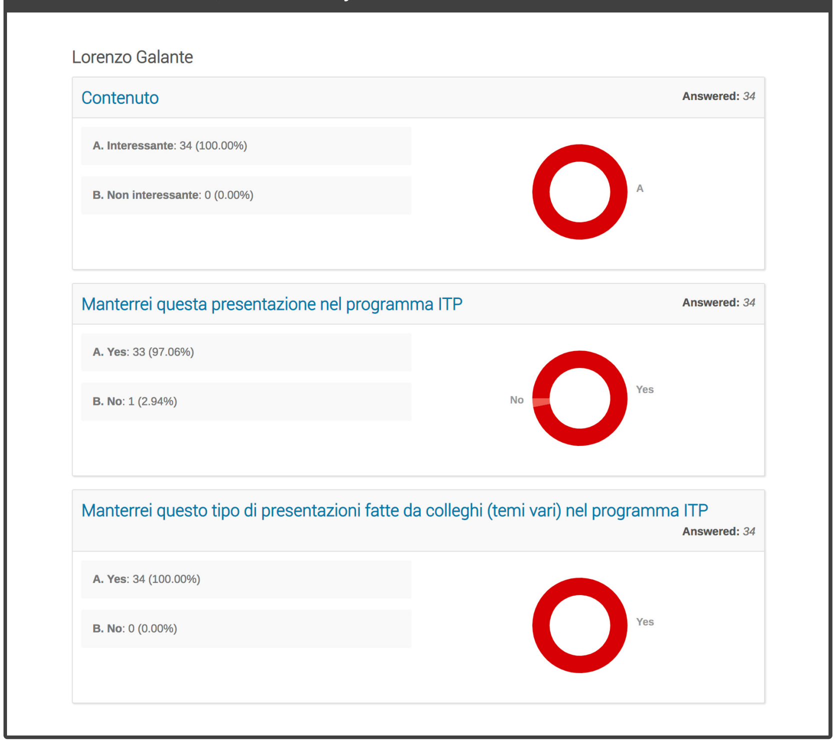
34 Italian secondary school teachers, were developed in the ParPLE environment. The list of the issues addressed during the talk is given in the following table (Tab. 3). All these topics were presented and deeply discussed in the previous chapter "ParPLE".

Topic	Educational environment
2D Collisions	The GA environment
Momentum from the tracks	The MMa application
Momentum conservation in particle collisions	The EAa application
The Energy, mass, momentum relativistic relation	The ESa application

Table 3: List of the ParPLE based courses.

The feedback of the 34 Italian teachers gathered by the ITP organisers was encouraging and in general very good. It is shown in the figures below and is also translated in English.

Feedback of 34 Italian Physics teachers - In Italian.



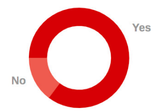
Feedback of 34 Italian Physics teachers - In Italian.

Riporterò tutto questo in classe?

Answered: 34

A. Yes: 29 (85.29%)

B. No: 5 (14.71%)



Perché SI / Perché NO ?

Answered: 34

Date una spiegazione della risposta che avete dato alla domanda precedente

Perché è un programma molto interessante

è facile da usare e avvicina ad argomenti nuovi

Ritengo Il Confronto fra insegnanti fondamentale, in particolare l'intervento ha mostrato uno strimento di lavoro brllisdimo(analisi dati di laboratorio di fisica moderna a livello studente!)

Applicabile ed volto ad una didattica laboratoriale, quelli che cercavo

Ci proverò di sicuro, perché credo anch'io che coinvolge i ragazzi attivamente nel loro apprendimento sia l'unica via

Molto interessante e accattivante per docenti e alunni, ma occorre prima una adeguata preparazione

Perché è uno dei canali migliori per favorire l'apprezzamento

Almeno in parte, ho trovato molto interessante la possibilità di poter analizzare dati sperimentali in maniera relativamente semplice pur senza avere a disposizione un vero laboratorio attrezzato per la fisica moderna. Mi piacerebbe poter vedere ulteriori esempi di applicazione dei temi della fisica moderna

Ci sono molti contenuti ed esperienze adattabili per i ragazzi

Perché non ho le competenze. Ma posso coinvolgere colleghi competenti della materia.

Credo che si possa sperimentare l'applicazione in classe

Riporterò in classe la metodologia proposta ,che ritengo innovativa e interessante, estendendola anche ad altri argomenti

Alce parti si', altre purtroppo richiederebbero tempi di lavoro incompatibili con il programma

Metodo semplice con competenze in vari ambiti

Si per sperimentare nuovi metodi didattici che mi sembrano molto efficaci

Trovo molto utile il confronto con studiosi di didattica che hanno insegnato anche negli istituti superiori. Ci sono degli aspetti della vita scolastica che si possono conoscere soltanto se si ha questo tipo di esperienza.

Si, per esperimenti semplici con raccolta dati.

Perché avvicina i ragazzi alla fisica sperimentale

Ottima applicazione dudattica

Mancanza di computer a disposizione degli alunni.

Approccio didattico innovativo

Proposte didattiche innovative spendibili per motivare i ragazzi a scuola

Cercherò di coinvolgere i ragazzi con i suggerimenti ascoltati.

Oltre alle lezioni teoriche è molto utile avere esempi di attività da proporre in classe. Gli esempi di buone pratiche non solo arricchiscono la propria metodologia ma consentono un confronto tra colleghi

Feedback of 34 Italian Physics teachers - In English.

Content

A. Interesting: 34 (100%)

B. Not interesting: 0 (0%)

I would keep this presentation in the ITP programme

A. Yes: 33 (97%)

B. No: 1 (3%)

I would keep colleagues talks in the ITP programme

A. Yes: 34 (100%)

B. No: 0 (0%)

Would I do these activities in my classroom?

A. Yes: 29 (85%)

B. No: 5 (15%)

Explain why. Give reasons to the answer to the previous question.

1. Because it is a very interesting programme.
2. It is easy to be used and brings closer to new topics.
3. I consider the sharing of ideas with other teachers fundamental, in particular in this talk beautiful tools were presented (modern physics data analysis at students level!)
4. I believe these tools can be used in the classroom, they are suitable for an active learning. I was looking for these kind of tools.
5. I will try them for sure, because I also believe that the active involvement in the learning process is the only way.
6. Very interesting and fascinating both for students and teachers.
7. It is a good channel to help students to appreciate this topics.
8. At least in part, I found very interesting the opportunity to analyse experimental data in a relatively simple way, even without a laboratory for modern physics. I would like to see further examples in which modern physics is applied to real data.

Feedback of 34 Italian Physics teachers - In English.

9. I found a lot of topics and practical activities suitable for the students.
10. Because I do not have the required skills. However I can involve other skilled teachers.
11. I believe that these applications could be tested in a classroom.
12. I will adopt the proposed methodology, which I believe to be innovative and interesting. I will also extend it to other subjects.
13. Some parts may be used in the classroom, others, unfortunately, would need too much time therefore are not compatible with the curriculum.
14. Easy method involving skills from different areas.
15. To try new educational approaches that I think would be effective.
16. I find very useful to share ideas with physics education researchers with an experience in secondary school teaching. Only if you have this kind of experience you can be aware of certain aspects of the school life.
17. Yes, simple experiments with data collection.
18. It brings students closer to the experimental physics
19. Great educational tool.
20. Lack of computers for the students.
21. Innovative educational approach.
22. Innovative educational proposals, useful to motivate students.
23. I will try to engage students with these proposals.
24. In addition to theoretical lessons it is very useful to have practical activities for the classroom. Best practises both enrich our methodology and foster brainstorming among colleagues.

PAINUC in the classroom

PAINUC in the classroom is a 30 hours course for secondary school students, taught to 16 Italian students of the "Liceo Majorana" in Moncalieri in October - November 2018. One of the primary goals was to reduce the gap between school and the world of the scientific research. Students were guided through a data analysis of a number ($\sim$ 1500) of particle collisions from a real experiment (PAINUC experiment). In this journey Special Relativity was the magnifying lens required to understand and interpret what was emerging from the analysis.

The analysis framework of this course is ROOT, an environment developed at CERN for the research in accelerator physics. The class introduces innovative educational aspects at several levels:

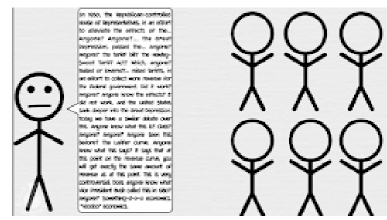


Figure 68: Teacher in front of a classroom.

Figure 69: The student at the centre of the learning process.

- **Educational**
  The class is designed to foster active learning processes. Both the teacher and student roles are reconsidered. The learner is moved towards the centre of the learning scenario while the teacher becomes a sort of tutor, who builds the environment where students will have the chance to shape their own knowledge. The instructor is no more in front of the classroom, is at the student's side, acting like someone to talk to and to share ideas with.

- **Teaching of physics**
  The course leads to a deeper understanding of the Special Relativity. The Einstein's theory becomes a tool to analyse and interpret experimental data from real particle physics experiment. The theory gets out of the textbook while entering the realm of experimental data from collision of relativistic particles. Form the encounter of theory and data a better understanding of the theory itself arises: a significant example of **3D teaching of Physics** (3DTP).

- **Scientific method awareness**
  A key aspect of the teaching of physics for the citizens of the future is to make them become aware of the difficulties of the scientific research. Science is not a dogma, i.e., according to the Oxford English Dictionary, "a set of principles laid down by an authority as incontrovertibly true". Science has its intrinsic hurdles, difficulties and uncertainties. Common aspects of Science are discussions and shared interpretations. With "PAINUC in the classroom" students will get closer to the scientific approach to experimental data, they will be asked to get a scientific result from the dataset working with exactly the same tools adopted by particle physicists.

In "PAINUC in the classroom" students work in groups with a tablet, a personal computer in which ROOT is installed, a C++ template to carry out the analysis and a web site (PhE[36]) where a set of video tutorials and written tutorial challenge them in a series of activities and enquiries.

[36] Lorenzo Galante. *PhE*. https://sites.google.com/view/physedu/

Results and conclusions reached by each group are collected in plenary discussions chaired by the instructor. Teacher lectures are very few in number and very short (nearly 15 minutes). The student learning is constantly monitored both via the plenary discussions and Google Forms which give real-time information about how the learning process evolves. This aspects will be discussed and presented in a later chapter.

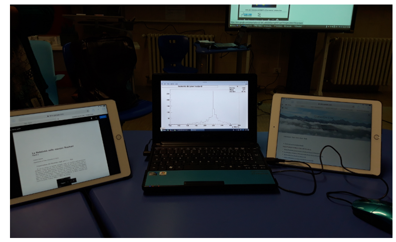


Figure 70: from left to right: a tablet showing the web site PhE, the PC where ROOT is installed, a tablet showing one the written tutorials.

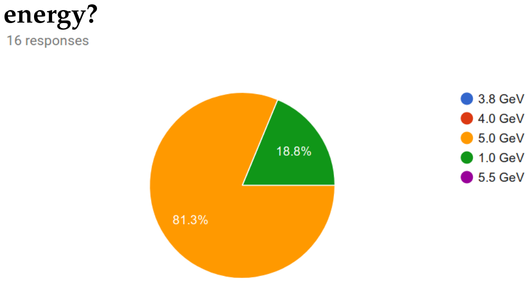
PAINUC in the classroom - Final test results

A final test (Google Form) was delivered to the participant in order to evaluate the effectiveness of the course. Here we show and briefly comment the results.

1. **What is an invariant quantity in Special Relativity?**
   14 correct answers out of 16: 87.5% correct.

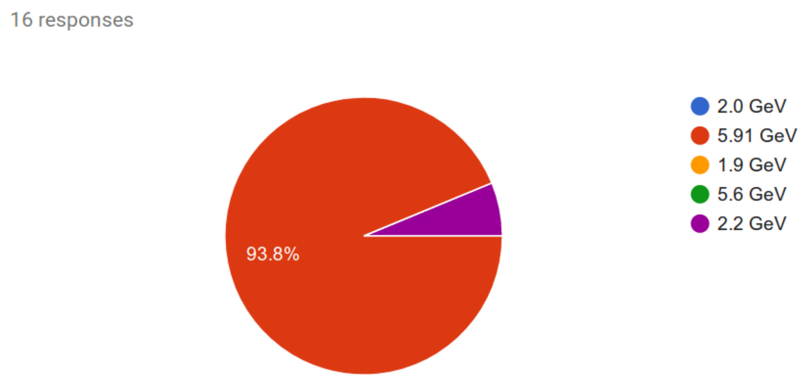
2. **For an observer a particle has 6 MeV of energy and 5.91 MeV of momentum. In general would these values be the same for an observer moving with respect to him?**
   12 correct answers out of 16: 75% correct.

3. **A proton ($m \sim 1\,GeV$) has an energy of 6 GeV. What is its kinetic energy?**



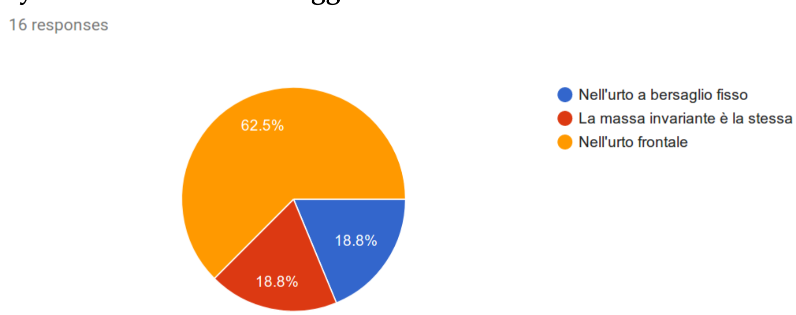
   81% of correct answers.

4. **Referring to the previous question, what is its momentum?**



93.8% of correct answers.

5. **Consider an head-on collision among two electrons at 1 TeV and compare it with a collision of an electron at 1 TeV with an electron at rest. In which case the mass of the system composed by the two electrons is bigger?**

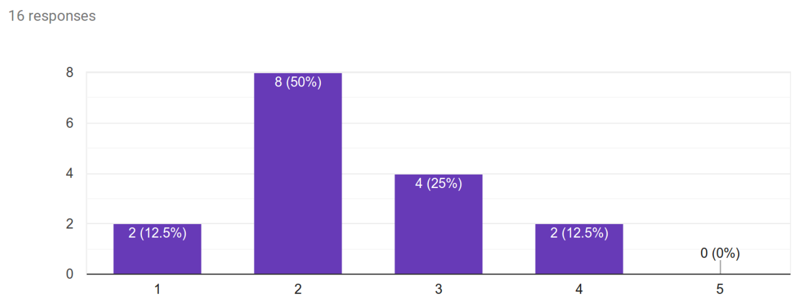


Blue = Fixed target collision; Red = The mass is the same; Orange = Head-on collision.
62.5% of correct answers.

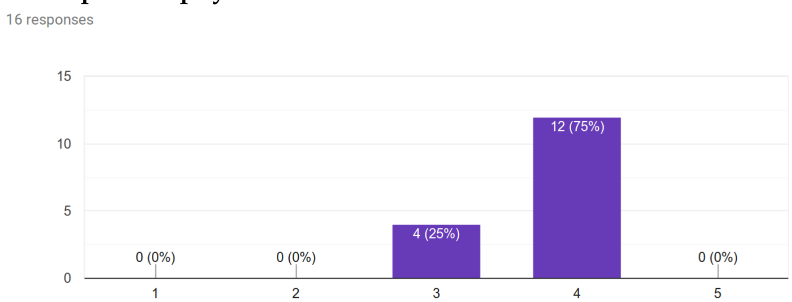
6. **The momentum of a pion is 217 MeV and its mass 139 MeV. Evaluate its speed and briefly explain the method you adopted.**
12 correct answers out of 16: 75% correct.

7. **A photon with an energy of 3 GeV interacts with a proton of 0.1 GeV of momentum. Assuming a collision angle of 180 degrees, evaluate the mass of the system.**
9 correct answers out of 16: 56.2% correct.

8. **From 1 to 5, how much did you feel able to understand a topic about particle physics before the course?**



9. **From 1 to 5, how much do you feel able to understand a topic about particle physics now that the course is over**



PAINUC in the classroom - Students feedback

The students were asked to fill a Google Form expressing an opinion about the class. This is what was reported in the original language.

Il corso è stato davvero emozionante. La possibilità di essere immersi in un mondo completamente nuovo e parzialmente ignoto è stata ispirante e divertente, emozionante e fonte continua di adrenalina. Essere guidati nel nostro percorso di crescita ha aiutato nello sviluppo nel nostro proprio pensiero e nel formulare idee nuove (sbagliate, a volte sbagliatissime) ma sempre in grado di facilitare la crescita e l'approfondimento di tutti.

Illuminante 💡

Molto interessante, permette agli studenti che partecipano di avere una visione diversa dalla fisica insegnata nelle ore scolastiche.
Molto divertente!

Molto interessante e ricco di spunti, forse un po' "deludente" la parte di analisi dati. Dal mio punto di vista lo avrei continuato e approfondito ancora.

Questo corso a mio parere è stato altamente istruttivo da un punto di vista fisico-scientifico ed ha contribuito ad aumentare la mia curiosità verso il mondo relativistico e particellare.

È stato molto interessante

É stato un corso molto interessante, anche se in alcuni momenti le parti di teoria sono risultate lunghe e noiose.

È stato un corso leggero e comprensibile in alcuni giorni rispetto ad altri, di per se interessante e sicuramente stimolante per appassionato della materia

Interessante ma complesso, ho avuto spesso difficoltà nel capire
Il corso mi è piaciuto per la chiarezza e la semplicità con cui venivano spiegate le cose, e per la capacità di suscitare interesse
Si é trattato un argomento interessante in modo da non risultare mai "pesante"
Decisamente più interessante delle mie aspettative purtroppo poco chiaro su alcuni punti svolti di fretta o per mancati prerequisiti
Gli argomenti effrontati sono stati molto interessanti. Sicuramente trattare argomenti del genere può aiutare a capire se il modo della fisica possa interessarci veramente
Il corso mi è piaciuto molto, in particolare la parte di analisi. Questo parte "di laboratorio" che mi ha permesso di capire meglio gli argomenti trattati, e anche se qualcuno non ha mai avuto esperienze in programmazione ciò non ha impedito, grazie alle basi fornite al'inizio.
È stato molto interessante, divertente e sorprendentemente abbastanza semplice da capire.
Interessante, il corso ha rivelato aspetti interessanti della fisica che non avevo ancora incontrato e approfondito altri già conosciuti.

The translation of the students' opinions is given hereafter.

> The course have been really exciting. The opportunity to be immersed in a completely new and partially unknown world has been inspiring and amusing, exciting and a continuing source of adrenaline. Being guided in our growth path helped us in developing our thought and in formulating new ideas (wrong, sometimes extremely wrong) but always useful to stimulate the growth of everyone.
>
> Illuminating.
>
> Very interesting, gives the opportunity to build a different perception of physics than what is taught during the school time. Very funny!
>
> Very interesting and full of ideas, maybe a bit "disappointing" the part devoted to the analysis of data. From my point of view I would have enlarged and deepened that part.
>
> highly instructive from the physics and scientific point of view, it also helped to increase my curiosity towards the realm of Relativity and particles physics.
>
> It has been very interesting.
>
> It has been a very interesting course, even if sometimes the theoretical parts were long and boring.
>
> It has been a light and easy to understand course in some days more than in others. Interesting and surely engaging for those who are passionate about the subject.
>
> Interesting but also complex, I often had difficulties in understanding.

I have appreciated the course for its clarity and simplicity and for the ability to arouse interest.

The subject was interesting and was dealt in such a way that it was never heavy.

Definitely more interesting than I expected, unfortunately unclear in some points tackled in a hurry or in some point where my prior knowledge was not appropriate.

The topics were very interesting. Addressing this subjects may definitely help in understanding our interest for physics.

I really have appreciated the course, especially the data analysis. This sort of "laboratory" helped me in a better understanding of the subject. Even if someone never had coding experiences, this was not an impediment thanks to the basis given at the beginning of the course.

It has been very interesting, funny and surprisingly fairly easy to understand.

Interesting, the course showed me some intriguing aspects of physics which were unknown, while others were already known.

"Particles Collisions among Special Relativity and Quantum Mechanics": Summer Physics Campus in Bardonecchia

On July 2019 a summer course was held in Bardonecchia for 40 students coming from different Italian schools. The age of the students ranged from 16 to 19 (from the third year of high school to the fifth). The course consisted of 4 sessions, each 2 hours long. It was a beautiful occasion to test the educational methodologies with an "enlarged" classroom (we can assume Italian classrooms to be composed by 30 students). The test was thus performed in more challenging conditions than usual.

The course was a series of short lectures alternated with experimental and investigation activities. Students were actively engaged in

- performing experiments with the Gravitational Accelerator (GA) verifying the momentum conservation law in two-dimension;
- investigating with the ESa application the relativistic energy relation
- investigating with the ESa2 application the mass of a system of two particles

- learning how to measure the momentum of a particle from its track and in measuring it with the interactive applications MFCMa and MMa.
- performing the analysis of an inelastic pion - Helium collision with the EAa application, discovering the missing momentum and the related missing particle.

They were lectured on the following topics:

- Overview of the PAINUC experiment
- How to build the GA
- The relation among the track of a coin and the velocity of the coin itself
- The four-momentum and the energy relation
- The four-momentum of a system of particles and the mass of a system
- The transferred momentum in an interaction and the force carrier
- The relation between the force carrier and the spatial scales of interaction
- The relation between momentum, radius of a track, charge and magnetic field.

The feedback of the students during the course was very good. Students asked a lot of questions, when asked to work by their own on some activity they were really focused on the goal they has to reach (Fig. 71), many of them talked with the teacher even during the coffee breaks, some of them explicitly expressed their appreciation for the things they were doing and learning.

At the end of the campus students were asked by the organiser to answer the question: "Did you find the course engaging and interesting?". The results emerging from the 37 answers is shown in figure 72 and confirm the extremely good impressions of the teacher.

Assessment of the performance of two interactive applications.

The work of 40 students gave the precious opportunity to collect some statistics on how the interactive applications designed to measure momentum from the curvature perform in a classroom environment. Two applications were tested: MMa (a sort of training framework in which students learn how to measure momentum and how to express it in eV units) and EAa (in which student analyse one

Figure 71: A picture portraying the students of the campus performing collisions with coins with the GA. They are pretty focused on the goal they have to reach.

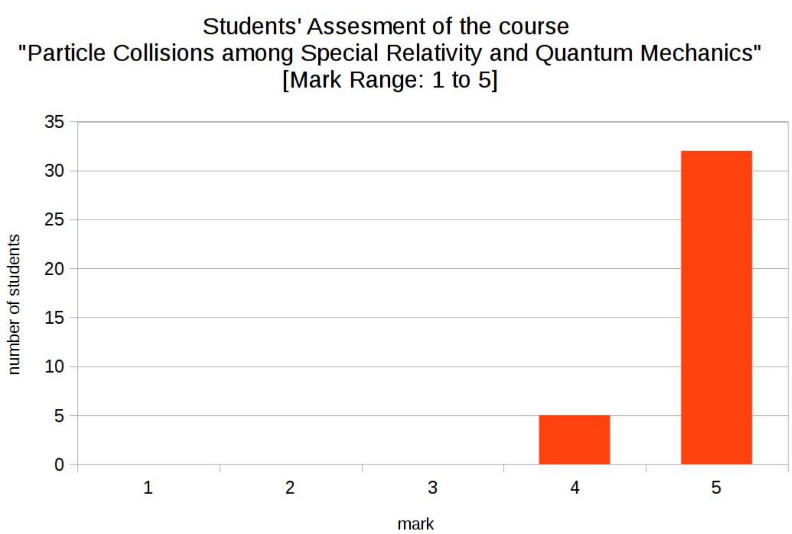


Figure 72: The students' assessment of the course.

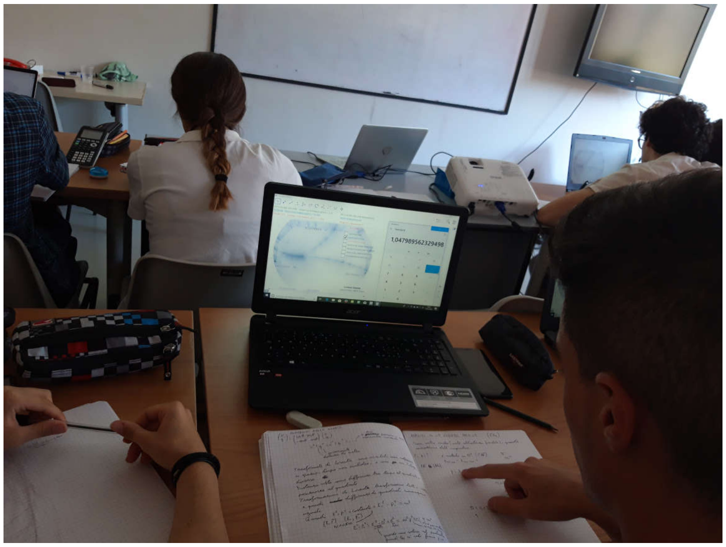


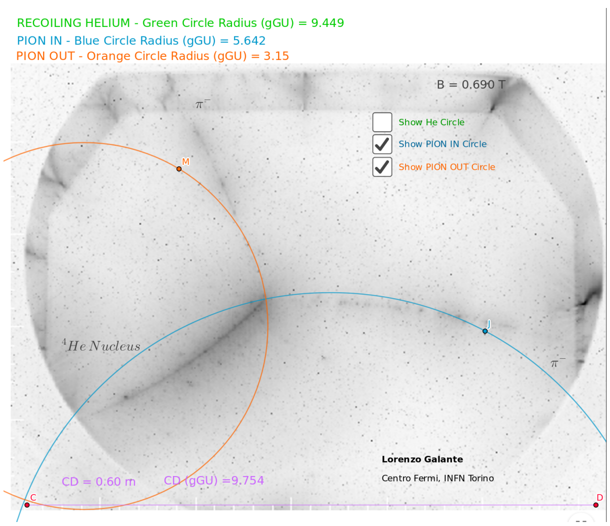
Figure 73: Students working on the analysis of the inelastic π, ^4He collision in the framework of the EAa application. Notice the notebook the student is consulting in order to evaluate the momentum.

Figure 74: Part of the classroom at work on the analysis of the inelastic event.

Figure 75: All the students portrayed in this picture are actively at work on the analysis and immersed in the EAa framework. Notice their napes, every teacher knows this is the classwork postural configuration. The difference is that in a classwork scenario students are working for a mark, here they are working for the sake of personal knowledge.

interesting two prong event from the PAINUC experiment, measure the momenta of the three particle involved, look for the conservation of momentum and discover a missing momentum). A very important information for the teacher willing to use these tools in his classes is what result he might expect from his students. From the data collected during this course we can report the following results.

1. **MMa Application.**



Students were supposed to work in groups of three with the MMa application (image above) which deals with a particular "training" event from the PAINUC experiment. They were instructed to evaluate the radius using the method presented in the previous chapter in the box "Useful tips to use the MMa".
The average momentum of the *incoming pion* was:

$< p_{in} >= 240\, MeV$

with a standard deviation:
$\sigma = 23\, MeV$.

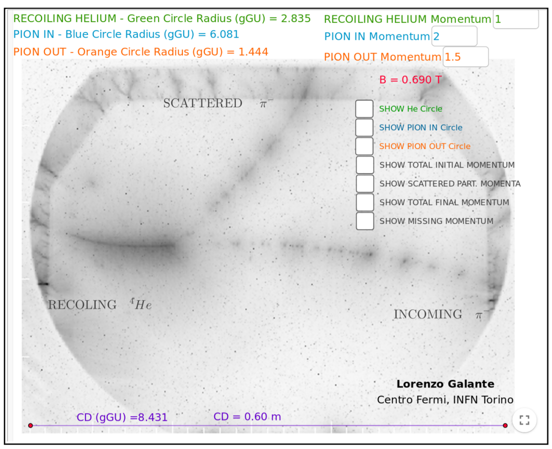
The average momentum of the *scattered pion* was:
$< p_{out} >= 174\, MeV$

with a standard deviation:
$\sigma = 28\, MeV$.

The average momentum of the *recoiling Helium* was:
$< p_{^4He} >= 251\, MeV$

with a standard deviation:
$\sigma = 33\, MeV$.

2. **EAa Application**



Student were asked to evaluate, if any, the missing momentum in this interaction. The average result was:
$< p_{miss} >= 229\, MeV$

with a standard deviation:
$\sigma = 14\, MeV$.

Two teachers bring ParPLE into the classroom

Two Italian physics teachers tested some of the ParPLE activities. The first one is Ugo Morra, a teacher of the Liceo "Nuzzi" in Andria (Puglia), the second one is Carlo Colombini, who teaches in the Liceo "Respighi" in Piacenza. Their work and considerations are very important and precious for the future development of the ParPLE environment. In this section we report what they have done and how they used ParPLE in their educational path.

