Even Orientations of Graphs

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Abstract

A graph G is 1-extendible if every edge belongs to at least one 1-factor of G. Let G be a graph with a 1-factor F. Then an even F-orientation of G is an orientation in which each F-alternating cycle has exactly an even number of edges directed in the same fixed direction around the cycle. In this paper, we examine the structure of 1-extendible graphs G which have no even F-orientation where F is a fixed 1-factor of G. In the case of graphs of connectivity at least four and k-regular graphs for $k \geq 3$ we give a complete characterization.

1 Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted. Most of our terminology is standard and can be found in many textbooks such as [2], [10] and [19].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and denote by (u, v) an edge with end-vertices u and v in G. An *orientation* $\vec{G}$ of G is an assignment of a direction to each edge of G.

A 1-factor F of G is said to *induce* a 1-factor of a subgraph H of G if $E(H) \cap E(F)$ is a 1-factor of H. Note that we will often identify F with $E(F)$.

Let F be a 1-factor of G. Then a cycle C is said to be *F-alternating* if $|E(C)| = 2|E(F) \cap E(C)|$. In particular, each F-alternating cycle has an even number of edges. An F-alternating cycle C in an orientation $\vec{G}$ of G is *evenly (oddly) oriented* if for either choice of direction of traversal around C, the number of edges of C directed in the direction of traversal is even (odd). Since C is even, this is clearly independent of the initial choice of direction around C. Let $\vec{G}$ be an orientation of G and F be a 1-factor of G. If every F-alternating cycle is evenly oriented then $\vec{G}$ is said to be an *even F-orientation of G*. On the other hand, if every F-alternating cycle is oddly oriented then $\vec{G}$ is said to be an *odd F-orientation of G*.

An F-orientation $\vec{G}$ of a graph G is *Pfaffian* if it is odd. It turns out that if $\vec{G}$ is a Pfaffian F-orientation then $\vec{G}$ is a Pfaffian F^*-orientation for all 1-factors F^* of G (cf.[10, Theorem 8.3.2 (3)]). In this case we simply say that G is *Pfaffian*. It is well known that every planar graph is Pfaffian and that the smallest non-Pfaffian graph is the complete bipartite graph $K_{3,3}$. The Petersen graph is a further example of a non-Pfaffian graph (cf. Lemma 2.7).

The literature on Pfaffian graph is extensive and the results often profound (see [17] for a complete survey). In particular, the problem of characterizing Pfaffian bipartite graphs was posed by Pólya [15]. Little [8] obtained the first such characterization in terms of a family of forbidden subgraphs. Unfortunately, his characterization does not give rise to a polynomial algorithm for determining whether a given bipartite graph is Pfaffian, or for calculating the permanent of its adjacency matrix when it is. Such a characterization was subsequently obtained independently by McCuaig [12, 13], and Robertson, Seymour and Thomas [16]. As a special case their result gives a polynomial algorithm, and hence a good characterization, for determining when a balanced bipartite graph G with adjacency matrix A is *det-extremal* i.e. it has $|det(A)| = per(A)$. For a structural characterization of det-extremal cubic bipartite graphs the reader may also refer to [18], [11], [13] and [6].

The problem of characterizing Pfaffian general graphs seems much harder. Nevertheless, there have been found some very interesting connections in terms of *bricks* and *near bipartite graphs* (cf. e.g. [7], [10], [14], [17], [20]).

A graph G is said to be 1-*extendible* if each edge of G is contained in at least one 1-factor of G. A subgraph J of a graph G is *central* if $G - V(J)$ has a 1-factor.

A 1-extendible non-bipartite graph G is said to be *near bipartite* if there exist edges e_1 and e_2 such that $G \backslash \{e_1, e_2\}$ is 1-extendible and bipartite.

The Pfaffian property which holds for odd F-orientations does not hold for even F-orientations. Indeed, the Wagner graph W is Pfaffian, so there is an odd orientation for each 1-factor. On the other hand, it has an even F_1-orientation and no even F_2-orientation where F_1 and F_2 are chosen 1-factors of W (cf. Lemma 2.5).

Since little is known about even F-orientations, the purpose of this paper is to achieve results helpful in this context. In particular, we examine the structure of 1-extendible graphs G which have no even F-orientation where F is a fixed 1-factor of G (cf. Theorem 3.8(i)). In the case of graphs of connectivity at least four and of k-regular graphs for $k \geq 3$ we give a characterization (cf. Theorem 3.8 points (ii) and (iii)).

2 Preliminaries

In order to state our results we need some preliminary definitions and properties.

We denote by $P(u,v)$ a uv-path $(u := u_0, u_1, \ldots, u_n =: v)$ and by $P(v,u)$ a vu-path $(v := u_n, u_{n-1}, \ldots, u_1, u_0 =: u)$. Suppose that u, v and w are distinct vertices of G and that $P(u,v)$ is a uv-path and $Q(v,w)$ is a vw-path such that $V(P(u,v)) \cap V(Q(v,w)) = \{v\}$. Then $P(u,v)Q(v,w)$ denotes the uw-path formed by the *concatenation* of these paths.

Definition 2.1 *Let $\vec{G}$ be an orientation of G. We define a $(0,1)$-function $\omega := \omega_{\vec{G}}$ on the set of paths and cycles of G as follows:*

(i) For any path $P := P(u,v) = (u_0, \ldots, u_n)$

$$\omega(P) := |\{i : [u_i, u_{i+1}] \in E(\vec{G}), 0 \leq i \leq n-1\}| \,(mod 2)$$

Note that $\omega(P(u,v)) \equiv \omega(P(v,u)) + n (mod 2)$;

(ii) For any cycle $C = (u_1, \ldots, u_n, u_1)$

$$\omega(C) := |\{i : [u_i, u_{i+1}] \in E(\vec{G}), 0 \leq i \leq n-1\}| \,(mod 2)$$

where the suffixes are integers taken modulo n.

We say that ω is the orientation function associated with *$\vec{G}$.*

In other words, for each path P (or cycle C), $\omega(P)$ (or $\omega(C)$) is the parity of the number of edges oriented consistently with $\vec{G}$.

As we have already noted, if n is even then $\omega(C)$ is independent of any cyclic rotation of the vertices of G. This is not the case when n is odd and so we have a slight abuse of notation in this case. Note also that when n is even, C is evenly oriented or oddly oriented if $\omega(C) = 0$ or $\omega(C) = 1$ respectively.

Suppose that $\vec{G}$ is an even (resp. odd) F-orientation of G where F is a fixed 1-factor of G. Then the orientation function ω associated with $\vec{G}$ is said to be an *even F-function* (resp. *odd F-function*).

Observe that when C is considered as a concatenation of paths, e.g.

$$C = (P_1(u_1, u_2)P_2(u_2, u_3), \ldots, P_n(u_n, u_1))$$

then

$$\omega(C) = \sum_{i=1}^{n}(P_i(u_i, u_{i+1})) \pmod 2$$

.

Definition 2.2 *Let G be a graph with a 1-factor F. Suppose that $\mathcal{A} := \{C_1, \ldots, C_k\}$ is a set of F-alternating cycles such that each edge of G is contained in exactly an even number of elements of $\mathcal{A}$. Then $\mathcal{A}$ is said to be a* zero-sum F-set.

We say that the zero-sum F-set is respectively an even F-set *or an* odd F-set *if k is even or odd.*

The following lemma and its corollary are very useful to our purpose.

Lemma 2.3 *[1] Let G be a graph with a 1-factor F and an odd zero-sum F-set $\mathcal{C}:= \{C_1, \ldots, C_k\}$. Suppose that $C_1, \ldots, C_{k_1}$ are oddly oriented and $C_{k_1+1}, \ldots, C_k$ are evenly F-oriented in an orientation $\vec{G}$ of G. Let $k_2 := k - k_1$ and $0 \leq k_i \leq k$ $(i = 1, 2)$. Then, G cannot have an even F-orientation or an odd F-orientation if either k_1 or k_2 is odd, respectively.*

Corollary 2.4 *[1] Let G be a graph with a 1-factor F and an odd F-set. Then G cannot have both an odd F-orientation and an even F-orientation.*

The *Wagner graph W* is the cubic graph having vertex set $V(W) = \{1, \ldots, 8\}$ and edge set $E(W)$ consisting of the edges of the cycle $C = (1, \ldots, 8)$ and the chords $\{(1, 5), (2, 6), (3, 7), (4, 8)\}$, see Figure 1.

Let C_1 and C_2 be cycles of G such that both include the pair of distinct independent edges $e = (u_1, u_2)$ and $f = (v_1, v_2)$. We say that e and f are *skew relative to C_1 and*

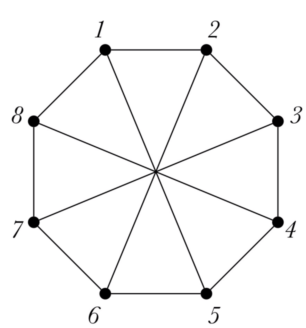


Figure 1: The *Wagner Graph W*

C_2 if the sequence (u_1, u_2, v_1, v_2) occurs as a subsequence in exactly one of these cycles. Equivalently, we may write, without loss of generality, $C_1 := (u_1, u_2, \ldots, v_1, v_2, \ldots)$ and $C_2 := (u_1, u_2, \ldots, v_2, v_1, \ldots)$ i.e. if the cycles C_1 and C_2 are regarded as directed cycles, the orientation of the pair of edges e and f occur differently.

Lemma 2.5 *Let $F_1 := \{(1,5), (2,6), (3,7), (4,8)\}$ and $F_2 := \{(1,2), (3,4), (5,6), (7,8)\}$ be 1-factors of the Wagner graph W. Set $e := (1,8)$ and $f := (4,5)$. Then the Wagner graph W satisfies the following:*

(i) W is 1-extendible.

(ii) $W - \{e, f\}$ is bipartite and 1-extendible (i.e. W is near bipartite).

(iii) W has an even F_1-orientation and an odd F_1-orientation.

(iv) W is Pfaffian.

(v) W has no even F_2-orientation.

(vi) There exists no pair of F_1-alternating cycles relative to which e and f are skew.

(vii) The edges e and f are skew relative to the F_2-alternating cycles $C_1 = (1, \ldots, 8)$ and $C_2 = (1, 2, 6, 5, 4, 3, 7, 8)$.

Proof. (i), (ii) and (vii) are easy to check.

(iii) The F_1-alternating cycles are $C_1 = (1, 2, 6, 5)$, $C_2 = (2, 3, 7, 6)$, $C_3 = (3, 4, 8, 7)$ and $C_4 = (4, 5, 1, 8)$. It is easy to check that the orientation $\vec{W}$:

$$E(\vec{W}) := \{[1,2], [2,3], [3,4], [4,5], [5,6], [6,7], [7,8], [8,1], [2,6], [1,5], [3,7], [4,8]\}$$

is an even F_1-orientation and that the orientation $\vec{W}$

$$E(\vec{W}) := \{[2,1], [2,3], [3,4], [4,5], [5,6], [6,7], [7,8], [1,8], [2,6], [7,3], [4,8], [1,5]\}$$

is an odd F_1-orientation.

(iv) As we already remarked in the introduction if W has an odd F_1-orientation then W has an odd F-orientation for every 1-factor F of G. Hence, from (iii) W is Pfaffian.

(v) The F_2-alternating cycles are:

$$C_1 = (1, 2, 3, 4, 5, 6, 7, 8)\,,\; C_2 = (1, 2, 6, 5, 4, 3, 7, 8)$$

$$C_3 = (1, 2, 3, 4, 8, 7, 6, 5)\,,\; C_4 = (3, 4, 8, 7)\,,\; C_5 = (1, 2, 6, 5)$$

It is easy to check that $\{C_1, C_2, C_3, C_4, C_5\}$ is an odd F_2-set and that $\vec{W}$ where

$$E(\vec{W}) = \{[1,2],[2,3],[3,4],[4,5],[5,6],[6,7],[7,8],[1,8],[5,1],[2,6],[7,3],[4,8]\}$$

is an odd F_2-orientation. Hence, from Corollary 2.4, W has no even F_2-orientation.

(vi) There is only one F_1-alternating cycle, namely $(4,5,1,8)$, which contains both e and f. □

Definition 2.6 *Let G be a bipartite graph with bipartition (X, Y). Set $X := \{x_1, x_2, \ldots, x_n\}$ and $Y := \{y_1, y_2, \ldots, y_n\}$. Let $F := \{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ be a 1-factor of G. Let $\vec{G}$ be the orientation of G defined by:*

$$E(\vec{G}) = \{[x_i, y_i] \mid i = 1, 2, \ldots, n\} \cup \{[y, x] \mid (y, x) \in E(G) \setminus F, x \in X, y \in Y\}$$

$\vec{G}$ is said to be the canonical F-orientation *of G. Clearly $\vec{G}$ is an even F-orientation.*

Note that if G is a bipartite graph containing a 1-factor then G has an even orientation: the canonical orientation. In this direction, the following results were shown by Carvalho, Lucchesi and Murty:

Lemma 2.7 *[4] The Petersen graph $\mathcal{P}$ has an even F-orientation for each 1-factor F of $\mathcal{P}$, but has no odd F_0-orientation, where F_0 is the prismatic 1-factor. Hence $\mathcal{P}$ is non-Pfaffian.*

Lemma 2.8 *[4] The complete bipartite graph $K_{3,3}$ has an even F-orientation but no odd F-orientation, where F is as in Defition (2.6). Hence, $K_{3,3}$ is non-Pfaffian.*

3 Main Results

As we have already said in the Introduction, since little is known about even F-orientations, the purpose of this paper is to achieve helpful results in this context. Recall that if G is a bipartite graph containing a 1-factor then G has an even orientation: the canonical orientation. We ask when graphs, not necessarily bipartite, have an even orientation. In particular, we examine the structure of 1-extendible graphs G which have no even F-orientation where F is a fixed 1-factor of G. (cf. Theorem 3.8).

