Counterfactuals as modal conditionals, and their probability

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ARTICLE INFO

ABSTRACT

In this paper we propose a semantic analysis of Lewis' counterfactuals. By exploiting the structural properties of the recently introduced boolean algebras of conditionals, we show that counterfactuals can be expressed as formal combinations of a conditional object and a normal necessity modal operator. Specifically, we introduce a class of algebras that serve as modal expansions of boolean algebras of conditionals, together with their dual relational structures. Moreover, we show that Lewis' semantics based on sphere models can be reconstructed in this framework. As a consequence, we establish the soundness and completeness of a slightly stronger variant of Lewis' logic for counterfactuals with respect to our algebraic models. In the second part of the paper, we present a novel approach to the probability of counterfactuals showing that it aligns with the uncertainty degree assigned by a belief function, as per Dempster-Shafer theory, to its associated conditional formula. Furthermore, we characterize the probability of a counterfactual in terms of Gärdenfors' imaging rule for the probabilistic update.

1. Introduction

Counterfactuals are subjunctive conditional statements about hypothetical situations phrased in the form "if [antecedent] were the case, then [consequent] would be the case" and where the antecedent is usually assumed to be false. The study of counterfactuals and counterfactual reasoning has been carried out in various fields, ranging from logic to philosophy of language and linguistics (see [51] and [32] and the more recent [11] and [46]), as well as social sciences and artificial intelligence.[1]

The present paper studies counterfactual statements from a logical and algebraic perspective by introducing a class of modal algebras which we call *Lewis algebras*.[2] We show that a slightly stronger variant of the logic for counterfactual defined by Lewis in [38] is sound and complete with respect to Lewis algebras. In addition, we present some new ideas and results

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[1] The literature on the subject is vast, and providing an exhaustive treatment of the applications of counterfactual reasoning in science is beyond the scope of this first section. We invite interested readers to consult [49] and the references therein for an interesting and comprehensive overview of the philosophical aspects of counterfactuals. The book [9], on the other hand, explores the use of counterfactuals in psychological and social sciences. Moving on to more AI-oriented approaches to counterfactuals, they will be discussed in what follows.

[2] The authors are aware that the term "Lewis algebras" has already been used in the literature to refer to *interior algebras* [6] (also known as *S4-algebras*) named after the logician C. I. Lewis, who initially proposed the modal logics S4 and S5. However, we believe that our chosen usage of the term will not cause any confusion.

on the uncertainty of counterfactuals by characterizing the imaging rule for updating an a priori probability defined on these structures.

Our modal algebraic perspective on counterfactual conditionals builds on, and it is inspired by, the algebraic framework of *boolean algebras of conditionals* recently developed in [19]. In addition to placing counterfactuals in the map of algebraic logic, our approach also enables us to introduce relational, Kripke-style, models which we call *Lewis frames*. These models naturally arise through the duality between algebras with operators and Kripke frames. Alongside Lewis algebras, the present paper aims to study this alternative semantics and present Lewis frames as an alternative to the typical semantics for counterfactuals provided by Lewis' sphere models.

Lewis algebras, Lewis frames, and their mutual inter-definability are employed in the second part of the paper to show relevant properties for the probability of counterfactuals and to establish a close connection with belief function theory, the mathematical models of Dempster-Shafer theory of evidence, and Gärdenfors' imaging rule for the probabilistic update.

In addition to these technical results, this paper aims to provide logicians working on counterfactuals with a new toolkit to study them using the powerful tools of algebra and algebraic modal logic. Furthermore, it aims to broaden the scope of applications of algebraic logic by bridging the gap between theoretical and applied research.

The best-known approach to counterfactuals originates from the philosophical literature, mainly due to the significant contributions by Lewis [38] and Stalnaker [48]. In the late '60s and early '70s, they initiated a productive and active research area focused on studying logics for counterfactuals in a broad sense. In the formal setting developed by Lewis and Stalnaker, a counterfactual statement of the form "if ψ were the case, then φ would have occurred" is usually denoted by $\psi \mathbin{\Box\!\!\rightarrow} \varphi$ and, according to their semantics, it is true in a possible world w if φ is true in all the *closest worlds* to w in which ψ is true. The rough intuition behind such interpretation is exemplified by referring to Lewis' own words [38, page 1]:

> 'If kangaroos had no tails, they would topple over' seems to mean something like this: in **any possible state of affairs** in which kangaroos have no tails, and **which resembles our actual state of affairs** as much as kangaroos having no tails permits it to, the kangaroos would topple over. I shall give a general analysis of counterfactual conditionals along these lines.

In artificial intelligence (AI), the interest in counterfactual reasoning has grown in slightly more recent times and from different perspectives. To start with, it is important to emphasize the early work of the symbolic-oriented community of AI, which can be traced back to the papers by Ginsberg [24] and Pearl [44]. In this stream, a special mention is surely due to the paper by Galles and Pearl [22] who formally presented a way to apply counterfactual reasoning to causality. This connection between counterfactuals and causality has been further investigated to explain causation [37,45], develop a logic for causal reasoning [3,7,28,22], and explore (recursive) causal models [29].

Counterfactuals have also drawn the attention of the community of subsymbolic AI. Indeed, counterfactual-like reasoning has inspired improvements in deep learning algorithms [41] and generative adversarial networks [42]. Currently, one of the most intriguing applications of counterfactuals concerns explainable AI (XAI), where they are adopted to increase the confidence in artificial neural networks by human users (see [10,40]).

The present paper aligns with the former stream and aims to put forward a *reductionist* perspective on counterfactuals. It aims to define counterfactuals by combining a normal necessity operator from modal logic [5] and the probabilistic conditional from recent work on boolean algebras of conditionals [19]. In the standard and other semantic accounts for counterfactuals (such that Fine's [18], Ciardelli's [11], Barbero's [3], Galles and Pearl's [22]) the counterfactual arrow $\mathbin{\Box\!\!\rightarrow}$ is taken as a primitive operator in the logical language. Truth conditions for $\mathbin{\Box\!\!\rightarrow}$ are hence spelled out in terms of *interventions*, *similarity relations*, *truthmakers*, and so on. In our account, however, the counterfactual operator is reduced to already known and familiar logical operators. This reduction conceptually offers a completely new perspective on counterfactuals from a more technical point of view. Lewis algebras and their dual Lewis frames allow us to re-elaborate the classical truth conditions of counterfactuals in terms of an accessibility relation over permutations of complete state descriptions (i.e. atoms of a boolean algebra of conditionals, see Subsection 2.2 for details), without appealing to any notion of similarity. This fact might suggest new interpretations for counterfactuals as, for instance, Kripke's translation of intuitionistic logic to S4 modal logic [34] shed light on the way intuitionism interprets the logical language.

In the second part of the paper we will address the problem of uncertain quantification of counterfactuals and relate it to the so-called *imaging rule* (or simply *imaging*, hereafter) for updating probabilities. The imaging rule is a method introduced by Lewis in [38] and further elaborated by Gärdenfors in [23] and others (see, e.g., [16,26]) for updating an a priori probability distribution on possible worlds when one learns (or imagines) that an event has occurred. Although various methods have been defined for assigning a probability to a counterfactual conditional, we still lack an answer to the question of how to interpret the probability of a counterfactual. Our reductionist account allows us to technically characterize the probability of a counterfactual and offer a new interpretation of it. In Theorem 7.4 we show that the probability of a counterfactual amounts to the value of a certain belief function and can be characterized in terms of Gärdenfors' imaging. It is important to point out again that Lewis algebras are a key tool for obtaining such results. Following the same reductionist philosophy that guided us in the first part of this paper, belief functions can be defined from a probability function and a normal modal operator [15,27,31].

Summary and structure of contributions. To make the rest of this section clear, let us specify that boolean algebras of conditionals are algebras containing formal expressions of the form $(a \mid b)$ and representing the conditional event "a given b".

Besides these basic expressions, these structures can also deal with compound boolean expressions like $(a \mid b) \wedge (c \mid d)$ or $\neg(a \mid b)$ that read, respectively, "a given b and c given d" and "it is not the case that a given b is true". The construction of such structures starts with any boolean algebra $\mathbf{A}$ of plain (unconditional) events and produces another boolean algebra $\mathcal{C}(\mathbf{A})$ that formalizes the conditionals that can be expressed from the events of $\mathbf{A}$. Therefore, for every initial boolean algebra $\mathbf{A}$, a Lewis algebra of $\mathbf{A}$ is a pair $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \Box)$ where $\mathcal{C}(\mathbf{A})$ is as above and $\Box : \mathcal{C}(\mathbf{A}) \to \mathcal{C}(\mathbf{A})$ is a normal necessity operator satisfying further properties that will be specified in Section 3.

The fact that Lewis algebras are special boolean algebras with operators allows us to regard them as Kripke frames through the mirror of Jónnson-Tarski duality [5]. By exploiting the dual relation between algebraic and relational models, we will show that every Lewis algebra results in a Lewisian original model for counterfactuals based on spheres, and conversely we will also show that every sphere model determines a Lewis algebra. Consequently, we will show that a (somewhat stronger version of) Lewis' logic for counterfactuals is sound and complete with respect to Lewis algebras and their associated relational frames.

In the second part of the paper, we will address the problem of uncertain quantification of counterfactuals and relate it to the so-called Lewis' *imaging rule* (or simply *imaging*, henceforth) for probability updating. This method was introduced by Lewis in [39] and then further elaborated by Gärdenfors in [23] and others (see for instance [16,26]), to update an a priori probability distribution P on possible worlds, upon learning (or *imaging*) that some event b has occurred. The intuition is that the a priori probability distribution P is updated to a new probability P_b by redistributing the masses $P(\alpha)$, for α being a possible world in which b is false, to those α^*'s that instead satisfy b. The way chosen to redistribute the probability masses from the possible words not satisfying b to those that instead satisfy it determines several specific definitions of imaging. One, for instance, is the usual Bayesian conditioning rule that uniformly distributes the mass of the non-model of b to those of b.

The boolean algebra $\mathcal{C}(\mathbf{A})$ that appears in the definition of a Lewis algebra $\mathcal{L}(\mathbf{A})$ allows us to define probabilities on the latter in a quite natural way. However, instead of updating P by another probability function P_b, we follow a suggestion made by Dubois and Prade in [14] and we will consider that the imaging by an event b, of an a priori P on $\mathbf{A}$, is made in terms of a belief function Bel_b defined on $\mathbf{A}$. Then, we show that such an update is faithfully recovered on a Lewis algebra $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \Box)$ by combining the so-called *canonical extension* μ_P of P to $\mathcal{C}(\mathbf{A})$ (cf. [19,21]) with the modal operator $\Box$. The probability $\mu_P(b \Box\!\!\rightarrow a)$ of the counterfactual $b \Box\!\!\rightarrow a$ can hence be defined as $\mu_P(\Box(a \mid b))$. As we will prove in Section 7, $\mu_P(\Box(a \mid b))$ coincides with $\mathrm{Bel}_b(a)$, i.e., the value that the belief imaged by b gives to the plain event a.

The paper is structured as follows: In the next Section 2 we first recall the basis of Lewis logic for counterfactuals and its semantics based on sphere models (Subsection 2.1), while Subsection 2.2 presents boolean algebras of conditionals and their main properties. The main focus of our paper, namely Lewis algebras, is the subject of Section 3. The details about these algebras and their necessary structural properties are respectively given in Subsections 3.1 and 3.2 to enhance readability. The logic of Lewis algebras, denoted as C1$^+$, is introduced in Section 4. Specifically, in Subsection 4.1 we formally define our logic and prove its soundness and completeness with respect to the proper subclass of sphere models that we refer to as *total*. Then, in Subsection 4.2, we focus on that semantics and show how to define *canonical models*. The relationship between Lewis algebras, their associated Kripke frames, and total sphere models is the topic that we develop in Section 5. There, in Subsections 5.1, 5.2 and 5.3, we respectively show how to define Lewis algebras from sphere models, sphere models from Lewis algebras, and we prove a completeness theorem for C1$^+$ with respect to Lewis algebras. The second part of the present paper, in which we study ways for the uncertain quantification of counterfactuals and the imaging rule based on our algebraic setting opens with Section 6 where we briefly recall how belief functions can be defined by combining probability measures and a normal necessity operator $\Box$. In Section 7, we then apply these ideas to define probability functions on Lewis algebras. In the same Section 7, we prove the main result (of the second part) showing that the probability of a counterfactual is the belief imaged by its antecedent and, in turn, it also coincides with the canonical extension, on Lewis algebras, of our modal representation of the counterfactual itself. We then end the paper with Section 8 where we recap our main contributions and also present remarks and ideas for improvements and we outline our future work.

To facilitate the reading of the paper, some technical proofs have been relegated to an appendix.

2. Preliminaries

For this initial section on preliminary notions, we assume that the reader is familiar with both Lewis' work on logics for counterfactuals and their semantics, universal algebra, and basics of finite boolean algebras. We invite the interest reader to consult [38] and [8] for more details.

2.1. Lewis logic for counterfactuals

Let $\mathscr{L}$ be the language of classical logic on finitely many variables $p_1, \ldots, p_k$ and in the usual signature $\wedge, \vee, \to, \bot, \top$. Formulas of classical logic will be denoted by lower-case Greek letters with possible subscripts $\varphi, \varphi_1, \psi, \psi_1$, etc. Furthermore, let $\Box\!\!\rightarrow$ be a new symbol for a binary connective that applies to pairs of classical formulas, so that, if φ, ψ are formulas, then $\psi \Box\!\!\rightarrow \varphi$ is a formula as well.

The connective $\Box\!\!\rightarrow$ is what Lewis ([38]) calls the *counterfactual conditional connective* that allows reading a formula $\psi \Box\!\!\rightarrow \varphi$ as "if it were the case that ψ, then it would be the case that φ".

Notation 1. To improve the readability and clarify the interpretation of Lewis' formulas in the algebraic language that will be presented in Section 3, for any pair of classical formulas φ, ψ, we will use the notation $(\varphi \pitchfork \psi)$ instead of $\psi \,\square\!\!\rightarrow \varphi$ for the counterfactual conditional "φ, if it were ψ".

More precisely, our language, which we refer to as $\mathscr{L}^+$, is $\mathscr{L} \cup \{\pitchfork\}$, where counterfactual formulas are defined in the following inductive way:

- every classical formula φ is a counterfactual formula;
- for every pair of classical formulas φ, ψ, for ψ satisfiable, $(\varphi \pitchfork \psi)$ is a *basic* counterfactual formula;
- the set $\mathfrak{CF}$ of counterfactual formulas is the smallest set of formulas including the above ones and being closed under classical connectives $\wedge, \vee, \rightarrow, \bot, \top$.

$\mathfrak{CF}$ denotes the set of counterfactual formulas, and we use the upper-case Greek letters $\Phi, \Psi, \dots$ to denote formulas in $\mathfrak{CF}$. Notice that Lewis' original language for counterfactuals allows nested occurrences of the binary operator $\square\!\!\rightarrow$ that, as already stressed, we denote by $\pitchfork$. In the present paper, we adopt the simplified language defined above that essentially does not allow for nested occurrences of $\pitchfork$. It is worth recalling that similar approaches to deal with simplified languages where some logical operators do not occur nested are quite common in logic and knowledge representation (see for instance [2,17,25,33]). Possible extensions of our language to encompass nested, yet controlled, occurrences of the counterfactual operator are discussed in Subsection 8.2. However, notice that our simplified language does not exclude us from formalizing the axioms of Lewis logic C1 as all of them can be expressed in $\mathfrak{CF}$.

Definition 2.1. Lewis' logic for counterfactual C1 is the logic induced by the following Hilbert-style axiomatization on $\mathfrak{CF}$

(A) $(\varphi \pitchfork \varphi)$;
(B) $((\varphi \pitchfork \psi) \wedge (\psi \pitchfork \varphi)) \rightarrow ((\chi \pitchfork \varphi) \leftrightarrow (\chi \pitchfork \psi))$;
(C) $(\varphi \pitchfork \varphi \vee \psi) \vee (\psi \pitchfork \varphi \vee \psi) \vee ((\chi \pitchfork \varphi \vee \psi) \leftrightarrow (\chi \pitchfork \varphi) \wedge (\chi \pitchfork \psi))$;
(D) $(\varphi \pitchfork \psi) \rightarrow (\psi \rightarrow \varphi)$;
(E) $(\varphi \wedge \psi) \rightarrow (\varphi \pitchfork \psi)$.

The following are the deduction rules of C1.

(T) $\vdash \varphi$ for each classical tautology φ;
(DT) $\psi_1, \dots, \psi_k \vdash \varphi$ for all classical tautology of the from $(\psi_1 \wedge \dots \wedge \psi_k) \rightarrow \varphi$;
(DC) $(\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \varphi \vdash ((\psi_1 \pitchfork \gamma) \wedge \dots \wedge (\psi_n \pitchfork \gamma)) \rightarrow (\varphi \pitchfork \gamma)$.

Lewis ([36,38]) introduced several semantics for the logic C1. In the present paper, we will be mainly concerned with the most popular ones, called *Lewis's sphere models*.

Definition 2.2. A *sphere model* is a triple $\Sigma = (I, \mathscr{S}, v)$ where I is a non-empty set and $\mathscr{S}$ is a function $\mathscr{S} \colon I \rightarrow \mathcal{P}(\mathcal{P}(I))$ that assigns, to each $i \in I$, a set $\mathscr{S}_i$ of subsets of I with the following properties: for all $i \in I$,

(S1) $\mathscr{S}_i$ is nested: for all $S, T \in \mathscr{S}_i$, either $S \subseteq T$ or $T \subseteq S$;
(S2) for all $S \in \mathscr{S}_i$, $i \in S$;
(S3) either $\bigcup \mathscr{S}_i = \varnothing$, or $\{i\} \in \mathscr{S}_i$.

Moreover, for each propositional variable p, $v(p)$ is a subset of I.

Given a sphere model $\Sigma = (I, \mathscr{S}, v)$, the map v is extended to compound formulas as follows:

- $v(\varphi \pitchfork \psi) = \{i \in I \mid v(\psi) \cap \bigcup \mathscr{S}_i = \varnothing, \text{ or } \exists S \in \mathscr{S}_i \, (\varnothing \neq (v(\psi) \cap S) \subseteq v(\varphi))\}$;
- $v(\neg \Phi) = I \backslash v(\Phi)$;
- $v(\Phi \wedge \Psi) = v(\Phi) \cap v(\Psi)$.

We write $i \Vdash \Phi$, if $i \in v(\Phi)$. A formula Φ is *valid* if for all sphere model $\Sigma = (I, \mathscr{S}, v)$ and all $i \in I$, $i \Vdash \Phi$.

Theorem 2.3 ([36]). *The logic* C1 *is sound and complete w.r.t. the class of sphere models. In particular, a formula is a theorem of* C1 *iff it is valid in every sphere model.*

The next example shows that the conditional excluded middle principle is not valid in C1.

Example 2.4 *(Conditional Excluded Middle).* It is easy to see that the principle of *conditional excluded middle* $(\varphi \pitchfork \psi) \vee (\neg \varphi \pitchfork \psi)$ is not valid in sphere models and hence, by Theorem 2.3 below, it is not a theorem of C1. Consider in fact a sphere model $(I, \mathscr{S}, v)$ such that:

- $I = \{i, j, k\}$;
- $\mathscr{S}_i = \{\{i\}, \{i, j, k\}\}$, with $S_1 = \{i\}$ and $S_2 = \{i, j, k\}$, $\mathscr{S}_j = \{j\}$, and $\mathscr{S}_k = \{k\}$;
- for some propositional variables p, q, $v(p) = \{j, k\}$, $v(q) = \{j\}$

Notice that $i \not\Vdash (q \pitchfork p)$ since $S_2 \cap v(p) \not\subseteq v(q)$ and $i \not\Vdash (\neg q \pitchfork p)$ since $S_2 \cap v(p) \not\subseteq v(\neg q)$. □

2.2. Boolean algebras of conditionals

In this second subsection, we recall basic notions and results from [19] that presents a construction for building a *boolean algebra of conditionals*, BAC for short, denoted by $\mathcal{C}(\mathbf{A})$, for any boolean algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \perp, \top)$. Intuitively, a BAC over $\mathbf{A}$ allows for the presence of *basic conditionals*, i.e. objects of the form $(a \mid b)$ for $a \in A$ and $b \in A' = A \setminus \{\perp\}$. These basic conditionals can be combined with the usual boolean operations, subject to certain constraints. In particular, a BAC must satisfy the following requirements:

(R1) For every $b \in A'$, the conditional $(b \mid b)$ will be the top element of $\mathcal{C}(\mathbf{A})$, while $(\neg b \mid b)$ will be the bottom;
(R2) Given $b \in A'$, the set of conditionals $A \mid b = \{(a \mid b) : a \in A\}$ will be the domain of a boolean subalgebra of $\mathcal{C}(\mathbf{A})$, and in particular when $b = \top$, this subalgebra will be isomorphic to $\mathbf{A}$;
(R3) In a conditional $(a \mid b)$ we can replace the consequent a by $a \wedge b$, that is, the conditionals $(a \mid b)$ and $(a \wedge b \mid b)$ represent the same element of $\mathcal{C}(\mathbf{A})$;
(R4) For all $a \in A$ and all $b, c \in A'$, if $a \leqslant b \leqslant c$, then the result of conjunctively combining the conditionals $(a \mid b)$ and $(b \mid c)$ must yield the conditional $(a \mid c)$.