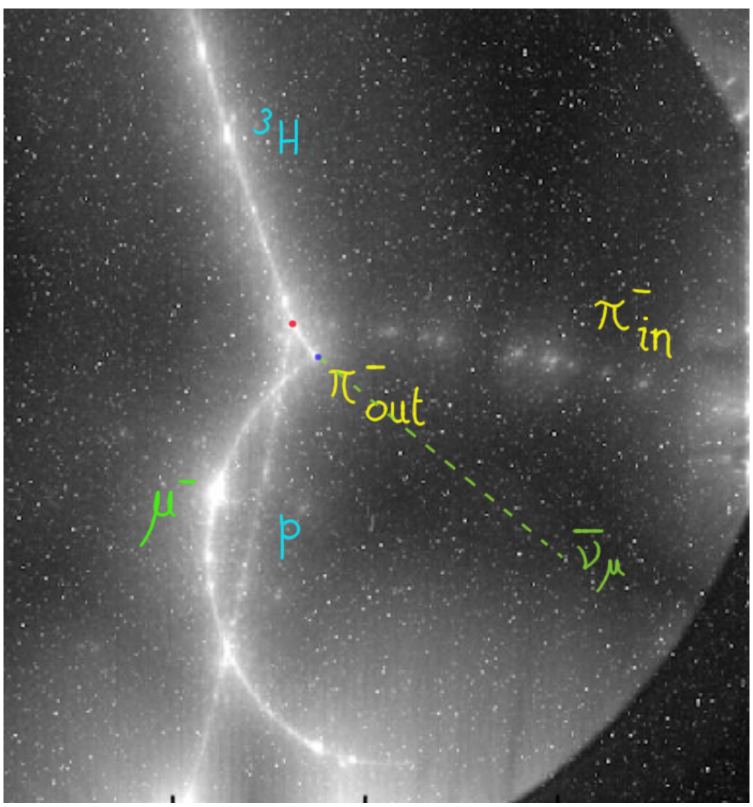
What was done at Liceo "Nuzzi"

Measuring the Lorentz $\gamma(v)$ factor of the PAINUC particles.

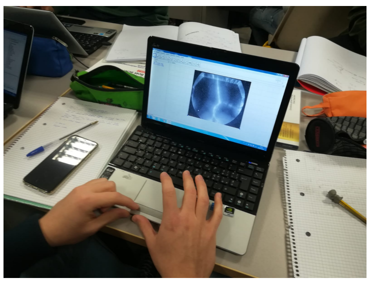
Number of students involved: 18
Class duration: 2h
Age of the students: 18
The class was taught during normal morning working time.

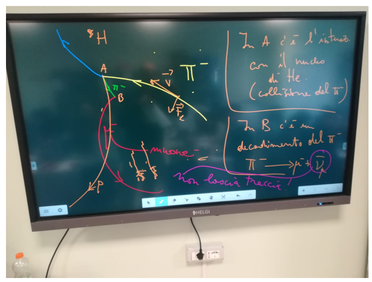
The entire activity was focused on one peculiar event of the PAINUC experiment, shown in the figure below. A negative pion enters the Helium chamber from the left. The collision with the helium nucleus takes place at the red vertex from which 3 particles come out: the scattered pion, and two positive particles, whose charge sign is recognised by the direction of rotation. The two positive systems are a proton and what remains from a ^4He nucleus which has lost one proton: a Tritium nucleus ^3H. As we may notice from the event image the scattered pion decays very soon, according to the most likely decay channel, in a negative particle (a muon) and an anti-neutrino :

$$\pi^- \to \mu^- + \bar{\nu}_\mu \,.$$



Students were asked to import the image in Geogebra and to fit the tracks with circles in order to evaluate the radius of the trajectory in Geogebra units. The radius was then converted in meters from the real dimension of the chamber. From the radius and the intensity of the magnetic field $\vec{B}$, perpendicular to the plane of the picture, students could evaluate the momentum p. Eventually form $p = m\gamma v$ they could evaluate the speed and the Lorentz γ factor. For example, from the ratio $\frac{p}{E}$ we get the speed in natural units, then it is easy to evaluate the Lorentz factor.





[Comments of the teacher:
While teaching this class i was gratified and surprised because students were very active.

Students were very involved in the activity, looking at the rotation direction they had to understand by their own the sign of the charge, furthermore they had to find a way to convert in meters the arbitrary units of the radii.]

What was done at Liceo "Respighi"

Momentum conservation in particle physics.

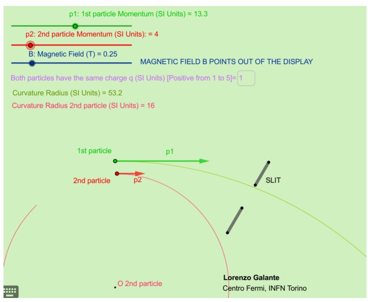
Number of students involved: 5
Class duration: 8h
Age of the students: 18
The class was an extracurricular activity.

The course was divided in 4 sessions (2 hours each), 5 students with a strong interest in physics coming from different classes were involved. The class was based on the on-line course available in ParPLE, therefore some tutorials from the PhE website were used. A summary of the course, commented by the teacher (comments in square brackets), is given below.

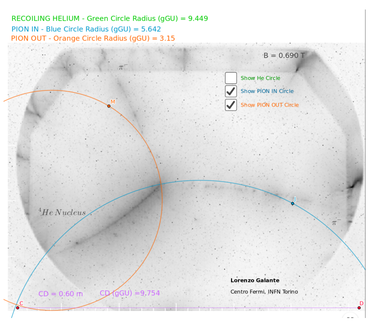
- **1*st* session.**

  General description of the PAINUC experiment. [Teacher Comment. *It was very important to give students a general overview, since the world of experimental particle physics was very distant from what they were used to.*]

  Connection between curvature of the track, magnetic field, electric charge and momentum. [*The MFCMa Geogebra application from ParPLE was used (figure below).*]

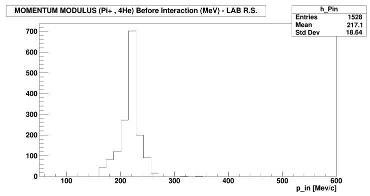
  

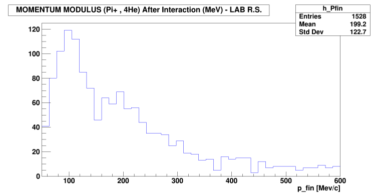
  Measure of the momentum from the radius of curvature. [*The MMa Geogebra application from ParPLE was used (figure below).*]

- **2*nd* session.**

  (Tutorials 4a, 4b - Part 4 of the on-line course on the PhE site). Students ran a ROOT programme without writing any line, using a pre-filled template to plot the distribution of the modulus of the total initial momentum and of the total final momentum of the PAINUC dataset ($\sim$ 1500 events).





[*Students were really engaged in this part of the class. The activity was interesting for them, it seems to be a good activity to me. I've appreciated to work in a Linux environment, but I would like to know more about this operative system. I have used a file with an already written code, one student was very interested in coding aspects, but I had no time to go deep in this direction. The analysis of the histograms was very challenging; were the plots built by students, probably it would have been even better.*]

- 3rd **session.**

  (Tutorial 5c - Part 5 of the on-line course on the PhE site). The essential kit of relativistic kinematics was given (the four-momentum $(E, \vec{p})$, the relativistic energy relation $E^2 = m^2 + p^2$, the kinetic energy T and invariants were introuced).

  (Tutorial 5d). Students carried out by themselves an enquiry on the relation $E^2 = m^2 + p^2$ working with the ESa application.

  [*Even if all the student had already addressed Special Relativity in curricular school time, no one got deep into the Energy, mass, momentum relation. Furthermore no one was aware of relativistic invariants. From this point of view the Energy Square activity was appreciated. The discussion about the results of the enquiry was carried out under the supervision of the teacher. As a teacher I had some problems during this phase of the class. Probably teacher should need some deeper training on these concepts before addressing this part.*]

- **4^{th} session.**

  [*Due to the lack of time this part was carried out in a hurry*.]

  (Tutorial 6d - Part 6 of the on-line course on the PhE site). Basic information on ionisation processes were discussed.

  (Tutorial 8a – Part 8 of the on-line course on the PhE site). The mass of a system of two particles was introduced.

  (Tutorial 9b - Part 9 of the on-line course on the PhE site). Students are engaged in the finding of the missing particle (the neutron knock-out reaction channel is discovered).

Final teacher comments.
Even if I have only partially carried out the on-line course, I believe that its educational potential is very big. I was surprised that 5 excellent students coming from different classes had studied Special Relativity especially by means of mechanical exercises for example on time dilation. This activity gives a deeper perspective and definitely engages them. Here some reflections about my personal experience:

1. Students have appreciated that data were from a real experiment researcher have worked and still are working on. Just this fact have flushed away the sensation of abstractness they feel studying physics in the usual curriculum.

2. This activity allows the study of Special Relativity in a live context.

3. It offers good interdisciplinary connections really appreciated by young students, for example with Information Technology.

1. Geogebra application like the Energetic Square application (ESa) and the one devoted to the momentum measurement (MMa) were appreciated as well. Also the application MFCMa (Magnetic Field Charge and Momentum application) was useful, some student told me that only using this interactive applications they have really understood the role of the magnetic field and of the charge on the particle trajectory.

2. A support would be useful both for teachers and for the most curious students. The questions asked in the enquiry activity 5d are really interesting but I'm not sure to have been a good tutor during the discussion. I would appreciate group of discussions with other teachers.

3. The momentum conservation law gets out from the usual problems with colliding carts and gets a central role.

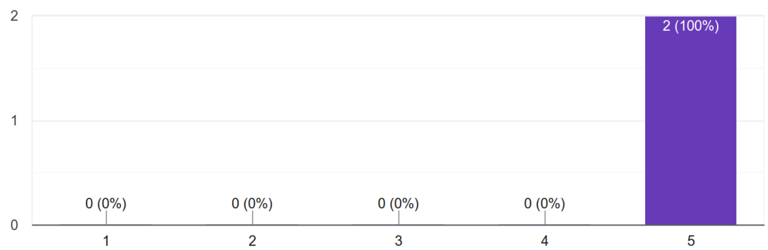
4. There is the opportunity to work with Inquiry Based Science Education methodologies. Students are invited to investigate and build their own knowledge working in a challenging environment, while the teacher acts like a tutor. However in order to offer this possibility to students the teacher has to become familiar with this topics first.

5. Students are asked to work a lot on these activities, but I have noticed that this required a huge effort to them. The educational methodology was different from the usual one, they were asked to change the role they were used to play. From the physics made of formulas to a research activity; students need time and the teacher should not take anything for granted, even in the apparently easy tasks.

The two teachers were asked to fill a form about their feelings during the class. Here we report their answers.

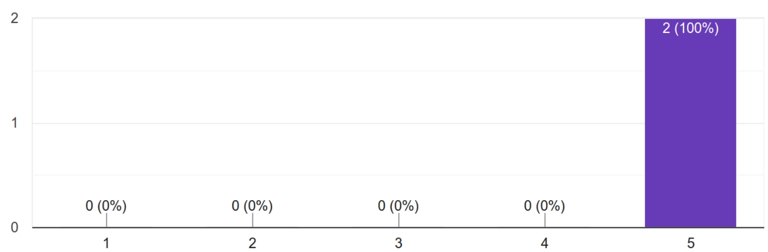
Evaluate student's appreciation to work, investigate and reason on their own

2 responses



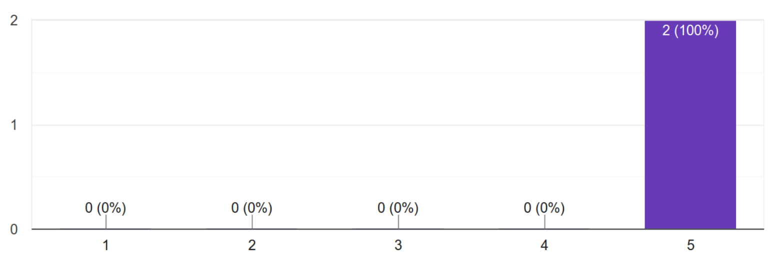
Evaluate your feelings

2 responses



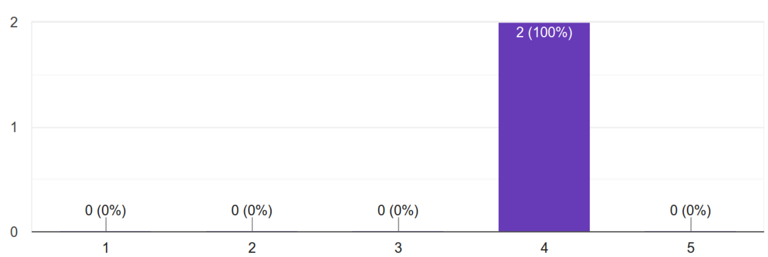
Evaluate student's engagement

2 responses



(Teacher's impressions immediately after the activity) Did the activity stimulate questions and discussions?

2 responses



The 17^{th} IPPOG Meeting at GSI Helmholtz Centre

In June 2019 we were ivited at the 17^{th} meeting of the International Particle Physics Outreach Group (IPPOG) to present the ParPLE environment. In that occasion we collected some feedback from the opinions of some of the organiser of the meeting and some IPPOG members who appreciated the possibility to build a low cost "particle" accelerator (the GA) on which student can study two-dimensional collisions and conservation of momentum as well as the interactive applications designed to introduce relativistic concepts and to analyse collision events from real experiments.

Teaching Quantum Mechanics

What is done in the Italian scientific Lyceum

In Italian textbooks Quantum Mechanics is generally presented according to the following sequence of topics:

- Black body radiation and Planck Hypothesis.
- Photons and Photoelectric effect.
- The Compton effect.
- The Bohr's Hydrogen atom.
- The de Broglie Hypothesis.
- The Heisenberg Principle.

Inquiry based activities to engage students in active collection of evidences, in autonomous drawing of conclusions and question formulation are almost completely absent. As a consequence the study of the subject is at high risk of being something students learn with a low level of understanding. The analogy with acoustic is barely used. Low-cost experiments, which would be useful to lead students toward a better understanding of fundamental concepts, are not proposed to the teacher.

This trend can be changed introducing some modifications in the pedagogical approach. In this chapter we propose and discuss a possible way to introduce Quantum Mechanics following a path that differs significantly from what is done in textbooks. A great effort is done in designing experimental and practical activities devoted to a better understanding of the key-concepts of the theory.

Some mathematical details are given, surely moving one step beyond what should be taught to students. This is done to give the teacher the opportunity to reach a general and global understanding of Quantum Mechanics, at least at introductory level. We strongly believe that starting from this work the teacher will be able to find and design his personal way for the teaching of such a beautiful part of Physics.

The Schrödinger equation

At the basis of Quantum Mechanics stands the Schrödinger equation.

A good introduction to Quantum Mechanics at secondary school level does not necessarily include this equation. However here we explore the possibility to present it to high-school students, without deriving it, using the mathematical knowledge covered by the Scientific Lyceum curriculum. As we will see this will allow us to gather some useful information about Quantum theory . The questions are:

1. Is it possible to introduce the Schrödinger equation and gather from it information on the Quantum Theory only working with the mathematical tools covered by the Italian Scientific Lyceum curriculum?
2. Is it worth it?

In this section we inquire into the first question also drawing some conclusion about the second. Anyway we would like to point out that, even if we may conclude that is better not to present the Schrödinger equation, this walk along this topic probably represents a useful set of information for the teacher who is going to teach Quantum Mechanics at secondary school level. It is our believe that the knowledge of the teacher, if possible, has to go beyond what is actually taught in the classroom.

The needed mathematical background

In this paragraph we report the list of the needed mathematical tools students should have in order to understand what is presented in this chapter.

- Functions
- Derivatives
- Complex numbers and their exponential form
- Integrals
- Familiarity with periodic functions like $sin(kx - \omega t)$, meaning of k (spatial frequency) and ω (angular frequency)

Notes on the Schrödinger's equation origins

Following this line of reasoning we begin with some consideration about how this equation was derived. Schrödinger was inspired by the existing connection between Physical Optics and Geometric Optics. In the first theory the wavelike nature of light is a first rank issue while in the second light moves along ray paths and the typical wave behaviours may be neglected. It is proved that Geometric Optic is the limiting case of Physical Optics as the wavelength approaches zero ($\lambda \rightarrow 0$)[37]. An easy way to understand this is considering the mathematical results we get when we introduce interference or diffraction at secondary school level. All the position y of the maximums and minimums of the patterns of light intensity after a single or multiple slit are proportional to the wavelength:

[37] Which means λ small with respect to the interaction, or spatial scale under study.

$$y \propto \lambda \tag{95}$$

Therefore, if $\lambda \rightarrow 0$ all the light intensity gets concentrated in a single spot (at $y = 0$) as we shall expect from a light ray. This is is something we might decide to highlight to students when we teach physical optics, it could come in handy when introducing Quantum Mechanics. The rigorous proof of this fact is a little bit more tricky and basically it consists of finding the solution for the d'Alembert wave equation in which the speed v of the wave is expressed by means of the refractive index $n = \frac{c}{v}$ which is supposed to be a slow varying function of the position in space $n = n(\vec{r})$. Then an approximation is carried out assuming the wavelength smaller than the typical dimension over which the refractive index changes significantly. Acting in this way one finds the *eikonal equation* which states that lights travels along rays which may be curved if the refractive index is not constant[38].

[38] H Goldstein. *Classical Mechanics*. Addison-Wesley, 1965

The Schröinger idea was that a similar situation could have occurred also for the classical mechanics which could have been a limiting case, as $\lambda \rightarrow 0$, of a wave mechanics yet to be discovered (Fig. 76). There was even another important clue supporting this way of thinking. The motion of a particle moving in a potential field V may be described in the Hamilton- Jacobi formalism with an equation which is substantially the same of the eikonal equation. Therefore there was also a bridge among Geometric Optics and Classical Mechanics. This is a quite technical consideration, that we may just use to understand what was known to a physicist of the first decades of the twentieth century. If we depict the situation as in figure 76, we can clearly notice the presence of a void tile asking for the discovery of a wave equation for a new wave mechanics.

We should add that in the same period the problem of the discrete spectrum of the Hydrogen atom was not explained by classic physics and that de Broglie was suggesting to interpret the quantisation of the Hydrogen energy levels by means of a wavelength λ associated to a particle. Therefore we may understand that there was a kind of

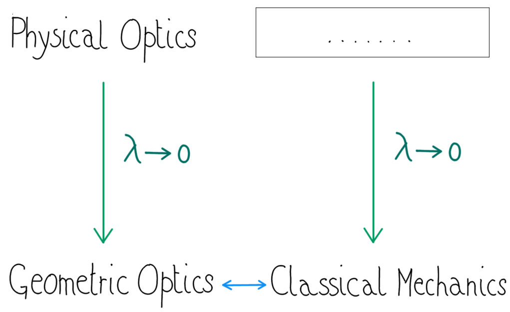


Figure 76: The knowledge situation in the first decades of the twentieth century.

pressure pushing the research towards the discovery of a theory to be placed on the missing tile of the scenario depicted before.

Bohr and de Broglie in a nutshell

As reminded in the previous section Schrödinger's work was not suspended on a vacuum, he was given a lot of guidance from the work of many physicists. We want to mention two important contributions regarding the unsolved problem of the atomic model: the Bohr idea to impose the quantisation of the angular momentum (1913)[39] and the de Broglie idea to associate a wavelength to a massive system like the electron (1923)[40]. Two where the unanswered questions about the atomic model:

[39] N Bohr. *On the Constitution of Atoms and Molecules, Part I*. Philosophical Magazine. 26 (151): 1-24. (1913)

[40] L de Broglie. *Ondes et quanta*. Comptes rendus de l'Academie des Sciences, vol. 177, pp. 507-510 (1923)

- According to the Rutherford model the electron's orbits around the nucleus should vary with continuity, thus suggesting a continuous emission spectrum coming from the atom itself. [This was in disagreement with the discrete nature of the observed atomic spectra (Fig. 77).]
- The accelerated motion of a charged particle around the nucleus would produce an electromagnetic radiation thus causing a continuous energy loss and, as a consequence, the atom instability.

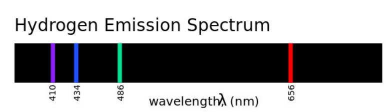


Figure 77: The discrete atomic spectrum of the Hydrogen atom.

The *Bohr's quantisation rule* was solving the problem of the discrete energy spectrum of the Hydrogen atom while still leaving unsolved the instability issue. He discovered that asking the angular momentum of the electron orbiting around the proton to be equal to integer multiples of $\hbar$ he was able to derive the correct formula for the discrete energy levels of the Hydrogen atom. We can see how it works, imposing the rule for the angular momentum L and reminding us the momentum expression for a circular orbit:

$$L = n\hbar \quad , \quad L = mvr \tag{96}$$

From these relations we find that the only allowed speeds are

$$v_n = \frac{n\hbar}{mr}\,. \tag{97}$$

Equating the Coulomb and the centripetal force we relate the speed to the orbit radius, so having the chance to derive the radii of the allowed orbits:

$$\begin{aligned} \tfrac{mv_n^2}{r} &= K\tfrac{e^2}{r_n^2} \\ r_n &= \tfrac{n^2\hbar^2}{me^2K}\,. \end{aligned} \tag{98}$$

We can get the de Broglie proposal in a complete non-historical way, just looking at the Bohr quantisation rule (96) and simply multiplying both members of the equation by 2π:

$$\begin{aligned} 2\pi p_n r_n &= nh \\ 2\pi r_n &= n\tfrac{h}{p_n} = n\lambda_n\,. \end{aligned} \tag{99}$$

in the second equation of the array (110) we are saying that for the nth energy level we have an orbit whose length is n times the nth wavelength. The French physicist derived his famous wavelength reasoning in a very different way. Just as a matter of curiosity his note on the Comptes Rendus de L'Academie starts with a relativistic consideration: the author associates a frequency to a particle according to the Einstein relation. The first equation of the paper is:

$$mc^2 = h\nu_0\,. \tag{100}$$

For those interested in understanding the de Broglie's initial conception of his waves we suggest the work of Lochak[41].

[41] G Lochak. *De Broglie's Initial Conception of De Broglie Waves.* In: Diner S., Fargue D., Lochak G., Selleri F. (eds) The Wave-Particle Dualism. Fundamental Theories of Physics (A New International Book Series on The Fundamental Theories of Physics: Their Clarification, Development and Application), vol 3. Springer)

Having discussed the origin of the Schrödinger equation and having considered two among the main aspects of the atomic research of the first two decades of the twentieth century we have a brief description of the historical scenario surrounding Schrödinger at the time he became challenged in the research for the equation of a new wave mechanics.

The equation

In 1926 Schrödinger published [42,43] two papers in which he presented an equation leading in a natural way to a discrete energy spectrum for the Hydrogen atom problem. The first paper begins with these words:

[42] E Schrödinger. *Quantisierung als Eigenwertproblem (Erste Mitteilung).* Ann. d. Physik 79 (1926) 361, a

[43] E Schrödinger. *Quantisierung als Eigenwertproblem (Zweite Mitteilung).* Ann. d. Physik 79 (1926) 489, b

"In this paper I wish to consider, first the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary

quantum conditions can be replaced by another postulate, in which the notion of 'whole numbers', merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the node-numbers of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules. The usual form of the latter is connected with the Hamilton-Jacobi differential equation, [...]".

The derivation of the equation is quite technical and even if we believe it is worth a deep study we will not enter in it. We may just say, in a very general and coarse way, that Schrödinger started from the Hamilton-Jacobi equation introducing in it the de Broglie relation. In a sense climbing upwards from classical mechanics and using the de Broglie relation as a foothold (Fig. 76). From the same words of the author we can appreciate the strength of its new proposal: the quantisation of the physical quantities is not imposed any more, it naturally descends from the solution of the equation in the same way it happens for the vibrating string equation. Today, the equation proposed by Schrödinger looks like this:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\,, \tag{101}$$

where V is the potential of the problem we are dealing with. In his paper after deriving it he solved it for the Hydrogen potential $\propto \frac{e^2}{r}$, reaching the right expression for the Hydrogen energy levels. Ψ is the wave function, the solution of the equation and it depends on the position x and the time t of the system we are dealing with. In the eraly years of quantum mechanics the physical meaning of the wave function was not clear at all for the entire physicist's community. According to the the current accepted interpretation (the *Copenhagen Interpretation*) its square modulus is a probability density dP/dx, meaning that the probability to find the system at a given time in the interval between x and x+dx may be evaluated in this way

$$\frac{dP}{dx}\cdot dx = |\Psi(x,t)|^2\cdot dx. \tag{102}$$

The probability to find the quantum system somewhere along the x axis is

$$\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\cdot dx\,, \tag{103}$$

and, of course has to be equal to 1.

> It is therefore clear that we are mainly interested in the $|\Psi(x,t)|^2$ quantity that gives us the idea of how the system at a given time is distributed over space.

From now on we will deal with the Schrödinger equation for a free system ($V = 0$) in one spatial dimension and we will have the chance to gather a lot of information working on it with the typical mathematical tools of the last year of Scientific Lyceum.

The free system equation

For a physical system moving freely in one dimension the equation becomes:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \quad (104)$$

We can easily recognise that this is a partial differential equation which is *linear* and *homogeneous*. It is linear because the unknown, the Ψ, and its derivatives appear only with a power of one, it is homogeneous because the constant term is zero.

Solutions of the free system equation

If we substitute the wave function

$$\Psi(x,t) = e^{i(kx-\omega t)} \quad (105)$$

in the equation (104) we discover that it may be considered a solution as long as we accept that

$$\hbar\omega = \frac{\hbar^2k^2}{2m}. \quad (106)$$

The equation that solved "in a natural way" the Hydrogen atom problem is asking a tribute. The first member of the request (106) is $\hbar\omega$, something physicists were already familiar with, since it is how they were used to write the energy E of a photon. The second member is the kinetic energy $p^2/2m$ since, from the de Broglie relation we know that $p = \hbar k$ ($p = \frac{h}{\lambda}$, $k = \frac{2\pi}{\lambda}$). Therefore the Schrödinger equation is asking to express the energy of a massive system with the same expression used for a photon. Physicists accepted this request.

Of course there are differences between a photon and a massive system. From Special relativity (c=1) we know that for a photon E=p, thus $\hbar\omega = p$. For a massive particle we are saying something different: $\hbar\omega = p^2/2m$.

We now have in our hands a solution of the free system equation, but before moving one step further let us discuss what does it represent. The next section is devoted to this issue.

What does $e^{i(kx-\omega t)}$ mean?

The wave function (105) may be represented as a vector in the complex plane (Fig. 78) with a phase $\alpha = kx - \omega t$ which depends both on position and time.

At every time t we can imagine one vector like the one depicted in figure 78 associated to each point of the x axis. Each one of these vectors will have a different phase (will be rotated with respect to the previous one) since x varies moving from one vector to the next. The length over which a complete rotation of the vector occurs is defined by the wave number $k = 2\pi/\lambda$, which fixes the value of the wavelength. Thus if, like in figure 79, we take a picture of our Ψ at a certain time t we will see a lot of arrows doing a complete 2π rotation along one wavelength λ .

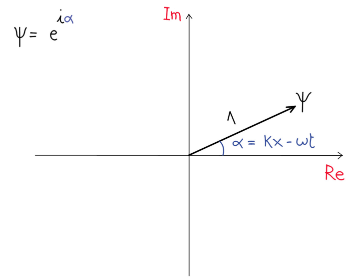


Figure 78: .

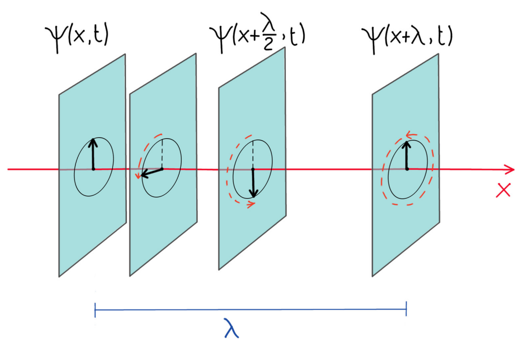


Figure 79: Rotation of the wave function as we move along the x-axis.

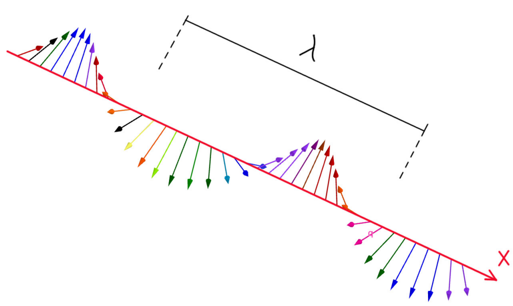


Figure 80: The wave function $e^{i(kx-\omega t)}$ along the x-axis at a fixed time t.

> Fixing the time t, we see a set of infinite vectors along the x-axis, each one rotated with respect with the previous. A complete rotation occurs after a wavelength λ determined by the wave number k (Fig. 79 and 80).

Now let us fix the x while leaving the time t free to flow (Fig. 81).

We are now focused on a single point of the x axis and, again, we see a vector who changes its phase (rotates) according to the ωt term. The time over which a complete rotation occurs is determined by the angular frequency $\omega = 2\pi/T$, which fixes the period (Fig. 81).

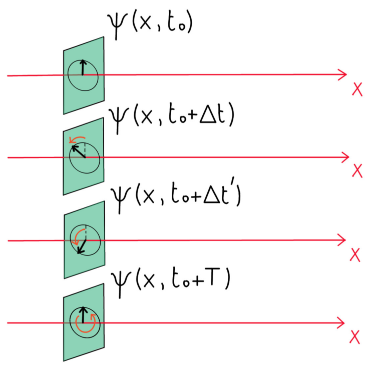


Figure 81: Rotation of the wave function along one time period T.

> Fixing the position x, we are focused on one single vector which rotates around the axis with angular speed ω. A complete rotation occurs after a period T determined by ω (Fig. 81).