However, before stating our main theorems, again, we need some additional notation.

Definition 3.1 *Let G be a graph and $H \leq G$. If G has a 1-factor F and $G \backslash V(H)$ has a 1-factor which is 1-extendible to F we say that H is F-*central.

Definition 3.2 *An* even subdivision *of a graph G is any graph G^* which can be obtained from G by replacing edges (u, v) of G by paths $P(u, v)$ of odd length such that $V(P(u, v)) \cap V(G) = \{u, v\}$.*

Note that, if F is a 1-factor of G then F induces, in a obvious way, a 1-factor F^* of G^* and conversely. For brevity, we will often blur the distinction between F and F^*.

Definition 3.3 *A graph G is said to be a* generalized Wagner graph *if*

(i) G is 1-extendible;

(ii) G has a subset $R := \{e, f\}$ of edges such that $G - R$ is 1-extendible and bipartite.

(iii) $G - R$ has a 1-factor F and F-alternating cycles C_1 and C_2 relative to which e and f are skew.

*The set of such graphs is denoted by $\mathcal{W}$. We define a $\mathcal{W}$-*factor *of $G \in \mathcal{W}$, a 1-factor of G satisfying Definition 3.3(iii).*

Remark 3.4 *(a) For example in Lemma 2.5, the Wagner graph $W \in \mathcal{W}$ and F_1 is not a $\mathcal{W}$-factor of W but F_2 is. Incidentally, it is easy to prove that if G is a cubic graph belonging to $\mathcal{W}$ with at most eight vertices then G is isomorphic to the Wagner graph. Thus the Wagner graph is the smallest graph in $\mathcal{W}$.*

(b) If we say that $G \in \mathcal{W}$ we will often assume the notation of Definition 3.3 i.e. that F is a $\mathcal{W}$-factor of G and R, C_1 and C_2 are as described in Definition 3.3(ii) and (iii) respectively.

(c) It is easy to see that Definition 3.3 implies that if $G \in \mathcal{W}$ then G is near bipartite. In particular, G is non-bipartite by Definition 3.3(iii).

Remark 3.5 *Let $G \in \mathcal{W}$. We use the notation of Definition 3.2 with G^* and F^* as defined therein. It is easy to prove that $G^* \in \mathcal{W}$ and that F^* is a $\mathcal{W}$-factor of G^*. The converse of this statement is also clearly true.*

Definition 3.6 *Let $n \geq 2$ be an integer. Let $\mathcal{W}(\leq n)$ denote the subset of $\mathcal{W}$ consisting of graphs G with maximum degree n. Moreover, we define $\mathcal{W}(n)$ to be the subset of $\mathcal{W}(\leq n)$ consisting of the graphs $G \in \mathcal{W}(\leq n)$ such that either*

(i) G is regular of degree n;

or

(ii) G is an even subdivision of such a graph (i).

Definition 3.7 *Suppose that $G \in \mathcal{W}(3)$. Then $G \in \mathcal{W}^*(3)$ if G is cubic and contains no proper central subgraph H such that H is an even subdivision of some element of $\mathcal{W}(3)$.*

Then, using this notation our main results are:

Theorem 3.8 *Let G be a 1-extendible graph containing a 1-factor F.*

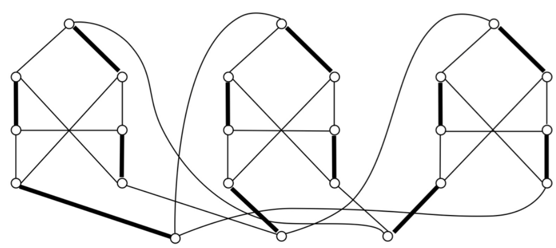
(i) Suppose that G has no even F-orientation, then G contains an F-central subgraph $H \in \mathcal{W}$ and F is a $\mathcal{W}$-factor of H.

(ii) Let $G \in \mathcal{W}$ such that $\kappa(G) \geq 4$, then G has no even F-orientation for some $\mathcal{W}$-factor F of G.

(iii) Let $G \in \mathcal{W}$ k-regular ($k \geq 3$), then G has no even F-orientation for some $\mathcal{W}$-factor F of G.

Theorem 3.9 *Let $G \in \mathcal{W}$. Suppose that G is a proper subgraph of some element of $\mathcal{W}^*(3)$. Then G is F-even.*

The proof of Theorem 3.8 is unfortunately very long. We begin by proving Theorem 3.8(i). In section 4 we discuss the structure of 1-extendible graphs (see [10]). In Sections 5 and 6 the structure of a possible minimal counterexample to Theorem 3.8(i) is examined. Then in Section 7 the proof of Theorem 3.8(i) is completed. In Section 9 we prove Theorem 3.8(ii) and (iii). Finally in Section 10 we prove Theorem 3.9.



$G \quad (F := \textbf{—})$

Figure 2: $G \notin \mathcal{W}$

Remark 3.10 *Note that the graph G in Figure 2 satisfies the conditions of Theorem 3.8(i). Such a graph G contains an F-central subgraph H where F is a $\mathcal{W}$-factor of H and of course $H \in \mathcal{W}$. However, $G \notin \mathcal{W}$.*

4 Structure of 1-extendible graphs

Let G be a 1-extendible graph A path of odd length in G whose internal vertices have degree two is called an *ear* of G. An *ear system* is a set $R = \{P_1, \ldots, P_n\}$ of vertex disjoint ears of G. Suppose that G has such an ear system. Then $G - R$ is the graph obtained from G by deleting all edges and the internal vertices of the constituent paths of R.

R is said to be *removable* if (i) $G - R$ is 1-extendible and (ii) there exists no proper subset R' of R such that $G - R'$ is 1-extendible.

Definition 4.1 *(cf. [10], [3]) Let G be a 1-extendible graph. An* ear decomposition *of G is a sequence $\mathcal{D}$ =(G_1, ..., G_r) of 1-extendible graphs G_i such that*

(i) $G_1 = K_2$, $G_r = G$;

(ii) $G_{i-1} = G_i - R_i$, for $2 \leq i \leq r$, where R_i is a removable ear system.

Theorem 4.2 *[10, Theorem 5.4.6] Let G be a 1-extendible graph and $\mathcal{D}$ =(G_1, ..., G_r) be an ear decomposition of G with $G_{i-1} = G_i - R_i$, for $2 \leq i \leq r$, where R_i is a removable ear system. Then, for each i, R_i has at most two ears.* □

We say that an ear system of size 1, size 2 is respectively a *single*, *double* ear. If $R = \{P\}$ is a removable single ear and P has length one with $E(P) = \{e\}$, then e is said to be a *removable edge*. If $R = \{P_1, P_2\}$ is a removable double ear and P_i has length one, $E(P_i) = \{e_i\}$, $i = 1, 2$, then $\{e_1, e_2\}$ is said to be a *removable doubleton*.

Definition 4.3 *Let F be a 1-factor of a 1-extendible graph G. Let $\mathcal{D}$ =(G_1, ..., G_r) be an ear decomposition of G such that $F_i := E(F) \cap E(G_i)$ is a 1-factor of G_i, $i = 1, \ldots, r$. Then $\mathcal{D}$ is said to be an F-*reducible *ear decomposition.*

Proposition 4.4 *Let F be a 1-factor of a 1-extendible graph G. Then there exists an F-reducible ear decomposition $\mathcal{D}$ =(G_1, ..., G_r) of G with $G_{i-1} = G_i - R_i$, where R_i is either a removable single ear or a removable double ear, $i = 2, \ldots, r$.*

Proof. We may assume that G is connected. $\mathcal{D}$ is constructed inductively.

Let $G_1 = K_2$ where $E(K_2) \subseteq E(F)$. Now suppose that for a fixed k, $2 \leq k \leq r$, there exists a sequence $\mathcal{D}_k$= (G_1, ..., G_k) of subgraphs G_i of G such that, for $2 \leq i \leq k$,

(i) $G_{i-1} = G_i - R_i$, where R_i is a removable ear system.

(ii) F_i is a 1-factor of G_i where $E(F_i) = E(F) \cap E(G_i)$.

Suppose that $G_k \neq G$. Select, if possible, e to be an edge of G which has exactly one end-vertex in G_k. Since G is 1-extendible there exists a 1-factor M of G which contains e. Adjoin to G_k the set R'_{k+1} of paths contained in $(M \backslash E(G_k)) \cup (F \backslash F_k)$. there exists at least one such path: the path containing e. Set $G'_{k+1} := \bigcup R'_{k+1}$. Then G'_{k+1} is 1-extendible since $F \cap E(G'_{k+1})$ and $M \cap E(G'_{k+1})$ are both 1-factors of G'_{k+1}. Now choose $R_{k+1} \subseteq R'_{k+1}$ so that R_{k+1} is removable. Again $F_{k+1} := E(F) \cap E(G_{k+1})$ is a 1-factor of G_{k+1}. Thus, by induction, $\mathcal{D} =(G_1, \ldots, G_r)$ is an ear decomposition of G with $G_{i-1} = G_i - R_i$, where R_i is a removable ear system, for $i = 2, \ldots, r$. Hence, from Theorem 4.2, R_i has at most two ears. Finally if e cannot be chosen with exactly one end in G_k then choose it so that e has both ends in G_k, and the proof then continues exactly as in the former case. □

Definition 4.5 *(i) Let G be a graph and $X \subseteq V(G)$. Let $\Delta(X)$ denote the set of edges with one end in X and the other in $V(G)\backslash X$. A* cut *in G is any set of the form $\Delta(X)$ for some $X \subseteq V(G)$.*

*(i) Suppose that G contains a 1-factor F. A cut $\Delta(X)$ is F-*tight *if $|\Delta(X) \cap F| = 1$. A cut is* tight *if it is F-tight for all 1-factors F of G. Let G be a graph G with a 1-factor and $v \in V(G)$, then every cut $\Delta(\{v\})$ in G is tight. These tight cuts are called* trivial *while all the other tight cuts are called* non-trivial*.*

(ii) Let $\Delta(X)$ be a non-trivial F-tight cut in a graph G where F is a 1-factor of G. Let G_1 and G_2 be obtained from G by identifying respectively all the vertices in X and all the vertices in $\bar{X} := V(G)\backslash X$ into a single vertex and deleting all resulting parallel edges. We say that G_1 and G_2 are the shores *of $\Delta(X)$. We denote by F_i the* 1-factor of G_i induced by F *(i=1,2).*

We now describe the Lovász [9] decomposition of 1-extendible graphs (cf. also [3]). Trivially we have:

Lemma 4.6 *[10],[3] Let $\Delta(X)$, $X \subseteq V(G)$ be a cut in a 1-extendible graph G. Then*

(i) if F is a 1-factor of G, F induces a 1-factor of both of the shores of $\Delta(X)$;

(ii) if $\Delta(X)$ is a tight cut then both of the shores of $\Delta(X)$ are 1-extendible. □

Definition 4.7 *A* brace *(respectively a* brick*) is a connected bipartite (respectively a connected non-bipartite) 1-extendible graph that has no non-trivial tight cuts.*

A Petersen brick *is a multigraph whose undelying simple graph is the Petersen graph.*

Definition 4.8 *A graph G is* bicritical *if G contains at least one edge and $G - u - v$ has a 1-factor for every pair of distinct vertices u and v in G.*

Lemma 4.9 *[5] Let G be a non-bipartite graph with at least four vertices. Then G is a brick if and only if G is 3-connected and bicritical.* $\square$

Let G be a 1-extendible graph with a non-trivial tight cut then, from Lemma 4.6, its two shores G_1 and G_2 are 1-extendible and both are smaller than G. If either G_1 or G_2 has a non-trivial tight cut this procedure can be repeated. The procedure can be repeated until a list of graphs which are either bricks or braces is obtained. This is known as the *tight cut decomposition procedure.*

Lemma 4.10 *[9], [3] Any two applications of the tight cut decomposition procedure yields the same list of bricks and braces, except for multiplicities of edges.* $\square$

Lemma 4.11 *[9], [3] Let G be a brick. If R is a removable doubleton then $G - R$ is bipartite.* $\square$

Recall that Tutte's 1-factor theorem states that *a graph G has a 1-factor if and only if $c_0(G - S) \leq |S|$ for every subset S of $V(G)$, where $c_0(G - S)$ denotes the number of odd components of $G - S$* (cf. e.g. [2]). A set $S \subseteq V(G)$ is said to be a *barrier* of G if $c_0(G - S) > |S|$. The empty set and singletons are said to be *trivial* barriers.