Notice that R4 encodes a sort of restricted chaining of conditionals and it is inspired by the chain rule of conditional probabilities: $P(a \mid b) \cdot P(b \mid c) = P(a \mid c)$ whenever $a \leqslant b \leqslant c$.

In mathematical terms, the formal construction of the algebra of conditionals $\mathcal{C}(\mathbf{A})$ is as follows. First, consider the free boolean algebra $\mathbf{Free}(A \mid A') = (Free(A \mid A'), \sqcap, \sqcup, \Rightarrow, \perp^*, \top^*)$ generated by the set $A \mid A' = \{(a \mid b) \mid a \in A, b \in A'\}$. For every element $t \in Free(A \mid A')$, we abbreviate $t \Rightarrow \perp^*$ by $\sim t$. Second, take the smallest congruence relation $\equiv_{\mathfrak{c}}$ on $\mathbf{Free}(A \mid A')$ such that:

(C1) $(b \mid b) \equiv_{\mathfrak{c}} \top^*$, for all $b \in A'$;
(C2) $(a_1 \mid b) \sqcap (a_2 \mid b) \equiv_{\mathfrak{c}} (a_1 \wedge a_2 \mid b)$, for all $a_1, a_2 \in A$, $b \in A'$;
(C3) $\sim (a \mid b) \equiv_{\mathfrak{c}} (\neg a \mid b)$, for all $a \in A$, $b \in A'$;
(C4) $(a \wedge b \mid b) \equiv_{\mathfrak{c}} (a \mid b)$, for all $a \in A$, $b \in A'$;
(C5) $(a \mid b) \sqcap (b \mid c) \equiv_{\mathfrak{c}} (a \mid c)$, for all $a \in A$, $b, c \in A'$ such that $a \leqslant b \leqslant c$.

Note that (C1)-(C5) faithfully account for the requirements R1-R4 where, in particular, (C2) and (C3) account for R2. Finally, the algebra $\mathcal{C}(\mathbf{A})$ is defined as follows.

Definition 2.5. For every boolean algebra $\mathbf{A}$, the *boolean algebra of conditionals* of $\mathbf{A}$ is the quotient structure

$$\mathcal{C}(\mathbf{A}) = \mathbf{Free}(A \mid A')/_{\equiv_{\mathfrak{c}}}.$$

To distinguish the operations of $\mathbf{A}$ from those of $\mathcal{C}(\mathbf{A})$, the following signature is adopted:

$$\mathcal{C}(\mathbf{A}) = (\mathcal{C}(A), \sqcap, \sqcup, \sim, \Rightarrow, \perp_{\mathfrak{c}}, \top_{\mathfrak{c}})$$

Since $\mathcal{C}(\mathbf{A})$ is a *quotient* of $\mathbf{Free}(A \mid A')$, its elements are equivalence classes. However, without danger of confusion, we will henceforth identify classes $[t]_{\equiv_{\mathfrak{c}}}$ with one of its representative elements, in particular, by t itself.

A basic observation is that if $\mathbf{A}$ is finite, $\mathcal{C}(\mathbf{A})$ is finite as well, and hence atomic. Indeed, if $\mathbf{A}$ is a boolean algebra with n atoms, i.e. $|\mathtt{at}(\mathbf{A})| = n$, it is shown in [19, Theorem 4.4] that the atoms of $\mathcal{C}(\mathbf{A})$ are in one-to-one correspondence with sequences $\langle \alpha_1, \ldots, \alpha_n \rangle$ of pairwise different atoms of $\mathbf{A}$ and of maximal length. Each of these sequences gives rise to an atom ω defined as the following conjunction of $n - 1$ basic conditionals:

$$\omega = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg \alpha_1) \sqcap \ldots \sqcap (\alpha_{n-1} \mid \neg \alpha_1 \wedge \ldots \wedge \neg \alpha_{n-2}). \tag{1}$$

It is clear that $|\mathtt{at}(\mathcal{C}(\mathbf{A}))| = n!$ and hence, although finite, $\mathcal{C}(\mathbf{A})$ is considerably larger than the original algebra $\mathbf{A}$.

Notation 2. Given the bijective correspondence between atoms of $\mathcal{C}(\mathbf{A})$ and strings of pairwise different atoms of $\mathbf{A}$ of maximal length recalled above, for every $\omega \in \mathrm{o}\mathbb{t}(\mathcal{C}(\mathbf{A}))$ and every $j = 1, \ldots, n$, we denote by $\omega[j]$ the atom of $\mathbf{A}$ that appears in the jth position in the (unique) string that defines ω by (1).

The next result ([19, Proposition 4.7]) provides a useful characterization for the atoms of $\mathcal{C}(\mathbf{A})$ that are below, in the lattice order of $\mathcal{C}(\mathbf{A})$, of a basic conditional of the form $(a \mid b)$. From now on (exactly as in [19]) we will use the same symbol $\leqslant$ to denote both the lattice order induced by the algebra $\mathbf{A}$ and the corresponding BAC, $\mathcal{C}(\mathbf{A})$.

Proposition 2.6. *For every $n \in \mathbb{N}$, if $\mathbf{A}$ is the boolean algebra with n atoms, for every basic conditional $(a \mid b) \in \mathcal{C}(\mathbf{A})$ and $\omega \in \mathrm{o}\mathbb{t}(\mathcal{C}(\mathbf{A}))$, $\omega \leqslant (a \mid b)$ iff $\omega[j] \leqslant a \wedge b$ and for all $i < j$, $\omega[i] \nleqslant b$.*

The following recaps some properties of BAC that will be used insofar and whose proof can be found in [19, Proposition 3.8].

Proposition 2.7. *In every BAC $\mathcal{C}(\mathbf{A})$ the following properties hold: for every $a \in A$ and $b, c \in A'$,*

1. $(a \wedge b \mid \top) \leqslant (a \mid b) \leqslant (b \to a \mid \top)$;
2. *if $a \leqslant b$ then $(a \mid c) \leqslant (b \mid c)$;*
3. $(a \mid b) \wedge (a \mid c) \leqslant (a \mid b \vee c)$.

3. Lewis algebras and their relational frames

As recalled in the introduction, our methodological approach is mainly algebraic; in this section, we introduce the main characters of the present paper that we call *Lewis algebras*. They consist of modal expansions of boolean algebras of conditionals. As we will show later on, an (almost immaterial) extension of Lewis logic C1 turns out to be sound and complete with respect to these algebraic models.

3.1. Lewis algebras and Lewis frames

Let us hence start by defining and studying some basic properties of the following modal expansions of boolean algebras for conditionals.

Definition 3.1. For every boolean algebra $\mathbf{A}$, a *Lewis algebra of $\mathbf{A}$* is a pair $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \square)$ where $\mathcal{C}(\mathbf{A})$ is the boolean algebra of conditionals of $\mathbf{A}$ and $\square$ is a unary operator on $\mathcal{C}(\mathbf{A})$ satisfying the following equations:

(L1) $\square(t \Rightarrow t') \leqslant (\square t \Rightarrow \square t')$ for all $t, t' \in \mathcal{C}(\mathbf{A})$;
(L2) $\square(a \mid \top) = (a \mid \top)$ for all $a \in A$;
(L3) $\square(a \mid a \vee b) \sqcup \square(b \mid a \vee b) \sqcup [\square(c \mid a \vee b) \Rightarrow \square((c \mid a) \sqcap (c \mid b))] = 1$ for all $a, b, c \in A$ with $a, b \neq \bot$.

In the remainder of the paper, we will mainly focus on finite Lewis algebras. However, the construction of $\mathcal{L}(\mathbf{A})$ is general and it applies to any initial boolean algebra $\mathbf{A}$. Indeed, given any boolean algebra $\mathbf{A}$, and being $\mathcal{C}(\mathbf{A})$ its boolean algebra of conditionals, there is always a unary operator $\square$ on $\mathcal{C}(\mathbf{A})$ that satisfies (L1), (L2) and (L3), namely, the identity map.

We will henceforth denote by $\mathbb{L}$ the class of all Lewis algebras $\mathcal{L}(\mathbf{A})$ for $\mathbf{A}$ being any finite boolean algebra. In other words

$\mathbb{L} = \{\mathcal{L}(\mathbf{A}) \mid \mathbf{A} \text{ is a finite boolean algebra}\}$.

To every finite boolean algebra $\mathbf{A}$ there correspond possibly several, non-isomorphic, Lewis algebras, each defined by different $\square$ operators, satisfying (L1), (L2), and (L3), which can be defined upon the conditional algebra $\mathcal{C}(\mathbf{A})$. It should be noted that, although conditions (L1), (L2), and (L3) are expressed by equations, the class of Lewis algebras is not a *variety of algebras*, meaning it is not an *equational class* (cf. [8]). This is because boolean algebras of conditionals do not form a variety, as pointed out in [19]. Before studying some basic properties of Lewis algebras, let us briefly comment on the equations that describe the unary operation $\square$.

Remark 3.2. The first equation (L1) is the algebraic version of the usual axiom (K) of modal logic. Also, recall that the top element of $\mathcal{C}(\mathbf{A})$ is $(\top \mid \top)$ and notice that $\square(\top \mid \top) = (\top \mid \top)$ is simply an instantiation of (L2). Therefore, the latter equation, together with (L1), ensures that the $\square$ of any Lewis algebra is normal (cf. [5]). Besides ensuring the normality of $\square$, (L2) has an interesting interpretation. In fact, when interpreting counterfactual statements $(a \,\phi\, b)$ as $\square(a \mid b)$ in Lewis algebras, (L2) can be read as "trivial counterfactuals are trivial conditionals", where the triviality comes from considering the sure event formula $\top$ as antecedent. Finally, (L3) is the most intricate condition for $\square$, and it arises from the equally

intricate axiom (C) of Lewis logic C1 as in Definition 2.1. However, it is worth noting that, compared to (C), the equation (L3) has a slightly simplified form. As we prove below in this section, this simplification has been made possible by the structural properties inherited from boolean algebras of conditionals. In Lemma 3.8 below and, more accurately in the next Subsection 3.2, we will take advantage of our algebraic realm to provide more insights into the meaning of (L3) and its effect on Lewis algebras.

As recalled in Subsection 2.2 every $\mathcal{C}(\mathbf{A})$ is a boolean algebra and hence every Lewis algebra $\mathcal{L}(\mathbf{A})$ is a *boolean algebra with operators*, BAO in brief (as usual in modal logic [5]). A first immediate consequence of this fact is that in every Lewis algebra $\mathcal{L}(\mathbf{A})$ we can define the modal operator $\Diamond$, dual to $\Box$, as customary: for all $t \in \mathcal{C}(\mathbf{A})$, $\Diamond t = \neg\Box\neg t$.

Proposition 3.3. *The following equations hold in every Lewis algebra $\mathcal{L}(\mathbf{A})$:*

1. $\Box(a \mid a) = \top_{\mathfrak{C}}$;
2. $(a \wedge b \mid \top) \leqslant \Box(a \mid b) \leqslant (b \rightarrow a \mid \top)$;
3. $\Diamond(a \mid \top) = (a \mid \top)$;
4. $\Box(a \mid b) \leqslant \Diamond(a \mid b)$;
5. $\Box((a \mid b) \wedge (a \mid c)) \leqslant \Box(a \mid b \vee c)$,

for every $a \in A$ and $b, c \in A'$.

Proof. Let us start noticing that equation (L1) implies, as usual, the monotonicity of $\Box$: for all $t, s \in \mathcal{L}(\mathbf{A})$, if $t \leqslant s$, then $\Box t \leqslant \Box s$. The same (L1) implies that $\Box(\top_{\mathfrak{C}}) = \top_{\mathfrak{C}}$.

1. By construction $(a \mid a)$ coincides, in $\mathcal{C}(\mathbf{A})$, with $\top_{\mathfrak{C}}$. Thus the claim follows from the above Remark 3.2.

2. By Proposition 2.7 (1), $(a \wedge b \mid \top) \leqslant (a \mid b) \leqslant (b \rightarrow a \mid \top)$. Then, by monotonicity of $\Box$, one has $\Box(a \wedge b \mid \top) \leqslant \Box(a \mid b) \leqslant \Box(b \rightarrow a \mid \top)$. Thus, equation (L2) implies $(a \wedge b \mid \top) \leqslant \Box(a \mid b) \leqslant (b \rightarrow a \mid \top)$.

3. By (L2), $\Diamond(a \mid \top) = \sim\Box\sim(a \mid \top) = \sim\Box(\neg a \mid \top) = \sim(\neg a \mid \top) = (\neg\neg a \mid \top) = (a \mid \top)$.

4. $\Box(a \mid b) \Rightarrow \Diamond(a \mid b) = \sim\Box(a \mid b) \sqcup \Diamond(a \mid b) = \Diamond\sim(a \mid b) \sqcup \Diamond(a \mid b) = \Diamond(\neg a \mid b) \sqcup \Diamond(a \mid b) = \Diamond(\neg a \vee a \mid b) = \Diamond(\top \mid b) = \Diamond(\top \mid \top) = (\top \mid \top)$, where the last equality follows from 3. Thus, $\Box(a \mid b) \leqslant \Diamond(a \mid b)$.

5. By Proposition 2.7 (3) and the monotonicity property of $\Box$. $\qquad\square$

Remark 3.4. We saw in Example 2.4 that the Conditional Excluded Middle $(a \pitchfork b) \vee (\neg a \pitchfork b)$ does not hold in Lewis sphere semantics. Analogously, that principle does not hold in Lewis algebras as well. Indeed, although $(a \mid b) \vee (\neg a \mid b)$ is valid in all boolean algebras of conditionals (see [19]), the presence of the modal $\Box$ makes it fail in some Lewis algebras. Indeed, as a counterfactual $(a \pitchfork b)$ is interpreted in an appropriate Lewis algebra as $\Box(a \mid b)$, it is enough to show that $\Box(a \mid b) \vee \Box(\neg a \mid b)$ fails in some $\mathcal{L}(\mathbf{A})$. Now, since $(\neg a \mid b) = \neg(a \mid b)$ is true in every boolean algebra of conditionals, the previous $\Box(a \mid b) \vee \Box(\neg a \mid b)$ becomes $\Box(a \mid b) \vee \Box\neg(a \mid b) = \Box(a \mid b) \vee \neg\Diamond(a \mid b)$. Thus, to get a counterexample of the Conditional Excluded Middle principle is enough to consider a Lewis algebra $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \Box)$ in which for some $(a \mid b) \in \mathcal{C}(\mathbf{A})$, $\Box(a \mid b) < \top$, $\Diamond(a \mid b) > \bot$ and $\neg\Diamond(a \mid b) < \neg\Box(a \mid b)$. Notice that this requirement is indeed compatible with what we proved in the above Proposition 3.3 (4).

A second immediate consequence of the fact that Lewis algebras are BAOs is that, from the well-known Jónsson-Tarski duality between (finite) BAOs and Kripke frames (see [5] and [35]), to each Lewis algebra $\mathcal{L}(\mathbf{A})$ it is uniquely associated a frame (W, R) in which

- W is the set $\mathfrak{at}(\mathcal{C}(\mathbf{A}))$ of atoms of $\mathcal{C}(\mathbf{A})$;
- R is the binary relation on $\mathfrak{at}(\mathcal{C}(\mathbf{A}))$ such that, for all $\omega, \omega' \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$, $\omega R \omega'$ iff $\forall t \in \mathcal{C}(\mathbf{A})$ if $\omega \leqslant \Box t$, then $\omega' \leqslant t$.

For every Lewis algebra $\mathcal{L}(\mathbf{A})$ we will say that the relational frame $(\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ defined as above is a *Lewis relational frame* (or simply a *Lewis frame*) which will be denoted by $\mathsf{F}_{\mathcal{L}(\mathbf{A})}$.

Definition 3.5. For every finite boolean algebra $\mathbf{A}$, for every element $t \in \mathcal{L}(\mathbf{A})$ and for every $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$, we say that ω *models* t and we write $\omega \Vdash t$ according to the next inductive definition:

1. if t is a basic conditional $(a \mid b) \in \mathcal{C}(\mathbf{A})$, $\omega \Vdash (a \mid b)$ if $\omega \leqslant (a \mid b)$ in $\mathcal{C}(\mathbf{A})$;
2. if $t = \Box(a \mid b)$, $\omega \Vdash \Box(a \mid b)$ if for all ω' such that $\omega R \omega'$, $\omega' \Vdash (a \mid b)$;
3. if $t = \sim s$, $\omega \Vdash \sim s$ if $\omega \nVdash s$;
4. if $t = s \wedge r$, $\omega \Vdash s \wedge r$ if $\omega \Vdash s$ and $\omega \Vdash r$;
5. if $t = s \vee r$, $\omega \Vdash s \vee r$ if $\omega \Vdash s$ or $\omega \Vdash r$.

Let us recall from Subsection 2.2 that, for every atom ω of $\mathcal{C}(\mathbf{A})$ there exists a unique string $\langle \alpha_1, \ldots, \alpha_n \rangle$ of atoms of $\mathbf{A}$ such that, for any basic conditional $(a \mid b) \in \mathcal{C}(\mathbf{A})$, $\omega \leqslant (a \mid b)$ iff for the first index j such that $\alpha_j \leqslant b$, then $\alpha_j \leqslant a$ as well.

Therefore, according to Definition 3.5, and adopting the notation of Section 2.2, $\omega \Vdash \Box(a \mid b)$ iff for all $\omega' \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ such that $\omega R \omega'$, $\omega'[j] \leqslant a \wedge b$ and for all $i < j$, $\omega'[i] \not\leqslant b$.

The following is an immediate consequence of a well-known and general result.

Fact 1. *For every finite boolean algebra* $\mathbf{A}$ *and for all* $t, s \in \mathcal{L}(\mathbf{A})$, *the equation* $t = s$ *holds in* $\mathcal{L}(\mathbf{A})$ *iff for all* $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$, $\omega \Vdash t$ *iff* $\omega \Vdash s$. *Thus, in particular, the classes of Lewis algebras and Lewis frames share the same tautologies written in the language of Lewis algebras.*

We shall see that the binary relation R of every Lewis frame satisfies some additional properties. To show that, some preliminary notions are needed.

Notation 3. For a boolean algebra $\mathbf{A}$ and $a \in A$, we indicate with $\llbracket a \rrbracket = \{\alpha \in \mathfrak{at}(\mathbf{A}) \mid \alpha \leqslant a\}$ the set of atoms of $\mathbf{A}$ that are below a. Furthermore, for every Lewis frame $F_{\mathcal{L}(\mathbf{A})} = (\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ and for every $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$, we will write $R[\omega]$ to denote $\{\omega' \in \mathfrak{at}(\mathcal{C}(\mathbf{A})) \mid \omega R \omega'\}$.

Definition 3.6. Given a Lewis frame $\mathcal{L}(\mathbf{A}) = (\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$, the *selection function* on $\mathcal{L}(\mathbf{A})$, is a map $f : \mathbf{A} \times \mathfrak{at}(\mathcal{C}(\mathbf{A})) \to \mathcal{P}(\mathfrak{at}(\mathbf{A}))$ defined as follows: for all $a \in \mathbf{A}$, for all $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$,

$$f(a, \omega) = \{\omega'[i] \in \mathfrak{at}(\mathbf{A}) \mid \omega R \omega' \text{ and } i \text{ is the smallest index s.t. } \omega'[i] \leqslant a\}$$

Remark 3.7. Note that, by the above definition of selection function, the semantic clauses for basic conditionals in Definition 3.5 (2) can be reformulated as:

(2') $\omega \Vdash \Box(a \mid b)$ iff $f(b, \omega) \subseteq \llbracket a \rrbracket$

The next lemma shows some properties of the selection function defined as in Definition 3.6 and the accessibility relation of a Lewis frame.

Lemma 3.8. *In every Lewis frame* $(\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$, *the binary relation* R *and the selection function* f *on* $(\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ *satisfy the following properties:*

1. $\forall \omega \exists \omega' \, (\omega R \omega')$ *(seriality)*
2. $\forall \omega, \omega' \, (\omega R \omega' \to (\omega[1] = \omega'[1]))$ *(centering)*
3. $\forall a \forall b \forall \omega, f(a \vee b, \omega) \subseteq \llbracket a \rrbracket$ or $f(a \vee b, \omega) \subseteq \llbracket b \rrbracket$ or $f(a, \omega) \cup f(b, \omega) \subseteq f(a \vee b, \omega)$ *(sphericity)*

Proof. 1. Assume, by contradiction, that the Lewis frame $(\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ does not satisfy seriality. In particular, let $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ such that, for all $\omega' \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ it is not the case that $\omega R \omega'$. Furthermore, let $a \in A$ be such that $\omega[1] \not\leqslant a$. Therefore, vacuously $\omega \Vdash \Box(a \mid \top)$, since no ω' is accessible from ω. On the other hand, $\omega \not\Vdash (a \mid \top)$ because, by Proposition 2.6 $\omega[1] \leqslant \top$, but $\omega[1] \not\leqslant a$ by assumption. Therefore, by Fact 1, one has that the equation $\Box(a \mid \top) = (a \mid \top)$ does not hold in $\mathcal{L}(\mathbf{A})$ which leads to a clear contradiction with the very definition of Lewis algebra.

