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ON SOME REFRACTION BILLIARDS

IRENE DE BLASI AND SUSANNA TERRACINI

ABSTRACT. The aim of this work is to continue the analysis, started in [10], of the dynamics of a point-mass particle P moving in a galaxy with an harmonic biaxial core, in whose center sits a Keplerian attractive center (e.g. a Black Hole). Accordingly, the plane \mathbb{R}^2 is divided into two complementary domains, depending on whether the gravitational effects of the galaxy's mass distribution or of the Black Hole prevail. Thus, solutions alternate arcs of Keplerian hyperbolæ with harmonic ellipses; at the interface, the trajectory is refracted according to Snell's law. The model was introduced in [11], in view of applications to astrodynamics.

In this paper we address the general issue of periodic and quasi-periodic orbits and associated caustics when the domain is a perturbation of the circle, taking advantage of KAM and Aubry-Mather theories.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we deal with the dynamics of a point-mass particle P moving in a galaxy with an harmonic biaxial core, in whose center sits a Keplerian attractive center (e.g. a Black Hole). In our setting, the plane \mathbb{R}^2 is divided into two complementary domains, depending on whether the gravitational effects of the galaxy's mass distribution or of the Black Hole prevail. We set the domain of influence of the the Black Hole's to be a generic regular domain $\mathbf{0} \in D \subset \mathbb{R}^2$, so that the particle moves on the plane governed by the potential

$$V(z) = \begin{cases} V_I(z) = \mathcal{E} + h + \frac{\mu}{|z|} & \text{if } z \in D \\ V_E(z) = \mathcal{E} - \frac{\omega^2}{2}|z|^2 & \text{if } z \notin D, \end{cases} \quad (1.1)$$

whith $\mathcal{E} + h, \mu, \omega > 0$, while the behaviour of the solution, when crossing the boundary ∂D , is ruled by a generalization of Snell's law (see §2.2) of refraction. Thus, as sketched in Figure 1, trajectories alternate arcs of Keplerian hyperbolæ with harmonic ellipses, with a refraction at the boundary ∂D . It is worthwhile noticing that, when also the domain D is radially symmetrical the system is integrable, resulting in a shift of the polar angle only depending on the shooting one. This paper deals with anisotropic perturbations of the circular domain, and possibly of the potentials.

The model was introduced in [11], for specific applications to astrodynamical problems; besides the motivations coming from Celestial Mechanics, as mentioned in [10], our class of systems are of interests in view of possible applications in engineering artificial optical devices [24]. Another model which presents strong analogies with the one described in this work is studied in [8], where the complex dynamics deriving by patching orbits coming from pure Kepler's and

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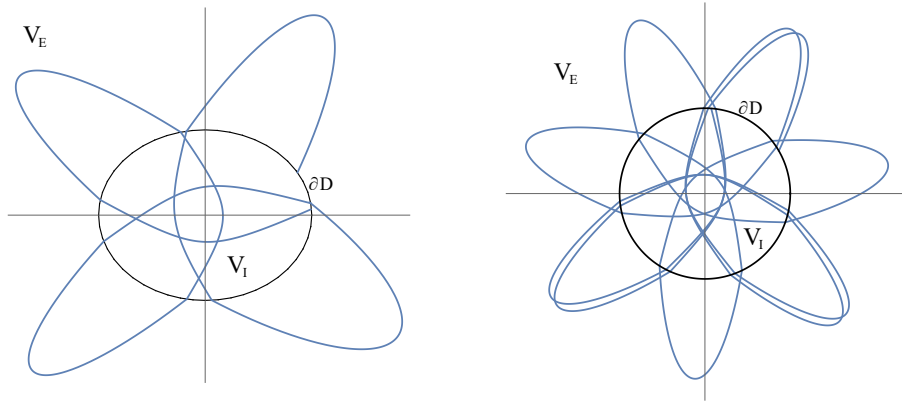


FIGURE 1. Examples of trajectories for $\mathcal{E} = 2.5$, $\omega = 1$, $h = 2$ and $\mu = 2$. Left: general trajectory for an elliptic domain with eccentricity $e = 0.6$. Right: quasi-periodic trajectory for a circular domain.

Stark's problem (the so-called *Sun-shadow dynamics*) is considered.

In [10], the authors addressed the problem of the stability of homotetic and brake solutions, depending on the values of the physical parameters $\mathcal{E}, h, \mu, \omega$, the curvature of the boundary and its distance from the Keplerian center. The analysis carried there shows an interesting and varied landscape of bifurcations of homothetic and brake periodic orbits.

The aim of this work is to continue the analysis started in [10], addressing the general issue of periodic and quasi-periodic orbits and associated caustics when the domain is a perturbation of the circle, taking advantage of both KAM and Aubry-Mather theories.

In order to tackle the problem, we shall consider the first¹ return map at the interface ∂D , after two consecutive (outer and inner) excursions, working at the zero energy level for the potential V as above. Shooting from boundary points, after performing a Levi-Civita regularization (cfr. [27]) of Kepler potential, our system looks like a billiard, with reflection at the boundary replaced by an outer excursion in between two refractions. We shall exploit the Lagrangian structure of the problem, building (locally) a generating function as the sum of an inner and outer contribution. Here comes a first problem, as these can not be globally defined. In addition, major difficulties arise from the return map not being globally defined, from the singularity of the attraction center and from a lack of twist condition (it can be in shown that, in certain regimes of the parameters, even the completely intergrable circular case admits at least a twist change, see Remark 4.7). Here are our main results.

Theorem 1.1 (Circular domains). *When D is the unit circle, there are action-angle coordinates $(\xi, I) \in \mathbb{R}/_{2\pi\mathbb{Z}} \times (-I_c, I_c)$, where ξ is the polar angle, such that we can express the first return map as a shift*

$$\begin{aligned} \mathcal{F} : \mathbb{R}/_{2\pi\mathbb{Z}} \times (-I_c, I_c) &\rightarrow \mathbb{R}/_{2\pi\mathbb{Z}} \times (-I_c, I_c), \\ (\xi_0, I_0) &\mapsto (\xi_1, I_1) = (\xi_0 + \bar{\theta}(I_0), I_0), \end{aligned} \tag{1.2}$$

¹The phrase *first return* may be confusing, since it is in fact a second crossing of the boundary; however, such apparent ambiguity will be clarified in §3.

where $\bar{\theta}(I) = f(I) + g(I)$ and

$$f(I) = \begin{cases} \operatorname{arccot} \left(\frac{\varepsilon - 2I^2}{I\sqrt{4\varepsilon - 2(2I^2 + \omega^2)}} \right) & \text{if } I \in (0, I_c) \\ 0 & \text{if } I = 0 \\ \operatorname{arccot} \left(\frac{\varepsilon - 2I^2}{I\sqrt{4\varepsilon - 2(2I^2 + \omega^2)}} \right) - \pi & \text{if } I \in (-I_c, 0) \end{cases}$$

and

$$g(I) = \begin{cases} 2 \arccos \left(\frac{2I^2 - \mu}{\sqrt{4(\varepsilon + h)I^2 + \mu^2}} \right) - 2\pi & \text{if } I \in (0, I_c) \\ 0 & \text{if } I = 0 \\ -2 \arccos \left(\frac{2I^2 - \mu}{\sqrt{4(\varepsilon + h)I^2 + \mu^2}} \right) + 2\pi & \text{if } I \in (-I_c, 0) \end{cases}$$

are real analytic functions in $(-I_c, I_c)$. For every $I \in (-I_c, I_c)$, except for a finite number (at most ten) of points, there holds $\bar{\theta}'(I) \neq 0$.

The critical value I_c corresponds to the action associated with the *total reflection* of the trajectory at the boundary, i.e. when the outgoing refracted trajectory becomes tangent to the boundary (see Remark 2.3) and is given in (3.17). In the circular case, the *rotation number* of the orbit is $\rho_I = \bar{\theta}(I) = f(I) + g(I)$. Depending whether this value is rational with 2π or not, the corresponding orbits are periodic or quasi-periodic. In both cases they determine an invariant curve and a pair of caustics, that is, smooth closed curves such that every trajectory which starts tangent to remains tangent after every passage in and out the domain D . Let us point out that caustics come in pair, respectively in D and in its complement. The proof of this Theorem is performed in §4.

Next we consider a perturbation D_ϵ of the domain whose boundary $\partial D_\epsilon = \operatorname{supp}(\gamma_\epsilon)$ is given by a radial deformation of the circle of the form

$$\gamma_\epsilon : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2 \quad \gamma_\epsilon(\xi) = (1 + \epsilon f(\xi; \epsilon)) e^{i\xi}, \quad (1.3)$$

where $f(\xi; \epsilon)$ is a suitably smooth function of $\mathbb{R}/2\pi\mathbb{Z} \times [-C, C]$ and ξ is the polar angle. Then, the first return map on the perturbed boundary can be extended as $\mathcal{F}(\xi, I; \epsilon)$ (see definition 5.18), where (ξ, I) are canonical variables defined in suitable neighbourhoods of such invariant curves. The following theorem resumes the results stated in Theorems 5.12, here considering a single invariant curve, and 5.21.

Theorem 1.2 (Persistence of invariant curves (KAM)). *Let $f \in \mathcal{C}^k$, with $k > 5$. Let us suppose that $\bar{\theta}'(I_0) \neq 0$, and assume $\rho_0 = \bar{\theta}(I_0)$ has a Diophantine ratio with 2π (see Definition 5.8). Then there exists $\bar{\epsilon}_{\rho_0}$ such that for every $\epsilon \in \mathbb{R}$, $|\epsilon| < \bar{\epsilon}_{\rho_0}$ the map $\mathcal{F}(\xi, I; \epsilon)$ admits a closed invariant curve of class C^1 with rotation numbers ρ_0 . Each of these invariant curves generates a pair of regular caustics.*

Two invariant curves with Diophantine rotation numbers border an invariant region for the map $\mathcal{F}(\xi, I; \epsilon)$, subject to the application of Poincaré-Birkhoff theorem and Aubry-Mather theory (see [1, 28, 29]). As the map is area-preserving, we only need to verify the twist condition.

This is a nontrivial issue, as the function $\bar{\theta}(I)$ may indeed change its monotonicity. This fact poses some technical difficulties but also gives rise to a richer phenomenology. We have the following result.

Theorem 1.3 (Existence of Aubry-Mather invariant sets). *Let $\bar{\rho}_- < \bar{\rho}_+$ be Diophantine rotation numbers, such that there are no critical values of the function $\bar{\theta}$ in the range $[\bar{\rho}_-, \bar{\rho}_+]$. Then there exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in \mathbb{R}$ with $|\epsilon| < \bar{\epsilon}$ and for every $\rho \in [\bar{\rho}_-, \bar{\rho}_+]$ the map $\mathcal{F}(\xi, I; \epsilon)$ admits at least one orbit with rotation number ρ . When $\rho = 2\pi \frac{m}{n}$ then for ϵ sufficiently small there are at least 2 (m, n) -orbits, namely, such that, denoted with $\{(\xi_k, I_k)\}_{k \in \mathbb{N}} = \{\mathcal{F}(\xi_0, I_0)\}_{k \in \mathbb{N}}$ the orbit generated by the initial point (ξ_0, I_0) , one has*

$$\forall k \in \mathbb{N} \quad (\xi_{k+n}, I_{k+n}) = (\xi_k + 2\pi m, I_k) \equiv_{2\pi} (\xi_k, I_k).$$

This statement can be easily deduced from Theorem 5.20; as we shall see there, however, the actual number of solutions can be larger, depending on the number of monotonicity changes of the twist $\bar{\theta}(I)$.

To conclude this preamble, let us remark that, although the computations are not explicitly performed here, with means of the same analytical tools and techniques other variations of the considered model, more similar to the one described in [11], can be investigated.

As an example, let us consider $\epsilon \in \mathbb{R}$ and a non-isotropic perturbation of the outer potential given by

$$\tilde{V}_E(z) = \mathcal{E} - \frac{\omega^2}{2}x^2 - \frac{(\omega + \epsilon)^2}{2}y^2,$$

where $z = (x, y) \in \mathbb{R}^2$. Taking, if necessary, $h \in \mathbb{R}$ instead of $h > 0$, one has that the dynamics induced by the potential

$$\tilde{V}(z) = \begin{cases} V_I(z) & \text{if } z \in D \\ \tilde{V}_E(z) & \text{if } z \notin D \end{cases} \quad (1.4)$$

with the usual Snell's refraction rule on the boundary is conservative if ∂D is the set of points in \mathbb{R}^2 such that the potentials differ by the same constant $A \in \mathbb{R}$, that is,

$$\partial D = \{z \in \mathbb{R}^2 \mid V_I(z) = \tilde{V}_E(z) + A\};$$

note that, if $A = 0$, the potential is continuous and the refraction reduces to a conservation of both the position and velocity on ∂D , leading to a C^1 junction between inner and outer arcs. For $\epsilon \in \mathbb{R}$ small, this system can be considered again as a small perturbation of the circular case.

1.1. Analogies and differences with Birkhoff billiards. In some sense, the model investigated in this work falls into the category of billiards and it is interesting to look for analogies. There are, however, fundamental differences: first of all the rays are curved by the gravitational force inside D ; moreover, reflection at the boundary is replaced by an excursion in the outer region in between two refractions. Similarly to billiards, our model can be described by an area preserving map of the cylinder, but, as we shall show, the twist condition may be violated even in the simplest case of a circular domain. It should be noted that refraction imposes a new constraint, because the interface can only be crossed outwards when the inner arc is transverse enough to the boundary (cfr Remark 3.2).

There is a wide literature on Birkhoff billiards, with recent relevant advances (see the book [33] and papers [22, 23, 20, 4]), including some cases of composite billiard with reflections and

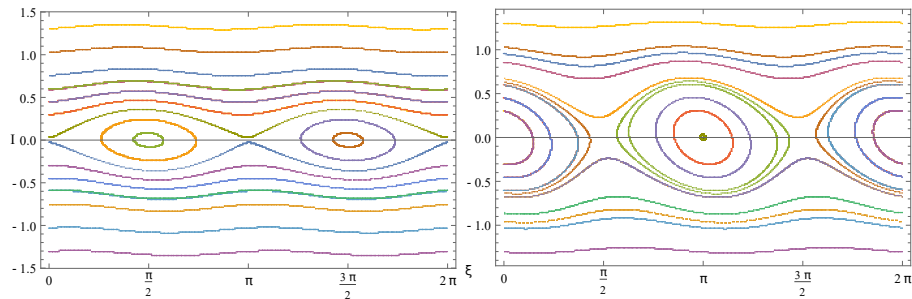


FIGURE 2. Orbits of the first return map F for $\mathcal{E} = 10$, $\omega = 1$, $h = 3$, $\mu = 44$ (left) and $\mu = 55$ (right). The shift in the stability of the homotetic equilibrium is evident, as well as the presence of invariant curves far from the ξ -axis in both cases.

refractions [3], also in the case of a periodic inhomogenous lattice [17]. Special mention should be paid to the work on magnetic billiards, where the trajectories of a charged particle in this setting are straight lines concatenated with circular arcs of a given Larmor radius [15, 16]. Let us add that, compared with the cases quoted above, additional difficulties arise because the corresponding return map is not globally well defined and from the singularity of the Kepler potential.

As a concrete example on how the considered model presents intrinsic analogies with classical billiards, as well as important differences, let us consider the case of an elliptic domain. In classical elliptic billiards of every eccentricity (see for example §4 in [33]), the straight orbit segment corresponding to the major axis determines always a saddle point for the associated billiard map, while the one coinciding with the minor axis is a center. A similar behaviour can be observed in our refractive model when the domain is an ellipse with small eccentricity, but with a fundamental difference: while this model admits the same equilibrium orbits of the classical billiard, their stabilities depend on the value of the physical parameter \mathcal{E} , h , ω , μ . This bifurcation phenomenon is derived in [10], and can be stated as follows:

Proposition 1.4 ([10], Corollary 1.3). *Let us consider an elliptic domain, with horizontal unitary major semiaxis and eccentricity e , and let \bar{z}_0 and $\bar{z}_{\pi/2}$ denote the horizontal and vertical homotetic periodic solutions of collision-reflection type. If $\frac{\sqrt{\mathcal{E}+h+\mu}}{\mu} < \frac{\sqrt{2\mathcal{E}-\omega^2}}{2\sqrt{2}\mathcal{E}}$, then, for small eccentricities, \bar{z}_0 is stable and $\bar{z}_{\pi/2}$ is unstable. Symmetrically, if $\frac{\sqrt{\mathcal{E}+h+\mu}}{\mu} > \frac{\sqrt{2\mathcal{E}-\omega^2}}{2\sqrt{2}\mathcal{E}}$, then, for small eccentricities, \bar{z}_0 is unstable and $\bar{z}_{\pi/2}$ is stable.*

Figure 2 shows the orbits of \mathcal{F} in the two cases of Proposition 1.4; one can notice that, far from the homotetic solutions, the system shows evidences of integrability.

As a final remark, let us highlight one more connection between classical Birkhoff billiards and our refraction model, related to the methods used to study the respective dynamics: in this work, the fundamental argument to prove Theorem 1.3 is the extension to small perturbations of the existence results obtained for a circular domain, which is completely integrable. Analogous reasonings are used for example in [2] and [23] to prove a local version of Birkhoff conjecture starting from small deformations respectively of a circular and an elliptic domain, both sharing the complete integrability property in the classic case.

2. PRELIMINARIES AND NOTATION

Let us consider a general fixed-ends differential problem of the type

$$\begin{cases} z''(s) = \nabla V(z(s)) & s \in [0, T] \\ \frac{1}{2}|z'(s)| - V(z(s)) = 0 & s \in [0, T] \\ z(0) = z_0, z(T) = z_1 \end{cases} \quad (2.1)$$

where $z_0, z_1 \in \mathbb{R}^2$ or some subset and $V(z)$ is regular and greater than zero almost everywhere. The solutions of problems of type (2.1) can be studied by taking advantage of the variational structure, which also incorporates the junction rule between the inner and outer geodesics.

This Section aims to provide the basic definitions and properties of the main analytical tools needed to study the dynamics described in §1, along with a derivation of the generalized Snell's law which connects the geodesic arcs. The latter, as well as a more detailed description of the results presented, can be found in Appendix A of [10].

2.1. Maupertuis functional and Jacobi length.

Definition 2.1. Given $z_0, z_1 \in \mathbb{R}^2$, we denote with $M([0, 1], z(t))$ the Maupertuis functional

$$M(z) = M([0, 1], z(t)) = \int_0^1 |\dot{z}(t)|^2 V(z(t)) dt$$

which is defined on the set

$$H_{z_0, z_1} = \{z(t) \in H([0, 1], \mathbb{R}^2) \mid z(0) = z_0, z(1) = z_1\}.$$

Furthermore, the Jacobi length is given by

$$L(z) = L([0, 1], z(t)) = \int_0^1 |\dot{z}(t)| \sqrt{V(z(t))} dt,$$

and is defined on the closure of $H_0 = \{z(t) \in H_{z_0, z_1} \mid |\dot{z}(t)| > 0, V(z(t)) > 0 \forall t \in [0, 1]\}$ in the weak topology of $H^1([0, 1], \mathbb{R}^2)$.

It can be shown (see for example [10, 31]) that the critical points at positive levels of $M(z)$ are reparametrizations of classical solutions of (2.1); moreover, the Hölder inequality, along with the energy conservations law, allow to prove that, if $z(t)$ is a critical point of M , then $L^2(z) = 2M(z)$: the search for solutions of (2.1) is hence equivalent to the study of the critical points of the Jacobi length, taking into account the different time scales of the problems.

Definition 2.2. We define:

- the **geodesic time** $t \in [0, 1]$ as the time parameter to be used in $M(z(t))$ and $L(z(t))$;
- the **cinetic time** $s \in [0, T]$ as the physical time parameter through which $z(s) = z(t(s))$ solves (2.1).

We denote with $\dot{} = d/dt$ and $\prime = d/ds$ respectively the derivatives with respect to the geodesic and the cinetic time.

The relation between the geodesic and the cinetic time is proved in [10] and is given by

$$\frac{d}{dt} = \frac{L}{\sqrt{2}V(z(t(s)))} \frac{d}{ds}, \quad (2.2)$$

where $L = |\dot{z}(t)|\sqrt{V(z(t))}$ is constant along the solutions of (2.1).

2.2. Generalized Snell's law. Besides the analysis of the inner and outer dynamics taken separately, the study of the dynamical system described in §1 is heavily influenced by the choice of a suitable junction law between two consecutive arcs through the interface ∂D . In our case, a local minimization argument leads to the determination of a connection rule which turns out to be a generalization for curved interfaces and non-Euclidean metrics of the classical Snell's refraction law.

In general, let us consider an inner potential V_I and an outer one V_E , along with their associated distances d_E and d_I , obtained by taking the metrics $(g_{E \setminus I})_{ij}(z) = V_{E \setminus I}(z)\delta_{ij}$. Fixed $z_I^0 \in D$ and $z_E^0 \notin D$ in a suitable neighborhood of ∂D , one can search for the point \bar{z} on the interface such that the sum of the distances $d_E(z_E^0, z) + d_I(z, z_I^0)$ is minimal. Denoted with $T(z)$ the unit tangent vector to ∂D in z , this extremality condition turns out to be equivalent to require

$$\sqrt{V_E(\bar{z})} \frac{\dot{z}_E(1)}{|\dot{z}_E(1)|} \cdot T(\bar{z}) = \sqrt{V_I(\bar{z})} \frac{\dot{z}_I(0)}{|\dot{z}_I(0)|} \cdot T(\bar{z}), \quad (2.3)$$

where $z_E(t)$ and $z_I(t)$ are respectively the geodesic arcs connecting \bar{z} with z_E^0 and z_I^0 parametrised by the geodesic time $t \in [0, 1]$. Equivalently, taking into account the relation between the geodesic and the cinetic time stated in (2.2), Equation (2.3) can be expressed as

$$\frac{1}{\sqrt{2}} z'_E(T_E) \cdot T(z) = \frac{1}{\sqrt{2}} z'_I(0) \cdot T(z),$$

where $T_E = s(1)$ is the cinetic ime value associated to $t = 1$. Geometrically, equation (2.3) represents the conservation of the tangent component of the velocity vector through the interface, and, if α_E and α_I are respectively the angles of $\dot{z}_E(1)$ and $\dot{z}_I(0)$ with the normal vector to ∂D in \bar{z} , it can be rephrased as

$$\sqrt{V_E(\bar{z})} \sin \alpha_E = \sqrt{V_I(\bar{z})} \sin \alpha_I; \quad (2.4)$$

clearly, the transition from the inside of the domain D can be treated in analogous way obtaining again (2.4).

Remark 2.3. *Since for every $z \in \mathbb{R}^2 \setminus \{0\}$ $V_E(z) < V_I(z)$, one has that the equation*

$$\alpha_I = \arcsin \left(\sqrt{\frac{V_E(z)}{V_I(z)}} \sin \alpha_E \right)$$

is always solvable, while

$$\alpha_E = \arcsin \left(\sqrt{\frac{V_I(z)}{V_E(z)}} \sin \alpha_I \right)$$

admits a solution if and only if $|\sqrt{V_I(z)/V_E(z)} \sin \alpha_I| \leq 1$, that is, iff

$$|\alpha_I| \leq \alpha_{I,crit} = \arcsin \left(\sqrt{\frac{V_E(z)}{V_I(z)}} \right). \quad (2.5)$$

Whenever condition (2.5) is satisfied, the passage from the inner to the outer arc is possible. Note that condition (2.5) depend globally on the physical parameters $\mathcal{E}, h, \mu, \omega$ and, locally, by $z \in \partial D$.

3. FIRST RETURN MAP

The dynamics described in Section 1 will be studied by considering the corresponding discrete dynamical system which keeps track of the crossings of the total orbit, obtained by the juxtaposition of outer and inner arcs, and the domain's boundary ∂D . Such a map is named *first return map* and is the analogous of the billiard map in the classical theory of Birkhoff billiards, see [32]: this Section is devoted to its construction, leaving to §4 and 5 the analysis of its good definition for particular domains.

Notation 3.1. From now on we will denote the differential problems connected to the outer and inner dynamics with the equivalent notations

$$\begin{cases} z''(s) = \nabla V_{E \setminus I}(z(s)) & s \in [0, T_{E \setminus I}] \\ \frac{1}{2}|z'(s)|^2 - V_{E \setminus I}(z(s)) = 0 & s \in [0, T_{E \setminus I}] \\ \text{boundary conditions} \end{cases} \Leftrightarrow \begin{cases} (HS_{E \setminus I})[z(s)] & s \in [0, T_{E \setminus I}] \\ \text{boundary conditions} \end{cases} \quad (3.1)$$

where $(HS_{E \setminus I})[z]$ denote the differential problem $z'' = \nabla V_{E \setminus I}(z(s))$, along with the corresponding energy conservation law. According to the scope of problem (3.1), the boundary conditions will prescribe either the initial position and velocity or the endpoints of $z(s)$.

Let us suppose that the boundary of the regular domain D defined in Section 1 can be parametrized by a regular closed curve $\gamma : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2$. Given some initial conditions $p_0^{(I)}, v_0^{(I)}, p_0^{(E)}$ and $v_0^{(E)}$, let us consider the solutions $z_I(s)$ and $z_E(s)$ of the two systems

$$\begin{cases} (HS_I)[z(s)] & s \in [0, T_I] \\ z_I(0) = p_0^{(I)}, z_I'(0) = v_0^{(I)} \end{cases} \quad \begin{cases} (HS_E)[z(s)] & s \in [0, T_E] \\ z_E(0) = p_0^{(E)}, z_E'(0) = v_0^{(E)} \end{cases} \quad (3.2)$$

for some $T_I, T_E > 0$.