Lemma 4.12 *[3, Theorem 1.5, Corollary 1.6]*

(i) Let G be a connected graph which contains a 1-factor. Then G is 1-extendible if and only if, for every non-empty barrier B of G, $G - B$ has no even components and no edge has both ends in B.

(ii) Every connected 1-extendible graph is 2-connected. $\square$

Definition 4.13 *(i) Suppose that B is a non-trivial barrier in a connected graph G. Suppose that H is a non-trivial odd component of $G - B$. Then $\Delta(V(H))$ is said to be a* barrier cut.

(ii) Let $\{u, v\}$ $(u \neq v)$ be a non-trivial barrier, 2-separation of a connected graph G . Let $G := G_1 \cup G_2$ where $G_1 \cap G_2 =< u, v >$ (i.e. the subgraph of G induced by u and v). Then $\Delta(V(G_i) - u)$, $\Delta(V(G_i) - v)$ are tight cuts. Such cuts are said to be 2-separation cuts *($G - \{u, v\}$ has exactly 2 components).*

Lemma 4.14 *[5], [3, Theorem 1.12] Suppose that G is a connected 1-extendible graph which contains a non-trivial tight cut. Then G has either a non-trivial barrier cut or a 2-separation cut.* $\square$

5 The structure of minimal counterexamples to Theorem 3.8(i)

Let G_0 be such that

(i) G_0 is a 1-extendible graph.

(ii) G_0 has no even F-orientation for some 1-factor F of G_0.

(iii) G_0 contains no F-central subgraph H such that $H \in \mathcal{W}$.

(iv) G_0 is as small as possible subject to (i), (ii) and (iii).

Then, if G_0 exists, it is a smallest counterexample to Theorem 3.8.

Lemma 5.1 *Let G_0 be a smallest counterexample to Theorem 3.8. Then G_0 is a non-bipartite graph and it is either* 3*-connected or each* 2*-separation is a barrier.*

Proof. G_0 is non-bipartite since otherwise G_0 has the canonical even F-orientation (see Definition 2.6).

By minimality G_0 is connected and, from Lemma 4.12(ii), G_0 is 2-connected.

Assume that G_0 has a 2-separation $\{u, v\}$ which is not a barrier. Write $G_0 := G_1 \cup G_2$ where $G_1 \cap G_2 := \{u, v\}$. Notice that, by definition, $|V(G_1)| = |V(G_2)| \equiv 0 \pmod 2$, and that G_1 and G_2 are both 1-extendible.

Let f_1 and f_2 be the edges of F incident with u and v respectively. There are two cases to consider:

Case *(i)*: $f_1 = f_2$.

Let $F_i := F \cap E(G_i)$. Then F_i is a 1-factor of G_i $(i = 1, 2)$. For $i = 1, 2$ assume that G_i has an even F_i-orientation $\vec{G_i}$ with associated even functions $\omega_i := \omega_{\vec{G_i}}$. We choose $\vec{G_i}$ so that $\omega_1(u, v) = \omega_2(u, v)$: this is possible since, if necessary, one can reverse all the orientations in, say, $\vec{G_1}$. Since $\{u, v\}$ is a 2-separation, $\vec{G_1}$ and $\vec{G_2}$ together induce an even F-orientation of G_0 with associated even function $\omega_1 \cup \omega_2$. This contradicts the definition of G_0.

Hence, without loss of generality, we may assume that G_1 has no even F_1-orientation. By the minimality of G_0, G_1 has an F_1-central subgraph H such that $H \in \mathcal{W}$. Then, it follows that H is an F-central subgraph of G_0 such that $H \in \mathcal{W}$. Again a contradiction by the minimality of G_0.

Case *(ii)*: $f_1 \neq f_2$.

Without loss of generality, we may assume that $f_1, f_2 \in E(G_1)$. Set

$$G_i^* := \begin{cases} G_i & \text{if } (u,v) \in E(G_0) \\ G_i + (u,v) & \text{if } (u,v) \notin E(G_0) \end{cases}, \; i = 1,2\,.$$

Then, again, since G_0 is 1-extendible and $\{u,v\}$ is a 2-separation, G_i^* is 1-extendible $(i = 1,2)$.

Set $F_1 := F \cap E(G_1)$ and $F_2 := F \cap E(G_2) \cup \{(u,v)\}$. Now assume that G_i^* has an even F_i-orientation $\vec{G_i^*}$ with associated even function ω_i $(i = 1,2)$. Reversing orientations as in Case (i), if necessary, we may assume that $\omega_1(u,v) = 1$ and $\omega_2(u,v) = 0$.

Suppose that C is any F-alternating cycle of G_0 such that C is not contained in G_i^* $(i = 1,2)$. Then u and v are both vertices of C since $\{u,v\}$ is a 2-separation. Hence

$$C := (P_1(u,v), P_2(v,u))\,,$$

where P_i is an F_i-alternating path in G_i $(i = 1,2)$.

Again C induces F_i-alternating cycles C_i in G_i^* where

$$C_1 := (u, P_1(u,v), v)$$

$$C_2 := (v, P_2(v,u), u)$$

and $\omega_i(C_i) = 0$, $i = 1,2$. Hence, setting $w := \omega_1 \cup \omega_2$,

$$\omega(C) = \omega_1(P_1(u,v)) + \omega_2(P_2(v,u)) =$$

$$= (\omega_1(P_1(u,v)) + \omega_1(v,u)) + (\omega_2(P_2(v,u)) + \omega_2(u,v)) =$$

$$= \omega_1(C_1) + \omega_2(C_2) = 0.$$

On the other hand, if C is contained in G_i^*, for some i, then $\omega(C) = \omega_i(C) = 0$ $(i = 1,2)$. In all cases $\omega(C) = 0$. Hence G_0 has an even F-orientation which is not the case.

Therefore, from cases (i) and (ii), we deduce that, for some $i = 1,2$, G_i^* has no even F_i-orientation.

Firstly assume that G_i^* has no even F_1-orientation. Then, by minimality, G_1^* has an F_1-central subgraph H_1 such that $H_1 \in \mathcal{W}$. Then, except in the case when $(u,v) \in E(H_1)$ and $(u,v) \notin E(G_0)$, H_1 is an F-central subgraph of G_0 such that $H_1 \in \mathcal{W}$. In the exceptional case, we replace $(u,v) \in E(H_1)$ by an F_2-alternating path $P(u,v)$ in G_2 to obtain an even subdivision H_1^* of H_1 such that H_1^* is an F^*-central subgraph of G_0 and $H_1^* \in \mathcal{W}$. Hence, using Definition 3.1 and Definition 3.2, again, in all cases minimality is contradicted.

Finally assume that G_1^* has an even F_1-orientation and G_2^* has no even F_2-orientation. The argument is almost identical as above but in the exceptional case when (u,v) is, by definition, in F_2, and $(u,v) \notin E(G_0)$. Now as above G_2^* has an F_2-central subgraph H_2 such that $H_2 \in \mathcal{W}$. We replace (u,v) in H_2 by an F_1-alternating path in G_1 to obtain an even subdivision H_2^* of H_2 such that H_2^* is an F^*-central subgraph of G_0 (see Definition 3.2) and $H_2^* \in \mathcal{W}$. Again minimality is contradicted.

Hence, if G_0 is not 3-connected each 2-separation is a barrier. $\square$

In the next lemma and subsequently, we use the notation of Definition 3.1 and Definition 3.2. We need the following definition:

Definition 5.2 *Let $e_0 \in E(G)$, we say that $e \in E(G)$ is* e_0-bad *if for all 1-factors L of G that contain e, L contains e_0. Thus e_0 itself is* e_0-bad.

Lemma 5.3 *Let $G \in \mathcal{W}$ and F be a $\mathcal{W}$-factor of G. Then G contains an F-central subgraph H such that $H \in \mathcal{W}(\leq 3)$. Moreover H is isomorphic to an even subdivision of K_4.*

Proof. We may assume that G is connected. Suppose firstly that $G \in \mathcal{W}(3)$. Without loss of generality $G - \{e, f\}$ is bipartite, with vertex bipartition $\{X, Y\}$ and e and f are skew relative to F-alternating C_1 and C_2. Set $e := (x_1, x_2)$ and $f := (y_1, y_2)$ where $x_i \in X$, $y_i \in Y$ $(i = 1, 2)$. Set

$$\begin{aligned} C_1 &= (x_1, x_2, P_2(x_2, y_2), P_1(y_1, x_1)) \\ C_2 &= (x_1, x_2, Q_2(x_2, y_1), Q_1(y_2, x_1)) \end{aligned}$$

Then we may choose $a_1, a_2 \in P_1$ and $b_1, b_2 \in P_2$ such that $Q_1(b_1, a_1)$ and $Q_2(b_2, a_2)$ are internally disjoint from C_1. Notice that $a_2, b_1 \in X$ and $a_1, b_2 \in Y$. Now if $a_1 < a_2$ in $P_1(y_1, x_1)$ and $b_2 > b_1$ in $P_2(x_2, y_2)$ (or if $a_2 < a_1$ in $P_1(y_1, x_1)$ and $b_2 < b_1$ in $P_2(x_2, y_2)$) then $C_1 \cup Q_1(a_1, b_1) \cup Q_2(b_2, a_2)$ gives the required H. So now assume that these cases do not arise.

Hence, without loss of generality, we may assume that $a_2 < a_1$ in $P_1(y_1, x_1)$ and $b_2 < b_1$ in $P_2(x_2, y_2)$ and furthermore that b_1 and b_2 are chosen so that

- $b_1 \in Q_1(y_2, x_1) \cap P_2(y_2, x_2)$ and subject to this choice b_1 is as large as possible in $Q_1(y_2, x_1)$ and
- $b_2 \in Q_2(x_2, y_1) \cap P_2(x_2, y_2)$ and subject to this choice b_2 is as large as possible in $Q_2(x_2, y_1)$.

Now choose y in $P_1(y_1, x_1)$ so that

(i) $y \in Q_1(y_2, x_1)$

(*ii*) if $v > y$ in $P_1(y_1, x_1)$, $v \notin Q_2(x_2, y_1)$

(*iii*) from (*i*) and (*ii*), y is as small as possible in $P_1(y_1, x_1)$.

Then choose $x \in Q_2(x_2, y_1) \cap P_1(y_1, x_1)$ so that $x < y$ in $P_1(y_1, x_1)$ and x is as large as possible.

Note that by choice $x \in X$, $y \in Y$ and $P_1(x, y)$ is internally disjoint from $Q_1 \cup Q_2$. Again $P_2(b_1, b_2)$ is internally disjoint from $Q_1 \cup Q_2$. Set

$$C_1^* := (x_1, P_2(x_2, b_2), Q_2(b_2, y_1), P_2(y_2, b_1), Q_1(b_1, x_1)),$$

and this case is symmetric to the one already studied with C_1^*,$P_1(x, y)$ and $P_2(b_1, b_2)$ taking respectively the roles of C_1, $Q_1(a_1, b_1)$ and $Q_2(b_2, a_2)$. Notice that now $b_2 < x$ in $Q_2(b_2, y_1)$ and $b_1 < y$ in $Q_1(b_1, x_1)$, $b_1 \in X$, $b_2 \in Y$. This give the required H.

Assume now that G contains a vertex u with $deg(u) \geq 4$. Since $deg(u) \geq 4$ there exists $e_0 := (u, v) \in E(G)$ such that $e_0 \notin C_1 \cup C_2 \cup F$. Since G is 1-extendible, $e_0 \in L_0$ for some 1-factor L_0 of G.

Let H be the graph obtained from G by deleting all e_0-bad edges. We show that $G^* \in \mathcal{W}$ and F^* is a $\mathcal{W}$-factor of H (see Definition 3.1 and Definition 3.2).

Step 1: $C_1 \cup C_2 \subseteq H$.

Let $e \in E(C_1 \cup C_2)$. If $u \in V(C_1 \cup C_2)$ then e is contained in a 1-factor L such that $e_0 \notin L$. So now suppose that $u \notin V(C_1 \cup C_2)$. If $e \in F$, then e is not e_0-bad, since $e_0 \notin F$. Thus, w.l.o.g , we may assume that $e \in E(C_1)$ and $e \notin F$. Let F_0 be the 1-factor derived from F by changing the "colours" of $E(C_1)$. Since $u \notin V(C_1 \cup C_2)$, $e_0 \in F_0$, and e is not e_0-bad.

Step 2: $H \in \mathcal{W}$.

Trivially C_1 and C_2 are skew relative to e and f in H since they are skew relative to e and f in G. Furthermore, since $C_1 \cup C_2 \subseteq H$, $H - \{e, f\}$ is bipartite.

Suppose $e \in E(H)$. Then e is not e_0-bad and hence there exists a 1-factor L of G such that $e \in L$ and $e_0 \notin L$. This, in turn, implies that each edge of L is not e_0-bad. Thus L is a 1-factor of H. Hence H is 1-extendible. Thus $H \in \mathcal{W}$, F^* is a $\mathcal{W}$-factor of H and $deg_H(u) = deg_G(u) - 1$.

