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Doctoral Thesis

# Properties of eigenfunctions of localization operators on modulation spaces

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Allegro
Insieme

DONN'ANNA, DONN'ELVIRA, ZERLINA, DON OTTAVIO E MASETTO

Trema, trema scellerato!
  Saprà tosto il mondo intero
  il misfatto orrendo e nero,
  la tua fiera crudeltà.
Odi il tuon della vendetta
  che ti fischia intorno intorno:
  sul tuo capo, in questo giorno,
  il suo fulmine cadrà.

DON GIOVANNI

È confusa la mia testa,
  non so più quel ch'io mi faccia,
  e un'orribile tempesta
  minacciando, oddio! mi va!
Ma non manca in me coraggio:
  non mi perdo o mi confondo.

LEPORELLO

È confusa la sua testa,
  non sa più quel ch'ei si faccia,
  e un'orribile tempesta
  minacciando, oddio! lo va!
Ma non manca in lui coraggio:
  non si perde o si confonde.

Più stertto
Insieme

DON GIOVANNI

Se cadesse ancora il mondo
  nulla mai temer mi fa!

LEPORELLO

Se cadesse ancora il mondo
  nulla mai temer lo fa!

Lorenzo Da Ponte, Wolfgang Amadeus Mozart. *Don Giovanni*, K. 527, Act I, Scene XXI, 1787.

# Contents

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# Symbols, abbreviations and conventions

- We use $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of natural numbers, whereas we adopt
$$\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}.$$
Given $d \in \mathbb{N}_0$, $\mathbb{R}^d$ and $\mathbb{C}^d$ are the $d$-dimensional Euclidean and complex space, with the convention that for $d = 0$ we have the singleton. We denote the set of extended complex numbers by
$$\overline{\mathbb{C}} \coloneqq \mathbb{C} \cup \{+\infty\} \cup \{-\infty\}.$$
By $\mathbb{Z}_N$, $N \in \mathbb{N}$, we mean the quotient group $\mathbb{Z}/N\mathbb{Z}$.
- If $\alpha \in \mathbb{N}^d$ or $\mathbb{N}_0^d$, then the *length of the multiindex* is denoted by $|\alpha| \coloneqq \alpha_1 + \ldots + \alpha_d$.
- Given a set $X$, the symbol $\#X$ denotes the cardinality of $X$.
- Given a set $X$ and a subset $E \subseteq X$, the characteristic function of $E$ is denoted by $\chi_E$, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in X \smallsetminus E$. The set $X$ is clear from the context.
- $A \lesssim B$ means that for given constants $A$ and $B$ there exists a constant $c > 0$ independent of $A$ and $B$ such that $A \leq cB$. We write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.
- If $A$ and $B$ are two sets, then $B^A$ denotes the set of functions from $A$ into $B$.
- Every vector space $X$ is supposed complex. If $X$ ia a topological vector space (TVS), we denote by $X'$ its topological dual defined as the set of *antilinear* continuous functional on $X$. The duality shall be denoted by
$${}_{X'}\langle \cdot, \cdot \rangle_X,$$
or simply $\langle \cdot, \cdot \rangle$. Whenever possible, the duality $\langle \cdot, \cdot \rangle$ is meant to extend the usual $L^2$-inner product, linear in the first argument and antilinear in the second one.
- If $X, Y$ are two TVSs, then $B(X, Y)$ is the set of all continuous and linear mappings from $X$ into $Y$. We use the notation $B(X) \coloneqq B(X, X)$.
- Given two vectors $x, y \in \mathbb{C}^d$ their inner product is
$$xy \coloneqq x \cdot y \coloneqq \sum_{i=1}^{d} x_i \overline{y_i},$$
the Euclidean norm of $x$ is
$$|x| \coloneqq \sqrt{\sum_{i=1}^{d} |x_i|^2}$$

and we write

$$x^2 := |x|^2 .$$

By restriction we have the inner product on $\mathbb{R}^d$.

- $\mathcal{S}(\mathbb{R}^d)$ denotes the class of Schwartz functions, the tempered distributions are represented by $\mathcal{S}'(\mathbb{R}^d)$.
- The normalization chosen for the Fourier transform on $L^1(\mathbb{R}^d)$ is the following:

$$\mathcal{F}f(\omega) := \hat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\omega}\, dx,$$

where $\omega \in \mathbb{R}^d$.

- $\mathcal{F}_\sigma$ stands for the symplectic Fourier transform on $L^1(\mathbb{R}^{2d})$ defined as

$$\mathcal{F}_\sigma F(x,\omega) := \int_{\mathbb{R}^{2d}} F(u,\xi)e^{-2\pi i(\omega u - \xi x)}\, du d\xi.$$

- In all chapters but 7, the symbol $\otimes$ denotes the tensor product of *functions*, i.e. if $f\colon X \to \mathbb{C}$ and $g\colon Y \to \mathbb{C}$ then

$$f \otimes g\colon X \times Y \to \mathbb{C}, (x,y) \mapsto f(x)g(y).$$

In Chapter 7 $f \otimes g$ denotes the *rank-one operator*. Namely, if $f,g \in L^2(\mathbb{R}^d)$, then

$$f \otimes g\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \psi \mapsto \langle \psi, g\rangle f,$$

where $\langle \cdot,\cdot \rangle$ is the usual sesquilinear inner product on $L^2(\mathbb{R}^d)$.

- The quasi-norm on $L^p_m(X)$ is denoted by $\|\cdot\|_{L^p_m}$ or $\|\cdot\|_{L^p_m(X)}$, accordingly to the need to highlight the underlying measure space. Similarly for $L^{p,q}_m(X \times Y)$.
- We set $1/\infty := 0$.
- We denoted a net of scalars by $(\alpha_i)_{i\in I} \subseteq \mathbb{C}$, whereas a net of vectors not in $\mathbb{C}$ is represented by $\{x_i\}_{i\in I} \subseteq X$. If the context is clear, we omit the index set, so that we write just $(\alpha_i)_i$ and $\{x_i\}_i$.
- $GL(\mathbb{R}^d)$ stands for the group of invertible, $d \times d$, real matrices.
- $\mathcal{J}^p := \mathcal{J}^p(L^2) := \mathcal{J}^p(L^2(\mathbb{R}^d))$, $0 < p \leq \infty$, denote the Schatten classes on $L^2(\mathbb{R}^d)$, where we put $\mathcal{J}^\infty := B(L^2(\mathbb{R}^d))$.
- If $\mathcal{G}$ is a locally compact abelian (LCA) group and $T$ a linear bounded operator on $L^2(\mathcal{G})$, we denote by $\sigma(T)$ the *spectrum* of $T$, that is the set $\{\lambda \in \mathbb{C} \,|\, T - \lambda I_{L^2} \,\text{is not invertible}\}$; in particular, the set $\sigma_P(T)$ denotes the *point spectrum* of $T$, that is

$$\sigma_P(T) := \{\lambda \in \mathbb{C} \,|\, \exists\, f \in L^2(\mathcal{G}) \smallsetminus \{0\} \,\text{such that}\, Tf = \lambda f\},$$

such an $f$ is called *eigenfunction of $T$ associated to the eigenvalue $\lambda$*.

- If $T \in B(L^2)$, we write $T \geq 0$ if $T$ is positive, i.e. $\langle Tf, f\rangle \geq 0$ for every $f \in L^2$.
- By $X \hookrightarrow Y$ we denote the continuous inclusion of a TVS $X$ into the TVS $Y$.

- If $X$ and $Y$ are quasi-normed spaces and $T: X \to Y$ is a linear operator, the notations

$$|||T|||_{X\to Y}\,, \qquad \|T\|_{\mathrm{Op}}\,, \qquad \|T\|_{B(X,Y)}$$

denotes the operator norm of $T$.

- If $\mathcal{G}$ ia locally compact abelian group, then

$$C(\mathcal{G}), \qquad C_0(\mathcal{G}), \qquad C_b(\mathcal{G})$$

are the sets of complex-valued functions on $\mathcal{G}$ which are: continuous, continuous and vanishing at infinity, continuous and bounded, respectively.

# Chapter 1

# Introduction

The main aim of this thesis work is threefold: to study of localization operators $A_a^{\psi_1,\psi_2}$, with a particular focus on decay and smoothness properties of their $L^2$-eigenfunctions [3, 8, 9, 11]; to define quasi-Banach modulation spaces on a locally compact abelian group $\mathcal{G}$ [8]; to establish newly named Feichtinger operators, introduced first in [62], as a suitable as well as easy to handle setting for quantum harmonic analysis [10].