In conclusion each vector is rotating with the same ω around the x axis and each vector is a little out of phase with respect to the previous one so that in one λ the initial orientation is restored. Therefore the situation is not static as we may think looking at figure 80 which is only one frame of a dynamic scenario. The vectors that are oriented upwards will rotate down and vice versa giving a net effect of a propagation along the x axis (Fig. 82) a sort of "ola" like the one people do in a stadium. If we focus on two consecutive humps, for example of the first plot of figure 82, we may say that, due to "time-rotation", in one period T both the humps will leave their initial position moving to the right. After one complete period of time since the vector will have completed the rotation two humps will appear exactly where they initially were. The net effect is that each hump gradually moves to the right occupying the position of the hump next to it. This also means that in T seconds the "ola" has travelled exactly one wavelength. From this we may conclude that the speed of propagation is

$$v = \frac{\omega}{k} \tag{107}$$

The wave function (105) is the so called *plane wave*, finally we have an idea of what it represents. We can start doing some consideration about its physical meaning.

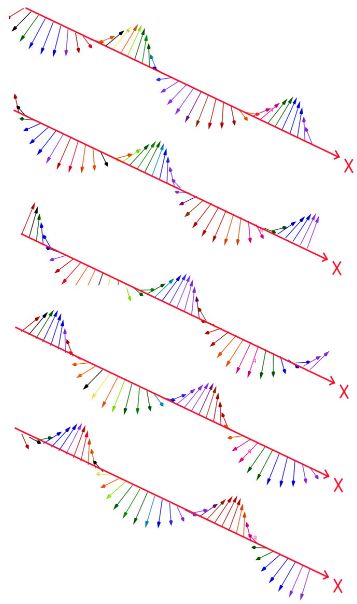


Figure 82: Each plot corresponds to an instant slightly later than the previous one. Propagation may be noticed.

Physical meaning of the plane wave Ψ

We have seen that the wave function

$$\Psi(x,t) = e^{i(kx-\omega t)} \tag{108}$$

is a solution of the Schrödinger equation for a free system and we have also discussed its mathematical meaning. Now we want to deal with the physical interpretation of such a solution; what kind of system does this function represent?

First of all we have to remind that the physics lies in the square modulus $|\Psi(x,t)|^2$ of the wave function, which, in our case, is

$$|e^{i(kx-\omega t)}|^2 = 1\,. \tag{109}$$

Here we encounter the first problem: this wave function gives a finite and constant probability density over all the space. The system this solution is describing has a flat probability distribution, can be everywhere in space. Of course this is something we do not like at all. We could also say that with such a probability density there is no way to make the integral (103) equal to 1, which is a crucial request for any probability density function.

Moreover there is a second issue that we have to consider. We would expect the wave function to propagate with the speed of the system p/m, but, as already pointed out, the propagation of this wave function occurs at a speed ω/k which is $p/2m$ (remember equation 106) :

$$\begin{aligned} \hbar\omega &= \frac{\hbar^2 k^2}{2m} \\ \frac{\omega}{k} &= \frac{\hbar k}{2m} \\ \frac{\omega}{k} &= \frac{p}{2m}\,. \end{aligned} \tag{110}$$

A plane wave Ψ has two problems:

1. Leads to a flat probability density, meaning that the particle is equally likely to be anywhere (space localisation problem).
2. Propagates with a speed equal to $p/2m$ which is different from the speed of the system.

In the next section we will address a special feature of the Schrödinger equation that will give us the possibility to solve both issues.

Superposition Principle and acceptable solutions

We have seen that a plane wave has at least two problems that lead us to discard it as a suitable candidate for representing a physical system. Now we want to discuss a mathematical aspect of the Schrödinger equation that allows us to solve both problems.

From the already observed linearity and homogeneity of the Schrödinger equation (notice that these features still hold for the complete equation (101)) we may prove a very important fact: given any set of solutions for the equation, any linear combination of the elements of this set still is a solution. This is called the *superposition principle*. To prove this we may consider the equation as a differential operator, S, which acts on the wave function, so that we may write the Schrödinger equation in this way:

Math Box - The Superposition Principle

$$\begin{aligned} S\Psi &= 0 \\ \text{where} \quad S &= i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \end{aligned} \tag{111}$$

If an operator is linear this relation holds:

$$S(a\,\Psi + b\,\Phi) = a\,S\Psi + b\,S\Phi\,. \tag{112}$$

For example, high-school students know that the derivative is a linear operator, since:

$$\frac{d}{dx}(a\,f(x) + b\,g(x)) = a\,\frac{d}{dx}f(x) + b\,\frac{d}{dx}g(x)\,. \tag{113}$$

Now it is easy to prove that if we know two solutions of the equation Ψ_1 and Ψ_2, due to the linearity and homogeneity any linear combination of them still is a solution. Since they are solutions we have:

$$S\Psi_1 = 0 \quad , \quad S\Psi_2 = 0\,. \tag{114}$$

On the other hand linearity and homogeneity guarantee that:

$$S(a\,\Psi_1 + b\,\Psi_2) = a\,S\Psi_1 + b\,S\Psi_2 = 0\,. \tag{115}$$

Equation (115) proves that the linear combination still is solution.

The superposition principle is what we need, if plane a wave is not an acceptable solution maybe a superposition of plane waves may solve at least the problem of the localisation in space. Students might already know, from what they have learned in acoustics, that the superposition of two sinusoidal functions may give rise to beats, a first example of waves which become localised in an infinite series of lumps. Consequently we are not acting completely blindly, we have a clue that suggests us to superimpose plane waves. Furthermore the mathematical model we are using (the equation) allows us to follows this guide line. Thus we are going to analyse in some detail what is a superposition of plane waves with a wave number (a momentum) dispersed around a central value k_0:

$$\Psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} e^{i(kx-\omega t)}dk\,. \tag{116}$$

In order to manage this integral we have to do some mathematics which, in a simplified version, is schematically expressed in the Mathematical Box below.

Math Box - The integral

The main features of the integral (116) may be reached in two separate steps.

- **1st Step.**

  The first step consists of understanding the shape of the function $\int_{-\Delta k}^{+\Delta k} e^{i\,kx} dk$ which is a sum over a certain interval of plane waves with different values of k. This integral is a simplified form of the original integral, but in it we have the essentials of what happens when we superimpose plane waves (in a sense it could be interpreted as the complete integral (116) evaluated at a well defined time: $t = 0$).

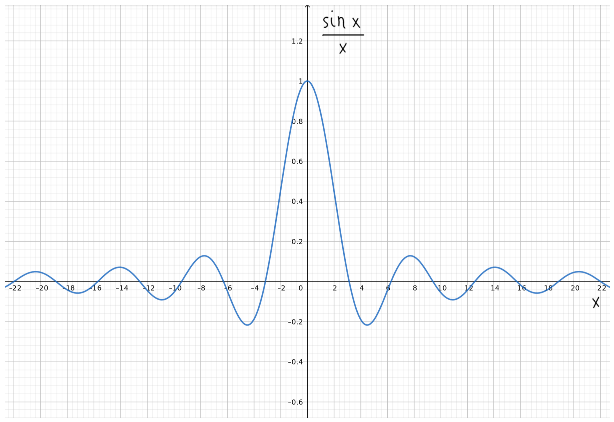
  We know that $e^{i\,kx} = cos(kx) + i \cdot sin(kx)$ and we may notice that since the sine is an odd function it will gave zero contribution to the integral, which then becomes:

  $$\int_{-\Delta k}^{+\Delta k} cos(kx)\, dk\,. \tag{117}$$

  The primitive function of $cos(kx)$ is $\frac{sin(kx)}{x}$, thus

  $$\int_{-\Delta k}^{+\Delta k} cos(kx)\, dk\, = 2 \cdot \frac{sin(\Delta k\, x)}{x} \tag{118}$$

  The resulting function of x is the so called *sinc* function and its shape is shown in the figure below.



The problem of the *space localisation* is then solved, we have understood that the wave function resulting from the (116) will be a *sinc* function which is significant only in a well localised space region.

- 2^{nd} **Step.**

  The second step consists of dealing with the propagation modalities of the wave function (116). We will address the problem in a smart way getting the right answer with few calculations. The sinc function has a huge peak that we will call the maximum. The maximum is given by the wave planes with different k that do not tend to interfere destructively. Actually this happens where the phase $\phi = (kx - \omega t)$ does not change significantly with k; in other words where

  $$\frac{d\phi}{dk} = 0 \quad \Longrightarrow \quad x - \frac{d\omega}{dk} t = 0\,. \tag{119}$$

  So we have discovered that the position of the maximum, and hence all the sinc function representing the integral, propagates along the axis according to the equation $x = \frac{d\omega}{dt} t$. A uniform motion is then associated to the wave function and the speed turns out to be $\frac{d\omega}{dk}$.

  Now we assume the amplitude of the integration interval $[k_0 - \Delta k, k_0 + \Delta k]$ to be small so that we may approximate the function $\omega(k)$ with the tangent line in k_0:

  $$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_0 \cdot (k - k_0) \tag{120}$$

  As a consequence the propagation speed of the wave function (116)) turns out to be $\left.\frac{d\omega}{dk}\right|_0 = \hbar k_0 / m = p_0 / m\,.$

The idea used to find the propagation speed was taken from the Bohm's book on quantum mechanics[44].

[44] D Bohm. *Quantum Theory.* Dover Publications, Inc. (1989)

> Surprisingly also the second problem, related to the propagation speed, is solved. If we define a wave function as a superposition of plane waves we have a probability density propagating with the speed p_0/m.

The Heisenberg Principle almost for free

Our discussion about a physical solution of the Schrödinger equation led us very close to the Uncertainty Principle. So we will introduce it here for the first time in a naive way, coming back to it later when we will also propose an educational environment where, via an acoustic analogy, students will have the opportunity to investigate it and better understand its physical meaning.

We have talked a lot of the wave function

$$\Psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} e^{i(kx-\omega t)} dk\,. \qquad (121)$$

which may be considered a good solution for the free system equation. We have seen that it is distributed in momentum[45] and we can evaluate the dispersion as being Δk. With our calculations we have directly experienced that it is the dispersion in momentum that originates the distribution in space of the system (that is to say where the sinc function is significantly important). We can easily evaluate the dispersion Δx of the space distribution of the system by finding the first zero of the sinc function (Fig. 83, equation 118):

[45] Since the wave number k is strictly related to the momentum p ($p = \hbar k$), for simplicity, we will refer to the wave number as a momentum.

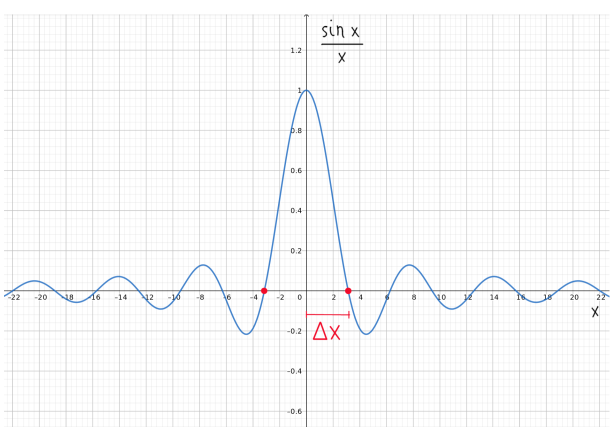


Figure 83: Space distribution of the wave function solution of the free Schrödinger equation. The dispersion Δx may be evaluated through the distance of the two zeros shown in the plot.

$$\Delta k \cdot x = \pi \qquad \Longrightarrow \quad x = \pi/\Delta k = \Delta x\,. \qquad (122)$$

We have the evidence that the product of the two dispersions is constant

$$\Delta x \cdot \Delta k = \pi \,, \tag{123}$$

the more the system is dispersed in momentum the less is dispersed in space. Remembering that $k = p/\hbar$ we can write equation (123) in this way:

$$\Delta x \cdot \Delta p = \frac{h}{2} \,. \tag{124}$$

Energy and operators in the Schrödinger equation

We can interpret the Schrödinger equation (101) as an equality between operators acting on the wave function Ψ:

$$\left(i\hbar \frac{\partial}{\partial t} \right) \Psi = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \right) \Psi \,, \tag{125}$$

We focus on the operator at the right-hand side of the equation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \,. \tag{126}$$

It is the sum of a spatial second derivative and a potential term that simply multiplies the wave-function. For dimensional reasons both terms are energies (the proof is in the Math Box).

Math Box - **Units of the space derivative operator**

$\hbar$ is $E \cdot t \implies$ dimensions are $J \cdot s$

$\frac{\partial}{\partial x}$ is $\frac{1}{l} \implies$ dimensions are $\frac{1}{m}$

therefore the dimensions of $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$ are $\frac{J^2 \, s^2}{kg \, m^2} = J$.

It is straightforward to see that the time derivative operator on the left-hand side,

$$i\hbar \frac{\partial}{\partial t} \quad , \tag{127}$$

has energy units too.

Thus we have proved the connection between the Schrödinger equation operators and energy. Now we may notice a strong similarity between the operator (126) and the expression for the total energy of a system:

$$E = \frac{\hbar^2 k^2}{2m} + V \,. \tag{128}$$

This consideration can lead us to this idea: associate to $p^2 = \hbar^2 k^2$ the operator $-\hbar^2 \partial^2 / \partial x^2$. Therefore we will say that the operator (126) will represent the total energy operator! Since the operator on the left-hand side of the (125) is substantially equated to the one on the right-hand side, it has to represent the total energy of a system too. We will call $i\hbar\,\partial/\partial t$ the Hamiltonian operator H.

We have found a set of rules to write the correct Schrödinger equation for a system:

1. Write the Hamiltonian of the system (the total energy),

$$H = \frac{p^2}{2m} + V. \tag{129}$$

2. Associate operators to the Hamiltonian and to the kinetic energy:

$$\begin{aligned} H &\to i\hbar\tfrac{\partial}{\partial t} \\ p^2 &\to -\hbar^2\tfrac{\partial^2}{\partial x^2}. \end{aligned} \tag{130}$$

3. Substitute the operators in the equation (129) and apply it to the wave-function Ψ.

This procedure leads to the correct equation!

Systems with constant total Energy

When the total energy is a constant E, the operator (126) acts on the wave-function like a multiplicative constant. Therefore, if we succeed in solving the equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\Psi = E\,\Psi\,, \tag{131}$$

we find the all the possible wave-functions representing the system with definite energy values and we also find the spectrum of the allowed energies. Of course the outcomes strongly depend on the potential V which defines the physical situation we are dealing with.

The equation (131) is called the *stationary Schrödinger equation*.

Solving the stationary Schrödinger equation for a given ptential V we find:

1. The set of the allowed energies of the system (*The Energy Spectrum*).

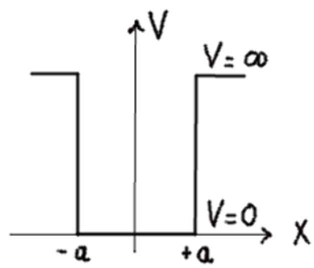
2. The wave-functions representing the system in each of the allowed constant energies. These wave-functions give us the probability density distribution in space of the system!

In the next sections we will deal with the simplest potential problem we can afford and we will find the solutions of the stationary Schrödinger equation. Furthermore we will discover that a simple experiment carried out at a coffee bar can lead us, by acoustic analogy, to the exact solutions. The example will be of great help in understanding the essential features of quantum systems bounded in a potential well. Moreover it will provide to the teacher a way to present such features in a relatively simple way.

System in a infinite potential well

We consider a system with mass m, bounded in a potential well $V(x)$ which is zero in the interval $]-a,+a[$ and infinite elsewhere (Fig. 84).

Figure 84: The infinite potential well.



Since the well is infinitely deep the system has no chance to be found outside, thus the outside solutions of the quantum problem are:

$$\Psi(x) = 0, \quad x > |\, a \,| \tag{132}$$

To discover the energy spectrum and the probability distribution of the system inside the well we write and try to solve the *stationary wave equation*. Since inside the well the potential is zero the equation becomes:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = E\,\Psi\,, \tag{133}$$

which we may rewrite in this way:

$$\frac{\partial^2\Psi}{\partial x^2} = -k^2\,\Psi\,, \tag{134}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$. Equation (134) substantially asks for wave-functions whose second derivative with respect to x is the opposite of the wave-function itself. The solutions are therefore well known at secondary school level[46] and are related to the trigonometric func-

[46] Here we are considering the fifth year of Scientific Lyceum for the Italian educational system.

tions $sin(kx)$ and $cos(kx)$, which are separately solutions of the (134). Therefore, as we have learned previously, the general solution will be a linear combination of them[47]:

$$\Psi(x) = A\,sin(kx) + B\,cos(kx). \quad (135)$$

[47] A possible pathway the teacher could choose to adopt in order to simplify a bit the mathematical scenario consists of saying that the solution is $sin(kx)$. This choice leads to the correct allowed wavelengths and to an energy spectrum which differs from the exact one by a constant factor 1/4. Some details are lost but the core of the physics remains intact.

Now we have to ask for the *continuity* of the solution, in other words we have to impose that at the boundaries (in $\pm a$) the wave-function (135) has to connect with the null wave-functions just outside the well:

$$\Psi(-a) = -A\,sin(ka) + B\,cos(ka) = 0 \quad (136)$$

$$\Psi(a) = A\,sin(ka) + B\,cos(ka) = 0$$

Since we are asking at the same time the sum and the difference of the same quantities to be null, we may conclude that both have to be zero[48]:

$$A\,sin(ka) = 0 \quad \cap \quad B\,cos(ka) = 0\,. \quad (137)$$

[48] We neglect the case $A = B = 0$ which would mean $\Psi = 0$ and hence absence of the physical system.

The conditions are respected if

c

$$B = 0\,, \quad ka = n\pi = 2n\,\pi/2\,, \quad n = 1, 2, ...$$
$$\cup$$
$$A = 0\,, \quad ka = (2n+1)\pi/2\,, \quad n = 0, 1, 2, ... \quad (138)$$

Merging together the two cases[49] we have the allowed values for k inside the potential well:

$$k = k_n = n\frac{\pi}{2a}\,, \quad (139)$$

[49] In the first case n starts from 1 since $n = 0$ would give $k = 0$ and hence a null wave-function.

with $n = 1, 2, ...$. The energy of the particle inside the well ($V(x) = 0$) is only kinetic, thus we have derived the energy spectrum of the bounded system which turns out to be a discrete spectrum, i.e. a *quantized spectrum*:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{8ma^2} \cdot n^2\,. \quad (140)$$

Moreover we have discovered the wave functions Ψ_E of the states with constant energy E. They are given by the equation (135) evaluated within the conditions (138).

We have derived a result in the physical scenario of a very simple potential $V(x)$, however it turns out that the quantization of the energy spectrum is a very general feature concerning every bounded quantum system.

Every quantum system bounded in any potential well exhibits the quantization of the Energy levels.

Therefore we may also explain why the energetic spectrum of the Hydrogen atom is quantized. In this situation the potential well is the Coulomb potential (Fig. 85).

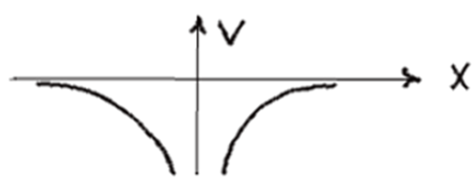


Figure 85: The Coulomb potential well for the Hydrogen atom.

It is also clear that

The discretization of the spectrum is originated by the imposed boundary conditions.

Eventually, you will have notice that

The mathematical concept behind the quantization of the energy spectrum of a quantum system is exactly the same we face in problems like acoustic waves trapped among the ends of a vibrating string or of a pipe.

The infinite square potential well in a nutshell

For the sake of clarity we summarise the main features related to the quantum problem we have encountered in the previous section.

- Schrödinger Stationary equation:

$$\frac{\partial^2 \Psi}{\partial x^2} = -k^2\,\Psi$$

- The stationary equation is a differential equation in the variable x, the solutions are functions of x, $\Psi(x)$.
- The solutions of the stationary equation, $\Psi_E(x)$, represent states with defined and constant energy E

- Quantization derives from the boundary conditions imposed by the presence of the potential well which confines the system in a finite region of space.
- Wave numbers are quantized:

$$k_n = \frac{n\pi}{2a} \quad \Longrightarrow \quad k_n \propto n$$

- Wavelengths are quantized:

$$\lambda_n \propto \frac{1}{n}$$

- Energies are quantized:

$$E_n = \frac{\hbar^2\pi^2}{8ma^2} \cdot n^2 \quad \Longrightarrow \quad E_n \propto n^2$$

- Frequencies are quantized: energy is proportional to ω ($E = \hbar\omega$) hence frequencies are quantized as well, with a proportionality to n^2.
- We just state, without proving it, that the wave-function $\Psi(x, t)$ is given by the product of the solution of the stationary equation $\Psi(x)$ and the exponential factor $e^{i\omega t} = e^{i\frac{E}{\hbar}t}$, where E is the energy of the stationary state:

$$\Psi(x, t) = \Psi_{E_n}(x) \cdot e^{i\frac{E_n}{\hbar}t} \tag{141}$$

Infinite potential well, vibrating strings and pipes

From the educational point of view it is of fundamental importance to notice that the stationary Schrödinger equation of the infinite potential well (134) is the same equation we encounter with the problem of a vibrating string fixed at both ends or, if we prefer, of a pipe opened at both ends. In these cases we have the d'Alembert equation:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0\,, \tag{142}$$

where $y(x, t)$, for example, represents the vertical displacement of a point of the string in position x (Fig. 86). In the case of the open ended pipe $y(x, t)$ may be the pressure displacement, i.e. the deviation from the atmospheric pressure along the pipe.

We look for harmonic solutions[50], for example:

$$y(x,t) = y(x) \cdot cos(\omega t)\,. \tag{143}$$

[50] We assume that each point of the string will oscillates with a given angular frequency ω, so that the time dependence of the solution will be given by the factor $cos(\omega t)$. The general solution will simply be a linear combination of harmonic solutions.

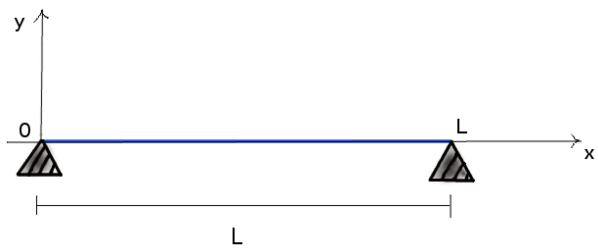


Figure 86: Vibrating string fixed at both ends.

Since the second derivative of $cos(\omega t)$ with respect to time is $-\omega^2\, cos(\omega t)$, the equation reduces to

$$\frac{\partial^2 y}{\partial x^2} = -\frac{\omega^2}{c^2}\, y \tag{144}$$

and finally, since for an acoustic wave $k = \omega/c$, to

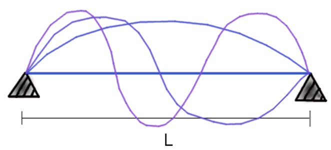
$$\frac{\partial^2 y}{\partial x^2} = -k^2\, y \tag{145}$$

which is formally identical to the quantum equation (134)! This circumstances lead to a mathematical and an educational consequence:

1. For the vibrating string or open ended pipe we expect a quantized energetic spectrum and the same dependence from n^2 we found in the quantum problem (140).
2. Students have the opportunity to gather a deep understanding of the quantisation of the energy levels investigating a vibrating string or an open ended pipe.

In this section we will prove the first statement, taking for granted the theory of vibrating strings or pipes. The next section will address how to move from experiments on a pipe to the quantisation of the energy levels for a bounded system.

Math Box - Vibrating String Basics



The basic set of equations describing the vibration modes of a string fixed at both ends are:

$$y(x;t) = A \cdot sin(kx) \cdot cos(\omega t) \tag{146}$$

$$L = n \cdot \frac{\lambda}{2} \quad \rightarrow \quad n = 1, 2, \ldots . \tag{147}$$

$$\nu_n = n \cdot \frac{c}{2L} = f_0 \cdot n \quad \rightarrow \quad n = 1, 2, \ldots . \tag{148}$$

The first one describes the shapes of the vibrating modes. The factor $sin(\omega t)$ tells us that every point of the string oscillates up and down with the "frequency" ω. The $sin(kx)$ factor determines the amplitude of the oscillation at each point, giving the well known shapes to the vibrating modes (remember that k defines the wavelength λ).
The second equation is originated by the boundary conditions, asking for the string to be fixed at both ends. A whole number of half wavelengths has to be exactly contained between the two ends. This rule determines the quantization of k and, as a consequence, of the angular frequency ω and of the frequency ν, see equation (148).

Math Box - The Energy Spectrum of a Vibrating String

We prove that the energy spectrum of a vibrating string depends on n^2, exactly as it happens for a quantum system trapped in a infinite square potential well.
The length of the string may be divided in a finite number of parts each one with a mass Δm. If we focus on the central point of one of these parts, we fix the x and the first two factors in the equation (146) behave as constants. The third factor represents the motion of an harmonic oscillator: $y(t) = cos(\omega t)$.

Thus the total energy of that part is:

$$\Delta e = \frac{1}{2}\Delta m v^2 + \frac{1}{2}k_{el.}y^2 \tag{149}$$

Where $k_{el.}$ is the elastic stiffness of the oscillator. Evaluating the velocity of the point from the law of motion $y(t)$ ad recalling that for the harmonic motion we have $\omega^2 = \frac{k_{el.}}{\Delta m}$, we may express the total energy as

$$\Delta e = \frac{1}{2}\Delta m\omega^2 sin^2(\omega t) + \frac{1}{2}\Delta m\omega^2 cos^2(\omega t) = \frac{1}{2}\Delta m\omega^2 . \tag{150}$$

For a vibrating string the allowed angular frequencies are

$$\omega_n = \frac{c\pi}{L} \cdot n\,, \tag{151}$$

therefore the energy spectrum turns out to be quantized:

$$\Delta e_n = \frac{c^2\pi^2\Delta m}{2L^2} \cdot n^2 . \tag{152}$$

Equation (188) represent the quantized energy of a single part of the string, however integrating Δe_n over all the length of the string, the n^2 dependence of the total energy E_n remains (see Appendix for the detailed calculation).

We can therefore write a summary of the main features concerning vibrating strings or open ended pipes which closely resembles what we wrote for the quantum system trapped in an infinite square potential:

- The d'Alembert equation for a vibrating string (or open ended pipe) is:
$$\frac{\partial^2 y}{\partial x^2} = -k^2\, y$$
- This equation is a differential equation in the variable x, therefore the solutions are functions of x: $y(x)$.
- The solutions $y_E(x)$, represent states with defined and constant energy E
$$y_E(x) = sin(kx)$$

- Quantization derives from the boundary conditions imposed by the presence of conditions that confine the system in a finite region of space (clamped ends of the strings, sudden open endings of the pipe, both causing backward reflections).
- Wave numbers are quantized:

$$k_n \propto n$$

- Wavelengths are quantized:

$$\lambda_n = \frac{2L}{n} \quad \Longrightarrow \quad \lambda_n \propto \frac{1}{n}$$

- Frequencies are quantized

$$f_n = \frac{c}{2L} \cdot n \quad \Longrightarrow \quad f_n \propto n$$

- Energies are quantized:

$$E_n \propto n^2$$

- The solutions (x, t) is given by the product of the solution of the d'Alembert equation for harmonic oscillations and the factor $cos(\omega t)$:

$$y(x, t) = y_{E_n}(x) \cdot cos(\omega_n t) = sin(k_n x) \cdot cos(\omega_n t)$$

A coffee with Schrödinger: introducing a quantum concept at the coffee bar.

As previously mentioned, the strong similarity between a quantum system (system in an infinite square potential) and a classical system (vibrating string or open ended pipe) gives to the teacher a great opportunity to introduce with an inquiry activity one of the major aspects of quantum mechanics: the *quantization of the energy levels*.

The Educational Idea

Working at a coffee bar with a straw (open ended pipe) students can carry out, without the need of a Physics Lab, a lot of investigations understanding the origin of the quantization, measuring the quantized frequency spectrum and even discovering its proportionality to n.

The **Learning Environment** would be:

1. Informal.
2. Based on autonomous inquiries of groups of students (Inquiry Based Science Education - IBSE - at a level of freedom chosen by the teacher).
3. Creative (students would be asked to design several aspects of their own experiment).

Two possible way to start the activity with a classroom.

- (During the curricular school-time)

  Start the lesson inside the classroom and, after a short introduction, surprise students inviting them to the canteen.

- (As an homework assignment)

  Ask students to form groups of two or three, and to meet in the afternoon in a coffee bar where to carry out the investigation.

We would like to point out that, at least in the Italian educational system, an homework like the one mentioned in the box above strongly differs from what the typical home assignment is. As already pointed out in the introduction to this work, generally the teaching of physics is based on lessons devoted to the introduction of a theoretical concept followed by a list of exercises designed to make students familiar with the topic. What we are suggesting proposing the experiment at a coffee bar is another example of 3DTP[51]. A good opportunity to create a contact among students, physics and data from a real experiment. This kind of activities are of great importance in the teaching of physics. Carrying out the required tasks student will face a lot of practical issues which are very common for every physicist, but that almost never appear in a teaching only based on the textbook. Just to mention one, student will unavoidably encounter the problem of the *background noise* generated by the environment in which they will carry out their measurements.

In the following box we will give a general description of the experiment with the straw[52].