The thesis follows on repetition, if necessary, of this argument. □

Theorem 5.4 *Let G_0 be a minimal counterexamples to Theorem 3.8. Then G_0 is 3-connected.*

Proof. Assume that G_0 is not 3-connected. Then, from Lemma 5.1, G_0 has a barrier $B = \{u, v\}$, $u \neq v$. Let H_1 and H_2 be the odd components of $G_0 - B$. From Lemma 4.12,

$G_0 - B$ has no even components and $(u, v) \notin E(G_0)$. Since G_0 is non-bipartite at least one of H_1 and H_2 is non-trivial. So assume that H_1 is non-trivial and suppose that (u, x_1), $(v, y_1) \in E(F)$, $x_1 \in V(H_1)$, $y_1 \in V(H_2)$. Write $X_i := V(H_i)$, $i = 1, 2$. Let G_1 and G_2 be the shores of $\Delta(X_i)$ (cf. Definition 4.5) where G_1 is obtained by contracting the vertices of $V(G_2)\backslash X_1$ to a vertex x and G_2 is obtained contracting the vertices of $V(G_0)\backslash X_2$ to a vertex y.

Set $F_1 := (F \cap E(H_1)) \cup \{(x, x_1)\}$ and $F_2 := (F \cap E(H_2)) \cup \{(y, y_1)\}$. Clearly F_i is a 1-factor of G_i $(i = 1, 2)$. From Lemma 4.6 both G_1 and G_2 are 1-extendible.

Since G_0 has no even F-orientation for some $i = 1, 2$, G_i has no even F_i-orientation. Indeed, suppose that G_i has an even orientation $\vec{G_i}$ with even F_i-orientation function ω_i, $i = 1, 2$. Set $K_1 := \Delta(X) = \{(x_i, x) : i = 1, \ldots, k_1\}$ and $K_2 := \Delta(Y) = \{(y_i, y) : i = 1, \ldots, k_2\}$. Moreover, suppose that C is an F-alternating cycle of G_0 such that (x_1, u) and (y_1, v) are edges of C. Then $C := (P_1(x_1, x_i), v, P_2(y_1, y_j), u)$, $2 \le i \le k_1$, $2 \le j \le k_2$, where P_i is and F_i-alternating path in H_i $(i = 1, 2)$.

We define an F-alternating function ω for G_0 as follows:

(i) if $(a, b) \in E(H_i)$ then $\omega(a, b) = \omega_i(a, b)$, $i = 1, 2$;

(ii) for edges of $E(G_0)\backslash E(H_1) \cup E(H_2)$ define

(1) $\quad \omega_1(x_i, x, x_1) + \omega_2(y_1, y, y_j) := \omega(x_1, u, y_j) + \omega(y_1, v, x_i)$.

Then, by definition of C, and using (1):

(2) $\quad \omega(C) = \omega(P_1(x_1, x_i)) + \omega(x_i, v, y_1) + \omega(P_2(y_1, y_j)) + \omega(y_j, u, x_1)$

$= \omega_1((P_1(x_1, x_i)) + \omega_1(x_i, x, x_1) + \omega_2(P_2(y_1, y_j)) + \omega_2(y_j, y, y_1)$

$= \omega_1(D_1) + \omega_2(D_2)$

where D_i is an F_i-alternating cycle in G_i. Hence $\omega(C) \equiv 0 \pmod 2$.

By (i) if C is an F-alternating cycle of G_0 not containing (x_1, u) or (y_1, v) then $\omega(C) \equiv 0 \pmod 2$.

This ends the proof that, since G_0 has no even F-orientation for some $i = 1, 2$, G_i has no even F_i-orientation.

Thus, we may assume that, say G_1, has no even F_1-orientation.

By the minimality of G_0, G_1 contains an F_1-central subgraph H such that $H \in \mathcal{W}$ and F_1 is a $\mathcal{W}$-factor of H. If $x \notin V(H)$ then G_0 contains H and H is central in G_0 and F is a $\mathcal{W}$-factor of H, thus contradicting the minimality of G_0. Hence $x \in V(H)$. By Lemma 5.3, we may assume that $2 \le deg_H(x) \le 3$.

Assume that $deg_H(x) = 3$ and $(x, x_i) \in E(H)$, $i = 1, 2, 3$. We may assume, without loss of generality, that either

$$(i)\, (u, x_1),\ (u, x_2),\ (v, x_3) \in E(G_0)$$

or

$$(ii)\,(u,x_1),\,(v,x_2),\,(v,x_3)\,\in\,E(G_0)$$

otherwise H again would contradict the minimality of G_0.

We consider case (i). Let L be a 1-factor of G_0 containing (v,x_3). Now replace the edge (x,x_3) in H by the path $P_1(u,x_3)$ contained in $F\cup L$ (disjoint from H_1) to again obtain a subgraph H^* of G_0 with the required properties. In case (ii), Let L be a 1-factor of G_0 containing (v,x_3). Now replace the edge (x,x_3) in H by the path $P_2(v,x_3)$ contained in $F\cup L$ (disjoint from H_1) to again obtain a subgraph H^* of G_0 with the required properties. Finally if $deg_H(x)=2$ then the proof of the existence of H^* is exactly the same as for case (ii).

In all cases we have a contradiction with the minimality of G_0. Hence G_0 is 3-connected. □

Lemma 5.5 *Suppose that G is a non-bipartite 1-extendible graph with a barrier cut B. Let $H_1,H_2,\ldots,H_n$ $(n\geq 2)$ be the odd components of $G-B$. Suppose that G has no even F-orientation where F is a 1-factor of G. Set $X_i:=V(H_i)$ and G_i to be the shore of $\delta(X_i)$ obtained by contracting $\overline{X_i}$ to a vertex y_i. Set $\delta(X_i)\cap F:=\{a_i,b_i\}$ where $a_i\in X_i$. Set $F_i:=(F\cap E(H_i))\cup\{a_i,y_i\}$. Then, for some i, $1\leq i\leq n$, G_i has no even F_i-orientation.*

Proof. The proof follows by induction, using the argument obtained in the proof of Theorem 5.4. □

Theorem 5.6 *Let G_0 be a minimal counterexample to Theorem 3.8(i). Then G_0 is a non-Petersen brick.*

Proof. By Lemma 2.7 G_0 is not the Petersen graph. By Lemma 5.1 and Theorem 5.4, G_0 is 3-connected and not bipartite. Now suppose that G_0 is not a brick. Then, by definition, G_0 has a non-trivial tight cut. Hence, by Lemma 4.14, G_0 has a barrier cut. So by Lemma 4.12 there exists a barrier B with odd components H_1, …, H_n $(n\geq 2)$ of G_0-B such that there are no even components and $E(B)=\emptyset$. Since G_0 is non-bipartite, using Lemma 5.5 and also its notation, w.l.o.g. we assume that H_1 is non-trivial and that G_1 has no even F_1-orientation. Therefore, by minimality, G_1 has a central subgraph H such that F_1 induced a 1-factor and H is an even subdivision of some graph in W. As in the proof of Theorem 5.4, using Lemma 5.3, we may also assume that $y_1\in V(H)$, $2\leq deg_H(y_1)\leq 3$ and $(y_1,a_1)\in E(H)$.

Firstly assume that $deg_H(y_1)=3$. Set $N_H:=\{x_{11},x_{12},x_{13}\}$ where $x_{11}=a_1$. Set $g_i:=(x_{1i},b_i)$ $i=1,2,3$ where $x_{11}=a_1$ and $g_1\in F$ (recall that F is a 1-factor of G_0). Up to relabelling we may set $B:=\{b_1,\ldots,b_n\}$. Write G_0^* for the multigraph obtained from G_0 by contracting each X_i to a single vertex x_i. Clearly G_0^* is a bipartite graph having the 1-factor $F^*:=\{(x_i,b_i)|i=1,\ldots,n\}$ induced by F. Let L_i be a 1-factor of G_0

which contains g_i, where $L_1 \equiv F$. Notice that, since B is a barrier cut, $|L_i \cap \Delta(X_i)| = 1$, $i = 1, \ldots, n$; $j = 1, 2, 3$. Set $g_i^* := (x_1, b_i)$, $i = 1, 2, 3$. Then, L_i induces naturally a 1-factor L_i^* of G_0^* which contains g_i^*, $i = 1, 2, 3$. Let $P_j := P_j(b_j, b_1)$ be the $b_j b_1$-path in $L_j^* \cup F^*$ (with first edge in F^*), $j = 2, 3$. Since $b_1 \in P_2 \cap P_3$, $P_2 \cap P_3 \neq \emptyset$. Now choose $u \in V(G_0^*)$ as follows:

(i) $u \in P_2 \cap P_3$;

(ii) $V(P_3(b_3, u) \cap P_2) = \{b_1, u\}$, (possibly $b_1 = u$).

By construction, $u \in B$ and there exist three internally disjoint F^*-alternating paths $Q_j^* := Q_j^*(u, b_j)$, $j = 1, 2, 3$ in G_0^* each of which has even length. Then, in G_0, we can construct three internally disjoint F-alternating paths $Q_j := Q_j(u, b_j)$ from Q_j^*, $j = 1, 2, 3$ as follows, suppose that $R_j^* := (y_1, x_i, y_2)$ is the subpath of Q_j^* containing x_i for some i, $1 \leq i \leq n$. We may assume that $(y_1, x_i) \in F^*$ and $(x_i, y_2) \in L_j^*$. Then there exist x_i, and x_{i2} in $V(H_i)$ such that $(y_1, x_{i1}) \in F$ and $(x_{i2}, y_j) \in L_j$. In Q_j^* we replace R_j^* by the path $(y_1, R(x_i, x_{i2}), y_2)$ where R is the $x_{i1}x_{i2}$-path contained in $(F \cup L_j) \cap E(H_i)$, $j = 1, 2$. Each of the paths P_j^* if of even length. So in this way, by iteration, we obtain the required paths $Q_j(u, b_j)$, $j = 1, 2, 3$. It follows that the graph H_0 defined by:

$$V(H_0) \;=\; (V(H) \backslash \{y_1\}) \cup \{u\}\,,$$

$$E(H_0) \;=\; E(H - y_1) \cup Q_1 \cup Q_2 \cup Q_3\,,$$

is a central subgraph of G_0 such that F induces a 1-factor of H_0 and $H_0 \in \mathcal{W}$.

We have assumed, for the sake of clarity, that if $B^* = \{b_1, b_2, b_3\}$ then $|B^*| = 3$. There is nothing to prove if $|B^*| = 1$ since H is already contained in G_0. If $|B^*| = 2$ the argument is contained in the case $|B^*| = 3$.

We observe that in all cases H_0 is contained in G_0 which contradicts the minimality of G_0. Hence G_0 is a non-Petersen brick. □

In the next theorem we use the notation of Definition 4.1 and 4.3:

Theorem 5.7 *Let G_0 be a minimal counterexample to Theorem 3.8(i). Then G_0 has an F-reducible ear decomposition $\mathcal{D} =(G_1, \ldots, G_n)$, ($n \geq 2$; $G_0 = G_n$), such that G_i has an even F_i-orientation ($i = 1, \ldots, n-1$) and either:*

(i) $G_{n-1} = G_0 - R$, where $R = \{e\}$ is a removable edge

or

(ii) $G_{n-1} = G_0 - R$, where $R = \{e_1, e\}$ is a removable doubleton and G_{n-1} is bipartite.

Proof. From Proposition 4.4 G_0 has an F-reducible ear decomposition $\mathcal{D} = (G_1, \ldots, G_n)$ with $G_n = G_0$ and $G_{i-1} = G_i - R_i$ where R_i is either a removable single ear or a removable double ear. Recall that $F_i = F \cap E(G_i)$. trivially G_1 $(= K_2)$ has an even F_1-orientation. Choose i, $1 \leq i \leq n$, as large as possible, so that G_i has an even F_i-orientation. By the minimality of G_0, $i = n - 1$. Since G_0 is a brick (see Theorem 5.6), G_0 is bicritical (cf. Lemma 4.9). Hence, R is either a removable edge or a removable doubleton. From Lemma 4.11, since G_0 is a brick, if R is a removable doubleton then $G_{n-1} = G_0 - R$ and G_{n-1} is bipartite. □

Remark 5.8 *In the next section, we get rid of case* (*i*) *of Theorem 5.7. Then we will be very close to proving the main Theorem 3.8(i).*

6 Theorem 5.7, Case (i)

We assume throughout this section that G_0 is a minimal counterexample to Theorem 3.8(i) and that G_0 has an F-reducible ear decomposition $\mathcal{D} = (G_1, \ldots, G_n)$, $(n \geq 2,\ G_0 = G_n)$ such that G_i has an even F_i-orientation $(i = 1, \ldots, n-1)$ and $G^* := G_{n-1} = G_0 - R$ where $R = \{e\}$ is a removable edge, i.e. we assume that Case (i) of Theorem 5.7 is true.

We now examine the structure of G_0 in even more detail and via a series of lemmas derive a contradiction. Our proof imitates the proof of [8, Theorem 1].

Let $\vec{G^*}$ be an even F-orientation of G^* with associated even F-function ω and let $e := (u, v)$.

Lemma 6.1 *There exist F-alternating paths $Q_1 := Q_1(u, v)$, $Q_2 := Q_2(u, v)$ in G^* such that $\omega(Q_1) \neq \omega(Q_2)$. Moreover, the first and last edges of Q_i $(i = 1, 2)$ belong to F.*

Proof. Since $\vec{G^*}$ is an even F-orientation if no such paths Q_1 and Q_2 exist, a suitable orientation of e would yield an even F-orientation of G_0.