2. By way of contradiction, assume that there exists ω, ω' such that $\omega R \omega'$, and $\omega[1] \neq \omega'[1]$. Let $a \in A$ be such that $\omega[1] \leqslant a$, but $\omega'[1] \not\leqslant a$ (for instance, take $a = \omega[1]$). Then, $\omega \Vdash (a \mid \top)$, but $\omega \not\Vdash \Box(a \mid \top)$ and hence, by the same argument used in the previous point, this is in contradiction with the fact that $\Box(a \mid \top) = (a \mid \top)$ holds in $\mathcal{L}(\mathbf{A})$, for any $a \in A$.

3. Assume by way of contradiction that there are $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ and $a, b \in \mathbf{A}$ such that $f(a \vee b, \omega) \not\subseteq \llbracket a \rrbracket$ and $f(a \vee b, \omega) \not\subseteq \llbracket b \rrbracket$ and $f(a, \omega) \cup f(b, \omega) \not\subseteq f(a \vee b, \omega)$. Then, by Remark 3.7, $\omega \not\Vdash \Box(a \mid a \vee b)$, $\omega \not\Vdash \Box(b \mid a \vee b)$. Moreover, since $f(a, \omega) \cup f(b, \omega) \not\subseteq f(a \vee b, \omega)$, we have that for some $\alpha \in f(a, \omega) \cup f(b, \omega)$, $\alpha \notin f(a \vee b, \omega)$. Without loss of generality, assume $\alpha \in f(a, \omega)$. Now, consider $c = \bigvee(f(a \vee b, \omega))$. Clearly $c \in \mathbf{A}$ and for all $x \in f(a \vee b, \omega)$, $x \leqslant c$ so that $x \in \llbracket c \rrbracket$, whence $f(a \vee b, \omega) \subseteq \llbracket c \rrbracket$. Therefore, by Remark 3.7, $\omega \Vdash \Box(c \mid a \vee b)$; however, since $\alpha \notin f(a \vee b, \omega)$ by assumption, then $\alpha \not\leqslant c$. And so, $\omega \not\Vdash \Box(c \mid a)$, hence $\omega \not\Vdash \Box(c \mid b) \sqcap \Box(c \mid a)$, thus $\omega \not\Vdash \Box(c \mid a \vee b) \Rightarrow \Box((c \mid b) \sqcap (c \mid a))$. Therefore, the condition (L3) does not hold in $(\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ contradicting the definition of Lewis algebra. $\square$

It follows from the proof of Lemma 3.8 that, indeed, seriality and centering are consequences of the fact that Lewis algebras satisfy (L2). Also, a partial version of the converse statement holds, as shown by the following.

Lemma 3.9. *Let* $\mathbf{A}$ *be a finite boolean algebra and* R *a binary relation on* $\mathfrak{at}(\mathcal{C}(\mathbf{A}))$. *If* R *satisfies seriality and centering, the algebra* $(\mathcal{C}(\mathbf{A}), \Box)$ *satisfies* $\Box(a \mid \top) = (a \mid \top)$.

Proof. Consider $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ such that $\omega \leqslant \Box(a \mid \top)$. Then, for all ω' such that $\omega R \omega'$, $\omega' \leqslant (a \mid \top)$. Since, by seriality, $R[\omega] \neq \varnothing$, take a $\omega' \in R[\omega]$. By assumption, it holds that $\omega' \leqslant (a \mid \top)$, thus $\omega'[1] \leqslant a$. By centering, $\omega'[1] = \omega[1]$, hence it

holds that $\omega[1] \leqslant a$. So, $\omega \leqslant (a \mid \top)$, therefore $\Box(a \mid \top) \leqslant (a \mid \top)$. Now, consider $\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$ such that $\omega \leqslant (a \mid \top)$. Then, $\omega[1] \leqslant a$. By seriality, $R[\omega] \neq \varnothing$. Then, take $\omega' \in R[\omega]$; by centering $\omega'[1] = \omega[1]$, then $\omega'[1] \leqslant a$. Therefore $\omega' \leqslant (a \mid \top)$. Since ω was taken arbitrarily in $R[\omega]$, then for all $\omega^* \in R[\omega]$, $\omega^* \leqslant (a \mid \top)$. Hence, $\omega \leqslant \Box(a \mid \top)$, and so $(a \mid \top) \leqslant \Box(a \mid \top)$. Therefore, $\Box(a \mid \top) = (a \mid \top)$. $\qquad\qquad\square$

The content of the two previous lemmas is hence recapped by the following.

Corollary 3.10. *Let* $\mathbf{A}$ *be any finite boolean algebra. Then the boolean algebra with operators* $(\mathcal{C}(\mathbf{A}), \Box)$ *satisfies* $\Box(a \mid \top) = (a \mid \top)$ *iff its corresponding frame* $(\mathbb{at}(\mathcal{C}(\mathbf{A})), R)$ *is such that* R *satisfies seriality and centering.*

Lemma 3.8 (3) provides a first description of equation (L3) on Lewis frames. However, for a more comprehensive characterization of (L3) that we anticipated in Remark 3.2 we will need to introduce additional combinatorial tools that will be developed in the next Subsection 3.2 below. Our desired characterization will be proved in Theorem 3.14 below.

3.2. A better understanding of the sphericity condition

Let us start fixing a generic frame $\mathsf{F} = (\mathbb{at}(\mathcal{C}(\mathbf{A})), R)$ based on the set of atoms of a boolean algebra of conditionals. Let ω be an atom of $\mathcal{C}(\mathbf{A})$ and, as in Notation 3, let $R[\omega]$ be the set $\{\omega_1, \ldots, \omega_k\}$ of elements of $\mathbb{at}(\mathcal{C}(\mathbf{A}))$ that are accessible via R from ω. For the sake of clarity and recalling that every atom of $\mathcal{C}(\mathbf{A})$ is uniquely identified with a string of length $|\mathbb{at}(\mathbf{A})| = n$ of atoms of the original algebra $\mathbf{A}$, let us denote by $\mathbf{R}[\omega]$ the $k \times n$ matrix whose generic element $a_{i,j}$ is $\omega_i[j] \in \mathbb{at}(\mathbf{A})$ (see Notation 2). Notice that, if the frame F satisfies seriality and centering, i.e., by Corollary 3.10 its associated algebra $(\mathcal{C}(\mathbf{A}), \Box)$ satisfies (L2), then $\mathbf{R}[\omega]$ is nonempty and its first column $\mathbf{c}_1$ has for supporting set $c_1 = \{\omega[1]\}$. From now on, we will always assume that F satisfies seriality and centering. Before introducing the main definition of this section, let us fix some additional notation:

- For every matrix $\mathbf{M}$, M denotes the set of its elements;
- If $\mathbf{C}$ and $\mathbf{D}$ are matrices with the same number of rows, then $\mathbf{C} \cdot \mathbf{D}$ denotes the juxtaposition of $\mathbf{C}$ and $\mathbf{D}$.

Moreover, for all $i = 1, \ldots, k$ and $j = 1, \ldots, n$ we will denote by:

- $\mathbf{r}_i$ the ith row of $\mathbf{R}[\omega]$ and by r_i the set of its elements;
- $\mathbf{c}_j$ the jth column of $\mathbf{R}[\omega]$, while c_j stands for the set of its elements.

For every $\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$, consider the general construction defined by the following steps:

Step 1. Take $\mathbf{c}_1$, the first column of $\mathbf{R}[\omega]$ and let $|c_1| = m_1$, i.e. assume that the first column of $\mathbf{R}[\omega]$ contains m_1 elements of $\mathbb{at}(\mathbf{A})$. Then, call $\mathbf{C}_1$ the submatrix of $\mathbf{R}[\omega]$ whose columns are, in the order, $\mathbf{c}_1, \ldots, \mathbf{c}_{m_1}$.

Step 2. If $m_1 = n$, necessarily $\mathbf{C}_1 = \mathbf{R}[\omega]$ since they have the same columns. Then, stop the construction. Otherwise consider the first column, in the usual order of $\{1, \ldots, n\}$, of $\mathbf{R}[\omega]$ that does not appear in $\mathbf{C}_1$, i.e., $\mathbf{c}_{m_1+1}$ and let $|c_{m_1+1}| = m_2$.
- If $m_1 + m_2 \geqslant n$, then take $\mathbf{C}_2$ to be the submatrix of $\mathbf{R}[\omega]$ with columns $\mathbf{c}_{m+1}, \ldots, \mathbf{c}_n$ and the construction stops.
- Otherwise, if $m_1 + m_2 < n$ define $\mathbf{C}_2$ to be the submatrix of $\mathbf{R}[\omega]$ with columns $\mathbf{c}_{m_1+1}, \ldots, \mathbf{c}_{m_1+m_2}$.

Step t. (inductively). If $m_1 + m_2 + \ldots + m_{t-1} = n$, $\mathbf{C}_1 \cdot \mathbf{C}_2 \cdots \mathbf{C}_{t-1} = \mathbf{R}[\omega]$. In such a case, stop the construction. Otherwise take the column $\mathbf{c}_{m_1+m_2+\ldots+m_{t-1}+1}$ and set $|c_{m_1+m_2+\ldots+m_{t-1}+1}| = m_t$.
- If $m_1 + m_2 + \ldots + m_{t-1} + m_t \geqslant n$ take $\mathbf{C}_t$ be the submatrix of $\mathbf{R}_\omega$ with columns $\mathbf{c}_{m_t+1}, \ldots, \mathbf{c}_n$ and stop the construction.
- Otherwise, if $m_1 + m_2 + \ldots + m_{t-1} + m_t < n$, define $\mathbf{C}_t$ to be the submatrix of $\mathbf{R}[\omega]$ with columns $\mathbf{c}_{m_1+m_2+\ldots+m_{t-1}+1}, \ldots, \mathbf{c}_{m_1+m_2+\ldots+m_{t-1}+m_t}$.

Since for every $\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$ the matrix $\mathbf{R}[\omega]$ has finitely many columns, the above procedure stops and it outputs a partition $\mathbb{C}_\omega = \{\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_t\}$ of $\mathbf{R}[\omega]$ into submatrices of the same.

Definition 3.11. Let $\mathsf{F} = (\mathbb{at}(\mathcal{C}(\mathbf{A})), R)$, $\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$ and $\mathbf{R}[\omega]$ be as above. Then we say that $\mathbf{R}[\omega]$ is *sliceable* if the partition $\mathbb{C}_\omega = \{\mathbf{C}_1, \ldots, \mathbf{C}_t\}$ is such that

$(*)$ for all $l = 1, \ldots, t$, calling $\mathbf{c}_l$ is the first column of $\mathbf{C}_l$, we have that for every row $\mathbf{r}$ of $\mathbf{C}_l$, $c_l = r$.

In such case, we also say that $\mathbb{C}_\omega$ is a *slice partition* of $\mathbf{R}[\omega]$. We say that F is *sliceable* if, for all $\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$, $\mathbf{R}[\omega]$ is sliceable.

To clarify the above definition and the introduced construction, let us consider the finite boolean algebra $\mathbf{A}$ with atoms $\{\alpha_1, \ldots, \alpha_6\}$ and let $\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$ be such that $\omega[1] = \alpha_1$. Let us consider the following two cases:
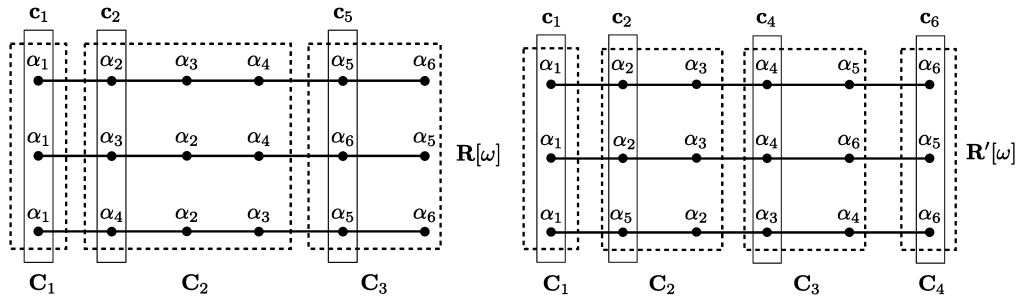
Fig. 1. On the left-hand-side, the sliceable matrix $\mathbf{R}[\omega]$ defined as in the above case 1; on the right-hand-side the matrix $\mathbf{R}'[\omega]$ that is not sliceable because, for instance, $c_2 = \{\alpha_2, \alpha_5\}$, but the first row $\mathbf{r}$ of the submatrix $\mathbf{C}_2$ contains α_3 and it does not contain α_5.

1. $R[\omega] = \{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle, \langle \alpha_1, \alpha_3, \alpha_2, \alpha_4, \alpha_6, \alpha_5 \rangle, \langle \alpha_1, \alpha_4, \alpha_2, \alpha_3, \alpha_5, \alpha_6 \rangle\}$
2. $R'[\omega] = \{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle, \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_5 \rangle, \langle \alpha_1, \alpha_5, \alpha_2, \alpha_3, \alpha_4, \alpha_6 \rangle\}$

Notice that for $\mathbf{R}[\omega]$ is the following 3×6 matrix:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_1 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_6 & \alpha_5 \\ \alpha_1 & \alpha_4 & \alpha_2 & \alpha_3 & \alpha_5 & \alpha_6 \end{pmatrix}$$

and $\mathbb{C}_\omega$ is the following:

- $\mathbf{C}_1 = \langle \alpha_1, \alpha_1, \alpha_1 \rangle$ and it satisfies the first condition of $(*)$;
- $\mathbf{C}_2$ has for columns $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$, $\langle \alpha_3, \alpha_2, \alpha_2 \rangle$ and $\langle \alpha_4, \alpha_4, \alpha_3 \rangle$. Its rows hence are $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$; $\langle \alpha_3, \alpha_2, \alpha_4 \rangle$ and $\langle \alpha_4, \alpha_2, \alpha_3 \rangle$. Each row is indeed a permutation of its first column $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ of $\mathbf{C}_2$ so satisfying $(*)$;
- $\mathbf{C}_3$ has columns $\langle \alpha_5, \alpha_6, \alpha_5 \rangle$ and $\langle \alpha_6, \alpha_5, \alpha_6 \rangle$. Again its rows are $\langle \alpha_5, \alpha_6 \rangle$, $\langle \alpha_6, \alpha_5 \rangle$ and $\langle \alpha_5, \alpha_6 \rangle$. So, $\mathbf{C}_3$ satisfies $(*)$ as its rows only contain α_5 and α_6.

Moreover $\mathbf{C}_1 \cdot \mathbf{C}_2 \cdot \mathbf{C}_3 = \mathbf{R}[\omega]$. Thus, $\mathbb{C}_\omega$ is a slice partition of $\mathbf{R}[\omega]$.

Now, let us consider the $R'[\omega]$ case. $\mathbf{R}'[\omega]$ is the following 3×6 matrix:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_6 & \alpha_5 \\ \alpha_1 & \alpha_5 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_6 \end{pmatrix}$$

and $\mathbb{C}_\omega$ is computed as follows:

- $\mathbf{C}_1$ has only one column $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$, so $m_1 = 1$;
- $\mathbf{C}_2$ is as follows: since $\mathbf{c}_2 = \langle \alpha_2, \alpha_2, \alpha_5 \rangle$, $|c_2| = 2 = m_2$, whence $\mathbf{C}_2$ has for columns $\mathbf{c}_2$ itself, plus $\langle \alpha_3, \alpha_3, \alpha_2 \rangle$. This fact already tells us that $\mathbb{C}_\omega$ is not a slice partition of $\mathbf{R}'[\omega]$ because the first row of $\mathbf{C}_2$ has for elements α_2 and α_3, while its first column also contains α_5.
- The first column of after $\mathbf{C}_2$ is hence $\mathbf{c}_4 = \langle \alpha_4, \alpha_4, \alpha_3 \rangle$. Again $|c_4| = 2 = m_3$ and hence $\mathbf{C}_3$ has for columns $\mathbf{c}_4$ and $\langle \alpha_5, \alpha_6, \alpha_4 \rangle$. Notice again that the first column of $\mathbf{C}_3$ does not contain α_5 that however appears in its first row.
- The next column of $\mathbf{R}'[\omega]$ after $\mathbf{C}_3$ is hence $\mathbf{c}_6 = \langle \alpha_6, \alpha_5, \alpha_6 \rangle$. Notice that $|c_6| = 2$ so that $m_1 + m_2 + m_3 + 2 = 7 > 6$ whence $\mathbf{C}_4$ has $\mathbf{c}_6$ as unique column and the procedure stops.

From what we observed in the construction of $\mathbf{C}_2$, $\mathbb{C}_\omega$ is not a slice partition of $\mathbf{R}'[\omega]$.

Fig. 1 is meant to clarify such property, graphically.

The next result is a direct consequence of the definition of sliceable matrix of the kind $\mathbf{R}[\omega]$ and, more precisely, of the construction of $\mathbb{C}_\omega$ described above.

Remark 3.12. Notice that it follows from the property $(*)$ of Definition 3.11 that $\mathbb{C}_\omega = \{\mathbf{C}_1, \ldots, \mathbf{C}_t\}$ is a slice partition of $\mathbf{R}[\omega]$ iff for all $l = 1, \ldots, t$, all rows of $\mathbf{C}_l$ are permutations of the first column $\mathbf{c}$ of $\mathbf{C}_l$.

The construction introduced insofar and the notion of sliceability allow us to fully characterize the sphericity property of a frame $\mathsf{F} = (\mathbb{ot}(\mathcal{C}(\mathbf{A})), R)$—recall Lemma 3.8 (3)—in terms of sliceable matrices. First, we need to show a preliminary technical result that will help in the proof.

Lemma 3.13. *Let* $F = (\mathfrak{alt}(\mathcal{C}(\mathbf{A})), R)$ *be a frame and let* $\omega \in \mathfrak{alt}(\mathcal{C}(\mathbf{A}))$ *such that* $\mathbf{R}[\omega]$ *is not sliceable. In particular let* l *be the first index, in the natural order, for which there exists a row* $\mathbf{r} \in \mathbf{C}_l$ *and* $c_l \neq r$. *Then the following properties hold:*

1. $|c_l| \geqslant 2$;
2. *There exists* $\alpha_{i,j} \in r$ *such that* $\alpha_{i,j} \notin c_l$;
3. *For all* $\alpha_{i,j}$ *as in (2), there exists* $\alpha_{x,y} \in c_l$ *such that* $y \neq j$ *and* $\alpha_{x,y}$ *does not appear in the jth row of* $\mathbf{R}[\omega]$ *before* $\alpha_{i,j}$.

Proof. (1) is trivial by construction of $\mathbb{C}_\omega$. As for (2), indeed, if $c_l \neq r$ it might be the case that there exists $\beta \in c_l$ and $\beta \notin r$. However, if this is the case, then, by cardinality argument and the construction of $\mathbf{C}_l$, there it must also be $\beta' \in r$ such that $\beta' \notin c_l$.

Hence, let us prove (3). By way of contradiction, assume that there exists $\alpha_{i,j} \in r$ that does not belong to c_l and for which, for all $\beta \in c_l$, they appear in the jth row before the ith column. Now, since $\mathbf{C}_l$ is the first submatrix of $\mathbf{R}[\omega]$ for which $(*)$ fails, all these β's cannot appear before $\mathbf{c}_l$ in the indexing of $\mathbf{R}[\omega]$, for otherwise there would exists a $\mathbf{C}_h$ with $h < l$ where $(*)$ fails contradicting the minimality of l. Thus, these β's must necessarily belong to $\mathbf{r}$ and before $\alpha_{i,j}$. This implies that $\mathbf{r}$ has length at least $|c_l| + 1$ as it contains all the elements of c_l plus $\alpha_{i,j}$ that does not belong to $\mathbf{c}_l$ by assumption. This is in contradiction with the construction of $\mathbb{C}_\omega$ and hence of $\mathbf{C}_l$. $\square$

Now we can prove the anticipated characterization result.

Theorem 3.14. *A frame* $F = (\mathfrak{alt}(\mathcal{C}(\mathbf{A})), R)$ *satisfies sphericity iff* F *is sliceable.*

Proof. See the Appendix. $\square$

We can hence conclude this section with the following useful corollary that summarizes the results contained in Theorem 3.14 above and Corollary 3.10.

Corollary 3.15. *A boolean algebras with operators* $(\mathcal{C}(\mathbf{A}), \square)$ *is a Lewis algebra iff its associated frame* $(\mathfrak{alt}(\mathcal{C}(\mathbf{A})), R)$ *is serial, centered, and sliceable.*

4. The logic of Lewis algebras

In this section, we focus on an extension of Lewis logic C1, denoted as C1$^+$, which captures the formulas of $\mathscr{L}^+$ that are valid in Lewis algebras. Specifically, the logic C1$^+$ will be introduced in the next Subsection 4.1 along with its sphere-based semantics: total sphere models. Then, in Subsection 4.2 we will demonstrate how to define canonical models from total models and establish the completeness of C1$^+$ with respect to these canonical models.