[51] 3-Dimensional Teaching of Physics (see Thesis Introduction).

[52] L Galante et al. *Acoustic with a bic pen*. La Fisica nella Scuola - AIF, XLVI, 2: 54-58, 2013

Description of the experiment

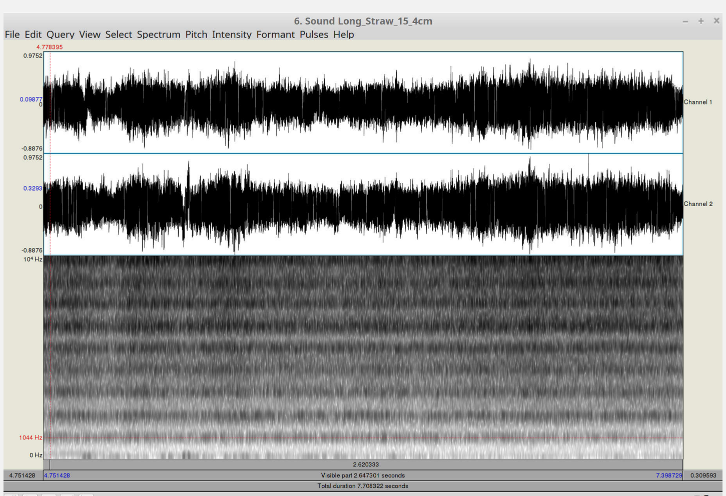
Needed material:

- a couple of straws with different lengths (for example $L_1 \sim 15{,}0\,cm$, $L_2 \sim 5{,}0\,cm$)
- One mobile (or one PC) with a sound recorder app.
- Free spectrum analyser software (we suggest, and we have used, Praat a free software for persona computers).

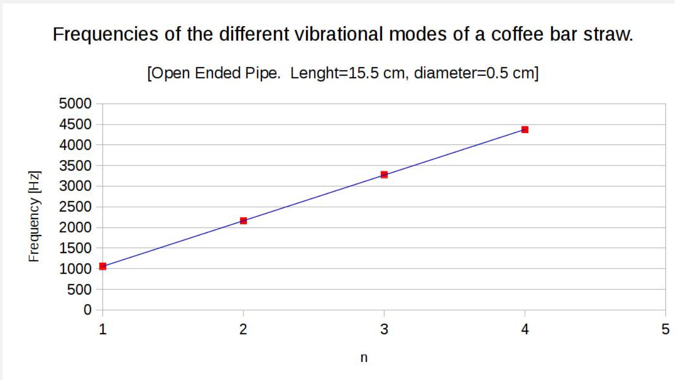
Blow gently over the opening of the long straw and record the sound with the mobile. The air column in the straw acts like the string closed at both ends: at each opening the pressure disturbance will be reflected backward. In the recorded sound we will find the main vibrating modes of the air column. In order to measure the quantized frequencies spectrum

$$\nu_n = n \cdot \frac{c}{2L} = f_0 \cdot n \quad \rightarrow \quad n = 1, 2, \ldots$$

we can perform a Fourier analysis with a free software, for example Praat. In the following figure the sound in the time domain (black upper plot) and the spectrogram (gray-scale lower plot) is shown for the long straw ($L = 15.4\,cm$). In the spectrogram the frequency is represented along the vertical axis in Hz, along the horizontal axis we have the time duration of the sound. Clicking somewhere on the spectrogram we get in return the corresponding y coordinate: the frequency.



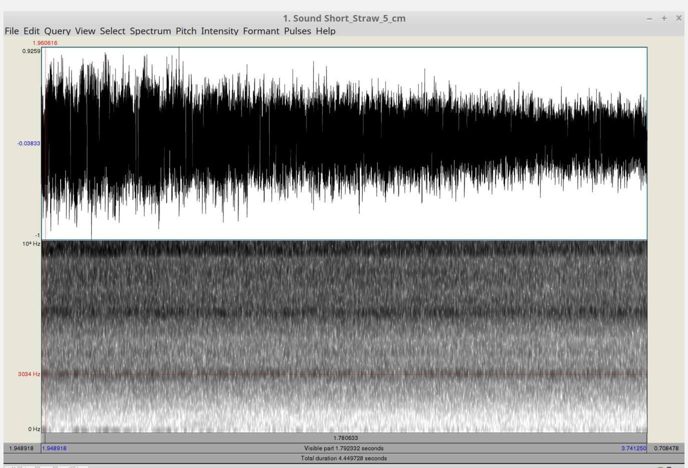
We have therefore the opportunity to clearly detect the quantization of the frequencies and even to measure the spectrum. In the plot below the first four frequencies of the spectrum are plotted as a function of n. It is evident that the expected relation $f_n = f_0\, n$ with $n = 1, 2, \ldots$ is respected.



If we perform the same experiment with the short straw we still get a quantized frequency spectrum (see image below), but this time the distance among equally spaced frequencies is bigger.
Students have the opportunity to see that, if $L \rightarrow \infty$, the distance among consecutive levels fades away: from a discrete spectrum we move to a continuum.

$$\Delta f = f_n - f_{n-1} = \frac{c}{2L},$$

Something that was expected, since for an infinite pipe the system is no more spatially bounded.



What can be learned form the straw experiment

In the following lines we summarise what students may learn from this experiment an how to use it to introduce the quantisation.

1. The physical system we are working with is an air column inside a pipe (the straw) opened at both ends.

2. Blowing on one open end of the pipe we put energy in the system.

3. (Experimental result). Frequencies are quantized ($f \propto n$) exactly as they are in a vibrating string fixed at both ends.

4. This means that the ends of an open ended pipe act like mirrors as the fixed ends of a clamped string.

5. When we put energy in the system, it stores[53] energy in a number of vibrational modes, each one with a well defined frequency which are simultaneously present in the system (superposition is at work!)[54]

6. (Frequency quantization in the pipe). It is important to stress that the vibrational modes in a pipe arise from two facts: the continuous reflection of the waves at the ends of the pipe and the boundary conditions asking for the solution to be zero at both the open ends. The superposition of the reflected waves moving back and forth can give the cancellation at the boundaries. The quantization rule is

$$L = n\frac{\lambda}{2}\,. \tag{153}$$

We can quickly prove this fact writing the equation of the pressure wave inside the pipe, that is the sum of a wave with a certain k (for example propagating leftwards) and of the corresponding reflected one:

$$y(x,\,t) = sin(kx - \omega t) - sin(-kx - \omega t)\,. \tag{154}$$

The reflected wave is represented by the second term, since it is symmetric to the incoming wave with respect to the x-axis we have to put a minus in front of it. The argument of the sine function is arranged so that the propagation is rightwards. The expression (154) may be rearranged as follows:

$$\begin{gathered} sin(kx - \omega t) + sin(kx + \omega t) = \\ sin(kx) \cdot cos(\omega t) - cos(kx) \cdot sin(\omega t) + \\ + sin(kx) \cdot cos(\omega t) + cos(kx) \cdot sin(\omega t) = \\ \\ = 2 \cdot sin(kx) \cdot cos(\omega t)\,. \end{gathered} \tag{155}$$

[53] We use the verb "store" because the continuous reflection of the perturbations confine the energy inside the pipe.

[54] Each mode is a possible solution of the equation modelling the physical scenario. The equation is the d'Alembert equation which is linear and homogeneous thus allowing linear combinations of any set of solutions.

Imposing the boundary conditions $y(0,t) = 0, \quad y(L,t) = 0$ we have:

$$k = n\frac{\pi}{L}, \quad \Longrightarrow \quad L = n\frac{\lambda}{2}. \tag{156}$$

7. (First contact with the quantum problem). Since the stationary Schrödinger equation for the quantum problem is formally identical to the equation for the pipe, we may expect the same situation to arise in the quantum scenario. The infinite potential well will force the wave-function $\Psi(x)$ to be null outside the boundaries hence, for continuity, also null at the boundaries! In straight analogy with the pipe or the string we may argue a quantization of wave-lengths to occur:

$$2a = n \cdot \frac{\lambda}{2} \quad \Longrightarrow \quad k = \frac{\pi}{2a} \cdot n\,.$$

8. (Second contact with the quantum problem). The quantum system is free inside the infinite square well, thus its energy is $E = \frac{\hbar^2 k^2}{2m}$. Thus from the quantization of the wave number We have the quantized spectrum of the quantum system:

$$E_n = \frac{\hbar^2 \pi^2}{8ma^2} \cdot n^2\,.$$

9. (Energy levels in a pipe). The energies of the vibrational modes of a pipe are quantised as well. As shown in equation (150), mathematical box at page 141, energies depend on ω^2 and $\omega \propto n$. Therefore again from the quantization of the wave number we have the energy quantization $E_n \propto n^2$.

10. (Conclusions). Both the open ended pipe and the quantum system in an infinite square well involve a system confined in a finite spatial region. Therefore in both cases the boundary conditions imply the same wave number quantization ($\propto n$) which in turn produces a discrete energy spectrum proportional to n^2.

Before leaving this section, one notice:

- If we want to compare the measured value of the fundamental frequency f_0 with the theoretical one $c/2L$, we have to consider the *end corrections* due to the open ends of the pipe. The length L of the straw has to be increased adding $0.6 * R$ (where R is the radius of the pipe) for each open end. Therefore in our case we have to add $2 \cdot 0.6R$ to the actual lenght of the straw. This means that even a sudden open end of a pipe is not an ideal mirror, it does not perfectly reflects the incoming wave backwards. It is like the reflection occurs in a point slightly outside the pipe.

The wave-function of the infinite square potential

Referring to the general solution of the equation for the infinite square potential well (135) and to the conditions imposed at the boundaries (138) we can find the analytical expression of the two wave-functions corresponding to the the first two energy levels:

$$\begin{aligned} n = 1 \quad k_1 = \tfrac{\pi}{2a} \quad &\Longrightarrow A = 0 \quad \Longrightarrow \\ \Psi_1(x) &= B\,cos(\tfrac{\pi}{2a}x). \\ n = 2 \quad k_2 = \tfrac{\pi}{a} \quad &\Longrightarrow B = 0 \quad \Longrightarrow \\ \Psi_2(x) &= A\,sin(\tfrac{\pi}{a}x). \end{aligned} \tag{157}$$

If we plot them (Fig. 87) we recognise they correspond to the first two vibrational modes of the pressure inside a pipe or of a vibrating string. However in quantum mechanics we are interested in the

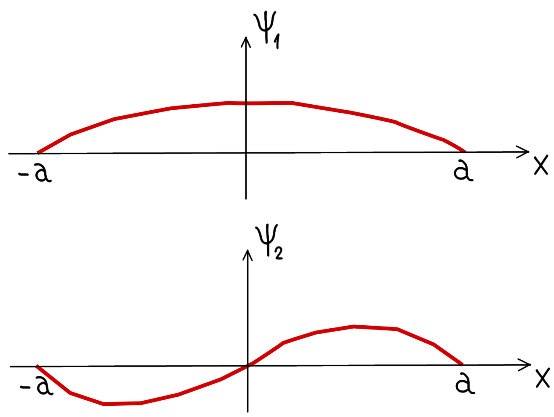


Figure 87: The wave-functions of the infinite potential square corresponding to the first two energy levels.

square modulus of the wave-functions, hence we focus our attention on figure 88.

Even if not proved, we have pointed out that the complete solution of a stationary quantum problem is the product of the solution (x) of the stationary equation and an exponential factor which express the time dependence of the wave-function (141).

$$\Psi(x, t) = \Psi(x) \cdot e^{iEt/\hbar} \tag{158}$$

When we compute the square modulus, since the modulus of the exponential factor is 1, the time dependence fades away. Therefore we must draw our attention to this aspect:

The square modulus $|\Psi(x,t)|^2$ of any stationary state does not depend on time. In other words, nothing oscillates at all. This remark should stimulate some reflections about the term "wave" used to define the mechanics of quantum systems.

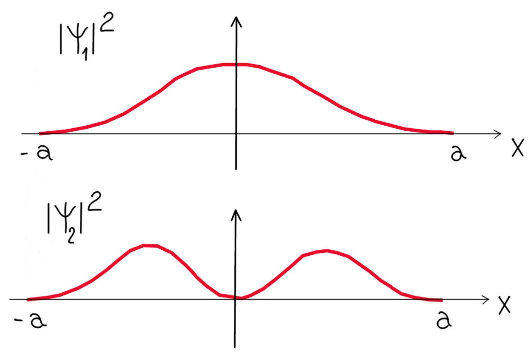


Figure 88: The square modulus of the wave-functions for the first two energy levels.

This is something that differs from what happens to standing waves in pipes or strings (Fig. 89). Considering a pipe, the "shape"of the pressure along it is given by the $sin(k_n x)$ factor (bold red lines), but each point oscillates up and down with the frequency of the vibrational mode according to the harmonic factor $cos(\omega_n t)$. The oscillation is produced by the continuous superposition of waves travelling back and forth due to reflections at the ends. This is something not occurring in the quantum scenario.

On the meaning of the wave-functions in the infinite well

We give a closer look to the $|\Psi(x,t)|^2$ of the infinite square potential (Fig. 90). For example we may examine the wave function corresponding to the first energy level E_1 (upper plot). Moreover we imagine a charged particle, say an electron, to be trapped in the well. The Copenhagen interpretation for the wave-function reads the $|\Psi_1|^2$ as the density probability to find the electron in whatever region inside the well. According to this point of view the area in figure 90 is the probability to find it in the region between x_1 and x_2. From this we may argue that there are chances to find the system almost everywhere inside the box. Of course, the system would be most likely to be found in the middle region, while the chances to meet it near the borders would be nearly zero. However sometimes the elec-

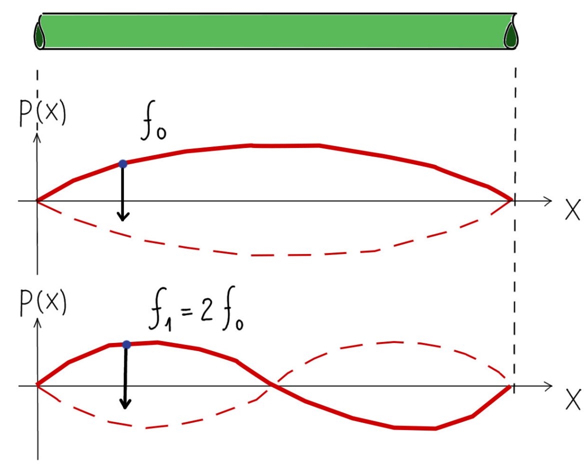


Figure 89: Standing waves in a pipe. Here we have an oscillation.

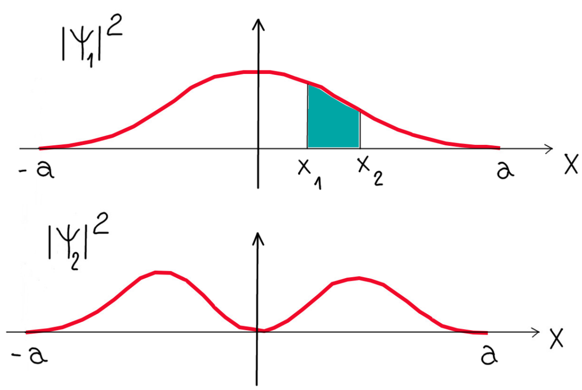


Figure 90: This area represents the probability to find the system somewhere in the region $[x_1, x_2]$.

tron will be in a certain position, other times in another. This point of view naturally leads us to think that the electron will experience accelerations (an electron with constant speed would certainly escape that limited region). Taking in mind that an accelerated charge emits electromagnetic radiation, thus loosing part of its energy, we come to the conclusion that this kind of interpretation leaves some open question. Just to make an example, the Hydrogen atom is the typical quantum system affected by this problematic situation.

> According to the Copenhagen interpretation, an electron in an infinite potential well should accelerate, hence loosing part of its energy by electromagnetic radiation. This fact conflicts with the idea of stationary states, i.e. states with well defined and constant energy.

The overall picture

Let us draw a picture of what we have learned up to now.

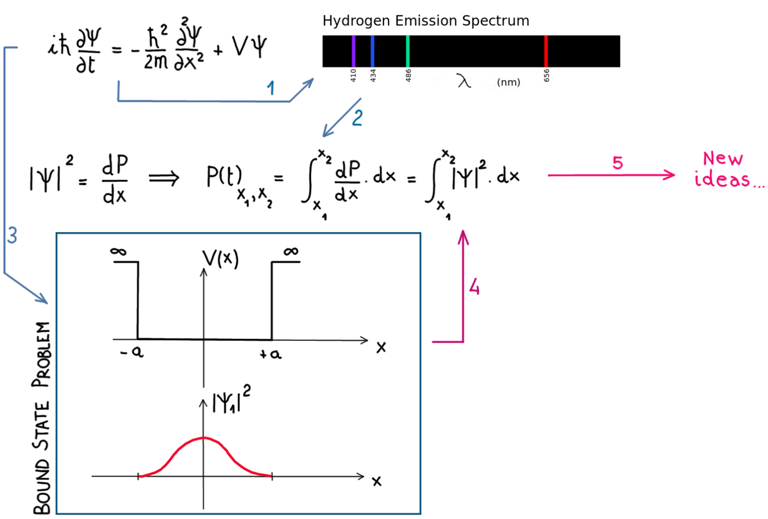


Figure 91: Scheme and flow of the events.

1. Our starting point was the *Schrödinger equation* whose great achievement was the explanation of the Hydrogen discrete Spectrum [arrow n.1].

2. The discussion about the meaning of the $\Psi(x, t)$ led to the axiomatic interpretation of a function whose square modulus represents the probability density distribution to find a system some-

where in space at given time (*Copenhagen Interpretation*) [arrow n.2].

3. Solving the Schrödinger *stationary equation* for a bound state problem (for example the infinite square potential well) we find solutions that lead us to constant energy states for charged systems localised in finite space regions. This generates unanswered questions about the physical possibility for these states to exist (energy loss due to radiation would be present) [arrow n.3].

What we would expect at this point is to move back to the interpretation that led to this situation [arrow n.4] and reconsider it, maybe with new reflections and ideas [arrow n.5].

An opportunity to switch on some critical thinking

When we teach is very comfortable to present things as if they were perfectly understood and established. This is a sort of mask we sometimes wear for a lot of reasons. It prevents unpleasant questions, it is energy saving since it does not require further thoughts and reflections, it strengthen our role giving a crystal image of what we teach and, in return, of ourselves. On the other side, in a lot of physics situations and theories we carry out approximations, we chose to adhere to accepted interpretations. In addition sometimes we repeat what is most commonly written in books without considering any more the possibility to carefully examine the meaning of what we are saying. This is a typical human behaviour.

Exactly for this reason it is important point out that such approach to the teaching risks to be less productive. Preventing ourselves, and hence our students, from a critical analysis of what we teach we originate a closed-loop transfer of knowledge. We teach something and students repeat what we teach. In this process both teachers and students play a passive role and, what is worst, innovation is put in danger both in the educational and the scientific field.

Quantum mechanics is a theory that probably more than others is affected by this problem. Is not by chance that we are often told:

"We cannot understand it."

"It is almost impossible to understand it, but it works."

Two commonly accepted statements providing formidable shields to everyone involved in the teaching of this theory. For this reason we believe that Quantum Mechanics offers a good opportunity to open discussions about its interpretation. We have already pointed out some problematic aspect of the theory, thus giving some food for thought. Along this chapter, devoted to the Quantum theory, we will identify other features that might be seriously considered and that might move our minds to some reflections.

Suggesting an interpretation for the wave-function

So far we have explored the Schrödinger equation in the framework of the Copenhagen interpretation and we have discussed a bit about the pros and cons. Now we propose a possible way to interpret the wave-function. We start with a summary of the general situation in the Copenhagen mindset, then we will discuss a possible change in the way of thinking.

The Copenhagen framework

- Usually we start writing the Hamiltonian which defines the problem we are dealing with,

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\,. \tag{159}$$

- Then we solve the stationary Schrödinger equation,

$$H\Psi = E\Psi\,. \tag{160}$$

- Eventually we find the solutions: the set of the wave-functions Ψ, one for each energy level of the spectrum.
- From the wave-functions we derive the square modulus $|\Psi|^2$.
- The $|\Psi|^2$ gives us the spatial probability distribution.

In the previous sections we have seen how the Copenhagen interpretation leads to a puzzling situation in which a charged system in a state with constant energy should accelerate. As already pointed out this situation is something that we should try to avoid, something that could bring us to look for different interpretations of the wave-function.

If we assume that $|\Psi|^2$ is the spatial distribution of the system itself, representing a quantum system as a whole, the problem related to the energy loss due to radiation disappears. The "electron in a potential well" becomes a new physical system which is distributed in space as described by the square modulus of the wave-functions. We no longer have something that can be found in a different region of the well each time we look for it. We have a new system distributed in space. A system that rises from the

Hamiltonian (159) of the problem. Solving the stationary Schrödinger equation (for brevity we should say "solving the Hamiltonian"[55]) we discover how this system is distributed in space. As already stated, the system is no longer an electron, is an electron in a potential well, a charge that interacts in a field. The Hamiltonian equation does no longer describe the particle, but a new state formed by the interaction with the field, moreover it does not say much about this new system, it does not tell us what it is. It only tells us about how it is distributed over space[56].

[55] This expressions underlines the fundamental role played by the Hamiltonian H in determining the solutions of the equations, thus determining the spatial distribution of the system.

[56] For those who already know about Quantum Mechanics it also tells us about how it is distributed in momentum. This aspect well be addressed later.

The scenario that emerges from this interpretation of the wavefunction leads us to think that

- Quantum systems are "solutions of the Hamiltonian H".
- The square modulus of the solutions ($|\Psi|^2$) tells us how the system distributed in space.
- For each Hamiltonian we may have a number of different solutions, one for each energy level of the spectrum (Fig. 88). Technically speaking each solution is a different *eigenstate* of the Hamiltonian.
- If the system is given a suitable amount of energy the next eigenstate can be produced.
- Our mathematical model of the system (the Hamiltonian) does not give information about what the system is, it only says how it is distributed in space (and in momentum) and what are the allowed energies for the system (energy spectrum).

We can apply this new framework to the Hydrogen atom. In this case the well is the Coulomb potential and the Hamiltonian we put in the Schrödinger equation involves two interacting particles: the proton and the electron. The solutions of the stationary equation no longer represent either the proton or the electron. They are giving us some information about a new system the *eigenstate* of the Hamiltonian. The solutions are not saying what this eigenstate is, however they are informing us about the spatial distribution of the system.

Within this new interpretation the problem of the constant energy states vanishes. The system is there, described by a $|\Psi_n(t)|^2$ which tells us how the system is distributed over the space at time t. Accelerating particles are no more considered!

Was it worth it?

With this paragraph we conclude the section dedicated to the Schrödinger equation. At the beginning we asked if it was possible, in principle, to introduce it using the mathematical knowledge covered by the Scientific Lyceum. We believe that our work moves one step toward a positive answer, even though we recognise that the involved mathematics is at the supremum of what can be actually done and understood in a classroom. Anyway we want to add two considerations about this topic:

1. A powerful opportunity for the teacher in order to move the classroom towards a deeper understanding is strictly correlated to the planning of the educational path. We are talking of the possibility to pave the way to quantum concepts planning a set of earlier interventions both in the Mathematics and Physics domains. In other words the teacher might anticipate a number of topics that might be useful later, when tackling quantum mechanics. We even suggest to reveal to students the aim of these interventions with statements like "...this fact will be very important when we will deal with quantum mechanics ... since will give us the opportunity to ..." or "..this mathematical aspect has to be clear since we will use it to model a wave that propagates ...". This attitude will help students to feel the presence of an educational programme. Moreover it will, at least partially, answer to the usual question "why should we study this?".

   We give some example to make this concept clear:

   - Deal with geometric transformations (dilations and translations) and their relation with the plot of a function, spend some time to explain the connection between the graph of $f(x)$ and $f(ax+b)$. Then we can substitute b with a time dependent term, therefore introducing time dependent translations, i.e. the propagation of a wave. This skill is extremely useful in a lot of situations encountered in this section.
   - Dedicate some time to discuss and listen to acoustic beats. This can be done in the physics curriculum (in acoustic or in the part devoted to the physics of waves) or in the mathematics curriculum when dealing with trigonometric functions. We should stress that superimposing two pure tones we can cancel the signal at some instants. Later, in the last year, we would have the opportunity to come back to the beats and extend what was carried out with two tones: we can solve the integral (118) and discuss its meaning in the framework of the superposition of pure tones.

- Introduce complex numbers and their exponential form.

2. Among all the topics addressed in this section the teacher has the chance to select small parts and design his own path.

Quantum mechanics in Action

Action is a quantity of great importance in Quantum Mechanics. The order of magnitude of the action of a system may predict if the system behaviour will be affected by quantum mechanics or not. In this paragraph we will show a way to introduce it at secondary school level (method inspired by the work of Cleon Teunissen [57]) and we will discuss briefly why the action determines the transition from classical to quantum mechanics.

[57] Cleon Teunissen. *Least Action Visualized*. http://www.cleonis.nl/

Integrals for breakfast

The action is an integral of a function over time, however in order to grasp its physical meaning, students do not necessarily need a deep knowledge about integration. For us the integral of a function will be the area of the region bounded by the graph of $f(t)$ and the t-axis. We will work with second degree polynomials, therefore we just need to know how to evaluate areas bounded by parabolas. Students can learn how to perform such calculations in a very "sweet" way. We can invite them to carry out the following activity.

- While having breakfast draw a square on the napkin (assuming its side has one unit length). Inside the square draw the parabola $y = t^2$ as shown in figure (92). An approximated plot for the parabola will be enough for our purposes. We just have to consider the stationary point in $(0, 0)$ and the passage of the curve through three other points.

- Take some brown sugar and spread it uniformly over the square.

- Take a picture and count the number n of grains in the area bounded by the parabola and the total number N of the grains inside the square.

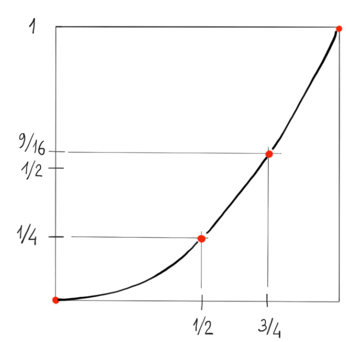
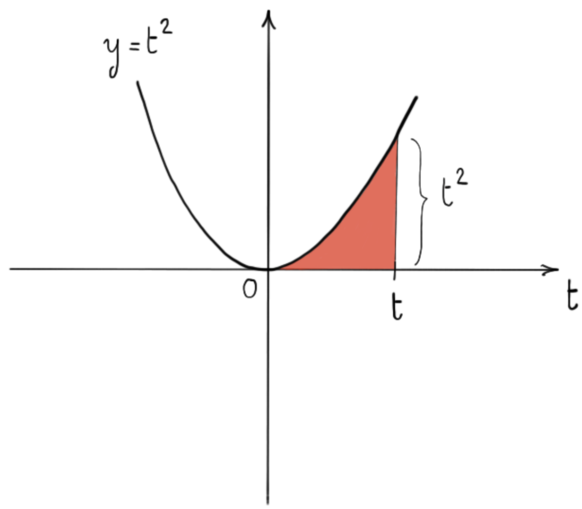


Figure 92: At breakfast draw this square on the napkin.

In this way students may find that the ratio $\frac{n}{N} \sim \frac{1}{3}$, thus discovering that the area of the parabola from $t = 0$ to t is one third of the area of the rectangle in which the arc is inscribed. From now on they will know how to evaluate the following integral, representing the area shown in figure (93).

$$A = \int_0^t t^2 \, dt = \frac{1}{3} t \cdot t^2 \, . \tag{161}$$

Figure 93: The area we need to evaluate.



Action definition

The action is the integral over time of the difference between the kinetic and potential energy of a body:

$$S = \int_{t_1}^{t_2} (T - V) dt \,. \tag{162}$$

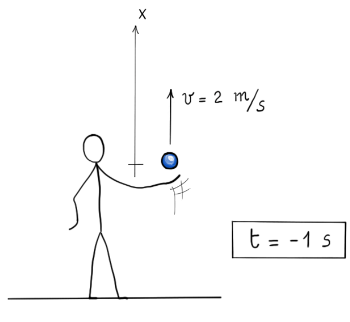
Both the kinetic energy T and the potential energy V of a body vary over time while the body is in motion, thus $T - V$ is a function of time and we can evaluate the bounded area represented by the integral (162).

The principle of Least Action

We want to show that if we evaluate the action integral (162) along the predicted Newtonian trajectory we get a minimum value. We will do it by means of an example, dealing with free fall in a uniform gravitational field.

Math Box - **Least Action in action**

We consider a body launched upwards, at time $t = -1s$, with initial speed $v_0 = 2\,m/s$. We assume the gravitational field to be $g = 2\,m/s^2$ (numbers are chosen in order to have simple calculations).