Since $e \notin F$, the first and last edges of Q_i $(i = 1, 2)$ must belong to F. □

Lemma 6.2 *The F-alternating paths Q_1 and Q_2 may be chosen in Lemma 6.1 so that there exist x_0, $y_0 \in V(Q_1) \cap V(Q_2)$ such that*

(i) $x_0 < y_0$ *in* Q_i $(i = 1, 2)$.

(ii) There exist paths $R_i := R_i(x_0, y_0)$ $(i = 1, 2)$ such that R_1 and R_2 are respectively equal to $Q_1 \backslash Q_2$ and $Q_2 \backslash Q_1$ (abusing notation slightly). The first and the last edges of R_i do not belong to F $(i = 1, 2)$.

(iii) $\omega(R_1) = 1$, $\omega(R_2) = 0$;

(iv) subject to (i), (ii) and (iii), $|E(Q_1(u, x_0)| + |E(Q_1(y_0, v)|$ *is a maximum.*

(v) $Q_2(u, v) = Q_1(u, x_0)R_2(x_0, y_0)Q_1(y_0, v)$.

Proof. Choose Q_1 and Q_2 as above and write $Q_1 :=Q_1(a_0,\ldots,a_k)$ and $Q_2 :=Q_2(b_0,\ldots,b_l)$, where $u = a_0 = b_0$, $v = a_k = b_l$. Let x be the smallest integer such that $a_x \neq b_x$. Since the first and the last edges of Q_i belong to F, $x \geq 2$ and $x \leq l - 2$, $x \leq k - 2$. Now choose Q_1 and Q_2 so that x is maximized. Let b_y be the first vertex of $Q_2(b_x, v)$ in $V(Q_1)$. By definition $y > x$. Set $R_1 := Q_1(a_{x-1}, b_y)$, $R_2 := Q_2(a_{x-1}, b_y)$, $x_0 := a_{x-1}$, $y_0 := b_y$. If $\omega(R_1) \neq \omega(R_2)$ then, without loss of generality, let $\omega(R_1) = 1$ and $\omega(R_2) = 0$. Finally, choose Q_2 such that $Q_2 = Q_1(u, x_0)R_2(x_0, y_0)Q_1(y_0, v)$.

Thus we assume that $\omega(R_1) = \omega(R_2)$. Let $Q_2^*(u, v) = Q_1(u, b_y)Q_2(b_y, v)$ and replace Q_2 by Q_2^* in the above argument. Then, by Lemma 6.1, the choice of Q_1, Q_2 and x is contradicted.

Now choose Q_1, Q_2, R_1 and R_2 as above to maximize $|E(Q_1(u, x_0)| + |E(Q_1(y_0, v)|$. This choice implies that $Q_2(u, v) = Q_1(u, x_0)R_2(x_0, y_0)Q_1(y_0, v)$.

Note that, since Q_1 and Q_2 are F-alternating paths, R_1 and R_2 are F-alternating paths with first and last edges not in F. □

We now examine G^* in more detail. Recall that $G^* = G_0 - e$ and that G^* is 1-extendible.

Lemma 6.3 *In* G^* *there exists an edge* f *in* $R_1 \backslash F$ *with the property that each* F*-alternating cycle containing* f *has a nonempty intersection with* R_2*. Furthermore,* f *is contained in at least one such cycle.*

Proof. Suppose that the Lemma is not true. Then for each $f = (a, b) \in R_1 \backslash F$ ($a < b$ in Q_1) there exists a path $P(x, y)$ ($y < a < b < x$ in Q_1) where P is internally disjoint from $Q_1 \cup Q_2$ and $C := Q_1(x, y)P(x, y)$ is an F-alternating cycle.

Since C is F-alternating and Q_1 is F-alternating, $Q_1(y, x)$ has first and last edge in F and $P(x, y)$ has first and last edge in $E(G^*)\backslash F$.

Let $f := e_1 = (u_1, y_0)$ where $u_1 < y_0$ in Q_1. From Lemma 6.2 and the definition of y_0, $e_1 \in R_1 \backslash F$. Choose a path $P_1(x_1, y_1)$, $y_1 < u_1 < y_0 < x_1$ in Q_1 where P_1 is internally disjoint from $Q_1 \cup Q_2$ and $C_1 := Q_1(y_1, x_1)P_1(x_1, y_1)$ is an F-alternating cycle in G^*. We choose x_1 and y_1 to minimize the length of $Q_1(u_1, y_1)$.

If $y_1 \in V(R_1)$, we repeat the procedure with y_1 playing the role of y_0. In the same way we choose y_2, x_2, $P_2(x_2, y_2)$ and $C_2 := Q_1(y_2, x_2)P_2(x_2, y_2)$ such that the length of $Q_1(u_2, y)$ is minimized. Because of the minimization of the lengths of $Q_1(u_i, y_i)$, $i = 1, 2$:

(i) $y_2 < y_1 < x_2 < y_0 < x_1$ in Q_1;

(ii) $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are disjoint.

We repeat this argument and continue to construct disjoint paths $P_i := P_i(x_i, y_i)$ and F-alternating cycles $C_i := Q_1(y_i, x_i)P_i(x_i, y_i)$, $(y_{i-1} < y_{i-2} < x_{i-1} < y_{i-3} < \ldots < x_2 < y_0 < x_1)$ until we reach an integer j such that $y_j \in Q(u, x_0)$ and $y_{j-1} \in R_1(x_0, y_0)$. Since C_j is F-alternating and the first and last edges of P_j do not belong to F, $y_j \neq x_0$.

Now let $\vec{G^*}$ be a fixed even F-orientation of G^* with associated even function ω. Since ω is even and C_i is an F-alternating cycle in G^*, $\omega(C_i) = 0$, for $i = 1, \ldots, j$. Hence

$$\sum_{i=1}^{j} \omega(Q_1(y_i, x_i)) + \sum_{i=1}^{j} \omega(P_i) \equiv 0 \pmod 2 . \tag{1}$$

Set

$$\begin{aligned} C := Q_1(y_j, x_0)R_2(x_0, y_0)Q_1(y_0, x_1)P_1(x_1, y_1)Q_1(y_1, x_2)P_2(x_2, y_2) \\ Q_1(y_2, x_3)P_3(x_3, y_3)\ldots P_{j-1}(x_{j-1}, y_{j-1})Q_1(y_{j-1}x_j)P_j(x_j, x_j) . \end{aligned} \tag{2}$$

By definition, C is an F-alternating cycle in G^* and therefore $\omega(C) = 0$. Hence, using Lemma 6.2(iii) and (2)

$$\omega(Q_1(y_j, x_0)) + \sum_{i=1}^{j} \omega(Q_1(y_{j-1}, x_i)) + \sum_{i=1}^{j} \omega(P_i) \equiv 0 \pmod 2 . \tag{3}$$

Since

$$\begin{aligned} Q_1(y_i, x_i) = Q_1(y_i, y_{i-1}) + Q_1(y_{i-1}, x_i), \\ \omega(Q_1(y_i, x_i)) \equiv \omega(Q_1(y_i, y_{i-1})) + \omega(Q_1(y_{i-1}, x_i)) \pmod 2 . \end{aligned} \tag{4}$$

Adding (1) and (3)

$$\begin{aligned} \sum_{i=2}^{j-1} (\omega(Q_1(y_i, x_i)) + \omega(Q_1(y_{i-1}, x_i))) + (\omega(Q_1(y_0, x_1)) + \omega(Q_1(y_1, x_1)) + \\ + \omega(Q_1(y_j, x_j)) + \omega(Q_1(y_j, x_0)) + \omega(Q_1(y_{j-1}, x_j))) \equiv 0 \pmod 2 . \end{aligned} \tag{5}$$

From (5), using (4)

$$\begin{aligned} \sum_{i=2}^{j-1} (\omega(Q_1(y_i, y_{i-1})) + \omega(Q_1(y_1, x_1)) + (\omega(Q_1(y_0, x_1)) + \omega(Q_1(y_j, x_j)) + \\ + \omega(Q_1(y_j, x_0)) + \omega(Q_1(y_{j-1}, x_j))) \equiv 0 \pmod 2 . \end{aligned} \tag{6}$$

i.e.

$$\omega(Q_1(y_{j-1}, y_1)) + \omega(Q_1(y_1, y_0)) + \omega(Q_1(x_0, y_{j-1})) \equiv 0 \pmod 2 .$$

i.e. $\omega(R_1) = 0$ which contradicts Lemma 6.2(iii). □

Lemma 6.4 *Case (i) of Theorem 5.7 is not possible.*

Proof. The result is proved by contradiction. Using Lemma 6.3 we can select an edge $f := (a, b)$ in $R_1 \backslash F$ and an F-alternating cycle C such that for some $z, x_1 \in V(Q_1)$, $z < a < b < x_1$ ($x_1 \neq y_0$) and $C := Q_1(z, x_1)P(x_1, z)$ where $P(x_1, z) \cap R_2(x_0, y_0) \neq \emptyset$.

Now choose $y_1 \in V(R_2)$ ($y_1 \neq y_0$) so that $P_1 := P(x_1, y_1)$ is edge-disjoint from R_2. Furthermore, choose x_1 and y_1 to minimize the length of $Q_2(u, y_1)$.

We repeat the argument of Lemma 6.3. In that Lemma we begin with the edge $e_1 = (u_1, y_0)$ where $u_1 < y_0$ in Q_1. We now start with the edge $e_2 := (u_1^*, y_1)$ in Q_2 where $e_2 \in R_2 \backslash F$. The edge e_2 plays the role of e_1 below.

As in Lemma 6.3 we construct disjoint F-alternating paths $P_i := P_i(x_i, y_i)$, $i = 1, \ldots, j$ such that

(i) P_i is edge disjoint from $Q_1 \cup Q_2$.

(ii) $x_1, y_j \in V(Q_1)$; $x_2 \in V(Q_2)$; $x_i \in V(R_2)$, $i = 2, \ldots, j$; $y_i \in V(R_2)$, $i = 1, \ldots, j-1$.

(iii) $y_0 < y_1 < x_3 < y_2 < x_4 < \ldots < x_j < y_{j-1} < x_0$ in $R_2(y_0, x_0)$; $y_0 < x_2 < y_1$ in $R_2(y_0, x_0)$ or $x_2 < y_0 < y_1$ in $Q_2(v, u)$.

Below, we assume that $y_0 < x_2 < y_1$ in $R_2(y_0, x_0)$ (the case when $x_2 < y_0 < y_1$ in $Q_2(v, u)$ is almost exactly the same; equation (12) must be adjusted in the case $i = 2$).

Set

$$C_i := R_2(y_i, x_i)P_i(x_i, y_i), \quad (i = 2, \ldots, j-1) \tag{7}$$

Then C_i is an F-alternating cycle.

Let $\vec{G^*}$ be a fixed even F-orientation of G^* with associated even function ω. Since ω is even, $\omega(C_i) = 0$. Hence, from (7),

$$\sum_{i=2}^{j-1} \omega(R_2(y_i, x_i)) + \sum_{i=2}^{j-1} \omega(P_i(x_i, y_i)) \equiv 0 \pmod 2 . \tag{8}$$

Case (a): $x_1, y_j \in V(R_1)$.

Set

$$C_0 := Q_1(y_j, x_1)P_1(x_1, y_1)R_2(y_1, x_2)P_2(x_2, y_2)R_2(y_2, x_3) \ldots R_2(y_{j-2}, x_{j-1}) \\ P_{j-1}(x_{j-1}, y_{j-1})R_2(y_{j-1}x_j)P_j(x_j, x_j) .$$

Then C_0 is an F-alternating cycle and $\omega(C_0) = 0$. Hence,

$$\sum_{i=1}^{j} \omega(P_i(x_i, y_i)) + \sum_{i=1}^{j-1} \omega(R_2(y_i, x_{i+1})) + \omega(Q_1(y_j, x_1)) \equiv 0 \pmod 2 . \tag{9}$$

Also (see Lemma 6.2 and its proof) because of the choice of Q_1, Q_2, R_1, R_2, x_0, y_0 and the maximality condition of Lemma 6.2(iv) (see Remark 6.5 below), $\omega(C_i^*) = 0$, $i = 1, 2$ where

$$C_1^* := Q_1(x_1, y_0)R_2(y_0, y_1)P_1(y_1, x_1)$$
$$C_2^* := Q_1(x_0, y_j)P_j(y_j, x_j)R_2(x_j, x_0)\,.$$

Hence

$$\omega(Q_1(x_1, y_0)) + \omega(R_2(y_0, y_1)) + \omega(P_1(y_1, x_1)) \equiv 0 \pmod 2\,, \tag{10}$$

and

$$\omega(Q_1(x_0, y_j)) + \omega(P_j(y_j, x_j)) + \omega(R_2(x_j, x_0)) \equiv 0 \pmod 2\,, \tag{11}$$

Adding (8), (9), (10) and (11), we obtain:

$$\begin{aligned}(\sum_{i=1}^{j-1} \omega(R_2(y_i, x_{i+1})) + \sum_{i=2}^{j-1} \omega(R_2(y_i, x_i))) + \omega(Q_1(y_j, x_1)) + \\ + \omega(Q_1(x_1, y_0)) + \omega(Q_1(x_0, y_j)) + \omega(R_2(y_0, y_1)) + \omega(R_2(x_j, x_0)) \equiv 0 \pmod 2\,.\end{aligned} \tag{12}$$

Since $R_2(y_i, x_i) = R_2(y_i, x_{i+1})R_2(x_{i+1}, x_i)$,(i=2,. . . , j-1), from (12):

$$\begin{aligned}\omega(R_2(x_j, y_1)) + (\omega(Q_1(x_0, y_j))) + \omega(Q_1(y_j, x_1)) + \omega(Q_1(x_1, y_0))) \\ + (\omega(R_2(y_1, y_0)) + 1) + (\omega(R_2(x_0, x_j)) + 1) \equiv 0 \pmod 2\,.\end{aligned}$$

i.e.

$$\omega(R_1) + \omega(R_2) \equiv 0 \pmod 2\,, \tag{13}$$

which contradicts Lemma 6.2(iii).