4.1. The logic C1$^+$ *and its sphere models*

Let us start by introducing a canonical way to evaluate the formulas of the language $\mathfrak{CF}$ in a Lewis algebra. In fact, as pointed out in [19] for the case of boolean algebras of conditionals, logical evaluations in this setting differ from evaluations in algebraic logic where variables are mapped in an algebra and compound formulas are evaluated by compositionality. Here, the notion of evaluation requires a slightly more tricky treatment, as the binary operator $\between$ has to be interpreted using a combination of the binary conditional operator $|$ and the unary modal operator $\square$.

Definition 4.1. Let ϕ be a formula in $\mathscr{L}^+$ and let $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \square)$ be any Lewis algebra. Then a $\mathcal{L}(\mathbf{A})$-*evaluation* of $\mathscr{L}^+$ is a map $e : \mathscr{L}^+ \rightarrow \mathcal{L}(\mathbf{A})$ satisfying the following conditions:

1. On classical formulas ϕ, e is a truth-functional classical valuation to A such that, if ϕ is satisfiable, then $e(\phi) \neq \bot$[3];
2. If $\phi = (\varphi \between \psi)$ is a basic counterfactual formula, $e(\varphi \between \psi) = \square(e(\varphi) \mid e(\psi))$;
3. If ϕ is a compound formula containing occurrences of $\between$, its truth value is computed by truth-functionality from its components and boolean operations.

We say that a formula ϕ is *valid* in a Lewis algebra $\mathcal{L}(\mathbf{A})$ (and we write $\mathcal{L}(\mathbf{A}) \models \phi$) if $e(\phi) = \top$ for every $\mathcal{L}(\mathbf{A})$-evaluation e. Further, ϕ is a $\mathbb{L}$-*tautology* if $\mathcal{L}(\mathbf{A}) \models \phi$ for all $\mathcal{L}(\mathbf{A}) \in \mathbb{L}$.

---

[3] Notice that our additional requirement for satisfiable formulas not to be evaluated in $\bot$, allows us to extend e to counterfactual formulas as in 2 below. Also, it forces the cardinality of the algebra $\mathbf{A}$ to be at least that of the Lindenbaum algebra of $\mathscr{L}$. More of this will be explained in Section 4.2 below.

Notice that in the above definition, with a slight abuse, we identify $\mathbf{A}$ with its isomorphic copy $\mathbf{A} \mid \top$ in $\mathcal{C}(\mathbf{A})$. This is the reason why e stands for both the evaluation of classical formulas in $\mathbf{A}$ (case 1 in the definition above) and the evaluation of purely counterfactual formulas in $\mathcal{L}(\mathbf{A})$ (cases 2 and 3).

It is now easy to show that the class $\mathbb{L}$ of Lewis algebras does not provide a complete semantics for Lewis original logic C1. To see this, recall that Lewis introduces a particular counterfactual connective that he calls the *might-counterfactual* (see [38]):

$$\varphi \diamondsuit\!\!\!| \, \psi := \neg(\neg\varphi \, \square\!\!\!| \, \psi)$$

Let us recall from [38] that, if $(I, \mathscr{S}, v)$ is a sphere model and $i \in I$, then $i \Vdash \varphi \diamondsuit\!\!\!| \, \psi$ if for all $S \in \mathscr{S}_i$, $v(\psi) \cap S \neq \varnothing$ implies that $v(\varphi) \cap v(\psi) \cap S \neq \varnothing$.

This connective is meant to represent conditional statements of the form *if [antecedent] were the case then [consequent] might be the case*. Lewis observes that, in his semantics and in the non-vacuous case, i.e. when the antecedent ψ is assumed to be satisfiable, the "would-counterfactual", $\varphi \, \square\!\!\!| \, \psi$, implies the corresponding "might-counterfactual", $\varphi \diamondsuit\!\!\!| \, \psi$ (see [38, p.80]). Lewis himself claimed that this principle is acceptable given his understanding of counterfactuals. Let us start by proving the following

Proposition 4.2. *The logic* C1 *does not prove* $(\varphi \, \square\!\!\!| \, \psi) \to (\varphi \diamondsuit\!\!\!| \, \psi)$.

Proof. By Theorem 2.3 it is sufficient to show that $(\varphi \, \square\!\!\!| \, \psi) \to \neg(\neg\varphi \, \square\!\!\!| \, \psi)$ fails in a sphere model. Consider the sphere model $\Sigma = (I, \mathscr{S}, v)$ such that for some propositional variable p, $v(p) = \varnothing$. Then, for an arbitrary $i \in I$, it is the case that $i \Vdash (\top \, \square\!\!\!| \, p)$ but $i \nVdash \neg(\bot \, \square\!\!\!| \, p)$. $\square$

However, by Proposition 3.3 (4) and noticing that for every $\mathcal{L}(\mathbf{A})$-evaluation e,

$$e(\varphi \diamondsuit\!\!\!| \, \psi) = e(\neg(\neg\varphi \, \square\!\!\!| \, \psi)) = \neg\square(\neg e(\varphi) \mid e(\psi)) = \neg\square\neg(e(\varphi) \mid e(\psi)) = \diamondsuit(e(\varphi) \mid e(\psi)),$$

$(\varphi \, \square\!\!\!| \, \psi) \to (\varphi \diamondsuit\!\!\!| \, \psi)$ is a $\mathbb{L}$-tautology. Therefore, Lewis algebras do not provide a complete semantics for C1. We hence need to consider the next axiomatic extension of it.

Definition 4.3. The logic C1$^+$ is the axiomatic extension of C1 obtained by the axiom

(D) $(\varphi \, \square\!\!\!| \, \psi) \to (\varphi \diamondsuit\!\!\!| \, \psi)$.

Before showing the completeness of C1$^+$ with respect to $\mathbb{L}$ and under $\mathcal{L}(\mathbf{A})$-evaluations, let us characterize the (proper) subset of sphere models that satisfy the above axiom (D). Let us start with the following definition.

Definition 4.4. Let $\mathscr{L}$ be the language of classical logic with n propositional variables. A sphere model $(I, \mathscr{S}, v)$ is said to be *total for* $\mathscr{L}$ (or simply *total* when $\mathscr{L}$ is clear) if, for all $i \in I$ and for all satisfiable formula $\varphi \in \mathscr{L}$,

$$\bigcup \mathscr{S}_i \cap v(\varphi) \neq \varnothing.$$

An immediate consequence of the definition of total sphere models is that, in a similar way to what we required in Definition 4.1, every satisfiable formula φ is mapped by v in a non-empty set.

Theorem 4.5. *A sphere model* $\Sigma = (I, \mathscr{S}, v)$ *satisfies (D) iff* Σ *is total. As a consequence,* C1$^+$ *is sound and complete with respect to total sphere models.*

Proof. Consider a total sphere model $\Sigma = (I, \mathscr{S}, v)$ and formulas φ, $\psi \in \mathscr{L}$ with ψ satisfiable. Take an arbitrary $i \in I$; by the totality condition we know that $\bigcup \mathscr{S}_i \cap v(\psi) \neq \varnothing$. Assume $i \Vdash (\varphi \, \square\!\!\!| \, \psi)$, then, by totality, there is a $S \in \mathscr{S}_i$ such that $v(\psi) \cap S \subseteq v(\varphi)$. This implies that for all $S \in \mathscr{S}_i$, if $v(\psi) \cap S \neq \varnothing$ then $v(\varphi) \cap v(\psi) \cap S \neq \varnothing$, hence $i \Vdash \varphi \diamondsuit\!\!\!| \, \psi$.

For the other direction, let us assume, by way of contradiction, that $\Sigma = (I, \mathscr{S}, v)$ is a total sphere model, but for some $i \in I$, $i \nVdash (\varphi \, \square\!\!\!| \, \psi) \to (\varphi \diamondsuit\!\!\!| \, \psi)$ for some $\varphi, \psi \in \mathscr{L}$ with ψ satisfiable. Then $i \Vdash \varphi \, \square\!\!\!| \, \psi$ and $i \nVdash \varphi \diamondsuit\!\!\!| \, \psi$. Since Σ is total, $v(\psi) \cap \bigcup \mathscr{S}_i \neq \varnothing$; hence we have that there are $S, S' \in \mathscr{S}_i$ such that $v(\psi) \cap S \subseteq v(\varphi)$ and $v(\psi) \cap S' \nsubseteq v(\varphi)$. However, $\mathscr{S}_i$ is nested, whence either $S \subseteq S'$ or $S' \subseteq S$, in both cases we reach a contradiction. $\square$

4.2. From total to canonical models

Let us start considering the finitely many propositional variables $p_1, p_2 \ldots, p_n$ on which our language $\mathscr{L}$ is defined. In every total sphere model $(I, \mathscr{S}, v)$, for all pair of variables $p_k \neq p_t$, one necessarily has that $v(p_k) \neq v(p_t)$. Indeed, if $v(p_k) = v(p_t)$ for some $k \neq t$, by definition of v, one would have

$$v(p_k \wedge \neg p_t) = v(p_k) \cap (I \backslash v(p_t)) = v(p_k) \cap (I \backslash v(p_k)) = \varnothing$$

and hence, although $p_k \wedge \neg p_t$ is clearly satisfiable, $v(p_k \wedge \neg p_t) \cap \mathscr{S}_i = \varnothing$, in contradiction with Definition 4.3. More precisely, the following holds.

Proposition 4.6. *Let $(I, \mathscr{S}, v)$ be a total model for $\mathscr{L}$. Then, for any pair of satisfiable and not logically equivalent formulas φ, ψ of $\mathscr{L}$, $v(\varphi) \nsubseteq v(\psi)$ and $v(\psi) \nsubseteq v(\varphi)$. In other words, for all φ, ψ that are satisfiable and not logically equivalent, there exist two distinct $i, j \in I$ such that $i \in v(\varphi) \backslash v(\psi)$ and $j \in v(\psi) \backslash v(\varphi)$.*

Proof. Assume, by contradiction, that for two satisfiable and not logically equivalent formulas φ, ψ of $\mathcal{L}$, $v(\varphi) \subseteq v(\psi)$. Then, since φ and ψ are satisfiable and not logically equivalent, $\varphi \wedge \neg \psi$ and $\psi \wedge \neg \varphi$ are satisfiable. However, $v(\varphi \wedge \neg \psi) = \varnothing$. Also, $v(\psi) \subseteq v(\varphi)$ implies $v(\psi \wedge \neg \varphi) = \varnothing$ contradicting the very definition of total model. $\square$

As an immediate consequence of the above result, we have the following.

Corollary 4.7. *For every total model $(I, \mathscr{S}, v)$ for $\mathscr{L}$ with n propositional variables, $|I| \geqslant 2^n$.*

Proof. Assume, by contradiction, that $|I| < 2^n$. In $\mathscr{L}$ there are exactly 2^{2^n} formulas, up to logical equivalence, and there are exactly 2^n valuations of $\mathscr{L}$ to $\{0, 1\}$. Then, and since $v(\bot) = \varnothing$, by a cardinality argument, there would be at least two satisfiable and not logically equivalent formulas φ and ψ for which $v(\varphi) = v(\psi)$. Thus, $(I, \mathscr{S}, v)$ would not be total for $\mathscr{L}$ by Proposition 4.6. Contradiction. $\square$

Now we want to show that every total model for $\mathscr{L}$ can be made in a *canonical model* for $\mathscr{L}$, i.e. a sphere model on the set $\Omega_{\mathscr{L}}$ of classical valuations from $\mathscr{L}$ to $\{0, 1\}$ that is also total for $\mathscr{L}$. To do so, let us start from a total model $(I, \mathscr{S}, v)$ for $\mathscr{L}$. Define the relation $\equiv$ on I as follows: for all $i, j \in I$,

$$i \equiv j \text{ iff } \forall \varphi \in \mathscr{L}, i \in v(\varphi) \text{ iff } j \in v(\varphi). \tag{2}$$

Let us point out the following: given a language $\mathscr{L}$ on n variables $p_1, \ldots, p_n$, let us denote by $\mathcal{M}$ the set of *minterms* of $\mathscr{L}$, that is to say, formulas of the form

$$\bigwedge_{i=1}^{n} p_i^*$$

where, for all i, p_i^* is either p_i or $\neg p_i$. Thus, $|\mathcal{M}| = 2^n$. Moreover, if $(I, \mathscr{S}, v)$ is total, for all minterms $m_i, m_j \in \mathcal{M}$, if $m_i \neq m_j$, then $v(m_i) \cap v(m_j) = \varnothing$. For otherwise, one would have $v(m_i \wedge m_j) = v(\bot) \neq \varnothing$. Moreover, $\bigcup_{i=1}^{2^n} v(m_i) = I$. In other words $\{v(m_i) \mid m_i \in \mathcal{M}\}$ is the finest partition of I.

The relation introduced in (2) is an equivalence on I and we will henceforth denote by $I_{\equiv}$ the quotient set $I/_{\equiv}$.

Proposition 4.8. *If $(I, \mathscr{S}, v)$ is a total sphere model for $\mathscr{L}$ and $\equiv$ the equivalence relation in (2), then for all $i, j \in I$, $i \equiv j$ iff there exists a (unique) mintem $m \in \mathcal{M}$ such that $i, j \in v(m)$.*

Proof. ($\Rightarrow$). Assume that $i \equiv j$. Then, by definition, for all formula φ, $i \in v(\varphi)$ iff $j \in v(\varphi)$. Assuming, by contradiction, that for all minterm m, either i or j does not belong to $v(m)$. Since $\{v(m) \mid m \in \mathcal{M}\}$ is a partition of I, we immediately have a contradiction.

($\Leftarrow$). Assume that $i \not\equiv j$. Therefore, there exists a formula φ such that either (1) $i \in v(\varphi)$ and $j \notin v(\varphi)$, or (2) $i \notin v(\varphi)$ and $j \in v(\varphi)$. Further, assume without loss of generality that φ is satisfiable. All satisfiable formulas of $\mathscr{L}$ are uniquely displayable as a join of minterms, that is to say, for all satisfiable formula φ, there are minterms $m_{\varphi_1}, \ldots, m_{\varphi_{k_\varphi}}$ such that

$$\varphi = \bigvee_{t=1}^{k_\varphi} m_{\varphi_t}.$$

Therefore, if (1) is the case, and since the $v(m)$'s are pairwise disjoint, there is a $1 \leqslant h \leqslant k_\varphi$ such that (1) happens for $v(m_{\varphi_h})$. Thus $j \notin v(m_{\varphi_h})$. If (2) is the case, reasoning as above, we see that $i \notin v(m_{\varphi_h})$. $\square$

Corollary 4.9. *Let $(I, \mathscr{S}, v)$ be a total model for $\mathscr{L}$. Then, $|I_{\equiv}| = 2^n$. More precisely, the map that associates to each $[i] \in I_{\equiv}$ the minterm m such that $v(m) = \{j \in I \mid j \in [i]\}$ is a bijection between $I_{\equiv}$ and $\mathcal{M}$.*

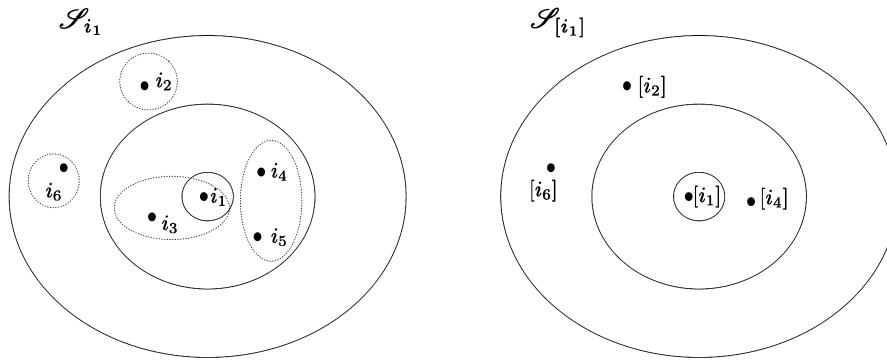**Proof.** Immediate from Proposition 4.8. $\square$

Fig. 2. The sphere systems $\mathscr{S}_{i_1}$ (on the left) and $\mathscr{S}_{[i_1]}$ (on the right). In the picture on the left, dashed circles represent the equivalence classes of $\equiv$ on I.

Since minterms from $\mathscr{L}$ are in bijection with the set $\Omega_{\mathscr{L}}$ of classical valuation of $\mathscr{L}$ to $\{0, 1\}$ and, in turn, to the set of atoms of the n-generated free algebra $\mathbf{L}$, the latter are in bijection with $I_{\equiv}$. Moreover, we can now define a valuation map $v_{\equiv} : \mathscr{L} \to \mathcal{P}(I_{\equiv})$ as follows: for every classical formula φ, $v_{\equiv}(\varphi) = [\![\varphi]\!]$, the set of classical models of φ. Indeed, for all φ,

$$v_{\equiv}(\varphi) = \{[i] \in I_{\equiv} : i \in v(\varphi)\}. \tag{3}$$

Now that we have defined $I_{\equiv}$ and $v_{\equiv}$, what is left to show is how sphere systems adapt to equivalence classes of worlds. To this end, let $i \in I$ and let $\mathscr{S}_i$ be the sphere system centered in i. Let us display as usual $\mathscr{S}_i$ as $\{S_1 = \{i\}, S_2, \ldots, S_t\}$ so that $S_1 \subset S_2 \subset \ldots \subset S_t$. Then, define

$$\mathscr{S}_{[i]} = \{(S_1)_{\equiv} = \{[i]\}, (S_2)_{\equiv} = \{[j_2] : j \in S_2\}, \ldots, (S_t)_{\equiv} = \{[j_t] : j_t \in S_t\}\}.$$

Notice that

1. For all $(S)_{\equiv} \in \mathscr{S}_{[i]}$, $[i] \in (S)_{\equiv}$;
2. $\{[i]\} \in \mathscr{S}_{[i]}$.

Therefore, if $\Sigma = (I, \mathscr{S}, v)$ is a total sphere model for $\mathscr{L}$, we can define $\Sigma_{\equiv} = (I_{\equiv}, \mathscr{S}_{\equiv}, v_{\equiv})$, where $\mathscr{S}_{\equiv}$ clearly stands for $\{\mathscr{S}_{[i]} : [i] \in I_{\equiv}\}$. Given (1) and (2) above, to prove that $\Sigma_{\equiv}$ is a sphere model, it is left to show that for all $[i] \in I_{\equiv}$, $\mathscr{S}_{[i]}$ is nested. This claim is shown in the next result where we will also show that $\Sigma_{\equiv}$ is total as well.

Proposition 4.10. *For a total sphere model $\Sigma = (I, \mathscr{S}, v)$, its associated canonical sphere model $\Sigma_{\equiv} = (I_{\equiv}, \mathscr{S}_{\equiv}, v_{\equiv})$ is a total sphere model.*

Proof. To show that $\Sigma_{\equiv} = (I_{\equiv}, \mathscr{S}_{\equiv}, v_{\equiv})$ is a sphere model, we must show that each system of spheres is nested. For $[i]_{\equiv} \in I_{\equiv}$ consider $(S)_{\equiv}, (S')_{\equiv} \in \mathscr{S}_{[i]}$. Since Σ is a sphere model, we have that either $S \subseteq S'$ or $S' \subseteq S$. If the former is the case, then, by definition of $(S)_{\equiv}$ and $(S')_{\equiv}$, we have that $(S)_{\equiv} \subseteq (S')_{\equiv}$. Analogously, if the latter is the case, then $(S')_{\equiv} \subseteq (S)_{\equiv}$.

Now, consider a satisfiable formula $\varphi \in \mathcal{L}$ and take any $[i] \in I$. We know that $\mathscr{S}_{[i]} = \{(S)_{\equiv} : S \in \mathscr{S}_i\}$. By the totality of Σ, we have that $\bigcup \mathscr{S}_i \cap v(\varphi) \neq \varnothing$, and so we immediately get that $\bigcup \mathscr{S}_{[i]} \cap v_{\equiv}(\varphi) \neq \varnothing$. Hence $\Sigma_{\equiv}$ is total. $\qquad \square$

The following example is meant to clary the "effect" of the equivalence relation $\equiv$ on any total sphere model for a language with two propositional variables.

Example 4.11. Let $\mathscr{L}$ be the language of classical propositional logic on two variables p and q. Let $(I, \mathscr{S}, v)$ be a total sphere model for $\mathscr{L}$ with $I = \{i_1, \ldots, i_6\}$ and consider $v(p \wedge q) = \{i_1, i_3\}$, $v(\neg p \wedge q) = \{i_4, i_5\}$, $v(p \wedge \neg q) = \{i_2\}$ and $v(\neg p \wedge \neg q) = \{i_6\}$ so that, according with Proposition 4.8, $[i_1] = [i_3] = \{i_1, i_3\}$, $[i_2] = \{i_2\}$, $[i_4] = [i_5] = \{i_4, i_5\}$ and $[i_6] = \{i_6\}$.

Furthermore, let $\mathscr{S}_{i_1} = \{\{i_1\}, \{i_1, i_3, i_4, i_5\}, \{i_1, i_2, i_3, i_4, i_5, i_6\} = I\}$. Therefore,

$$\mathscr{S}_{[i_1]} = \{\{[i_1]\}, \{[i_1], [i_4]\}, \{[i_1], [i_2], [i_4], [i_6]\} = I_{\equiv}\}.$$

Fig. 2 provides a graphical explanation.