We shall see how these three equally important issues overlap naturally. The techniques used in Chapter 3 in order to study $L^2$-eigenfunctions of $A_a^{\psi_1,\psi_2}$ require the operator to be continuously defined between quasi-Banach modulation spaces on $\mathbb{R}^d$, cf. Theorem 3.2.1, so that in order to extend the result to any LCA group $\mathcal{G}$ (Theorem 5.3.3 from [8]) it is necessary to define modulation spaces on $\mathcal{G}$ in the quasi-Banach setting. Moreover, localization operators arise in the context of quantum harmonic analysis also. In fact, they can be written as

$$A_a^{\psi_1,\psi_2} = a \star (\psi_2 \otimes \psi_1),$$

where: $\psi_2 \otimes \psi_1$ is the rank one operator on $L^2(\mathbb{R}^d)$ $f \mapsto \langle f, \psi_1\rangle \psi_2$, $\star$ is the convolution defined between (generalized) functions and operators, see below and in particular Chapter 7.
We shall focus on each of these issues in a dedicated section of the present chapter.

Besides the main scope of the thesis, we characterize also the symbol class

$$S^m(\mathbb{R}^{2d}) \coloneqq \{\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \,|\, |\partial^\alpha \sigma(z)| \leq C_\alpha (1+|z|^2)^{m/2},\ \alpha \in \mathbb{N}_0^{2d},\ z \in \mathbb{R}^{2d}\},$$

introduced by Sjöstrand in [127], in terms of Gabor matrix of $\mathrm{Op}_\tau(\sigma)$ and study continuity properties for such operators with symbol $\sigma$ in such class [7]. For reader's sake, we summarize the main result and address to Chapter 4, for $m \in \mathbb{R}$ fixed the following properties are equivalent:

*(i)* $\sigma \in S^m\left(\mathbb{R}^{2d}\right)$.

*(ii)* $\sigma \in \mathcal{S}'\left(\mathbb{R}^{2d}\right)$ *and for every* $s \geq 0$, $0 < q \leq \infty$, *there exists a function* $H_\tau \in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$, *with*

$$\|H_\tau\|_{L^q_{\langle\cdot\rangle^s}} \leq C, \quad \forall \tau \in [0,1];$$

*such that*

$$|\langle \mathrm{Op}_\tau\left(\sigma\right) \pi\left(z\right) g, \pi\left(u\right) g\rangle| \leq H_\tau(u-z) \langle \mathcal{T}_\tau(z,u)\rangle^m, \qquad \forall u, z \in \mathbb{R}^{2d}.$$

See (2.212) for the definition of the transformation $\mathcal{T}_\tau$ and (2.7) for $\langle\cdot\rangle^s$.

Operators $\mathrm{Op}_\tau(\sigma)$, considered on $\mathbb{R}^d$ or on $\mathcal{G}$ for $\tau = 0$ only, shall be used through out all the thesis work: they play a key role in the study of $A_a^{\psi_1,\psi_2}$, see e.g. the boundedness results on modulation spaces Theorem 3.1.2, 3.3.1 and 5.2.17, as well as the statements about $L^2$-eigenfunction of $\mathrm{Op}_\tau(\sigma)$, Proposition 3.1.4 and 3.1.5, Theorem 3.3.5 and Proposition 5.2.18; they are paid particular attention in Chapter 7 where we consider them in the setting of quantum harmonic analysis.

## 1.1 New quasi-Banach modulation spaces on $\mathcal{G}$ LCA group

Banach modulation spaces over a LCA group $\mathcal{G}$ (see Assumption 2.2.2 for the hypothesis made in this work) were introduced by H. G. Feichtinger in early '80s [56], they are set of distributions characterized by a common decay in time as well as in frequencies. The framework most taken into account is the Euclidean one, i.e. $\mathcal{G} = \mathbb{R}^d$, and in this case their definition goes as follows. For $p, q \geq 1$ and $m$ "suitable" weight on $\mathbb{R}^d$ (see Chapter 2 for more details about weight functions) we set

$$M_m^{p,q}(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \,|\, V_g f \in L_m^{p,q}(\mathbb{R}^{2d})\}$$

and

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}}\,.$$

Above, $g$ is a fixed nonzero function in $\mathcal{S}(\mathbb{R}^d)$, usually called *widow*, and $V_g f$ is the short-time Fourier transform (STFT) of $f$ w.r.t. $g$ which can be written formally as

$$V_g f(x,\omega) := \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega t} \overline{g(t-x)}\, dt, \qquad \forall (x,\omega) \in \mathbb{R}^{2d}.$$

For $p = 1 = q$ we have the so-called Feichtinger algebra which can be equivalently described in this way:

$$\mathcal{S}_0(\mathbb{R}^d) := M^{1,1}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \,|\, V_g f \in L^1(\mathbb{R}^{2d})\},$$

for some $g \in L^2(\mathbb{R}^d) \smallsetminus \{0\}$ and obvious norm. The space $\mathcal{S}_0(\mathbb{R}^d)$ is a Banach algebra and can be easily generalized to any LCA group:

$$\mathcal{S}_0(\mathcal{G}) := \{f \in L^2(\mathcal{G}) \,|\, V_g f \in L^1(\mathcal{G} \times \widehat{\mathcal{G}})\},$$

with $\widehat{\mathcal{G}}$ the dual group of $\mathcal{G}$, $V_g f(x,\omega) := \int_{\mathcal{G}} f(t) \overline{\langle \omega, t\rangle g(t-x)}\, dt$, $(x,\omega) \in \mathcal{G} \times \widehat{\mathcal{G}}$, $\langle \omega, t\rangle := \omega(t)$. The Feichtinger algebra $\mathcal{S}_0(\mathcal{G})$ allows us to give one of the possible definitions for (Banach) modulation spaces over $\mathcal{G}$:

$$M_m^{p,q}(\mathcal{G}) := \{f \in \mathcal{S}_0'(\mathcal{G}) \,|\, V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})\}, \qquad 1 \leq p, q \leq \infty,$$

for some $g \in \mathcal{S}_0(\mathcal{G}) \smallsetminus \{0\}$, the norm is the natural one as for the Euclidean case.

In [56] H. G. Feichtinger proved that $M_m^{p,q}(\mathcal{G})$, with $1 \leq p, q \leq \infty$, are Banach spaces, whose norm does not depend on the window $g$, in the sense that different window functions in $\mathcal{S}_0(\mathcal{G})$ (or $\mathcal{S}(\mathbb{R}^d)$) yield equivalent norms.

The modulation spaces $M_m^{p,q}(\mathbb{R}^d)$, with $0 < p < 1$ or $0 < q < 1$, were introduced in 2004 by Y.V. Galperin and S. Samarah in [75] and then studied in [104, 119, 149]. Their definition is the same as above with $0 < p, q \leq \infty$ and in [75] it is shown that they are quasi-Banach spaces and almost every property which appears in the Banach case arises in the indexes' range $(0, 1)$

as well. E.g., they do not depend on the window chosen in order to compute the quasi-norm. There are thousands of papers involving modulation spaces with indices $1 \le p, q \le \infty$, whereas very few works deal with the quasi-Banach case $0 < p < 1$ or $0 < q < 1$. Indeed, many properties related to the latter case are still unexplored, such as duals for $M^{p,q}_m(\mathbb{R}^d)$ with at least one index strictly below 1 and a generic weight $m$.

## New contributions

The technique used by Y.V. Galperin and S. Samarah in [75] to construct quasi-Banach modulation spaces on $\mathbb{R}^d$ cannot be adapted to the general case of a LCA group. Indeed, in order to prove the fundamental independence from the window, they exploit some properties of entire functions.
The paper [8], by E. Cordero and the author, overcome this difficulty by getting inspiration from the idea of H. G .Feichtinger and K. Gröchenig in [58], and view modulation spaces on $\mathcal{G}$ as particular coorbit spaces over the Heisenberg group $\mathcal{G} \times \widehat{\mathcal{G}} \times \mathbb{T}$, $\mathbb{T}$ being the complex torus. However, the coorbit theory proposed by H. G. Feichtinger and K. Gröchenig in their works [58, 59, 60] is not suitable for the quasi-Banach case. E.g., it requires the continuous embedding into $L^1_{loc}$, which would prohibit to take into account the spaces $L^p$ with $p < 1$. The right construction is provided by another coorbit theory which suits the quasi-Banach spaces, it was started by H. Rauhut in [119] and developed by F. Voigtlaender in his Ph.D. thesis [147]. Thanks to this new theory, we are able to give a good definition of modulation spaces on LCA groups. In fact, we prove that they are quasi-Banach spaces independent from the window chosen in order to compute the quasi-norm (see below).
Exploiting quasi-lattices, Gabor frame expansions are provided, see Theorem 5.2.13.
In the spirit of [9], new convolution relations for such spaces are proved as well, cf. Proposition 2.5.19 and Proposition 5.2.14. We address the reader to Chapter 5 for all details.
Concretely, we shall see that $M^{p,q}_m(\mathcal{G})$ with $0 < p, q \le \infty$ can be described as

$$M^{p,q}_m(\mathcal{G}) \coloneqq \{f \in \mathcal{S}'_0(\mathcal{G}) \,|\, V_g f \in W(L^\infty, L^{p,q}_m)(\mathcal{G} \times \widehat{\mathcal{G}})\},$$

the quasi-norm is inherited from the Wiener Amalgam space $W(L^\infty, L^{p,q}_m)(\mathcal{G} \times \widehat{\mathcal{G}})$ and it is

$$\|f\|_{M^{p,q}_m} \coloneqq \|V_g f\|_{W(L^{p,q}_m)} \coloneqq \|\mathsf{M}_Q V_g f\|_{L^{p,q}_m},$$

where $Q \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ is a suitable unit neighbourhood and

$$\mathsf{M}_Q V_g f(x, \omega) \coloneqq \operatorname*{ess\,sup}_{(u,\xi) \in (x,\omega)+Q} |V_g f(u, \xi)|, \qquad \forall (x, \omega) \in \mathcal{G} \times \widehat{\mathcal{G}},$$

is called the maximal function of $V_g f$.