Fixed $p_0 \in \partial D$, $v_0 \in \mathbb{R}^2$ such that it points towards the exterior of D , we want to describe (supposing that it exists) the orbit obtained by the juxtaposition of an exterior orbit z_E and the subsequent inner orbit z_I , namely, the function $z_{EI}(s) : [0, T_E + T_I] \rightarrow \mathbb{R}^2$ defined by

$$z_{EI}(s) = \begin{cases} z_E(s) & s \in [0, T_E] \\ z_I(s) & s \in [T_E, T_E + T_I] \\ z_E^{(1)}(s) & s = T_E + T_I, \end{cases} \quad (3.3)$$

where, defined

$$\begin{aligned} \tilde{p}_1 &= z_E(T_E), & \tilde{p}_2 &= z_I(T_E + T_I), \\ u_1 &= \frac{z'_E(T_E)}{|z'_E(T_E)|}, & u'_1 &= \frac{z'_I(T_E)}{|z'_I(T_E)|}, & u_2 &= \frac{z'_I(T_E + T_I)}{|z'_I(T_E + T_I)|}, & u'_2 &= \frac{z'^{(1)}_E(T_E + T_I)}{|z'^{(1)}_E(T_E + T_I)|}, \end{aligned}$$

the arcs $z_E, z_I, z_E^{(1)}$ are solutions of either the outer or inner problem and the the initial conditions of each branch satisfy the Snell's law. More precisely,

$$\begin{cases} (HS_E)[z_E(s)] & s \in [0, T_E] \\ z_E(s) \notin D, \tilde{p}_1 \in \partial D & s \in (0, T_E) \\ z_E(0) = p_0, z'_E(0) = v_0 \end{cases} \quad (3.4)$$

$$\begin{cases} (HS_I)[z_I(s)] & s \in [T_E, T_E + T_I] \\ z_I(s) \in D, \tilde{p}_2 \in \partial D & s \in (T_E, T_E + T_I) \\ z_I(T_E) = \tilde{p}_1, \sqrt{V_E(\tilde{p}_1)}u_1 \cdot e_1 = \sqrt{V_I(\tilde{p}_1)}u'_1 \cdot e_1 \end{cases} \quad (3.5)$$

$$\begin{cases} (HS_E)[z_E^{(1)}(s)] & s \in [T_E + T_I, T_E + T_I + \tilde{T}] \\ z_E^{(1)}(s) \notin D & s \in (T_E + T_I, T_E + T_I + \tilde{T}) \\ z_E^{(1)}(T_E + T_I) = \tilde{p}_2, \sqrt{V_E(\tilde{p}_2)}u'_2 \cdot e_2 = \sqrt{V_I(\tilde{p}_2)}u_2 \cdot e_2, \end{cases} \quad (3.6)$$

for some $T_E, T_I, \tilde{T} > 0$ and where e_1 and e_2 are the unit vectors tangent to ∂D respectively in \tilde{p}_1 and \tilde{p}_2 .

Once defined the concatenation of arcs $z_{EI}(s)$, the first return map is given by the iteration map which expresses $(p_1, v_1) = (z_{EI}(T_E + T_I), z'_{EI}(T_E + T_I))$ as a function of (p_0, v_0) in a suitable set of coordinates.

Recalling that we set $\partial D = \text{supp}(\gamma)$, with γ regular closed curve in $\mathbb{R}/2\pi\mathbb{Z}$, let us suppose that $p_0 = z_{EI}(0) = \gamma(\xi_0) \in \partial D$ is the starting point of the first outer branch of $z_{EI}(s)$: then, denoting with $t(\xi_0)$ and $n(\xi_0)$ respectively the tangent and the outward-pointing normal unit vectors to γ in p_0 , the initial velocity v_0 can be expressed as $v_0 = \sqrt{2V_E(\gamma(\xi_0))}(\cos \alpha_0 n(\xi_0) + \sin \alpha_0 t(\xi_0))$, where $\alpha_0 \in [-\pi/2, \pi/2]$ is the angle between v_0 and $n(\xi_0)$, positive if $v \cdot t(\xi_0) \geq 0$ and negative otherwise. Then, once ξ_0 is fixed, the vector v_0 is completely determined by α_0 . We can then consider the map

$$F : \mathbb{R}/2\pi\mathbb{Z} \times [-\pi/2, \pi/2] \rightarrow \mathbb{R}/2\pi\mathbb{Z} \times [-\pi/2, \pi/2], \quad (\xi_0, \alpha_0) \mapsto (\xi_1, \alpha_1) = (\xi_1(\xi_0, \alpha_0), \alpha_1(\xi_0, \alpha_0)), \quad (3.7)$$

where the pair (ξ_1, α_1) completely determines $(z_{EI}(T_E + T_I), z'_{EI}(T_E + T_I))$.

Remark 3.2. *In general, one can not guarantee that F is well defined, as the existence of ξ_1 and α_1 as described depends strongly on the interaction between the outer and inner arcs with the domain's boundary. The conditions for F to be well defined are essentially two:*

- (i) *the existence and uniqueness of the outer and inner arcs for fixed endpoints on ∂D ;*
- (ii) *the good definition of the refraction from the inside to the outside of D . As a matter of fact, from Remark 2.3 one has that, while the refraction exterior-interior is always possible, an inner arc can be refracted in an outer one if and only if, denoted with β_1*

the angle between $z'_I(T_E + T_I)$ and the inward-pointing normal unit vector of γ in $\gamma(\xi_1)$, $|\beta_1| < \beta_{crit} = \arcsin\left(\sqrt{V_E(\gamma(\xi_1))/V_I(\gamma(\xi_1))}\right)$.

In other words, condition (ii) is verified if the inner Keplerian arcs z_I are transverse enough to the domain's boundary ∂D . This obstruction to the good definition of F may be circumvented by considering a suitable prolonging according to which, whenever $|\beta_1| > \beta_{crit}$, the test particle returns back in the interior of the domain with an angle $\beta = \beta_1$. This extension, corresponding to the so-called total reflection, is somehow suggested by physical intuition and by analogy with the classical Birkhoff billiards, as well as the traditional Snell's law for light rays. However, as this approach leads to technical difficulties due to the passage to the tangent case for $|\beta_1| = \beta_{crit}$ and the consequent loss in regularity, in this work such extension is not considered, and, by definition, the map F will be well defined only when $|\beta_1| < \beta_{crit}$ for every inner arc.

As we will see in Sections 4.1 and 5.1, this is true if D is a disk or a perturbation of the latter, provided $\alpha_0 \neq \pm\pi/2$.

3.1. Variational approach. Choosing the right set of canonical variables, the first return map can be expressed in a variational form, which allows to take advantage of a wide range of powerful theoretical instruments, coming from KAM and Aubry-Mather theories, which allow to tackle the case of small perturbations of a circular domain D_0 (the latter is discussed in §4, while the perturbational approach is the subject of §5). Following e.g. [18], we can consider the *generating function*

$$S : \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}, \quad S(\xi_0, \xi_1) = S_E(\xi_0, \tilde{\xi}) + S_I(\tilde{\xi}, \xi_1) = d_E(\gamma(\xi_0), \gamma(\tilde{\xi})) + d_I(\gamma(\tilde{\xi}), \gamma(\xi_1)), \quad (3.8)$$

where d_E and d_I are defined as in §2 and, according to Snell's law, the intermediate point $\tilde{p} = \gamma(\tilde{\xi})$ is such that $\tilde{\xi}$ is a critical point for the function $f(\xi_0, \xi, \xi_1) = d_E(\gamma(\xi_0), \gamma(\xi)) + d_I(\gamma(\xi), \gamma(\xi_1))$ with ξ_0 and ξ_1 fixed. In other words, $\tilde{\xi}$ is a solution of

$$\partial_b S_E(\xi_0, \tilde{\xi}) + \partial_a S_I(\tilde{\xi}, \xi_1) = 0, \quad (3.9)$$

where ∂_a and ∂_b denote respectively the partial derivatives with respect to the first and second variable. In general, one can not guarantee the global good definition of the generating function $S(\xi_0, \xi_1)$, which depends strongly on the specific geometry of the domain and on the values of the physical parameters $\mathcal{E}, h, \mu, \omega$. On the other hand, the local definition of $S(\xi_0, \xi_1)$ needs a nondegeneration condition: under the assumption of existence and uniqueness of the inner and outer arcs, which ensures the differentiability of $d_E(p_0, p_1)$ and $d_I(p_0, p_1)$ separately, let us consider $(\xi_0, \tilde{\xi}, \xi_1) \in (\mathbb{R}/2\pi\mathbb{Z})^3$ such that (3.9) is satisfied. If

$$\partial_{\tilde{\xi}}(\partial_b S_E(\xi_0, \tilde{\xi}) + \partial_a S_I(\tilde{\xi}, \xi_1)) = \partial_b^2 S_E(\xi_0, \tilde{\xi}) + \partial_a^2 S_I(\tilde{\xi}, \xi_1) \neq 0, \quad (3.10)$$

then locally around ξ_0, ξ_1 one can express $\tilde{\xi} = \tilde{\xi}(\xi_0, \xi_1)$ as a function of the endpoints. Moreover, one has

$$\begin{aligned} \partial_{\xi_0} \tilde{\xi}(\xi_0, \xi_1) &= -\frac{\partial_{ab} S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1))}{\partial_b^2 S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1)) + \partial_a^2 S_I(\tilde{\xi}(\xi_0, \xi_1), \xi_1)}, \\ \partial_{\xi_1} \tilde{\xi}(\xi_0, \xi_1) &= -\frac{\partial_{ab} S_I(\tilde{\xi}(\xi_0, \xi_1), \xi_1)}{\partial_b^2 S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1)) + \partial_a^2 S_I(\tilde{\xi}(\xi_0, \xi_1), \xi_1)}. \end{aligned} \quad (3.11)$$

If (3.10) holds, the generating function is well defined locally around ξ_0 and ξ_1 , an one can define the *canonical actions* associated to the system and to the above coordinates by the relations

$$I_0 = -\partial_{\xi_0} S(\xi_0, \xi_1), \quad I_1 = \partial_{\xi_1} S(\xi_0, \xi_1); \quad (3.12)$$

note that, when $S(\xi_0, \xi_1)$ is well defined, the same is true also for the actions as functions of the angles ξ_0 and ξ_1 .

In order to define a first return map in the new canonical action-angle variables, one needs to express ξ_1 and I_1 as functions of ξ_0 and I_0 : this is possible if a second nondegeneracy condition holds. Let us consider $\xi_0, \xi_1, \tilde{\xi}(\xi_0, \xi_1)$ such that (3.10) holds, and define I_0 as in (3.12): if

$$\partial_{\xi_1} (I_0 + \partial_{\xi_0} S(\xi_0, \xi_1)) \neq 0, \quad (3.13)$$

one can find $\xi_1 = \xi_1(\xi_0, I_0)$ as a function of the initial action-angle variables. In particular, making use of (3.9) and (3.11), condition (3.13) translates in

$$\frac{\partial_{ab} S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1)) \partial_{ab} S_I(\tilde{\xi}(\xi_0, \xi_1), \xi_1)}{\partial_b^2 S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1)) + \partial_a^2 S_I(\tilde{\xi}(\xi_0, \xi_1), \xi_1)} \neq 0, \quad (3.14)$$

which is well defined in view of (3.10). If (3.14) holds, one can then find two neighborhoods $[\xi_0 - \lambda_{\xi_0}, \xi_0 + \lambda_{\xi_0}]$ and $[I_0 - \lambda_{I_0}, I_0 + \lambda_{I_0}]$ such that the new *local first return map*

$$\begin{aligned} \mathcal{F} : [\xi_0 - \lambda_{\xi_0}, \xi_0 + \lambda_{\xi_0}] \times [I_0 - \lambda_{I_0}, I_0 + \lambda_{I_0}] &\rightarrow \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}, \\ (\xi_0, I_0) &\mapsto (\xi_1(\xi_0, I_0), I_1(\xi_0, I_0)), \end{aligned} \quad (3.15)$$

where $I_1(\xi_0, I_0) = \partial_{\xi_1} S(\xi_0, \xi_1(\xi_0, I_0))$, is well defined.

The switch from a *Lagrangian* approach adopted by using the original first return map F and an *Hamiltonian* one involving the generating function and the canonical action-angle variables is crucial as it will allow, in Section 5, to take advantage of the results coming from KAM and Aubry-Mather theories to derive the existence of orbits with prescribed rotation numbers in the case of small perturbations of a circular domain (see §5).

In the circular case, which will be analyzed in details in §4, both the potentials and the domain are centrally symmetric: a consequence of the subsequent invariance under rotations is that the nondegeneracy conditions (3.10) and (3.14) will result in fact equivalent, and S , denoted in this case with S_0 , will be well defined almost everywhere.

As a final remark, one can observe that (2.4), along with (3.12) and (3.9), provides a general relation between the actions I_0, I_1 and the angles α_0, α_1 defined above: in particular,

$$\begin{aligned} I_0(\xi_0, \xi_1) &= -\partial_{\xi_0} (S_E(\xi_0, \tilde{\xi}) + S_I(\tilde{\xi}, \xi_1)) = -\partial_a S_E(\xi_0, \tilde{\xi}) = \\ &= \frac{1}{\sqrt{2}} z'_E(0) \cdot \frac{\dot{\gamma}(\xi_0)}{|\dot{\gamma}(\xi_0)|} = \sqrt{V_E(\gamma(\xi_0))} \sin \alpha_0 \\ I_1(\xi_0, \xi_1) &= \partial_{\xi_1} (S_E(\xi_0, \tilde{\xi}) + S_I(\tilde{\xi}, \xi_1)) = \partial_b S_I(\tilde{\xi}, \xi_1) = \\ &= \frac{1}{\sqrt{2}} z'_I(T_E + T_I) \cdot \frac{\dot{\gamma}(\xi_1)}{|\dot{\gamma}(\xi_1)|} = \sqrt{V_I(\gamma(\xi_0))} \sin \alpha'_1 = \sqrt{V_E(\gamma(\xi_1))} \sin \alpha_1. \end{aligned} \quad (3.16)$$

Eqs.(3.16) provides a natural boundary for the values that the actions can assume, which, in a inhomogeneous case, depend on ξ_0 and ξ_1 : in particular

$$|I_0| \leq \sqrt{V_E(\gamma(\xi_0))} \quad \text{and} \quad |I_1| \leq \sqrt{V_E(\gamma(\xi_1))},$$

where the equalities correspond to the tangent case $\alpha_{0\setminus 1} = \pm\pi/2$.

In the case of a circular domain, the bound on the actions is uniform and given by

$$I_0, I_1 \in \left[-\sqrt{\mathcal{E} - \frac{\omega^2}{2}}, \sqrt{\mathcal{E} - \frac{\omega^2}{2}} \right] = [-I_c, I_c]. \quad (3.17)$$

For reasons which will be clear in §4, related to the good definition of the map \mathcal{F} , we will not consider the tangent case, restricting ourselves to the open interval $(-I_c, I_c)$.

4. THE UNPERURBED CASE: CIRCULAR DOMAIN

Let us suppose that D is a disk of radius 1, and denote it with D_0 : in this particular case, both the potentials and the domain are centrally symmetric, and, as a consequence, the system is integrable. In particular, it is possible to find the explicit expression of the first return map in action-angle variables, which in this case is denoted with \mathcal{F}_0 , as it will be done in Section 4.3.

4.1. Good definition of d_E and d_I . As already pointed out in Remark 3.2, the preliminary condition for the generating function $S(\xi_0, \xi_1)$ to be well defined is that the inner and outer distances $d_E(p_0, p_1)$ and $d_I(p_0, p_1)$ are differentiable with respect to their variables. This is true when the inner and outer arcs connecting two points on ∂D are (locally) unique, as Theorems 4.1 and 4.2 yield for the circular domain. The proofs of these theorems are given, along with some preliminary results leading to them, in Appendix A.

Theorem 4.1. *If $\mathcal{E} > \omega^2$, for every $p_0, p_1 \in \partial D_0$ such that $|p_0 - p_1| < 2$, there exists $T > 0$ and a unique $z(s; p_0, p_1) : [0, T] \rightarrow \mathbb{R}^2$ solution of the fixed ends problem*

$$\begin{cases} (HS_E)[z(s)] & s \in [0, T] \\ |z(s)| > 1 & s \in (0, T) \\ z(0) = p_0, \quad z(T) = p_1. \end{cases} \quad (4.1)$$

Moreover, $z(s; p_0, p_1)$ is of class C^1 with respect to variations of the endpoints.

The analogous of Theorem 4.1 for the inner dynamics states the existence and uniqueness of the Keplerian arc connecting two points with an additional topological constraint. To define it, it is necessary to recall and adapt the classical concept of winding number to open arcs which connects points on the circle; to do that, we use a definition which recalls the one given in [31]. In general, let $p_0, p_1 \in \partial D_0$ two non-antipodal points on the circle, that is, $|p_0 - p_1| < 2$, and a regular, simple curve $\alpha : [a, b] \rightarrow \mathbb{R}^2$ such that $\alpha(a) = p_0$ and $\alpha(b) = p_1$ and $\alpha(s) \neq \mathbf{0}$ for every $s \in [a, b]$. Let us then consider a closed curve Γ_α which is equal to $\alpha(s)$ for $s \in [a, b]$ and then follows the shortest arc of ∂D_0 which connects p_1 to p_0 . We can then define the winding number associated to the curve α with respect to $0 \in \mathbb{R}^2$ as

$$I(\alpha([a, b]), 0) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{dz}{z}. \quad (4.2)$$

Theorem 4.2. *For every $p_0, p_1 \in \partial D_0$, $|p_0 - p_1| < 2$, there is a unique $T > 0$ and a unique solution $z(s; p_0, p_1)$ of*

$$\begin{cases} (HS_I)[z(s)] & s \in [0, T] \\ |z(s)| < 1 & s \in (0, T) \\ z(0) = p_0, z(T) = p_1 \end{cases} \quad (4.3)$$

such that $z(s; p_0, p_1)$ is of class C^1 with respect to p_0 and p_1 and:

- if $p_0 = p_1$, $z(s; p_0, p_0)$ is an ejection-collision solution;
- if $p_0 \neq p_1$, $z(s; p_0, p_1)$ is a classical solution of (4.3) such that $|\text{Ind}(z([0, T]), 0)| = 1$. If $p_1 \rightarrow p_0$, $z(s; p_0, p_1)$ tends to the ejection-collision solution $z(s; p_0, p_0)$.

Moreover, there is $0 < C < 1$ such that, for every p_0, p_1 as above,

$$-p_0 \cdot \frac{z'_1(0; p_0, p_1)}{|z'_1(0; p_0, p_1)|} > C \quad \text{and} \quad p_1 \cdot \frac{z'_1(T; p_0, p_1)}{|z'_1(T; p_0, p_1)|} > C. \quad (4.4)$$

The quantity C depends on the physical parameters $\mathcal{E} + h, \mu$ and on $|p_0 - p_1|$. In particular, it tends to 1 when $\mathcal{E} \rightarrow \infty$ or $|p_0 - p_1| \rightarrow 0$

The estimates given by (4.4) are crucial to ensure that the inner arcs are transversal to ∂D_0 as much as needed: this is necessary for the first return map to be well defined, since, according to Remark 3.2, it is clear that these arcs can not be tangent to the interface.

4.2. Study of the map \mathcal{F}_0 . Once the good definition and differentiability of the distances $d_I(p_0, p_1)$ and $d_E(p_0, p_1)$ is ensured, one can eventually consider the generating function introduced in Section 3.1 and given by

$$S_0(\xi_0, \xi_1) = S_{E,0}(\xi_0, \tilde{\xi}) + S_{I,0}(\tilde{\xi}, \xi_1) = d_E(\gamma_0(\xi_0), \gamma_0(\tilde{\xi})) + d_I(\gamma_0(\tilde{\xi}), \gamma_0(\xi_1)),$$

where $\gamma_0 : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2$ denotes the circle of radius 1, and investigate the associated nondegeneracy conditions (3.10) and (3.14). Although the complete analysis on its good definition will be done after the derivation of the explicit formulation of the associated first return map \mathcal{F}_0 in Section 4.3, the central symmetry of the circular case allows to give some preliminary informations. As both the outer and inner systems are invariant under rotations, the associated generating functions can be expressed as univariate functions depending on the angle spanned by the arc; more precisely,

$$S_0(\xi_0, \xi_1) = \tilde{S}_0(\xi_1 - \xi_0) = \tilde{S}_{E,0}(\tilde{\xi} - \xi_0) + \tilde{S}_{I,0}(\xi_1 - \tilde{\xi}).$$

Going through the same analysis described in general in Section 3.1, the intermediate coordinate $\tilde{\xi}$ is implicitly determined as a function of ξ_0 and ξ_1 by the relation

$$\partial_{\tilde{\xi}}(\tilde{S}_{E,0}(\tilde{\xi} - \xi_0) + \tilde{S}_{I,0}(\xi_1 - \tilde{\xi})) = 0$$

if (3.10) is verified. In the circular case, the latter translates in

$$\tilde{S}''_{E,0}(\tilde{\xi} - \xi_0) + \tilde{S}''_{I,0}(\xi_1 - \tilde{\xi}) \neq 0. \quad (4.5)$$

If (4.5) holds, one has

$$\partial_{\xi_0} \tilde{\xi} = \frac{\tilde{S}''_{E,0}(\tilde{\xi} - \xi_0)}{\tilde{S}''_{E,0}(\tilde{\xi} - \xi_0) + \tilde{S}''_{I,0}(\xi_1 - \tilde{\xi})}, \quad \partial_{\xi_1} \tilde{\xi} = \frac{\tilde{S}''_{I,0}(\xi_1 - \tilde{\xi})}{\tilde{S}''_{E,0}(\tilde{\xi} - \xi_0) + \tilde{S}''_{I,0}(\xi_1 - \tilde{\xi})},$$

and the canonical actions are defined by

$$\begin{aligned} I_0 &= -\partial_{\xi_0} \tilde{S}_0(\xi_1 - \xi_0) = \tilde{S}'_{E,0}(\tilde{\xi}(\xi_0, \xi_1) - \xi_0), \\ I_1 &= \partial_{\xi_1} \tilde{S}_0(\xi_1 - \xi_0) = \tilde{S}'_{I,0}(\xi_1 - \tilde{\xi}(\xi_0, \xi_1)). \end{aligned} \quad (4.6)$$

The first equation in (4.6) defines implicitly $\xi_1 = \xi_1(\xi_0, I_0)$, and, as a consequence, the map \mathcal{F}_0 , if (3.14) holds, that is, if

$$\partial_{\xi_1}(I_0 - \tilde{S}'_{E,0}(\tilde{\xi}(\xi_0, \xi_1) - \xi_0)) = -\frac{\tilde{S}''_{E,0}(\tilde{\xi}(\xi_0, \xi_1) - \xi_0)\tilde{S}''_{I,0}(\xi_1 - \tilde{\xi}(\xi_0, \xi_1))}{\tilde{S}''_{E,0}(\tilde{\xi}(\xi_0, \xi_1) - \xi_0) + \tilde{S}''_{I,0}(\xi_1 - \tilde{\xi}(\xi_0, \xi_1))} \neq 0. \quad (4.7)$$

As a final remark, note that the validity of conditions (4.5) and (4.7) are strongly related to the *twist condition* (see [21, 18, 1]) associated to the map \mathcal{F}_0 , defined as $\partial_{I_0}\xi_1 \neq 0$. As a matter of fact, one has

$$\partial_{I_0}\xi_1 = \frac{\tilde{S}''_{E,0}(\tilde{\xi}(\xi_1 - \xi_0) - \xi_0) + \tilde{S}''_{I,0}(\xi_1 - \tilde{\xi}(\xi_1 - \xi_0))}{\tilde{S}''_{E,0}(\tilde{\xi}(\xi_0, \xi_1) - \xi_0)\tilde{S}''_{I,0}(\xi_1 - \tilde{\xi}(\xi_1 - \xi_0))} :$$

one can then say that, if \mathcal{F}_0 is given, the twist condition is equivalent to require the nondegenerations (4.5) and (4.7) to be true.

4.3. Explicit formulation of \mathcal{F}_0 . When the domain D is circular, the first return map $F : (\xi_0, \alpha_0) \mapsto (\xi_1, \alpha_1)$ can be explicitly determined: in this case, the nondegeneracy given through (4.5) and (4.7) can be investigated in the equivalent form given by the twist condition. The boundary ∂D_0 can be parametrized as $\gamma(\xi) = (\cos \xi, \sin \xi)$, with $\xi \in \mathbb{R}/2\pi\mathbb{Z}$, and the symmetry properties of the potentials V_E and V_I and the isotropy of the Snell's law on a circular domain imply that F is of the form

$$F(\xi_0, \alpha_0) = (\xi_1(\xi_0, \alpha_0), \alpha_1(\xi_0, \alpha_0)) = (\xi_0 + \bar{\theta}(\alpha_0), \alpha_0); \quad (4.8)$$

in other words, the first return map on the circle reduces to a conservation of the velocity variable α and a shift in the angle ξ of a suitable quantity $\bar{\theta}$ which depends only on the physical parameters of the problem and on α_0 . The Jacobian matrix $DF((\xi_0, \alpha_0))$ can be then expressed for every pair $(\xi_0, \alpha_0) \in \mathbb{R}/2\pi\mathbb{Z} \times (-\pi/2, \pi/2)$ as

$$DF(\xi_0, \alpha_0) = \begin{pmatrix} 1 & \frac{\partial \bar{\theta}}{\partial \alpha_0}(\alpha_0) \\ 0 & 1 \end{pmatrix}.$$

From the above considerations, we have that $\bar{\theta}(\alpha_0) = \bar{\theta}_E(\alpha_0) + \bar{\theta}_I(\alpha_0)$, where $\bar{\theta}_E(\alpha_0)$ and $\bar{\theta}_I(\alpha_0)$ represent the excursions in the angles due respectively to the outer and the inner arcs of the orbit $z_{EI}(s)$.