Also notice that the sphere system $\mathscr{S}_{i_3} = \{\{i_3\}, \{i_1, i_3, i_4, i_5\}, \{i_1, i_2, i_3, i_4, i_5, i_6\} = I\}$ is such that $\mathscr{S}_{[i_3]} = \mathscr{S}_{[i_1]}$. In other words the map that assigns, for every $i \in I$, $\mathscr{S}_i \mapsto \mathscr{S}_{[i]}$ is not injective in general.

Furthermore, the following holds.

Theorem 4.12. *For a total sphere model $\Sigma = (I, \mathscr{S}, v)$, consider its associated canonical model $\Sigma_\equiv = (I_\equiv, \mathscr{S}_\equiv, v_\equiv)$. For all formulas $\Phi \in \mathfrak{CF}$, and for all $i \in I$ we have:*

$$i \Vdash \Phi \text{ iff } [i] \Vdash \Phi$$

As a consequence, the logic $\mathsf{C1}^+$ is complete with respect to canonical models.

Proof. The proof can be easily obtained by induction on the complexity of Φ. We show the (unique interesting) case $\Phi = (\varphi \pitchfork \psi)$ for exemplification.

($\Rightarrow$) Assume $i \Vdash (\varphi \pitchfork \psi)$, then, by totality, there is a $S \in \mathscr{S}_i$ such that $\varnothing \neq v(\psi) \cap S \subseteq v(\varphi)$. Now, by the construction of $\Sigma_\equiv$, consider $S_\equiv = \{[j] \mid j \in S\}$, $v_\equiv(\psi) = \{[h] \mid h \Vdash \psi\}$, and $v_\equiv(\varphi) = \{[k] \mid k \Vdash \varphi\}$. By induction hypothesis, we have that for all $k \in v(\psi)$, $[k] \Vdash \psi$, and so, for all $[k] \in v_\equiv(\psi) \cap S_\equiv$, $[k] \Vdash \psi$. Analogously, since $v(\varphi)_\equiv = \{[j] \mid j \Vdash \varphi\}$, by induction hypothesis, we have that for all $j \in v(\varphi)$, $[j] \Vdash \varphi$. Now, by assumption $\varnothing \neq v(\psi) \cap S$ and so, by construction and what we just proved above, it holds that $\varnothing \neq v_\equiv(\psi) \cap S_\equiv$; analogously, since $v(\psi) \cap S \subseteq v(\varphi)$, we have that $v(\psi)_\equiv \cap (S)_\equiv \subseteq v(\varphi)_\equiv$. Hence, $[i] \Vdash (\varphi \pitchfork \psi)$.

($\Leftarrow$) is similar to the previous one and left to the reader. $\square$

5. Getting Lewis algebras from sphere models, and back

The intuitive connection between Lewis algebras, sphere models, and Lewis frames that has been briefly presented in the above section is made clear here. In this section, we show how to construct sphere models from of Lewis algebras and vice-versa. We will restrict our attention to a proper subclass of spheres—we will call them *universal*—that are defined as follows.

Definition 5.1. A sphere model $(I, \mathscr{S}, v)$ is *universal* if it satisfies the following property:

(F) for all $i \in I$, $\bigcup \mathscr{S}_i = I$.

In the next Subsection 5.1 we present a way to define universal sphere systems from Lewis algebras. To do so, we first need to go through a finer description of Lewis frames and provide a more informative characterization of the effect that the property of sphericity described in the above Lemma 3.8 (3) has on them. Then, in Subsection 5.2, we go backward and define Lewis algebras from sphere systems. Finally, in Subsection 5.3, we will apply these ideas and constructions, together with the results of the previous Section 4, to prove that the logic $\mathsf{C1}^+$ is sound and complete with respect to Lewis algebras.

5.1. From Lewis algebras to universal sphere models

Now we are in a position of showing how to define sphere models from of Lewis algebras, through Lewis frames. The idea can be roughly outlined as follows: given a Lewis algebra $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \square)$, consider its dual Lewis frame $\mathsf{F}_{\mathcal{L}(\mathbf{A})} = (\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ and pick an element $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$. By Corollary 3.15, $R[\omega]$ is nonempty and for all $\omega' \in R[\omega]$, $\omega[1] = \omega'[1] \in \mathfrak{at}(\mathbf{A})$. Thus, we can convert $R[\omega]$ in a sphere system $\mathscr{S}_{\omega[1]}$ on $\mathfrak{at}(\mathbf{A})$, centered in $\omega[1]$ thanks to the fact that $\mathsf{F}_{\mathcal{L}(\mathbf{A})}$ is sliceable (see Definition 3.11). However, to get a Lewis sphere model $(I, \mathscr{S}, e)$, and for the construction to be well-defined, a sphere system $\mathscr{S}_i$ must be associated with each $i \in I$ (and only one!) on which $\mathscr{S}_i$ is centered. On the other hand, in the dual frame $\mathsf{F}_{\mathcal{L}(\mathbf{A})}$ there are several atoms of $\mathcal{C}(\mathbf{A})$, say $\omega', \omega'', \ldots, \omega^{(n)}$, such that $\omega'[1] = \omega''[1] = \ldots = \omega^{(n)}[1]$ and with possibly different accessible worlds. For this reason, to obtain only one sphere system centered in one atom of $\mathbf{A}$, to each Lewis algebra $\mathcal{L}(\mathbf{A})$, and hence to its Lewis frame $\mathsf{F}_{\mathcal{L}(\mathbf{A})}$, we need to adopt a *choice function* χ that selects, for each α atom of $\mathbf{A}$, an (unique) atom ω of $\mathcal{C}(\mathbf{A})$ among those for which $\omega[1] = \alpha$. Formally,

$$\chi : \mathfrak{at}(\mathbf{A}) \to \mathfrak{at}(\mathcal{C}(\mathbf{A})), \tag{4}$$
$$\alpha \mapsto \omega, \text{ such that } \omega[1] = \alpha.$$

In detail, let $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \square)$ be a Lewis algebra and let $\chi : \mathfrak{at}(\mathbf{A}) \to \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ be a choice function as in (4). Define $\Sigma^\chi_{\mathcal{L}(\mathbf{A})}$ to be the system

$$\Sigma^\chi_{\mathcal{L}(\mathbf{A})} = (\mathfrak{at}(\mathbf{A}), \mathscr{S})$$

where:

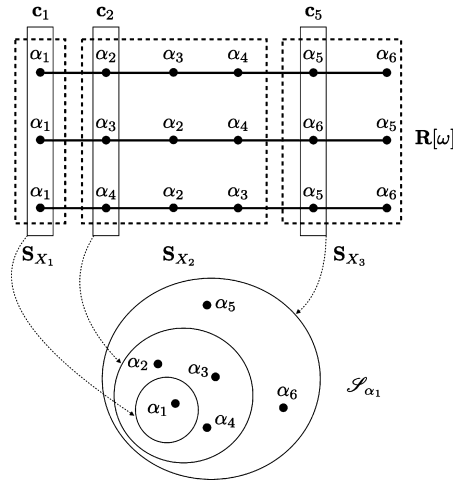1. $\mathfrak{at}(\mathbf{A})$ is the set of atoms of the original algebra $\mathbf{A}$;

Fig. 3. A graphical representation of the sphere systems $\mathscr{S}_{\alpha_1}$ defined by the matrix $\mathbf{R}[\omega]$, where $\omega[1] = \alpha_1$, determined by a Lewis algebra.

2. $\mathscr{S} = \{\mathscr{S}_\alpha\}_{\alpha \in \mathbb{at}(\mathbf{A})}$ and, for all $\alpha \in \mathbb{at}(\mathbf{A})$, let $\chi(\alpha)$ be the (unique) atom of $\mathcal{C}(\mathbf{A})$ selected by χ (its first component is α). Then, let $\mathbf{R}[\chi(\alpha)]$ be the matrix built from $R[\chi(\alpha)]$, and let $\mathbf{c}_1, \ldots, \mathbf{c}_t$ be its columns that determine the slice partition $\{\mathbf{C}_1, \ldots, \mathbf{C}_t\}$ of $\mathbf{R}[\chi(\alpha)]$ as ensured by Theorem 3.14. Then, define

$$\mathscr{S}_\alpha = \{\{c_1\}, \{c_1 \cup c_2\}, \ldots, \{c_1 \cup c_2 \cup \ldots \cup c_t\}\}.$$

The construction is exemplified in Fig. 3.

Proposition 5.2. *For every Lewis algebra $\mathcal{L}(\mathbf{A})$ and every choice function $\chi : \mathbb{at}(\mathbf{A}) \to \mathbb{at}(\mathcal{C}(\mathbf{A}))$, the system $\mathscr{S}$ is a sphere system, i.e., for all $\alpha \in \mathbb{at}(\mathbf{A})$, $\mathscr{S}_\alpha$ is centered in α, nested, and $\alpha \in S$ for all $S \in \mathscr{S}_\alpha$. Furthermore, it satisfies that $\bigcup \mathscr{S}_\alpha = \mathbb{at}(\mathbf{A})$, i.e., $(\mathbb{at}(\mathbf{A}), \mathscr{S})$ is universal.*

Proof. The proof is straightforward by the construction of $\mathscr{S}_\alpha$. In particular, the fact that $\mathscr{S}_\alpha$ is centered in α is implied by the centered property of $\mathsf{F}_{\mathcal{L}(\mathbf{A})}$ and nested by definition. As for the extra property that $\bigcup \mathscr{S}_\alpha = \mathbb{at}(\mathbf{A})$, it depends on the fact that each $\mathscr{S}_\alpha$ is nonempty because $\mathsf{F}_{\mathcal{L}(\mathbf{A})}$ is serial, plus the characterization of the atoms of $\mathcal{C}(\mathbf{A})$ as complete sequences of atoms of $\mathbf{A}$. □

To conclude the construction, let e be an $\mathcal{L}(\mathbf{A})$-evaluation as in Definition 4.1. Thus, we define v from the variables of $\mathscr{L}$ to $\mathcal{P}(\mathbb{at}(\mathbf{A}))$ in the usual way: for every variable p, $v(p) = \{\alpha \in \mathbb{at}(\mathbf{A}) : \alpha \leqslant e(p)\}$. Notice that, since $\mathcal{P}(\mathbb{at}(\mathbf{A}))$ is isomorphic to $\mathbf{A}$, we can directly define $v(p) = e(p)$ without danger of confusion and any loss of generality. Also, notice that v extends in a straightforward way to compound classical formulas. Moreover, if ψ is a satisfiable formula, for all $\alpha \in \mathbb{at}(\mathbf{A})$, $\bigcup \mathscr{S}_\alpha \cap v(\psi) \neq \varnothing$ because of condition (1) of Definition 4.1 and the fact that $(\mathbb{at}(\mathbf{A}), \mathscr{S})$ is universal. Therefore, $\Sigma_{\mathcal{L}(\mathbf{A})}^\chi = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ is universal and total for $\mathscr{L}$. Moreover the following holds.

Proposition 5.3. *Let $\mathcal{L}(\mathbf{A})$ be a Lewis algebra, let e be an $\mathcal{L}(\mathbf{A})$-evaluation, $\chi : \mathbb{at}(\mathbf{A}) \to \mathbb{at}(\mathcal{C}(\mathbf{A}))$ a choice function and $\Sigma_{\mathcal{L}(\mathbf{A})}^\chi$ the sphere model defined as above. Then, for every formula Φ of $\mathscr{L}^+$, and every $\alpha \in \mathbb{at}(\mathbf{A})$, $\chi(\alpha) \leqslant e(\Phi)$ iff $\alpha \Vdash e(\Phi)$.*

Proof. See the Appendix. □

5.2. From total sphere models to Lewis algebras

We now show how to reverse the construction defined above and hence exhibit how to define Lewis algebras starting from total sphere models.

Let us hence fix a sphere model $\Sigma = (I, \mathscr{S}, v)$ that is total for $\mathscr{L}$ and let $\Sigma_\equiv = (I_\equiv, \mathscr{S}_\equiv, v_\equiv)$ be the canonical sphere model built from Σ as in Subsection 4.2. Recall that $\Sigma_\equiv$ is universal and that $I_\equiv$ can (and will be) identified with the set $\Omega_{\mathscr{L}}$ of all classical evaluations of $\mathscr{L}$ to $\{0, 1\}$. Thus, if $\mathscr{L}$ has n propositional variables, $|I_\equiv| = |\Omega_{\mathscr{L}}| = 2^n$. With no surprises, we fix $\mathbf{A} = \mathcal{P}(I_\equiv)$, the powerset of I. Thus, up to isomorphism, $\mathbf{A}$ is the Lindenbaum-Tarski algebra of classical logic for $\mathscr{L}$ and $\mathbb{at}(\mathbf{L}) = \{\alpha_1, \ldots, \alpha_{2^n}\}$. We will henceforth identify $I_\equiv$ with $\mathbb{at}(\mathbf{L})$. For every $\alpha \in \mathbb{at}(\mathbf{L})$, we consider the set

of all *maximal paths* from α, the center of $\mathscr{S}_\alpha$, along the spheres of $\mathscr{S}_\alpha$, defined as follows: first of all, let us display $\mathscr{S}_\alpha$ as usual as a collection of subsets of $\mathfrak{at}(\mathbf{L})$, i.e.,

$$\mathscr{S}_\alpha = \{S_1, S_2, \ldots, S_t\},$$

with $S_1 = \{\alpha\}$. Then, a maximal path (a *path* henceforth) through $\mathscr{S}_\alpha$ is a list

$$\pi = \langle \alpha, s_{2,1}, \ldots, s_{2,|S_2 \setminus \{i\}|}, \ldots, s_{t,1}, \ldots s_{t,|S_t \setminus \bigcup_{j<t} S_j|} \rangle,$$

for $s_{2,1}, \ldots, s_{2,|S_2 \setminus \{i\}|} \in S_2 \setminus \{i\}$, and $\ldots, s_{t,1}, \ldots s_{t,|S_t \setminus \bigcup_{j<t} S_j|} \in S_t \setminus \bigcup_{j<t} S_j$.

Since $\Sigma_\equiv$ is universal, each path has length $|\mathfrak{at}(\mathbf{L})|$. For every $\alpha \in \mathfrak{at}(\mathbf{L})$, let $\mathrm{Path}(\mathscr{S}_\alpha)$ be the (finite) collection of all the paths that can be defined through $\mathscr{S}_\alpha$. Notice that, for each $\pi \in \mathrm{Path}(\mathscr{S}_\alpha)$, $\pi[1] = \alpha$.

In other words, each path contains, with no repetition, all the elements of $\mathfrak{at}(\mathbf{L})$, whence $\{\mathrm{Path}(\mathscr{S}_\alpha) : \alpha \in \mathfrak{at}(\mathbf{L})\}$ is a subset of atoms of $\mathcal{C}(\mathbf{L})$.

Finally, let us define $R \subseteq \mathfrak{at}(\mathcal{C}(\mathbf{L})) \times \mathfrak{at}(\mathcal{C}(\mathbf{L}))$ by the following stipulation: for all $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{L}))$,

$$R[\omega] = \mathrm{Path}(\mathscr{S}_{\omega[1]}). \tag{5}$$

In other words, all atoms of $\mathcal{C}(\mathbf{L})$ whose first component is α access through R to all the atoms that are paths through $\mathscr{S}_\alpha$. Thus, we call F_Σ the frame $(\mathfrak{at}(\mathcal{C}(\mathbf{L})), R)$. Let us prove the following.

Proposition 5.4. *For every total sphere model* $\Sigma = (I, \mathscr{S}, v)$, $\mathrm{F}_\Sigma = (\mathfrak{at}(\mathcal{C}(\mathbf{L})), R)$ *is a Lewis frame and hence its associated algebra* $\mathcal{L}(\mathbf{L}) = (\mathcal{C}(\mathbf{L}), \square)$ *is a Lewis algebra.*

Proof. By Corollary 3.15, we only need to show that $\mathrm{F}_\Sigma = (\mathfrak{at}(\mathcal{C}(\mathbf{L})), R)$ is serial, centered, and sliceable. The first two properties are indeed easy to show because, for all $\alpha \in \mathfrak{at}(\mathbf{L})$, there is a nonempty $\mathscr{S}_\alpha \in \mathscr{S}$ that is centered in $\{\alpha\}$. Thus, let us show that F_Σ is sliceable.

For that, recall from (5) that for all $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{L}))$, $R[\omega] = \mathrm{Path}(\mathscr{S}_{\omega[1]})$ so that the matrix $\mathbf{R}[\omega]$ can be displayed as follows:

- The first column $\mathbf{c}_1$ of $\mathbf{R}[\omega]$ is just $\langle \omega[1], \omega[1], \ldots, \omega[1] \rangle$;
- Take now the second column $\mathbf{c}_2$. This is made of all the second elements of each path that belong to $\mathrm{Path}(\mathscr{S}_{\omega[1]})$. In other words, the set c_2 of elements appearing in such column is $S_2 \setminus S_1$.
- Moreover, each row of $\mathbf{C_2}$ is just a permutation of the elements in $\mathbf{c}_2$. In other words, adopting the notation introduced in the above Subsection 5.1, $C_2 = c_2$.
- The same argument then applies to the first new column $\mathbf{c}_{|S_2 \setminus S_1|+1}$, and hence $C_{|S_2 \setminus S_1|+1} = c_{|S_2 \setminus S_1|+1}$ until reaching the end of $\mathbf{R}[\omega]$.

Thus, $\{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_{|S_2 \setminus S_1|+1}, \ldots, \mathbf{C}_{|S_t \setminus S_{t-1}|+1}\}$ is a slice partition of $\mathbf{R}[\omega]$. A similar partition can be determined for all $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ and hence F_Σ is sliceable. $\square$

In [38], Lewis showed how to define, for every sphere model $(I, \mathscr{S}, v)$, a system $(I, \{\preceq_i\}_{i \in I}, v)$ where, for every $i \in I$, $\preceq_i$ is a total preorder on I that is induced by the sphere system $\mathscr{S}_i \in \mathscr{S}$. For the reader who is familiar with total preorder models, it is worth noting that a similar construction, leading to Lewis frames and consequently to Lewis algebras from total sphere systems, could have been performed by first considering the total preorder model associated with a total sphere model and then directly defining maximal paths of the form $\mathrm{Path}(\preceq_i)$ from (an appropriate quotient of) each total preorder $(I, \preceq_i)$.

To exemplify the presented construction, we apply it to the example introduced in Fig. 3.

Example 5.5. Consider a universal sphere system on $I = \{1, 2, \ldots, 6\}$ and $\mathscr{S}_1$ displayed (see Fig. 4) as

$$\mathscr{S}_1 = \{S_1 = \{1\}, S_2 = \{1, 2, 3, 4\}, S_3 = \{1, 2, 3, 4, 5, 6\}\}.$$

The 12 possible paths that we can consider in $\mathscr{S}_1$, and that we write on the atoms $\alpha_1, \ldots, \alpha_6$ of $\mathbf{A}$, have the following generic form

$$\pi = \langle \alpha_1, \underbrace{\alpha_{s_{2,1}}, \alpha_{s_{2,2}}, \alpha_{s_{2,3}}}_{\in S_2 \setminus S_1}, \underbrace{\alpha_{s_{3,1}}, \alpha_{s_{3,2}}}_{\in S_3 \setminus (S_2 \cup \{\alpha_1\})} \rangle$$
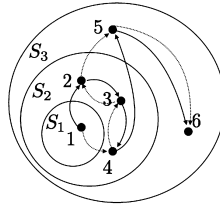
Fig. 4. The nested spheres of $\mathscr{S}_1 = \{S_1 = \{1\}, S_2 = \{1, 2, 3, 4\}, S_3 = \{1, 2, 3, 4, 5, 6\}\}$ and two paths through it that respectively corresponds to the atoms $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$ (solid arrows) and $\langle \alpha_1, \alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_6 \rangle$ (dotted arrows).

As we did also at the end of the previous Subsection 5.1, let us now show how to define a $\mathcal{L}(\mathbf{L})$-evaluation e from the evaluation v of the starting total model Σ. This is indeed quite immediate because, as done in Subsection 5.1, for every propositional variable p, define $e(p) = v_{\equiv}(p) = [p] \in \mathbf{L}$. Again this definition maps variables into $\mathbf{L} = \mathcal{P}(I_{\equiv}) = \mathcal{P}(\Omega_{\mathscr{L}})$. Moreover, if ψ is any classical satisfiable formula, $e(\psi) = v_{\equiv}(\psi)$ is not empty because $\Sigma_{\equiv}$ is total. In the following result, and thanks to the identification $I_{\equiv} = \mathbb{ot}(\mathbf{L})$, we bijectively index the atoms of $\mathbf{L}$ with the elements $[i]$ of $I_{\equiv}$.

Proposition 5.6. *Let* $\Sigma = (I, \mathscr{S}, v)$ *be a total sphere model for* $\mathscr{L}$ *and let* Φ *be a formula of* $\mathscr{L}^+$ *and* $\mathcal{L}(\mathbf{L}) = (\mathcal{C}(\mathbf{L}), \square)$ *the Lewis algebra determined by* Σ. *Then* $i \Vdash \Phi$ *iff for all* ω *such that* $\omega[1] = \alpha_{[i]}$, $\omega \leqslant e(\Phi)$.