The presented new construction recaptures all the previous definitions of modulation spaces. Indeed, $M^{p,q}_m(\mathcal{G})$ as above coincides with

(*i*) $M^{p,q}_m(\mathbb{R}^d)$ for every $0 < p, q \le \infty$, as defined in [56, 75];

(*ii*) $M^{p,q}_m(\mathcal{G})$ for $1 \le p, q \le \infty$ and every $\mathcal{G}$ [56].

Recapturing the already known definitions entails the following equivalence of quasi-norms:

$$\|V_g f\|_{L^{p,q}_m} \asymp \|V_g f\|_{W(L^{p,q}_m)}.$$

Of course, one inequality is always true, namely

$$\|V_g f\|_{L^{p,q}_m} \lesssim \|V_g f\|_{W(L^{p,q}_m)}.$$

The other way round instead is in general an open problem which is part of a wider issue arising from coorbit theory, see [119, 146]. Here we are able to give a small contribution in this direction, we prove that if $\mathcal{G}$ is discrete or compact then

$$\|V_g f\|_{L^{p,q}_m} \gtrsim \|V_g f\|_{W(L^{p,q}_m)}$$

for every $0 < p, q \leq \infty$, see Lemma 5.1.38.

## 1.2 Localization operators

Localization operators $A_a^{\psi_1,\psi_2}$ on $\mathbb{R}^d$ arise from pure and applied mathematics in connection with various areas of research. Given a *symbol* $a$ and *windows* $\psi_1, \psi_2$ the operator $A_a^{\psi_1,\psi_2}$ is defined by the formal integral

$$A_a^{\psi_1,\psi_2} f(t) := \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\psi_1} f(x,\omega) M_\omega T_x \psi_2(t)\, dx d\omega,$$

where: translation and modulation operators are defined respectively as $T_x f(t) := f(t-x)$ and $M_\omega f(t) := e^{2\pi i \omega t} f(t)$ and $V_{\psi_1} f$ is the short-time Fourier transform of $f$ with respect to $\psi_1$. Depending on the field of application, these operators are known under the names of Wick, anti-Wick or Toeplitz operators, as well as wave packets, Gabor or short-time Fourier transform multipliers. We will introduce them by means of time-frequency analysis, a branch of modern harmonic analysis which deals with how to describe a function simultaneously in time and frequency. It originates in the early development of quantum mechanics by H. Weyl, E. Wigner and J. von Neumann around 1930, and in the theoretical foundation of signal analysis by the engineer D. Gabor in 1946. However in 1980 the time-frequency analysis became an independent mathematical field thanks to the work of G. Janssen. In the presented framework, localization operators are a mathematical tool to define a restriction of modified signals, according to the engineering lexicon, to a region of the phase space. Their first introduction as *anti-Wick* operators is due to Berezin, in 1971. As a physicist, he introduced them by means of a quantization rule $a \mapsto A_a$, acting from a symbol $a$ defined on a phase space to an operator $A_a$ acting on a suitable Hilbert space. The symbol $a$ is called *anti-Wick* symbol, while the corresponding operator $A_a$ is referred to as the *anti-Wick* operator associated to the symbol $a$. However we point out that some authors talk about *Wick quantization* rather than anti-Wick.
The terminology *localization operators* appears for the first time in 1988, in a paper by I. Daubechies [37]. She introduced these operators as a generalisation of the anti-Wick ones to localize a signal both in time and frequency. Her primary motivations were applications in optics and signal analysis. For instance, localization operators could be used to filter out noise from given (noisy) signals. Since then they have been extensively investigated.

The generalization to any LCA group $\mathcal{G}$ is quite straightforward. Namely, we replace $\mathbb{R}^{2d}$ with the phase-space $\mathcal{G} \times \widehat{\mathcal{G}}$ and the modulation operator becomes $M_\omega f(t) := \langle \omega, t \rangle f(t)$, where $\omega \in \widehat{\mathcal{G}}$. Of course, windows $\psi_1, \psi_2$ and symbol $a$ shall belong to suitable function and distribution spaces on $\mathcal{G}$ and $\mathcal{G} \times \widehat{\mathcal{G}}$, respectively.

### New contributions

In first place, we pay particular attention to $L^2$-eigenfunctions of $A_a^{\psi_1,\psi_2}$ defined on $\mathbb{R}^d$ or, more generally, any LCA group $\mathcal{G}$. Namely, we are able to show Theorem 3.2.1 and 5.3.3 which are roughly summarized in the following item.

**Theorem 1.2.1.** *Let $0 < p < \infty$ and $a \in M^{p,\infty}(\mathcal{G} \times \widehat{\mathcal{G}})$. Consider $\psi_1, \psi_2$ suitable nonzero windows on $\mathcal{G}$. Suppose that $\sigma_P(A_a^{\psi_1,\psi_2}) \smallsetminus \{0\} \neq \varnothing$ and $\lambda \in \sigma_P(A_a^{\psi_1,\psi_2}) \smallsetminus \{0\}$. Then any eigenfunction $f \in L^2(\mathcal{G})$ with eigenvalue $\lambda$ satisfies*

$$f \in \bigcap_{\gamma>0} M^\gamma(\mathcal{G}).$$

The above result was first obtained in [9] by E. Cordero, F. Nicola and the author in the case $\mathcal{G} = \mathbb{R}^d$. The general case just presented was published in [8] by E. Cordero and the author. Notice that the new modulation spaces $M_m^{p,q}(\mathcal{G})$ with $0 < p, q \leq \infty$, mentioned in the previous section, are used here.

Theorem 3.2.9 and 3.3.5 are results about smoothness for $L^2$-eigenfunctions of $A_a^{\psi_1,\psi_2}$ on $\mathbb{R}^d$. Briefly, they states that, if $A_a^{\psi_1,\psi_2}$ has suitable windows $\psi_1, \psi_2$ and symbol $a$, then any $L^2$-eigenfunction $f$ of $A_a^{\psi_1,\psi_2}$ is in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ or in the Gelfand-Shilov spaces $\mathcal{S}^{(\gamma)}(\mathbb{R}^d)$. For reader's sake, we state the first mentioned result. In what follows we adopt the polynomial weight

$$v_s(x) := (1 + |x|^2)^{\frac{s}{2}}, \qquad \forall x \in \mathbb{R}^d, \quad s \in \mathbb{R}.$$

**Theorem 1.2.2.** *Consider a symbol $a \in M^\infty_{v_s\otimes 1}(\mathbb{R}^{2d})$, for some $s > 0$, and non-zero windows $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ is an eigenfunction of the localization operator $A_a^{\psi_1,\psi_2}$, that is $A_a^{\psi_1,\psi_2} f = \lambda f$, with $\lambda \neq 0$, then $f \in \mathcal{S}(\mathbb{R}^d)$.*

The above result was presented in [9] whereas the improvement to the Gelfand-Shilov class was obtained in [11] by N. Teofanov and the author. Roughly speaking, in order to have $f \in \mathcal{S}^{(\gamma)}(\mathbb{R}^d)$ we need to consider the (sub-)exponential weights

$$w_s^\gamma(x) := e^{s|x|^{1/\gamma}}, \qquad \forall x \in \mathbb{R}^d, \quad s, \gamma > 0,$$

rather than the polynomial $v_s$. We address the reader to Section 3.3 of Chapter 3 for a precise formulation of the previous statement.
We highlight that whenever we speak of eigenfunctions for $A_a^{\psi_1,\psi_2}$, the localization operator will be always guaranteed to be compact on $L^2$.