Outer shift. The outer shift has been already computed as an additional result in the proof of Theorem 4.1 in Appendix A, and is equal to

$$\theta_E(\alpha) = \begin{cases} \theta_E^+(\alpha) = \operatorname{arccot} \left(\frac{\omega^2}{(2\mathcal{E} - \omega^2) \sin(2\alpha)} + \cot(2\alpha) \right) & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ \theta_E^-(\alpha) = \operatorname{arccot} \left(\frac{\omega^2}{(2\mathcal{E} - \omega^2) \sin(2\alpha)} + \cot(2\alpha) \right) - \pi & \text{if } \alpha < 0. \end{cases} \quad (4.9)$$

Inner shift. As the system is invariant under rotations, without loss of generality let $p_0 = (1, 0)$ to be the initial point of the inner orbit and, denoted by v_0 its initial velocity, let $\beta_0 \in (-\pi/2, \pi/2)$ the angle between v_0 and the inward-pointing radial unit vector, namely, $-p_0$; then $v_0 = \sqrt{2(\mathcal{E} + h) + 2\mu}(-\cos \beta_0, \sin \beta_0) = (v_x, v_y)$. The inner Cauchy problem is then given by

$$\begin{cases} z''(s) = -\frac{\mu}{|z(s)|^3}z(s), & s \in [0, T] \\ z(0) = p_0, z'(0) = v_0. \end{cases} \quad (4.10)$$

Unlike the outer case, for the inner Keplerian orbit it is not possible to decouple the 2-dimensional system into two one-dimensional systems in the variables (x, y) ; we shall rely on other classical techniques (see [9]) which require the passage in polar coordinates: consider the functions $r(s) \in \mathbb{R}^+$ and $\theta(s) \in \mathbb{R}/2\pi\mathbb{Z}$ such that $z(s) = r(s)e^{i\theta(s)}$. From the conservation of the angular momentum, we have that

$$r(s)^2\theta'(s) = \text{const} = k = |p_0||v_0|\sin \beta_0 = \sqrt{2\mathcal{E} + 2h + 2\mu}\sin \beta_0, \quad (4.11)$$

while the energy conservation law implies

$$\mathcal{E} + h = \frac{1}{2}(r'(s)^2 + r(s)^2\theta'(s)^2) - \frac{\mu}{r(s)} \Rightarrow r'(s) = -\sqrt{2(\mathcal{E} + h) - \frac{k^2}{r^2(s)} + \frac{2\mu}{r(s)}}, \quad (4.12)$$

where the sign depend by the fact that, according to the chosen initial conditions, $r(s)$ is decreasing. Taking together (4.11) and (4.12), one has then

$$d\theta = -\frac{k}{r^2\sqrt{2(\mathcal{E} + h) - \frac{k^2}{r^2} + \frac{2\mu}{r}}}dr. \quad (4.13)$$

The classical results for the two-body problem ensure that, for positive energies, the Kepler problem is unbounded, and $r(t)$ reaches its unique minimum r_p at a time $s_p > 0$. The value of r_p is given by (see [9])

$$r_p = \frac{k^2}{\mu} \left(1 + \sqrt{1 + \frac{2(\mathcal{E} + h)k^2}{\mu^2}} \right)^{-1}. \quad (4.14)$$

If we denote with θ_p the polar angle of the pericenter and consider the initial conditions given by (4.10), taking into account the simmetry of $r(s)$ with respect to s_p , we have that the inner shift angle is given by

$$\bar{\theta}_I = 2\theta_p, \quad (4.15)$$

where θ_p can be obtained by integration from (4.13):

$$\theta_p = \int_0^{\theta_p} d\theta = -k \int_1^{r_p} \frac{dr}{r^2\sqrt{2(\mathcal{E} + h) - \frac{k^2}{r^2} + \frac{2\mu}{r}}} = \frac{k}{|k|} \int_1^{\frac{1}{r_p}} \frac{du}{\sqrt{\frac{2(\mathcal{E} + h)}{k^2} - u^2 + \frac{2\mu}{k^2}u}}$$

setting $x = u - \mu/k^2$, $x_0 = 1 - \mu/k^2$ and $x_1 = \frac{\mu}{k^2}\sqrt{1 + \frac{2(\mathcal{E} + h)k^2}{\mu^2}}$:

$$\theta_p = \frac{k}{|k|} \int_{x_0}^{x_1} \frac{dx}{\sqrt{\frac{2(\mathcal{E} + h)}{k^2} + \frac{\mu^2}{k^4} - x^2}}.$$

Finally, defining $y = x \left(\frac{2(\mathcal{E}+h)}{k^2} + \frac{\mu^2}{k^4} \right)^{-1/2}$, $y_0 = (k^2 - \mu) (2(\mathcal{E} + h)k^2 + \mu^2)^{-1/2}$ and $y_1 = 1$:

$$\theta_p = \frac{k}{|k|} \int_{y_0}^{y_1} \frac{dy}{\sqrt{1-y^2}} = \frac{k}{|k|} \arccos y_0 = \frac{k}{|k|} \arccos \left(\frac{k^2 - \mu}{\sqrt{2(\mathcal{E} + h)k^2 + \mu^2}} \right). \quad (4.16)$$

Casting together (4.11), (4.15) and (4.16), one obtains

$$\bar{\theta}_I(\beta_0) = \begin{cases} \bar{\theta}_I^+(\beta_0) = 2 \arccos \left(\frac{(2\mathcal{E}+2h+2\mu) \sin \beta_0^2 - \mu}{\sqrt{4(\mathcal{E}+h)(\mathcal{E}+h+\mu) \sin \beta_0^2 + \mu^2}} \right) - 2\pi & \text{if } \beta_0 \geq 0 \\ \bar{\theta}_I^-(\beta_0) = -2 \arccos \left(\frac{(2\mathcal{E}+2h+2\mu) \sin \beta_0^2 - \mu}{\sqrt{4(\mathcal{E}+h)(\mathcal{E}+h+\mu) \sin \beta_0^2 + \mu^2}} \right) + 2\pi & \text{if } \beta_0 < 0, \end{cases} \quad (4.17)$$

where the shift is such that $\bar{\theta}_I(\beta_0) \in (-\pi, \pi)$ for every $\beta_0 \in (-\pi/2, \pi/2)$. Note that

$$\begin{aligned} \lim_{\beta_0 \rightarrow 0^+} \bar{\theta}_I^+(\beta_0) &= 0 = \lim_{\beta_0 \rightarrow 0^-} \bar{\theta}_I^-(\beta_0), \\ \lim_{\beta_0 \rightarrow 0^+} \frac{d}{d\beta_0} \bar{\theta}_I^+(\beta_0) &= -\frac{4(\mathcal{E} + h + \mu)}{\mu} = \lim_{\beta_0 \rightarrow 0^-} \frac{d}{d\beta_0} \bar{\theta}_I^-(\beta_0) \end{aligned}$$

and then $\bar{\theta}_I \in C^1(-\pi/2, \pi/2)$.

Total shift and properties of the overall trajectories. The total shift angle $\bar{\theta}(\alpha_0)$ is computed by taking the sum of the outer and the inner shifts and taking into account the transition laws for the velocities across the interface ∂D_0 . In particular, if $\tilde{\alpha}$ and $\tilde{\beta}$ denote respectively the angles with the normal unit vector of the outer and the inner velocities of an orbit crossing the interface in a point $\tilde{p} \in \partial D_0$, from (2.4) one has

$$\sqrt{\mathcal{E} - \frac{\omega^2}{2} |\tilde{p}|^2} \sin \tilde{\alpha} = \sqrt{\mathcal{E} + h + \frac{\mu}{|\tilde{p}|}} \sin \tilde{\beta};$$

in the particular case of a circular domain, the Snell's law is uniform over all the points of ∂D_0 , and the initial and final angles with the radial direction are equal for every brach of the orbit. Performing in (4.17) the substitution $\sin \beta_0 = \sqrt{(2\mathcal{E} - \omega^2)/(2(\mathcal{E} + h + \mu))} \sin \alpha_0$, one obtains the total shift

$$\bar{\theta}(\alpha_0) = \begin{cases} \bar{\theta}_E^+(\alpha_0) + \bar{\theta}_I^+(\alpha_0) & \text{if } \alpha_0 > 0 \\ 0 & \text{if } \alpha_0 = 0 \\ \bar{\theta}_E^-(\alpha_0) + \bar{\theta}_I^-(\alpha_0) & \text{if } \alpha_0 < 0, \end{cases}$$

where $\bar{\theta}_E^+(\alpha_0)$ and $\bar{\theta}_E^-(\alpha_0)$ are given by (4.9) and

$$\bar{\theta}_I^+(\alpha_0) = 2 \arccos \left(\frac{(2\mathcal{E} - \omega^2) \sin \alpha_0^2 - \mu}{\sqrt{2(\mathcal{E} + h)(2\mathcal{E} - \omega^2) \sin \alpha_0^2 + \mu^2}} \right) = -\bar{\theta}_I^-(\alpha_0).$$

The map is continuous and differentiable with respect to α_0 , and

$$\begin{aligned} \frac{d}{d\alpha_0} \bar{\theta}(0) &= \lim_{\alpha_0 \rightarrow 0^+} \frac{d}{d\alpha_0} \bar{\theta}(\alpha_0) = \lim_{\alpha_0 \rightarrow 0^-} \frac{d}{d\alpha_0} \bar{\theta}(\alpha_0) = \\ &= \frac{2\mathcal{E} - \omega^2}{\mathcal{E}} - \frac{2\sqrt{2}\sqrt{\mathcal{E} + h + \mu}\sqrt{2\mathcal{E} - \omega^2}}{\mu}. \end{aligned}$$

Passing to the canonical coordinates (ξ, I) , the axisymmetry of the potentials and the isotropy of Snell's law on the circle translates in the conservation of the quantity I both in the endpoints and the transition point $\tilde{\xi}$. The first claim is a straightforward consequence of Eq.(4.8), while to prove the conservation of the action across the intermediate point one needs to consider the actions I^E and I^I associated to $S_{E,0}$ and $S_{I,0}$ separately:

$$\begin{aligned} I_1^E(\xi_0, \xi_1) &= \partial_b S_{E,0}(\xi_0, \tilde{\xi}) = \sqrt{V_E(\gamma(\tilde{\xi}))} \sin \beta_0 = \sqrt{V_I(\gamma(\tilde{\xi}))} \sin \beta_1 \\ I_0^I(\xi_0, \xi_1) &= -\partial_a S_{I,0}(\tilde{\xi}, \xi_1) = \sqrt{V_I(\gamma(\tilde{\xi}))} \sin \beta_1. \end{aligned} \quad (4.18)$$

Since on the circle $\alpha_0 = \beta_0$ and $\beta_1 = \alpha_1'$, we have that for every $\xi_0, \xi_1 \in \mathbb{R}/2\pi\mathbb{Z}$

$$I_0(\xi_0, \xi_1) = I_1^E(\xi_0, \xi_1) = I_0^I(\xi_0, \xi_1) = I_1(\xi_0, \xi_1) \equiv I(\xi_0, \xi_1). \quad (4.19)$$

Moreover, from (3.16) one has that in the circular case the global domain of definition of the actions does not depend on the points ξ_0, ξ_1 , that is

$$I_0, I_1 \in \left(-\sqrt{\mathcal{E} - \frac{\omega^2}{2}}, \sqrt{\mathcal{E} - \frac{\omega^2}{2}} \right) = (-I_c, I_c) = \mathcal{I}.$$

Taking into account Eq.(4.8), the definitions of $\bar{\theta}_E$ and $\bar{\theta}_I$ and the relations (3.16), (4.19), in the new set of canonical coordinates $(\xi, I) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathcal{I}$ we can express the first return map as

$$\begin{aligned} \mathcal{F} : \mathbb{R}/2\pi\mathbb{Z} \times \mathcal{I} &\rightarrow \mathbb{R}/2\pi\mathbb{Z} \times \mathcal{I}, \\ (\xi_0, I_0) &\mapsto (\xi_1, I_1) = (\xi_0 + \bar{\theta}(I_0), I_0) = (\xi_0 + f(I_0) + g(I_0), I_0), \end{aligned} \quad (4.20)$$

where

$$f(I) = \begin{cases} \operatorname{arccot} \left(\frac{\mathcal{E} - 2I^2}{I\sqrt{4\mathcal{E} - 2(2I^2 + \omega^2)}} \right) & \text{if } I \in (0, I_c) \\ 0 & \text{if } I = 0 \\ \operatorname{arccot} \left(\frac{\mathcal{E} - 2I^2}{I\sqrt{4\mathcal{E} - 2(2I^2 + \omega^2)}} \right) - \pi & \text{if } I \in (-I_c, 0) \end{cases}$$

and

$$g(I) = \begin{cases} 2 \arccos \left(\frac{2I^2 - \mu}{\sqrt{4(\mathcal{E} + h)I^2 + \mu^2}} \right) - 2\pi & \text{if } I \in (0, I_c) \\ 0 & \text{if } I = 0 \\ -2 \arccos \left(\frac{2I^2 - \mu}{\sqrt{4(\mathcal{E} + h)I^2 + \mu^2}} \right) + 2\pi & \text{if } I \in (-I_c, 0) \end{cases}$$

are C^1 functions in \mathcal{I} .

Remark 4.3. *Direct computations show that for every $\mathcal{E} > \omega^2, h > 0, \mu > 0$ and for every $I \in \mathcal{I}$ one has $f'(I) > 0$ and $g'(I) < 0$: the outer and inner shifts are then invertible in \mathcal{I} , and one can define the inverse functions $\tilde{f}(\theta) = f^{-1}(I)|_{I=I(\theta)}$ and $\tilde{g}(\theta) = g^{-1}(I)|_{I=I(\theta)}$. From the regularity of both f and g , we have that \tilde{f} and \tilde{g} are of class C^1 in the respective domains. In particular,*

$$f(\mathcal{I}) = (-\pi, \pi), \quad g(\mathcal{I}) = (-\bar{\theta}, \bar{\theta}),$$

$$\bar{\theta} = 2\pi - 2 \arccos \left(\frac{2\mathcal{E} - \omega^2 - \mu}{\sqrt{2(\mathcal{E} + h)(2\mathcal{E} - \omega^2) + \mu^2}} \right). \quad (4.21)$$

Moreover,

$$\begin{aligned} \tilde{\xi} = \xi_0 + f(I) &\Leftrightarrow I = \tilde{f}(\tilde{\xi} - \xi_0) \equiv I_1^E(\xi_0, \tilde{\xi}), \\ \xi_1 = \tilde{\xi} + g(I) &\Leftrightarrow I = \tilde{g}(\xi_1 - \tilde{\xi}) \equiv I_0^I(\tilde{\xi}, \xi_1). \end{aligned} \quad (4.22)$$

Lemma 4.4. *For every $I \in \mathcal{I}$, except for a finite number of points, $f'(I) + g'(I) \neq 0$.*

Proof. Direct computations lead to

$$\begin{aligned} f'(I) + g'(I) &= \frac{\sqrt{2}(2\mathcal{E}^2 - (\mathcal{E} + 2I^2)\omega^2)}{\sqrt{2\mathcal{E} - \omega^2 - 2I^2(\mathcal{E}^2 - 2\omega^2I^2)}} - \frac{8(\mathcal{E} + h)I^2 + 4(\mathcal{E} + h)\mu + 4\mu^2}{\sqrt{\mathcal{E} + h + \mu - I^2(4(\mathcal{E} + h)I^2 + \mu^2)}} = \\ &= \frac{A(I^2)}{B(I^2)} - \frac{C(I^2)}{D(I^2)}. \end{aligned} \quad (4.23)$$

Since for every $I \in \mathcal{I}$ we have that $A(I^2), B(I^2), C(I^2), D(I^2) > 0$,

$$f'(I) + g'(I) = 0 \Leftrightarrow X = I^2 \in [0, I_c^2) \text{ is a solution of } p(x) = 0,$$

where $p(x) = A^2(x)D^2(x) - B^2(x)C^2(x)$. As $p(x)$ is a real polynomial of degree 5 in X , one can have at most ten values of $I \in \mathcal{I}$ such that $f'(I) + g'(I) = 0$. \square

We define $\bar{\mathcal{I}} = \{I \in \mathcal{I} \mid f'(I) + g'(I) = 0\}$ as the set of the critical points of the function $f + g$.

Proposition 4.5. *The generating function $S_0(\xi_0, \xi_1)$ is well defined in $\mathbb{R}_{/2\pi\mathbb{Z}} \times \mathbb{R}_{/2\pi\mathbb{Z}}$ except for a finite number of pairs (ξ_0, ξ_1) in the quotient space $(\mathbb{R}_{/2\pi\mathbb{Z}} \times \mathbb{R}_{/2\pi\mathbb{Z}}) / \sim$, where $(\xi_0, \xi_1) \sim (\xi'_0, \xi'_1) \Leftrightarrow \xi_1 - \xi_0 = \xi'_1 - \xi'_0$.*

Proof. For $S_0(\xi_0, \xi_1) = S_{E,0}(\xi_0, \tilde{\xi}) + S_{I,0}(\tilde{\xi}, \xi_1)$ to be well defined, one needs to verify condition (3.10). From the definition of the actions

$$\begin{aligned} \partial_{\tilde{\xi}}(\partial_b S_{E,0}(\xi_0, \tilde{\xi}) + \partial_a S_{I,0}(\tilde{\xi}, \xi_1)) &= \partial_{\tilde{\xi}}(I_1^E(\xi_0, \tilde{\xi}) - I_0^I(\tilde{\xi}, \xi_1)) = \partial_{\tilde{\xi}} I_1^E(\xi_0, \tilde{\xi}) - \partial_{\tilde{\xi}} I_0^I(\tilde{\xi}, \xi_1) = \\ &= \partial_{\tilde{\xi}} \tilde{f}(\tilde{\xi} - \xi_0) - \partial_{\tilde{\xi}} \tilde{g}(\xi_1 - \tilde{\xi}) = \left(\frac{1}{f'(I)} + \frac{1}{g'(I)} \right)_{I=I(\xi_0, \xi_1)} = \\ &= \left(\frac{f'(I) + g'(I)}{f'(I)g'(I)} \right)_{I=I(\xi_0, \xi_1)} \end{aligned}$$

which is zero if and only if $\xi_1 - \xi_0 \in (f + g)(\bar{\mathcal{I}})$. \square

Proposition 4.6. *For every $(\xi_0, I_0) \in \mathbb{R}_{/2\pi\mathbb{Z}} \times (\mathcal{I} \setminus \bar{\mathcal{I}})$ the first return map \mathcal{F}_0*

- (1) *is conservative;*
- (2) *satisfies the twist condition*

$$\frac{\partial \xi_1}{\partial I_0}(\xi_0, I_0) \neq 0.$$

Proof. The conservativity of \mathcal{F}_0 is a direct consequence of the variational formulation of the problem: when $\tilde{\xi}$ is well defined, we have (expressing $\xi_1 = \xi_1(\xi_0, I_0)$)

$$\begin{aligned}\partial_{\xi_0}\xi_1 &= -\frac{\partial_a^2 S_0(\xi_0, \xi_1)}{\partial_{ab} S_0(\xi_0, \xi_1)}, & \partial_{I_0}\xi_1 &= -\frac{1}{\partial_{ab} S_0(\xi_0, \xi_1)} \\ \partial_{\xi_0}I_1 &= \partial_{ab} S_0(\xi_0, \xi_1) - \frac{\partial_b^2 S_0(\xi_0, \xi_1)\partial_a^2 S_0(\xi_0, \xi_1)}{\partial_{ab} S_0(\xi_0, \xi_1)}, & \partial_{I_0}I_1 &= -\frac{\partial_b^2 S_0(\xi_0, \xi_1)}{\partial_{ab} S_0(\xi_0, \xi_1)},\end{aligned}$$

where, from (4.22),

$$\partial_{ab} S(\xi_0, \xi_1) = \partial_{\xi_0 \xi_1} S(\xi_0, \xi_1) = \partial_{ab} S_I(\tilde{\xi}, \xi_1) \partial_{\xi_0} \tilde{\xi} = -\frac{\tilde{g}'(\xi_1 - \tilde{\xi}) \tilde{f}'(\tilde{\xi} - \xi_0)}{\tilde{f}'(\tilde{\xi} - \xi_0) + \tilde{g}'(\xi_1 - \tilde{\xi})}$$

is well defined and different from zero for every $(\xi_0, I_0) \in \mathbb{R}_{/2\pi\mathbb{Z}} \times \mathcal{I} \setminus \bar{\mathcal{I}}$. Whenever \mathcal{F}_0 is well defined, the determinant of its Jacobian matrix is

$$D_{(\xi_0, I_0)} \mathcal{F}_0 = \partial_{\xi_0} \xi_1 \partial_{I_0} I_1 - \partial_{I_0} \xi_1 \partial_{\xi_0} I_1 = 1,$$

thus \mathcal{F}_0 is conservative.

As for the twist condition, we have $\partial_{I_0} \xi_1 = f'(I_0) + g'(I_0)$, which is nonzero whenever $I_0 \notin \bar{\mathcal{I}}$. \square

Summarizing the previous results, we can then conclude that the set $\mathcal{I} \setminus \bar{\mathcal{I}}$ is the finite union of open intervals (at most eleven, but possibly the whole $(-I_c, I_c)$ if $\bar{\mathcal{I}} = \emptyset^2$), in which \mathcal{F}_0 is well defined, conservative and satisfies the twist condition with constant sign.

Remark 4.7. *Locally around $\pm I_c$ and 0 the sign of $\partial_{I_0} \xi_1$ can be easily determined as a function of the physical parameters $\mathcal{E}, \omega, h, \mu$: as a matter of fact, one has*

$$\lim_{I \rightarrow I_c^-} \partial_{I_0} \xi_1 = \lim_{I \rightarrow -I_c^+} \partial_{I_0} \xi_1 = +\infty$$

and

$$\partial_{I_0} \xi_1(\xi_0, 0) = \frac{2\sqrt{\mathcal{E} - \frac{\omega^2}{2}}}{\mathcal{E}} - \frac{4\sqrt{\mathcal{E} + h + \mu}}{\mu},$$

then, for every $\mathcal{E} > \omega^2, h > 0, \mu > 0$

- $\exists \bar{I} \in (0, I_c)$ such that for every $I \in \mathcal{I}$ with $|I| > \bar{I}$ it results $\partial_{I_0} \xi_1 > 0$;
- if $\frac{2\sqrt{\mathcal{E} - \frac{\omega^2}{2}}}{\mathcal{E}} > \frac{4\sqrt{\mathcal{E} + h + \mu}}{\mu}$ (resp. $\frac{2\sqrt{\mathcal{E} - \frac{\omega^2}{2}}}{\mathcal{E}} < \frac{4\sqrt{\mathcal{E} + h + \mu}}{\mu}$), $\exists \bar{I} \in (0, I_c)$ such that for every $I \in (-I_c, I_c)$ with $|I| < \bar{I}$ one has $\partial_{I_0} \xi_1 > 0$ (resp. $\partial_{I_0} \xi_1 < 0$);
- additionally, if $\frac{2\sqrt{\mathcal{E} - \frac{\omega^2}{2}}}{\mathcal{E}} < \frac{4\sqrt{\mathcal{E} + h + \mu}}{\mu}$, the derivative $\partial_{I_0} \xi_1$ admits at least a change of sign, which corresponds to a change of twist for the map \mathcal{F}_0 .

²Numerical investigations shows that this case is consistent, in the sense that there are values of the parameters \mathcal{E}, h, μ and ω such that the sign of $f' + g'$ is constant (for example $\mathcal{E} = 2.5, \omega = 2, \mu = 2$ and $h = 2$).

4.4. Periodic solutions on the circle. Once the general properties of \mathcal{F}_0 on the circle are defined, we can pass to the study of its orbits. To this end, given $(\xi_0, I_0) \in \mathbb{R}_{/2\pi\mathbb{Z}} \times \mathcal{I} \setminus \tilde{\mathcal{I}}$, let us define the orbit of (ξ_0, I_0) as the sequence of the iterates $\{(\xi_k, I_k)\}_{k \in \mathbb{Z}} = \{\mathcal{F}_0^k(\xi_0, I_0)\}_{k \in \mathbb{Z}}$.

Definition 4.8. *The rotation number³ associated to (ξ_0, I_0) through \mathcal{F}_0 is given by*

$$\rho(\xi_0, I_0) = \lim_{k \rightarrow \infty} \frac{\xi_k - \xi_0}{k}. \quad (4.24)$$

In the circular case, one can easily see that for every (ξ_0, I_0) for which \mathcal{F}_0 is well defined one has $\rho(\xi_0, I_0) = \bar{\theta}(I_0)$.

As the action I_0 is preserved on the circle, we have that, taking into account the phase space $(\xi, I) \in \mathbb{R}_{/2\pi\mathbb{Z}} \times \mathcal{I}$, the straight lines $\mathbb{R}_{/2\pi\mathbb{Z}} \times \{I_0\}$ are invariant for the dynamics induced by \mathcal{F}_0 . We can then distinguish between two types of orbits:

- if $\bar{\theta}(I_0)/2\pi = p/q \in \mathbb{Q}$, then

$$(\xi_q, I_q) = (\xi_0 + 2\pi p, I_0) \equiv_{2\pi} (\xi_0, I_0);$$

in this case, we say that the point (ξ_0, I_0) and the associated orbit are (p, q) -periodic;

- if $\bar{\theta}(I_0)/2\pi \notin \mathbb{Q}$, then for all $\xi_0 \in \mathbb{R}_{/2\pi\mathbb{Z}}$ the orbit with initial point (ξ_0, I_0) is dense in $\mathbb{R}_{/2\pi\mathbb{Z}} \times \{I_0\}$.

A particular class of fixed points for \mathcal{F}_0 is given by the ejection-collision solutions, which form an invariant line of periodic points of period one defined on $\mathbb{R}_{/2\pi\mathbb{Z}} \times \{0\}$. Taking advantage of the continuity of the function $f + g$ on \mathcal{I} , one can state the following existence result.

Proposition 4.9. *Given $C = \bar{\theta} - \pi$, where $\bar{\theta}$, as in (4.21), depends only on the physical parameters $\mathcal{E}, h, \mu, \omega$, for every $\rho \in (-C, C)$ there are two values $I^\pm \in (-I_c, I_c)$ of the actions such that, for every $\xi_0 \in \mathbb{R}_{/2\pi\mathbb{Z}}$, $\rho(\xi_0, I_0^\pm) = \rho$.*

In particular, for every $p, q \in \mathbb{Z}$ such that $-C < 2\pi p/q < C$, there are $I_\pm^{(p,q)} \in (-I_c, I_c)$ such that for every $\xi_0 \in \mathbb{R}_{/2\pi\mathbb{Z}}$ the points $(\xi_0, I_+^{(p,q)})$ and $(\xi_0, I_-^{(p,q)})$ are (p, q) -periodic.

In the circular case, the existence of two orbits of all the rotation numbers is a simple consequence of the continuity of the total shift function. As it will be analysed in §5, a deformation of the boundary ∂D_0 breaks the symmetry of the system: in general, the first return map will be not integrable anymore and more sophisticated tools should be used to retrieve, at least partially, analogous existence results. In this framework, the persistence of the twist condition under small perturbations of the boundary will play a crucial role, and this is the reason why, although not immediately used, this nondegeneracy condition has been investigated in the circular case.

Under particular assumptions on the physical parameters, one can prove the existence of a second type of fixed points different from the ones which correspond to the ejection-collision solutions:

Proposition 4.10. *Fixed $\mathcal{E} > \omega^2 > 0$, let us define*

$$\bar{\mu} = \frac{4\mathcal{E} + \sqrt{8\mathcal{E}^3(4\mathcal{E} - \omega^2)}}{2\mathcal{E} - \omega^2} > 2\mathcal{E} - \omega^2, \quad \bar{h} = \frac{2\mathcal{E} - \omega^2}{8\mathcal{E}^2} \mu^2 - (\mathcal{E} + \mu).$$

³With an abuse of notation, in Section 5.2 we will use the same definition to identify the rotation number of the *lift* of a map of the annulus $\mathbb{R}_{/2\pi\mathbb{Z}} \times [a, b]$, that is, its periodic extension to $\mathbb{R} \times [a, b]$.