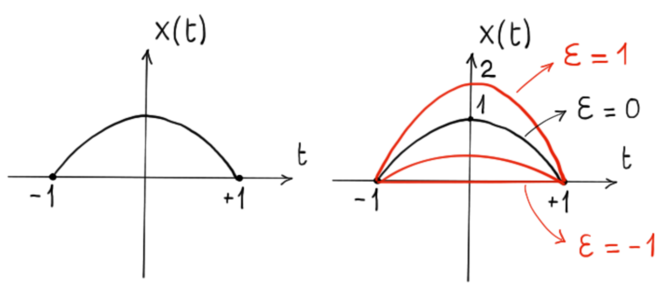


The motion equation is:

$$x(t) = -t^2 + 1 \tag{163}$$

The plot of the Newtonian trajectory in space-time is shown in the image below. We generate a family of still parabolic trajectories around the Newtonian one:

$$x(t) = (1+\epsilon)\,(-t^2+1) \quad \in [-1, 1] \tag{164}$$



and we compute the action integral along them:

$$S = \int_{-1}^{1} (T - V) dt \,. \tag{165}$$

Both T and V change over time between $t = -1$ and $t = 1$. We thus evaluate the area bounded by these curves and then we subtract the results:

$$S = \int_{-1}^{1} T dt - \int_{-1}^{1} V dt = S_T - S_V\,. \tag{166}$$

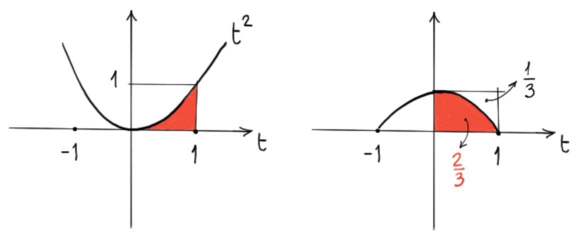
Since $v = -2\,(1+\epsilon)\,t$ and $T = \frac{1}{2} m v^2$,

$$S_T = \int_{-1}^{1} 2(1+\epsilon)^2 t^2 dt = 2(1+\epsilon)^2 \cdot \int_{-1}^{1} t^2 dt\,. \tag{167}$$

On the other hand $v = mgx = 2x$, therefore

$$S_V = 2(1+\epsilon) \int_{-1}^{1} (-t^2+1) dt\,. \tag{168}$$

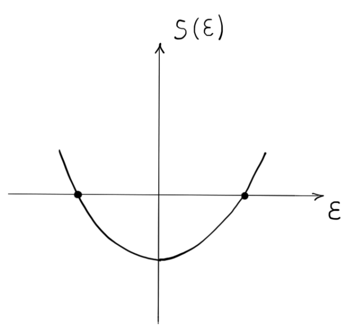
Both integrals can be solved looking at the plots below.



$$S_T = \frac{4}{3}\,(1+\epsilon)^2\,, \quad S_V = \frac{8}{3}\,(1+\epsilon)\,. \tag{169}$$

Evaluating $S_T - S_V$ we get:

$$S(\epsilon) \propto (1+\epsilon)(\epsilon - 1) \tag{170}$$



hence we see that the Newtonian trajectory in spacetime ($\epsilon = 0$) is the one that minimise S! This is a general property, that may be proved to hold for every system in a far more rigorous way.

Action units

The dimensions of the action are energy multiplied by time.

$$\text{Action} \rightarrow \textit{Energy} \cdot \textit{Time} \tag{171}$$

It turns out that the action of a system [58] is related to the product of its spatial dimensions L (the spatial range covered by the system) and its momentum dimensions p (the momentum range). For example for a vibrating system (mass-spring system) L is the amplitude of the spatial oscillation, p the amplitude of the momentum oscillation (p varies from a minimum to a maximum value). Therefore we can also reach a rough estimate of the action of a system by simply multiplying these two quantities.

[58] A Cuppari, G Rinaudo, O Robutti, and P Violino. *Gradual introduction of some aspects of quantum mechanics in a high school curriculum*. IOP Publishing, Physics Education, 1997, vol. 32, n. 5, 302-308

$$\text{Action} \sim p \cdot L \tag{172}$$

So the order of magnitude of the action of a system may e represented by the area covered in the phase space x, p. Dimensionally speaking it can be easily verified that the relation (172) is true. On the other hand, if we want to test it mathematically we would need to solve some integrals.

Quantum systems and Action

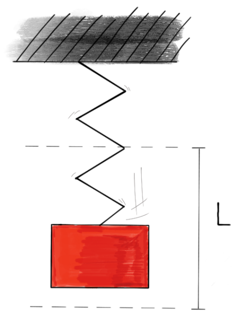
As we have discussed previously, Classic Mechanics may be considered a limiting case of Quantum Mechanics as $\lambda \rightarrow 0$, in analogy with the relation between Physical Optics and Geometric Optics. If we consider the De Broglie equation $p = h/\lambda$, we discover that

$$\begin{gathered} \lambda \rightarrow 0 \quad \Longrightarrow \quad p \rightarrow \infty \\ \text{hence} \quad S \rightarrow \infty\,. \end{gathered} \tag{173}$$

So we may conclude that if the action of a system is big Classic Mechanics may be a good theory to deal with. On the other hand, for small values of the action Quantum Mechanics is needed. In conclusion we may ask students to evaluate the action of typical classical system (a mass-spring system) and a typical quantum system (the Hydrogen atom).

Math Box - Action of a classical and of a quantum system

The Action of a mass-spring system



Frequency $\sim 1\,Hz$, Period $\sim 1\,s$, $L \sim 0.5\,m$, $m \sim 0.1$ kg.

$$S \sim pL = mv\,L \tag{174}$$

In one period the mass covers $2L = 1$m, hence $v = 1$ m/s.

$$S \sim 0.05\,J\,s \tag{175}$$

The Action of the Hydrogen atom

The evaluation of the action for the Hydrogen atom is quite easy if we take in mind the Bohr's quantisation rule (96)
$L = n\hbar$.
According to the (172) the angular momentum L is a direct estimate of the action, thus the action is of the order of magnitude of the Planck constant:

$$S \sim 10^{-35}\,J\,s \tag{176}$$

The quantization of the electromagnetic radiation: light is made of photons

In this section we will show how to introduce the concept of photons to secondary school students by means of a quite simple experiment requiring only a white sheet of paper and a digital camera. The idea relies on the simple consideration that if light is composed of particles, when it strikes the pixels of a digital sensor it should present the typical statistical properties of grains spread over a square grid. Moreover the number of photons (grains) recorded on each pixel (square of the grid) should follow the Poisson distribution.

The simple idea is that if we succeed in showing this fact taking digital pictures, we also succeed in convincing students that light comes in packets we call photons.

Spreading a number of points on a square grid we will count a number of them in each cell. The counts distribution may be simulated and histogrammed, as shown in figure 94, by means of a C++ code or may be derived by direct experimentation, simply spreading salt (or brown sugar) grains over a square grid. The Poisson distribution $P(n)$ gives the probability to have n counts in a cell of the grid. The distribution depends on a single parameter: the mean value of the counts, λ.

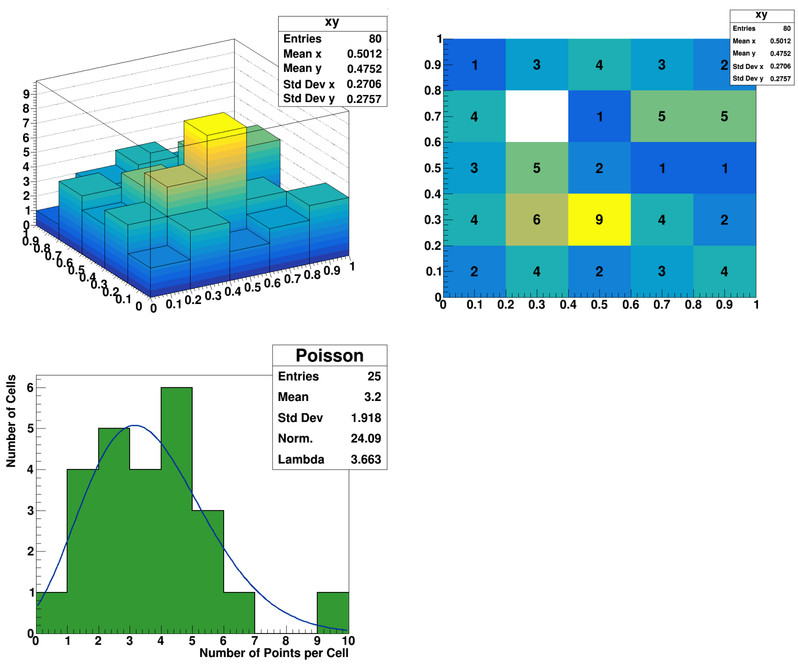


Figure 94: C++ Simulation for the random distribution of points on a 5x5 square grid.

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda} . \tag{177}$$

In the simulation presented in figure 94 we have 80 points randomly distributed over the 5x5 grid, the mean value for the counts is nearly 3. Just to be clear, the second column of the green histogram tells us that in 4 out of 25 cases a we had 1 point in a cell. The curve that fits the histogram is the Poisson function multiplied by a normalisation factor ("Norm" in the histogram box). As we can easily verify, both with reiterated simulations or direct experimentation, the standard deviation σ is the square root of the mean value λ:

$$\sigma = \sqrt{\lambda} . \tag{178}$$

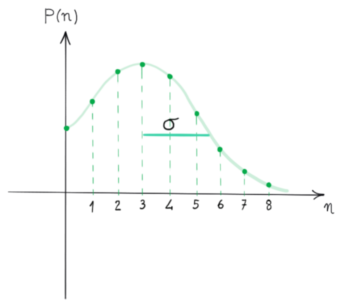


Figure 95: The Poisson distribution and its standard deviation.

If we consider the points on the grid as photons "falling" on the pixel matrix of the sensor of a digital camera we may establish a correspondence among the mean value of the counts on each cell and the signal recorded on each pixel. On the other hand the standard deviation σ will correspond to the signal fluctuations.

$\lambda \rightarrow$ SIGNAL
$\sigma \rightarrow$ FLUCTUATION

The ratio between the standard deviation and the mean value specifies the relative importance of the fluctuation over the signal. If we compare figures 96, 97, showing two simulations with different values of the signal (3 for the first one, 80 for the second one), we can discover that fluctuations are mostly important for low signal situations. A fact that is confirmed by the ratio σ/λ which becomes smaller and smaller as the signal increases:

$$\frac{\sigma}{\lambda} = \frac{Noise}{Signal} = \frac{1}{\sqrt{N}} \tag{179}$$

This means that Poissonian Fluctuations are more easily resolved in a Low Signal scenario.

This is a useful tip for the design of our simple experiment.

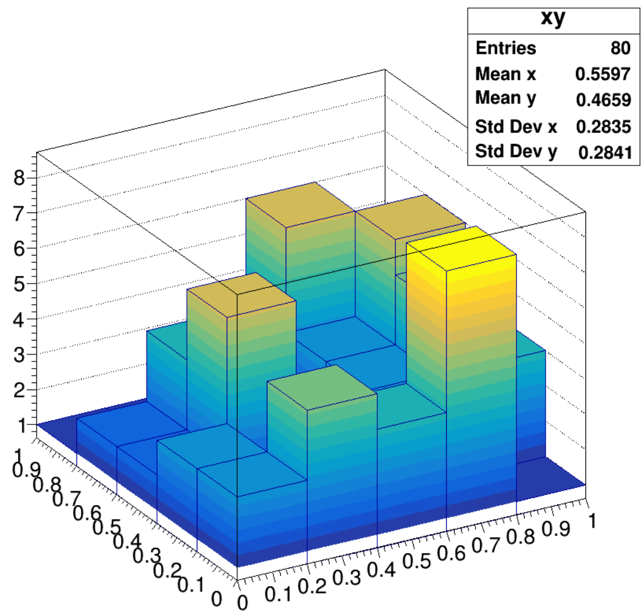


Figure 96: Low Signal simulation, 80 points spread over the 5x5 grid.

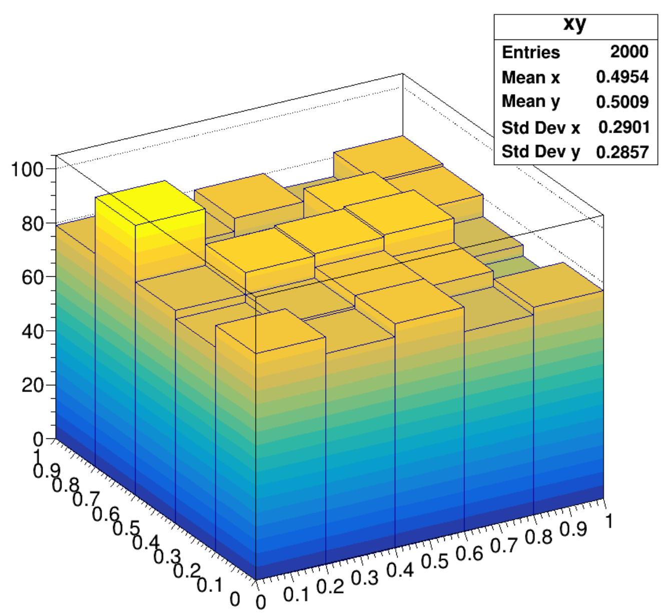


Figure 97: High Signal simulation, 2000 points spread over the 5x5 grid..

The experiment with the digital camera

In order to show that light impinges on the sensor matrix as a shower of photons, we take a series of pictures of a white sheet of paper gradually reducing the time exposure[59]. Therefore moving from a high signal to a low signal scenario. If we manage to measure the signal λ and the noise σ, we have the opportunity to check if, in the low luminosity images, the relation among these quantities is Poissonian. The figure below (Fig. 98) shows three pictures of the white sheet taken with different shutter speeds (from the picture n. 1 with the highest signal to the darkest one, picture n. 3). For each

[59] Maintaining fixed all the other camera settings.

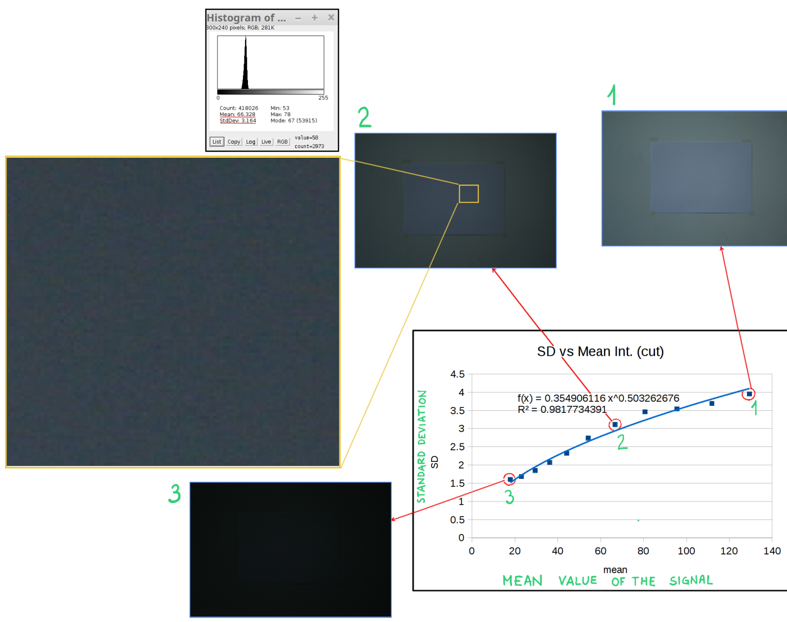


Figure 98: The plot of the relation among the standard deviation of the intensity distribution and the mean value.

picture we consider a box (see the yellow one inside picture n. 2) and we evaluate the distribution of the signal intensities for all the pixels of the box. These operation may be carried out by a free software, Image-J [60]. The resulting distribution (see top-left part in Fig. 98) returns the mean value of the signal of the pixels in the box (λ) and the standard deviation (σ).

Collecting such values for a dozen of pictures we build the plot showing some points of the function $\sigma(\lambda)$. These data may be fitted with a power law ($y = a \cdot x^b$). The data collected in our experiment, carried out with a Canon 350D camera, show a proportionality between the standard deviation and the square root of the signal ($b \sim 0.5$) (178),

[60] ImageJ. *ImageJ, open source image processing program designed for scientific multidimensional images*. https://imagej.net/Welcome

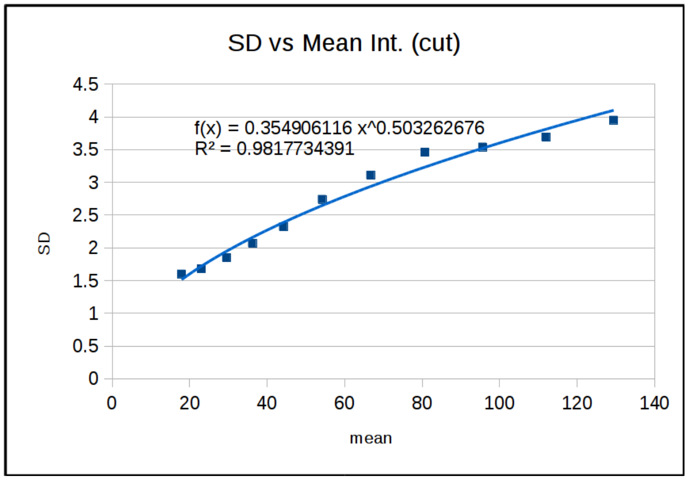


Figure 99: The plot showing the 'fluctuation' (the standard deviation) as a function of the 'signal' (the mean value).

thus proving the Poissonian distribution of the photons on the pixels of the sensor!

We may notice that the proportionality factor is not one, a fact that has a quite technical explanation we are not going to develop here. We just say that the camera does not directly counts the number of photons impinging the sensor. The photons collected by a pixel (N_{ph}) ere converted in electrons stored in a potential well, these electrons generate a voltage then transformed in a digital number (N_{ADU}). Eventually the number of photons N_{ph} is related to N_{ADU} by a factor given by the quantum efficiency Q (the ratio between the number of photons collected on a pixel and the number of electrons stored) multiplied by the gain g (the ratio between the number of electrons and N_{ADU}). The proportionality factor arising from the data analysis ($a \sim 0.35$) is the product Qg.

The Close Encounter approach to the Heisenberg Principle

The Uncertainty Principle is one of the milestones marking the break-up with former theories describing reality at macroscopic level. Its physical meaning has been deeply discussed and currently various approaches have not yet come to a common description. Here we propose an inquiry based approach to the principle, based on the acoustic analysis of a musical theme form the Steven Spielberg's movie "Close Encounters". The 'Close Encounters' approach[61] introduces the principle as an intrinsic property of a quantum system, as something that holds regardless of any measurement process. Furthermore, it gives the possibility to carry out a discussion about the

[61] L Galante et al. *Close Encounters with Heisenberg: uncertainty in the secondary school*. Physics Education - (IOP Publishing), vol. 54, n.1., 2018

physical meaning of the Principle itself.

The interested teacher may be introduced to 'Close Encounters' method on the site "*PHE - Teaching Quantum Mechanics*" in the section devoted to this topic (Fig. 100). In the site the educational path is divided in three steps. The site guides the teacher (or the entire classroom) providing tutorials, videos, a series of inquiry and practical activities and forms. The site structure and, therefore, the educational method will be described in the following sections.

Close Encounters - Step 1

To fully grasp the physical meaning of the Heisenberg Uncertainty Principle (UP) we firstly need to become familiar with three key concepts:

1. What is a *Distribution*.
2. What is the *Dispersion* of a distribution.
3. The possibility to represent a physical system in different *Domains*[62].

[62] We might symbolise these concepts with a triple 'D': DDD.

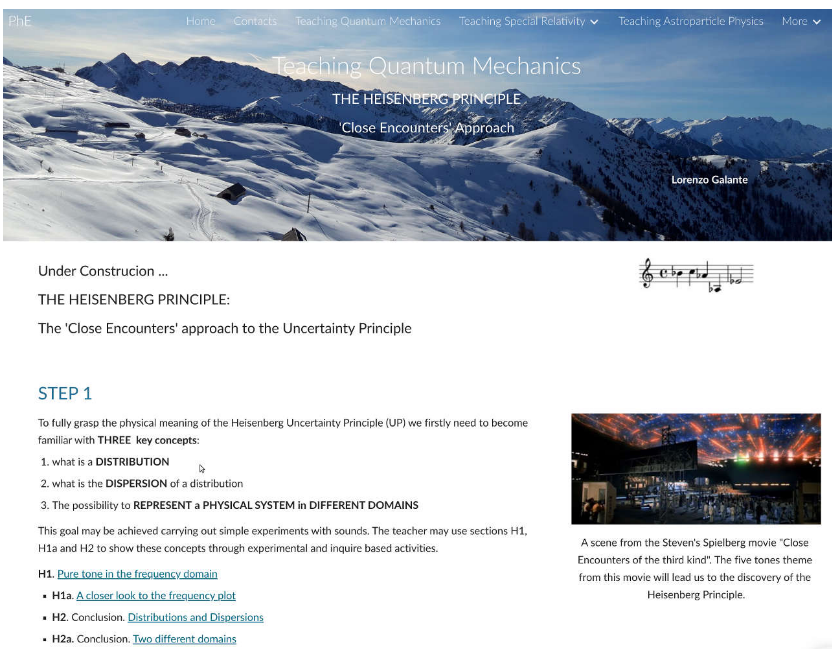
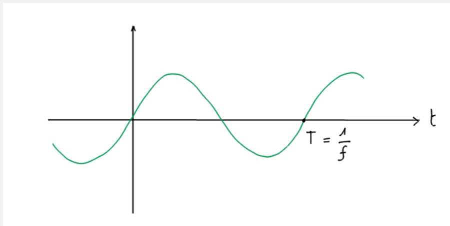


Figure 100: The section *Teaching Quantum Mechanics* of the PhE Site devoted to the 'Close Encounters' approach

Step 1 is entirely dedicated to this task. The goal may be achieved carrying out simple experiments with sounds, thus providing the classroom with an inquiry environment where they can carry out their own explorations and draw their own conclusions. In the following boxes we will show what is done on the site.

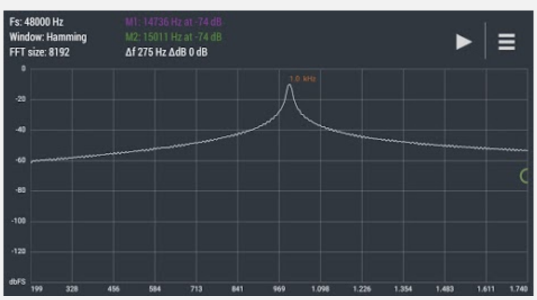
Pure tones in the Frequency Domain

Clicking on a link of the site students may listen to a pure tone ($f = 1000 Hz$), a sinusoidal sound with a well defined frequency f.



They are asked to explore the sound in the frequency domain, with a Spectrum Analyser installed on their mobile phones. Two free applications are suggested: Android Spectrum Analyzer PRO (for Android devices) and SpectrumView Free (for Apple devices). Some basic explanation about Spectrum Analysers is given, then their exploration guided by an on-line form starts. The questions are:

- What Frequency plot do you expect from the analysis of a pure tone?
- Run your analyser and zoom the frequency axis between $0\,Hz$ and $2000\,Hz$. Play the pure tone. When the spectrum of the sound appears, pause the application in order to freeze the frequency plot on your display (see figure below). What is the frequency of the pure tone you were playing?

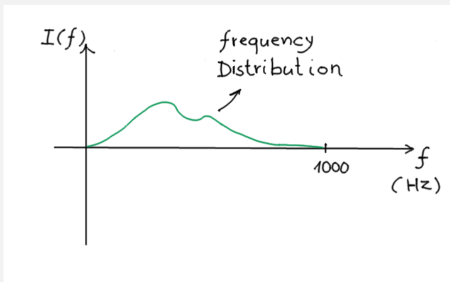


With this activity the teacher has the opportunity to discuss the possibility to represent a sound both in the time and in the frequency domains. Furthermore, comparing the expected frequency plot for a pure tone with the measured one, a preliminary discussion about distributions may be started.

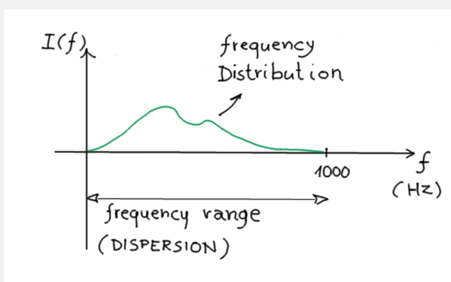
Once the frequency distribution of a sound has been experienced, students face a section of the site in which distributions in the frequency domain are presented in some more detail.

Distributions and Dispersions

Sounds are composed by a complex of frequencies. Each sound has a different distribution in f. The shape of the distribution tells us what frequencies are composing the sound and their importance, the intensity $I(f)$.

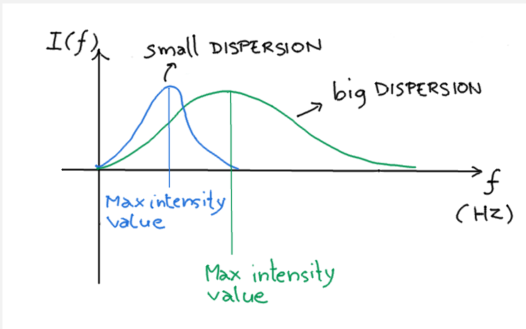


An important feature is the dispersion of the distribution. Without moving into mathematical details we can say that the dispersion is the length of the frequency range in which the distribution is significantly high. Dispersions may be represented by the symbol Δf.



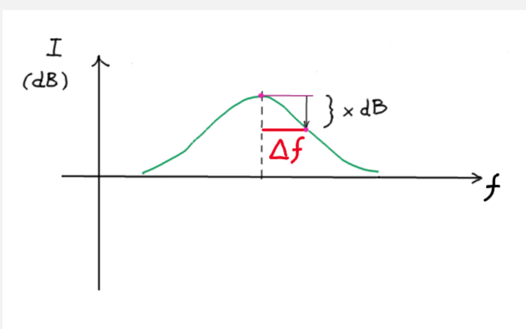
The dispersion gives an idea of the spread of the distribution along the x-axis. Generally physical systems are dispersed in many different domains. A sound may be dispersed in time, t, or in frequency, f. A body may be dispersed in space. For example think to a string fixed at both ends. It is made of many parts distributed along one spatial dimension. If the string vibrates, its parts have different speeds, zero at the ends, maximum at the centre (if the first vibrational mode is active). Therefore the system is distributed even in the speed domain, v.

A gas at temperature T is distributed in the speeds of the molecules (Maxwell distribution) and, for example, this is extremely important for the fusion processes in the Sun.



Rough estimation of the dispersion

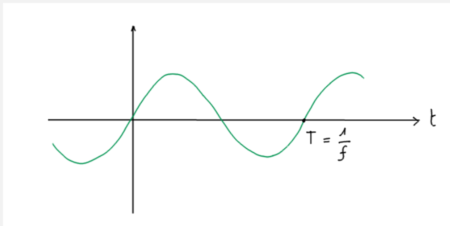
In case of peaked shapes of the distribution, a rough calculation of the dispersion may be achieved evaluating the distance along the x-axis (Hz in our case) between the peak and a point x dB below. If we want to compare dispersions of different distributions we may use this method, always keeping in mind that the amount of decibels (x dB) has to be the same for every distribution we want to analyse.



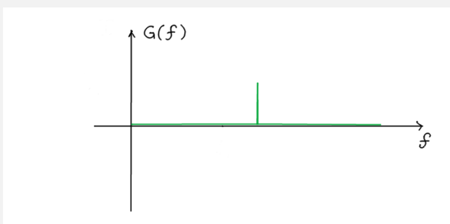
The same system may thus be described in different domains.

Two different domains

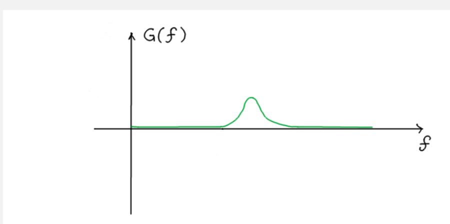
A sound perceived by our ear may be represented as a function of time: a pressure changing over time. We have considered a pure tone that may be represented by a sinusoidal function over time: the pressure at our ear varies up and down with a sinusoidal trend.



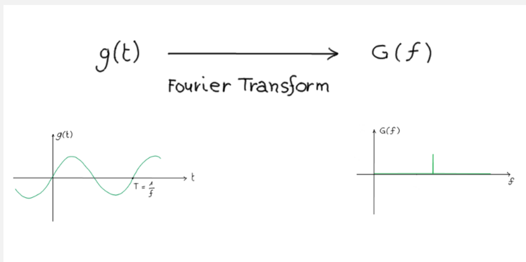
But the same sound may also be represented as a function of frequency. We have seen how an ideal pure tone should be represented in the frequency domain as well as the frequency plot of a real pure tone generated by the speakers of our computer.
An ideal pure tone may be represented in the frequency domain as a function $G(f)$: a very sharp peak at the frequency f.



A real pure tone generated by the speakers of the computer is represented in the frequency domain as a function G(f) which is a distribution of frequencies peaked around the frequency f.



Therefore we have two different variables (t and f) and two different functions: $g(t)$ and $G(f)$.
$g(t)$ and $G(f)$ represent the same physical system (a sound in our case) in two different domains.
Since both functions describe the same system, it is not difficult to understand that they are connected to each other. The mathematical calculation allowing to transform $g(t)$ in the corresponding $G(f)$ is called *Fourier Transform* (FT).