In Chapter 6 we take into account localization operators $A_a^{\psi_1,\psi_2}$ with symbols of type $a = 1 \otimes m$, with $m$ defined on $\mathbb{R}^d$, and study the equality

$$A_{1\otimes m}^{\psi_1,\psi_2} = T_{m_2} \qquad \text{on} \quad \mathcal{S}(\mathbb{R}^d), \quad M^1(\mathbb{R}^d), \quad \text{or} \quad L^2(\mathbb{R}^d).$$

I.e., we study under which conditions a localization operator $A_{1\otimes m}^{\psi_1,\psi_2}$ can be written as a Fourier multiplier $T_{m_2}$. Such a problem was addressed in [3] by E. Cordero, H. G. Feichtinger, N. Schweighofer, P. Balasz and the author. For sake of clarity, we report the related main result Theorem 6.2.1. By $\mathcal{I}$ we mean the reflection operator $\mathcal{I}f(t) := f(-t)$.

**Theorem 1.2.3.** *Fix multiplier symbols $m, m_2 \in \mathcal{S}'(\mathbb{R}^d)$ (resp. $m, m_2 \in M^\infty(\mathbb{R}^d)$) and windows $\psi_1, \psi_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$). Then the equality*

$$A_{1\otimes m}^{\psi_1,\psi_2} = T_{m_2} \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d) \ (\text{resp. } M^1(\mathbb{R}^d))$$

*holds if and only if*

$$m_2 = m * \mathcal{F}^{-1}(\mathcal{I}\psi_2 * \bar{\psi_1}) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \ (\text{resp. } M^\infty(\mathbb{R}^d)).$$

*The same conclusions hold under the following assumptions:*
*(i) The symbols $m, m_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(\psi_1, \psi_2)$ in $\mathcal{S}'(\mathbb{R}^d) \times$*

*$\mathcal{S}(\mathbb{R}^d)$ (resp. $M^\infty(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$);*
*(ii) The symbols $m, m_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(\psi_1, \psi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ (resp. $M^1(\mathbb{R}^d) \times M^\infty(\mathbb{R}^d)$).*

In Section 6.5 of Chapter 6 also the finite discrete setting of $\mathbb{Z}_N$ is taken into account. In this case, we talk about finite Gabor multipliers $\mathsf{G}_a^{g_1,g_2}$ and linear time invariant filters $H$ rather than localization operators $A_a^{\psi_1,\psi_2}$ and Fourier multipliers $T_m$.

## 1.3 Feichtinger operators in quantum harmonic analysis

Translations, convolutions and Fourier transform of functions dwell at the very heart of classical harmonic analysis. R. Werner in his work [150] introduced analogues notions for operators instead of functions, we recall them briefly. Given an operator $T \in B(L^2(\mathbb{R}^d))$ its translation at $z = (x, \omega) \in \mathbb{R}^{2d}$ and its involution are meant to be

$$\alpha_z(T) \coloneqq \pi(z) T \pi(z)^* \qquad \text{and} \qquad \check{T} \coloneqq \mathcal{I} T \mathcal{I},$$

recall that $\mathcal{I} f(t) \coloneqq f(-t)$ and $\pi(z) \coloneqq M_\omega T_x$. Such definitions allow to introduce the subsequent crucial notions, given $a \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{J}^1$, trace class on $L^2(\mathbb{R}^d)$, the convolution between a function and an operator or between two operators are set to be

$$a \star S \coloneqq S \star a \coloneqq \int_{\mathbb{R}^{2d}} a(z) \alpha_z(S)\, dz \qquad \text{and} \qquad S \star T(z) \coloneqq \operatorname{tr}\left(S \alpha_z(\check{T})\right),$$

where $z = (x, \omega) \in \mathbb{R}^{2d}$ and the integral has to be understood in weak sense. The operator $\star$ enjoys all the expected properties of a convolution, i.e. it is commutative and associative. Moreover, it interacts nicely with the usual convolution $*$, see Chapter 7 for details. The equivalent of the Fourier transform is given by the so-called Fourier-Wigner transform

$$\mathcal{F}_W S(z) \coloneqq e^{-\pi i x \omega} \operatorname{tr}(\pi(-z) S).$$

When we consider a rank-one operator $S = f \otimes g$, where $f, g \in L^2(\mathbb{R}^d)$ and it acts on $L^2(\mathbb{R}^d)$ as $\psi \mapsto \langle \psi, g \rangle f$, the previous definitions boil down to well-known objects. For sake of clarity we mention the following remarkable identities [108]:

$$A_a^{\psi_1,\psi_2} = a \star (\psi_2 \otimes \psi_1), \quad (f \otimes f) \star (\mathcal{I} g \otimes \mathcal{I} g) = |V_g f|^2, \quad \mathcal{F}_W(f \otimes g) = A(f, g),$$

where $A(f, g)(x, \omega) = e^{\pi i x \omega} V_g f(x, \omega)$ is the cross-ambiguity function of $f$ and $g$.

In [103] M. Keyl, J. Kiukas and R. Werner introduced and studied Schwartz operators $\mathfrak{S}$. Namely, $\mathfrak{S}$ is the set of those pseudo-differential operators with Weyl symbol in $\mathcal{S}(\mathbb{R}^{2d})$, by $\mathfrak{S}'$ it is denoted the collection of pseudo-differential operators with Weyl symbol in $\mathcal{S}'(\mathbb{R}^{2d})$. The authors of [103] were able to turn $\mathfrak{S}$ into a Fréchet space such that its topological dual is $\mathfrak{S}'$, this allowed them to define convolutions and Fourier(-Wigner) transform also on $\mathfrak{S}'$ using duality. Schwartz operators behave with respect to $\star$ and $\mathcal{F}_W$ like the Schwartz class with the usual convolution $*$ and Fourier transform $\mathcal{F}$. We cite some of the main result of [103] which show two things: the notions recalled at the beginning of this section are not valid just for operators in $B(L^2(\mathbb{R}^d))$ or $\mathcal{J}^1$ and distributions in $L^1(\mathbb{R}^{2d})$; $\mathfrak{S}$ is the very counterpart in quantum harmonic analysis of $\mathcal{S}(\mathbb{R}^d)$ in classical harmonic analysis.
If we consider $S, T \in \mathfrak{S}$, $A \in \mathfrak{S}'$, $b \in \mathcal{S}(\mathbb{R}^{2d})$ and $a \in \mathcal{S}'(\mathbb{R}^{2d})$, then

$$S \star T \in \mathcal{S}(\mathbb{R}^{2d}), \quad S \star A \in \mathcal{S}'(\mathbb{R}^{2d}), \quad b \star S \in \mathfrak{S}, \quad a \star S,\, b \star A \in \mathfrak{S}'.$$

Moreover, the Fourier-Wigner transform can be extended to a topological isomorphism from $\mathfrak{S}'$ onto $\mathcal{S}'(\mathbb{R}^{2d})$. We address the reader to [103] for details and proofs, in particular Section 5.

## New contributions

Just as for the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, to handle the Fréchet space of Schwartz operators $\mathfrak{S}$ and its dual can be quite cumbersome. One can have a hint of this by looking at the proofs contained in [103]. It is known that the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^d)$ reveals to be a valid alternative to $\mathcal{S}(\mathbb{R}^d)$ and, being a Banach space, easier to work with. In this spirit, F. Luef and the author in [10] consider a space of operators introduced in [62] and named it the *space of Feichtinger operators*. Such a space shall be defined in Chapter 7 as follows

$$\begin{aligned}\mathbb{S}_0 :=\{S\colon \mathcal{S}_0'(\mathbb{R}^d) \to \mathcal{S}_0(\mathbb{R}^d) \,|\, S \text{ is linear, continuous and} \\ \text{maps norm bounded w-}*\text{ convergent sequences in } \mathcal{S}_0'(\mathbb{R}^d) \\ \text{into norm convergent sequences in } \mathcal{S}_0(\mathbb{R}^d)\}\end{aligned}$$

and endowed with the norm of $B(\mathcal{S}_0'(\mathbb{R}^d), \mathcal{S}_0(\mathbb{R}^d))$. H. G. Feichtinger and M. S. Jakobsen in [62] proved that $\mathbb{S}_0$ is a Banach space and a Banach algebra under composition of operators, it is isomorphic to $\mathcal{S}_0(\mathbb{R}^{2d})$ through the map $T \mapsto K_T$, where $K_T$ is the integral kernel of $T$, see Theorem 7.2.3. We highlight that the isomorphism given by $T \mapsto K_T$ shall be heavily used trough out all Chapter 7. Indeed, it will be usually convenient to work on the kernel $K_T$ rather than directly on the operator $T$. The dual of Feichtinger operators ig given by

$$\mathbb{S}_0' := B(\mathcal{S}_0(\mathbb{R}^d), \mathcal{S}_0'(\mathbb{R}^d)),$$

cf. Theorem 7.2.2 and 7.2.3. In the same fashion of what done for $\mathfrak{S}$, we shall prove that translations, convolutions and the Fourier-Wigner transform make sense for elements in $\mathbb{S}_0$ and $\mathbb{S}_0'$. Moreover, if we consider $S, T \in \mathbb{S}_0$, $A \in \mathbb{S}_0'$, $b \in \mathcal{S}_0(\mathbb{R}^{2d})$ and $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$, we have

$$S \star T \in \mathcal{S}_0(\mathbb{R}^{2d}), \quad S \star A \in \mathcal{S}_0'(\mathbb{R}^{2d}), \quad b \star S \in \mathbb{S}_0, \quad a \star S,\, b \star A \in \mathbb{S}_0'.$$