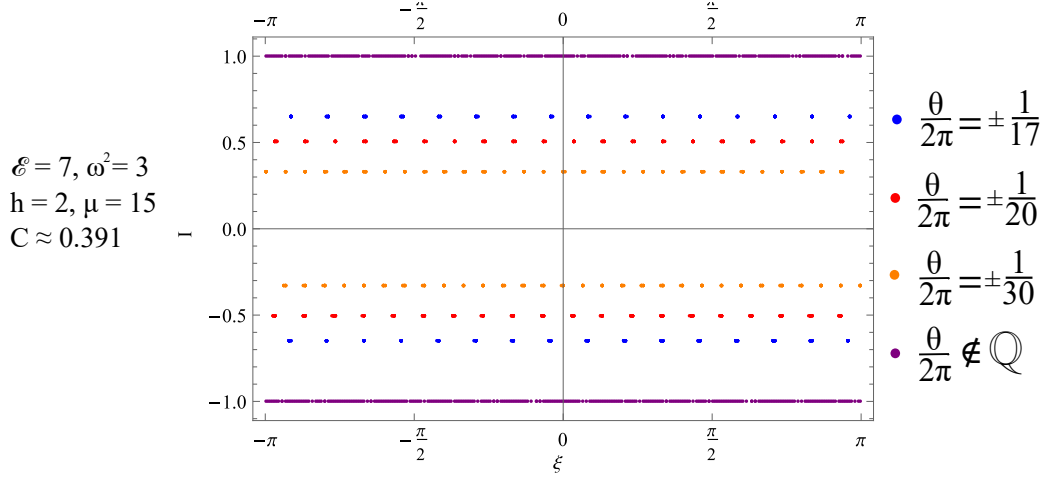


FIGURE 3. Examples of periodic and non-periodic orbits on the circle in the phase space $(\xi, I) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathcal{I}$.

If $(\mu > \bar{\mu} \wedge h > \bar{h})$ or $(2\mathcal{E} - \omega^2 < \mu \leq \bar{\mu} \wedge h > 0)$ there is $\bar{I}^{(1)} \in (0, I_c)$ such that for every $\xi_0 \in \mathbb{R}/2\pi\mathbb{Z}$ the points $(\xi_0, \bar{I}^{(1)})$ and $(\xi_0, -\bar{I}^{(1)})$ are non-homotetic fixed points of \mathcal{F} .

Proof. Recalling that $f(0) + g(0) = 0$, $\lim_{I \rightarrow I_c^-} f(I) + g(I) = \bar{\theta} + \pi$ and Eq.(4.23), from direct computations one has that

- fixed $\mathcal{E} > \omega^2 > 0$ and $\mu > 0$,

$$f'(0) + g'(0) < 0 \iff h > \bar{h};$$

- fixed $\mathcal{E} > \omega^2 > 0$,

$$\bar{h} > 0 \iff \mu > \bar{\mu} \quad \text{and} \quad \bar{\theta} + \pi > 0 \iff \mu > 2\mathcal{E} - \omega^2.$$

If $(\mu > \bar{\mu} \wedge h > \bar{h})$ or $(2\mathcal{E} - \omega^2 < \mu \leq \bar{\mu} \wedge h > 0)$, we have then that $f'(0) + g'(0) < 0$ and $\lim_{I \rightarrow I_c^-} f(I) + g(I) > 0$: as a consequence, there exists $\bar{I}^{(1)} > 0$ such that $f(\bar{I}^{(1)}) + g(\bar{I}^{(1)}) = 0 = f(-\bar{I}^{(1)}) + g(-\bar{I}^{(1)})$, and then for every $\xi_0 \in \mathbb{R}/2\pi\mathbb{Z}$, $(\xi_0, \pm\bar{I}^{(1)})$ are fixed points of \mathcal{F} . Given that $\bar{I}^{(1)} \neq 0$, these points are not homotetic (see Figure 4). \square

4.5. Caustics for the unperturbed case. A question of great interest in the study of billiards is that of caustics, which plays a key role in the determining the regions of the plane where the orbits can access. A caustic is a smooth closed curve Γ such that every trajectory which is tangent to Γ in a point remains tangent to the latter after every passage in and out the domain D . The issue of the existence of caustics in standard billiards ([25, 33]) and its variants ([16]) has been widely studied; in particular, in the framework of a standard convex billiard D , Lazutkin used the KAM approach to prove that, if ∂D is sufficiently smooth (of class C^{553} in the original paper [25], later improved to C^6 by Douady in [14]), then there exists a discontinuous family of caustics in a small neighborhood of ∂D .

The aim of this Section is to extend the concept of caustic to our refractive model in the circular case: in view of the presence of two distinct dynamics inside and outside the domain D , one

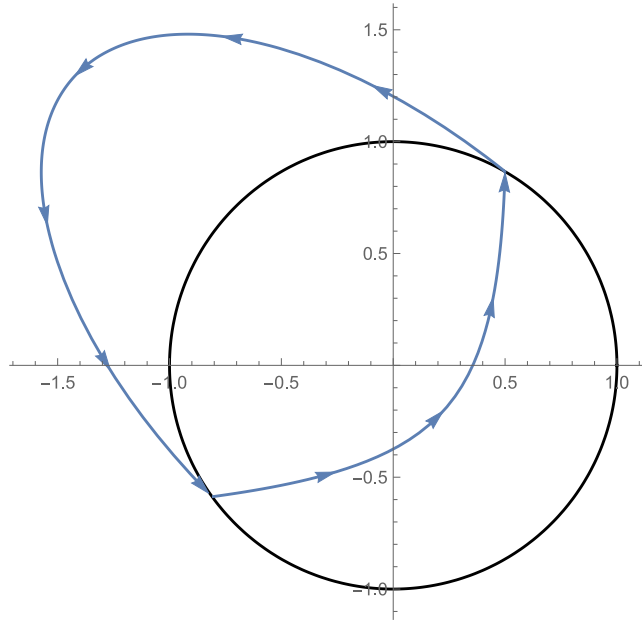


FIGURE 4. Example of a non-homotetic fixed point for \mathcal{F} , with $\mathcal{E} = 7, \omega^2 = 3, h = 2, \mu = 15$. In this case, with reference to Proposition 4.10, $\bar{\mu} = 41.6287$.

shall search for two of such curves, which can be studied separately. Moreover, by the central symmetry typical of the circular case, it is reasonable to foresee that the inner and outer caustics are circles of suitable radii depending on the action I_0 .

Theorem 4.11. *For every $\mathcal{E}, h, \omega, \mu > 0, \mathcal{E} > \omega^2$, given $I_0 \in (-I_c, I_c)$:*

- *the exterior caustic $\Gamma_E(\zeta; I_0)$ is given by the locus of the apocenters of the outer ellipses, namely,*

$$\Gamma_E : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad \Gamma_E(\zeta; I_0) = R_E(\cos \zeta, \sin \zeta),$$

$$R_E = \frac{\sqrt{\mathcal{E} + \sqrt{\mathcal{E}^2 - 2I_0^2\omega^2}}}{\omega};$$

- *the interior caustic $\Gamma_I(\zeta; I_0)$ is the locus of the pericenters of the inner Keplerian hyperbolæ. In particular,*

$$\Gamma_I : [0, 2\pi] \rightarrow \mathbb{R}^2 \quad \Gamma_I(\zeta; I_0) = R_I(\cos \zeta, \sin \zeta),$$

$$R_I = \frac{p}{1 + e},$$

where

$$p = \frac{2I_0^2}{\mu}, \quad e = \sqrt{1 + \frac{4I_0^2(\mathcal{E} + h)}{\mu^2}}.$$

In general, following [33] and [6], one shall give the following characterization for the caustic: take an orbit for our dynamical system and suppose that one of its (interior or exterior) branches

is implicitly defined through the relation

$$G(x, y; \xi) = 0, \quad (4.25)$$

where $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class C^2 in all the variables and ξ acts as a parameter (for example, it could denote the polar angle of the initial point of the branch, its pericenter or apocenter). The caustic Γ can be then seen as the envelope of the family of curves defined by (4.25) varying $\xi \in [0, 2\pi]$, that is, the set of points $(x_0(\xi), y_0(\xi))$ satisfying

$$\begin{cases} G(x, y; \xi) = 0 \\ \partial_\xi G(x, y; \xi) = 0 \end{cases}. \quad (4.26)$$

By means of the implicit function theorem, it is straightforward that if

$$\nabla_{(x,y)} G(x, y; \xi) \nparallel \nabla_{(x,y)} \partial_\xi G(x, y; \xi) \text{ on the solutions of (4.26),} \quad (4.27)$$

then (4.26) defines a regular curve $\Gamma(\xi) = (x_0(\xi), y_0(\xi))$.

The proof of Theorem 4.11 relies on the evaluation of (4.26) in the particular cases of the inner and outer dynamics: in the case of circular domains, the solutions of such system can be computed explicitly.

Outer caustic. Given $p_0 = (p_x, p_y) = e^{i\xi_0}$, $v_0 = (v_x, v_y) \in \mathbb{R}^2$, $|v_0| = \sqrt{2\mathcal{E} - \omega^2}$, from the proof of Theorem 4.1 (see Appendix A), one has that the solution of

$$\begin{cases} (HS_E)[z(s)] & s \in [0, T_E] \\ z(0) = p_0, z'(0) = v_0 \end{cases} \quad (4.28)$$

can be parametrized as

$$(x(s), y(s)) = \left(p_x \cos(\omega s) + \frac{v_x}{\omega} \sin(\omega s), p_y \cos(\omega s) + \frac{v_y}{\omega} \sin(\omega s) \right).$$

If, as in §3, $\alpha \in (-\pi/2, \pi/2)$ denotes the angle between p_0 and v_0 , recalling the definition of canonical action (3.16) one has

$$v_0 = \sqrt{2\mathcal{E} - \omega^2}(\cos \alpha p_0 + \sin \alpha t_0) = \sqrt{2\mathcal{E} - \omega^2 - 2I_0^2} p_0 + \sqrt{2}I_0 t_0,$$

where $t_0 = ie^{i\xi}$ is the tangent unit vector to ∂D_0 in p_0 . Since in the circular case the action I_0 is constant along the orbits, it can be treated as a parameter in \mathcal{I} . Additionally, consider the non-homotetic case, that is, suppose $I_0 \neq 0$ (the case $I_0 = 0$ can be easily analysed separately, leading to the same result).

Taking the function $r^2(s) = x^2(s) + y^2(s)$, by direct computations one has

$$a^2 = \max_{s \in [0, \frac{2\pi}{\omega}]} r^2(s) = \frac{\mathcal{E} + \sqrt{\mathcal{E}^2 - 2I_0^2\omega^2}}{\omega^2} > 0$$

$$b^2 = \min_{s \in [0, \frac{2\pi}{\omega}]} r^2(s) = \frac{\mathcal{E} - \sqrt{\mathcal{E}^2 - 2I_0^2\omega^2}}{\omega^2} > 0.$$

In the reference frame whose axes coincide with the ellipse's ones, denoted with $R(O, x'', y'')$, the outer arc can be then implicitly defined as a segment of the conic

$$G_{E,0}(x'', y'') = \frac{x''^2}{a^2} + \frac{y''^2}{b^2} - 1 = 0.$$

Denoting by ζ the polar angle of one of the apocenter points, all the solutions of (4.28) with $\angle(p_0, v_0) = \alpha$ are then implicitly defined by

$$G_E(x, y; \zeta) = \frac{(x \cos \zeta + y \sin \zeta)^2}{a^2} + \frac{(y \cos \zeta - x \sin \zeta)^2}{b^2} - 1 = 0$$

where $\zeta \in [0, 2\pi]$ is treated as a parameter.

As

$$\partial_\zeta G_E(x, y; \zeta) = \frac{\sqrt{\mathcal{E} - 2I_0^2 \omega^2}}{I_0^2} ((x^2 - y^2) \sin(2\zeta) - 2xy \cos(2\zeta)),$$

the explicit formulation of (4.26) for the outer arcs is then, for $\zeta \neq k\frac{\pi}{2}$, $k = 0, 1, 2, 3$

$$\begin{cases} \frac{(x \cos \zeta + y \sin \zeta)^2}{a^2} + \frac{(y \cos \zeta - x \sin \zeta)^2}{b^2} - 1 = 0 \\ \frac{\sqrt{\mathcal{E} - 2I_0^2 \omega^2}}{I_0^2} \sin(2\zeta) (x + y \cot \zeta) (x - y \tan \zeta); \end{cases} \quad (4.29)$$

note that, in the degenerate cases $\zeta = k\frac{\pi}{2}$, from $\partial_\zeta G_E(x, y; \zeta) = 0$ one obtains $x = 0$ or $y = 0$. The solutions of (4.29) are the ellipse's apocenters and pericenters: since the outer arc of the considered dynamical systems involves only the first apocenter, the only admissible solution of (4.29) is given by

$$(\bar{x}(\zeta), \bar{y}(\zeta)) = \frac{\sqrt{\mathcal{E} + \sqrt{\mathcal{E}^2 - 2I_0^2 \omega^2}}}{\omega} (\cos \zeta, \sin \zeta),$$

which describes, for $\zeta \in [0, 2\pi]$, the caustic $\Gamma_E(\zeta; I_0)$ as the circle of radius R_E of Theorem 4.11. Although the caustics for the circular domain are completely determined, let us investigate the nondegeneracy condition (4.27), which will be generalized for small perturbations of D_0 in Section 5.3. By direct computations, one has

$$\begin{aligned} \nabla_{(x,y)} G_E(x, y; \zeta)|_{(\bar{x}, \bar{y})} &= 2a \frac{\sqrt{\mathcal{E} - 2I_0^2 \omega^2}}{I_0^2} (-\cos \zeta, -\sin \zeta), \\ \nabla_{(x,y)} \partial_\zeta G_E(x, y; \zeta)|_{(\bar{x}, \bar{y})} &= 2a \frac{\sqrt{\mathcal{E} - 2I_0^2 \omega^2}}{I_0^2} (\sin \zeta, -\cos \zeta), \end{aligned}$$

leading to

$$\nabla_{(x,y)} G_E(x, y; \zeta)|_{(\bar{x}, \bar{y})} \perp \nabla_{(x,y)} \partial_\zeta G_E(x, y; \zeta)|_{(\bar{x}, \bar{y})}. \quad (4.30)$$

Inner caustic. Let us now consider the inner problem

$$\begin{cases} (HS_I)[z(s)] & s \in [0, T_I] \\ z(0) = p_0, \quad z'(0) = v_0, \end{cases} \quad (4.31)$$

and denote with $\alpha = (\pi/2, 3\pi/2)$ the angle between p_0 and v_0 . Recalling (A.23) and (A.24), and given that

$$k = |p_0 \wedge v_0| = |p_0| |v_0| \sin \alpha = \sqrt{2} I_0,$$

one has that the polar equation of the Keplerian inner arc is

$$r = \frac{p}{1 + e \cos f},$$

with

$$p = \frac{2I_0^2}{\mu}, \quad e = \frac{\sqrt{\mu^2 + 4I_0^2(\mathcal{E} + h)}}{\mu}.$$

Choosing the reference frame $R(O, x'', y'')$ where the pericenter is on the positive branch of the x -axis, the inner Keplerian arc is expressed by

$$G_I(x'', y'') = (e^2 - 1)x''^2 - y''^2 - 2pe x'' + p^2 = 0, \quad x'' \leq \frac{p}{e + 1}, \quad (4.32)$$

where the inequality condition expresses the choice of the branch of the hiperbola whith the concavity in the direction of the central mass.

As in the outer case, denoting with ζ the polar angle of the pericenter, one has that all the Keplerian hyperbolæ with central mass μ , energy $\mathcal{E} + h$ and angular momentum $k = \sqrt{2}I_0$ are given by

$$G_I(x, y; \zeta) = (e^2 - 1)(x \cos \zeta + y \sin \zeta)^2 - (y \cos \zeta - x \sin \zeta)^2 - 2pe(x \cos \zeta + y \sin \zeta) + p^2 = 0, \\ x \cos \zeta + y \sin \zeta \leq \frac{p}{e + 1},$$

with $\zeta \in [0, 2\pi]$. The system (4.26) in the inner case becomes then

$$\begin{cases} (e^2 - 1)(x \cos \zeta + y \sin \zeta)^2 - (y \cos \zeta - x \sin \zeta)^2 - 2pe(x \cos \zeta + y \sin \zeta) + p^2 = 0 \\ 2e(y \cos \zeta + x \sin \zeta)(p - e x \cos \zeta + e y \sin \zeta) = 0 \end{cases} \quad (4.33)$$

with the additional condition $(x \cos \zeta + y \sin \zeta) \leq p/(1 + e)$. Problem (4.31) admits the unique solution

$$(\bar{x}(\zeta), \bar{y}(\zeta)) = \frac{p}{1 + e}(\cos \zeta, \sin \zeta)$$

which corresponds to the position of the pericenter of the corresponding Keplerian arc, taking ζ as a parameter. The inner caustic $\Gamma_I(\zeta, I_0)$ is then expressed by a circle with radius R_I as in Theorem 4.11.

It is straightforward to verify that the nondegeneracy condition (4.27) is verified: from

$$\nabla_{(x,y)} G_I(x, y; \zeta)|_{(\bar{x}, \bar{y})} = -2p(\cos \zeta, \sin \zeta), \quad \nabla_{(x,y)} \partial_\zeta G_I(x, y; \zeta)|_{(\bar{x}, \bar{y})} = \frac{2ep}{1 + e}(\sin \zeta, -\cos \zeta),$$

one has

$$\nabla_{(x,y)} G_I(x, y; \zeta)|_{(\bar{x}, \bar{y})} \perp \nabla_{(x,y)} \partial_\zeta G_I(x, y; \zeta)|_{(\bar{x}, \bar{y})}.$$

5. PERTURBATIONS OF THE CIRCLE

Many of the results obtained in the circular case, although significant in themselves, can be generalized to non-circular smooth domains, provided they are close enough to D_0 in a way which will be soon specified. This extension can be performed by means of classical perturbation theory, as well as of more sophisticated results such as KAM and Aubry-Mather theorems (see [18, 29, 21, 28, 1]).

To this end, let us consider a class of domains D_ϵ whose boundary $\partial D_\epsilon = \text{supp}(\gamma_\epsilon)$ is given by a radial deformation of the circle of the form

$$\gamma_\epsilon : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2 \quad \gamma_\epsilon(\xi) = (1 + \epsilon f(\xi; \epsilon)) e^{i\xi}, \quad (5.1)$$

where $f(\xi; \epsilon)$ is a smooth function of $\mathbb{R}/2\pi\mathbb{Z} \times [-C_\epsilon, C_\epsilon]$, with $C_\epsilon > 0$ arbitrarily high; note that, from the choice of the parametrization of γ_ϵ , the variable ξ still represents the polar angle of ξ . This Section aims to analyze the generating function S_ϵ , with particular emphasis to its good definition and nondegeneracy properties, and the associated first return map \mathcal{F}_ϵ , whose orbits, when possible, will be studied in terms of their rotation numbers.

5.1. Global existence of the outer and inner arcs for the perturbed dynamics. As for the circular case (see Section 4.1), the generating function associated to the so-called *perturbed dynamics*, that is, the dynamics induced by the potential (1.1) inside and outside the perturbed domain D_ϵ , is given by

$$S(\xi_0, \xi_1; \epsilon) = d_E(\gamma_\epsilon(\xi_0), \gamma_\epsilon(\tilde{\xi})) + d_I(\gamma_\epsilon(\tilde{\xi}), \gamma_\epsilon(\xi_1)),$$

where $\gamma_\epsilon(\tilde{\xi})$ is the passage point of $z_{EI}(s)$, as defined in §3, through ∂D_ϵ .

A preliminary passage to discuss the good definition of S_ϵ as a whole is to ensure that the functions $d_E(p_0, p_1)$ and $d_I(p_0, p_1)$ are differentiable as functions of $p_0, p_1 \in \partial D_\epsilon$, namely, that the inner and outer dynamics admit a unique geodesic arc joining p_0 and p_1 .

In view of the results of Section 4.1 and Appendix A, this follows from the continuous dependence of the solutions of the fixed ends problems (4.1) and (4.3) with respect to p_0 and p_1 . To fix the notation, let us denote with $z_{E \setminus I}(s; p_0, p_1; 0)$ the respective solutions in the unperturbed circular case, where the last variable refers to $\epsilon = 0$.

Remark 5.1. *Focusing on the outer problem, from the continuous dependence on p_0 and p_1 of the solution $z_E(s; p_0, p_1; 0)$ defined in Theorem 4.1, along with the invariance of the system under rotations, there exists $\rho_E > 0$ such that for every $p_0, p_1, \tilde{p}_0, \tilde{p}_1$ satisfying $|p_0 - p_1| < 2$ and $|\tilde{p}_0 - p_0|, |\tilde{p}_1 - p_1| < \rho_E$ one finds $T > 0$ and a unique solution $z_E(s; \tilde{p}_0, \tilde{p}_1)$ of the problem*

$$\begin{cases} (HS_E)[z(s)] & s \in [0, T] \\ z(0) = \tilde{p}_0, \quad z(T) = \tilde{p}_1. \end{cases} \quad (5.2)$$

For computational reasons, we require $\rho_E < 1$, and set

$$\mathcal{S}_{\rho_E} = \bigcup_{p_0 \in \partial D_0} B_{\rho_E}(p_0) = \{p \in \mathbb{R}^2 \mid \text{dist}(p, \partial D_0) < \rho_E\}. \quad (5.3)$$

Proposition 5.2. *There exists $\delta > 0$ such that for every $\tilde{p}_0, \tilde{p}_1 \in \mathcal{S}_{\rho_E}$ with $|\tilde{p}_0 - \tilde{p}_1| < \delta$ there is $T > 0$ and a unique $z_E(s; \tilde{p}_0, \tilde{p}_1)$ solution of the fixed-ends problem*

$$\begin{cases} (HS_E)[z(s)] & s \in [0, T] \\ z(0) = \tilde{p}_0, \quad z(T) = \tilde{p}_1. \end{cases} \quad (5.4)$$

Proof. It is sufficient to set $\delta < 2(1 - \rho_E)$. Denoting in polar coordinates $\tilde{p}_0 = r_0 e^{i\theta_0}$ and $\tilde{p}_1 = r_1 e^{i\theta_1}$, consider $p_0 = e^{i\theta_0}$ and $p_1 = e^{i\theta_1}$: we have then $|\tilde{p}_0 - p_0|, |\tilde{p}_1 - p_1| < \rho_E$, and

$$|p_0 - p_1| \leq |\tilde{p}_0 - p_0| + |\tilde{p}_0 - \tilde{p}_1| + |\tilde{p}_1 - p_1| < \delta + 2\rho_E < 2, \quad (5.5)$$

then, by Remark 5.1, the thesis is proved. \square

Theorem 5.3. *There are $\bar{\delta}_E > 0, \bar{\epsilon}_E > 0$ such that for every $\epsilon \in \mathbb{R}$ and for every $\xi_0, \xi_1 \in \mathbb{R}/2\pi\mathbb{Z}$ with $|\xi_0 - \xi_1| < \bar{\delta}_E$ and $|\epsilon| < \bar{\epsilon}_E$ there is $T > 0$ and a unique function $z_E(s; \gamma_\epsilon(\xi_0), \gamma_\epsilon(\xi_1)) \equiv z_E(s; \xi_0, \xi_1; \epsilon)$ which is a classical solution of*

$$\begin{cases} (HS_E)[z(s)] & s \in [0, T] \\ z(0) = \gamma_\epsilon(\xi_0), \quad z(T) = \gamma_\epsilon(\xi_1). \end{cases} \quad (5.6)$$

Proof. The claim is true if $\bar{\delta}_E < \delta/2$ and $\bar{\epsilon}_E < \min\{1/(\|f\|_\infty + \|\partial_\xi f\|_\infty), \rho_E/\|f\|_\infty\}$. For, fixed $\epsilon \in \mathbb{R}$ such that $|\epsilon| < \bar{\epsilon}_E$, one has that for every $\xi \in \mathbb{R}/2\pi\mathbb{Z}$

$$|\dot{\gamma}_\epsilon(\xi)| = |\epsilon \partial_\xi f(\xi, \epsilon) e^{i\xi} + (1 + \epsilon f(\xi, \epsilon)) i e^{i\xi}| \leq 1 + |\epsilon| (\|\partial_\xi f\|_\infty + \|f\|_\infty) < 2.$$

If $\xi_0, \xi_1 \in \mathbb{R}/2\pi\mathbb{Z}$ are such that $|\xi_0 - \xi_1| < \bar{\delta}_E < \delta/2$, defining $\tilde{p}_0 = \gamma_\epsilon(\xi_0), \tilde{p}_1 = \gamma_\epsilon(\xi_1), p_0 = e^{i\xi_0}$ and $p_1 = e^{i\xi_1}$:

$$\begin{aligned} |\tilde{p}_0 - p_0| &= |\epsilon| |f(\xi_0)| \leq |\epsilon| \|f\|_\infty < \rho_E, \\ |\tilde{p}_1 - p_1| &< \rho_E, \\ |\tilde{p}_0 - \tilde{p}_1| &< \|\dot{\gamma}_\epsilon\|_\infty |\xi_0 - \xi_1| < 2|\xi_0 - \xi_1| < \delta, \end{aligned} \quad (5.7)$$

then the hypotheses of Proposition 5.2 hold and the claim is true. \square

Passing to the inner dynamics, let us observe that the definition of winding number given in (4.2) can be extended to regular, simple curves joining points which are in a sufficiently small tubular neighborhood of ∂D_0 : if \mathcal{S}_ρ is defined as in (5.3), $\tilde{p}_0 = r_0 e^{i\theta_0}, \tilde{p}_1 = r_1 e^{i\theta_1} \in \mathcal{S}$ with $|\theta_0 - \theta_1| \neq \pi$ and $\tilde{\alpha}(s)$ is such that $\tilde{\alpha}(a) = \tilde{p}_0, \tilde{\alpha}(b) = \tilde{p}_1$, one can construct the closed curve $\Gamma_{\tilde{\alpha}}$ by following $\tilde{\alpha}$ in $[a, b]$, then the shortest arc of $\partial B_{r_1}(0)$ joining \tilde{p}_1 and $\tilde{p}'_0 = r_1 e^{i\theta_0}$ and finally the ray of $e^{i\theta_0}$ from \tilde{p}'_0 to \tilde{p}_0 (see Figure 5). If ρ is sufficiently small, the definition (4.2) can be straightforwardly extended to $\tilde{\alpha}$.

Theorem 5.4. *There exist $\bar{\epsilon}_I > 0, \bar{\delta}_I > 0$ and $C > 0$ such that for every $\epsilon \in \mathbb{R}, \xi_0, \xi_1 \in \mathbb{R}/2\pi\mathbb{Z}$ satisfying $|\xi_0 - \xi_1| < \bar{\delta}_I$ and $|\epsilon| < \bar{\epsilon}_I$ there exists a unique $T(\xi_0, \xi_1) \equiv T > 0$ and a unique solution $z_I(s; \xi_0, \xi_1; \epsilon)$ of*

$$\begin{cases} (HS_I)[z(s)] & s \in [0, T] \\ z(s) \in D_\epsilon & s \in (0, T) \\ z(0) = \gamma_\epsilon(\xi_0), \quad z(T) = \gamma_\epsilon(\xi_1) \end{cases} \quad (5.8)$$

with the following properties:

- if $\xi_0 \neq \xi_1$, then $z(s; \xi_0, \xi_1; \epsilon)$ is a classical solution of (5.8) such that $|I(z([0, T]), 0)| = 1$;
- if $\xi_0 = \xi_1$, $z(s; \xi_0, \xi_0; \epsilon)$ is an ejection-collision solution.