Proof. See the Appendix. $\qquad \square$

Let us conclude this subsection by noting some additional properties that will be used in Section 7. Specifically, it is worth remarking that in this subsection, to define a Lewis algebra from a total sphere model, it has been necessary to pass through a canonical model because their set of worlds is identifiable with the set of atoms (i.e., logical valuations) of the Lindenbaum-Tarski algebra on which our construction is based. Therefore, the same construction of a Lewis algebra $\mathcal{L}(\mathbf{A})$ can be performed starting from a total sphere model Σ whose set of possible worlds is $\mathbb{ot}(\mathbf{A})$ itself. The same construction we presented for the case of canonical models and defining Lewis algebras of the form $\mathcal{L}(\mathbf{L})$ can be adapted to this scenario.

Proposition 5.7. *Let* $\mathbf{A}$ *be a finite boolean algebra and let* $(\mathbb{ot}(\mathbf{A}), \mathscr{S}, v)$ *be a total sphere model. Then, the relation R defined as in* (5) *on* $\mathbb{ot}(\mathbf{A})$ *is serial, centered, and sliceable. As a consequence, the algebra* $(\mathcal{C}(\mathbf{A}), \square)$, *where* $\square$ *is defined by R is a Lewis algebra.*

Also notice that for every formula Φ from $\mathscr{L}^+$ and $\alpha \in \mathbb{ot}(\mathbf{A})$, one has that $\alpha \Vdash \Phi$ iff $\omega \leqslant e(\Phi)$ for every $\omega \in \mathbb{ot}(\mathcal{C}(\mathbf{A}))$ such that $\omega[1] = \alpha$ and where e is the $\mathcal{L}(\mathbf{A})$-evaluation defined mapping the variables of $\mathscr{L}$ as $e(p) = v(p)$. Thus, the following also easily holds.

Corollary 5.8. *Let* $\mathbf{A}$ *be a finite boolean algebra, let* $\Sigma = (\mathbb{ot}(\mathbf{A}), \mathscr{S}, v)$ *be a total sphere model for* $\mathscr{L}$ *and let* Φ *be a formula of* $\mathscr{L}^+$. *Also let* $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \square)$ *be the Lewis algebra determined by* Σ. *Then for all* $\alpha \in \mathbb{ot}(\mathbf{A})$, $\alpha \Vdash \Phi$ *iff for all* ω *such that* $\omega[1] = \alpha$, $\omega \leqslant e(\Phi)$.

5.3. Completeness

Recall from Subsection 4.2 that to every total sphere model Σ we can associate a canonical model $\Sigma_{\equiv} = (I_{\equiv}, \mathscr{S}_{\equiv}, v_{\equiv})$. Now, by Proposition 5.4 and the above theorem, $\Sigma_{\equiv}$ determines a unique Lewis algebra having, for a domain, the boolean algebra of conditionals built from $\mathcal{P}(I_{\equiv})$, where $I_{\equiv}$ is identifiable with the set $\Omega_{\mathscr{L}}$ of all classical evaluations of $\mathscr{L}$ to $\{0, 1\}$. In other words, and up to isomorphism, $\mathcal{P}(I_{\equiv})$ is the Lindenbaum-Tarski algebra $\mathbf{L}$ of classical logic for the language $\mathscr{L}$ with respect to which Σ and $\Sigma_{\equiv}$ are total. Moreover, by (3), the evaluation $v_{\equiv}$ actually maps, via such identification, every classical formula φ to its equivalence class $[\varphi] \in \mathbf{L}$. This observation justifies the following.

Definition 5.9. A Lewis algebra is said to be *canonical for* $\mathscr{L}$ (or simply *canonical*) if it is in the form $(\mathcal{C}(\mathbf{L}), \square)$ where $\mathbf{L}$ is the Lindenbaum-Tarski algebra of classical logic for $\mathscr{L}$. The *canonical* evaluation of $\mathscr{L}^+$ to $(\mathcal{C}(\mathbf{L}), \square)$ is that one assigning each variable p of $\mathscr{L}$ to its equivalence class $[p] \in \mathbf{L}$ and that, in particular, evaluates $(\varphi \looparrowright \psi)$ to $\square([\varphi] \mid [\psi])$.

We have now all the needed results and constructions allowing us to prove soundness and completeness of the logic $\mathsf{C1}^+$ (defined in the previous Section 4) with respect to Lewis algebras and canonical evaluations.

Theorem 5.10. *The logic* $\mathsf{C1}^+$ *is sound and complete with respect to canonical Lewis algebras and canonical evaluations.*

Proof. Soundness holds in general as all axioms and rules of $\mathsf{C1}^+$ holds in every Lewis algebra $\mathcal{L}(\mathbf{A})$ and under any $\mathcal{L}(\mathbf{A})$-evaluation. Thus, they hold in particular in canonical algebras under canonical evaluations.

As for completeness, assume that ϕ is a formula from $\mathscr{L}^+$ such that $\mathsf{C1}^+ \not\vdash \phi$. Then, by Theorem 4.5 there exists a sphere model Σ that is total for $\mathscr{L}$ that does not satisfy ϕ and by Theorem 4.12 also $\Sigma_\equiv$ does not satisfy ϕ. Since $\Sigma_\equiv$ has a universal frame, by Proposition 5.6, its associated algebra $\mathcal{L}(\mathbf{L}) = (\mathcal{C}(\mathbf{L}), \square)$ does not satisfy ϕ. Finally, from what we just remarked above, the Lewis algebra associated with $\Sigma_\equiv$ is canonical, and the evaluation that does not validate ϕ is canonical as well. This concludes the proof. $\qquad\square$

6. Imaging probabilities as belief functions and probability on modal algebras

In this section, we introduce and review the basic properties of imaging probabilities and belief functions that are required for Section 7. Before getting into the details, let us briefly recap that the *imaging rule* (hereafter referred to as *imaging*) is a technique introduced by Lewis in [39] for updating an a priori probability distribution P over possible worlds (in algebraic terms, atoms of a boolean algebra) upon learning that an event b has occurred. Classical Bayesian conditioning is in fact a special form of imaging where the probabilities assigned to the atoms that are not below b are redistributed uniformly among the atoms that are below b.

In the next Subsection 6.1 we review some key properties of imaging and place them in our algebraic framework. We also present an alternative approach to imaging inspired by a remark made by Dubois and Prade in [14, §3.B]. Briefly, their proposal is to consider the updated mass outside the realm of Bayesian probability and treat it as a mass function in the sense of Dempster-Shafer theory of evidence. We will adopt this approach and therefore need to recall how to handle masses and belief functions, which are the mathematical models used in Dempster-Shafer theory of evidence, within a modal algebraic framework. This will be discussed in Subsection 6.2.

6.1. A non-Bayesian imaging

We have just recalled that the imaging rule is a method introduced by Lewis to redistribute an a priori probability, upon learning that a certain event has occurred. In this rule, if α is a world that does not satisfy b, its probability is added to that of the unique world (as per Lewis' original formulation) α' that satisfies b and is the closest[4] world to α. This idea has been further extended by Gärdenfors to a more general setting where the set of worlds closest to α is not necessarily a singleton [23]. In such cases, if $\alpha_1, \ldots, \alpha_t$ are the closest worlds to α that satisfy b, the probability $P(\alpha)$ is shared among these α_i's in proportion to their closeness.

More formally and contextualizing these ideas within the bounds of the framework of the present paper, let $\mathbf{A}$ be a (finite) boolean algebra with atoms $\alpha_1, \ldots, \alpha_n$ and $(\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ be a sphere model on $\mathbb{at}(\mathbf{A})$. Also, let P be a probability distribution on $\mathbb{at}(\mathbf{A})$ and $b \in A'$. Let us assume, without loss of generality that $b = \bigvee_{i=1}^{t} \alpha_i$ (with $t \leqslant n$), while $\neg b = \bigvee_{j=t+1}^{n} \alpha_j$ so that $\alpha_i \leqslant b$ for all $i = 1, \ldots, t$ and $\alpha_j \not\leqslant b$ for all $j = t+1, \ldots, n$. For every $j = t+1, \ldots, n$ let $C_b(\alpha_j) = \{\alpha_{j_1}, \ldots, \alpha_{j_{k_j}}\}$ be set of the closest atoms to α_j that are below b, i.e., for all j, the sphere system $\mathscr{S}_j = \{\{\alpha_j\}, S_2, \ldots, S_p\}$ is such that

1. there is $S_h \in \mathscr{S}_j$, $\alpha_{j_1}, \ldots, \alpha_{j_{k_j}} \in S_h \setminus \bigcup_{l=1}^{h-1} S_l$;
2. $\alpha_{j_1}, \ldots, \alpha_{j_{k_j}} \leqslant b$;
3. for all $\alpha \in \bigcup_{l=1}^{h-1} S_l$, $\alpha \not\leqslant b$.

Gärdenfors' imaging rule defines, starting from the original distribution P, the updated P_b assigning $P_b(\alpha_j) = 0$ for each $j = t+1, \ldots, n$, and for all $i = 1, \ldots, t$, $P_b(\alpha_i) = P(\alpha_i)$ if for all $j = t+1, \ldots, n$, $\alpha_i \neq \alpha_{j_l}$, $P_b(\alpha_i) = P(\alpha_i) + \lambda_{j_l} P(\alpha_{j_l})$ if there exists $j = t+1, \ldots, n$ such that $\alpha_i = \alpha_{j_l}$ and where the parameters λ_{j_l} are such that $\sum_{l=1}^{k_j} \lambda_{j_l} = 1$.

Besides Gärdenfors' definition, imaging can be approached in a formally different way. To explain this second idea, let us recall the following quote by Dubois and Prade taken from [14, §3.B] and whose notation has been adapted to our current setting.

> [...] Instead of sharing $P(\alpha)$ among the atoms $\alpha_i \leqslant b$, a less committed update is to allocate $P(\alpha)$ to b itself (and none of its subsets). In that case, the imaging process produces a basic probability assignment in the sense of Dempster's view of belief functions. But this type of update is not consistent with Bayesian probabilities because the result of imaging is a family of probability distributions, not a unique one.

In other words, one can think of defining P_b by reassigning the probability of the atom α to that element of the algebra that is defined as the join of the atoms below b and that are more similar to α, i.e., $C_b(\alpha)$ adopting the above notation. Fig. 5 is intended to provide a graphical comparison between these two approaches.

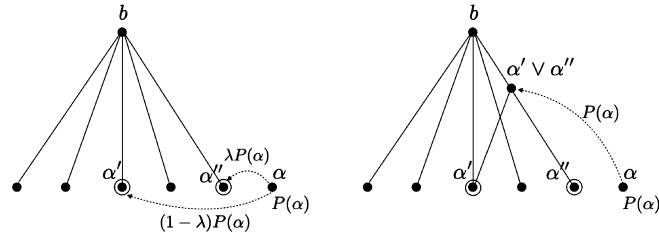[4] Of course, *closest* with respect to a given system of spheres.

Fig. 5. A slice of the boolean algebra with 6 atoms: an element b by which we apply the imaging rule, an atom α that is not below b, and the two atoms α' and α'' (circled) that are below b and are the closest atoms to α, according to some sphere model. On the left-hand side, Gändenfors imaging: the probability $P(\alpha)$ is proportionally added to the probabilities of the atoms α' and α''. On the right-hand side, following Dubois and Prade, the whole mass $P(\alpha)$ is allocated to $\alpha' \vee \alpha''$.

In the rest of this paper, we will indeed follow this latter approach inspired by Dubois and Prade that, at best of the authors' knowledge, has not been further explored in the literature. Further details will be provided in Section 7, where we apply it to our specific setting.

6.2. Probabilities, modalities, and belief functions on finite boolean algebras

We assume that the reader is acquainted with probability functions on boolean algebras and, in the atomic case, with probability distributions on their atoms. This section is intended to introduce and discuss belief and mass functions, the mathematical models of Dempster-Shafer theory of evidence [12,47] and their interplay with probabilities and normal modal operators.

Belief functions on boolean algebras can be introduced in several equivalent ways. Probably, the most common definition is the following (see [12,13,30]).

Definition 6.1. A *belief function* on a boolean algebra $\mathbf{A}$ is a map Bel: $\mathbf{A} \to [0, 1]$ satisfying the following properties:

(B1) $\text{Bel}(\top) = 1$;

(B2) $\text{Bel}\left(\bigvee_{i=1}^{n} a_i\right) \geqslant \sum_{i=1}^{n} \sum_{\{J \subseteq \{1,\dots,n\}: |J|=i\}} (-1)^{i+1} \text{Bel}\left(\bigwedge_{j \in J} a_j\right)$ for $n = 1, 2, 3, \dots$ and $a_1, \dots, a_n \in A$.

A belief function Bel is said to be *normalized* if $\text{Bel}(\bot) = 0$.

In the same way probabilities (over finite algebras) can be characterized in terms of distributions on atoms belief functions on boolean algebras are fully described in terms of *mass functions* as follows. Let $\mathbf{A}$ be any finite boolean algebra. A mass function on $\mathbf{A}$ is a map m that assigns to each element a of A, a real number such that $m(\bot) = 0$ and $\sum_{a \in A} m(a) = 1$. Given a mass function m on $\mathbf{A}$, the map

$$\text{Bel}(a) = \sum_{b \leqslant a} m(b) \tag{6}$$

is a belief function and every belief function on $\mathbf{A}$ arises in this way. The elements of $\mathbf{A}$ that have a strictly positive mass are called *focal elements* and they are those that contribute to defining Bel as in (6).

Besides the axiomatic approach to belief functions we recalled in the above Definition 6.1, there exists a way to introduce these uncertain models that is more appropriate for us and that consists in defining Bel by combining a probability function P and a normal modal operator $\Box$. Although this latter approach has been followed by several scholars [27,31] (see also the more recent papers [15,25]), it is less known. It is hence convenient to recall its pillars.

Let $(\mathbf{A}, \Box)$ be a finite boolean algebra with a normal operator $\Box$ satisfying

(D) $\Box x \leqslant \Diamond x$

and let $(\mathfrak{at}(\mathbf{A}), R)$ be its associated dual (serial) Kripke frame. As usual, for all $\alpha \in \mathfrak{at}(\mathbf{A})$, let us denote by $R[\alpha] = \{\alpha' \in \mathfrak{at}(\mathbf{A}): \alpha R \alpha'\}$. Notice that for every $\alpha \in \mathfrak{at}(\mathbf{A})$, $R[\alpha]$ can be identified with the element $\bigvee R[\alpha]$ of $\mathbf{A}$. Also observe that, since R is serial, $R[\alpha] \neq \varnothing$ (and hence $R[\alpha] > \bot$ once regarded as element of $\mathbf{A}$), for all α.

For every probability $P: \mathbf{A} \to [0, 1]$, let us define $m_P: \mathbf{A} \to [0, 1]$ as follows: for all $a \in A$,

$$m_P(a) = \sum_{R[\alpha]=a} P(\alpha). \tag{7}$$

Proposition 6.2. *Let **A** be a finite boolean algebra and P a probability function on **A**. Then the map $m_P : \mathbf{A} \to [0, 1]$ is a normalized mass function. In other words, $\sum_{a \in A} m_P(a) = 1$ and $m_P(\bot) = 0$.*

Proof. By seriality, for all $\alpha \in \mathfrak{at}(\mathbf{A})$, $R[\alpha] \neq \emptyset$. Thus, $m_P(\bot) = 0$ by definition of m_P. Indeed, a necessary condition for $m_P(a) > 0$, and hence for a to be focal, is that $a = R[\alpha]$ for some $\alpha \in \mathfrak{at}(\mathbf{A})$.

Now, consider the binary relation $\sim$ on $\mathfrak{at}(\mathbf{A})$ defined by $\alpha \sim \alpha'$ iff $R[\alpha] = R[\alpha']$. That is an equivalence relation, $[\alpha]_\sim$ indicates the class of α under $\sim$, while $\mathfrak{at}(\mathbf{A})_\sim$ will stand for the quotient set, so that $m_P(R[\alpha]) = \sum_{\alpha' \in [\alpha]_\sim} P(\alpha')$. Therefore

$$\sum_{\alpha \in \mathfrak{at}(\mathbf{A})} m_P(R[\alpha]) = \sum_{[\alpha]_\sim \in \mathfrak{at}(\mathbf{A})_\sim} \sum_{\alpha' \in [\alpha]_\sim} P(\alpha') = \sum_{\alpha \in \mathfrak{at}(\mathbf{A})} P(\alpha) = 1. \qquad \square$$

The following is meant to clarify the above result.

Example 6.3. Let **A** be the boolean algebra with five atoms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, P a probability on **A**, and $\Box$ the modal operator on **A** whose associated accessibility relation R is

$$R = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), (\alpha_2, \alpha_2), (\alpha_2, \alpha_3), (\alpha_3, \alpha_4), (\alpha_4, \alpha_4), (\alpha_5, \alpha_2), (\alpha_5, \alpha_3)\}.$$

Notice that R is serial. Indeed all $R[\alpha]$ are not empty as $R[\alpha_1] = R[\alpha_2] = R[\alpha_5] = \{\alpha_2, \alpha_3\}$ and $R[\alpha_3] = R[\alpha_4] = \{\alpha_4\}$. Thus, $\alpha_1 \sim \alpha_2 \sim \alpha_5$ and $\alpha_3 \sim \alpha_4$. By (7),

$$m_P(\{\alpha_2, \alpha_3\}) = \sum_{R[\alpha] = \{\alpha_2, \alpha_3\}} P(\alpha) = P(\alpha_1) + P(\alpha_2) + P(\alpha_5)$$

and

$$m_P(\{\alpha_4\}) = \sum_{R[\alpha] = \{\alpha_4\}} P(\alpha) = P(\alpha_3) + P(\alpha_4).$$

Moreover, for all $a \neq \alpha_2 \vee \alpha_3$ and $a \neq \alpha_4$, $m_P(a) = 0$. Thus,

$$\sum_{a \in A} m_P(a) = m_P(\{\alpha_2, \alpha_3\}) + m(\{\alpha_4\}) = \sum_{i=1}^{5} P(\alpha_i) = 1.$$

Summing up, by Proposition 6.2, every probability on a finite boolean algebra **A** and every modal operator $\Box$ on **A** satisfying (D), determine a mass function m_P on **A** and hence a belief function $\mathrm{Bel}_P : \mathbf{A} \to [0, 1]$ defined as usual from m_P: for all $a \in A$,

$$\mathrm{Bel}_P(a) = \sum_{b \leqslant a} m(b).$$

Therefore the following immediately holds.

Corollary 6.4. *Let $(\mathbf{A}, \Box)$ be a finite boolean algebra with an operator satisfying (D). Let $(\mathfrak{at}(\mathbf{A}), R)$ be the Kripke frame associated to $(\mathbf{A}, \Box)$. For every probability P on **A**, the map Bel_P on **A** such that for all $a \in A$,*

$$\mathrm{Bel}_P(a) = \sum_{b \leqslant a} \sum_{R[\alpha] = b} P(\alpha) \qquad (8)$$

*is a belief function on **A** whose focal elements are all of the form $R[\alpha]$ for some $\alpha \in \mathfrak{at}(\mathbf{A})$.*

There is another, yet equivalent, way to define a belief function from a probability P and a modal operator $\Box$ satisfying (D) on **A**. It consists in defining $\mathrm{Bel}^P : A \to [0, 1]$ as follows: for all $a \in A$,

$$\mathrm{Bel}^P(a) = P(\Box a). \qquad (9)$$

Indeed, the fact that $\Box$ is normal and satisfies (D) is enough to prove that Bel^P is a normalized belief function. Moreover, the following holds:

Proposition 6.5. *Let $(\mathbf{A}, \Box)$ be a finite boolean algebra with operators satisfying (D) with associated Kripke frame $(\mathfrak{at}(\mathbf{A}), R)$, and let P be a probability on **A**. Then, $\mathrm{Bel}_P = \mathrm{Bel}^P$.*

Proof. Recall that for all $a \in A$,

$$\Box a = \bigvee \{\alpha \in \mathbb{at}(\mathbf{A}) : \forall \alpha' \in \mathbb{at}(\mathbf{A}), (\alpha R \alpha') \Rightarrow (\alpha' \leqslant a)\} = \bigvee \{\alpha \in \mathbb{at}(\mathbf{A}) : R[\alpha] \leqslant a\}. \tag{10}$$

Thus, by (8), (9) and (10) we can prove that

$$\sum_{b \leqslant a} \sum_{R[\alpha]=b} P(\alpha) = P\left(\bigvee \{\alpha \in \mathbb{at}(\mathbf{A}) : R[\alpha] \leqslant a\}\right).$$

Indeed, by the additivity of P, $P\left(\bigvee \{\alpha \in \mathbb{at}(\mathbf{A}) : R[\alpha] \leqslant a\}\right) = \sum_{R[\alpha] \leqslant a} P(\alpha)$. The latter clearly equals $\sum_{b \leqslant a} \sum_{R[\alpha]=b} P(\alpha)$. Thus, $\mathrm{Bel}_P(a) = P(\Box a) = \mathrm{Bel}^P(a)$. $\square$

7. Imaging, belief and canonical extensions on Lewis algebras

After the above brief degression on belief functions, probabilities, and normal modal operators, we now return to our formal framework to demonstrate how the preceding results allow us to approach the uncertain quantification of counterfactuals and represent the non-Bayesian imaging rule described in Subsection 6.1, using probabilities on Lewis algebras.