The analogy with the results obtained in [103] for $\mathfrak{S}$ is evident; to see how the convolutions are technically defined, e.g. $\mathbb{S}_0 \star \mathbb{S}_0'$, we address the reader to Definition 7.2.18 and subsequent items. As well as the Feichtinger algebra is Fourier invariant, we have that

$$\mathcal{F}_W \colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$$

is a topological isomorphism and it can be extended to a topological isomorphism from $\mathbb{S}_0'$ onto $\mathcal{S}_0'(\mathbb{R}^{2d})$.

We shall introduce a continuum of time-frequency representations depending on $\tau \in [0,1]$, namely the $\tau$-short-time Fourier transforms

$$V_g^\tau f(x,\omega) = e^{2\pi i \tau x\omega} V_g f(x,\omega),$$

and the $\tau$-Wigner distribution of an operator $S$ with integral kernel $K_S$:

$$W_\tau S(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i t\omega} K_S(x+\tau t, x-(1-\tau)t)\, dt.$$

For rank-one operators $S = f \otimes g$, we recapture the cross-$\tau$-Wigner distribution $W_\tau(f \otimes g) = W_\tau(f,g)$. We shall introduce a dependence on $\tau \in [0,1]$ also in the Fourier-Wigner transform, so that we will talk about Fourier-$\tau$-Wigner transform:

$$\mathcal{F}_{W_\tau} S(x,\omega) := e^{-2\pi i(1-\tau)x\omega}\, \mathrm{tr}(\pi(-x,-\omega)S).$$

The well-known spreading representation will be interpreted as a mapping from functions into operators and a dependence on $\tau \in [0,1]$ shall be imposed as well,

$$\mathrm{SR}^\tau \colon a \mapsto \int_{\mathbb{R}^{2d}} a(x,\omega) e^{-2\pi i \tau x\omega} \pi(x,\omega)\, dx d\omega$$

is the $\tau$-spreading representation operator.
In Chapter 7 we show how these objects are naturally related to each other, see e.g. Lemma 7.1.9, and that $\mathbb{S}_0$ is a fruitful setting where to consider them. To give a flavour of the main results, we summarize Theorem 7.2.6 and 7.2.6 as follows. Recall that the $\tau$-quantization of $a$ if formally given by

$$\mathrm{Op}_\tau(a) f(t) := \int_{\mathbb{R}^{2d}} e^{2\pi i (t-x)\omega} a((1-\tau)t + \tau x, \omega) f(x)\, dx d\omega.$$

**Theorem 1.3.1.** *Let $\tau \in [0,1]$ and*

$$(X,Y) = (\mathcal{J}^1, L^2(\mathbb{R}^{2d})), (\mathbb{S}_0, \mathcal{S}_0(\mathbb{R}^{2d})),$$

*so that*

$$(X',Y') = (\mathcal{J}^\infty, L^2(\mathbb{R}^{2d})), (\mathbb{S}_0', \mathcal{S}_0'(\mathbb{R}^{2d})).$$

*Then the following mappings are linear and continuous*

$$\mathrm{Op}_\tau \colon Y' \to X', \qquad W_\tau \colon X \to Y$$

*and $\mathrm{Op}_\tau$ is the Banach space adjoint of $W_\tau$: $\mathrm{Op}_\tau = W_\tau^*$,in the sense that*

$$_{Y'}\langle a, W_\tau S\rangle_Y = {}_{X'}\langle \mathrm{Op}_\tau(a), S\rangle_X$$

*for all $a \in Y'$ and $S \in X$.*

Moreover, in Corollary 7.2.8 we show that $W_\tau \colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$ is a topological isomorphism which inverse is given by $\mathrm{Op}_\tau \colon \mathcal{S}_0(\mathbb{R}^{2d}) \to \mathbb{S}_0$.
Analogous results hold true for the Fourier-$\tau$-Wigner transform $\mathcal{F}_{W_\tau}$ and the $\tau$-spreading representation $\mathrm{SR}^\tau$. In the following item we report Corollary 7.2.9.

**Theorem 1.3.2.** *Let $\tau \in [0,1]$. Then the following mappings are linear and continuous*

$$\mathrm{SR}^\tau \colon \mathcal{S}_0'(\mathbb{R}^{2d}) \to \mathbb{S}_0', \qquad \mathcal{F}_{W_\tau} \colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$$

*and $\mathrm{SR}^\tau$ is the Banach space adjoint of $\mathcal{F}_{W_\tau}$: $\mathrm{SR}^\tau = \mathcal{F}_{W_\tau}^*$, in the sense that*

$$_{\mathcal{S}_0'}\langle a, \mathcal{F}_{W_\tau} S\rangle_{\mathcal{S}_0} = {}_{\mathbb{S}_0'}\langle \mathrm{SR}^\tau a, S\rangle_{\mathbb{S}_0}$$

*for all $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $S \in \mathbb{S}_0$.*
*Moreover $\mathcal{F}_{W_\tau} \colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$ is a topological isomorphism which inverse is given by $\mathrm{SR}^\tau \colon \mathcal{S}_0(\mathbb{R}^{2d}) \to \mathbb{S}_0$.*

Eventually, in the last section of the chapter, a characterization for $\mathfrak{S}$ is given in terms of weighted classes of Feichtinger operators $\mathbb{M}^1_s$. Namely

$$\mathfrak{S} = \bigcap_{s \geq 0} \mathbb{M}^1_s,$$

see Theorem 7.3.6. As a consequence, in the spirit of [95], a sufficient condition for an operator to be in $\mathfrak{S}$ is provided.

## 1.4 Structure of the thesis

This thesis work is structured as follows. Chapter 2 contains the preliminaries which are required along all the text. The subsequent chapters contains the results of the papers [3, 7, 8, 9, 10, 11]. We chose to present them following the order under which they were written.
Chapter 3 presents the results about decay and smoothness for $L^2$-eigenfunction of localization operators $A_a^{\psi_1,\psi_2}$ on $\mathbb{R}^d$ [9] as well as the generalization to the Gelfand-Shilov setting [11].
In Chapter 4 we characterize the symbol class $S^m(\mathbb{R}^{2d})$ in terms of the Gabor matrix decay of $\mathrm{Op}_\tau$ [7].
Chapter 5 is devoted to the definition of quasi-Banach modulation spaces on LCA groups, the study of their main properties and of Kohn-Nirenberg operators $\mathrm{Op}_0(\sigma)$ on such spaces as well as of eigenfunctions of localization operators $A_a^{\psi_1,\psi_2}$. The results were published in [8].
Chapter 6 addresses the problem of writing a localization operator with symbol only in the frequencies $A_{1\otimes m}^{\psi_1,\psi_2}$ as Fourier multiplier [3]. Also the finite discrete setting of $\mathbb{Z}_N$ is taken into account.
Eventually, Chapter 7 deals with Feichtinger operators in quantum harmonic analysis and the various interpretations of the $\tau$-quantization $\mathrm{Op}_\tau$ as Banach space adjoint between suitable Banach spaces [10].

# Chapter 2

# Preliminaries

In the present chapter we recall and collect notations, backgrounds and some preliminary results which are shared by and exploited in chapters from 3 to 7.

In Section 2.1 we report the definition of quasi-norm, $r$-norms, quasi-normed and and quasi-Banach spaces. Basic theory which steams from them if briefly recalled. The main references are [44, 147].