In any case, $z(s; \xi_0, \xi_1; \epsilon)$ is of class C^1 with respect to variations of ξ_0 and ξ_1 and, if we define

$$\beta_0 = \angle(\dot{\gamma}_\epsilon(\xi_0), z'(0; \xi_0, \xi_1; \epsilon)), \quad \beta_1 = \angle(\dot{\gamma}_\epsilon(\xi_1), z'(T; \xi_0, \xi_1; \epsilon)), \quad (5.9)$$

one has that $|\beta_0| > C$ and $|\beta_1| > C$.

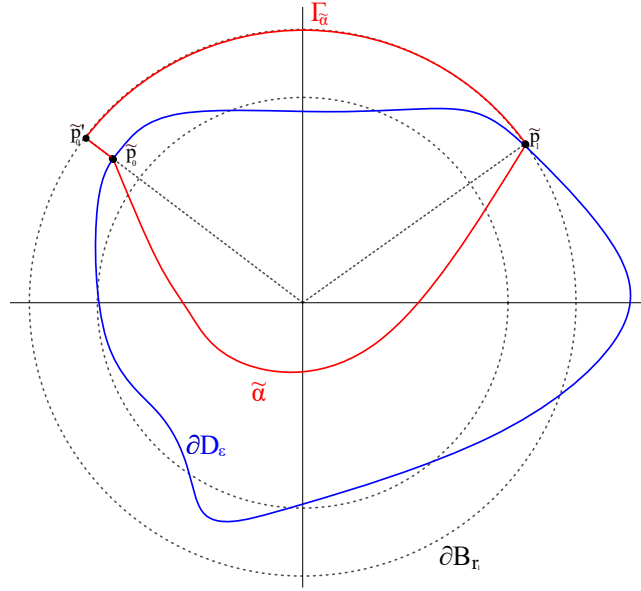


FIGURE 5. Curve $\Gamma_{\tilde{\alpha}}$ for the computation of the winding number in the perturbed case.

Proof. As in the case of the outer dynamics, in view of Theorem 4.2, there exists $0 < \rho_I < 1$ such that, choosing $\delta < 2(1 - \rho_I)$, for every $\tilde{p}_0 = r_0 e^{i\theta_0}, \tilde{p}_1 = r_1 e^{i\theta_1} \in \mathcal{S}_{\rho_I}$ the problem

$$\begin{cases} (HS_I)[z(s)] & s \in [0, T] \\ z(0) = \tilde{p}_0, z(T) = \tilde{p}_1 \end{cases}$$

admits a unique solution $z_I(s; \tilde{p}_0, \tilde{p}_1)$, with $T > 0$ depending on the endpoints. Moreover, possibly reducing ρ_I , defining $p_0 = e^{i\theta_0}$ and $p_1 = e^{i\theta_1}$ and considering $z_I(s; p_0, p_1)$ introduced by Theorem 4.2, one has that $z_I(s; \tilde{p}_0, \tilde{p}_1)$ and $z_I(s; p_0, p_1)$ have the same winding number. Choosing then $\epsilon_1 < \min \left\{ \frac{\rho_I}{\|f\|_\infty}, \frac{1}{\|f\|_\infty + \|\partial_\xi f\|_\infty} \right\}$ and $\bar{\delta}_I = \delta < 2(1 - \rho_I)$, for any $\xi_0, \xi_1 \in \mathbb{R}/2\pi\mathbb{Z}$ with $|\xi_0 - \xi_1| < \delta$ and any $|\epsilon| < \epsilon_1$, following the proof of Theorem 5.3, the existence and uniqueness of the solution $z_I(s; \xi_0, \xi_1; \epsilon)$ of problem (5.8) is ensured, with preservation of the winding number with respect to the unperturbed case.

From the regularity of γ_ϵ with respect to ξ and of $z(s; \tilde{p}_0, \tilde{p}_1)$ with respect to the endpoints, we have that $z(s; \xi_0, \xi_1; \epsilon)$ is of class C^1 in the variables ξ_0 and ξ_1 .

To prove the transversality properties of $z_I(s; \xi_0, \xi_1; \epsilon)$, let us observe that by Proposition A.30 and the differentiability of $z(s) \equiv z(s; \xi_0, \xi_1; \epsilon)$, as well as by the invariance under rotations of the system, there is $C_1 > 0$, possibly lower than C , such that for every ξ_0, ξ_1 satisfying the existence hypotheses, defined \tilde{p}_0 and \tilde{p}_1 as above, one has

$$1 \geq -\frac{\tilde{p}_0}{|\tilde{p}_0|} \cdot \frac{z'(0)}{|z'(0)|} > C_1, \quad 1 \geq \frac{\tilde{p}_1}{|\tilde{p}_1|} \cdot \frac{z'(T)}{|z'(T)|} > C_1. \quad (5.10)$$

Let us now consider $\alpha_1 = \angle(\tilde{p}_1, z'(T))$ (if $\alpha_0 = \angle(-\tilde{p}_0, z'(0))$ we proceed analogously): setting $C_2 = \arccos(C_1) \in (0, \pi/2)$, we have $|\alpha_1| < C_2$, and, taking $\beta_1 = \angle(\dot{\gamma}_\epsilon(\xi_1), z'(T))$, one has

$$\beta_1 = \angle(\dot{\gamma}_\epsilon(\xi_1), \tilde{p}_1^\perp) + \angle(\tilde{p}_1^\perp, z'(T)), \quad (5.11)$$

where $\tilde{p}_1^\perp = i\tilde{p}_1$. Then we have that

$$\begin{aligned} |\beta_1| &\geq \left| \angle(\tilde{p}_1^\perp, z'(T)) \right| - \left| \angle(\dot{\gamma}_\epsilon(\xi_1), \tilde{p}_1^\perp) \right|, \\ \left| \angle(\tilde{p}_1^\perp, z'(T)) \right| &> \frac{\pi}{2} - C_2 \equiv C_3 \in \left(0, \frac{\pi}{2}\right). \end{aligned} \quad (5.12)$$

To estimate $\angle(\dot{\gamma}_\epsilon(\xi_1), \tilde{p}_1^\perp)$, let us observe that

$$\left| \sin \left(\angle(\dot{\gamma}_\epsilon(\xi_1), \tilde{p}_1^\perp) \right) \right| = \frac{|\dot{\gamma}_\epsilon(\xi_1) \wedge \tilde{p}_1^\perp|}{|\dot{\gamma}_\epsilon(\xi_1)| |\tilde{p}_1^\perp|}, \quad (5.13)$$

where $\tilde{p}_1^\perp / |\tilde{p}_1^\perp| = ie^{i\xi_1}$ and $\dot{\gamma}_\epsilon(\xi_1) = \epsilon \partial_\xi f(\xi_1, \epsilon) e^{i\xi_1} + (1 + \epsilon f(\xi_1, \epsilon)) ie^{i\xi_1}$. If $|\epsilon| < \epsilon_1$,

$$\begin{aligned} |\dot{\gamma}_\epsilon(\xi_1)| &= \sqrt{(\epsilon \partial_\xi f(\xi_1, \epsilon))^2 + (1 + \epsilon f(\xi_1, \epsilon))^2} \geq 1 - |\epsilon| \|f\|_\infty > 1 - \rho_I \\ \Rightarrow \left| \sin \left(\angle(\dot{\gamma}_\epsilon(\xi_1), \tilde{p}_1^\perp) \right) \right| &= \frac{|\epsilon| |\partial_\xi f(\xi_1, \epsilon)|}{|\dot{\gamma}_\epsilon(\xi_1)|} < \frac{|\epsilon|}{1 - \rho_I} |\partial_\xi f(\xi_1, \epsilon)|. \end{aligned} \quad (5.14)$$

If we consider $C_4 > 0$ such that $0 < \arcsin(C_4) < C_3$, setting $\epsilon < \min \left\{ \epsilon_1, \frac{C_4(1-\rho_I)}{\|\partial_\xi f\|_\infty} \right\} = \bar{\epsilon}_I$:

$$\left| \sin \left(\angle(\dot{\gamma}_\epsilon(\xi_1), \tilde{p}_1^\perp) \right) \right| < C_4 \Rightarrow |\beta_1| > C_3 - \arcsin(C_4) \equiv \bar{C} > 0. \quad (5.15)$$

Recalling the definitions which lead to $\bar{\epsilon}$ and \bar{C} , it is clear that they do not depend on ξ_0 nor ξ_1 , but only on ρ and the global properties of f .

Finally, the condition $z(s) \in D_\epsilon$ for $s \in (0, T)$ follows from the smallness of ϵ and the transversality of $z(s)$ with respect to the perturbed domain ∂D_ϵ , which ensures that $z(s)$ do not intersect twice the domain's boundary in a neighborhood of $z(T)$. \square

Remark 5.5. *Using the same transversality argument described in details for the inner dynamics, one can prove that, if ϵ is small enough and ξ_0, ξ_1 sufficiently close, the solution $z_E(s; \xi_0, \xi_1; \epsilon)$ of (5.6), whose existence is ensured by Theorem 5.3, is such that $z_E(s; \xi_0, \xi_1; \epsilon) \notin \bar{D}_\epsilon$ for $s \in (0, T)$.*

5.2. Invariant sets for \mathcal{F}_ϵ . The good definition of the distances $d_E(p_0, p_1)$ and $d_I(p_0, p_1)$ for $p_0, p_1 \in \partial D_\epsilon$ allows to consider the associated generating function

$$S(\xi_0, \xi_1; \epsilon) = S_E(\xi_0, \tilde{\xi}; \epsilon) + S_I(\tilde{\xi}, \xi_1; \epsilon) = d_E(\gamma_\epsilon(\xi_0), \gamma_\epsilon(\tilde{\xi})) + d_I(\gamma_\epsilon(\tilde{\xi}), \gamma_\epsilon(\xi_1)). \quad (5.16)$$

When well defined, $S(\xi_0, \xi_1; \epsilon)$ has the same regularity of $f(\xi, \epsilon)$ as a function of both the angle variables ξ_0, ξ_1 and the perturbative parameter ϵ .

This Section aims to prove that, under suitable assumptions, the results proved for the circle regarding the twist condition and the existence of invariant sets with prescribed rotation numbers (see Section 4.4) can be extended to the perturbed dynamics as described in (5.1).

Recalling the notation of Section 4, $\bar{\mathcal{I}}$ is the finite set in $\mathcal{I} = (-I_c, I_c)$ for which \mathcal{F}_0 is not well defined, with $I_c = \sqrt{\mathcal{E} - \omega^2/2}$. To highlight the dependence on ϵ , from now on we will use the notation $\mathcal{F}_0(\xi_0, I_0) \equiv \mathcal{F}(\xi_0, I_0; 0)$.

Proposition 5.6. *Let $[a, b] \subset \mathcal{I} \setminus \bar{\mathcal{I}}$, and suppose that, in (5.1), $f \in C^k(\mathbb{R}/2\pi\mathbb{Z} \times \mathcal{I})$ with $k \geq 2$. Then there exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in \mathbb{R}$, $|\epsilon| < \bar{\epsilon}$, the perturbed first return map*

$$\mathcal{F}(\xi_0, I_0; \epsilon) = (\xi_1(\xi_0, I_0; \epsilon), I_1(\xi_0, I_0; \epsilon))$$

is well defined and of class $C^{k-2}(\mathbb{R}/2\pi\mathbb{Z} \times [a, b])$. Moreover, $\mathcal{F}(\cdot, \cdot, \epsilon)$ is conservative and twist.

Proof. Let us consider $[a, b] \subset \mathcal{I} \setminus \bar{\mathcal{I}}$, and, with reference to (4.20), define

$$K = \{(\xi_0, \xi_0 + \bar{\theta}(I_0)) \mid \xi_0 \in \mathbb{R}/2\pi\mathbb{Z}, I_0 \in [a, b]\} :$$

in view of Propositions 4.5 and 4.6, the generating function $S(\xi_0, \xi_1; 0)$ is well defined and infinitely many differentiable in K , and the same holds for $\mathcal{F}(\xi_0, I_0; 0)$ in $\mathbb{R}/2\pi\mathbb{Z} \times [a, b]$. Moreover, K is a compact subset of the torus $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. In particular, one has that the quantity

$$\partial_b^2 S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1; 0); 0) + \partial_a^2 S_I(\tilde{\xi}(\xi_0, \xi_1; 0), \xi_1; 0), \quad (5.17)$$

with $\tilde{\xi}(\xi_0, \xi_1; 0)$ such that $\partial_b S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1; 0); 0) + \partial_a S_I(\tilde{\xi}(\xi_0, \xi_1; 0), \xi_1; 0) = 0$, is different from 0 and has always the same sign for $(\xi_0, \xi_1) \in K$.

Let us now fix $(\bar{\xi}_0, \bar{\xi}_1) \in K$: by the implicit function theorem, there are two neighborhoods $A_{\bar{\xi}_0}, A_{\bar{\xi}_1}$ respectively of $\bar{\xi}_0$ and $\bar{\xi}_1$, a quantity $\bar{\epsilon}(\bar{\xi}_0, \bar{\xi}_1) > 0$ and a unique function $\tilde{\xi}(\xi_0, \xi_1; \epsilon)$, defined in $A_{\bar{\xi}_0} \times IA_{\bar{\xi}_1} \times [-\bar{\epsilon}(\bar{\xi}_0, \bar{\xi}_1), \bar{\epsilon}(\bar{\xi}_0, \bar{\xi}_1)]$, such that the refraction law

$$\partial_b S_E(\xi_0, \tilde{\xi}(\xi_0, \xi_1; \epsilon); \epsilon) + \partial_a S_I(\tilde{\xi}(\xi_0, \xi_1; \epsilon), \xi_1; \epsilon) = 0$$

holds also in the perturbed case. Moreover, the function $\tilde{\xi}$ is of class C^{k-1} in all its variables. As a consequence, the generating function $S(\xi_0, \xi_1; \epsilon)$ is well defined in $A_{\bar{\xi}_0} \times A_{\bar{\xi}_1} \times [-\bar{\epsilon}(\bar{\xi}_0, \bar{\xi}_1), \bar{\epsilon}(\bar{\xi}_0, \bar{\xi}_1)]$. Varying $(\bar{\xi}_0, \bar{\xi}_1) \in K$, the family

$$\{A_{\bar{\xi}_0} \times A_{\bar{\xi}_1} \mid (\bar{\xi}_0, \bar{\xi}_1) \in K\}$$

is a covering of K such that, if $(A_{\bar{\xi}_0} \times A_{\bar{\xi}_1}) \cap (A_{\bar{\xi}'_0} \times A_{\bar{\xi}'_1}) \neq \emptyset$, then $\tilde{\xi}(\xi_0, \xi_1; \epsilon)$ coincide in the intersection. Since K is compact, there exists a finite sequence $(\bar{\xi}_0^{(i)}, \bar{\xi}_1^{(i)})_{i=1}^N$ such that

$$K \subset \bigcup_{i=1}^N A_{\bar{\xi}_0^{(i)}} \times A_{\bar{\xi}_1^{(i)}}.$$

Setting $\bar{\epsilon}' = \min_{i=1, \dots, N} \bar{\epsilon}(\bar{\xi}_0^{(i)}, \bar{\xi}_1^{(i)})$, one has that for every $(\xi_0, \xi_1) \in K$ and every $\epsilon \in \mathbb{R}$ such that $|\epsilon| < \bar{\epsilon}'$, the perturbed generating function $S(\xi_0, \xi_1; \epsilon)$ is well defined and of class C^{k-1} . In such set one can define the canonical actions

$$I_0(\xi_0, \xi_1; \epsilon) = -\partial_{\xi_0} S(\xi_0, \xi_1; \epsilon), \quad I_1(\xi_0, \xi_1; \epsilon) = \partial_{\xi_1} S(\xi_0, \xi_1; \epsilon),$$

and, by the definition of K , one has that for every $(\xi_0, \xi_1) \in K$, $I_0(\xi_0, \xi_1; 0) \in [a, b]$. Fixing $\bar{\xi}_0 \in \mathbb{R}/2\pi\mathbb{Z}$ and $\bar{I}_0 \in [a, b]$, set $\bar{\xi}_1 = \xi_1(\bar{\xi}_0, \bar{I}_0; 0)$: from the proof of Proposition 4.6, one has that $\partial_{\xi_1} (\bar{I}_0 + \partial_{\xi_0} S(\bar{\xi}_0, \bar{\xi}_1; 0)) \neq 0$, and then, varying $(\bar{\xi}_0, \bar{I}_0)$ in the compact rectangle $[0, 2\pi] \times [a, b]$, one can apply the same reasoning used before to find $0 < \bar{\epsilon} < \bar{\epsilon}'$ such that for every $\xi_0 \in [0, 2\pi]$, $I_0 \in [a, b]$ and $|\epsilon| < \bar{\epsilon}$ the function $\xi_1(\xi_0, I_0; \epsilon)$ is well defined and of class C^{k-2} . Extending $\xi_1(\xi_0, I_0; \epsilon)$ by periodicity for $\xi_0 \in \mathbb{R}/2\pi\mathbb{Z}$, one has then that, for every $\epsilon > 0$, $|\epsilon| < \bar{\epsilon}$, the perturbed first return map

$$\mathcal{F}(\xi_0, I_0; \epsilon) = (\xi_1(\xi_0, I_0; \epsilon), I_1(\xi_0, I_0; \epsilon)),$$

where $I_1(\xi_0, I_0; \epsilon) = I_1(\xi_0, \xi_1(\xi_0, I_0; \epsilon); \epsilon)$, satisfies the claim in terms of good definition and regularity. The conservativity is a straightforward consequence of the existence of the perturbed

generating function, while the twist property depends on the existence of $\xi_1(\xi_0, \xi_1; \epsilon)$, since

$$\frac{\partial \xi_1}{\partial I_0} = \left(\frac{\partial I_0}{\partial \xi_1} \right)^{-1} = -\frac{1}{\partial_{\xi_0 \xi_1} S(\xi_0, \xi_1; \epsilon)}.$$

□

Remark 5.7. *Proposition 5.6 remains valid if we ask weaker regularity hypotheses on $f(\xi, \epsilon)$. In particular, if f is of class C^k in ξ and is continuous, along with all its k ξ -derivatives, in ϵ , one can find $\bar{\epsilon} > 0$ such that, for $|\epsilon| < \bar{\epsilon}$, the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ is of class C^{k-2} in $\mathbb{R}_{/2\pi\mathbb{Z}} \times [a, b]$ and continuous in ϵ , and the same holds for all its $k - 2$ derivatives.*

The map $\mathcal{F}(\xi_0, I_0; \epsilon)$, whose existence under suitable conditions and for subsets of $\mathbb{R}_{/2\pi\mathbb{Z}} \times \mathcal{I}$ is ensured by Proposition 5.6, can be expressed in the form

$$\mathcal{F}(\xi_0, I_0; \epsilon) = \begin{cases} \xi_1 = \xi_0 + \bar{\theta}(I_0) + F(\xi_0, I_0; \epsilon) \\ I_1 = I_0 + G(\xi_0, I_0; \epsilon) \end{cases} \quad (5.18)$$

where F and G are of class C^{k-2} in all the variables and

$$\|F\|_{C^{k+2}} \xrightarrow{\epsilon \rightarrow 0} 0, \quad \|G\|_{C^{k-2}} \xrightarrow{\epsilon \rightarrow 0} 0.$$

We can now prove the existence of particular orbits with prescribed rotation number for \mathcal{F}_ϵ . We will make use of *KAM Theorem* in the finitely differentiable version of Moser (cfr [29]); before stating the Theorem, let us now give some preliminary definitions.

Definitions 5.8. *Let $s \geq 1$ and $f(\xi, I)$ of class C^s in $\mathbb{R}_{/2\pi\mathbb{Z}} \times [a, b]$. The s -th derivative norm of f is given by*

$$|f|_s = \sup \left| \left(\frac{\partial}{\partial I} \right)^{m_1} \left(\frac{\partial}{\partial \xi} \right)^{m_2} f(\xi, I) \right|, \quad m_1 + m_2 \leq s.$$

Let us now consider $\mathcal{F}(\xi_0, I_0) = (\xi_1(\xi_0, I_0), I_1(\xi_0, I_0))$ a given map on the annulus $\mathbb{R}_{/2\pi\mathbb{Z}} \times [a, b]$. We say that \mathcal{F} has the *intersection property* if for any closed curve α near the circle, that is, of the form

$$\alpha(\xi_1) = (\xi_1, f(\xi_1))$$

where f is 2π -periodic and with f' small, one has

$$\text{supp}(\alpha) \cap \text{supp}(\mathcal{F}(\alpha)) \neq \emptyset.$$

If \mathcal{F} is area-preserving, the intersection property is straightforwardly verified (see [5]). Finally, given $\sigma > 0$, define

$$\mathcal{D}(\sigma) = \{ \omega \in \mathbb{R} \mid \forall n, m \in \mathbb{Z}, n > 0, |n\omega - m2\pi| \geq \sigma n^{-3/2} \}$$

the set of *Diophantine numbers* with respect to the constant $\sigma/2\pi$ and the exponent $5/2$. According to [18], given

$$\mathcal{D} = \bigcup_{\sigma > 0} \mathcal{D}(\sigma),$$

one has that \mathcal{D} is dense in $[0, 1]$, and by extension in every closed interval of \mathbb{R} . As a consequence, for every $[c, d] \subset \mathbb{R}$ and every $\rho \in (c, d) \cap \mathcal{D}$ there exists $\sigma_\rho > 0$ such that

$$\forall \sigma < \sigma_\rho \quad \rho \in (c + \sigma, d - \sigma) \cap D(\sigma).$$

Let us point out that, although we consider \mathcal{D} in accordance to the original statement by Moser in [29], in all the following results such set can be replaced by the set of all Diophantine numbers of exponent $\tau > 2$, that is,

$$\tilde{\mathcal{D}} = \bigcup_{\substack{\sigma > 0 \\ \tau > 2}} \left\{ \omega \in \mathbb{R} \mid \forall n, m \in \mathbb{Z}, n > 0, \left| \frac{\omega}{2\pi} - \frac{m}{n} \right| \geq \frac{\sigma}{n^\tau} \right\}$$

Theorem 5.9. (KAM Theorem, [29]) *Let $a, b \in \mathbb{R}$ such that $0 < a < b$ and $b - a \geq 1$, and let*

$$\mathcal{F}_0(\xi_0, I_0) = \begin{cases} \xi_1 = \xi_0 + \bar{\theta}(I_0) \\ I_1 = I_0 \end{cases}$$

be a map on the annulus $\mathbb{R}/_{2\pi\mathbb{Z}} \times [a, b]$; suppose that there is $c_0 \geq 1$ such that

$$c_0^{-1} \leq \frac{\partial \bar{\theta}}{\partial I_0}(I_0) \leq c_0.$$

Moreover, let

$$\mathcal{F}(\xi_0, I_0) = \begin{cases} \xi_1 = \xi_0 + \bar{\theta}(I_0) + F(\xi_0, I_0) \\ I_1 = I_0 + G(\xi_0, I_0) \end{cases}$$

a perturbation of \mathcal{F}_0 that satisfies the intersection property.

Fixed $\sigma > 0$ and $s \geq 1$, there are $\delta_0 = \delta_0(c_0, \sigma, s) > 0$ and an integer $l = l(s) > 0$ such that, if

- (1) $|F|_0 + |G|_0 < \delta_0$,
- (2) F and G are of class $C^l(\mathbb{R}/_{2\pi\mathbb{Z}} \times [a, b])$ and $|\bar{\theta}|_l + |F|_l + |G|_l < c_0$,

then \mathcal{F} admits a closed invariant curve

$$\begin{cases} \xi = u + p(u) \\ I = \bar{I} + q(u), \end{cases} \quad (5.19)$$

with $\bar{I} \in [a, b]$, which induces a mapping

$$u_1 = u_0 + \bar{\theta}(\bar{I}) \quad (5.20)$$

and such that p and q are 2π -periodic functions in the parameter u with s continuous derivatives and

$$|p|_s + |q|_s < \sigma. \quad (5.21)$$

Moreover, for every $\omega \in (\bar{\theta}(a) + \sigma, \bar{\theta}(b) - \sigma) \cap \mathcal{D}(\sigma)$ there exists an invariant curve of the form (5.19) with rotation number $\bar{\theta}(\bar{I}) = \omega$.

Remark 5.10. *The rotation number of the invariant curve (5.19) can be derived from the mapping (5.20) as follows: let us take $u_0 \in \mathbb{R}$ and consider the sequence $\{u_n\}_{n \in \mathbb{N}}$ produced by*

(5.20), which is trivially given by $u_n = u_0 + n\bar{\theta}(\bar{I})$. The orbit of \mathcal{F} generated by (5.20) and lying in the invariant curve (5.19) is then $\{(\xi_n, I_n)\}_{n \in \mathbb{N}}$, with

$$\begin{cases} \xi_n = u_0 + n\bar{\theta}(\bar{I}) + p(u_0 + n\bar{\theta}(\bar{I})) \\ I_n = \bar{I} + q(u_0 + n\bar{\theta}(\bar{I})). \end{cases} \quad (5.22)$$

From the definition (4.24) and given that p is bounded, one can easily compute the rotation number associated to the initial condition $(\xi_0, I_0) = (\xi(u_0), I(u_0))$ through the map \mathcal{F} as

$$\rho(\xi_0, I_0) = \lim_{n \rightarrow \infty} \frac{\xi_n - \xi_0}{n} = \lim_{n \rightarrow \infty} \bar{\theta}(\bar{I}) + \frac{p(u_0 + n\bar{\theta}(\bar{I})) - p(u_0)}{n} = \bar{\theta}(\bar{I}).$$

Remark 5.11. Although in the original paper [29] for $s = 1$ the minimal number of continuous derivatives required for the application of Theorem (5.9) is $l = 333$, Rüssman and Hermann reduced this number to $l = 5$ and then to $l > 3$ (see [30] and [19]). For this reason, and in view of Proposition 5.6, we require the normal perturbation $f(\xi, \epsilon)$ to be of class $C^k(\mathbb{R}/2\pi\mathbb{Z} \times [-\bar{\epsilon}, \bar{\epsilon}])$, with $k > 5$: as a consequence, $\mathcal{F}(\xi_0, I_0; \epsilon) \in C^{k'}(\mathbb{R}/2\pi\mathbb{Z} \times [a, b] \times [-\bar{\epsilon}, \bar{\epsilon}])$, with $k' > 3$, and the invariant curves, if existing, are of class $C^1(\mathbb{R}/2\pi\mathbb{Z})$.