Case (b): $x_1 \in V(R_1)$, $y_j \in V(Q_1(u, x_0))$.

The only difference from Case (a) is that now C_2^* is an F-alternating cycle and hence $\omega(C_2^*) = 0$, simply because ω is an even function.

Case (c): $x_1 \in V(Q_1(y_0, v))$, $y_j \in V(R_1)$.

This is the same as Case (b) up to a relabelling.

Case (d): $x_1 \in V(Q_1(y_0, v))$, $y_j \in V(Q_1(u, x_0))$.

This is the same as Case (a) except that now $\omega(C_i^*) = 0$, $i = 1, 2$, simply since ω is an even function. □

Remark 6.5 *Note that $\omega(C_i^*) = 0$, $i = 1, 2$, by the maximality condition in Lemma 6.2(iv). For instance, consider the cycle C_1^* and new paths $Q_1^* := Q_1(u, v)$ and $Q_2^* := Q_1(u, x_1)P_1(x_1, , y_1)R_2(y_1, y_0)Q_1(y_0, v)$ with $R_1^* := Q_1(x_1, y_0)$, $R_2^* := P_1(x_1, y_1)R_2(y_1, y_0)$. By maximality $\omega(R_1^*) = \omega(R_2^*)$ i.e. $\omega(Q_1(x_1, y_0)) = \omega(P_1(x_1, y_1)) + \omega(y_1, y_0))$ (mod 2). Since for odd length paths $P(u, v)$, $\omega(P(u, v)) + \omega(P(u, v)) \equiv 1$ (mod 2), we have $\omega(Q_1(x_1, y_0)) + \omega(R_2(y_0, y_1)) + \omega(P_1(y_1, x_1)) \equiv 0$ (mod 2).*

7 Proof of Theorem 3.8(i)

Let G_0 be a minimal counterexample to Theorem 3.8(i). From Theorem 5.7 and Lemma 6.4, $G^* = G_0 - R$ where $R = \{e_1, e_2\}$ is a removable doubleton and G^* is bipartite. Also F is a fixed 1-factor of G_0 such that $R \cap F = \emptyset$ and such that G_0 has no even F-orientations.

Let $\vec{G^*}$ be the canonical even F-orientation of G^* with associated even function ω (cf. Definition 2.6). Assume that there does not exist cycles C_1 and C, relative to which e_1 and e_2 are skew. Let $e_1 = (x_1, x_2)$ and $e_2 = (y_1, y_2)$, $x_i \in X$ and $y_i \in Y$ $(i = 1, 2)$ and (X, Y) is a bipartition of G^*. W.l.o.g., any cycle C containing e_1 and e_2 is of the form

$$C := (x_1, x_2, P_1(x_2, y_2), y, y_1, P_2(y_1, x_1)). \tag{1}$$

Since $\vec{G^*}$ is canonical, $\omega(P_1) = 1$ and $\omega(P_2) = 0$. Now define an F-alternating function ω_0 on G_0 as follows:

(i) if $(x, y) \in E(G_0^*)$, $\omega_0(x, y) = \omega(x, y)$;

(ii) $\omega_0(x_1, x_2) = 0$, $\omega_0(y_2, y_1) = 1$.

Then ω_0 extends ω which itself is even. Hence, if C is any cycle such that $R \cap E(C) = \emptyset$ then $\omega_0(C) = 0$. If $R \cap E(C) \neq \emptyset$ then $R \subseteq E(C)$ and C has the form of (1). Then

$$\omega_0(C) := \omega_0(x_1, x_2) + \omega_0(P_1) + \omega_0(y_2, y_1) + \omega_0(P_2) \equiv 0\,, (\text{mod}\,2).$$

Hence, $\omega_0(C) = 0$ for all F-alternating cycles C. Thus G_0 has an even F-orientation which is not true. Hence G_0 does have cycles C_1 and C_2 relative to which e_1 and e_2 are skew. Hence G_0 has a central subgraph H $(H = G_0)$ such that F is a 1-factor of H and H is an even subdivision of a graph in $\mathcal{W}$. This contradicts the definition of G_0. □

8 Preliminaries to Theorem 3.8(ii), (iii) and to Theorem 3.9

In this section we introduce important tools, which will be useful in the proofs of the theorems.

Definition 8.1 *(Weights)*

Let $\overrightarrow{G}$ be an F-orientation of the graph G. Let w be an additive $(0, 1)$-function defined on the directed edges of $E(\overrightarrow{G})$ as follows:

Let $\overrightarrow{P} \equiv (u_1, u_2, \ldots, u_n)$ denote an orientation of the F-alternating path $P(u_1, u_n)$. The "opposite orientation" of $\overrightarrow{P}$ is denoted by $\overleftarrow{P}$. Now define a function w^* as follows. Set

$$w^*(u_i, u_{i+1}) = \begin{cases} 1 & \textit{if} \quad \overrightarrow{(u_i, u_{i+1})} \\ \\ 0 & \textit{if} \quad \overleftarrow{(u_i, u_{i+1})}, \end{cases} \quad , \ 1 \leq i \leq n$$

and $w^*(P) \equiv \sum_{i=1}^{n-1} w^*(u_i, u_{i+1})$. *Similarly if* C *is the* F*-alternating cycle* $C := (u_1, u_2, \ldots, u_n, u_1) = (P, u_1)$ *set* $w^*(C) := w^*(P) + w^*(u_n, u_1)$. *Finally set* $w \equiv w^*(mod\, 2)$.

We shall say that w *is the* weight of the orientation $\overrightarrow{G}$. □

Lemma 8.2 *Let* w *be the weight functions of* $\overrightarrow{G}$, $G \in \mathcal{W}$, *where* $\overrightarrow{G}$ *is an* F*-orientation of* G. *Let* $C = (u_1, u_2, \ldots, u_n, u_1)$ *be an* F*-alternating cycle. Then, for* $1 \leq i \leq n - 1$ (i *modulo* n)

$$w(u_1, u_2, \ldots, u_i) \equiv w(u_i, u_{i+1}, \ldots, u_n, u_1)\ (\,mod\, 2)$$

if and only if C *is evenly oriented.*

Proof. Follows immediately from the definitions. □

Lemma 8.3 *Let* $\overrightarrow{G}$ *be an even* F*-orientation of* G, $G \in \mathcal{W}$, *with weight function* w. *Let* P *and* Q *be* F*-alternating paths of odd length:*

$$P := (u_1, u_2, \ldots, u_k)$$
$$Q := (v_1, v_2, \ldots, v_l)$$

where $u_1 = v_1$, $u_k = v_l$ *and* $(u_1, u_2) \in F$. *Then* $w(P) \equiv w(Q)$.

Proof. Notice that since $(u_1, u_2) \in F$ and k and l are both even, $E(P \cup Q) \subset E(G) - \{e, f\}$. From Lemma 8.2, if $E(P) \cap E(Q) = \emptyset$ the result immediately follows. So now assume that this is not the case.

Assume that the result is false. Choose P and Q so that $w(\overrightarrow{P}) \equiv w(\overrightarrow{Q})$ and such that $|V(P) \cap V(Q)|$ is minimal. Choose j as small as possible such that $u_j \in V(P) \cap V(Q)$ ($j \geq 1$). Then, form above, $j < k$. And choose i so that $i \leq j - 1$, $P(u_i, u_j) \subset Q$ and i is as small as possible. From Lemma 8.2

$$w(P(u_1, u_i)) = w(\overleftarrow{Q}(v_n, v_1)) \tag{2}$$

where $v_n = u_i$.

Now replace u_1 by $u_i(= v_n)$ in the above argument and replace the paths P and Q by $P(u_i, u_k)$ and $Q(v_m, v_k)$ respectively, using (2) and in minimality we obtain a contradiction. □

Lemma 8.4 *Let $G \in \mathcal{W}$ and $u, v \in V(G)$. Let $\overrightarrow{G}$ be an F-orientation of G. Suppose that G contains a path $P(u, v)$:*

$$P(u,v) :=(u_1, u_2, \ldots, u_k)\,;\ (u_1, u_2) \in E(G) \setminus F$$
$$u_1 = u, u_k = v \text{ and } k \geq 2,\ k \text{ even.}$$

Then $\overrightarrow{G}$ contains an F-alternating path $P^*(u_1, u_k)$.

Proof. Choose $P := P(u, v)$ so that $|E(P) \setminus F|$ is as small as possible. Now choose i as large as possible such that $P(u_1, u_i)$ is F-alternating. Set $e_0 = (u_i, u_{i+1})$ and choose C_0 to be an F-alternating cycle containing e_0. Then $P \cup C$ contains a path $P^* := P^*(u, v)$ such that $|E(P^*) \setminus F| < |E(P) \setminus F|$ which is a contradiction. □

Lemma 8.5 *Let $G \in \mathcal{W}$. Let $u, v \in V(G)$. Let $\overrightarrow{G}$ be an F-orientation of G. Let $P(u, v)$ be an F-alternating path in $\overrightarrow{G}$ with first and last edges in F. Then if $Q(u, v)$ is any F-alternating path*

$$w(P) = w(\overleftarrow{Q}).$$

Proof. This follows from Lemmas 8.3 and 8.4. □

Definition 8.6 *(Splitting an edge)*

Let G be a cubic graph and $e_0 = (a, b) \in E(G)$. Suppose that $N(a) = \{b, b_1, b_2\}$, $N(b) = \{a, a_1, a_2\}$ and $N(a) \cap N(b) = \emptyset$. Set $R_1 := \{(a_1, b_1), (a_2, b_2)\}$ and $R_2 := \{(a_1, b_2), (a_2, b_1)\}$. An e_0-splitting of G *is a multigraph G^* such that:*

(i) $V(G^*) = V(G) \backslash \{a, b\}$;

(ii) $E(G^*) = E(G - a - b) \cup R$, *where* $R = R_i$ *for some* $i \in \{1, 2\}$.

Note that, we abuse notation slightly in Definition 8.6(ii): for instance if $(a_1, b_1) \in E(G)$ and $R = R_1$ then (a_1, b_1) is a multiple edge in G^*.

Definition 8.7 *(Special vertices and edges, e-splittings)*

Suppose that $G \in \mathcal{W}(3)$ and F is a $\mathcal{W}$-factor for G. Let $G - \{e, f\}$ be bipartite and $e = (x_1, x_2)$, $f = (y_1, y_2)$. Then we say that x_i, y_i $(i = 1, 2)$ are special vertices *and that e and f are* special edges.

Suppose that $e_0 = (x, y) \in F$, $x \in X$, $y \in Y$ and y is not special, where (X, Y) is a bipartition of $G - R$ with $R = \{e, f\}$. Suppose that there exists a special vertex u which is adjacent to either x or y. Then any e_0-splitting G^* is said to be a special e_0-splitting. *The converse construction where two edges e_1 and e_2 (one of which is incident to a special vertex) are* glued *together will be called a* special $\{e_1, e_2\}$-glueing.

Lemma 8.8 *Suppose that the 3-regular graphs $G \in \mathcal{W}(3)$ has no non-trivial F-tight cut of size three (see Definition 4.5), where F is a $\mathcal{W}$-factor for G. Then there exist a special e_0-splitting G^* of G such that G^* is a graph.*

Proof. If G^* contains no multiple edges then $R_i \cap E(G) = \emptyset$ for some i $(i = 1, 2)$. Otherwise, if (x_1, y_3) and (x_3, y_3) are both edges of G, then $\{x_1, x_3, x, y, y_3\}$ is an F-tight cut. If (x_1, y_4) and (x_3, y_4) are both edges of G, then $\{x_1, x_3, x, y, y_4\}$ is an F-tight cut. If (x_3, y_3) and (x_3, y_4) are both edges of G, then $\{x_3, x, y_3, y_4, y\}$ is an F-tight cut. Finally, since x_1 has degree 3 and $(x_1, x_2) \in E(G)$, at most one of (x_1, y_3) and (x_1, y_4) is an edge. It follows that $R_i \cap E(G) = \emptyset$ for some i $(i = 1, 2)$ which is a contradiction. $\square$

9 Proof of Theorem 3.8(ii) and (iii)

Notation 9.1 *Suppose that $G \in \mathcal{W}$ (see Figure 3 and set $\ell := \kappa(G)$. Let $S = \{w_1, w_2, \ldots, w_\ell\}$ be a separating set. Let $G \backslash S := G_1 \cup G_2$ where $e \in E(G_1)$ and $f \in E(G_2)$. Suppose that $V(G_1) \setminus \{e\} := X_1 \cup Y_1$ and $V(G_2) \setminus \{f\} := X_2 \cup Y_2$ where $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.*

Set $F_i := F \cap E(S, X_i \cup Y_i)$ $(i = 1, 2)$ and finally set $F_{i1} := F_i \cap E(S, X_i)$ and $F_{i2} := F_i \cap E(S, Y_i)$ $(i = 1, 2)$.