Section 2.2 concerns solid quasi-Banach function (QBF) spaces $Y$ on locally compact Hausdorff (LCH) groups $G$ and a summary of the relative coorbit theory developed by F. Voigtlaender in his Ph.D. thesis [147]. In particular, Subsection 2.2.1 contains definitions and assumptions made on weight functions through all the present thesis work, both in the general setting of a LCH group $G$ and on $\mathbb{R}^d$. In the latter situation, particular weights are considered, such as e.g. polynomial weights $v_s$ and (sub-)exponential ones $w_k^\gamma$. Subsection 2.2.2 reports definitions and main properties of solid QBF spaces $Y$ and associated discrete sequence spaces $Y_d$, along with relatively separated families and BUPUs; Subsection 2.2.3 deals with Wiener amalgam spaces $W_Q(Y)$ with local component $L^\infty(G)$. We point out that, for sake of simplicity, we shall stick to the situation where the space $W_Q(Y)$ is independent of the window subset $Q$, although more general scenarios can be taken into account. Subsection 2.2.4 shows weighted Lebesgue mixed-norm space $L^{p,q}_m$ [12] in the perspective of the preceding subsection and provides proofs for a numbers of results which seem to be folklore, for no proof was available to author's knowledge. See in particular lemma 2.2.26 and Proposition 2.2.27, which were both published in [8]. Eventually, a comparison with the theory developed by H. G. Feichtinger and K. Gröchenig in [58] is made, see Theorem 2.2.39 and the above remark.

The main tools of time-frequency analysis (TFA) on Euclidean space $\mathbb{R}^d$ are listed in Section 2.3. Here we find the definitions and main properties of time-frequency shifts (TFS) $\pi(x,\omega)$, short-time Fourier transform (STFT) $V_g f$, cross-$\tau$-Wigner distribution $W_\tau(f,g)$. In Subsection 2.3.2 we list different and equivalent ways of representing linear and continuous operators from the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$: as integral operator with kernel, as pseudo-differential operator and as continuous superposition of TFS-s. In Subsection 2.3.3 we generalize $\pi(x,\omega)$, $V_g f$, $W_0(f,g) = R(f,g)$ to any LCA group $\mathcal{G}$. Moreover, the important Structure Theorem 2.3.23 [96] and the class of special test functions $\mathcal{S}_\mathcal{C}(\mathcal{G})$ [88] are reported. Eventually, the specific choice $\mathcal{G} = \mathbb{Z}_N$ is taken into account. Under the identification $\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$, see (2.104), we look at: STFT, spreading and matrix (kernel) representation of linear mappings from $\mathbb{C}^N$ into itself, discrete Fourier transform $\mathcal{F}_N \colon \mathbb{C}^N \to \mathbb{C}^N$, discrete two dimensional Fourier transform $\mathsf{F}_2 \colon \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$, Kronecker delta function $\delta$, Dirac comb $\text{Ш}_{\alpha,\beta}$ and discrete symplectic Fourier transform $\mathsf{F}_s \colon \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$. The main references are [18, 35, 82, 88, 117].

A brief recap of frame theory in Hilbert spaces in given in Section 2.4. Gabor frames, along with analysis and synthesis operators, and some of their properties are shown both on $\mathbb{R}^d$ and $\mathcal{G}$ LCA group. In particular, in Subsection 2.4.2 we introduce quasi-lattices on $\mathcal{G}$ as done in [88] in order to build Gabor frames. Once more, we consider the particular case of $\mathcal{G} = \mathbb{Z}_N$.

Modulation spaces are introduced and discussed in Section 2.5. In particular, modulation spaces $M^{p,q}_m$ with indexes $0 < p, q \leq \infty$ on $\mathbb{R}^d$ are shown in Subsection 2.5.1, whereas $M^{p,q}_m$ with $1 \leq p, q \leq \infty$ on any $\mathcal{G}$ LCA group are treated in Subsection 2.5.2. There we also list some properties of the Feichtinger algebra $\mathcal{S}_0(\mathcal{G}) = M^1(\mathcal{G})$. In particular, in this section we present the proof of some new convolution relations on modulation spaces on $\mathbb{R}^d$ which were published in [9], see Proposition 2.5.19. This result has been improved to any $\mathcal{G}$ LCA group in [8], see Proposition 5.2.14 in Chapter 5. Some results about inclusion relations and an equivalent semi-discrete quasi-norm on $M^{p,q}_m(\mathbb{R}^d)$ taken from [7] are reported. See Proposition 2.5.20, 2.5.21 and Corollary 2.5.22. Proposition 2.5.23 was present in [3]. We address the reader to [35, 56, 82, 88].

Section 2.6 recalls various function spaces which shall be used in many of the subsequent chapters. In particular we shall briefly define: Wiener amalgam spaces $W(L^p, L^q_m)(\mathbb{R}^d)$ with indexes $0 < p, q \leq \infty$, Besov spaces $B^{p,q}_s(\mathbb{R}^d)$ [145], the class of smooth symbols $S^m(\mathbb{R}^{2d})$ [127], weak $L^{r,\infty}$ spaces [145]. Original results here presented and both published in [7] are: Lemma 2.6.8, which generalizes a characterization of Hörmander's class $S^0_{0,0}$ proved in [87, Lemma 6.1], and the subsequent Lemma 2.6.9.

The main operators which are used in the present thesis, and some of which are also object of main results illustrated in subsequent chapters, are defined in Section 2.7. We shall see: localization operator $A^{\psi_1,\psi_2}_a$ both on $\mathbb{R}^d$ and $\mathcal{G}$, Gabor multipliers $G^{g_1,g_2}_a$ on $\mathbb{R}^d$ and $\mathsf{G}^{g_1,g_2}_a$ on $\mathbb{Z}_N$, pseudo-differential operators $\mathrm{Op}_\tau(\sigma)$ and Born-Jordan operators $\mathrm{Op}_{BJ}(\sigma)$ on $\mathbb{R}^d$, Kohn-Nirenberg operators $\mathrm{Op}_0(\sigma)$ on $\mathcal{G}$ LCA group, Fourier multiplier $T_m$ on $\mathbb{R}^d$ and linear time invariant filters $H$ on $\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$ (see (2.104)).

Eventually, Section 2.8 revisits Sections 2.3, 2.5 and 2.7 in the framework of Gelfand-Shilov spaces $\mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ and $\Sigma^\gamma_\tau(\mathbb{R}^d)$, which are treated in Subsection 2.8.1. For some references about the Gelfand-Shilov setting we address, e.g. to [76, 118, 141, 143]. Proposition 2.8.13 was published by N. Teofanov and the author in [11], it extends the convolutions for modulation spaces presented in Proposition 2.5.19 [9]. Also Lemma 2.8.15 and Proposition 2.8.16 appeared for the first time in [11].

## 2.1 Quasi-normed and quasi-Banach spaces

In this section, we report the definition of quasi-norm, $r$-norm, quasi-normed and and quasi-Banach space. These notions will be used through all further chapters, since we shall always tackle the quasi-Banach case, whenever possible. In particular, the so-called Aoki-Rolewicz Theorem 2.1.3 is exploited heavily in Chapter 5. We address the reader to [44, 147] for further references.

**Definition 2.1.1.** *Let $X$ be a vector space. An application $\|\cdot\|_X : X \to [0, +\infty)$ is called* ***quasi-norm*** *if:*

*(i)* $\|x\|_X = 0$ *if and only if* $x = 0_X$*, for every* $x \in X$*;*

*(ii)* $\|\alpha x\|_X = |\alpha| \, \|x\|_X$*, for very* $x \in X$ *and* $\alpha \in \mathbb{C}$*;*

*(iii) There exists a constant* $C \geq 1$ *such that for every* $x, y \in X$

$$\|x + y\|_X \leq C \left(\|x\|_X + \|y\|_X\right);$$

*such a $C$ is called* ***triangle constant****.*

*The pair* $(X, \|\cdot\|_X)$ *is called* ***quasi-normed space****. Two quasi-norms* $\|\cdot\|_{1,X}$ *and* $\|\cdot\|_{2,X}$ *on* $X$ *are called* ***equivalent*** *if there exist two constants* $C_1, C_2 > 0$ *such that for every* $x \in X$

$$C_1 \|x\|_{1,X} \le \|x\|_{2,X} \le C_2 \|x\|_{1,X} .$$

*In this case we write* $\|\cdot\|_{1,X} \asymp \|\cdot\|_{2,X}$.
*Given* $0 < r \le 1$*, an application* $\|\cdot\|_X : X \to [0, +\infty)$ *which satisfies* $(i)$ *and* $(ii)$ *and such that*

$$\|x + y\|_X^r \le \|x\|_X^r + \|y\|_X^r , \qquad \forall x, y \in X,$$

*is called* $r$***-norm****.*

If the triangle constant can be chosen to be 1, then we recover the definition of norm and normed space. Trivially, a 1-norm is just a norm.