Theorem 5.12. Let us suppose that $\bar{\theta}'(I_0) > 0$ in $[a, b]$, and take $\rho_0, \rho_1 \in (\bar{\theta}(a), \bar{\theta}(b)) \cap \mathcal{D}$. Then there exists $\bar{\epsilon}_{\rho_0 \rho_1}$ such that for every $\epsilon \in \mathbb{R}$, $|\epsilon| < \bar{\epsilon}_{\rho_0 \rho_1}$ the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ defined in (5.18) admits two closed invariant curves of class C^1 with rotation numbers ρ_0 and ρ_1 .

Proof. To verify the hypotheses of Theorem 5.9, let us choose $C > (b - a)^{-1}$ such that $\rho'_0 = \rho_0/C, \rho'_1 = \rho_1/C \in \mathcal{D}$ (such C exists for the density of \mathcal{D} in \mathbb{R}), and consider the canonical change of coordinates

$$\begin{cases} \xi' = \frac{\xi}{C}, & I' = C I. \end{cases}$$

Expressing \mathcal{F}_0 and \mathcal{F}_ϵ in the new variables, one obtains the rescaled problem

$$\tilde{\mathcal{F}}_0(\xi'_0, I'_0) = \begin{cases} \xi'_1 = \xi'_0 + \Theta(I'_0) \\ I'_1 = I'_0 \end{cases} \quad \tilde{\mathcal{F}}(\xi'_0, I'_0; \epsilon) = \begin{cases} \xi'_1 = \xi'_0 + \Theta(I'_0) + \tilde{F}(\xi'_0, I'_0; \epsilon) \\ I'_1 = I'_0 + \tilde{G}(\xi'_0, I'_0; \epsilon), \end{cases}$$

where $I' = CI \in [a', b'] = C[a, b]$, $b' - a' > 1$, and

$$\begin{aligned} \Theta(I'_0) &= \frac{\bar{\theta}\left(\frac{I'_0}{C}\right)}{C} = \frac{\bar{\theta}(I_0)}{C}, & \tilde{F}(\xi'_0, I'_0; \epsilon) &= \frac{1}{C} F\left(C\xi'_0, \frac{I'_0}{C}; \epsilon\right) = \frac{1}{C} F(\xi_0, I_0; \epsilon), \\ \tilde{G}(\xi'_0, I'_0; \epsilon) &= C G\left(C\xi'_0, \frac{I'_0}{C}; \epsilon\right) = C G(\xi_0, I_0; \epsilon) \end{aligned}$$

are defined for $\xi'_0 \in \mathbb{R}/2\pi\mathbb{Z}$, $I'_0 \in [a', b']$ and $|\epsilon| < \bar{\epsilon}$ (see Proposition 5.6). Note that the monotonicity and convergence properties

$$\partial_{I'_0} \Theta(I'_0) > 0, \quad \|\tilde{F}\|_{C^{k-2}} \xrightarrow{\epsilon \rightarrow 0} 0, \quad \|\tilde{G}\|_{C^{k-2}} \xrightarrow{\epsilon \rightarrow 0} 0 \quad (5.23)$$

hold also for the rescaled map, as well as the conservativity. Moreover, as $\Theta(a') = \bar{\theta}(a)/C$ and $\Theta(b') = \bar{\theta}(b)/C$, one finds $\sigma > 0$ such that $\rho'_0, \rho'_1 \in (\Theta(a') + \sigma, \Theta(b') - \sigma) \cap \mathcal{D}(\sigma)$.

Since $\tilde{\mathcal{F}}$ is conservative, it satisfies the intersection property, and, given that $\Theta \in C^1([a', b'])$, there is $c_0 > 1$ such that

$$\forall I'_0 \in [a', b'] \quad c_0^{-1} \leq \Theta'(I'_0) \leq c_0.$$

Fixed $s = 1$, let us consider $l = l(s)$ as in Theorem 5.9, and, eventually taking a higher c_0 , suppose $c_0 > |\Theta|_l$. By Theorem 5.9, there exists $\delta_0 = \delta_0(c_0, \sigma, s) > 0$ such that, if (1) and (2) hold for $\tilde{\mathcal{F}}$, then the existence of the two invariant orbits for the rescaled problem is ensured. From (5.23), one can choose $0 < \bar{\epsilon}_{\rho_0\rho_1} < \bar{\epsilon}$, such that for every $\epsilon \in [-\bar{\epsilon}_{\rho_0\rho_1}, \bar{\epsilon}_{\rho_0\rho_1}]$

$$|\tilde{F}|_0 + |\tilde{G}|_0 < \delta_0 \quad \text{and} \quad |\tilde{F}|_l + |\tilde{G}|_l < c_0 - |\Theta|_l,$$

then the hypotheses of Theorem 5.9 hold and the invariant curves obtained for $\tilde{\mathcal{F}}$ can be reparametrized to be invariant curves for \mathcal{F} . In particular, such curves have rotation number ρ_0 and ρ_1 : for example, let us consider the invariant curve for the rescaled problem with rotation number ρ'_0 , which, in view of Theorem 5.9, can be expressed as

$$\begin{cases} \xi'_0 = u' + \tilde{p}(u') \\ I'_0 = \bar{I}' + \tilde{q}(u') \end{cases} \quad \text{with mapping } u'_1 = u'_0 + \Theta(\bar{I}') = u'_0 + \rho'_0.$$

Returnig to the original coordinates and setting $u = C u'$, one gets the rescaled invariant curve

$$\begin{cases} \xi_0 = u + p(u) \\ I_0 = \bar{I} + q(u) \end{cases} \quad \text{with mapping } u_1 = u_0 + \bar{\theta}(\bar{I}) = u_0 + \rho_0,$$

with $p(u) = C\tilde{p}(u/C)$ and $q(u) = C^{-1}\tilde{q}(u/C)$. □

In the phase space (ξ, I) , the curves obtained in Theorem 5.12 can be identified as the graphs of functions of the form $I_\rho(\xi; \epsilon) \in C^1(\mathbb{R}/2\pi\mathbb{Z})$: fixing $\epsilon \in (-\bar{\epsilon}_{\rho_0\rho_1}, \bar{\epsilon}_{\rho_0\rho_1})$, let us consider for example the closed invariant curve of \mathcal{F} of rotation number ρ_0 , which can be expressed, according to (5.19) and (5.20), as

$$\begin{cases} \xi_{\rho_0}(u; \epsilon) = u + p(u; \epsilon) \\ I_{\rho_0}(u; \epsilon) = \bar{I} + q(u; \epsilon) \end{cases} \quad \text{with } \bar{\theta}(\bar{I}) = \rho_0. \quad (5.24)$$

From the boundedness of p asserted in (5.21), if σ is small enough (e.g. $\sigma < 1$) the quantity

$$\partial_u \xi(u; \epsilon) = 1 + \partial_u p(u; \epsilon) \quad (5.25)$$

is always positive: one can then invert the first equation in (5.24) obtaining $u(\xi)$, which is differentiable. As a consequence, one can parametrize the curve (5.24) as the graph of the C^1 function

$$I_{\rho_0} : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}, \quad I_{\rho_0}(\xi; \epsilon) = I_{\rho_0}(u(\xi); \epsilon). \quad (5.26)$$

Remark 5.13. *Taking σ sufficiently small and a suitable $\bar{\epsilon}_{\rho_0\rho_1}$, one can find invariant curves of \mathcal{F} which are arbitrarily close to the unperturbed orbits $\mathbb{R}/2\pi\mathbb{Z} \times \{\bar{I}\}$ in the plane (ξ, I) . Then, as $\epsilon \rightarrow 0$, the functions $I_\rho(\xi; \epsilon)$ which define the invariant curves in the perturbed phase space tend in norm $C^1(\mathbb{R}/2\pi\mathbb{Z})$ to the constant functions \bar{I}_ρ with $\bar{\theta}(\bar{I}_\rho) = \rho$.*

Moreover, Theorem 5.9 can be extended to negative twist maps, leading to the existence result for invariant curves as stated in Theorem 1.2

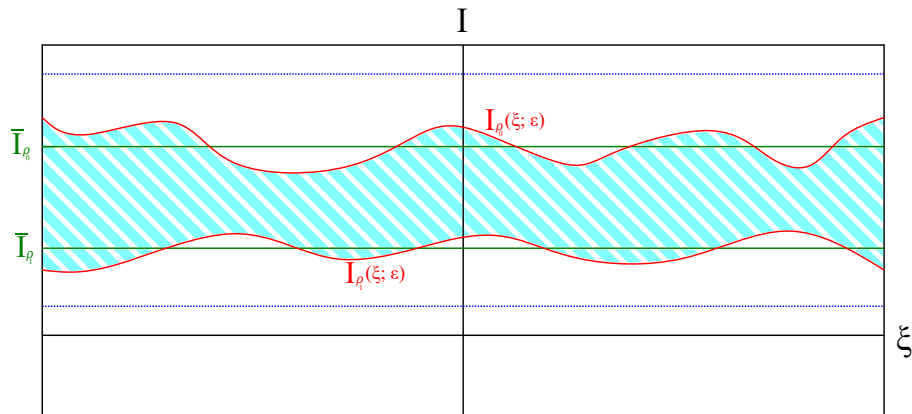


FIGURE 6. Sketch of the perturbed dynamics in the region described by Proposition 5.6 and Theorem 5.12 in the phase plane (ξ, I) . Red: the invariant curves of Diophantine rotation numbers ρ_0 and ρ_1 , which are deformations of the unperturbed invariant straight lines $I = \bar{I}_{\rho_0}$, $I = \bar{I}_{\rho_1}$ (green) such that $\bar{\theta}(\bar{I}_{\rho_0}) = \rho_0$ and $\bar{\theta}(\bar{I}_{\rho_1}) = \rho_1$. In the striped region the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ is area-preserving and twist. The blue dashed lines denote two singular action values for the unperturbed dynamics (i.e. $I \in \bar{\mathcal{I}}$).

As sketched in Figure 6, the presence of two invariant curves with irrational rotation number leads to a *confinement* in the dynamics of $\mathcal{F}(\xi_0, I_0; \epsilon)$, where, in view of Proposition (5.6), the map is conservative and twist. More precisely, the set

$$\mathcal{A} = \{(\xi, I) \in \mathbb{R}/_{2\pi\mathbb{Z}} \times \mathbb{R} \mid I_{\rho_0}(\xi; \epsilon) \leq I \leq I_{\rho_1}(\xi; \epsilon)\} \quad (5.27)$$

is invariant under \mathcal{F}_ϵ , as well as its boundaries.

In the unperturbed dynamics, the existence of periodic orbits of any rotation number in a suitable interval is the simple consequence of the continuity of the total shift $\bar{\theta}(I)$; when $\epsilon \neq 0$, one can not take advantage of the explicit formulation of the perturbed map, then this strategy is no longer suitable. Nevertheless, the broad properties of the map, such as its conservativity and the existence of the invariant curves ensure Theorem 5.12, enable the use of more sophisticated topological results, where the existence of orbits with prescribed rotation number is ensured under more general assumptions: this is the case of the Poincaré-Birkhoff theorem, here presented in the version of [18].

Theorem 5.14 (Poincaré-Birkhoff). *Let \mathcal{F} an area preserving map on the annulus $\mathbb{R}/_{2\pi\mathbb{Z}} \times [c, d]$ which preserves the boundaries and $\mathcal{F}_{\mathbb{R}}$ its lift on $\mathbb{R} \times [c, d]$. Suppose that $\mathcal{F}_{\mathbb{R}}$ satisfies the boundary twist condition, that is, the restrictions of $\mathcal{F}_{\mathbb{R}}$ to each boundary component $u_- = \mathbb{R} \times \{c\}$ and $u_+ = \mathbb{R} \times \{d\}$ have rotation numbers ρ_{\pm} with $\rho_- < \rho_+$ (the case $\rho_+ < \rho_-$ is analogous). If $2\pi \frac{m}{n} \in [\rho_-, \rho_+]$ and m, n are coprime, then \mathcal{F} has at least two (m, n) -orbits.*

Remark 5.15. *Theorem 5.14 can be extended to conservative maps which preserve invariant strips in $\mathbb{R}/_{2\pi\mathbb{Z}} \times \mathbb{R}$ whose boundaries are fixed by \mathcal{F} and are graphs of C^1 functions over the*

ξ -axis. Let us take the set \mathcal{A} defined in 5.27 and consider $\Phi_0(\xi; \epsilon), \Phi_1(\xi; \epsilon), \bar{I}_0, \bar{I}_1$ such that

$$\partial_\xi \Phi_0(\xi; \epsilon) = I_{\rho_0}(\xi; \epsilon), \quad \partial_\xi \Phi_1(\xi; \epsilon) = I_{\rho_1}(\xi; \epsilon), \quad \bar{\theta}(\bar{I}_{\rho_0}) = \rho_0, \quad \bar{\theta}(\bar{I}_{\rho_1}) = \rho_1,$$

choosing Φ_1 and Φ_0 such that $\Phi_1(\xi; 0) = \bar{I}_1 \xi$ and $\Phi_0(\xi; 0) = \bar{I}_0 \xi$. For ϵ sufficiently small, consider the quantity

$$A(\epsilon) = \int_0^{2\pi} I_{\rho_1}(\xi; \epsilon) - I_{\rho_0}(\xi; \epsilon) d\xi = \Phi_1(2\pi; \epsilon) - \Phi_0(2\pi; \epsilon) - (\Phi_1(0; \epsilon) - \Phi_0(0; \epsilon)) > 0,$$

and, noted that $A(0) = (\bar{I}_1 - \bar{I}_0) 2\pi$, define the change of coordinates

$$\Psi(\xi, I; \epsilon) = \begin{cases} \xi' = \frac{2\pi}{A(\epsilon)} (\Phi_1(\xi; \epsilon) - \Phi_0(\xi; \epsilon)) \\ I' = \frac{A(\epsilon)}{2\pi} \left(\frac{I - I_{\rho_0}(\xi; \epsilon)}{I_{\rho_1}(\xi; \epsilon) - I_{\rho_0}(\xi; \epsilon)} + \frac{\bar{I}_0}{\bar{I}_1 - \bar{I}_0} \right). \end{cases}$$

From direct computations, one has that:

- (i) $\Psi(\xi, I; \epsilon)$ is C^1 in all its variables and for every fixed $\epsilon \in \mathbb{R}$ $\det(D_{(\xi, I)} \Psi(\xi, I; \epsilon)) = 1$: hence, Ψ defines a canonical change of variables;
- (ii) for $\epsilon = 0$, $\Psi(\xi, I; 0) = Id$;
- (iii) Ψ maps the horizontal boundaries of \mathcal{A} , that is, $\{(\xi, I_{\rho_0}(\xi; \epsilon)) \mid \xi \in \mathbb{R}/_{2\pi\mathbb{Z}}\}$ and $\{(\xi, I_{\rho_1}(\xi; \epsilon)) \mid \xi \in \mathbb{R}/_{2\pi\mathbb{Z}}\}$ respectively into the straight lines

$$I = I'_0 = \frac{A(\epsilon)}{2\pi} \frac{\bar{I}_0}{\bar{I}_1 - \bar{I}_0} > 0 \quad \text{and} \quad I = I'_1 = \frac{A(\epsilon)}{2\pi} \frac{\bar{I}_1}{\bar{I}_1 - \bar{I}_0} > I'_0;$$

- (iv) for every $\xi \in \mathbb{R}/_{2\pi\mathbb{Z}}$ and every fixed ϵ , one has $\Phi_1(\xi + 2\pi; \epsilon) - \Phi_0(\xi + 2\pi; \epsilon) = \Phi_1(\xi; \epsilon) - \Phi_0(\xi; \epsilon) + A(\epsilon)$, and then

$$\begin{aligned} \xi'(\xi + 2\pi) &= \frac{2\pi}{A(\epsilon)} (\Phi_1(\xi + 2\pi; \epsilon) - \Phi_0(\xi + 2\pi; \epsilon)) = \\ &= \frac{2\pi}{A(\epsilon)} (\Phi_1(\xi; \epsilon) - \Phi_0(\xi; \epsilon) + A(\epsilon)) = \xi'(\xi) + 2\pi; \end{aligned}$$

- (v) ξ' is strictly increasing in ξ , while I' is 2π -periodic in ξ .

Globally, Ψ maps the \mathcal{F}_ϵ -invariant set \mathcal{A} into the straight line $\mathcal{B} = \mathbb{R}/_{2\pi\mathbb{Z}} \times [I'_0, I'_1]$ preserving the orientation and the boundaries. One can then consider the map $\bar{\mathcal{F}}_\epsilon : \mathcal{B} \rightarrow \mathcal{B}$ such that $\bar{\mathcal{F}}_\epsilon \circ \Psi = \Psi \circ \mathcal{F}_\epsilon$, namely, such that $\bar{\mathcal{F}}_\epsilon = \Psi \circ \mathcal{F}_\epsilon \circ \Psi^{-1}$. It can be proved that $\bar{\mathcal{F}}_\epsilon$ preserves the rotation number of the corresponding orbits of \mathcal{F} : for a (m, n) -periodic orbit, it is a simple consequence of (iv), as, taken $\{(\xi_k, I_k)\}_{k \in \mathbb{N}}$ (m, n) -periodic for \mathcal{F}_ϵ and defined for every $k \in \mathbb{N}$ $(\xi'_k, I'_k) = \Psi(\xi_k, I_k)$, one has

$$(\xi'_{k+n}, I'_{k+n}) = \Psi(\xi_{k+n}, I_{k+n}) = \Psi(\xi_k + 2\pi m, I_k) = (\xi'_k + 2\pi m, I'_k).$$

Let us now take a \mathcal{F}_ϵ -orbit with rotation number $\rho \in \mathbb{R}$ parametrized, according to Mather's definition in [28], by $(\xi_k, I_k) = (\psi_1(t_k), \psi_2(t_k))$ such that

$$\begin{aligned} t_{k+1} &= t_k + \rho, \quad F(\xi_k, I_k) = (\xi_{k+1}, I_{k+1}) = (\psi_1(t_k + \rho), \psi_2(t_k + \rho)), \\ (\psi_1(t + 2\pi), \psi_2(t + 2\pi)) &= (\psi_1(t) + 2\pi, \psi_2(t)), \end{aligned}$$

with $\psi_1 : \mathbb{R} \rightarrow \mathbb{R}$ a weakly order preserving map (not necessarily continuous). Now, setting $(\xi'_k, I'_k) = \Psi(\xi_k, I_k) = \Psi(\psi_1(t_k), \psi_2(t_k))$, one has

$$\bar{\mathcal{F}}_\epsilon(\xi'_k, I'_k) = \Psi \circ F(\xi_k, I_k) = \Psi(\xi_{k+1}, I_{k+1}) = (\xi'_{k+1}, I'_{k+1}),$$

and, defined $(\tilde{\psi}_1(t), \tilde{\psi}_2(t)) = \Psi(\psi_1(t), \psi_2(t))$,

$$(\tilde{\psi}_1(t + 2\pi), \tilde{\psi}_2(t + 2\pi)) = (\tilde{\psi}_1(t) + 2\pi, \tilde{\psi}_2(t)),$$

leading to the conclusion that the $\bar{\mathcal{F}}_\epsilon$ -orbit $\{(\xi'_k, I'_k)\}_{k \in \mathbb{N}}$ has period ρ . Note that for ϵ sufficiently small $\bar{\mathcal{F}}_\epsilon$ satisfied the hypotheses of Theorem 5.14, as for $\epsilon = 0$ the identity map is trivially twist. As the preservation of the rotation number holds also for Ψ^{-1} , given a twist map on invariant sets of the type \mathcal{A} which preserves the horizontal boundaries one can pass to the strip \mathcal{B} and use Theorem 5.14 to prove the existence of (m, n) -periodic orbits for $\bar{\mathcal{F}}_\epsilon$; returning then to the map \mathcal{F}_ϵ , this translates to the existence of (m, n) -periodic orbits for the original map.

Making use of Theorem 5.14, one can prove, under suitable conditions on the perturbation, the existence of periodic orbits for the dynamics induced by the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ with $\epsilon \neq 0$. We recall that in §4 we denoted with $\mathcal{I} \setminus \bar{\mathcal{I}}$ the set of well definition of the unperturbed map $\mathcal{F}(\xi_0, I_0; 0)$ and we proved that it is the finite union of open intervals in \mathbb{R} . In particular, the set of the singular points $\bar{\mathcal{I}}$ is composed by the critical points of the C^1 function $\bar{\theta}(I)$ (see Proposition 4.6), and one can set

$$\mathcal{I} \setminus \bar{\mathcal{I}} = \bigcup_{i=1}^N A_i$$

with $N > 0$ (possibly $N = 1$) and A_i open intervals in \mathcal{I} . In the following, to ensure the good definition of the perturbed map in a compact set, a finite union of closed intervals in $\mathcal{I} \setminus \bar{\mathcal{I}}$ will be fixed, : in particular, we fix $a_i, b_i \in \mathcal{I}$ such that $\forall i \in \{1, \dots, N\}$

$$\begin{aligned} & [a_i, b_i] \subset A_i \text{ and, if } \bar{\theta}_- = \min_i \{\bar{\theta}(a_i), \bar{\theta}(b_i)\}, \bar{\theta}_+ = \max_i \{\bar{\theta}(a_i), \bar{\theta}(b_i)\}, \\ & \bar{\theta}_-^{(i)} = \min\{\bar{\theta}(a_i), \bar{\theta}(b_i)\}, \text{ and } \bar{\theta}_+^{(i)} = \max\{\bar{\theta}(a_i), \bar{\theta}(b_i)\}, \text{ one has} \\ & \bigcup_{i=1}^N [\bar{\theta}_-^{(i)}, \bar{\theta}_+^{(i)}] = [\bar{\theta}_-, \bar{\theta}_+]. \end{aligned} \tag{5.28}$$

Note that, by the continuity of $\bar{\theta}$, such sets $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ exist.

Proposition 5.16. *Let $a_i, b_i \in \mathcal{I}$ as in (5.28), and fix $\rho_\pm^{(i)} \in \mathcal{D}$ such that for every $i = 1, \dots, N$ one has $\bar{\theta}_-^{(i)} < \rho_-^{(i)} < \rho_+^{(i)} < \bar{\theta}_+^{(i)}$. Then there exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in \mathbb{R}, |\epsilon| < \bar{\epsilon}$, and for every $m, n \in \mathbb{Z}$ coprime, $n > 0$, with $2\pi \frac{m}{n} \in (\rho_-^{(i)}, \rho_+^{(i)})$ for some $i \in \{1, \dots, N\}$, the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits at least $2k$ (m, n) -orbits, where k is the number of the pairs $(\rho_-^{(i)}, \rho_+^{(i)})$ such that $\rho_-^{(i)} < 2\pi \frac{m}{n} < \rho_+^{(i)}$.*

Proof. According to Theorem 5.12, for every pair $\rho_-^{(i)}, \rho_+^{(i)}$ there is $\bar{\epsilon}_{\rho_\pm}^{(i)}$ such that for every $|\epsilon| < \bar{\epsilon}_{\rho_\pm}^{(i)}$ the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits two orbits of rotation numbers $\rho_-^{(i)}$ and $\rho_+^{(i)}$. Moreover, the perturbed map is conservative and twist between these two orbits. Setting $\bar{\epsilon} = \min_{i \in \{1, \dots, N\}} \bar{\epsilon}_{\rho_\pm}^{(i)}$, one has that if ϵ is such that $|\epsilon| < \bar{\epsilon}$ all the orbits of rotation numbers $\rho_\pm^{(i)}$ are preserved and

the perturbed map in between is well defined and conservative.

Fixing $\epsilon \in \mathbb{R}$, $|\epsilon| < \bar{\epsilon}$, if m, n are such that $2\pi\frac{m}{n} \in (\rho_-^{(i)}, \rho_+^{(i)})$, then by Theorem 5.14 the perturbed map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits at least 2 (m, n) -orbits. As this reasoning can be repeated whenever a pair $(\rho_-^{(i)}, \rho_+^{(i)})$ is such that $\rho_-^{(i)} < 2\pi\frac{m}{n} < \rho_+^{(i)}$, the claim is true. \square

Proposition 5.16 claims the existence of a unique treshold value of ϵ under which the presence of periodic orbits of prescribed rotation numbers in a certain set is guaranteed. Another slightly different approach is proposed in Proposition 5.17, where, fixed m, n such that $2\pi\frac{m}{n}$ lies in a suitable interval which does not depend on prefixed boundary rotation numbers $\rho_{\pm}^{(i)}$, one can find a treshold $\bar{\epsilon}_{mn}$, depending on m, n , such that for every $|\epsilon| < \bar{\epsilon}_{mn}$ the presence of the corresponding (m, n) -orbit is ensured.

Proposition 5.17. *Given $\{a_i\}_{i=1}^N$, $\{b_i\}_{i=1}^N$, $\{\bar{\theta}_+^{(i)}\}_{i=1}^N$, $\{\bar{\theta}_-^{(i)}\}_{i=1}^N$ as in (5.28), let $m, n \in \mathbb{Z}$ coprime, $n > 0$, such that $2\pi\frac{m}{n} \in (\bar{\theta}_-^{(i)}, \bar{\theta}_+^{(i)})$ for some $i \in \{1, \dots, N\}$. Then $\exists \bar{\epsilon}_{mn} > 0$ such that for every $\epsilon \in \mathbb{R}$, $|\epsilon| < \bar{\epsilon}_{mn}$ the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits at least $2k$ (m, n) -orbits, where k is the number of intervals (a_i, b_i) such that $2\pi\frac{m}{n}$ is between $\bar{\theta}_-^{(i)}$ and $\bar{\theta}_+^{(i)}$.*

Proof. By the density of \mathcal{D} in every bounded interval, one can find $\rho_{\pm} \in \mathcal{D}$ with $\bar{\theta}_-^{(i)} < \rho_- < 2\pi\frac{m}{n} < \rho_+ < \bar{\theta}_+^{(i)}$. From Theorem 5.12, one can find $\bar{\epsilon}_{mn} = \bar{\epsilon}_{\rho_{\pm}}$ such that, if ϵ is such that $|\epsilon| < \bar{\epsilon}_{mn}$, the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits two orbit with rotation numbers ρ_{\pm} , and it is conservative between them. Applying again Theorem 5.14, the claim follows. \square

Extending the discussion beyond perodic orbits, one may search for more general class of invariant sets. KAM theory allowed us to claim the persistence of orbits with Diophantine rotation numbers within certain ranges, while Poincaré-Birkhoff theorem extended the existence result to periodic number with 2π -rational numbers between them. The Aubry-Mather theory allows to move further, providing the existence of orbits of the perturbed map of every prescribed rotation number in suitable subsets of \mathbb{R} .