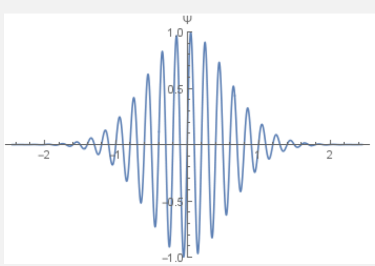


Close Encounters - Step 2

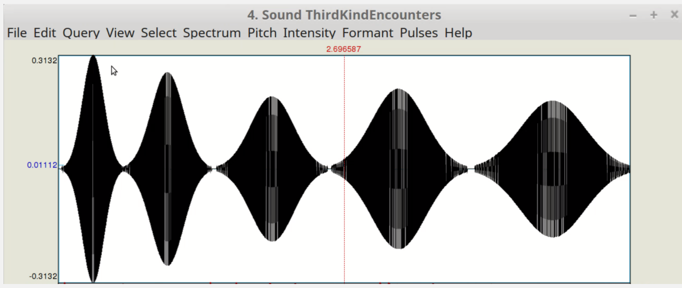
Now we are ready to move towards the discovery on an uncertainty relation in acoustics. Students are asked to work with the five tones theme from the Steven Spielberg's movie "Close Encounters of the third kind", the theme used by humans to establish a communication channel with the alien spaceship. For our purposes we have recorded a motif with the same five notes, each one with a different time duration.

The Close Encounters' theme Analysis

Each of the five pure tones has been modulated in time with a Gaussian function and mathematically synthesised with the software Mathematica. Five modulations, each with a different standard deviation, in order to have a set of five differently dispersed sounds.

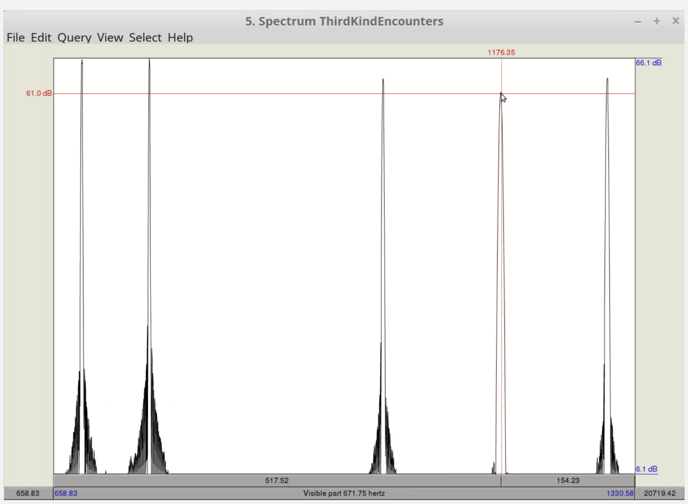


(The sine curve modulated by a gaussian function)



(The theme as a function of time)

Evaluating the spectrum of the motif and measuring for each note the frequency and the time dispersions, Δf and Δt, we get a set of five data pairs showing the uncertainty relation $\Delta f \cdot \Delta t = \text{const.}$. In the following lines we will show how to carry out the analysis.

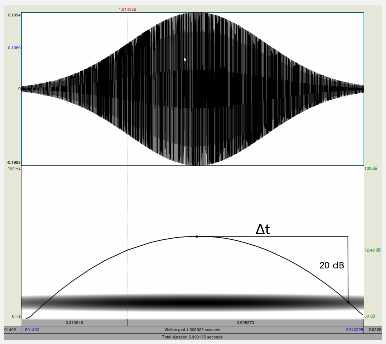


(The Spectrum of the theme)

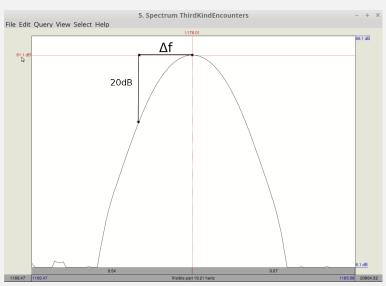
We suggest to carry out the analysis of the five-note tonal phrase with the free software Praat[63].

[63] Boersma P and Weenink D. *Praat: doing phonetics by computer*. http://www.praat.org/, 2018

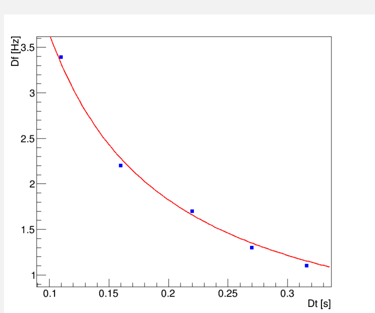
Praat allows to analyse sounds both in the time and frequency domain. The available tools for the time analysis allow to measure the five time dispersions Δt of each note. In the figure below we zoomed in a single note and estimated the dispersion Δt by measuring the time distance between the top of the intensity profile and the point 20 dB under the top level. We acted on the intensity line (lower plot in figure) expressed in dB. A Click on any point of the plot returns the intensity level (in dB) and the corresponding time value (in seconds).



In a similar way we zoomed in a single spectrum peak and measured the frequency dispersion Δf as the frequency distance between the the top of the spectrum and the point 20 dB below.



This process, repeated five times, gives the data table 4, showing the constant value of the dispersions product. Data was also fitted with a one parameter function $xy = k$; the resulting plot shows a good agreement of the data set with the mathematical model (the parameter value in this case is $k = 0.365$).



Δt	Δf	$\Delta t \cdot \Delta f$
(s)	(Hz)	
0.11	3.4	0.37
0.16	2.2	0.35
0.22	1.7	0.37
0.27	1.3	0.35
0.32	1.1	0.35

Table 4: The data set.

On the website a set of three short videos directly explains how to analyse the musical motif with the software Praat.

Close Encounters - Step 3

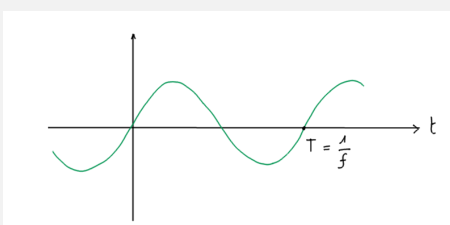
So far students should be aware that, at least in acoustics,

Uncertainty Relations involve dispersions of a system in two different domains.

This is a crucial consideration that we want to extend to the Heisenberg Uncertainty Principle. In this section we show how, according to our educational approach. We report what is proposed on the website.

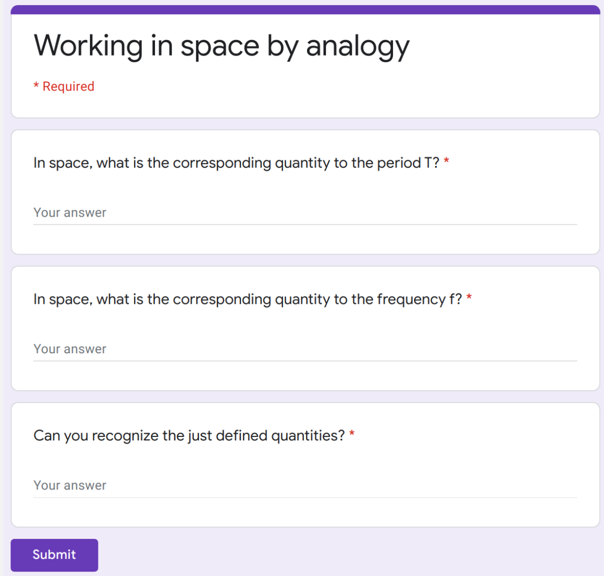
From the Uncertainty Relations in acoustics to the Heisenberg Principle

We have worked in the time and frequency domains. Two quantities that have a specific relation: an amount of time T automatically gives a frequency $f = 1/T$.



Now we may ask if there are other similar variables, related in the same way. For example, we can work with a space variable, like x. From now on we will move from the time-frequency domain to a new domain involving space, x, instead of t.

Working in space, by analogy, fill the following Form.



Working in space by analogy

* Required

In space, what is the corresponding quantity to the period T? *

Your answer

In space, what is the corresponding quantity to the frequency f? *

Your answer

Can you recognize the just defined quantities? *

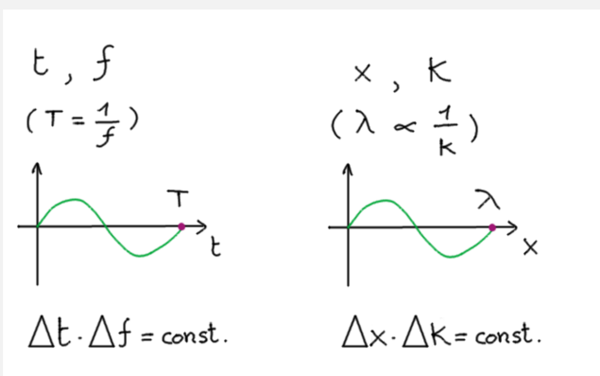
Your answer

Submit

The form should stimulate students to discover that two variables corresponding to the pair (t, f) exist: x and $1/\lambda$. In $1/\lambda$ the classroom may recognise a quantity proportional to the wave number k.

$$k = \frac{2\pi}{\lambda} \tag{180}$$

Hence from now on we will consider the pair (x, k).



From the mathematical point of view there is no difference between working with (t, f) or with (x, k), the relations among the variables are exactly the same. So we expect an Uncertainty relation to hold also in the new domain:

$$\Delta k \, \Delta x = \text{const.} \tag{181}$$

We have come to an Uncertainty Relation (181) between the pair of variables (x, k). This relation tells us that the more a system is dispersed in space the less is dispersed in the other variable.

Now we want to better understand what k represents. We have to say that in 1927 was carried out an experiment showing that the momentum p of electrons could be expressed through a wavelength λ[64]. The relation being:

$$p = \frac{h}{\lambda}\,. \tag{182}$$

The experiment was performed sending an electron beam against a Nickel crystal and observing diffraction patterns generated by the scattered electrons.

[64] Davisson C J and Germer L H. *Reflection of electrons by a crystal of nickel*. Proceedings of the National Academy of Sciences of the United States of America. 14 (4): 317322, 1928

According to this relation p is proportional to k:

$$p \propto h\,k\,. \tag{183}$$

It can be proved that the wave function describing a quantum system in space is related to the wave function in momentum by the Fourier Transforms exactly as it happens in acoustics with sounds in time and frequency. This is strictly related to the fundamental equations of quantum mechanics and acoustics (the Schrödinger and the D'Alembert equations), both linear and homogeneous, thus supporting the superposition principle (see page 129). This fact allows us to extend the uncertainty relation we discovered in acoustic to the Heisenberg UP for quantum systems. Therefore, since $\Delta k = \frac{\Delta p}{k}$, eventually we have:

$$\Delta p\,\Delta x \propto h\,. \tag{184}$$

Dispersion not Precision

The introduction of the UP with the "Close Encounter" method offers the possibility to engage critical thinking. In our opinion this is a crucial point in the learning process of young students, in order to avoid a blind belief in what is taught and written on textbooks. In the proposed approach the deltas (Δt , Δf) are dispersions of the physical system instead of measurement uncertainties. We can consider each note of the musical theme as a system spread over time exactly as we consider a ball as a system dispersed in space. Undoubtedly it is not difficult to understand that the measurement uncertainty δx in defining the position of the ball is not related with its spatial dispersion Δx. To be clear, we know that in order to define the position of the system we can choose a reference point P and measure it with a certain precision δx, conceptually different from the dispersion Δx (Fig. 101).

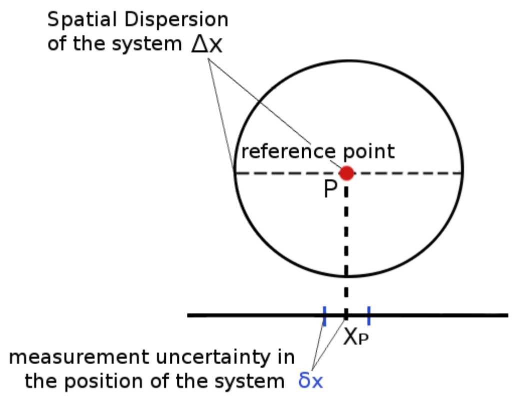


Figure 101: The measurement uncertainty (δx) is different from the dispersion of the system (Δx).

In the "Close Encounter" approach we have a situation similar to the one described in figure 101, just transposed in time. It is important to point out that

> we are not dealing at all with measurement uncertainties but with dispersions of a unique system in two different domains.

The dispersion Δf of each sound has a meaning very close to the spatial dispersion of the ball: each sound may be considered as composed of many harmonics with frequencies spanning a range Δf. All these aspects are then mapped to the quantum context via the analogy that is based on the fact that both in acoustics and in quantum mechanics the Fourier Transform connects the functions describing the system in two different domains.

We thus arrive to a meaning of the Heisenberg's Principle which differs from the following statement that is often written:

> It is impossible to measure at the same time and with the desired precision the position and the momentum of a particle. These uncertainties are related by an inverse proportionality.

Our proposal rather leads to this statement:

> A quantum system has a dispersion in position that is inversely proportional to the dispersion in momentum.

This is the true nature of the UP. The principle involves the product of the standard deviations (i.e. dispersions) of two wave functions in two "non-commuting" variables. The section "*Bound states and the Uncertainty Principle*" in the Appendix is devoted to show this in a simple case of a quantum system whose spatial dispersion is given by a Gaussian function.

The UP and how it is taught in the Italian Secondary School

As we have already stated the Heisenberg UP does not involves precision and disturbance of a measurement [65,66]. It concerns with an intrinsic property of a quantum system, it holds regardless of any measurement on the system. Therefore we think that any effort to introduce it involving a measuring processes to define the position of a particle (for example the detection of a scattered photon) should be avoided. Some authors [67], although choosing the measurement approach, highlight the fact that also an ideal measuring device (sending a single photon on the system), would cause a disturbance on the measured system. Even if this approach does not involve anymore the imperfection of the apparatus, it is still linking the UP to a measurement process, thus losing the possibility to reach the deep root of the principle itself. The UP is also often addressed through the analysis of the single slit experiment. This approach gives the valuable possibility to engage students in hands-on activities performing a simple and low-cost experiment with a laser. However, the strategy does not fulfil the expected needs. Again we lose the chance to introduce the principle as a quantum intrinsic aspect. The slit experiment shows properties arising from the interaction of the particle and the slit, beyond any doubt, is a scattering process. The formulation of the Heisenberg's Principle that we can achieve working on the electron diffraction pattern is a consequence of an interaction rather than an intrinsic property of the electron. Thus, at least, it should be pointed out that the obtained relation is not referring to the particle but to the system < electron + slit >. In a sense the slit experiment is again a measuring process. Common assertions such as "the spatial uncertainty Δx of the electron passing the slit is given by the slit width" sound bizarre. Students should accept that the Δx of the electron changes sharply when passing the slit, thus figuring a sudden change in the spatial dimensions of the electron, without any given explanation.

Figure 102) gives a good idea of how the topic is addressed at least in three different textbooks among the mainly adopted in the Italian secondary school.

[65] Robertson H P. *The Uncertainty Principle*. Phys. Rev. 34, 163, 1929

[66] Mahler D H Hayat A Soudagar Y Rozema L A, Darabi A and Steinberg A M. *Violation of Heisenberg's measurement-disturbance relationship by weak measurements*. Phys. Lett. Rev. 109, 100404, 2012

[67] Valentzas A and Halkia K. *The 'Heisenberg's Microscope' as an example of using thought experiments in teaching physics theories to students of the upper secondary school.* Res Sci Educ 41:525539, 2011

Classroom Test of the Close Encounter approach

The Close Encounter approach to the Uncertainty Principle was tested in one class of the classical Lyceum "Cavour" in Torino. A

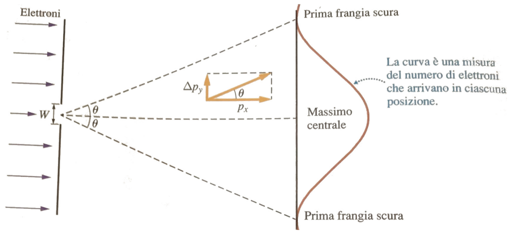


Figure 102: How the UP is usually presented in the Italian textbook.

school whose educational curriculum is mainly devoted to classical studies: philosophy, Latin, Greek and Literature. Physics is taught in the last three years of the five years curriculum (2 hours per week, 66 hours per year). In the last year students are introduced to electromagnetism and the basic principles of modern physics.

The course was a 6 hours programme, divided in three meetings (one per week in April - May 2019). The class was guided through all the steps previously described and was engaged in all the inquiry based activities. The PhE site was the framework in which students were supposed to move in order to understand the Heisenberg Principle. Therefore the course was held in a computer lab so that each group (formed by 2 students) had the opportunity to work directly on the dedicated site.

The lecturing delivered by the instructor was limited to short introductions of the activities and to short introductions of key concepts. The instructor also collected the results obtained by students during their investigations. Students achievements and understanding was monitored through Google Forms filled by the participants and through plenary discussions. It was instructor's task and concern to collect the main ideas, to stimulate discussions and, if possible, to convey the entire class to a common interpretation. The course was held by an external instructor, the physics teacher of the class followed the course with his pupils. Both the physics teacher and the students were asked to answer few questions about the course in order to have a feedback about the adopted educational approach. Two surveys were prepared, one for learners and one for an expert in the teaching of physics: the physics teacher. Here we report the results of both surveys in original language and in English.

Students' Feedback - In Italian.

Hai seguito un corso sul principio di Heisenberg, che cosa ti ha lasciato questa esperienza?

11 responses

È stata un'esperienza costruttiva che mi ha aperto nuove prospettive riguardo allo studio della fisica.

Innanzitutto una conoscenza approfondita sull'argomento in particolare. La voglia di approfondire sempre, anche gli argomenti meno piacevoli. Quello che mi ha stupito è la chiarezza con cui queste conoscenze sono rimaste fissate nella mia memoria, nonostante io non le abbia più ripassate (come le altre materie) sarei in grado anche adesso di esporle con molta chiarezza (non mi succede mai)

La cosa più interessante è stato senza dubbio il modo in cui l'insegnante ci ha fatto affrontare l'argomento, cioè adogmaticamente. Ho trovato difficile partecipare in modo critico all'analisi di argomenti poco noti e muovermi con una certa sicurezza in ambiti inconsueti, ma l'esperienza è stata sicuramente innovativa e originale.

Qualcosa della fisica quantistica

Ho visto come la fisica sia molto più vicina a noi di quanto di solito pensiamo. Ho scoperto la bellezza di imparare sperimentando.

A capire in generale cos'è il principio di Heisenberg

Mi ha fatto capire che tramite lo studio della fisica l'uomo ha ancora molto da scoprire dalla natura

L'odio per la fisica

Curiosità sulla quantistica

Mi ha lasciato nuove conoscenze è un nuovo approccio alla fisica

Ho appreso qualcosa sul principio di Heisenberg, ma non in maniera così significativa

In certe fasi del corso sei stato coinvolto attivamente in attività pratiche di misura e di analisi dei risultati. A volte ti è stato chiesto di giungere autonomamente a delle conclusioni. Come valuti questo tipo di attività?

11 responses

Molto interessante e innovativa perchè aumenta l'autonomia degli studenti.

La trovo molto interessante soprattutto per il fatto che mi son sentita pienamente partecipe e non è stata la solita lezione di fisica frontale. Siamo giunti a delle conclusioni teoriche partendo da delle rilevazioni pratiche che è il lavoro di un fisico che si rispetti

L'analisi dei risultati e l'interpretazione dei grafici hanno permesso di trarre delle conclusioni che mi hanno fatto capire che i risultati della fisica sono molto meno lontani dalla realtà di quanto non sembri nello studio consueto sul libro.

Utili

Sono attività molto interessanti; aiutano a mettersi in gioco per imparare. È difficile però svolgere tali attività in una classe perché ogni studente ha tempi diversi.

Utile per imparare meglio a ragionare da soli

Questo tipo di attività secondo me è molto utile per riuscire a padroneggiare un argomento

Molto utile e proficuo, se avessi avuto le conoscenze di base necessarie.

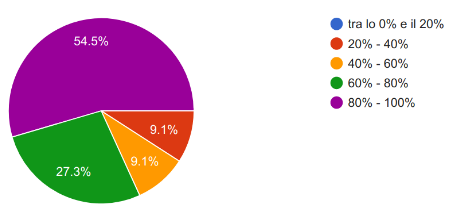
Positiva

L'ho trovata molto interessante e utile

Ottima

Sei riuscito a seguire il corso e comprendere gli aspetti essenziali del percorso?

11 responses



Durante questo corso è cambiato qualcosa per quel che riguarda il tuo rapporto con la Fisica?

11 responses

Non è mai stata la mia materia preferita, ma dopo questo corso ho imparato a rivalutarla e ad apprezzarla al meglio proprio perché ho potuto constatare che non si tratta dello studio di fenomeni poi così astratti ed estranei a noi

Mi ha sicuramente ricordato che la fisica è legata all'esperienza reale e compare in fenomeni come quelli acustici con cui veniamo a contatto ogni giorno. E' come se la fisica si fosse avvicinata alla vita: in realtà anche la professoressa sottolinea sempre come si traduce una regola a livello pratico, perché noi studenti tendiamo istintivamente ad astrarre le regole quasi fossero staccate dall'esperienza.

Non tanto

No, ma ho scoperto aspetti nuovi di questa materia, che comunque mi appassiona.

No nulla

sì, ho scoperto che ci sono molti modi per approcciarsi con la Fisica

No, ha confermato che non fa per me.

No, ha confermato che non fa per me.

È cresciuta la mia curiosità

Si, mi ha fatto capire che non devo fermarmi a ciò che c' scritto sui libri, ma approfondire e andare oltre

No

Qual è l'aspetto del corso che più ti è piaciuto?

6 responses

La presenza di attività di analisi pratiche.

La dedizione del professore che, nonostante la sua preparazione, ha saputo rendere un argomento insidioso così semplice. Ha fornito anche molti dettagli e si è mostrato davvero interessato all'insegnarci, oltre alla materia, anche un'importante lezione: dubitare sempre e ed esporre qualsiasi cosa al nostro giudizio critico

Il fatto che pur essendo lasciati liberi di raccogliere dati venissimo sempre guidati e assistiti e aiutati a ragionare.

La parte più pratica

La possibilità di arrivare io stessa a una conclusione del ragionamento

Arrivare alla teoria dalla pratica

Qual è l'aspetto del corso che meno ti è piaciuto?

4 responses

Che sia stato così breve

Ho incontrato difficoltà nell'utilizzo dell'analizzatore di spettro.

Non lo so

La pretesa che a tutti dovesse interessare la fisica quantistica.

Hai dei consigli a proposito delle modalità con cui proporre il corso a degli studenti come te?

3 responses

Credo che questo metodo sia utile, forse eviterei le prime due ore del sabato mattina. Per il resto credo che il professore si sia dimostrato in grado di saper catturare l'attenzione dei suoi allievi e di mantenerla alta durante tutte le ore del corso

Il corso è stato forse troppo breve, sarebbe stato bello approfondirlo perché personalmente mi ha interessato molto.

Fare più incontri

Students' Feedback - In English.

- You have attended a course on the Heisenberg Principle, what have you gained from this experience?
  - It has been an enriching experience, that has opened new perspectives in the study of physics.
  - First of all a deep knowledge about the topic. The desire to always go deeper, even on less attractive topics. What have surprised me is the clear way in which this things have been fixed in my mind. Even though I have not reviewed them (as usually happens with other subjects) I would be able to clearly expose them (it never happens to me).
  - Indeed, the most interesting aspect was the methodology adopted by the instructor, an undogmatic approach. I had difficulties in being critical, since we were dealing little known topics. However the experience was surely innovative and original.
  - Something about Quantum Physics
  - I have seen that physics is closer to us than we think. I have discovered the beauty of learning by inquiry.
  - To understand what the Heisenberg Principle is
  - It made me understand that mankind still has a lot to discover from nature.
  - My hate for physics
  - It has stimulated curiosity about physics
  - I have gained new knowledge and a new approach to physics
  - I have learned something about the Heisenberg Principle, but not so significantly.

- In some stages of the course you were directly engaged in measurement activities and data analysis. Sometimes you were asked to reach your own conclusions. How do you estimate this kind of approach?
  - Very interesting and innovative, since it increases the students' autonomy.
  - Very interesting mainly because I felt myself part of the process, it was not the usual frontal lecture. We reached theoretical conclusions working on experimental measurements. This is the job of a self-respecting physicist.
  - The data analysis and the plots interpretation allowed us to draw conclusions. This made me understand that physics is not far from reality as we might think reading books.
  - Useful
  - These are very interesting activities; they stimulate to get in the game to learn. However they are difficult to be implemented, since each students has a different pace.
  - Useful for a better learning and for the autonomous reasoning.
  - The activity was useful for mastering the topic.
  - Positive
  - Interesting and useful
  - Very good
- Did you manage to understand the essential aspects of the course?
  - 54.5% of the students understood among 80% and 100%
  - 27.3% of the students understood among 60% and 80%
  - 9.1% of the students understood among 40% and 60%
  - 9.1% of the students understood among 20% and 40%

- Something has changed in your relationship with physics?
  - Physics is not my favourite subject, however this course helped me in rethink my feelings and in appreciating it. The reason is that I have discovered that it is not an abstract discipline.
  - This course helped me in remembering that physics deals with our real world, for example it explains acoustic phenomena always present in our life. It is as if physics had become closer to our life. Something that our teacher always remembers us, since we often tend to look for abstract rules.
  - Not so much
  - No, however I have discovered that a lot of different methods to approach physics exist. Nothing has changed.
  - No, it confirmed that physics is something not for me.
  - My curiosity has increased.
  - Yes, it helped me in understanding that we have to go beyond what is written on textbooks, we have to go deeper.
  - No.
- What is the feature of the course you liked most?
  - The activity of data analysis.
  - The commitment of the teacher, which was able to explain in an easy way such a difficult topic. He was really interested in teaching both the content and the stimulus to activate our critical thinking.
  - Even if we were free to collect data we were guided and helped in our reasoning.
  - The practical activities.
  - The opportunity to draw conclusions by my own.
  - Get the theory from practical activities.

- What did you least liked ?
  - The course was too short.
  - I had difficulties in working with the spectrum analyser.
  - The claim that everyone should be interested in quantum mechanics.
- Do you have something to suggest about how to present the course to young students?
  - I believe the method is useful, probably I would avoid the first two hours of the Saturday morning.
  - It probably was too short. It would have been nice to move even deeper in this topics, since I was really interested in the subject.
  - Do more meetings.

Teaching General Relativity

The modern theory of gravity is the General Relativity proposed by Albert Einstein in 1915. As we will discuss, his idea to explain gravitation deeply leans on geometric concepts. In this booklet we will explore this fascinating theory through a series of stages. In each one we will have the chance to build an hand-on model and to carry out explorations on it. This will help us to visualise and understand the main important concepts of this beautiful theory.

This is the result of a work experience in spring 2019 with the group of Ute Kraus and Corvin Zahn at the University of Hildesheim in Germany. They have developed an original approach to the teaching of General Relativity[68,69] based on the Regge Calculus[70]. My work in collaboration with them was to design and write a short booklet to present their educational method to high-schools both in Italy and Germany.

[68] U Kraus C Zahn. *Sector models—A toolkit for teaching general relativity: I. Curved spaces and spacetimes*. European Journal of Physics, vol. 35, n. 5., 2014

[69] U Kraus C Zahn. *Sector models—a toolkit for teaching general relativity: II. Geodesics*. The European Journal of Physics, vol. 40, n. 1., 2019

[70] T Regge. *General Relativity without Coordinates*. Nuovo Cimento, XIX, 3, 559., 1961

Curvature in 2D

In this section we will learn what the curvature of a surface is and how to recognise it in all the surfaces that surround us. Later, going forward on our journey to General Relativity, we will also learn how to measure it.

We can start building a very common surface: the sphere. What we need is to print the image in figure (103), cut the three shapes, and join together six edges with six pieces of adhesive tape (Fig. 104). Before joining the edges we recommend to fold the faces along the horizontal lines. What we will get (Fig. 105) is a spherical cap. To be more precise we will have in our hands an *approximating surface*: a spherical cap approximated by 9 flat sectors. Thus:

> 2D Surfaces may be represented by *approximating surfaces*.

To explore the curvature of the sphere we will adopt a *Flattening*

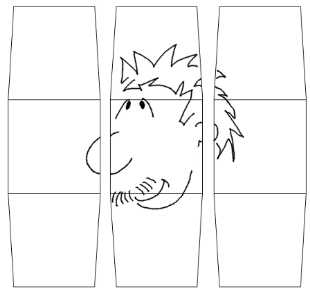


Figure 103: Print this image to build a spherical cap.

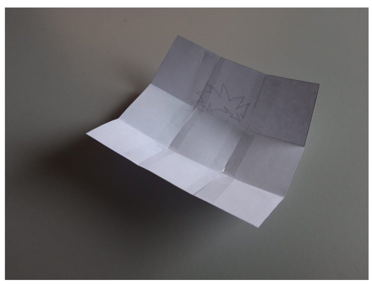


Figure 104: Six pieces of adhesive tape and some folding to build the surface.

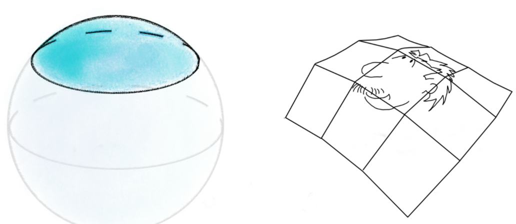


Figure 105: A spherical cap (left) and its *approximating surface* (right).

Technique: we will flatten the spherical cap on a plane. In order not to damage the paper model, we need to cut some edges in advance (Fig. 106). Now we can flatten the *approximating surface* on the desk. As a result the surface tears, leaving empty spaces among the separated edges.