Set $|X_i| = s_i$, $|Y_i| = t_i$; $k_{ij} := |E(w_i, X_j \cup Y_j)|$ $(i = 1, 2, 3; j = 1, 2)$.

Proof. (Theorem 3.8(ii))

Suppose that $G \in \mathcal{W}$ and $\kappa(G) = 4$. By Lemma 5.3, G contains an F-central subgraph $H \in \mathcal{W}(\leq 3)$ which is isomorphic to an even subdivision of K_4 (cfr. Figure 4).

In Figure 4 P_1 and P_2 $(P_1 \cap P_2 = \emptyset)$ denote paths $P_1 := P_1(a_1, b_2)$, $P_2 := P_2(b_1, a_2)$, $a_1, b_1 \in X$ and $a_2, b_2 \in Y$. We set $e := (u_1, u_2)$ and $f := (v_1, v_2)$. The skew cycles C_1 and C_2 are

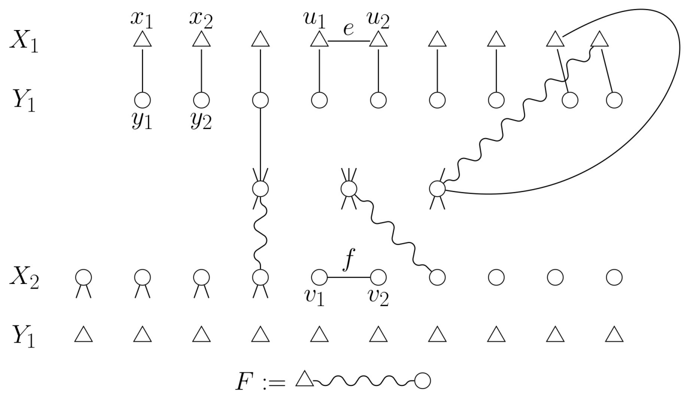


Figure 3: Illustration of Notation 9.1

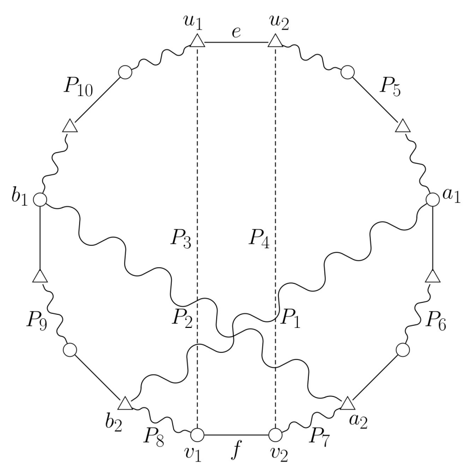


Figure 4: Even subdivision of K_4

$$C_1 := (u_1, u_2, \ldots, a_1, \ldots, a_2, \ldots, v_2, v_1, \ldots, b_2, b_1, \ldots, u_1)$$
$$C_2 := (u_1, u_2, \ldots, a_1, P_1, b_2, \ldots, v_1, v_2, \ldots, a_2, P_2, b_1, \ldots, u_1).$$

Since $\kappa(G) = 4$ there exist disjoint paths $P_3^* := P_3^*(u_1, v_1)$ and $P_4^* := P_4^*(u_2, v_2)$ (or we can relabel say u_1 and v_1). Now from Lemma 8.4 there exist F-alternating paths $P_3 := P_3(u_1, v_1)$ and $P_4 := P_4(u_2, v_2)$. Now let us suppose that G has an F-even orientation $\overrightarrow{G}$ with weight function w. Set (again see Figure 4)

$$P_5 := P_5(u_2, \ldots, a_1) \quad P_6 := P_6(a_1, \ldots, a_2) \quad P_7 := P_7(a_2, v_2)$$
$$P_8 := P_8(v_1, \ldots, b_2) \quad P_9 := P_9(b_2, \ldots, b_1) \quad P_{10} := P_{10}(b_1, u_1)$$

Now using Lemmas 8.3 and 8.5 and set $w_i := w(P_i)$ $(i = 1, 2, \ldots, 10)$

$$\begin{aligned} w(u_1,u_2) + w_5 + w_6 + w_7 + w(v_2,v_1) + w_8 + w_9 + w_{10} &\equiv 0 \quad (mod \quad 2) \\ w(u_1,u_2) + w_5 + w_1 + \overline{w_8} + w(v_1,v_2) + \overline{w_7} + w_2 + w_{10} &\equiv 0 \quad (mod \quad 2) \\ w_3 + w_8 + w_9 + w_{10} &\equiv 0 \quad (mod \quad 2) \\ w_5 + w_6 + w_7 + \overline{w_4} &\equiv 0 \quad (mod \quad 2) \\ w_3 + w_8 + \overline{w_1} + \overline{w_5} + w_4 + \overline{w_7} + w_2 + w_{10} &\equiv 0 \quad (mod \quad 2) \end{aligned}$$

where $\overline{w_i} := 1 + w_i$ $(i = 1, 2, \ldots, 10)$

Adding this equation:

$$w_5 + w(v_1,v_2) + w_8 + w_1 + w_4 \equiv 0 \quad (mod \quad 2)$$

which is a contradiction. □

Lemma 9.2 *Let $G \in \mathcal{W}$ and G be regular of degree k $(k \geq 3)$. Suppose that $\kappa(G) = 2$. Then G is not F-even.*

Proof. Suppose that $G \in \mathcal{W}$, $\kappa(G) = 2$ and G k-regular $(k \geq 3)$. Suppose that G is F-even. Set $S := \{w_1, w_2\}$ where S is a separating set. There are (several cases) to consider.

Case 1 ($|F_{11}| + |F_{12}| \equiv 0\,(mod\,2)$) In these cases $G \notin \mathcal{W}$ since the skewness condition of $\mathcal{W}$ is contradicted.

Case 2 ($|F_{11}| = 1, |F_{12}| = 0$) In this case $s_1 = t_1 + 1$ and $ks_1 - k_{11} - 2 = kt_1 - k_{12}$. Hence $k = 2 + k_{11} + k_{12}$ and $k_{11} = k - 1$, $k_{12} = 1$. Set $E_0 = \{(w_1, y_1), (w_2, x_2)\}$ where $y_1 \in Y_1$, $x_2 \in X_2$, $w_1 \in X$, $w_2 \in Y$. Then E_0 is an edge-cut. Let $G_1 \setminus E_0 = H_1 \dot{\cup} H_2$ and set $H_1^* := H_1 + (w_1, x_2)$ and $H_2^* := H_2 + (y_1, w_2)$. Clearly $H_i^* \in \mathcal{W}$ for some $i \in \{1, 2\}$. Furthermore we can choose H_i^* to be as small as possible. In particular we can choose H_i^* so that $\kappa(H_i^*) \geq 3$ and H_i^* is k-regular $(k \geq 3)$. So by Case 1, H_i^* is not F-even which implies G is not F-even which is not true.

Case 3 ($|F_{11}| = 2$) In this case $s_1 = t_1 + 2$ and $s_1k - k_{11} - k_{12} - 2 = t_1k$. Hence $2k = k_{11} + k_{12} + 2$. Hence $k_{11} = k_{12} = k - 1$. Set $E_0 = \{(w_1, y_1), (w_2, y_2)\}$ where $y_i \in Y_2$, $w_i \in X$ $(i = 1, 2)$. This E_0 is an edge-cut and we repeat the argument of Case 2. □

Proof. (Theorem 3.8(iii)) Let $G \in \mathcal{W}$ be regular of degree k (≥ 3). Assume that G is F-even. From Theorem 3.8(ii) and Lemma 9.2, $\kappa(G) = 3$.

We use the terminology of the introduction and of Notation 9.1. Thus $S = \{w_1, w_2, w_3\}$ is a separating set. By symmetry we may assume that $|S \cap X| \geq 2$. We now prove, with this assumption, that

$$|S \cap X| \equiv |F_{11}| + |F_{12}| \quad (mod \quad 2) \tag{1}$$

Set $k_{ij} := |E(w_i, X_, \cup Y_j|$ $(i = 1, 2, 3;\ j = 1, 2)$. There are two cases

Case 1 ($|S \cap X| = 3$) Since $|S \cap (V(C_1) \cup V(C_2))| \geq 2$, $|F_{12}| \leq 1$. Suppose that $|F_{12}| = 0$. Then $|F_{22}| = 3$. Hence $t_2 = s_2 + 3$. Hence $t_2k - 2 - (k_{21} + k_{22} + k_{23}) = s_2k$. Hence $3k = 2 + (k_{21} + k_{22} + k_{23}) \leq 3k - 1$. Hence $|F_{12}| = 1$ and, by definition, $|F_{11}| = 0$. Hence (1) is satisfied.

Case 2 ($|S \cap X| = 2$) Suppose that $w_1, w_2 \in X$ and $w_3 \in Y$. As in the previous case $|F_{12}| \leq 1$. By definition $|F_{11}| \leq 1$.

Suppose that $|F_{12}| = 0$. Assume that $F_{11} = 1$. Then $t_2 = s_2 + 2$. Hence $s_2k - k_{23} = t_2k - 2 - k_{21} - k_{22}$. Hence $2k = 2 + k_{21} + k_{22} - k_{23} \leq 2k - 1$. Hence if $|F_{12}| = 0$, $|F_{11}| = 0$ and (1) is satisfied.

Finally suppose that $|F_{12} = 1|$. Assume that $|F_{11}| = 0$. Then this is impossible since $|S \cap (V(C_1) \cup V(C_2))| \geq 2$. Hence $|F_{11}| = 1$.

Hence in all cases (1) is satisfied.

Now we assume that $|S \cap X| = 2$ and $|F_{11}| = |F_{12}| = 1$ (see (1) above. We follow the method of proof of Theorem 3.8(ii). By Lemma 5.3, G contains an F-central subgraph $H \in \mathcal{W}(\leq 3)$ which is isomorphic to an even subdivision of K_4 (See Figure 3 in the proof of Theorem 3.8(ii).

We may assume (see Figure 5 without loss of generality that there exist edges $w_1, y_{11} \in E(G) \setminus F$ and $(w_2, x_{13}) \in F_1$, $y_{11} \in Y_1$, $x_{13} \in X_1$, belonging to $E(C_1) \cup E(C_2)$. Also there exists an edge $(w_2, y_{12}) \in F$, $y_{12} \in Y_1$.

Suppose that e and y_{12} are in different components of G_1. If y_{12} and x_{13} are in the same component then since $x_{13} \in V(C_1) \cup V(C_2)$, y_{12} and e are in the same component which is a contradiction.

So now assume that x_{13} and y_{12} belong to different components of G_1. Consider the component of G_1 containing x_{13} (which must clearly contain e) having sides $X_{12} \subseteq X_1$, $Y_{12} \subseteq Y_1$, $s_{12} := |X_{12}|$ and $t_{12} := |Y_{12}|$. Then $s_{12} = t_{12}+1$ and $s_{12}k-2-k^*_{23} = t_{12k}-k^*_{21}-k^*_{22}$ where $k^*_{ji} := |E(w_i, X_{12} \cup Y_{12}|$. Hence

$$k^*_{21} + k^*_{22} = 2 + k^*_{23} - k \leq 1.$$

Hence there exists a proper subset of S which separates K form $G \setminus K$ which is a contradiction. Hence x_{13} and y_{12} belong to the same component of G_1 which also includes e.

Without loss of generality, there exist paths $P^*_{13} := P^*_{13}(u_1, y_{12})$ and $P^*_{14} := P^*_{14}(u_2, y_{12})$

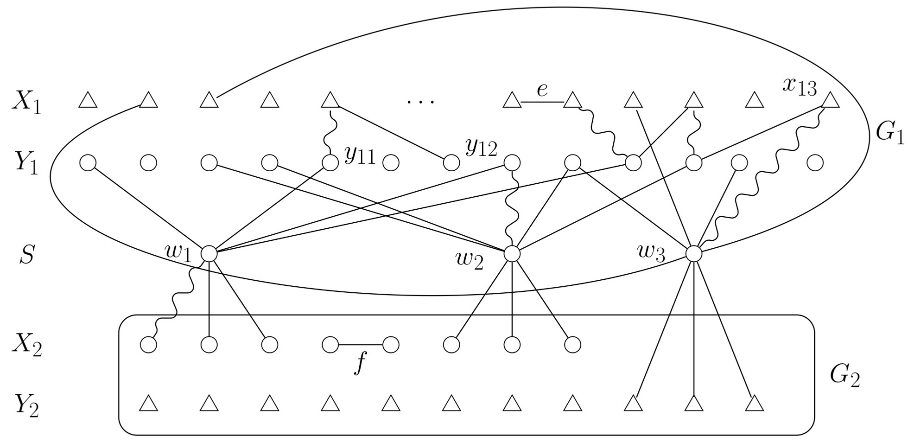


Figure 5: Illustration for the proof of Theorem 3.8(iii)

in $G_1 \setminus \{e\}$ with both paths having their final edges in $E(G) \setminus F$ (note that the notation may be chosen so that u_1 and u_2 or v_1 and v_2 may be interchanged). Similarly in $G \setminus G_1$ there exist paths $P^*_{23} := P^*_{23}(w_2, v_1)$ and $P^*_{24} := P^*_{24}(w_2, v_2)$. Finally set $P^*_3 := P^*_{13}P^*_{23}$ and $P^*_4 := P^*_{14}P^*_{24}$ and continue exactly as in Theorem 3.8(ii).