Let us first recall the general method of defining probability functions from a probability distribution on a set of models of a logical language (cf. [43] for a general reference and [22] for the special case of the probability of counterfactuals described within causal models). Within our formal setting based on the language $\mathscr{L}^+$, we introduce the following definition which explains what the *probability of a counterfactual* means in our context.

Definition 7.1. Let $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ be a sphere model on the set of atoms of a given boolean algebra $\mathbf{A}$ and let P be a probability on $\mathbf{A}$. For every counterfactual formula $(\varphi \pitchfork \psi)$ of $\mathscr{L}^+$, we define its *probability* as the value

$$\beta(\varphi \pitchfork \psi) = \sum_{\alpha \Vdash (\varphi \pitchfork \psi)} P(\alpha).$$

Now, let us describe in more detail the non-Bayesian imaging rule that we briefly discussed in the previous Subsection 6.1 and let us fix a finite boolean algebra with atoms $\{\alpha_1, \ldots, \alpha_n\}$ and a positive probability P on $\mathbf{A}$. Also, let $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ be a total sphere model and let b be an element of A'. For the sake of clarity, and as we did in Subsection 6.1, let us display b as the join of the atoms of $\mathbf{A}$ below it and assume, without loss of generality, that those are the first t atoms of $\mathbf{A}$ so that $b = \bigvee_{i=1}^{t} \alpha_i$ and $\neg b = \bigvee_{j=t+1}^{n} \alpha_j$.

For every $j = t + 1, \ldots, n$, let $C_b(\alpha_j)$ be the set of closest atoms, among $\alpha_1, \ldots, \alpha_t$, to α_j as determined by a spheres model Σ. Without danger of confusion and for all $j = t + 1, \ldots, n$, we denote by the same symbol $C_b(\alpha_j)$ that element of $\mathbf{A}$ being the join of the elements of $C_b(\alpha_j)$.

Now, define the map $m_b : A \to [0, 1]$ as follows: for all $a \in A$,

$$m_b(a) = \begin{cases} P(\alpha_i) & \text{if } a = \alpha_i, \ i = 1, \ldots, t, \\ P(\alpha_j) & \text{if } a = C_b(\alpha_j), \ j = t+1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Notice that $m_b(\alpha_j) = 0$ for all $j = t + 1, \ldots, n$ and the probability of each such α_j has been transferred, as a whole, to $C_b(\alpha_j) \in A$. By construction, it is immediate to see that $\sum_{a \in A} m_b(a) = 1$ and, in particular $\sum_{a \leqslant b} m_b(a) = 1$. Thus, m_b is a mass function on $\mathbf{A}$ that induces a belief function Bel_b on $\mathbf{A}$ defined as in (6): for all $a \in A$, $\mathrm{Bel}_b(a) = \sum_{c \leqslant a} m_b(c)$. For every pair of element $a, b \in A$ with $b \neq \bot$, we call $\mathrm{Bel}_b(a)$ the *belief of a imaged by b*.

Our first result of this final section shows that the probability of a counterfactual $(a \pitchfork b)$ given as in Definition 7.1 coincides with the just defined belief of a imaged by b. To prove it, let us first define a notion of *selection function* that is similar to the one given in Definition 3.6, but is contextualized in the setting of sphere models.

Definition 7.2. Let $\mathbf{A}$ be a finite boolean algebra and let $(\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ be a sphere model on the set of atoms of $\mathbf{A}$. Then we define the *selection function* $f : \mathbb{at}(\mathbf{A}) \times A \to A$ as follows: for every $\beta \in \mathbb{at}(\mathbf{A})$ and $a \in A$, if S is the first sphere of $\mathscr{S}_\beta$ (with respect to the total order of set-inclusion) that contains at least an atom $\alpha \leqslant a$, then

$$f(\beta, a) = \bigvee (S \cap [\![a]\!]).$$

In other words, $f(\beta, a)$ is that element of A defined as the disjunction of those atoms of $\mathbf{A}$ that are model of a and that appear first, in the order of spheres, in the sphere system $\mathscr{S}_\beta$ centered in β.

Notice that f is indeed well-defined because every sphere model is universal in the sense of Definition 5.1, whence for all β a first sphere $S \in \mathscr{S}_\beta$ that contains a model of a, always exists.

The next fact is indeed easy to prove and it immediately follows from the very definitions of the notions involved.

Fact 2. *Let* **A** *be any finite algebra,* $(\mathtext{o}\mathfrak{t}(\mathbf{A}), \mathscr{S}, v)$ *be a sphere model and* $(a \pitchfork b)$ *be a counterfactual. Then the following conditions hold:*

1. *For all* $\beta \in \mathtext{o}\mathfrak{t}(\mathbf{A})$, $\beta \Vdash (a \pitchfork b)$ *iff* $f(\beta, b) \leqslant a$;
2. *For every positive probability* P *on* **A** *and for all* $c \in A$, $m_b(c) = \sum_{f(\beta, b) = c} P(\beta)$, *where* m_b *is defined as in (11).*

We are now in the position of proving the anticipated result that connects the probability of a counterfactual $(a \pitchfork b)$ with the belief of a imaged by b.

Theorem 7.3. *For every finite boolean algebra* **A***, spheres system* $(\mathtext{o}\mathfrak{t}(\mathbf{A}), \mathscr{S}, v)$*, and positive probability* P *on* **A***, for all* $a \in A$ *and* $b \in A'$,

$$\mathrm{Bel}_b(a) = \beta(a \pitchfork b).$$

Proof. By definition, $\beta(a \pitchfork b) = \sum_{\alpha \Vdash (a \pitchfork b)} P(\alpha)$. By Fact 2 (1), the latter equals $\sum_{f(\alpha, b) \leqslant a} P(\alpha)$ that in turn is equal to $\sum_{c \leqslant a} \sum_{f(\alpha, b) = c} P(\alpha)$. Now, by Fact 2 (2), $\sum_{f(\alpha, b) = c} P(\alpha) = m_a(c)$. Thus, we can conclude that

$$\beta(a \pitchfork b) = \sum_{c \leqslant a} \sum_{f(\alpha, b) = c} P(\alpha) = \sum_{c \leqslant a} m_a(c) = \mathrm{Bel}_b(a). \qquad \square$$

Now, let us turn our attention to Lewis algebras and their probability functions. First of all, recall from [19] that every positive probability P on a finite boolean algebra **A** can be canonically extended to $\mathcal{C}(\mathbf{A})$, the boolean algebra of conditionals of **A**, by the following stipulation: pick any $\omega \in \mathtext{o}\mathfrak{t}(\mathcal{C}(\mathbf{A}))$ and recall that ω is uniquely associated to a complete list of atoms $\langle \alpha_1, \ldots, \alpha_n \rangle$ of **A**. Indeed,

$$\omega = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1) \sqcap \ldots \sqcap (\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2}).$$

Then, define the *canonical extension* of P to ω to be

$$\mu_P(\omega) = P(\alpha_1) \cdot \frac{P(\alpha_2)}{P(\neg\alpha_1)} \cdot \ldots \cdot \frac{P(\alpha_{n-1})}{P(\neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2})}.$$

By [19, Lemma 6.8] the map μ_P is a positive probability distribution on $\mathtext{o}\mathfrak{t}(\mathcal{C}(\mathbf{A}))$ and hence it induces, as usual, a positive probability function on $\mathcal{C}(\mathbf{A})$ that we will indicate by the same symbol.

Now, let $\Sigma = (\mathtext{o}\mathfrak{t}(\mathbf{A}), \mathscr{S}, v)$ be a total sphere model on the atoms of the original algebra **A**. By Proposition 5.7, $\mathscr{S}$ determines a normal operator $\square$ on $\mathcal{C}(\mathbf{A})$ such that $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \square)$ is a Lewis algebra. With this machinery at hand, we hence have another way to define the probability of a counterfactual $(a \pitchfork b)$, interpreted in $\mathcal{L}(\mathbf{A})$ by $\square(a \mid b)$, as

$$\mathrm{Pr}(a \pitchfork b) = \mu_P(\square(a \mid b)). \tag{12}$$

By Proposition 6.5, the map $\mathrm{Bel}_{\mu_P} : t \in \mathcal{C}(\mathbf{A}) \mapsto \mu_P(\square t)$ is hence a belief function on $\mathcal{C}(\mathbf{A})$. Moreover, if $\mathcal{F}_{\mathcal{L}(\mathbf{A})} = (\mathtext{o}\mathfrak{t}(\mathcal{C}(\mathbf{A})), R)$ is the Lewis frame associated to $\mathcal{L}(\mathbf{A})$, by Corollary 6.4 above we have that, for all $t \in \mathcal{C}(\mathbf{A})$,

$$\mathrm{Bel}_{\mu_P}(t) = \sum_{s \leqslant t} \sum_{R[\omega] = s} \mu_P(\omega) = \sum_{R[\omega] \leqslant t} \mu_P(\omega). \tag{13}$$

The next result is the main outcome of this final section, demonstrating that the two uncertain quantifications of counterfactuals, namely the one of Definition 7.1 and the one in (12), are equivalent. This result, as stated in Theorem 7.3, confirms that the probability of a counterfactual $(a \pitchfork b)$ is equal to the belief of the consequent a imaged by the antecedent b.

Theorem 7.4. *For every positive probability* P *on a finite boolean algebra* **A** *and every total sphere model* $\Sigma = (\mathtext{o}\mathfrak{t}(\mathbf{A}), \mathscr{S}, v)$ *it holds that, for every* $a \in A$ *and* $b \in A'$,

$$\mathrm{Bel}_{\mu_P}(a \mid b) = \mu_P(\square(a \mid b)) = \beta(a \pitchfork b) = \mathrm{Bel}_b(a).$$

Proof. By the above (13), $\mathrm{Bel}_{\mu_P}(a \mid b) = \sum_{R[\omega] \leqslant (a \mid b)} \mu_P(\omega)$ and hence, since $R[\omega] \leqslant (a \mid b)$ iff $\omega \leqslant \square(a \mid b)$, one has that

$$\mathrm{Bel}_{\mu_P}(a \mid b) = \sum_{\omega \leqslant \square(a \mid b)} \mu_P(\omega).$$

Now, by Corollary 5.8, $\omega \leqslant \square(a \mid b)$ iff $\omega[1] \Vdash (a \pitchfork b)$ in the sphere model Σ. Thus, the above can be rewritten as

$$\text{Bel}_{\mu_P}(a \mid b) = \sum_{\alpha \Vdash (a \pitchfork b)} \sum_{\omega[1]=\alpha} \mu_P(\omega).$$

By [19, Lemma 6.12 (i)], $\sum_{\omega[1]=\alpha} \mu_P(\omega) = \mu_P(\alpha \mid \top) = P(\alpha)$. Thus, by Theorem 7.3, one finally has that

$$\text{Bel}_{\mu_P}(a \mid b) = \sum_{\alpha \Vdash (a \pitchfork b)} P(\alpha) = \beta(a \pitchfork b) = \text{Bel}_b(a).$$

This hence settles our claim. □

Remark 7.5 *(Getting rid of positive probabilities).* The assumption made in the above Theorem 7.4 concerning the positivity of the probability function P can be relaxed and we can assume to start directly with a *conditional probability* C on **A**, i.e., a function $C : A \times A' \to [0, 1]$ satisfying the usual axioms that can be found in e.g. [30, Definition 3.2.3]. Indeed, as proved in [21], if C is a conditional probability on **A**, the canonical extension μ_C on $\mathcal{C}(\mathbf{A})$ can be defined just by assigning to the generic atom $\omega = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1) \sqcap \ldots \sqcap (\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2})$ of $\mathcal{C}(\mathbf{A})$, the value

$$\mu_C(\omega) = C(\alpha_1 \mid \top) \cdot C(\alpha_2 \mid \neg\alpha_1) \cdot \ldots \cdot C(\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2}).$$

As it is proved in [21, Proposition 3], the map μ_C defined as above is a probability distribution on $\mathbb{at}(\mathcal{C}(\mathbf{A}))$, i.e. $\sum_{\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A}))} \mu_C(\omega) = 1$ and hence it naturally induces a unique finitely additive probability measure, still denoted by μ_C, on the algebra $\mathcal{C}(\mathbf{A})$.

The key difference between starting with a positive probability P and a conditional probability C consists in the fact that while the canonical extension μ_P is positive as well, the extension μ_C might give 0 probability to some atom ω of $\mathcal{C}(\mathbf{A})$. Thus, this latter might be regarded as a more general way to approach probability functions on $\mathcal{C}(\mathbf{A})$. Notice that the main result presented in this section, namely Theorem 7.4, clearly holds if we define μ_C, instead of μ_P and, in particular, the function Bel_{μ_C} still is a belief function on $\mathcal{C}(\mathbf{A})$. Also, the proof of the previous theorem can be straightforwardly arranged to prove that, if C is a conditional probability of **A**, and $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ is a total sphere model, then it holds that, for every $a \in A$ and $b \in A'$,

$$\text{Bel}_{\mu_C}(a \mid b) = \mu_C(\Box(a \mid b)) = \beta(a \pitchfork b) = \text{Bel}_b(a).$$

There, the probability P on **A** that defines β as in Definition 7.1 and Bel_b on **A** clearly is $C(\cdot \mid \top)$, i.e., the unconditional probability determined by C and that fixes the sure event $\top$ as antecedent.

8. Final remarks and future work

In this paper we have presented a logical and algebraic approach to counterfactuals aiming to provide new insights into their representation in terms of modal conditionals. In particular, we introduced a class of boolean algebras with operators, named Lewis algebras, consisting of a boolean algebra of conditionals [19] and a normal modal operator $\Box$ satisfying suitable properties. In this way, we have shown that a counterfactual statement of the form $(a \pitchfork b)$ can be expressed using the modal conditional formula $\Box(a \mid b)$. On the algebraic side, we obtained two main results: firstly, a general method to associate Lewis' sphere models with Lewis algebras and vice-versa; second, the introduction of a logic for counterfactuals, denoted C1$^+$, whose algebraic models precisely correspond to Lewis algebras. Furthermore, we focused on addressing the problem of uncertain quantification of counterfactual formulas in an appropriate way. In particular, we showed that the probability of a counterfactual $(a \pitchfork b)$ can be described both by the belief of a imaged by b, and by the canonical extension of $\Box(a \mid b)$.

In addition to the technical results we have discussed, we believe that the methodology used in the present paper, which combines counterfactual reasoning and modal logic, has the potential to foster a deeper connection and cross-fertilisation between these well-established areas of research. In the following subsections we will provide final remarks and discuss possible research directions that we believe may be of particular interest.

8.1. Weakenings and extensions

The logic of Lewis algebras, denoted C1$^+$ in this paper, extends the Lewis logic C1 by a modal axiom very similar the typical axiom (D) of modal logic, $(\varphi \pitchfork \psi) \to (\varphi \diamondsuit \psi)$. It is worth noting that this axiom holds in Lewis algebras due to certain properties inherited from boolean algebras of conditionals (as seen in the proof of Proposition 3.3 (4)). In other words, (D) holds in any Lewis algebra as a consequence of our approach to counterfactuals as modal conditionals. Still semantically, (D) describes the fact that atoms of BACs (and hence possible worlds in Lewis frames) are *complete* lists of classical valuations, while its failure would imply that they are only partial lists

Beyond C1 and C1$^+$, the literature on conditional and counterfactual reasoning encompasses a wide class of other logical formalisms (see for instance [38, p. 131]). Of particular interest are the logics that Lewis calls C0 and C2, between which C1 lies (called V, VCS and VC, respectively, in the terminology of [38]).

- The logic C0 is the weakest logic in the hierarchy described in [38, p. 131]. Semantically, the sphere models with respect to which C0 is sound and complete are those which, compared with C1 spheres, have more than one element in the inner sphere. It follows by Lemma 3.9 that their algebraic models could be obtained by removing equation (L2) from the definition of Lewis algebras.
- The logic C2, extensively studied by Stalnaker in [48], coincides with Adam's logic for indicative conditionals [1], rather than counterfactuals. On the algebraic level, models of C2 (or, more precisely, of a variant of C2 requiring possible worlds to be permutations of classical valuations) could be obtained from Lewis algebras by strengthening (L2) to require $(a \mid b) = \Box(a \mid b)$. Consequently, any Lewis algebra $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \Box)$ would turn out to be (term-)equivalent to the underlying BAC, $\mathcal{C}(\mathbf{A})$. The logic having boolean algebras of conditionals as algebraic models have been studied in [19] and denoted by LBC, for *Logic of Boolean Conditionals*. Establishing whether C2 and LBC are equivalent in any sense remains an open problem.

Finally, let us comment on what we discussed in Sections 6 and 7. In particular, Theorem 7.4 shows that the probability of a counterfactual statement of the form $(a \between b)$ is captured by the belief function Bel_{μ_P} of the conditional object $(a \mid b)$. It must be noted that based on a general result established in [19] (namely [19, Theorem 6.13]) within the realm of boolean algebras of conditionals, the canonical extension μ_P of a positive probability function P satisfies that $\mu_P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$, indicating that the value assigned by μ_P to the conditional object $(a \mid b)$ coincides with the conditional probability of "a given b". However, when dealing with counterfactual formulas, demonstrating that $\mathrm{Bel}_{\mu_P}(a \mid b)$ coincides with (any form of defining, see [14] for an overview) the conditional belief of the event "a given b" remains an open problem and it seems to be far from trivial.

8.2. On nested conditionals

At the beginning of our algebraic analysis of counterfactual conditionals, we restricted the language to disallow nesting of the conditional symbol "$\mid$". This limitation was justified by the fundamental construction of boolean algebras of conditionals, which, as originally defined in [19], only allows us to represent elementary objects of this nature.

However, the same construction of boolean algebras of conditionals does permit the representation of nested conditionals by applying the construction multiple times to an algebra like $\mathcal{C}(\mathbf{A})$. We will now provide some intuitions on how this generalized construction can be established and discuss a couple of intriguing questions that naturally arise in this context.

Let $\mathbf{A}$ be a boolean algebra and let $\mathcal{C}(\mathbf{A})$ be its boolean algebra of conditionals as in Definition 2.5. The algebra $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ is, by construction, the quotient $\mathbf{Free}(\mathcal{C}(A) \mid \mathcal{C}(A)')/_{\equiv_{\mathfrak{C}}}$ where $\mathcal{C}(A) \mid \mathcal{C}(A)'$ is the set of expressions $(c_1 \mid c_2)$, where $c_1, c_2 \in \mathcal{C}(\mathbf{A})$ and $c_2 \neq \bot$, and $\equiv_{\mathfrak{C}}$ is the congruence defined as in Subsection 2.2, but instantiated on elements of $Free(\mathcal{C}(A) \mid \mathcal{C}(A)')$. Now, c_1 and c_2, as elements of $\mathcal{C}(\mathbf{A})$, are boolean conditionals, i.e., boolean expressions constructed using basic conditionals like $(x \mid y)$ for $(x, y) \in A \times A'$. For instance, if $c_1 = (a_1 \mid b_1)$ and $c_2 = (a_2 \mid b_2)$, the pair (c_1, c_2) represents the *nested* conditional expression $((a_1 \mid b_1) \mid (a_2 \mid b_2))$. Once the construction is completed, there are no inherent reasons to prevent further applications, allowing for an increase in the depth of nesting for the conditional operator.

In the case of Lewis algebras, and hence counterfactual conditionals, a similar intuition reveals that we can repeat the construction of $\mathcal{L}(\mathbf{A})$ to handle different depths of nested counterfactuals. This is possible because the construction leading to a conditional algebra $\mathcal{C}(\mathbf{A})$ does not require the original structure $\mathbf{A}$ to be boolean and it can be defined starting with a boolean algebra with operators, including Lewis algebras. Let us clarify this claim by outlining the necessary steps for this specific construction. Let $\mathbf{A}$ be any boolean algebra and let us proceed as follows:

1. First of all, we define a Lewis algebra $\mathcal{L}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}), \Box)$ built upon $\mathbf{A}$.

2. Now, let $(\mathcal{L}(A) \mid \mathcal{L}(A)')$ be the set of formal expressions like $(l_1 \mid l_2)$ for $l_1 \in \mathcal{L}(A)$ and $l_2 \in \mathcal{L}(A)'$. For instance, if $l_1 = \Box(a_1 \mid b_1)$ and $l_2 = \Box(a_2 \mid b_2)$, $(l_1 \mid l_2) = (\Box(a_1 \mid b_1) \mid \Box(a_2 \mid b_2))$ belongs to $(\mathcal{L}(A) \mid \mathcal{L}(A)')$.

3. Let $\mathbf{Free}(\mathcal{L}(A) \mid \mathcal{L}(A)')$ be the free boolean algebra generated by the above pairs and let $\equiv_{\mathfrak{C}}$ be as in Subsection 2.2 and instantiated on elements of $Free(\mathcal{L}(A) \mid \mathcal{L}(A)')$. Thus, $\mathcal{C}(\mathcal{L}(\mathbf{A}))$ is defined as $\mathbf{Free}(L(A) \mid L(A)')/_{\equiv_{\mathfrak{C}}}$.