Should we repeat the same process with a flat surface, for example a sheet of paper, nothing would happen when forcing it to stick to the desk. We have thus discovered that, when flattened, surfaces may behave in different ways: if they are flat they adhere perfectly to the plane of the desk, if they are like a sphere they tear.

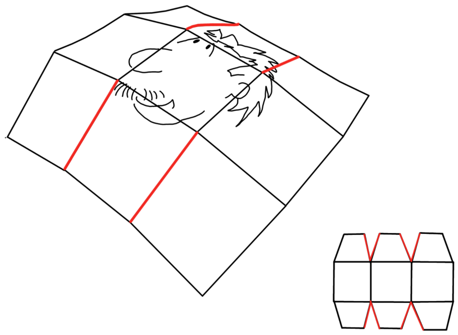


Figure 106: Cut the red edges to flatten the approximating surface on a plane. The surface will tear (bottom-right).

The Flattening technique is a tool to distinguish between flat and sphere-like surfaces.

We will now explore another surface. Print the the image in figure 107 and join the six edges: you will have in your hands an *approximating surface* of the saddle (Fig. 108). How will it behave if flattened?

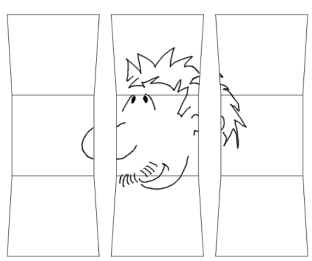


Figure 107: Print this image to build a saddle.

Cut the edges shown in the figure and flatten the surface. What we see differs again from how flat surfaces behave, but it also differs from the sphere. We do not have empty regions among the separated edges, on the contrary, we have overlapping sections.

We have discovered a third type of surface: the saddle-like surface. We can distinguish among this surfaces with the flattening technique, but the revealed feature is what mathematicians call the Curvature.

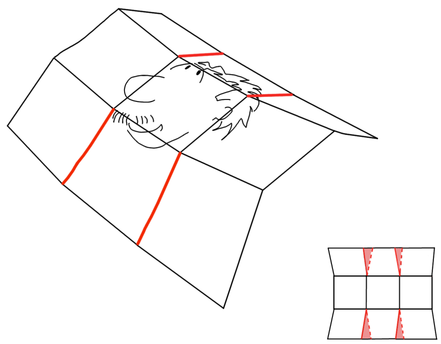


Figure 108: A Flattened saddle will overlap.

- Surfaces that once flattened perfectly adhere to the plane are zero curvature surfaces

  ($K = 0$).

- Surfaces that tear are positive curvature surfaces

  ($K > 0$).

- Surfaces that overlap are negative curvature surfaces

  ($K < 0$).

A number, the curvature K, can distinguish among the three types of 2D surfaces. We have a well defined technique to evaluate the sign of this number, later we will learn how to assign a value to it.

Inhabitants of their own space

So far we have seen three different kinds of surfaces (the flat one, the sphere, the saddle) and we have built approximating surfaces of some of them. We should now be able to recognise them looking at the great variety of surfaces embedded in the space around us. However looking at embedded surfaces is going to become unsuitable, as soon as we will move from 2D to 3D objects. Dealing with the theory of gravity we will be interested in considering the 3D space around a Black Hole and we would have the nasty surprise that is not possible to embed a 3D curved space in our flat space: we should need extra dimensions! This is something we can guess with an analogy at lower dimensions: if we were inhabitants of a flat world we would have no problem to embed a flat object in our space (Fig. 109), but no chance to do the same with a curved surface like the sphere (Fig. 110).

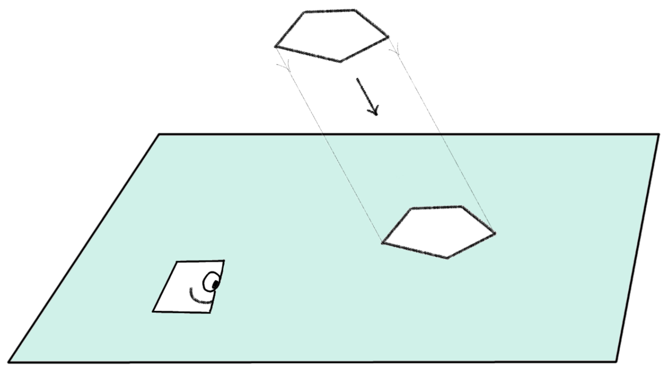


Figure 109: A flat surface can be embedded on a plane.

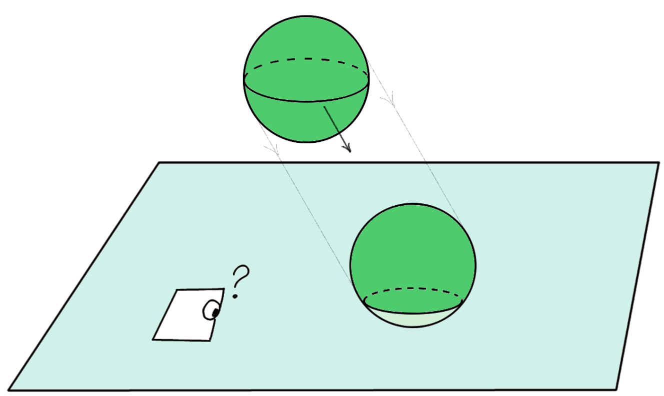


Figure 110: A sphere, a curved surface, can not be embedded in a flat world.

In the same way

> we can not embed 3D curved spaces in our 3D flat world. As a consequence we can not see, draw and even imagine them.

Luckily we have an opportunity to bypass the problem. Again, an example at lower dimensions will help us. A saddle is a curved surface that can not be hosted in a flat world (Fig.111), however a flat being has the chance to perceive its curvature. Let's see how.

1. We can build an *approximating surface* of the saddle.
2. The approximating surface can be flattened, so becoming a *Sector Model* of the curved surface.
3. The *Sector Model* can be embedded in the flat world.
4. The flat being can discover the overlapping and the curvature of the surface. Sadly he will never be able to see the saddle, draw it or even imagine it.

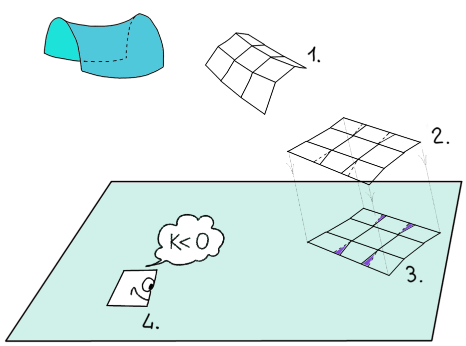


Figure 111: With *Sector Models* any curved surface can be perceived by a 2D being.

This is true, in general, for any 2D surface.

> With the *Sector Model* of a 2D curved surface we can discover its curvature working on a plane.

> A *Sector Model* is the flattened version of the approximating surface (Fig. 112).

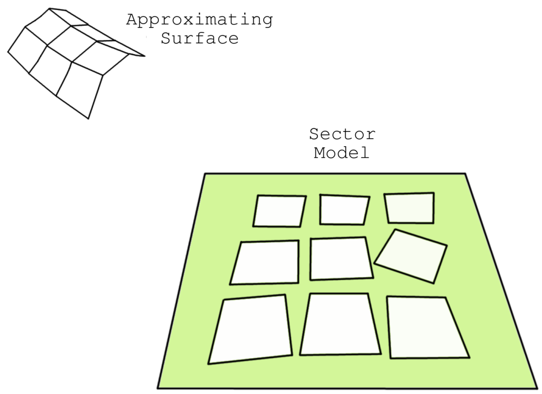


Figure 112: A *Sector Model* of a Saddle. To flatten the approximating surface we take from it each flat sector and we dispose it on a plane.

We will call this method *Sector Model Approach*. It will be crucial for us when we will try to have an idea of the space surrounding a gravitational source. At that point we will have the same problem of the being of figure (111).

As three-dimensional being, we will be forbidden to embed a curved 3D space in our 3D flat world.

Our only chance? Come up with a workaround. The same we adopted for the flat being. Here we present the general idea, in the following chapters we will use it and details will be deeply discussed.

1. We build an *approximating space* of the curved 3D space by means of polyhedra (before they were polygons) joined together sharing faces (polygons were sharing edges).

2. The approximating space can be "flattened", so becoming a *Sector Model* of the curved 3D space. To flatten it we just have to take each polyhedron apart and dispose it in our flat space, as shown in figure (112), in a 2D example.

3. The *Sector Model* can be embedded in our 3D flat world.

4. Joining the faces of the polyhedra, the flat 3D being can find out the tearing or overlapping of the *Sector Model* and therefore the curvature of the space.

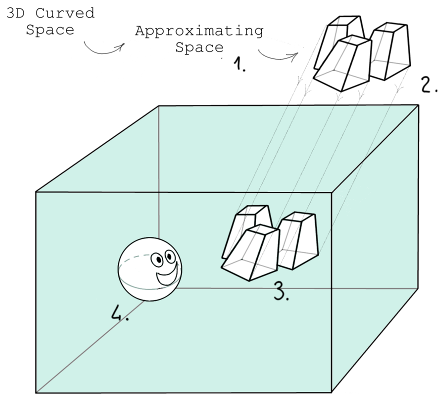


Figure 113: With a *Sector Model* of a curved 3D space we can find out its curvature.

3D Flat space

In the previous section we have repeatedly used terms like "3D flat world" or "our flat space". Without going in deep details we would like to give some explanation of the concept behind. The task is not that easy since, as already stated, we are unable to imagine a curved 3D space. However we will tell what distinguish a flat 3D space from a curved one. On a 3D flat space, for example, parallel lines always maintain the same distance. This is not going to happen on curved spaces: accordingly to their curvature parallel lines will become closer and closer or more and more distant.

Further explorations

Sometimes exercises may be transformed in further investigations. In this paragraph we give some example of small explorations that could be carried out by students.

1. An inhabitant of an outer world measures the sectors shown in figure (114), what can you say about the curvature of his world? [The colours of the edges tells you how to join the edges]

2. Determine the curvature of a cylinder

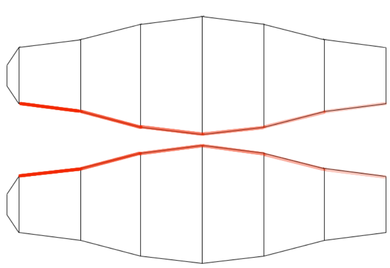


Figure 114: Sector reported by the inhabitant of an outer world.

Curvature with Sector Models

What is the space around a Black Hole like? We can try to answer to this question using Sector Models. We will start a trip through curvature of 3D spaces. At the end of this fascinating journey we will find the Einstein field equation, which is there ...waiting for us.

Curvature in 2D

In principle, finding the curvature of a surface at a point P with Sector Models is quite simple. Working just on a small region of the surface, we can follow this "*Curvature Hunter*" procedure:

1. Go to the surface and draw a grid of polygons sharing the vertex P and completely surrounding it (Fig. 115).
2. Take measures of the lengths of the edges of the polygons.
3. Go back home and, according to the measures, reproduce the polygons on scale on a flat surface (this step corresponds to the building of a *Sector Model* of the surface around P).
4. On **a flat surface** dispose the sectors around the common vertex P and try to join their edges.
5. Look for tearing or overlapping.

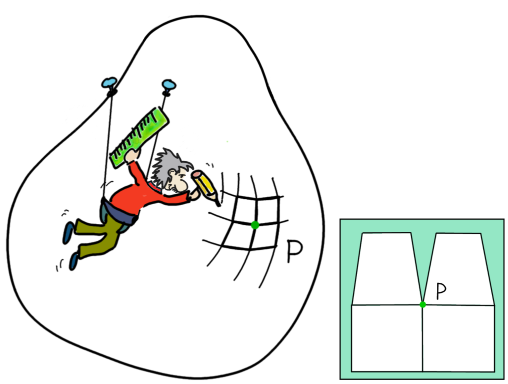


Figure 115: The *Curvature Hunter* Procedure. Reach the surface and bring back home measures to build a Sector Model of the surface.

If the polygons do not cover the whole angle around P (tearing) the curvature K is positive, if they cover more than the full angle (overlapping), K is negative. If neither one of those things happens the surface is flat.

2D to 3D extension

We have to keep in mind that our goal is to analyse the three-dimensional space around a Black-Hole. So we are interested in extending to 3D the *Curvature Hunter* procedure we have outlined for 2D surfaces. In a way we have to wear special "3D glasses" in order to properly modify the 2D process. Looking at 2D objects with these lenses we will see the corresponding 3D geometrical entity (Fig. 116).

A vertex will look like an edge, and edge like a face, a face like a block. With this in mind we can now extend the *Curvature Hunter* procedure.

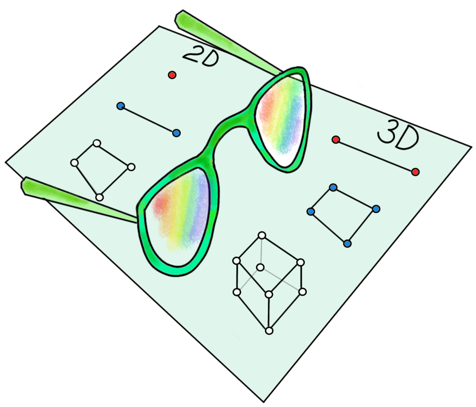


Figure 116: The "3D glasses" connect a 2D object into its corresponding 3D object.

- A VERTEX corresponds to an EDGE,
- An EDGE to a FACE
- A FACE to a BLOCK.

Curvature in 3D

Now let's look at the 2D *Curvature Hunter* procedure with our new "3D glasses". It changes a bit, but the logical basis remains the same.

With 2D surfaces we were used to select a vertex P and surround it with a number of polygons. In a 3D space the vertex becomes an edge as well as a polygon a block, thus we need to select an edge L and to draw around it a grid of blocks completely surrounding it (Fig. 117 shows this procedure in our space).

Like before, is time to take measures of our grid of blocks in order to build a 3D *Sector Model* of that region of 3D space, meaning that

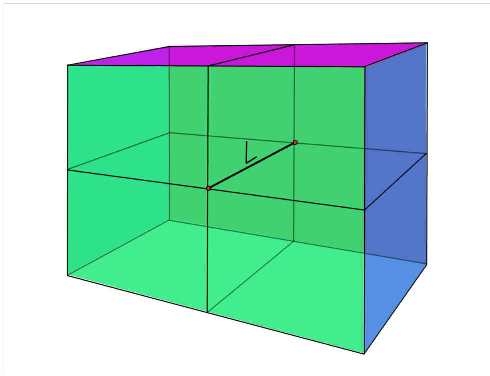


Figure 117: An edge L in a 3D Flat space, surrounded by blocks.

we want reproduce the blocks on scale in our 3D flat space. Once the *Sector Model* is ready, in a our **flat 3D space** we have to dispose the blocks along the common edge L trying to join their faces. Eventually, look for empty spaces among the faces (Fig. 118), in analogy with the tearing of 2D surfaces, or for blocks that penetrate one inside the other (Fig.119), in analogy with the overlapping.

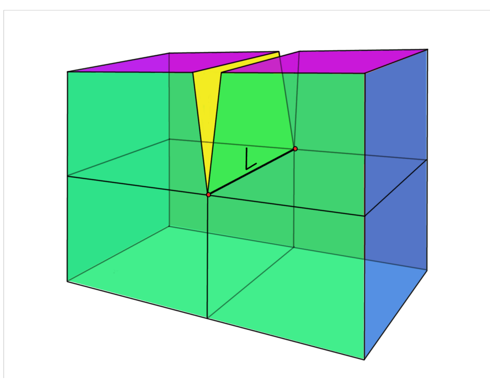


Figure 118: Tearing Blocks.

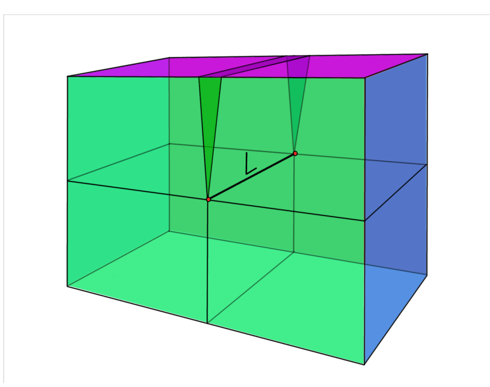


Figure 119: Overlapping Blocks.

In the first eventuality the curvature of the investigated 3D space is positive, in the second negative.

However, if we take a moment and focus our attention on the edge L, some questions come up. In three dimensions an edge can be oriented along the three fundamental directions: x, y, z. Therefore, to evaluate the 3D curvature we have to choose three different edges, L_x, L_y, L_z originating from the same point (Fig. 120) and for each one we have to carry out our procedure.

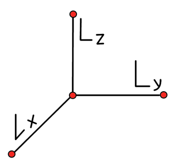


Figure 120: The three directions along which we can choose the edge.

Almost naturally we have discovered that

> the 3D curvature is no more a number, is a mathematical quantity with three components!

The Black Hole geometry

The Black-Hole Sector Model

We now have a basic idea about how to detect the curvature of a 3D space, it is time to put into practice our knowledge. It is time to consider a Black Hole (BH)!

First we have to build a *Sector Model* of the space around it, thus we have to send there a messenger. He is asked to stay far from the horizon of the BH, to "build" a grid of blocks around three edges like those shown in figure 120 and to take measures of the grid (Fig. 121). With his measures we can reproduce the BH blocks on scale and

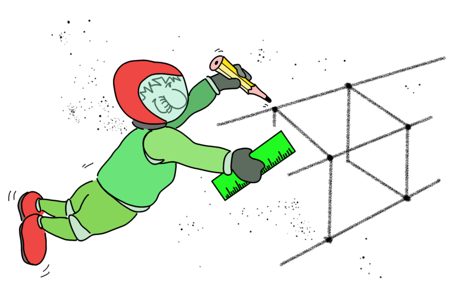


Figure 121: Building a grid of blocks around a BH.

dispose them in our flat 3D space, exactly as shown in figure 122.

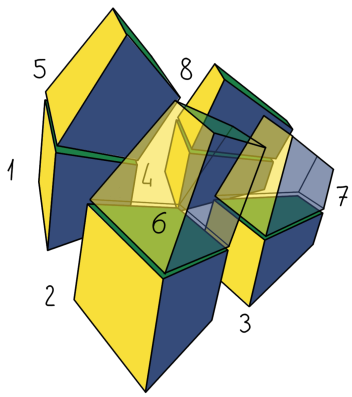


Figure 122: The *Sector Model* of the space around a BH.

To reproduce these blocks print the cut-out sheet (Fig. 123) and assemble eight blocks.

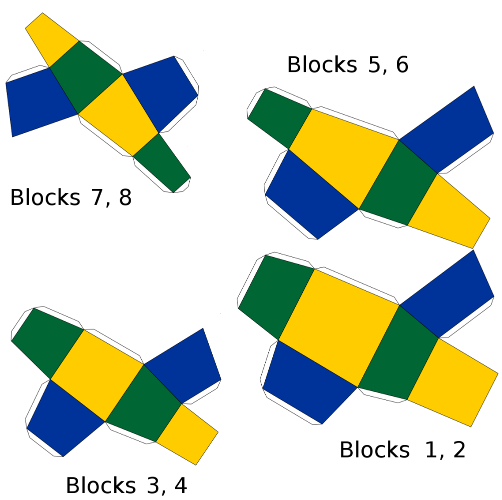


Figure 123: The cut-out sheet to assemble the blocks.

The BH blocks. A closer look

Figure 124 defines the main directions of the blocks. Once you have this blocks in your hands the first thing that you may notice is that they cannot be arranged in order to fill our space without gaps: this is the first clue that the space around a BH is not flat. However, with a BH of the appropriate mass in the centre of the model, the blocks, the way they are, would fit without gaps.

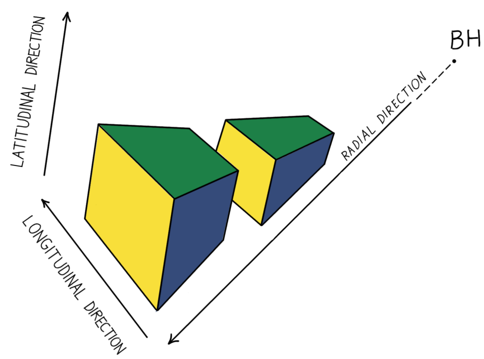


Figure 124: The main directions of the blocks.

The BH curvature

Now we want to detect the three components of the curvature. As discussed before we have to work along three different edges originating from the same point, the radial (Fig. 125), the longitudinal (Fig. 126) and, eventually, the latitudinal one (Fig. 127).

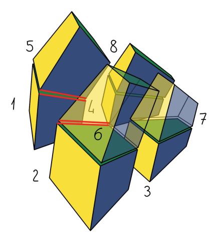


Figure 125: The radial edge.

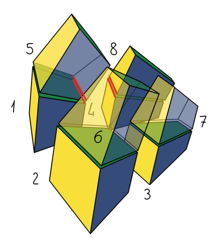


Figure 126: The longitudinal edge.

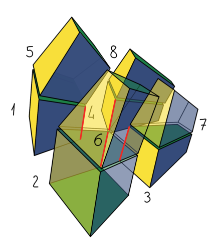


Figure 127: The latitudinal edge.

To find the curvature along the radial direction we have to dispose the four blocks 1, 2, 5, 6, along the common edge (oriented in the radial direction) trying to join their faces. Figure (128) shows the result of this attempt, the common radial edge is perpendicular to the page and is highlighted by the red dot. The blocks do not completely fill the space around the edge.

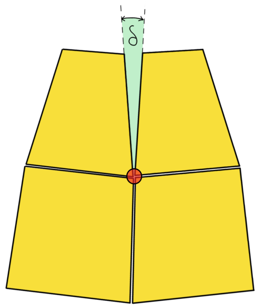


Figure 128: Along the radial direction the space is not completely filled.

The first component of the curvature around a BH is positive.
$K_1 > 0$.

Considering the longitudinal direction, we try to dispose the four blocks 1, 5, 4, 8, along the common edge. But what we find is that is impossible. In order to do that the blocks should penetrate one inside the other. What we can do is shown in figure (129): we may join the faces of three out of four blocks, sliding the last one along the face it should share until it touches the block in front of it. The angle shown in figure is equal to the "overlapping" angle, if it was possible to penetrate the block. In conclusion the blocks fill more space than they have available.

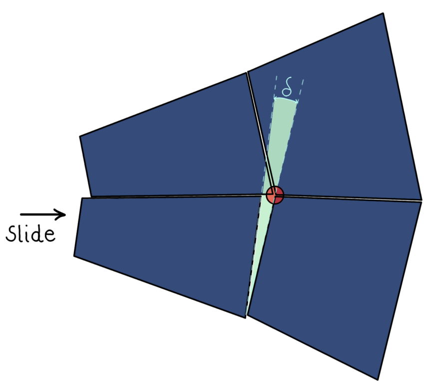


Figure 129: Along the longitudinal direction we have an overlapping.

The second curvature component around a BH is negative.
$K_2 < 0$.

Similar situation if we align four blocks (1, 2, 3, 4) along the latitudinal edge (114).

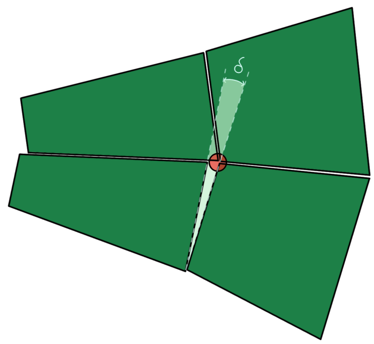


Figure 130: Along the latitudinal direction we find an overlapping.

The third curvature component around a BH is negative.
$K_3 < 0$.

We have learned how to reveal the curvature of a 3D space. Basically we have defined blocks disposed around a common edge in the curved 3D space and reproduced them on scale in our flat space. This procedure led us to discover that 3D curvature is no more a number, but a mathematical quantity defined by components.

Geometry and gravitation

We started to unveil the space geometry around a BH, which is an incredible result if we consider how we obtained it, simply working with paper blocks. But what is even more surprising is that we now have the opportunity to grasp the essence of General Relativity. We have enough elements to establish a connection among Gravity and Geometry.

Physical effects of the negative curvature around a BH

So far we have worked with eight blocks representing the curved space around a BH, figure 131 involves a bigger number of blocks and layers so giving a more complete description. Now we can clearly see that both the green and blue faces of our blocks form planes: we will call them *Equatorial Planes*. Figures 129 and 130 tell us what happens on regions of these surfaces. The overlapping blocks resemble quite closely the situation we have with the overlapping sectors of a saddle. In this sense we will say that

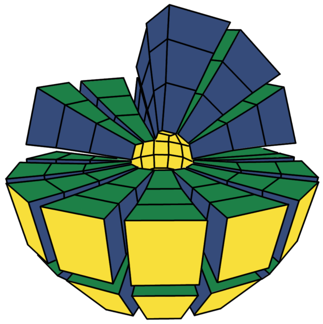


Figure 131: The sector model of a BH.

the negative curvature components k_2 and k_3 are the curvatures of the green and blue equatorial planes.

Now we have all the pieces we need to understand the basic idea of the General Relativity, it is just a matter of putting them together.

1. **Paper model of a Saddle**

Building a paper model of a negative curvature surface is a matter of few minutes. With two sheet of paper and two strips of adhesive tape we can quickly build a saddle (Fig. 132). Cut the angle

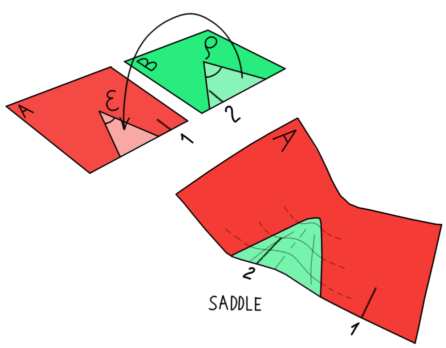


Figure 132: Paper model of a saddle.

ϵ from the sheet A and cut a bigger angle ρ from B. Then, with the help of the tape, substitute ϵ with ρ. What you get is a saddle. Having the foresight to draw two parallel segments, one on ρ and one on the sheet A, we have a saddle with two parallel lines. We will use this segments later, now it is important to notice that the red regions of our model are flat. As we have learned in the previous chapter, if we flatten them on a plane they will perfectly stick to it. However there is a region that does not adhere to a plane:

the part that contains the vertex of the angle ρ. Around the vertex we have deliberately inserted a bigger angle with respect to the angle which is up to a flat surface. Flattening this zone will result in an overlapping (try it!).

> Our paper model has negative curvature in a region concentrated around the vertex, elsewhere it is flat!

2. **Straight Lines on Negative Curvature Surfaces.**

We want to study how parallel straight lines behave on a negative curvature surface. What we have to do is to extend the two parallel segments we have previously traced. We need to make sure that our extended lines are straight, which means they do not deviate either to the right or to the left. To fulfil this requirement we will use a ruler that will flatten the surface along the line we want to extend (Fig. 133).

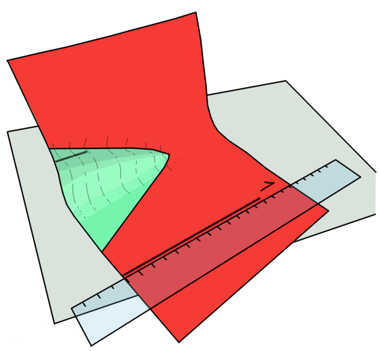


Figure 133: The rule flattens the paper model so that we can extend the line.

The result is shown in figure (134). Line 1 lies on a flat region of our model and is separated from line 2 by the negative curvature region. We may conclude that:

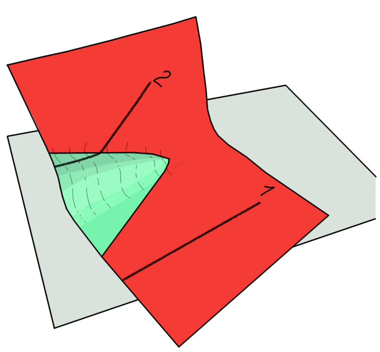


Figure 134: The two lines separated by a negative curvature region.

Straight lines initially parallel, if separated by regions with negative curvature, diverge!

3. **Straight Lines on the BH Equatorial Planes**

The green plane around a BH is a surface with negative curvature, and we have just learned something about these surfaces! Therefore, why not repeat the same experiment we made on the saddle?

We draw a straight line on a flat region of the plane (line 1, Fig. 135), where the BH effects are negligible. Then we move closer to our gravitational source, ready to draw line 2, that starts parallel to line 1 and is straight. But, wait a moment, they are initially parallel straight lines separated by a negative curvature region so they must diverge. As shown in figure we are obliged to draw a line that changes direction around the BH.

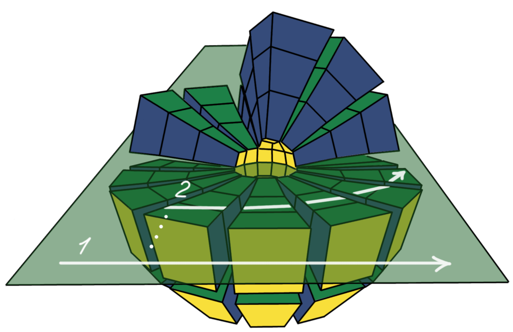


Figure 135: Line 1 is a straight line far away from the BH. Line 2 is a straight line initially parallel to line 1 and closer to the BH.

It is extremely important at this point to be as clear as possible about how to consider line 2.