**Remark 2.1.2.** *The following observations are borrowed from [147, Remark 2.1.2], see therein for related computations.*

- $(i)$ *Given an* $r$*-norm* $\|\cdot\|_X : X \to [0, +\infty)$*, on account of the convexity of* $t \mapsto t^{\frac{1}{r}}$ *on* $[0, +\infty)$*, for every* $x, y \in X$ *we have*

$$\|x + y\|_X \le 2^{\frac{1}{r}-1} \left(\|x\|_X + \|y\|_X\right) .$$

  *Therefore every* $r$*-norm is a quasi-norm with triangle constant* $C = 2^{\frac{1}{r}-1}$*;*

- $(ii)$ *If* $0 < s \le r \le 1$*, then any* $r$*-norm is also an* $s$*-norm.*

The following fundamental result will be repeatedly used in Chapter 5.

**Theorem 2.1.3** (Aoki-Rolewicz)**.** *If* $(X, \|\cdot\|_{1,X})$ *is a quasi-normed space, then there is* $0 < r \le 1$ *such that the mapping defined by*

$$\|x\|_{2,X} := \inf \left\{ \left( \sum_{i=1}^{n} \|x_i\|_{1,X}^r \right)^{\frac{1}{r}} \mid n \in \mathbb{N},\, x_1, \ldots, x_n \in X,\, x = \sum_{i=1}^{n} x_i \right\},$$

*for every* $x \in X$*, is an* $r$*-norm equivalent to* $\|\cdot\|_{1,X}$.

About the previous theorem, we address the reader to [79, Exercise 1.4.6] and [44, Chapter 2, Theorem 1.1].

**Definition 2.1.4.** *Let* $(X, \|\cdot\|_X)$ *be a quasi-normed space. A sequence* $\{x_n\}_n \subseteq X$ *is called* ***Cauchy*** *if*

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad | \quad \|x_i - x_j\|_X < \varepsilon \quad \forall i, j > N_\varepsilon .$$

*If every Cauchy sequence in* $X$ *is convergent, then* $(X, \|\cdot\|_X)$ *is called* ***quasi-Banach space****.*

**Lemma 2.1.5.** ([147, Lemma 2.1.5]) *Let* $(X, \|\cdot\|_X)$ *a quasi-normed space. Given* $x \in X$ *and* $\varepsilon > 0$*, we define the* ***ball of radius*** $\varepsilon$ ***centred at*** $x$ *as*

$$B_\varepsilon(x) := \{y \in X \mid \|x - y\|_X < \varepsilon\}.$$

*The collection of all the subsets* $E \subseteq X$ *such that*

$$\forall x \in E \quad \exists \varepsilon > 0 \, : \, B_\varepsilon(x) \subseteq E$$

*is a topology on $X$. It is called the* **topology induced by** $\|\cdot\|_X$ *and turns $X$ into a TVS. Equivalent quasi-norms induce the same topology. If $\|\cdot\|_{1,X}$ is an $r$-norm, for some $0 < r \leq 1$, equivalent to $\|\cdot\|_X$, then*

$$d\colon X \times X \to [0, +\infty), (x, y) \mapsto \|x - y\|_{1,X}^r$$

*is a metric on $X$ which induces the same topology as $\|\cdot\|_X$.*

**Remark 2.1.6.** (*i*) *On account of Lemma 2.1.5, if $(X, \|\cdot\|_X)$ is quasi-Banach and $\|\cdot\|_{1,X}$ is an $r$-norm, for some $0 < r \leq 1$, equivalent to $\|\cdot\|_X$, then $(X, \|\cdot\|_{1,X})$ is quasi-Banach too;*

(*ii*) *If $(X, \|\cdot\|_X)$ is quasi-normed space which does not admit an equivalent norm, then pathological behaviours may occur. For example: the unit ball may not be open; the quasi-norm can be not continuous w.r.t. the topology it induces, not even Borel measurable. For concrete examples, we address the reader to [147, Remark 2.1.9]. Moreover, quasi-normed spaces are in general not convex. In order to point out one more difference w.r.t. the Banach case, we mention that the topological dual of $L^p(\mathbb{R})$, $0 < p < 1$, is $\{0\}$, see [79, Theorem 1.4.1 (i)].*

Eventually we recall the following result from [147].

**Lemma 2.1.7.** ([147, Lemma 2.1.6]) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed spaces. The a linear mapping $T\colon X \to Y$ is continuous if and only if it is bounded, i.e. if*

$$|||T|||_{X \to Y} := \sup_{\substack{x \in X, \\ \|x\|_X \leq 1}} \|Tx\|_Y < +\infty.$$

## 2.2 Solid quasi-Banach function spaces on $G$ and coorbit theory

The present section summarizes the construction of coorbit spaces $\mathsf{Co}(Y)$, when $Y$ is a solid quasi-Banach function space on a locally compact Hausdorff group $G$, even not abelian. This theory was first developed by H. Rauhut in [119] and technically fixed and deepened by F. Voigtlaender in his Ph.D. thesis [147]. In the end we shall highlight the differences with the original theory for Banach spaces by H. G. Feichtinger and K. Gröchenig, see [58, 59, 60].
We mention that an exposition and treatment of the named coorbit theory is now available also in the recent article [146] from J. T. van Velthoven and F. Voigtlaender, where the requirements on the weights are lightened up. However, due to the time when the work [8] was written, we shall stick to the first version presented in [147]. Moreover, on account of the objects of our particular setting, this makes no difference.

From now on we make the following assumptions.

**Assumptions 2.2.1.** *A topological group $G$, not necessarily abelian, is always assumed to be locally compact Hausdorff (LCH) and $\sigma$-compact. The group law on $G$ is represented as multiplication.*

**Assumptions 2.2.2.** *We use the letter $\mathcal{G}$ for a locally compact, abelian, Hausdorff, $\sigma$-compact, second-countable group.*
*We shall simply refer to it as a LCA group.*
*The group law on $\mathcal{G}$, as on any abelian group, is represented as addition and $e$ stands for the identity. $\widehat{\mathcal{G}}$ denotes the dual group of $\mathcal{G}$. Latin letters such as $x$, $y$ and $u$ denote elements in $\mathcal{G}$*

*whereas all the characters in $\widehat{\mathcal{G}}$, except the identity $\hat{e}$, are indicated by Greek letters like $\xi$, $\omega$ and $\eta$. For the evaluation of a character $\xi \in \widehat{\mathcal{G}}$ at a point $x \in \mathcal{G}$ we write*

$$\langle \xi, x \rangle \coloneqq \xi(x).$$

*We use boldface Latin letters to denote point in the phase-space:*

$$\mathbf{x} = (x, \xi), \mathbf{y} = (y, \eta), \mathbf{u} = (u, \omega) \in \mathcal{G} \times \widehat{\mathcal{G}}.$$

*Similarly, boldface Greek ones $\boldsymbol{\xi}, \boldsymbol{\omega}, \boldsymbol{\eta}$ stand for elements of $\widehat{\mathcal{G}} \times \mathcal{G}$. The Haar measure on $\mathcal{G}$ is denoted by $dx$ and $d\xi$ stands for the dual Haar measure on $\widehat{\mathcal{G}}$. Hence the Fourier transform defined as*

$$\mathcal{F}f(\xi) \coloneqq \hat{f}(\xi) \coloneqq \int_{\mathcal{G}} f(x)\overline{\langle \xi, x \rangle}\, dx, \qquad \forall \xi \in \widehat{\mathcal{G}},$$

*is an isometry from $L^2(\mathcal{G})$ onto $L^2(\widehat{\mathcal{G}})$.*

Given $x \in G$ and a function $f$ on $G$, we denote ***left*** and ***right translation operators*** by

$$L_x f(y) \coloneqq f(x^{-1}y), \qquad R_x f(y) \coloneqq f(yx). \tag{2.1}$$

If we are dealing with $\mathcal{G}$, i.e. the group is abelian, we shall adopt the common notation

$$T_x \coloneqq L_x.$$

Whenever a measure on $G$ is involved, it is understood to be the left Haar measure. We shall not list systematically the known properties for the spaces introduced in the sequel, but rather recall them when necessary. The reader is invited to consult [147, Chapter 2] for an exhaustive treatment.

### 2.2.1 Weights

In what follows we state what we mean, and which assumptions are made, by weight functions in the present thesis work. We refer to [35, 82, 84, 147].