There are now two other cases to consider. In fact these cases basically duplicate the first case:

Case A ($|S \cap X| = 3$, $|F_1| = 1$, $|F_{11}| = 0$) Suppose that $(w_2, y_{12}) \in F_1$, $y_{12} \in Y_1$ (see Figure 6)

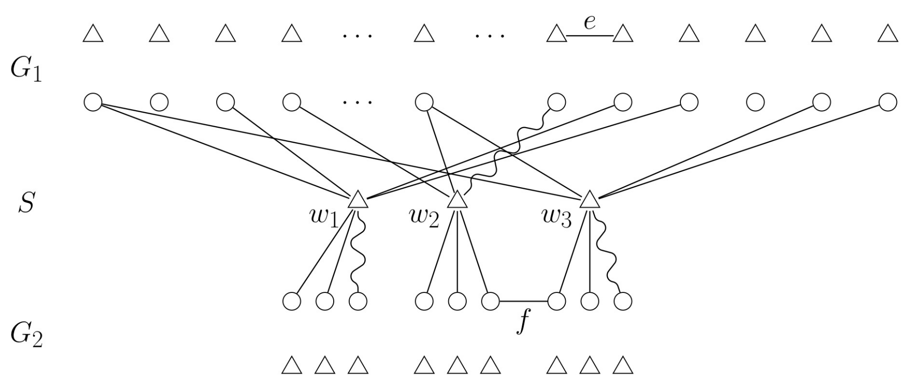


Figure 6: Illustration for the proof of Case A in Theorem 3.8(iii)

Suppose that y_{12} and e are in different components of G_1. Then $t_2 = s_2$ and $s_2k - 2 = t_2k - k^*_{21} - k^*_{22} - k^*_{23}$. Hence $k^*_{21} + k^*_{22} + k^*_{23} = 2$. Hence the component K, say, containing e in G_1 is separated from $G \setminus K$ by a proper subset of S which is a contradiction.

CASE B ($|S \cap X| = 2$, $|F_{11}| = |F_{12}| = 0$)

Suppose that $(w_3, x_{3i}) \in E(G) \setminus F$, $x_{3i} \in X_1$, $i = 1, 2, \ldots, \ell$ (see Figure 7)

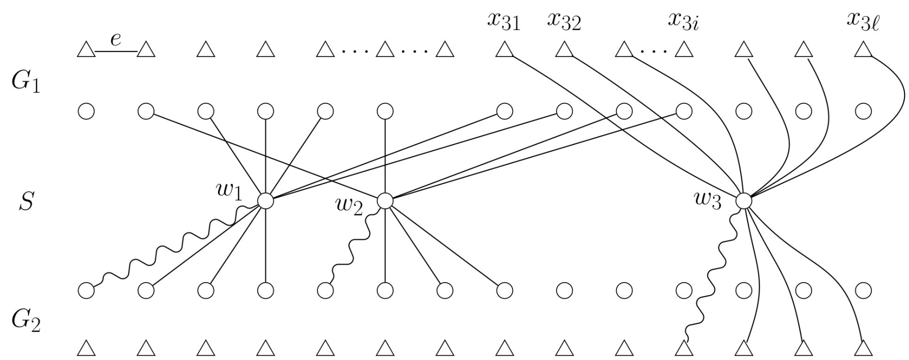


Figure 7: Illustration for the proof of CASE B in Theorem 3.8(iii)

Now suppose that there is no component in G_1 containing both e and x_{3i} for any $i \in \{1, 2, \ldots, \ell\}$. Again this would imply that the component K containing e in G_1 is separated from $G \setminus K$ by a proper subset of S which is the final contradiction. □

Example 9.3 *We give a concrete example illustrating Theorem 3.8(iii). The graph G in Figure 8 is such that $G \in \mathcal{W}$, $\kappa(G) = 3$ and G is 4-regular G has a separating set $S = \{w_1, w_2, w_3\}$ where $w_1 \in X$, $w_2, w_3 \in Y$.*

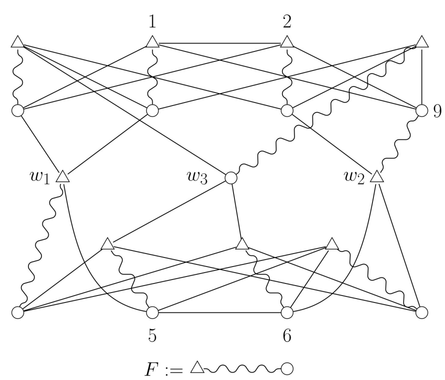


Figure 8: G

In Figure 9 the graph G_0 is an F-central even subdivision of G and $\overrightarrow{G_0}$ is an F-orientation. We use the labelling of Figure 8 except now w_2 and w_3 are relabelled 10 *and* 8 *respectively.*

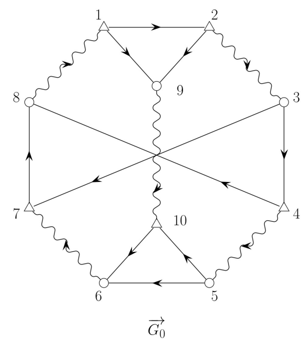


Figure 9: G_0 = an F-central even subdivision of G

G_0 has F-alternating cycles:

$$\begin{array}{ll} C_1 := (1,2,3,4,5,6,7,8,1), & C_2 := (1,9,10,6,7,8,1), \\ C_3 := (2,9,10,5,4,3,2), & C_4 := (1,9,10,5,4,8,1), \\ C_5 := (2,9,10,6,7,3,2), & C_6 := (1,2,3,7,6,5,4,8,1). \end{array}$$

Then $\{C_1, C_2, \ldots, C_6\}$ is a zero-sum set and in this set C_6 is the only evenly F-oriented cycle. This proves that G_0 is not F-even and hence G is not F-even. □

10 Proof of Theorem 3.9

Definition 10.1 *(Almost F-even)*
Suppose that $G \in \mathcal{W}$ and G is not F-even. If for each $e_0 \in E(G) \setminus (F \cup \{e, f\})$, $G - e_0$ is F-even then G is said to be almost F-even.

Proof. (of Theorem 3.9)

Let $G \in \mathcal{W}(3)$ then G is not F-even (Theorem 3.8 (i)). If G is a graph satisfying the conditions of Theorem 3.9 then G is a proper subgraph of some graph $G_0 \in \mathcal{W}(3)$ that is almost F-even.

Choose G_0 as small as possible such that $G_0 \in \mathcal{W}(3)$ and G_0 is not almost F-even. Hence there exists $e_0 \in E(G_0) \setminus (F \cup \{e, f\})$ such that $G_0 - e_0$ is F-even.

Select a special e^*-splitting G_0^* of G_0 (see 8.6) where $e^* = (a_2, b_2)$. We use the notation of Figure 10.

We recall that $G_0^* := (G_0 \setminus \{a_2, b_2\}) \cup \{e_1, e_2\}$ where $e_1 = (a_1, b_1)$ and $e_2 = (a_3, b_3)$. Furthermore $G_0^* \in \mathcal{W}(3)$ and G_0^* is cubic.

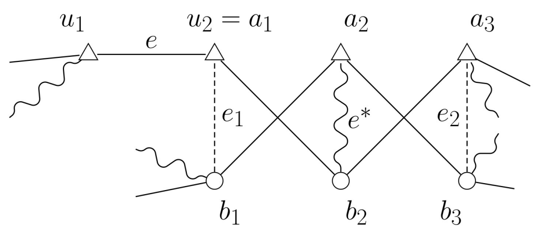


Figure 10: Notation for a special e^*-splitting G_0^* of G_0

Since $G_0 \in \mathcal{W}(3)$, $G_0^* \in \mathcal{W}(3)$. Otherwise there is an F-even subdivision of G_0^* which gives rise to a central subgraph H as described in Definition 3.7. Since e_1 is incident to e and $P_2(a_3, b_3)$ (see below) exists H must be contained in G_0 which is not possible.

By the minimality of G_0, G_0^* is almost F-even.

Let $f_0^* \in E(G_0^*)$. Let $\overrightarrow{G_0^* \setminus \{f_0^*\}}$ be an F-even orientation of $G_0^* \setminus \{f_0^*\}$ with weight function w^*. Set

$$\begin{array}{ll} P_1 := (a_1, b_2, a_2, b_1)\,; & P_2 := (a_3, b_2, a_2, b_3) \\ P_3 := (a_1, b_2, a_2, b_3)\,; & P_4 := (b_1, a_2, b_2, a_3) \end{array} \tag{2}$$

We now show that this orientation induces an F-even orientation $\overrightarrow{G_0^* \setminus \{f_0^*\}}$ of $G_0^* \setminus \{f_0^*\}$ with weight function w defined as follows:

$$w(e^*) := w^*(e^*)\,; \forall e^* \in E(G_0^*) \setminus \{f_0^* \cup \{e_1, e_2\}\} \tag{3}$$

$$\begin{array}{c} w(P_i) := w^*(e_i)\,,\, i = 1, 2 \text{ if } f_0^* \notin \{e_1, e_2\} \\ \text{if } f_0^* = e_i \text{ define } w(P_i) := w^*(e_j); j \neq i, i \in \{1, 2\}. \end{array} \tag{4}$$

It is clear that $w(C^*)$ is F-even for all cycles C^* in $G_0^* \setminus (\{f_0^*\} \cup \{e_1, e_2\})$. Furthermore if $E(C^*) \cap \{e_1, e_2\} = e_i \quad (i \in \{1, 2\})$ it is clear from (3) that $w(C)$ is F-even where if for example if $i = 1$

$$C = (a_1, b_1, P_0, a_1)$$

then

$$C^* = (a_1, P_1, P_0, a_1).$$

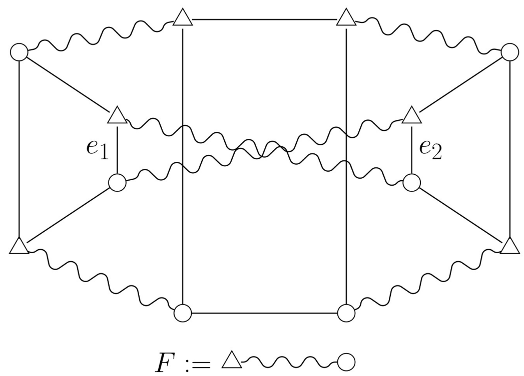


Figure 11: $G \in \mathcal{W}(3)$ but $G \notin \mathcal{W}^*(3)$

One small twist here is if $|E(C^*) \cap \{e_1, e_2\}| = 2$. This case is covered in fact by the following argument.

Finally we must consider the cycles C in $G_0 \setminus \{f_0^*\}$ such that $E(C) \cap \{P_3, P_4\} = \emptyset$. Suppose that

$$C_1 := (a_1, P_3, b_3, Q_1, a_1) \tag{5}$$

and

$$C_2 := (b_1, P_4, a_3, Q_2, a_3) \tag{6}$$

Now since C_1 and C_2 are F-even

$$w(P_3) := w^*(Q_1) \tag{7}$$

$$w(P_4) := w^*(Q_1) \tag{8}$$

Then, from Lemma 8.3, the definitions of $w(P_i)$ $(i = 3, 4)$ is independent of the choice of Q_i $(i = 1, 2)$.

The case mentioned above when $|E(C^*) \cap \{e_1, e_2\}| = 2$ gives rise to two cycles as in (5) and (6).

Now since w^* is an additive $(0, 1)$-function equations (4), (7) and (8) have a solution (we have four equations in the four unknowns in $E(P_3) \cup E(P_4)$). So w is a weight function for some even F-orientation $\overrightarrow{G_0 \setminus \{f_0^*\}}$ of $G_0 \setminus \{f_0^*\}$. This is true for all f_0^*

in $E(G_0) \setminus (E(P_3) \cup E(P_4) \setminus \{e^*\})$. However this case is easy to deal with. If we set $f_0^* := (B_3, a_3)$ then this case is exactly the same for the case when $f_0^* := e_2$ and so on.

We have thus shown that $\overrightarrow{G_0 \setminus \{f_0^*\}}$ is F-even and G_0 is almost F-even which is a contradiction. □

Remark 10.2 *The graph G in Figure 11 is such that $G \in \mathcal{W}(3)$ but $G \notin \mathcal{W}^*(3)$ since $G \setminus \{e_1, e_2\}$ is an even subdivision of $W \in \mathcal{W}(3)$. This example illustrates why in Theorem 3.9 we need the restriction that $G \in \mathcal{W}^*(3)$.* □

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