First, line 2 it is a straight line (does not deviate either to right or to the left).

Second, if we compare the directions of line 2 far before and far after the BH (thus in flat regions) we can notice a deflection.

In this sense, we can say:

Straight lines traced on any equatorial plane around a BH are bent around the BH.

The word "bend" could lead to think that the line is not straight, but this is not true. The line is a straight line of a curved space!

The word "bend" has to be interpreted as explicitly pointed out in the second consideration about line 2. You will probably hear or read a lot about *bending of light* from a gravitational source, *gravitational lensing* or *light deflection*. We do not intend to demonise this expressions that we are also going to use, we just want to be sure you know what do they mean.

4. **Newtonian Gravity**

Why planets are kept in their orbits? Newton's idea was that massive bodies attract each other with a force. This force determines the gravitational bending of the trajectories of massive bodies (Fig. 136). We are talking about the Newton's law of universal Gravitation, that for centuries has explained the motion of celestial bodies in the heavens.

> In the Newtonian theory of Gravity the bending of the trajectory of a massive body is explained by means of an attractive Force.

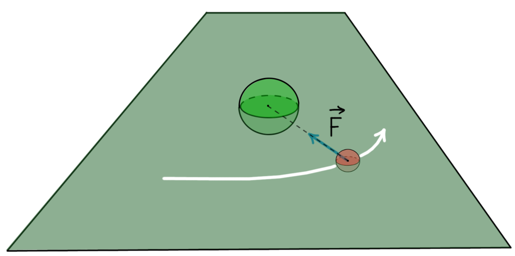


Figure 136: The Gravitational Force.

5. **Geometry and Gravitation**

Two similar situations, coming from different domains, capture our attention. One belongs to the domain of Geometry, the other to the Newtonian Theory of Gravity.

- *From Geometry* we have straight lines changing direction near a BH.
- *From Newton's Theory of Gravity* we have the bending of the trajectories of massive bodies near a gravitational source.

We have the opportunity to transform Gravity from a force exchange theory to a pure geometrical theory. We can state that all the bodies near a BH move along the straight lines of the curved

3D space. The gravitational bending of the trajectories remains, the force acting on the bodies vanishes. This is a beautiful and elegant result, which removes a problem of the Newton's theory and also formulate an unexpected prediction. The issue about the Newtonian point of view is the so called *action at a distance* problem. In few words, it is not clear how two bodies separated by a vacuum can act on each other. Conceiving the theory from a geometrical perspective, the force is no more needed and the problem is solved. Gravity becomes a matter of the spatial environment in which a body happen to move. A mass curves the surrounding space, particles move along the straight lines of the curved space.

The unexpected prediction concerns light. Photons are massless particles, therefore from a Newtonian point of view their trajectories should not be affected by gravitational forces which directly depend on mass. However, the geometrical nature of Gravity can be extended also to photons. There is no reason for the photons to behave differently from any other particle. The prediction is astonishing: light should be deflected by Gravity!

We have reached one of the main concepts of General Relativity:

> It is the geometry that tells matter and light how to move.

However, we need to say that in our discussion we have completely neglected time. Nearly ten years before his theory of General Relativity, Einstein showed that time and space are two physical quantities extremely correlated. He did this in the theory of Special Relativity, where the natural environment for the description of particles motion is space-time. So, actually

> It is space-time geometry that tells matter and light how to move.

In other words, we would need to take into account a four dimensional space, adding one temporal direction to the three spatial directions, thus having to consider curvature components related to time.

How a Black Hole looks like

April the 10th 2019, the Event Horizon Telescope (EHT) Collaboration presents its first results: an image of the supermassive BH in galaxy M87. We take the leap to ask ourselves how a BH looks like. To answer this question we will apply what we have just learned about light travels in a curved space.

Figure 137 shows three different regions that surround a BH, let's have a look together.

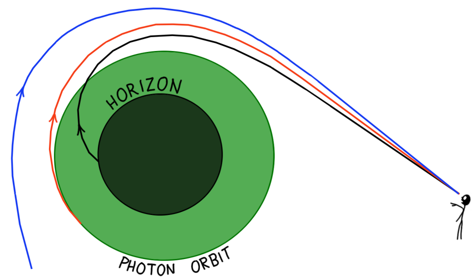


Figure 137: Three regions surrounding a BH.

The inner one (dark grey) is a zone from which neither matter nor light can escape, its surface is called *Event Horizon*. A photon emitted by the observer along the black line would follow a straight line in the curved space leading it directly to the Event Horizon. If we trace the trajectory backwards (see arrow on the black track) we can argue that no light from the BH is expected to reach the observer along that line. The second circle represents the *Photon Orbit* sphere: photons can move along this surface without getting closer or away from the BH. However, this orbit is unstable. A slight perturbation towards the BH will put the photon on his way towards the Event Horizon. The red line is the path of a photon emitted by the observer that enters and remains in the photon orbit. A path slightly above escapes the BH (see the blue line for example), a track slightly below ends its journey in the BH. Reversing the direction and moving towards the observer we can conclude that light coming from above the red line may reach the observer, light below will never be detected: the red track separates regions that can send light to the observer from regions that cannot. Therefore, the angle between two diametrically opposite red lines (Fig. 138) defines a black region surrounding the BH. This region is called the *Shadow* of the BH.

> The *Shadow* of a BH is black region, bigger than the Horizon, surrounding the BH.

A disc of matter spiralling towards the BH forms the so called *Accretion Disc*. This is the way a BH increases its mass over time. The accretion disc of a BH emits light at different wavelengths, giving us

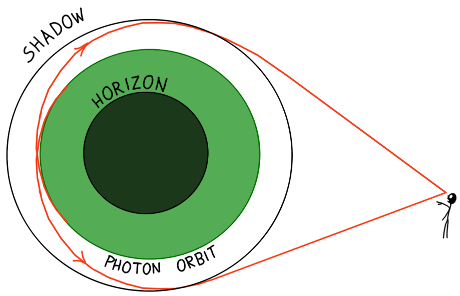


Figure 138: The shadow of a BH.

the possibility to detect it with appropriate telescopes.

In Figure 139 we see how photons coming from two points of the accretion disc can reach the observer. They are moving along an equatorial plane of the BH, their path is affected by the negative curvature we have already discussed. From the observer point of view, the points A and B are behind the BH, anyhow he will see them! The small box in figure 139 and figure 140 finally show how a BH would look like for an observer who looks at it from an edge-on line of sight.

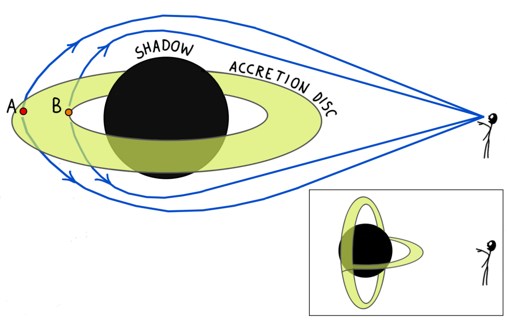


Figure 139: The path of two photons coming from the accretion disc behind the BH. In the box, how the observer sees the BH.

Figure 141 shows the Event Horizon Telescope picture of the BH in the centre of galaxy M87. The black region is the Shadow of the BH, the surrounding glow is the accretion disc. As you may guess, in this case, the line of sight of the Earth is not edge-on to the accretion disc, this fact explains why we do not see a bar crossing the shadow. The accretion disc is only 17 degrees tilted with respect to the line of sight of our telescopes. However this small angle introduces an asymmetry in the velocities of the spiralling gas: the lower part has a velocity component directed towards us, the upper part a component receding from us. Thus for a relativistic effect called *Doppler Beaming* the lower part appears more brighter than the upper one.

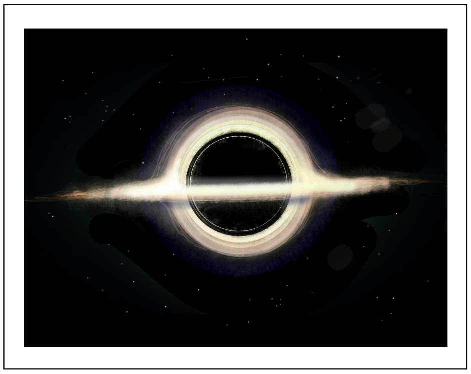


Figure 140: Artist impression of the accretion disc of a BH. The Black region is the shadow of the BH.

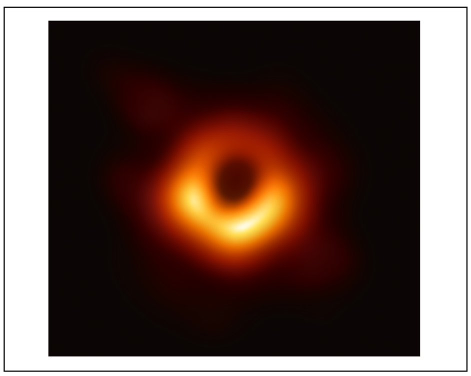


Figure 141: The Event Horizon Telescope picture of the BH in the centre of galaxy M87.

Einstein's Field equations

Einstein Field Equations (EFE) connect curvature components in a point of space-time to the energy density content at that point. We would like to express this concept writing a very general and only symbolic equation:

$$R(t, \vec{x}) \;=\; T(t, \vec{x}) \tag{185}$$

Where R represents a quantity related to the curvature at a point and T a quantity expressing the energy density content at the same point. As we have already learned Curvature has components and so does T. In fact there are different ways to store energy in a certain volume, each way forms a different component of T. One component of T is related to the mass content in a volume, in this case the T component is the mass density ρ. Another is given by the pressure, which directly is an energy density. Since both R and T have components actually equation (185) is a set of equations, one for each type of component. Solving this set of equations, we can theoretically discover how space-time is curved around (but also inside) a gravitational source. Given an energy density distribution in space (matter, pressure, ...) we can derive the curvature. Therefore equation (185) is the mathematical relation behind the famous Wheeler's statement:

"Matter tells space how to curve".

In this section we will deal with the equation which involves the mass density at a point in space-time. The equation is written below, for us it will be the representative of the EFE[71].

[71] $C = \frac{8\pi G}{c^2}$, c is the speed of light, G is the gravitational constant.

$$k_1 + k_2 + k_3 = C \cdot \rho \,. \tag{186}$$

The sum of the three spatial components of the curvature is proportional to the mass density. This means, for example, that outside our BH where we have a vacuum ($\rho = 0$) the sum of the curvatures adds to zero, a situation which is perfectly compatible with the result we got with the *sector models* of the BH. Even if the blocks were built with paper it was clear that k_1 was positive, while k_2 and k_3 were both negative. Furthermore, since the curvature components are proportional to the *deficit angle* δ shown in figures 128, 129, 130, we have a bigger k_1 value (bigger δ, if compared with the other two) which is cancelled by two smaller negative contributions (k_2 and k_3). In conclusion, in the space around a BH:

1. Wee are in a region in which there is no matter, $\rho = 0$.
2. The three spatial curvature components k_i are not null ($k_1 > 0$, $k_2 < 0$, $k_3 < 0$).
3. The sum of the three spatial curvature components is zero ($\sum k_i = 0$).

Someone might ask what is the difference between the situation we have around a BH and a scenario in which we have no gravitational source at all and we are in a complete vacuum. Indeed, also in a complete vacuum ρ would be zero and, since the EFE still hold, we would have $\sum k_i = 0$.

The answer to this question is that in the latter case the three components of the curvature would be zero ($k_i = 0$, $i = 1, 2, 3$), there would be no curvature at all. Instead outside a source the components would only add to zero, being individually not null!

In Newtonian Gravity the situation is similar, we have the Poisson equation [72] that connects the spatial derivatives of the gravitational field $\vec{C}$ to the mass density ρ:

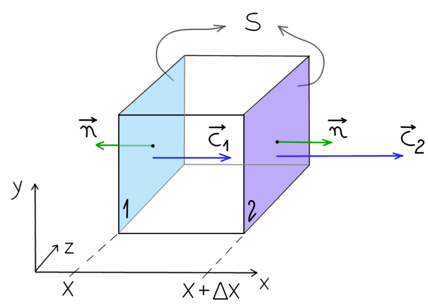
[72] For the derivation of this equation see Math Box - 1.

$$\left(\frac{dC_x}{dx} + \frac{dC_y}{dy} + \frac{dC_z}{dz}\right) \propto \rho \tag{187}$$

Therefore even in the classical theory outside a gravitational source ρ is zero, the three derivatives add to zero, while the field components and their derivatives are individually not null.

Since in certain conditions the equations of General Relativity have to reduce to the classical equations of gravity, we have that our Einstein Equation (186) will reduce to the Poisson equation. Therefore we have a strong clue suggesting that the curvature terms k_i are strictly related to the derivatives of the gravitational field! This definitely explains why outside a source the curvature components are present even if they add up to zero.

Math Box - 1. The Poisson Equation



Consider a gravitational field $\vec{C}$ whose components along the x direction are shown in blue in the figure: C_1 is the component on the surface 1, C_2 on the surface 2. The vector $\vec{n}$ is the unit vector perpendicular to the surfaces. We want to evaluate the total flux along the two surfaces S.

$\Phi_{tot} = \Phi_1 + \Phi_2 = -C_1 \cdot S + C_2 \cdot S$

C_1 and C_2 in general are slightly different. If Δx is small enough, we can write:

$C_2 = C_1 + \frac{dC_x}{dx} \cdot \Delta x$,

so the total flux becomes:

$\Phi_{tot} = -C_1 \cdot S + C_1 \cdot S + \frac{dC_x}{dx} \cdot \Delta x \cdot S = \frac{dC_x}{dx} \cdot \Delta V$.

If we extend our computation to the complete surface of the cube we have:

$\Phi_{tot} = (\frac{dC_x}{dx} + \frac{dC_y}{dy} + \frac{dC_z}{dz})\Delta V$.

Now we apply the Gauss Theorem:

$\Phi_{tot} \propto \Delta m$

$(\frac{dC_x}{dx} + \frac{dC_y}{dy} + \frac{dC_z}{dz})\Delta V \propto \Delta m \quad \Rightarrow \quad (\frac{dC_x}{dx} + \frac{dC_y}{dy} + \frac{dC_z}{dz}) \propto \rho$.

The spatial derivatives of the gravitational field $\vec{C}$ are connected to the mass density ρ.

Appendix

Energy spectrum of a vibrating string

To prove that the energy spectrum of a vibrating string is quantized we have to integrate over all the string length the expression representing the energy of a single part:

$$\Delta e_n = \frac{c^2 \pi^2 \Delta m B^2}{2L^2} \cdot n^2 \,. \tag{188}$$

In order to perform this calculation we have to substitute Δm with the product $\rho \cdot dx$, where ρ is the linear mass density of the string. We also have to remember that x, the variable of integration, is hidden in the constant B, which is:

$$B = A \, sin(kx) \,. \tag{189}$$

Since the wavelength is quantized $\lambda_n = 2L/n$, the wave number is quantized too $k_n = \frac{\pi}{L} n$. For the sake of simplicity we will neglect some of the constant factors in (188), therefore moving from discrete to continuous (number of the parts $\to \infty$) the integral we have to solve is:

$$E_n = \int_0^L de_n \quad \propto \quad n^2 \rho \int_0^L sin^2(n \frac{\pi}{L} x) \cdot dx \,. \tag{190}$$

We can carry out the variable substitution

$$\frac{n\pi}{L} x = z \quad \to \quad dx = \frac{L}{n\pi} dz \,, \tag{191}$$

so that the integral becomes:

$$\frac{n^2 \rho L}{n\pi} \int_0^{n\pi} sin^2(z) \cdot dz = \frac{n^2 \rho L}{n\pi} \Big|_0^{n\pi} (z - sin\, z \cdot cos\, z) = \tag{192}$$

$$= \frac{n^2 \rho L}{n\pi} \cdot n\pi = n^2 \rho L \,.$$

The Energy of the vibrating string is therefore quantized and the dependence from n^2 is the same we have for the spectrum of the quantum system in a infinite square well:

$$E_n = \frac{A^2 c^2 \pi^2}{2L^2} m \cdot n^2 \,. \tag{193}$$

The Delta Function

A delta function $\delta(k' - k)$ is a function that is zero everywhere except in one point $k = k'$, where it has an infinite value. We can define it with this integral

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(k'-k)x} dx = \delta(k' - k) \quad . \tag{194}$$

Here we will show why the expression (**??**) is a delta. First of all we notice that the exponential inside the integral is a unit vector in the complex plane, its phase being $\alpha = (k' - k)x$ (Fig. 142).

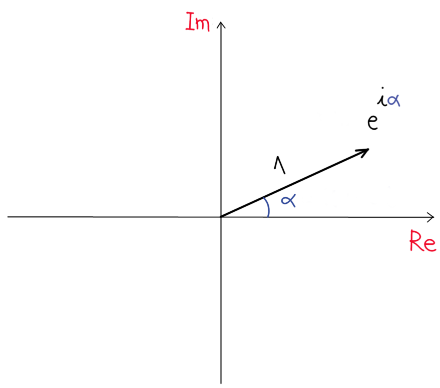


Figure 142: A vector in the complex plane and its exponential notation.

If $k' \neq k$ the the phase will vary uniformly with x and the integral will be sum of unit vectors each one with the phase just a bit bigger than the previous one. So, displacing vectors head to tail, is easy to understand that the sum will be zero (Fig. 143). Else, if $k' = k$ the phase will be zero, each vector will be laying along the x axis and the sum will diverge to ∞.

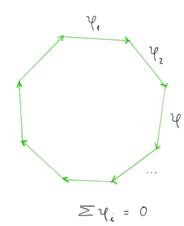


Figure 143: Head to tail sum of unit vectors with uniformly varying phase.

Delta function area

We want to show that the area under the delta function is 1. The delta can be thought of as the derivative of a step function: $\delta(k' - k) = D_{k'}[\theta(k')]$.

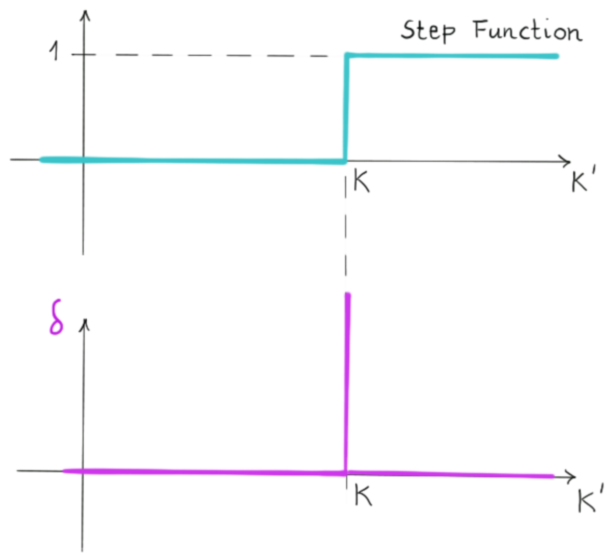
Hence the area of the delta becomes:

$$\text{AREA} = \int_{-\infty}^{+\infty} \delta(k' - k) dk' = \int_{-\infty}^{+\infty} D_{k'}[\theta(k')] dk' = \theta(k')\Big|_{-\infty}^{+\infty} =$$

$$= \theta(+\infty) - \theta(-\infty) = 1\,. \tag{195}$$

That's why the delta is also called a distribution.

Figure 144: Delta function as derivative of a step function.



Three ways to introduce the gaussian integral

We suggest three way to introduce the gaussian integral,

$$\int_{-\infty}^{+\infty} e^{-x^2} dx\,, \tag{196}$$

to high-school students. The first goes through a Monte Carlo simulation, easy to do with a spreadsheet. The basic idea is to fill with random points a rectangle, well visible in the image (fig.145), and to count the spots inside the gaussian area and those inside the rectangle: the ratio of these two numbers is proportional to the ratio of the areas. The simulation, done with 570 points and repeated 5 times, gives an average estimate of the integral of 1.77, very close to $\sqrt{\pi}$, the exact value. Another possibility is based on the evaluation of the area of the triangle OAB shown in figure 146. One edge is the tangent line in the inflection point of the gaussian function. This method leads to a value of 1.72. The last approach follows the Poisson idea. It is a very nice way to get the exact value of the integral.

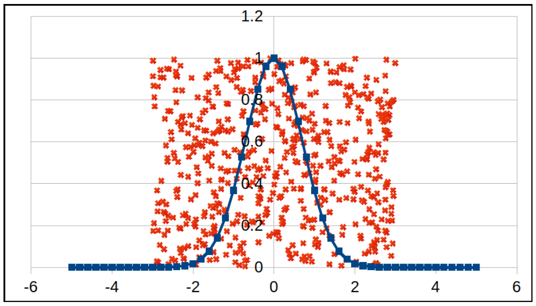


Figure 145: Montecarlo simulation to estimate the guassian integral (rectangle dimensions: 6x1).

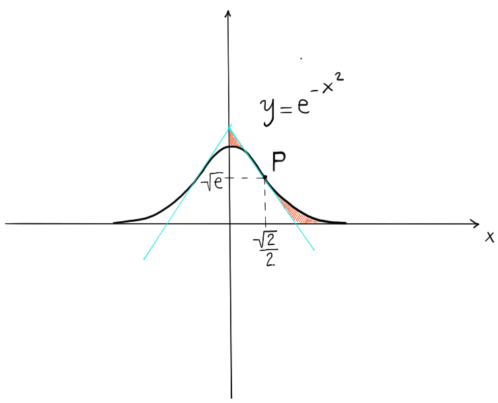


Figure 146: The triangle tangent to the gaussian function in its inflection points.

The Fourier Transform in Quantum Mechanics

We have seen that if a plane wave with momentum k is a solution of the free particle equation, then any linear combination of plane waves with different values of k will also be a solution. A fact that may be expressed with this statement:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) e^{ikx} dk \,. \tag{197}$$

The factor $A(k)$ represents the amplitude of each plane wave involved in the sum. Higher values of $A(k)$ mean higher contributions of that plane wave to the wave functions[73]

The situation is similar to the one we have with vectors:

$$\vec{a} = a_1 \vec{d}_1 + a_2 \vec{d}_2 + a_3 \vec{d}_3 + ... \tag{198}$$

A vector can be expressed as sum of components along different directions. The $A(k)$ is akin to the value of the component a_i, the plane wave e^{ikx} is the "direction" $\vec{d}_i$ of the specific component.

[73] The term $\frac{1}{\sqrt{2\pi}}$ is part of the plane wave definition $\varphi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$. It comes from the normalising conditions.

We have no problem in understanding that each component a_i is the projection of the vector along the i^{th} direction, i.e. the inner product between the vector $\vec{a}$ and W$\vec{d}_i$W

$$a_i = \vec{a} \cdot \vec{d}_i. \tag{199}$$

We can act in a similar way for the "components" $A(k)$ of the wave function. We may say that $A(k)$ is the projection of Ψ along the plane wave with momentum k: the inner product. In this case the inner product is expressed in this way:

$$(\varphi_k, \Psi) = \text{projecion of } \Psi \text{ along } \varphi_k \tag{200}$$

Where the exact mathematical definition of this symbols is:

$$(\varphi_k, \Psi) = \int_{-\infty}^{+\infty} \varphi_k^* \cdot \Psi \quad dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi \cdot e^{-ikx} \quad dx. \tag{201}$$

It looks difficult, but we will be soon convinced that there is a reason for this. We already know that (φ_k, Ψ) should be $A(k)$, therefore we may evaluate it and see if it leads to the correct result:

$$\begin{aligned} (\varphi_k, \Psi) &= \text{projection of } \Psi \text{ on } \varphi_k \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} \left(\int_{-\infty}^{+\infty} A(k') e^{ik'x} dk' \right) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(k') e^{i(k'-k)x} dx dk' . \end{aligned} \tag{202}$$

The integral over x in the expression (202) is a delta function (see page 218) thus the expression becomes

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k'-k)x} dx = \delta(k' - k) \tag{203}$$

and we have

$$(\varphi_k, \Psi) = \int_{-\infty}^{+\infty} A(k') \delta(k' - k) dk' = A(k). \tag{204}$$

A result we may explain taking in mind the shape of the delta function and the value of the area beneath. In conclusion, equation (204) shows that the proper way to define the inner product in quantum mechanics is given by relation (201).

We have thus derived two important equations:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) \cdot e^{ikx} dk \quad . \tag{205}$$

and

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x) \cdot e^{-ikx} dx \quad . \tag{206}$$

Meaning that:

We can associate to a quantum system two wave functions, $\Psi(x)$ that defines the system in the spatial domain and $A(k)$, that defines the system in the momenta domain.

Therefore we have two equivalent ways to describe our system. We may choose to work in the spatial domain or in the momentum domain, furthermore we have a well defined process to move from one domain to the other and vice versa: equations (205) and (206). They are known as the **Fourier Transform** equations. As we have seen, the Heisenberg Uncertainty principle is strictly correlated to this situation: the spatial wave functions $\Psi(x)$ and the momentum wave functions $A(k)$ are connected by a Fourier Transform.

Bound states and the Uncertainty Principle

To move a little more into a concrete physical situation, in this section we will consider a system free to move along a straight line and whose spatial wave function is a Gaussian function (one dimensional problem). As we will discuss later, something similar happens when a particle is in a bound state.

$$\Psi(x) = Ne^{-\frac{x^2}{a^2}}. \tag{207}$$

We will perform some initial calculations splitting them in subsections.

Normalisation of Ψ.

First of all the area under the the density function has to be 1, this requirement will define N:

$$\begin{aligned} (\Psi, \Psi) = 1 = |N|^2 \int_{-\infty}^{+\infty} e^{-\frac{2x^2}{a^2}} dx \\ = |N|^2 \frac{a}{\sqrt{2}} \int_{-\infty}^{+\infty} e^{-z^2} dz = |N|^2 \frac{a}{\sqrt{2}} \sqrt{\pi} \quad . \end{aligned} \tag{208}$$

To solve the integral we made the following substitution and used the formula in the footnote[74]

$$\begin{aligned} \frac{\sqrt{2}x}{a} &= z \\ dx &= \frac{a}{\sqrt{2}}dz \quad . \end{aligned} \tag{209}$$

In order for Ψ to be a "good" wave function we have to impose $N = \sqrt[4]{\frac{2}{a^2\pi}}$.

[74] $\int_{-\infty}^{+\infty} e^{-z^2} dz = \sqrt{\pi}$. In a previous section of this Appendix you may find three way to introduce this integral to high-school students.

Figure 147: Gaussian probability density function in the positions space.

The Fourier Transform of Ψ: $A(k)$.

We have learned that we can shift from the x to k domain with a Fourier transform. In order to find how the momenta of the system are distributed, $A(K)$ we are going to evaluate the Fourier transform of the Ψ .

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x) \cdot e^{-ikx} dx \\ &= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(\frac{x^2}{a^2} + ikx)} dx \end{aligned} \tag{210}$$

To solve this integral we have to transform the exponent in a square of a binomial:

$$A(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(\frac{x}{a} + \frac{ika}{2})^2} e^{-\frac{k^2a^2}{4}} dx \quad . \tag{211}$$

Making the substitution

$$\begin{aligned} \frac{x}{a} + \frac{ika}{2} &= z \\ dx &= adz \end{aligned} \tag{212}$$

The integral becomes

$$\begin{aligned} A(k) &= \frac{Na}{\sqrt{2\pi}} e^{-\frac{k^2a^2}{4}} \int_{-\infty}^{+\infty} e^{-z^2} dz \\ &= \frac{Na}{\sqrt{2\pi}} e^{-\frac{k^2a^2}{4}} \sqrt{\pi} \quad , \end{aligned} \tag{213}$$

considering the value of the normalising factor N we have:

$$A(k) = \sqrt[4]{\frac{a^2}{2\pi}} \cdot e^{-\frac{k^2a^2}{4}} \quad . \tag{214}$$

$$\Psi(x) = \sqrt[4]{\frac{2}{a^2\pi}} \cdot e^{-\frac{x^2}{a^2}}. \tag{215}$$

If we look at the same time at the $\Psi(x)$ and $A(k)$ we will reach an important result:

> *the wave function $A(k)$ associated to a gaussian $\Psi(x)$ is still a gaussian function with standard deviation inversely proportional to the one belonging to $\Psi(x)$.*

$$\sigma_x \propto a \quad \sigma_k \propto \frac{1}{a} \tag{216}$$

Clearly, this is also valid for the square modulus. Avoiding further calculations, we can say that the product of the standard deviations in the two domains is constant:

$$\sigma_x \cdot \sigma_k = \text{constant} \quad . \tag{217}$$

Again we find that any confined quantum system undergoes the UP: the more is dispersed in space the less in momentum.

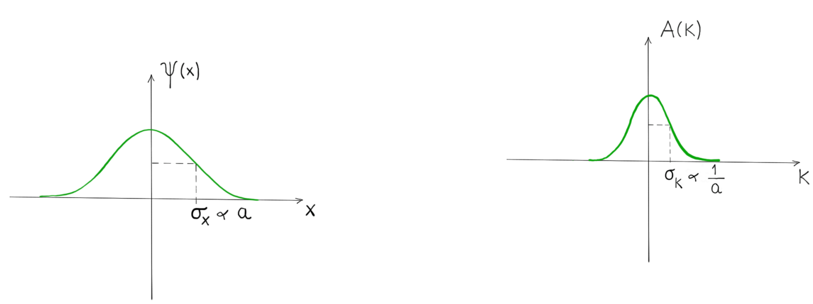


Figure 148: Wave functions in the two domains for a "Gaussian" bounded system.

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