Theorem 5.18 (Aubry-Mather on the compact annulus). *Let \mathcal{F} an area and orientation-preserving twist homeomorphism of the annulus $\mathbb{R}/2\pi\mathbb{Z} \times [a, b]$ which preserves $\mathbb{R}/2\pi\mathbb{Z} \times \{a\}$ and $\mathbb{R}/2\pi\mathbb{Z} \times \{b\}$, and define ρ_a and ρ_b as the rotation numbers of the two boundary components. Then for every $\rho \in [\rho_a, \rho_b]$ there exists at least an orbit for \mathcal{F} with rotation number ρ . In particular:*

- if $\rho = m/n \in \mathbb{Q}$, such orbit is periodic of period n ;
- if $\rho \notin \mathbb{Q}$, the orbit rotates either on a closed continuous curve or on a Cantor set.

In any case, the orbits with the same rotation number belong to a common invariant set Γ_{ρ} , called Mather set, which is a subset of the graph of a Lipschitz-continuous function over the ξ -axis.

We refer to [1, 21, 28] for the definition of Mather set and for a thorough discussion on the Aubry-Mather theory.

Remark 5.19. *As in the case of Poincaré-Birkhoff Theorem 5.14, with the same reasoning also Aubry-Mather Theorem can be extended to maps on invariant sets of the type \mathcal{A} defined in 5.27.*

Making use of the same arguments used in the proofs of Propositions 5.16 and 5.17, one can take advantage of Theorem 5.18 to state these existence results in a more general way.

Theorem 5.20. *Let $\{a_i\}_{i=1}^N$, $\{b_i\}_{i=1}^N$, $\{\bar{\theta}_-\}^N$, $\{\bar{\theta}_+\}^N$ as in (5.28). Then:*

- *letting $\rho_{\pm}^{(i)} \in \mathcal{D}$ as in Proposition 5.16, there exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in \mathbb{R}$ with $|\epsilon| < \bar{\epsilon}$ and for every $\rho \in \mathbb{R}$ such that $\rho \in [\bar{\rho}_-^{(i)}, \bar{\rho}_+^{(i)}]$ for some $i \in \{1, \dots, N\}$ the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits k orbits with rotation number ρ , with k defined as in Proposition 5.16;*
- *for every $\rho \in \mathbb{R}$ such that $\rho \in [\bar{\theta}_-^{(i)}, \bar{\theta}_+^{(i)}]$ for some $i \in \{1, \dots, N\}$ there is $\bar{\epsilon}_\rho > 0$ such that for every ϵ with $|\epsilon| < \bar{\epsilon}_\rho$ the map $\mathcal{F}(\xi_0, I_0; \epsilon)$ admits k orbits with rotation number ρ , where k is defined as in Proposition 5.17.*

In both cases, if $\rho = 2\pi \frac{m}{n}$ then for ϵ sufficiently small there are at least $2k$ (m, n) -orbits, where k is defined suitably according to the cases.

5.3. Caustics for the perturbed case. The persistence of invariant curves with Diophantine rotation numbers ensured by the KAM theorem has important consequences for the existence of caustics in the perturbed dynamics. As a matter of fact, for such invariant tori (which are dense in the phase space) it is possible to find, although not explicitly, the inner and outer caustics also for small perturbations of the circular domain D_0 .

Theorem 5.21. *Let $\xi_0 \in [0, 2\pi]$, $I_0 \in \mathcal{I} \setminus \bar{\mathcal{I}}$ such that $\theta(I_0) \in \mathcal{D}$. Then there exists $\bar{\epsilon} > 0$ such that for every $|\epsilon| < \bar{\epsilon}$ there are $\Gamma_E(\xi; \epsilon, \theta(I_0))$, $\Gamma_I(\xi; \epsilon, \theta(I_0))$ respectively outer and inner caustics related to the perturbed orbit of rotation number $\theta(I_0)$.*

The proof of Theorem 5.21 relies on showing that, for ϵ small enough, system (4.26) evaluated both for the outer and inner dynamics admits a unique solution for each $\xi \in [0, 2\pi]$, which defines a regular and closed curve. To prove that, it is worthwhile to derive the form of $G_{E \setminus I}(x, y; \xi)$ for a perturbed domain.

Outer dynamics. Let us consider $\xi \in [0, 2\pi]$, $p_0 = \gamma_\epsilon(\xi)$ and $v_0 \in \mathbb{R}^2$ such that $|v_0| = \sqrt{2V_E(p_0)}$ and $\alpha = \angle(p_0, v_0) \in (-\pi/2, \pi/2)$.

To fix the notation, recall the definition of $\gamma_\epsilon(\xi) = (1 + \epsilon f(\xi, \epsilon))e^{i\xi} = \rho(\xi; \epsilon)e^{i\xi}$: as the perturbation of the circle is only in the normal direction, the curve's parameter ξ still represents the polar angle of the point $\gamma_\epsilon(\xi)$. We want to find the Cartesian equation of the outer elliptic arc of initial conditions p_0 and v_0 .

Following the same reasoning of Appendix A and denoting with $(x(s), y(s))$ the parametrization of such ellipse, its maximal and minimal distances from the origin can be then computed as

$$\begin{aligned} a^2 &= \max_{s \in [0, 2\pi/\omega]} r^2(s) = A + \frac{\mathcal{E}}{\omega^2} = \frac{\mathcal{E} + \sqrt{(\mathcal{E} - \omega^2|p_0|^2)^2 + \omega^2(p_0 \cdot v_0)^2}}{\omega^2} \\ b^2 &= \min_{s \in [0, 2\pi/\omega]} r^2(s) = -A + \frac{\mathcal{E}}{\omega^2} = \frac{\mathcal{E} - \sqrt{(\mathcal{E} - \omega^2|p_0|^2)^2 + \omega^2(p_0 \cdot v_0)^2}}{\omega^2} : \end{aligned} \quad (5.29)$$

in the reference frame $R(O, x'', y'')$ whose axes coincide with the ellipse's ones the latter is then implicitly defined by the equation

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1.$$

Let us now search for the angle $\bar{\beta}$ such that the rotated ellipse

$$\frac{(x' \cos \bar{\beta} + y' \sin \bar{\beta})^2}{a^2} + \frac{(y' \cos \bar{\beta} - x' \sin \bar{\beta})^2}{b^2} = 1$$

passes from $p_0 = |p_0|(1, 0)$ in $R(O, x', y')$: one has to solve the equation

$$\frac{|p_0|^2 \cos^2 \bar{\beta}}{a^2} + \frac{|p_0|^2 \sin^2 \bar{\beta}}{b^2} - 1 = 0 \Rightarrow \sin^2 \bar{\beta} = \frac{b^2}{a^2 - b^2} \left(\frac{a^2}{|p_0|^2} - 1 \right) \geq 0.$$

Denoting by (v'_x, v'_y) the components of v_0 in $R(O, x', y')$, one has that

$$\sin \bar{\beta} = \begin{cases} -\frac{b}{\sqrt{a^2 - b^2}} \sqrt{\frac{a^2}{|p_0|^2} - 1} & \text{if } v'_y < 0 \\ \frac{b}{\sqrt{a^2 - b^2}} \sqrt{\frac{a^2}{|p_0|^2} - 1} & \text{if } v'_y > 0 \end{cases} \Rightarrow \cos \bar{\beta} = \frac{a}{\sqrt{a^2 - b^2}} \sqrt{1 - \frac{b^2}{|p_0|^2}}. \quad (5.30)$$

Returning to the original frame $R(O, x, y)$, one can then retrieve the Cartesian equation of the outer arc as

$$G_E(x, y; \xi, \epsilon) = \frac{(x \cos(\xi + \bar{\beta}) + y \sin(\xi + \bar{\beta}))^2}{a^2} + \frac{((y \cos(\xi + \bar{\beta}) - x \sin(\xi + \bar{\beta}))^2}{b^2} - 1 = 0 \quad (5.31)$$

Once obtained the general expression for an ellipse of initial conditions p_0 and v_0 , we shall return to the framework of our perturbed problem. Let us then consider $I_0 \in \mathcal{I} \setminus \bar{\mathcal{I}}$ such that $\theta(I_0)$ is Diophantine: from Theorem 5.9 there exists $\bar{\epsilon}^{(1)} > 0$ such that, if $|\epsilon| < \bar{\epsilon}^{(1)}$, we can define $I(\xi; \epsilon)$ invariant curve in the plane (ξ, I) for the perturbed map \mathcal{F}_ϵ such that $I(\xi; 0) \equiv I_0$ and with rotation number $\theta(I_0)$. Moreover, $I(\xi; \epsilon)$ is continuous in ϵ and differentiable in ξ , with $\partial_\xi I(\xi; \epsilon)$ continuous in ϵ : as a consequence, since $\theta(I_0) \in \mathcal{D}$ implies $I_0 \neq 0$, possibly reducing $\bar{\epsilon}^{(1)}$ one can assume that $I(\xi; \epsilon)$ has always the same sign of I_0 .

For the caustic of the orbit associated to $(\xi, I(\xi; \epsilon))$ to be well defined, it is necessary that the system

$$\begin{cases} G_E(x, y; \xi, \epsilon) = 0 \\ \partial_\xi G_E(x, y; \xi, \epsilon) = 0 \end{cases} \quad (5.32)$$

defines implicitly a unique curve $\Gamma_E(\xi; \epsilon)$ for $\xi \in [0, 2\pi]$, that is, that x and y can be expressed as functions of (ξ, ϵ) globally defined for $\xi \in [0, 2\pi]$. As already pointed out in Section 4.5, from the implicit function theorem the local existence of $\Gamma_E(\xi; \epsilon)$ is then ensured by requiring the nondegeneracy condition

$$\nabla_{(x,y)} G_E(x, y; \xi, \epsilon) \nparallel \nabla_{(x,y)} \partial_\xi G_E(x, y; \xi, \epsilon) \quad (5.33)$$

on the solutions of (5.32).

Lemma 5.22. *If $I_0 \in \mathcal{I} \setminus \bar{\mathcal{I}}$ is such that $\theta(I_0)$ is Diophantine, then there is $\bar{\epsilon}^{(2)} > 0$ such that, if $|\epsilon| < \bar{\epsilon}^{(2)}$, then $G_E(x, y; \xi, \epsilon)$ is continuous in ϵ , differentiable in ξ and such that $\partial_\xi G_E(x, y; \xi, \epsilon)$ is continuous in ϵ .*

Proof. Recalling (5.31), the proof of the Lemma relies on showing that all the quantities involved in the definition of $G_E(x, y; \xi, \epsilon)$, namely, $a^{-2}, b^{-2}, \cos \bar{\beta}$ and $\sin \bar{\beta}$ are continuous in ϵ , differentiable in ξ and with derivative continuous in ϵ , provided the latter is small enough. Starting from a^{-2} and b^{-2} , from (5.29) it is clear that the expression of $p_0 \cdot v_0$ as a function of

ξ and ϵ is needed. Recalling the definition (5.1), denoted with $t(\xi; \epsilon)$ and $n_e(\xi; \epsilon)$ the tangent and the outward-pointing normal unit vectors to γ_ϵ in p_0 , one has that

$$v_0 = \sqrt{2\mathcal{E} - \omega^2 \rho(\xi; \epsilon)^2} (\cos \alpha n_e(\xi; \epsilon) + \sin \alpha t(\xi; \epsilon)).$$

Expliciting $\cos \alpha$, $\sin \alpha$, $t(\xi; \epsilon)$, $n_e(\xi; \epsilon)$ and setting for simplicity $\rho \equiv \rho(\xi; \epsilon)$, $\rho' \equiv d\rho(\xi; \epsilon)/d\xi$ and $I(\xi; \epsilon) \equiv I$, one obtains

$$\begin{aligned} v_0 &= \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{\rho^2 + \rho'^2}} \begin{pmatrix} \sqrt{2}I (\rho' \cos \xi - \rho \sin \xi) + \sqrt{2\mathcal{E} - \omega^2 \rho^2 - 2I^2} (\rho' \sin \xi + \rho \cos \xi) \\ \sqrt{2}I (\rho' \sin \xi + \rho \cos \xi) + \sqrt{2\mathcal{E} - \omega^2 \rho^2 - 2I^2} (\rho \sin \xi - \rho' \cos \xi) \end{pmatrix} \\ \Rightarrow p_0 \cdot v_0 &= \frac{\rho(\xi; \epsilon)}{\sqrt{\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon)}} \left(\rho(\xi; \epsilon) \sqrt{2\mathcal{E} - \omega^2 \rho^2(\xi; \epsilon) - 2I^2(\xi; \epsilon)} + \sqrt{2}I(\xi; \epsilon) \rho'(\xi; \epsilon) \right), \end{aligned} \quad (5.34)$$

which has the desired continuity and differentiability properties provided ϵ is small enough. This implies that a^2 and b^2 have the same properties. Moreover, it is trivial that $a^2 > 0$ and, since for $\epsilon = 0$

$$b_{|\epsilon=0}^2 = \frac{\mathcal{E} - \sqrt{\mathcal{E}^2 - 2\omega^2 I_0^2}}{\omega^2} > 0,$$

by the continuity of b with respect to ϵ we have also $b^2 > 0$ for ϵ small enough. Applying the same reasoning, we can infer $\sqrt{a^2 - b^2} > 0$.

Going back to (5.30), $\cos \bar{\beta}$ is then continuous and differentiable, and the same conclusion holds for $\sin \bar{\beta}$ if one can ensure that v'_y has the same sign for all the points of the orbit $(\xi, I(\xi; \epsilon))$. From (5.34), in the plane $R(O, x', y')$ one has

$$v'_y = \frac{1}{\sqrt{\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon)}} \left(\rho(\xi; \epsilon) \sqrt{2}I(\xi; \epsilon) - \rho'(\xi; \epsilon) \sqrt{2\mathcal{E} - \omega^2 \rho^2(\xi; \epsilon) - 2I^2(\xi; \epsilon)} \right),$$

which for $\epsilon = 0$ translates in

$$v'_{y|\epsilon=0} = \sqrt{2}I_0 \neq 0.$$

Taking again advantage of the continuity of v'_y with respect ϵ , we can finally ensure that for ϵ small enough the thesis is proved. \square

Proposition 5.23. *If $I_0 \in \mathcal{I} \setminus \bar{\mathcal{I}}$ is such that $\theta(I_0) \in \mathcal{D}$, then there exists $\bar{\epsilon}_E$ such that for $|\epsilon| < \bar{\epsilon}_E$ the caustic $\Gamma_E(\xi; \epsilon, \theta(I_0))$ is globally well defined.*

Proof. As the nondegeneracy condition (5.33) holds for $\epsilon = 0$ (cfr. (4.30)), from Lemma 5.22, for every $\bar{\xi} \in [0, 2\pi]$ there exists $\bar{\epsilon}^{(2)}(\bar{\xi})$ such that for every $|\epsilon| < \bar{\epsilon}^{(2)}(\bar{\xi})$ condition (5.33) is satisfied. By the implicit function theorem, there are $\lambda_\xi(\bar{\xi}), \lambda_\epsilon(\bar{\xi}) > 0$ such that the curve $(x(\xi; \epsilon), y(\xi; \epsilon))$ solution of (5.32) is well defined in $R(\bar{\xi}) = (\bar{\xi} - \lambda_\xi(\bar{\xi}), \bar{\xi} + \lambda_\xi(\bar{\xi})) \times (-\lambda_\epsilon(\bar{\xi}), \lambda_\epsilon(\bar{\xi}))$. For the uniqueness of the solution, if $\bar{\xi}_1$ and $\bar{\xi}_2$ are such that $R(\bar{\xi}_1) \cap R(\bar{\xi}_2) \neq \emptyset$, the curve coincides in such intersection. As $[0, 2\pi]$ is compact, it is possible to find $N > 0$, $\{\bar{\xi}_1, \dots, \bar{\xi}_N\} \subset [0, 2\pi]$ such that

$$[0, 2\pi] \subset \bigcup_{i=1}^N (\bar{\xi}_i - \lambda_\xi(\bar{\xi}_i), \bar{\xi}_i + \lambda_\xi(\bar{\xi}_i)),$$

then, setting

$$\bar{\epsilon}_E = \min_{i \in \{1, \dots, N\}} \lambda_\epsilon(\bar{\xi}_i),$$

for every $\epsilon > 0$ such that $|\epsilon| < \bar{\epsilon}_E$ the curve $\Gamma_E(\xi; \epsilon, I_0) = (x(\xi; \epsilon), y(\xi; \epsilon))$ is globally well defined in $[0, 2\pi]$. \square

Inner caustics. Following the same reasoning applied for the outer caustic, let us consider the inner problem

$$\begin{cases} z''(s) = -\frac{\mu}{|z(s)|^3} z(s), & s \in [0, T_I] \\ \frac{1}{2}|z'(s)|^2 - \mathcal{E} - h - \frac{\mu}{|z(s)|} = 0 & s \in [0, T_I] \\ z(0) = p_0, \quad z'(0) = v_0 \end{cases}$$

by fixing $p_0 = |p_0|e^{i\xi}$, $v_0 = \sqrt{2(\mathcal{E} + h + \mu/|p_0|)}e^{i\theta_v}$ such that $\theta_v - \xi \in (\pi/2, 3\pi/2)$. This last assumption, which is done to guarantee that the hyperbola points inward a circle of radius $|p_0|$, can be ensured for ϵ small enough and suitable bounds on $I(\xi)$. Rotating again the reference frame $R(O, x, y)$ by an angle $-\xi$, we obtain $R(O, x', y')$ such that $p_0 = |p_0|(1, 0)$.

Recalling (4.32), in the reference frame $R(O, x'', y'')$ were the hiperbola's pericenter lies on the positive half of the x -axis, its Cartesian equation is given by:

$$(e^2 - 1)x''^2 - y''^2 - 2pex'' + p^2 = 0 \text{ with } x \leq \frac{p}{e+1},$$

where

$$p = \frac{k^2}{\mu}, e = \frac{\sqrt{\mu^2 + 2(\mathcal{E} + h)k^2}}{\mu}, \quad k = |p_0 \wedge v_0|.$$

To find the corresponding equation in the reference frame $R(O, x', y')$, one can search again for the angle $\bar{\delta}$ such that the arc defined by

$$\begin{aligned} (e^2 - 1)(x' \cos \bar{\delta} + y' \sin \bar{\delta})^2 - (y' \cos \bar{\delta} - x' \sin \bar{\delta})^2 - 2ep(x' \cos \bar{\delta} + y' \sin \bar{\delta}) + p^2 &= 0, \\ x' \cos \bar{\delta} + y' \sin \bar{\delta} &\leq \frac{p}{e+1} \end{aligned} \quad (5.35)$$

passes from $p_0 = |p_0|(1, 0)$. Solving (5.35) with $x' = |p_0|$ and $y' = 0$, one obtains

$$\cos \bar{\delta} = \frac{p - |p_0|}{e|p_0|},$$

which is in $[-1, 1]$ if we take non-degenerate hyperbolæ. Referring to v'_y as the vertical component of v_0 in $R(O, x', y')$, one has then

$$\sin \bar{\delta} = \begin{cases} \frac{(e^2-1)|p_0|^2 + 2p|p_0| - p^2}{e|p_0|} & \text{if } v'_y > 0 \\ -\frac{(e^2-1)|p_0|^2 + 2p|p_0| - p^2}{e|p_0|} & \text{if } v'_y < 0 \end{cases}.$$

Returning to the original reference frame $R(O, x, y)$, one obtains then the Cartesian equation for the inner Keplerian arc

$$\begin{aligned} G_I(x, y; \xi\epsilon) &= (e^2 - 1)(x \cos(\bar{\delta} + \xi) + y \sin(\bar{\delta} + \xi))^2 - (y \cos(\bar{\delta} + \xi) - x \sin(\bar{\delta} + \xi))^2 + \\ &\quad - 2pe(x \cos(\bar{\delta} + \xi) + y \sin(\bar{\delta} + \xi)) + p^2 = 0 \\ x \cos(\bar{\delta} + \xi) + y \sin(\bar{\delta} + \xi) &\leq \frac{p}{e+1}. \end{aligned} \quad (5.36)$$

Note that, with reference to the polar angles ξ and θ_v , the angle $\bar{\delta}$ can be also expressed as

$$\bar{\delta} = \frac{\sin(\theta_v - \xi)}{|\sin(\theta_v - \xi)|} \arccos \left(\frac{p - |p_0|}{e|p_0|} \right). \quad (5.37)$$

As in the case of the outer dynamics, the global good definition of the inner caustic $\Gamma_I(\xi; \epsilon, \theta(I_0))$ depends on proving that $G_I(x, y; \xi, \epsilon)$ differentiable in ξ and that both G_I and $\partial_\xi G_I$ are continuous in ϵ .

Lemma 5.24. *If $I_0 \in \mathcal{I} \setminus \bar{\mathcal{I}}$ is such that $\theta(I_0) \in \mathcal{D}$, then there is $\bar{\epsilon}^{(3)} > 0$ such that, if $|\epsilon| < \bar{\epsilon}^{(3)}$, then $G_I(x, y; \xi, \epsilon)$ is continuous in ϵ , differentiable in ξ and such that $\partial_\xi G(x, y; \xi, \epsilon)$ is continuous in ϵ .*

Proof. As in the case of Lemma 5.22, one needs to prove the desired regularity properties on the quantities p and e , as well as $\sin \bar{\delta}$ and $\cos \bar{\delta}$. As all these quantities depend on $k = |p_0 \wedge v_0|$, let us find the expression of the angular momentum as a function of ξ . As already done in §3, let us now denote with α the angle between v_0 and the inward-pointing normal unit vector to γ_ϵ in p_0 , which we indicate with $n_i(\xi)$; then referring to (3.16) and using the same notation of Lemma 5.22, we have

$$\begin{aligned} v_0 &= \sqrt{2 \left(\mathcal{E} + h + \frac{\mu}{\rho} \right)} (\sin \alpha t(\xi) + \cos \alpha n_i(\xi)) = \\ &= \frac{1}{\sqrt{\rho^2 + \rho'^2}} \left(\sqrt{2} I (\rho' \cos \xi - \rho \sin \xi) - \sqrt{2(\mathcal{E} + h + \mu/\rho - I^2)} (\rho' \sin \xi + \rho \cos \xi) \right) \\ &\quad \left(\sqrt{2} I (\rho' \sin \xi + \rho \cos \xi) + \sqrt{2(\mathcal{E} + h + \mu/\rho - I^2)} (\rho' \cos \xi - \rho \sin \xi) \right) \end{aligned}$$

And, since $p_0 = \rho e^{i\xi}$,

$$\begin{aligned} k &= |p_0 \wedge v_0| = \frac{\sqrt{2}\rho(\xi; \epsilon)}{\sqrt{\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon)}} \left(I(\xi; \epsilon)\rho(\xi; \epsilon) + \rho'(\xi; \epsilon) \sqrt{\mathcal{E} + h + \frac{\mu}{\rho(\xi; \epsilon)} - I^2(\xi; \epsilon)} \right) \\ p &= \frac{k^2}{\mu} = \frac{2\rho^2(\xi; \epsilon)}{\mu(\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon))} \left(I(\xi; \epsilon)\rho(\xi; \epsilon) + \rho'(\xi; \epsilon) \sqrt{\mathcal{E} + h + \frac{\mu}{\rho(\xi; \epsilon)} - I^2(\xi; \epsilon)} \right)^2 \\ e &= \sqrt{1 + \frac{4(\mathcal{E} + h)\rho^2(\xi; \epsilon)}{(\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon))\mu^2} \left(I(\xi; \epsilon)\rho(\xi; \epsilon) + \rho'(\xi; \epsilon) \sqrt{\mathcal{E} + h + \mu/\rho(\xi; \epsilon) - I^2(\xi; \epsilon)} \right)^2} \end{aligned}$$

The regularity of p and e is then ensured whenever $\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon) \neq 0$, which is true for ϵ small enough. As for $\sin \bar{\delta}$ and $\cos \bar{\delta}$, from (5.37) one can infer that the requested regularity is ensured if $\sin(\theta_v - \xi)$ has always the same sign on the orbit $(\xi, I(\xi; \epsilon))$. As in the case of the outer orbit, this is a consequence of the continuity of $\rho(\xi; \epsilon)$, $\rho'(\xi; \epsilon)$ and $I(\xi; \epsilon)$ with respect to ϵ . Denoting with θ_{n_i} the polar angle of $n_i(\xi)$, from the definition of α one has $\theta_v - \xi = \theta_{n_i} - \xi - \alpha$,

and then

$$\begin{aligned}
\sin(\theta_v - \xi) &= \sin(\theta_{n_i} - \xi) \cos \alpha + \cos(\theta_{n_i} - \xi) \sin \alpha = \\
&= \frac{1}{\rho(\xi; \epsilon) \sqrt{\mathcal{E} + h + \frac{\mu}{\rho(\xi; \epsilon)}}} \left(\sqrt{\mathcal{E} + h + \frac{\mu}{\rho(\xi; \epsilon)}} - I^2(\xi; \epsilon) |n_I(\xi) \wedge \gamma_\epsilon(\xi)| - I(\xi; \epsilon) \gamma_\epsilon(\xi) \cdot n_i(\xi) \right) = \\
&= \frac{\rho'(\xi; \epsilon) \sqrt{\mathcal{E} + h + \frac{\mu}{\rho(\xi; \epsilon)}} - I^2(\xi; \epsilon) + \rho(\xi; \epsilon) I(\xi; \epsilon)}{\sqrt{\rho^2(\xi; \epsilon) + \rho'^2(\xi; \epsilon)} \sqrt{\mathcal{E} + h + \frac{\mu}{\rho(\xi; \epsilon)}}}.
\end{aligned}$$

For $\epsilon = 0$, $\sin(\theta_v - \xi)|_{\epsilon=0} = I_0 / \sqrt{\mathcal{E} + h + \mu} \neq 0$, then, if ϵ is small enough, $\sin(\theta_v - \xi)$ has always the same sign of I_0 , and $\sin \bar{\delta}$, $\cos \bar{\delta}$ are differentiable in ξ and continuous in ϵ , with derivative continuous in ϵ . \square

Making use of Lemma 5.24 and following the same reasoning used in the proof of Proposition 5.23, it is possible to prove the existence of a well-defined inner caustic $\Gamma_I(\xi; \epsilon, \theta(I_0))$ related to the invariant curve for the map \mathcal{F}_ϵ with rotation number $\theta(I_0)$.

Proposition 5.25. *If $I_0 \in \mathcal{I} \setminus \bar{\mathcal{I}}$ is such that $\theta(I_0) \in \mathcal{D}$, then there exists $\bar{\epsilon}_I$ such that for $|\epsilon| < \bar{\epsilon}_I$ the caustic $\Gamma_I(\xi; \epsilon, \theta(I_0))$ is globally well defined.*

APPENDIX A. PROOFS OF THEOREMS 4.1 AND 4.2

A.1. Outer arcs. In the case of the inner arcs, the proof of the existence and uniqueness Theorem 4.1 lies on the direct computation of the solutions of problem 4.1 for fixed p_0 and p_1 . Moreover, the results obtained in the following proof are used in Section 4.3 to give the explicit expression of the outer shift in the unperturbed case.