4. By construction $\mathcal{C}(\mathcal{L}(\mathbf{A}))$ is a boolean algebra of conditionals. Thus, one can now consider $\mathcal{L}(\mathcal{L}(\mathbf{A}))$ to be a Lewis algebra $(\mathcal{C}(\mathcal{L}(\mathbf{A})), \hat{\Box})$ where $\hat{\Box}$ is a normal necessity operator, in general independent on the above $\Box$, on the boolean algebra $\mathcal{C}(\mathcal{L}(\mathbf{A}))$ and satisfying (L1), (L2) and (L3) of Definition 3.1. As we briefly showed right after the definition of Lewis algebras, such a necessity operator always exists.

The generalized constructions briefly presented above define algebras that are reasonable candidates to serve as models for a logic of conditionals and counterfactuals in a language that permits controlled nesting of these operators. However, the increased complexity of these languages raises several issues that need to be addressed before we can assert that these structures effectively capture nested conditionals and counterfactuals. Among these issues, we believe the following two are of key importance:

(Q1) Are the properties forced by the congruence $\equiv_{\mathfrak{C}}$ enough for nested conditionals, or do we have to require some further equation to be satisfied in this more general case? For instance, an equation that we believe should hold among nested conditionals is the following $((a \mid b) \mid \top_{\mathfrak{C}}) = ((a \mid \top) \mid (b \mid \top))$. Does this equality hold in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$, or does it have to be required by extending $\equiv_{\mathfrak{C}}$?

(Q2) In defining $\mathcal{L}(\mathcal{L}(\mathbf{A}))$, we must take care of the interaction of two modal operators: an inner and an outer one. How do these necessity operators interact with each other? What kind of Kripke models arise in this multi-modal context?

8.3. Future research directions

Future research should begin with a more in-depth logical investigation of the framework presented above. Our focus has been on providing a semantics, based on Lewis algebras, for a variant of Lewis's logic C1, which he claims to be *the correct logic of counterfactuals*. However, Lewis introduced a wide range of logics of conditionals (see [38]). Exploring whether the algebraic methodology developed in the present paper can be applied to these other systems is, in our opinion, an interesting line of research that we will aim at investigating in our future work. As we briefly mentioned earlier, by dropping axiom (L2) we could retrieve the logic C0$^+$ (i.e. C0+(D)). It is interesting to notice that our approach offers a unique perspective on the logic of conditionals: stronger or weaker logics than C1$^+$ can be explored using well-established modal logic techniques. Indeed, stronger or weaker constraints on the normal operator $\Box$ might induce different logics of conditionals. Therefore, it would be interesting to explore which logic of conditionals emerges from imposing some of the axioms commonly adopted in modal logic. This work would help to clarify the connection between modal logic and counterfactual reasoning.

Another interesting topic is the study of logics weaker than C0. This latter corresponds to the weakest system considered by Lewis and its semantics consists of sphere models satisfying the "nestedness" condition (which corresponds to axiom (L3) in the framework of Lewis algebras) without any additional constraint. By dropping axiom (L3), we can obtain weaker logics than those considered by Lewis. To the best of the authors knowledge, a semantical exploration of these weaker systems has only been very recently initiated by some authors [50], using neighborhood semantics. The framework we introduced in this paper, could be easily applied to deepen this semantical exploration and provide intuitive algebraic models for those logics.

The open problem just discussed in the previous Subsection 8.1 will surely deserve to be further investigated. Part of our future research in the area of uncertain conditional measures will focus on determining if the present approach can be further generalized to other frameworks. Preliminary results in this direction are contained in [20] for the case of possibility and necessity measures. However, extending them to the case of belief functions and plausibility measures, or imprecise probabilities is still an ongoing research stream that seems to be far from having a straightforward solution.

To extend the present algebraic setting to the case of nested conditionals, according to what we just discussed in Subsection 8.2, questions (Q1) and (Q2) will surely need to receive an answer.

Another interesting line of research concerns counterfactuals and nonmonotonic reasoning. Indeed, our formalized counterfactuals exhibit clear nonmonotonic behavior, as in Lewis algebra it might hold that $\Box(a \mid b) \not\leqslant \Box(a \mid b \wedge c)$. Hence, we are not allowed to strengthen the antecedent of a counterfactual and maintain soundness. This property is not surprising, as it has already been observed for conditionals in [19]. However, the results of Section 7 and in particular Theorem 7.4, suggest that a new perspective on counterfactuals and nonmonotonic reasoning could be approached following [4] where the authors apply belief function theory to deal with several systems of nonmonotonic reasoning.

Finally, a more philosophical-oriented future work will be directed towards understanding the intuitive interpretation of the modality $\Box$ that we used in this paper to represent counterfactuals in combination with the conditional operator "$\mid$". Indeed, although we managed to grasp some intuition behind the axioms of Lewis algebras (recall Remark 3.2), a complete understanding of how to read an expression of the kind $\Box(a \mid b)$ is still lacking, yet we believe to be of interest and worth investigating.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:
Tommaso Flaminio reports financial support, administrative support, and travel were provided by European Commission. Tommaso Flaminio reports financial support was provided by Spain Ministry of Science and Innovation. Stefano Bonzio reports administrative support was provided by Francesco Severi National Institute of Higher Mathematics National Group of Algebra and Geometric Structures and their Applications.

Data availability

No data was used for the research described in the article.

Acknowledgements

Appendix A. Technical proofs

Proof of Theorem 3.14. ($\Rightarrow$) By contraposition, assume that there is $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ for which $\mathbf{R}[\omega]$ does not admit a slice partition. Furthermore, assume that $\mathbf{C}_l$ is the first submatrix of $\mathbf{R}[\omega]$ for which $c_l \neq r$ for a row $\mathbf{r}$ of $\mathbf{C}_l$. Then, by Lemma 3.13 (1) and (2) we know that c_l contains (at least) two different elements $\alpha_s \neq \alpha_p$ and there exists $\alpha_{i,j}$ such that $\alpha_{i,j} \in r$ but it does not appear in c_l. Then, since $\mathbf{c}_l$ is the first column of $\mathbf{R}[\omega]$ that witness the failure of $(*)$, $\alpha_{i,j}$ does not belong to any other column that precedes $\mathbf{c}_l$ in the order of columns of $\mathbf{R}[\omega]$. Moreover, by Lemma 3.13 (3), there exists an $\alpha_{x,y} \in c_l$ such that $y \neq j$ and $\alpha_{x,y}$ does not appear in the jth row of $\mathbf{R}[\omega]$, before the ith column, i.e., $\alpha_{x,y}$ does not appear in the jth row before $\alpha_{i,j}$.

Then, let $a = (\alpha_{x,y} \vee \alpha_{i,j})$ and $b = \bigvee(c_l \backslash \{\alpha_{x,y}\})$. Notice that $[\![a]\!] = \{\alpha_{x,y}, \alpha_{i,j}\}$ and $[\![b]\!] = c_l \backslash \{\alpha_{x,y}\}$. Let f be the selection function as in Definition 3.6. Notice that,

$$f(a \vee b, \omega) = \{\omega'[i] \in \mathfrak{at}(\mathbf{A}) \mid \omega R \omega' \text{ and } i \text{ is the smallest index s.t. } \omega'[i] \leqslant a \vee b\} = c_l.$$

Clearly $f(a \vee b, \omega) \nsubseteq [\![a]\!]$ since $\alpha_{i,j} \notin c_l$. Also $f(a \vee b, \omega) \nsubseteq [\![b]\!]$ since $\alpha_{x,y} \notin [\![b]\!]$. Moreover, $\alpha_{i,j} \in f(a, \omega)$ since $\alpha_{x,y}$ does not appear before on the same j-th row as $\alpha_{i,j}$, and $\alpha_{i,j}$ does not appear anywhere in $\mathbf{R}[\omega]$ before $\mathbf{c}_l$. Hence, $f(a, \omega) \cup f(b, \omega) \nsubseteq f(a \vee b, \omega)$.

($\Leftarrow$) Assume that F is sliceable, and hence for all $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$, $\mathbb{C}_\omega = \{\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_t\}$ is a slice partition of $\mathbf{R}[\omega]$. We now prove that for all $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ and $a, b \in A$, $f(a \vee b, \omega) \nsubseteq [\![a]\!]$ and $f(a \vee b, \omega) \nsubseteq [\![b]\!]$ implies that $f(a, \omega) \cup f(b, \omega) \subseteq f(a \vee b, \omega)$. Notice that the conditions $f(a \vee b, \omega) \nsubseteq [\![a]\!]$ and $f(a \vee b, \omega) \nsubseteq [\![b]\!]$ are equivalent to the existence of $\alpha, \beta \in \mathfrak{at}(\mathbf{A})$ such that $\alpha, \beta \in f(a \vee b, \omega)$ and $\alpha \in [\![a]\!]$ and $\alpha \notin [\![b]\!]$, and $\beta \in [\![b]\!]$ but $\beta \notin [\![a]\!]$, i.e., $\alpha \leqslant a \wedge \neg b$ and $\beta \leqslant b \wedge \neg a$. We distinguish two cases:

(1) α, β are in the same $\mathbf{C}_l \in \mathbb{C}_\omega$, more precisely, $\alpha, \beta \in C_l$. This means that α and β appears in each row of $\mathbf{C}_l$ by Remark 3.12. Now, $\alpha, \beta \in f(a \vee b, \omega)$, implies $f(a \vee b, \omega) \subseteq C_l$. Indeed, by way of contradiction, assume there is $\gamma \in f(a \vee b, \omega)$ such that $\gamma \notin C_l$. If $\gamma \in C_z$ for some $z > l$, by definition of the selection function f, α or β would appear before γ on the same row, contradicting the assumption that $\gamma \in f(a \vee b, \omega)$. Thus, assume that $\gamma \in C_z$ for some $z < l$. In this case, by Remark 3.12 again, γ appears in each row of $\mathbf{C}_z$, and so, it must appears before α (β), in the same rows as α (β). This implies that $\alpha \neq \gamma$ and $\beta \neq \gamma$ and $\alpha, \beta \notin f(a \vee b, \omega)$, contradicting our assumption. So, $f(a \vee b, \omega) \subseteq C_l$. By an analogous reasoning and using the fact that $\alpha \in [\![a]\!]$ and $\alpha \notin [\![b]\!]$, and $\beta \in [\![b]\!]$ but $\beta \notin [\![a]\!]$, we can show that $f(a, \omega) \subseteq C_l$ and $f(b, \omega) \subseteq C_l$.

Let $\mathbf{c}_l$ be the first column of $\mathbf{C}_l$ so that $f(a, \omega) \subseteq c_l = C_l$, $f(b, \omega) \subseteq c_l = C_l$, and $f(a \vee b, \omega) \subseteq c_l = C_l$. Hence, $f(a, \omega), f(b, \omega), f(a \vee b, \omega)$ are all subsets of c_l. Therefore, it cannot exists $\gamma \in f(a \vee b, \omega)$ but $\gamma \notin f(a, \omega)$ and $\gamma \notin f(b, \omega)$ because if $\gamma \leqslant a \vee b$, then $\gamma \leqslant a$, $\gamma \leqslant b$ and $\gamma \in c_l$. And this implies $f(a, \omega) \cup f(b, \omega) \subseteq f(a \vee b, \omega)$.

(2) α, β are in two different submatrices in $\mathbb{C}_\omega$, i.e. for some $\mathbf{C}_l \neq \mathbf{C}_p \in \mathbb{C}_\omega$, $\alpha \in C_l$, and $\beta \in C_p$. Thus, $\beta \notin C_l$. By contradiction, we show that this case cannot hold. Indeed, and without loss of generality, assume $l < p$, then, by definition of $\mathbb{C}_\omega$, α appears in each row of $\mathbf{C}_l$. By reasoning analogously to the case (1) above, we can show that $f(a \vee b, \omega) \subseteq C_l$. This leads to a contradiction with the fact that $\beta \in f(a \vee b, \omega)$ but $\beta \notin C_l$. $\square$

Proof of Proposition 5.3. As usual let us denote by $\mathsf{F}_{\mathcal{L}(\mathbf{A})} = (\mathfrak{at}(\mathcal{C}(\mathbf{A})), R)$ the Lewis frame (dually) associated to the Lewis algebra $\mathcal{L}(\mathbf{A})$. Let us prove the claim by induction on the complexity of Φ and in particular, let us show the claim for the case $\Phi = \varphi \pitchfork \psi$. In particular, let $e(\varphi) = a$ and $e(\psi) = b$ so that $e(\Phi) = \square(a \mid b)$ in $\mathcal{L}(\mathbf{A})$.

($\Rightarrow$) Let $\omega \in \mathfrak{at}(\mathcal{C}(\mathbf{A}))$ such that $\chi(\alpha) = \omega$ and $\omega \leqslant \square(a \mid b)$ and hence $\varnothing \neq f(b, \omega) \subseteq [\![a]\!]$. By Corollary 3.15, the matrix set of matrices $\mathbb{C}_\omega = \{\mathbf{C}_1, \ldots, \mathbf{C}_t\}$ is a slice partition of $\mathbf{R}[\omega]$. Now, notice that there exists $1 \leqslant l \leqslant t$ such that

(i) $f(b, \omega) \subseteq C_l$, i.e., $C_l \cap [\![b]\!] \neq \varnothing$;
(ii) for all $1 \leqslant j < l$, $[\![b]\!] \cap C_j = \varnothing$;
(iii) for all $\gamma \in C_l$, $\gamma \Vdash b \to a$, that is to say $\varnothing \neq [\![b]\!] \cap C_l \subseteq [\![a]\!]$.

By definition and construction of the sphere model $\Sigma^\chi_{\mathcal{L}(\mathbf{A})}$ there exists $S \in \mathscr{S}_{\omega[1]} = \mathscr{S}_\alpha$ such that $S = C_l$. Thus by (i)–(iii), $\varnothing \neq S \cap [\![b]\!] \subseteq [\![a]\!]$, i.e., $\alpha \Vdash (a \pitchfork b)$.

($\Leftarrow$) Let us assume that $\alpha \Vdash (a \pitchfork b)$, so that there exists $S \in \mathscr{S}_\alpha$ such that $\varnothing \neq [\![b]\!] \cap S \subseteq [\![a]\!]$. As above, let us call ω the atom $\chi(\alpha)$, whence $\mathscr{S}_\alpha = \mathscr{S}_{\omega[1]}$. Again, let $\mathbf{C}_1 \cdots \mathbf{C}_t$ be the slice partition of $\mathbf{R}[\omega]$ as endured by Corollary 3.15. Then, by construction, there exists $1 \leqslant k \leqslant t$ such that $S = \bigcup_{i=1}^k C_i$ and hence

$$\varnothing \neq [\![b]\!] \cap \bigcup_{i=1}^k C_i \subseteq [\![a]\!]. \tag{A.1}$$

Now, if there exists $u > k$ such that $f(b, \omega) \subseteq C_u$ then we would have that for all $1 \leqslant j \leqslant u$, $C_j \cap [\![b]\!] = \varnothing$ contradicting (A.1). Thus, there must exist $1 \leqslant z \leqslant k$ such that $f(b, \omega) \subseteq C_z$. Indeed it holds that $f(b, \omega) \subseteq C_z \cap [\![b]\!]$. Thus, by (A.1),

$$f(b, \omega) \subseteq \llbracket b \rrbracket \cap \bigcup_{i=1}^{k} C_i \subseteq \llbracket a \rrbracket.$$

In other words, $\omega \leqslant \square(a \mid b)$. □

Proof of Proposition 5.6. By induction on the complexity of the formulas. The base case is the only informative one. Assume $\Phi = \square(\varphi \mid \psi)$. Observe, once again, that we can identify $\mathcal{P}(I)$ and $\mathbf{L}$, $\mathbf{L} = \mathcal{P}(I)$, via the isomorphism

$$\iota : \mathbf{L} \to \mathcal{P}(I)$$

$$[\varphi] \mapsto v(\varphi)$$

to simplify the notation, for every $X \in \mathcal{P}(I)$, we will henceforth write $X^{\mathbf{L}}$ instead of $\iota^{-1}(X)$. Notice that each $i \in I$ is univocally associated to a minterm m_i (see Proposition 4.8 and Corollary 4.9), and so, for any $X \in \mathcal{P}(I)$,

$$X^{\mathbf{L}} = \left[\bigvee_{i \in X} m_i \right]$$

Hence, we will identify elements of I with atoms of $\mathbf{L}$ and, in general, elements of $\mathcal{P}(I)$ with elements of $\mathbf{L}$. Now consider a selection function $f_\Sigma : I \times \mathcal{P}(I) \to I$ defined on Σ such that for all $i \in I$ and $X \in \mathcal{P}(I)$:

$$f_\Sigma(i, X) = \min\{S \in \mathscr{S}_i \mid S \cap X \neq \varnothing\} \cap X$$

Intuitively, $f_\Sigma(i, X)$ selects the closest possible worlds to i belonging to X. Hence, the semantics of counterfactuals with respect to total sphere models can be reformulated as:

$$i \Vdash (\psi \pitchfork \varphi) \text{ if and only if } f_\Sigma(i, v(\varphi)) \subseteq v(\psi). \tag{A.2}$$

Notice that, by the isomorphism mentioned above, (A.2) can be rewritten as

$$i \Vdash (\psi \pitchfork \varphi) \text{ if and only if } (f_\Sigma(i, v(\varphi)))^{\mathbf{L}} \leqslant [\psi]). \tag{A.3}$$

Now, consider the Lewis frame $F_\Sigma = ((\mathcal{C}(\mathbf{L})), R)$ induced by Σ and take the selection function f on F_Σ as in Definition 3.6. Notice that for every element a of $\mathbf{L}$ and all atoms ω of $\mathcal{C}(\mathbf{L})$, $f(a, \omega)$ can be identified with the element $(f(a, \omega))^{\mathbf{L}}$ of $\mathbf{L}$. Hence, the semantics of counterfactuals on F_Σ can be reformulated once again as it follows: for all $\omega \in at(\mathcal{C}(\mathbf{L}))$,

$$\omega \leqslant e(\psi \pitchfork \varphi) \text{ if and only if } (f([\varphi], \omega))^{\mathbf{L}} \leqslant [\psi]. \tag{A.4}$$

Now, we will prove that for all formulas φ in $\mathscr{L}$ such that φ is satisfiable, it holds that

$$\text{for all } i \in I \text{ and all } \omega \text{ such that } \omega[1] = i, f_\Sigma(i, v(\varphi)) = f([\varphi], \omega) \tag{A.5}$$

To prove this, consider any $i \in I$ and $\omega \in (\mathcal{C}(\mathbf{L}))$ such that $\omega[1] = i$. Since $\mathscr{S}_i = \{S_1, \ldots, S_t\}$, by definition of f_Σ, there will be a least index l such that $f_\Sigma(i, v(\varphi)) \subseteq S_l$. More formally, by definition of f_Σ, there is a $S_l \in \mathscr{S}_i$ such that for all $k < l$, $f_\Sigma(i, v(\varphi)) \notin S_k$ and $v(\varphi) \cap S_k = \varnothing$ and $S_l \cap v(\varphi) \neq \varnothing$. By the construction put forward in Subsection 5.2 that leads from sphere models to Lewis frame, we know that there is a slice partition $\mathbb{C}_\omega = \{\mathbf{C}_1, \ldots, \mathbf{C}_t\}$ of $\mathbf{R}[\omega]$ such that for all $1 \leqslant m \leqslant t$, $C_m = S_m$. Hence, there is C_l such that $v(\varphi) \cap C_l \neq \varnothing$ and for all $k < l$, $C_k \cap v(\varphi) = \varnothing$. Namely, $C_l \cap v(\varphi) = f_\Sigma(i, v(\varphi))$. Moreover, by sliceability, we also know that $C_l = c_l^1$ where $\mathbf{c}_l^1$ is the first column of the submatrix $\mathbf{C}_l$. Notice, also, that rows of $\mathbf{C}_l$ are permutations of elements of c_l^1, hence we have that:

$$f([\psi], \omega) = \{\omega'[j] \in (\mathbf{L}) \mid \omega R \omega' \text{ and } j \text{ is the smallest index s.t. } \omega'[j] \leqslant [\psi]\} = C_l \cap v(\varphi). \tag{A.6}$$

This equality holds since, by the definition of $\mathbb{C}_w$ and by the isomorphism between $\mathbf{L}$ and $\mathcal{P}(I)$, there is no element x in $\mathbf{R}[\omega]$ appearing before $\mathbf{c}_l^1$ and such that $x \leqslant [\varphi]$. Moreover, we know that $C_l = c_l^1 \cap v(\varphi) \neq \varnothing$ and that rows of $\mathbf{C}_l$ are permutations of elements of $\mathbf{c}_l^1$.

As a corollary of the above equality (A.6), we get that (A.5) holds. Hence, by (A.2), (A.3), (A.4) and (A.5) we have that:

$$i \Vdash (\psi \pitchfork \varphi) \Leftrightarrow f_\Sigma(i, v(\varphi)) \subseteq v(\psi) \Leftrightarrow f_\Sigma(i, [\varphi])^{\mathbf{L}} \leqslant [\psi] \Leftrightarrow f([\varphi], \omega)^{\mathbf{L}} \leqslant [\psi] \Leftrightarrow \omega \leqslant e(\psi \pitchfork \varphi). \qquad \square$$

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