Proof of Theorem 4.1. Fix $p_0 = e^{i\theta_0} \in \partial D_0$, and, given $\alpha \in (-\pi/2, \pi/2)$, consider the Cauchy problem

$$\begin{cases} z''(s) = -\omega^2 z(s), \\ z(0) = p_0, z'(0) = v_0 = \sqrt{2\mathcal{E} - \omega^2} e^{i\theta_0 + \alpha}, \end{cases} \quad (\text{A.1})$$

whose solution $z(s; p_0, v_0)$ is an ellipse whose parameters depend on the initial conditions and can be decoupled as

$$z(s) = (x(s), y(s)) = (p_{0,x} \cos \omega s + \frac{v_{0,x}}{\omega} \sin \omega s, p_{0,y} \cos \omega s + \frac{v_{0,y}}{\omega} \sin \omega s) \quad (\text{A.2})$$

Since $\alpha \in (-\pi/2, \pi/2)$, the orbit is exterior to D in a neighborhood of $s = 0$. Let $s_1 > 0$ the first positive instant for which $z(s_1; p_0, v_0) \in \partial D_0$ again, and define $p_1 = e^{i\theta_1} = p_1(p_0, \alpha) = z(s_1; p_0, v_0)$. As the system is invariant under rotations, the shift θ_E from θ_0 to θ_1 and s_1 depend only on the direction of v_0 with respect to the radial direction, i.e. on α . We can then fix $p_0 = \bar{p}_0 = (1, 0)$, and we have $\theta_E(\alpha) = \theta_1$. The solution $z(s; \bar{p}_0, v_0)$ simplifies as

$$z(t) = (x(s), y(s)) = \left(\cos \omega s + \frac{v_x}{\omega} \sin \omega s, \frac{v_y}{\omega} \sin \omega s \right), \quad (\text{A.3})$$

from which one has

$$r^2(s) = x^2(s) + y^2(s) = \frac{\omega^2 - \mathcal{E}}{\omega^2} \cos(2\omega s) + \frac{v_x}{\omega} \sin(2\omega s) + \frac{\mathcal{E}}{\omega^2} = A \cos(2\omega s + \bar{\alpha}) + \frac{\mathcal{E}}{\omega^2}, \quad (\text{A.4})$$

with $A \in \mathbb{R}$ and $\bar{\alpha} \in [0, 2\pi)$ such that

$$\begin{cases} A \cos \bar{\alpha} = \frac{\omega^2 - \mathcal{E}}{\omega^2}, & A \sin \bar{\alpha} = -\frac{v_x}{\omega}, \end{cases} \quad (\text{A.5})$$

and, since $v_x > 0$,

$$\begin{aligned} \cot \bar{\alpha} &= \frac{\mathcal{E} - \omega^2}{\omega v_x} \Rightarrow \bar{\alpha} = \operatorname{arccot} \left(\frac{\mathcal{E} - \omega^2}{\omega v_x} \right) \in \left(0, \frac{\pi}{2} \right), \\ \cos \bar{\alpha} &= \frac{\mathcal{E} - \omega^2}{\sqrt{\omega^2 v_x^2 + (\mathcal{E} - \omega^2)^2}}, & \sin \bar{\alpha} &= \frac{\omega v_x}{\sqrt{\omega^2 v_x^2 + (\mathcal{E} - \omega^2)^2}} \\ A &= -\frac{v_x}{\omega \sin \bar{\alpha}} = -\frac{\sqrt{\omega^2 v_x^2 + (\mathcal{E} - \omega^2)^2}}{\omega^2} < 0. \end{aligned} \quad (\text{A.6})$$

The time $s_1 > 0$ is such that $\rho(s_1) = 1$ and is given by $s_1 = (\pi - \bar{\alpha})/\omega$: if $y(s_1) \neq 0$ (namely, $\alpha \neq 0$), the polar angle θ_1 of the point p_1 is given by

$$\theta_1 = \begin{cases} \operatorname{arccot} \left(\frac{x(s_1)}{y(s_1)} \right) & \text{if } \alpha > 0, \\ \operatorname{arccot} \left(\frac{x(s_1)}{y(s_1)} \right) - \pi & \text{if } \alpha < 0, \end{cases} \quad (\text{A.7})$$

where we took into account that, for $\alpha < 0$, $\bar{\theta}_E \in [\pi, 2\pi]$, then one has to take the second determination of arccot .

Direct computations of the homotetic solution (corresponding to $\alpha = 0$) and equation (A.6), along with the definition of θ_E , lead finally to

$$\theta_E(\alpha) = \begin{cases} \theta_E^+(\alpha) = \operatorname{arccot} \left(\frac{\omega^2}{(2\mathcal{E} - \omega^2) \sin(2\alpha)} + \cot(2\alpha) \right) & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ \theta_E^-(\alpha) = \operatorname{arccot} \left(\frac{\omega^2}{(2\mathcal{E} - \omega^2) \sin(2\alpha)} + \cot(2\alpha) \right) - \pi & \text{if } \alpha < 0. \end{cases} \quad (\text{A.8})$$

If $\mathcal{E} > \omega^2$, the function $\theta_E(\alpha)$ is of class C^1 in $(-\pi/2, \pi/2)$ and assumes all the values in $(-\pi, \pi)$. Moreover,

$$\frac{d\theta_E}{d\alpha}(\alpha) = \frac{(2\mathcal{E} - \omega^2)(2\mathcal{E} - \omega^2 + \omega^2 \cos(2\alpha))}{2\mathcal{E}(\mathcal{E} - 2\omega^2) - (2\mathcal{E} - \omega^2)\omega^2 \cos(2\alpha)} > 0 \text{ for all } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (\text{A.9})$$

From the inverse function theorem, there exist a unique function $\alpha : (-\pi, \pi) \rightarrow (-\pi/2, \pi/2)$, $\theta_1 \mapsto \alpha(\theta_1)$ such that for every θ_1 we have

$$p_1 = e^{i\theta_1} = z \left(s_1(\alpha(\theta_1)); \bar{p}_0, \sqrt{2\mathcal{E} - \omega^2} e^{i\alpha(\theta_1)} \right). \quad (\text{A.10})$$

Moreover, $\alpha(\theta_1) \in C^1(-\pi, \pi)$.

Fixing now $p_0, p_1 \in D_0$ such that $|p_0 - p_1| < 2$, we have that $|\theta_0 - \theta_1| < \pi$, then problem

$$\begin{cases} z''(s) = -\omega^2 z(s), \\ z(0) = p_0, z'(0) = v_0 = \sqrt{2\mathcal{E} - \frac{\omega^2}{2}} e^{i(\theta_0 + \alpha(\theta_1 - \theta_0))}, \end{cases} \quad (\text{A.11})$$

admits the unique solution $z(s; p_0, p_1)$. If we define $T = s_1$ as above, we have that $z(T; p_0, p_1) = p_1$ and $|z(s)| > 1$ for every $s \in (0, T)$, while the energy conservation law is ensured by the choice of v_0 . Moreover, by the differentiable dependence on the initial conditions of the Cauchy problem and the fact that $\alpha(\theta_1)$ is of class C^1 , one can conclude that $z(s; p_0, p_1)$ is differentiable as a function of its endpoints. \square

A.2. Inner arcs. Unlike the outer case, the inner Kepler problem presents a singularity in the origin, which should be treated with more sophisticated strategies. To this end, techniques such as the Levi-Civita regularisation and more general results from Riemannian Geometry are used.

The Levi-Civita regularization technique consists in a change both in the temporal parameter and the spatial coordinates, in order to remove the singularity of Kepler-type potentials.

Lemma A.1. *Let $p_0, p_1 \in \mathbb{R}^2 \setminus \{0\}$. The fixed end problem*

$$\begin{cases} (HS_I)[z(s)], & s \in [0, T] \\ z(0) = p_0, \quad z(T) = p_1 \end{cases} \quad (\text{A.12})$$

is conjugated to the Levi-Civita problem

$$\begin{cases} \ddot{w}(\tau) = \Omega^2 w(\tau) & \tau \in [0, \tilde{T}] \\ \frac{1}{2} |\dot{w}(\tau)|^2 - \frac{\Omega^2}{2} |w(\tau)|^2 - E = 0, & \tau \in [0, \tilde{T}] \\ w(0) = w_0, \quad w(\tilde{T}) = w_1 \end{cases} \quad (\text{A.13})$$

with $\Omega^2 = 2(\mathcal{E} + h)$, $E = \mu$, $w_0^2 = p_0$, $w_1^2 = p_1$ and $\tau = \tau(s)$ such that $\frac{d\tau}{ds} = \frac{1}{2|z(s)|}$.

Proof. Let us consider the reparametrisation $s = s(\tau)$ such that $\frac{d}{ds} = \frac{1}{2r} \frac{d}{d\tau}$ with $r = |z(s)|$. Then, denoting with the dot the derivation with respect to τ ,

$$\frac{d^2}{ds^2} = \frac{1}{4} \left(-\frac{1}{r^3} \dot{r} \frac{d}{d\tau} + \frac{1}{r^2} \frac{d^2}{d\tau^2} \right) : \quad (\text{A.14})$$

the first and second equations in (A.12) can be expressed with respect to τ as

$$r\ddot{z} - \dot{r}\dot{z} + 4\mu z = 0, \quad \frac{1}{8r^2} |\dot{z}|^2 = \mathcal{E} + h + \frac{\mu}{r}. \quad (\text{A.15})$$

Considering now the new spatial variable in the complex plane $\mathbf{C} \simeq \mathbb{R}^2$ given by $z = w^2$, we have from (A.15) :

$$\frac{1}{2} |\dot{w}|^2 - \frac{\Omega^2}{2} |w|^2 - E = 0, \quad r\ddot{w} + w(2E - |\dot{w}|^2) = 0 \Rightarrow \ddot{w} = \Omega^2 w. \quad (\text{A.16})$$

\square

We will refer to the time variable τ as the Levi-Civita time, and to the new reference system as the Levi-Civita plane; the original time and coordinate space will be called *physical* time and plane. Moreover, following up on Notation 3.1, we will denote the first two lines of system (A.13) with the abbreviation $(HS_{LC})[w]$.

Remark A.2. From Lemma A.1 we have that, if $w(\tau; w_0, w_1)$ is a solution of (A.13), then $z(s; p_0, p_1) = (w(\tau; w_0, w_1)|_{\tau=\tau(s)})^2$ is a solution of (A.12) with endpoints $p_0 = w_0^2$ and $p_1 = w_1^2$. On the other hand, as the complex square determines a double covering of $\mathbb{R}^2 \setminus \{0\}$, if we fix $p_0 = r_0 e^{i\theta_0}, p_1 = r_1 e^{i\theta_1} \in \mathbb{R}^2 \setminus \{0\}$, with $p_0 \neq p_1$, and $z(z, p_0, p_1)$ is a solution of (A.12), we can find two distinct solutions of (A.13) such that $(w(\tau; w_0, w_1)|_{\tau=\tau(s)})^2 = z(s; p_0, p_1)$. For, there are two pairs of points in the Levi-Civita plane, namely, $w_0^\pm = \pm\sqrt{r_0}e^{i\theta_0/2}$ and $w_1^\pm = \pm\sqrt{r_1}e^{i\theta_1/2}$, such that $w_{0\setminus 1}^\pm = p_{0\setminus 1}$. Supposing that problem (A.13) admits the solutions $w(\tau; w_0^-, w_1^-)$ and $w(\tau; w_0^+, w_1^+)$, it is straightforward that $w(\tau; w_0^+, w_1^+) = -w(\tau; w_0^-, w_1^-)$ and $w(\tau; w_0^+, w_1^-) = -w(\tau; w_0^-, w_1^+)$: passing to the physical plane, we have then two distinct solutions of (A.12), given by $(w(\tau; w_0^-, w_1^-)|_{\tau=\tau(s)})^2$ and $(w(\tau; w_0^+, w_1^+)|_{\tau=\tau(s)})^2$. If instead $p_0 = p_1 = r_0 e^{i\theta_0}$, the solution given by $w(\tau; w_0^-, w_1^-)$ collapses into a single point: there is only one solution of (A.13) conjugated to $z(s; p_0, p_1)$, and it can be computed explicitly by choosing $w_0 = -\sqrt{r_0}e^{i\theta_0/2}$ and $w_1 = \sqrt{r_0}e^{i\theta_0/2}$:

$$w(\tau, w_0, w_1) = \sqrt{r_0} \sqrt{\frac{2E}{\Omega}} \sinh(\Omega(\tau - \tau_0)) e^{i\theta_0/2}, \quad \tau_0 = \frac{1}{\Omega} \operatorname{arcsinh}\left(\sqrt{\frac{\Omega}{2E}}\right) = \frac{T}{2}, \quad (\text{A.17})$$

which corresponds to an ejection-collision solution $z(s; p_0, p_0)$ parallel to the direction $e^{i\theta_0}$.

To find solutions of (A.12) we can then search for solutions of (A.13) with suitable endpoints. We will prove that the fixed ends problem in the Levi-Civita plane admits a unique solution for every pair of points in \mathbb{R}^2 : to this purpose, we need to introduce some known results from Riemannian Geometry.

The solutions of (A.13) can be seen as reparametrizations of geodesic curves which connect w_0 and w_1 in the Riemannian manifold $(\mathbb{R}^2, \tilde{g})$, with the metric \tilde{g} given by the metric tensor

$$\tilde{g}_{ij} = e^{\sigma(w)} \delta_{ij}, \quad \sigma(w) = \ln\left(\frac{\Omega^2}{2}|w|^2 + E\right) > 0 \quad (\text{A.18})$$

with $i, j \in \{1, 2\}$: if we prove the existence of a unique geodesic in (\mathbb{R}, \tilde{g}) which connects w_0 and w_1 , our claim follows straightforwardly. Once we have verified that its hypotheses hold, this will follow from the Cartan-Hadamard theorem: to retrieve its statement, along with the definition of all the involved quantities, we refer to [26] and [7]. Taking into account (A.18), one can prove that, given $w \in \mathbb{R}^2$, the sectional curvature of $T_w \mathbb{R}^2$ at w is given by

$$K(w) = -\frac{\Delta\sigma(w)}{e^{\sigma(w)}} = -\frac{\Omega^2 E}{\left(\frac{\Omega^2}{2}|w|^2 + E\right)} < 0. \quad (\text{A.19})$$

As \mathbb{R}^2 is simply connected, to apply the Cartan-Hadamard theorem one has then to verify that the conformal metric (M, \tilde{g}) is complete. In view of Hopf-Rinow Theorem, this is equivalent to show that $(M, d_{\tilde{g}})$ is complete as a metric space, where the distance $d_{\tilde{g}}(w_0, w_1)$ is defined as the infimum of all the lengths in $(M; \tilde{g})$ of piecewise C^1 curves which connect w_0 and w_1 . To prove this last assertion, we shall take advantage on the following Lemma, stated as part of Theorem 1 in [12, 13].

Lemma A.3. Let (M, g) be a non-compact and complete Riemannian manifold and $A : M \rightarrow (0, \infty)$ a positive function. We denote a conformally transformed metric on M by $\tilde{g} = g/A^2$. If

A grows at most linearly towards g -infinity on M , that is,

$$\forall x_0 \in M \exists c_1, c_2 > 0 \text{ such that } \forall x \in M \quad A(x) \leq c_1 d_g(x_0, x) + c_2, \quad (\text{A.20})$$

then (M, \tilde{g}) is complete.

In our regularised system, g is the usual Euclidean metric on \mathbb{R}^2 , then $A = \left(\sqrt{\frac{\Omega^2}{2}|w|^2 + E} \right)^{-1}$, and (A.20) is trivially verified with $c_1 = 1$ and $c_2 = 1/\sqrt{\mu}$, as

$$A(w) = \frac{1}{\sqrt{\frac{\Omega^2}{2}|w|^2 + E}} \leq \frac{1}{\sqrt{\mu}} \leq |w_0 - w| + \frac{1}{\sqrt{\mu}}$$

for every $w_0, w \in \mathbb{R}^2$. The Cauchy-Hadamard theorem can be then applied, and as a consequence the existence and uniqueness of an orbit for any pairs of points in the Levi-Civita plane can be proved.

Proposition A.4. *For every $w_0, w_1 \in \mathbb{R}^2 \exists |w(\tau; w_0, w_1)$ solution of (A.13) for some $\tilde{T} > 0$. Moreover, $w(\tau; w_0, w_1)$ is of class C^1 with respect to variations of w_0 and w_1 .*

Taking together Proposition A.4 and Remark A.2, we can then pass to the physical plane, obtaining the below existence theorem.

Theorem A.5. *For every $p_0, p_1 \in \partial D_0$ with $p_0 \neq p_1$ there are exactly two classical solutions $z_0(s; p_0, p_1)$ and $z_1(s; p_0, p_1)$ of problem (A.12) for some $T_0, T_1 > 0$, which are of class C^1 with respect to p_0 and p_1 .*

If instead $p_0 = p_1$, there is a unique solution of (A.12), which is ejection-collision.

The previous results ensure the existence and differentiability of exactly two orbits (one in the case of ejection-collision solutions) which join two different points in ∂D_0 ; on the other hand, to gain transversality properties an explicit expression of the two solutions is needed. Classical arguments of Celestial Mechanics (see [9]) allow us to obtain their Cartesian equations: let us consider the first equation of Problem (A.12), namely, $z''(s) + \frac{\mu}{|z(s)|^3} z(s) = 0$, and express it in polar coordinates, obtaining

$$\begin{cases} r''(s) - r(s)\theta'(s) = -\frac{\mu}{r^2(s)}, \\ r(s)\theta''(s) - 2r'(s)\theta'(s) = 0. \end{cases} \quad (\text{A.21})$$

The second equation express the conservation of the modulus of the angular momentum

$$k = r^2(s)\theta'(s) = |z(s) \wedge z'(s)| = |z(0) \wedge z'(0)|. \quad (\text{A.22})$$

Taking into account the uniqueness of the ejection-collision solution for $p_0 = p_1$, we can state that $k = 0 \Leftrightarrow p_0 = p_1$: assuming $p_0 \neq p_1$, we have then $k \neq 0$.

The polar equation associated to a solution of $z''(s) + \frac{\mu}{|z(s)|^3} z(s) = 0$ having energy $\mathcal{E} + h$ and angular momentum k is

$$r(f) = \frac{p}{1 + e \cos f}, \quad (\text{A.23})$$

where $f \in [0, 2\pi)$ is traditionally called *true anomaly* and

$$p = \frac{k^2}{\mu}, \quad e = \sqrt{1 + \frac{2k^2(\mathcal{E} + h)}{\mu^2}} > 1. \quad (\text{A.24})$$

With a suitable choice of a reference frame denoted with $\mathcal{R}(0, x', y')$, we can see that equation (A.23) represents the branch of particular hyperbolæ with the concavity facing the origin. More precisely, if $a = \mu / [2(\mathcal{E} + h)]$:

- if we choose $(x', y') = (r \cos f, r \sin f)$, we obtain that for every $s \in \mathbb{R}$

$$(x'(s), y'(s)) \in \mathcal{H}_1 = \{(x, y) \in \mathbb{R}^2 \mid (x - ae)^2(e^2 - 1) - y^2 = a^2(e^2 - 1), x \leq a(e - 1)\} \quad (\text{A.25})$$

- if instead $(x', y') = (-r \cos f, r \sin f)$, we have that $\forall s \in \mathbb{R}$

$$(x'(s), y'(s)) \in \mathcal{H}_0 = \{(x, y) \in \mathbb{R}^2 \mid (x + ae)^2(e^2 - 1) - y^2 = a^2(e^2 - 1), x \geq a(1 - e)\}. \quad (\text{A.26})$$

Proposition A.6. *For every $p_0 = e^{i\theta_0}, p_1 = e^{i\theta_1} \in \partial D_0, p_0 \neq p_1$, the two solutions $z_0(s; p_0, p_1)$ and $z_1(s; p_0, p_1)$ of (A.12) are such that:*

- for every $s \in [0, T_0]$

$$z_0(s; p_0, p_1) \in \mathcal{H}_0(p_0, p_1) = \{(x', y') \in \mathbb{R}^2 \mid (x' + ae_0)^2(e_0^2 - 1) - y'^2 = a^2(e_0^2 - 1), x' \geq a(1 - e_0)\},$$

- for every $s \in [0, T_1]$

$$z_1(s; p_0, p_1) \in \mathcal{H}_1(p_0, p_1) = \{(x', y') \in \mathbb{R}^2 \mid (x' - ae_1)^2(e_1^2 - 1) - y'^2 = a^2(e_1^2 - 1), x' \leq a(e_1 - 1)\},$$

where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \alpha_0 & \sin \alpha_0 \\ -\sin \alpha_0 & \cos \alpha_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \alpha_0 = \frac{\theta_1 + \theta_0}{2}, \quad (\text{A.27})$$

$$e_0 = \frac{-x_0 + \sqrt{4a^2 + 4a + x_0^2}}{2a}, \quad e_1 = \frac{x_0 + \sqrt{4a^2 + 4a + x_0^2}}{2a}, \quad x_0 = \cos \left(\frac{\theta_1 - \theta_0}{2} \right). \quad (\text{A.28})$$

Proof. As the system is invariant under rotations, it is sufficient to prove the claims for p_0, p_1 symmetric with respect to the x -axis, namely, $\bar{p}_0 = e^{-i\beta}, \bar{p}_1 = e^{i\beta}, \beta \in (0, \pi)$: any other cases can be treated as this one after a rotation of angle α_0 .

For $p_0 = \bar{p}_0$ and $p_1 = \bar{p}_1$, we have $\alpha_0 = 0$ and $x_0 = \cos \beta$: direct computations shows that e_0 and e_1 as defined in (A.28) are the only values of eccentricity such that $\bar{p}_0, \bar{p}_1 \in \mathcal{H}_0$ and $\bar{p}_0, \bar{p}_1 \in \mathcal{H}_1$. The thesis follows from the uniqueness of the two solutions of (A.12). \square

Remarks A.7. • *One can easily verify that, according to Definition 4.2, for every $p_0, p_1 \in \partial B_1(0), p_0 \neq p_1, |p_0 - p_1| < 2$,*

$$\text{Ind}(z_0[0, T_0], 0) = 0 \quad \text{and} \quad |\text{Ind}(z_1[0, T_1], 0)| = 1. \quad (\text{A.29})$$

- *For $p_1 \rightarrow p_0$, the solution $z_1(s; p_0, p_1)$ tends to the ejection-collision solution $z(s; p_0, p_0)$ in the direction of p_0 . For, $p_1 \rightarrow p_0$ implies $e_1 \rightarrow 1$ and $k_1 \rightarrow 0$.*
- *It is clear that for every $s \in (0, T_1)$ we have that $|z(s; p_0, p_1)| < 1$.*

The analytic expressions of the traces of $z_0(s; p_0, p_1)$ and $z_1(s; p_0, p_1)$ allow to derive the transversality properties of the latter; as we are interested to the solution which converges to the ejection-collision one for $p_1 \rightarrow p_0$, we will focus on the transversality of $z_1(s; p_0, p_1)$ (an analogous result can be obtained for $z_0(s; p_0, p_1)$).

Proposition A.8. *There is $0 < C < 1$ such that for every $p_0, p_1 \in \partial B_1(0)$, $|p_0 - p_1| < 2$, we have*

$$-p_0 \cdot \frac{z'_1(0; p_0, p_1)}{|z'_1(0; p_0, p_1)|} > C \quad \text{and} \quad p_1 \cdot \frac{z'_1(T_1; p_0, p_1)}{|z'_1(T_1; p_0, p_1)|} > C. \quad (\text{A.30})$$

Proof. If $p_0 = p_1$, the ejection-collision solution is orthogonal to ∂D_0 , then the claims are trivially true. Let us assume $p_0 \neq p_1$. As in Proposition A.6, it is sufficient to prove the claims for $p_0 = e^{-i\beta} = (x_0, -\sqrt{1-x_0^2})$, $p_1 = e^{i\beta} = (x_0, \sqrt{1-x_0^2})$, with $\beta \in (0, \pi/2)$. The positive branch of \mathcal{H}_1 can be parametrized as $y(x) = \sqrt{e_1^2 - 1} \sqrt{(x + ae_1)^2 - a^2}$, with e_1 as in (A.28). Writing $p_1 = p_1(x_0) = (x_0, y(x_0))$ and $v_1 = (1, \partial_x y(x_0))$, we can express the cosine of the angle between p_1 and v_1 as a function

$$c(x_0) \equiv p_1 \cdot \frac{v_1}{|v_1|} = e_0 \sqrt{\frac{2a + x_0(x_0 + \sqrt{4a^2 + 4a + x_0^2})}{2 + 4a}}, \quad (\text{A.31})$$

which is strictly increasing for $x_0 \in [0, 1]$ and such that $c(0) = \sqrt{\frac{1+a}{1+2a}} = C < 1$ and $c(1) = 1$: this prove the claim for p_1 and $s = T_1$. The same estimate holds for $-p_0$ and $s = 0$, taking into account that $z'_1(0, p_0, p_1)$ points inward the domain D . \square

Remark A.9. *The value of C depends on the physical parameters of the problem: in particular, with reference to the proof of Proposition A.8, one has*

$$c(0) = \sqrt{\frac{1+a}{1+2a}} = \sqrt{\frac{\mathcal{E} + h + \mu/2}{\mathcal{E} + h + \mu}},$$

which tends to 1 when $\mathcal{E} + h \rightarrow \infty$. As a consequence, one can control the transversality of $z_1(s; p_0, p_1)$ by acting on the value of the total inner energy. This fact is of particular importance in view of Remark 3.2, since for the first return F to be well defined one needs that the angle $\beta_1 = \angle(p_1, v_1)$ is such that

$$|\sin \beta_1| \leq \sqrt{\frac{V_E(p_1)}{V_I(p_1)}} = \sqrt{\frac{\mathcal{E} - \omega^2/2}{\mathcal{E} + h + \mu}}. \quad (\text{A.32})$$

If $\mathcal{E} + h$ is such that $\sqrt{1-C^2} < \sqrt{(\mathcal{E} - \omega^2/2)/(\mathcal{E} + h + \mu)}$, and this is true if $\sqrt{\mu/2} < \sqrt{\mathcal{E} - \omega^2/2}$, Eq.(A.32) is satisfied.

Moreover, since for $p_0 \rightarrow p_1$ the arc $z_1(s; p_0, p_1)$ tends to the ejection-collision solution, the lower bound C can be controlled also by choosing the endpoint to be close enough.

Taking together Theorem A.5, Propositions A.6 and A.8 and Remark A.7, we can finally state the final existence and uniqueness theorem for the unperturbed boundary.

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