**Definition 2.2.3.** *A **weight on** $G$ is a measurable function $m\colon G \to (0, +\infty)$. A weight $v$ on $G$ is said to be **submultiplicative** if*

$$v(xy) \leq v(x)v(y), \qquad \forall x, y \in G. \tag{2.2}$$

*Given two weights $m$ and $v$ on $G$, $m$ is said to be **left-moderate w.r.t.** $v$ if*

$$m(xy) \lesssim v(x)m(y), \qquad \forall x, y \in G, \tag{2.3}$$

*it is **right-moderate w.r.t.** $v$ if*

$$m(xy) \lesssim m(x)v(y), \qquad \forall x, y \in G. \tag{2.4}$$

*If a weight $m$ is both left- and right-moderate w.r.t. $v$, we simply say that it is **moderate w.r.t.** $v$ or $v$**-moderate.***
*Consider $v$ submultiplicative weight on $G$ which is also even, bounded from below and satisfies the Gelfand-Raikov-Shilov (GRS) condition, i.e.*

$$\begin{aligned} v(x) = v(x^{-1}) &\quad \forall x \in G, \\ \exists c > 0 \, : \, v(x) \geq c &\quad \forall x \in G, \\ \lim_{n \to +\infty} v(x^n)^{\frac{1}{n}} = 1 &\quad \forall\, x \in G, \end{aligned}$$

*then the **class of weights on** $G$ **moderate w.r.t.** $v$ is denoted as follows:*

$$\mathcal{M}_v(G) := \{m\,\text{weight on}\,G\,|\,m\,\text{is}\,v\text{−moderate}\}\,. \tag{2.5}$$

*Two weights* $w_1, w_2$ *on* $G$ *are said to be **equivalent** if* $w_1 \asymp w_2$*, i.e.*

$$w_1(x) \lesssim w_2(x) \lesssim w_1(x), \qquad \forall x \in G.$$

**Remark 2.2.4.** *The GRS condition is not always needed, but it is necessary in order to have Gabor frames for* $L^2(\mathcal{G})$*. The related result is Theorem 2.4.26 in which* $v$ *is a weight on the abelian group* $\mathcal{G} \times \widehat{\mathcal{G}}$*, hence the GRS condition has the form*

$$\lim_{n\to+\infty} v(n\mathbf{x})^{\frac{1}{n}} = 1, \quad \forall\, \mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}.$$

**Theorem 2.2.5.** ([92, Theorem 2.1.4]) *Let* $v$ *be a submultiplicative weight on* $G$*. Then* $v$ *is bounded on compact sets.*

#### 2.2.1.1 Weights on $\mathbb{R}^d$

Here we list some specific weights which shall be explicitly used in the following chapters, in particular in Chapter 3. Namely, we define the Japanese brackets $\langle\cdot\rangle$, the weight of polynomial type $v_s$ and (super/sub-)exponential weights $w_k^\gamma$. Some technical results from [11] are reported, in particular Lemma 2.2.13 shall be exploited in order to prove Theorem 3.3.5. Finally, weights $m_u^\tau$ are defined and they will be needed in Chapter 3 as well.

**Remark 2.2.6.** *When dealing with the abelian group* $G = \mathbb{R}^d$*, we see that if* $m \in \mathcal{M}_v(\mathbb{R}^d)$*, then*

$$v^{-1}(x) \lesssim m(x) \lesssim v(x), \qquad \forall x \in \mathbb{R}^d.$$

*It follows that* $v(x) \lesssim m^{-1}(x) \lesssim v^{-1}(x)$ *for every* $x \in \mathbb{R}^d$*. This, together with Theorem 2.2.5 gives:* $m, 1/m \in L^\infty_{loc}(\mathbb{R}^d)$*.*

**Definition 2.2.7.** *We define the **Japanese bracket of** $x \in \mathbb{R}^d$ to be*

$$\langle x\rangle := \sqrt{1+|x|^2}. \tag{2.6}$$

*Let* $s \in \mathbb{R}$*, the **polynomial weight** $v_s$ is defined to be*

$$v_s(x) := \langle x\rangle^s = (1+|x|^2)^{\frac{s}{2}}, \qquad \forall x \in \mathbb{R}^d. \tag{2.7}$$

*We say that a weight* $m$ *on* $\mathbb{R}^d$ *has **at most polynomial growth (at infinity)** if there exist* $C > 0$ *and* $s > 0$ *such that*

$$m(x) \le C v_s(x), \qquad \forall x \in \mathbb{R}^d. \tag{2.8}$$

*Let* $k, \gamma > 0$*, and define*

$$w_k^\gamma(x) := e^{k|x|^{1/\gamma}}, \qquad \forall x \in \mathbb{R}^d. \tag{2.9}$$

*If* $\gamma > 1$ *the above functions are called **subexponential weights**, when* $\gamma = 1$ *we have the **exponential weights** and write* $w_k$ *instead of* $w_k^1$*, for* $0 < \gamma < 1$ *we call them **superexponential weights**.*

Sometimes we shall use the expression $w_k^\gamma$ for $k = 0$ also, with obvious meaning.

**Remark 2.2.8.** *(i) The polynomial weight $v_s$ fails to be submultiplicative for $s \geq 0$ due to* ***Peetre's inequality*** *(which is sharp):*

$$\langle x+y\rangle^s \leq 2^s \langle x\rangle^s \langle y\rangle^{|s|}, \qquad \forall x, y \in \mathbb{R}^d, \quad s \in \mathbb{R}.$$

*However the function $2^s\langle x\rangle^s$, for $s \geq 0$, is a submultiplicative weight equivalent to $v_s$. By abuse of notation we denote by $\mathcal{M}_{v_s}(\mathbb{R}^d)$ the class of weights which are $v_s$-moderate weights. We leave as an exercise for the reader to prove that it coincides with $\mathcal{M}_{2^s v_s}(\mathbb{R}^d)$. We also observe that, for $s < 0$, $v_s$ is $v_{|s|}$-moderate due to Peetre's inequality;*

*(ii) Observe that* ***(sub-)exponential weights*** *$w_k^\gamma$, $\gamma \geq 1$, are submultiplicative;*

*(iii) Both $v_s$ and $w_k^\gamma$, with $s, k \geq 0$ and $\gamma \geq 1$, are even and fulfil the GRS condition.*

**Definition 2.2.9.** *Let $\gamma > 0$ and define*

$$\begin{aligned}
\mathscr{P}_E(\mathbb{R}^d) &:= \{m \text{ weight on } \mathbb{R}^d \mid m \text{ is } v\text{-moderate for some submultiplicative } v\},\\
\mathscr{P}_{E,\gamma}(\mathbb{R}^d) &:= \{m \text{ weight on } \mathbb{R}^d \mid m \text{ is } w_k^\gamma\text{-moderate for some } k > 0\},\\
\mathscr{P}^0_{E,\gamma}(\mathbb{R}^d) &:= \{m \text{ weight on } \mathbb{R}^d \mid m \text{ is } w_k^\gamma\text{-moderate for every } k > 0\}.
\end{aligned}$$

For $0 < \gamma_2 < \gamma_1$ we have

$$\mathscr{P}^0_{E,\gamma_1} \subseteq \mathscr{P}_{E,\gamma_1} \subseteq \mathscr{P}^0_{E,\gamma_2} \subseteq \mathscr{P}_E.$$

Moreover, for $0 < \gamma < 1$ we have $\mathscr{P}_E = \mathscr{P}_{E,\gamma} = \mathscr{P}^0_{E,\gamma}$; see [19, Remark 2.6] and [144]. In the next lemma we show that if $m \in \mathscr{P}_E$, then it is $w_k$-moderate fore some $k > 0$ large enough. This implies $\mathscr{P}_E = \mathscr{P}_{E,1}$.

**Lemma 2.2.10.** ([11, Lemma 2.1]) *Let $m \in \mathscr{P}_E$. Then $m$ is $w_k$-moderate fore some $k > 0$.*

*Proof.* The lemma is folklore ([84, 19, 143, 142]). For the sake of completeness we report a self-contained proof following [84]. By the hypothesis, we may assume that $m$ is moderate with respect to some continuous $v_0 > 0$ (cf. [35, 84, 139]): $m(x+y) \leq Cv_0(x)m(y)$, $x, y \in \mathbb{R}^d$. It follows that $\sup_{|t|\leq 1} Cv_0(t) = e^a$ for some $a \in \mathbb{R}$. For any given $x, y \in \mathbb{R}^d$ we choose $n \in \mathbb{N}$ such that $n-1 < |x| \leq n$. Then for all $x$ and $y$ in $\mathbb{R}^d$

$$\begin{aligned}
m(x+y) &= m\left(n\frac{x}{n} + y\right) \leq Cv_0\left(\frac{x}{n}\right) m\left((n-1)\frac{x}{n} + y\right)\\
&\leq C^2 v_0^2\left(\frac{x}{n}\right) m\left((n-2)\frac{x}{n} + y\right)\\
&\leq \dots\\
&\leq \left(Cv_0\left(\frac{x}{n}\right)\right)^n m(y) \leq e^{an} m(y)\\
&< e^{a(|x|+1)} m(y) = e^a e^{a|x|} m(y).
\end{aligned}$$

The claim follows for $k > \max(0, a)$. □

We remark that $\mathscr{P}_E$ contains the weights of ***polynomial type***, i.e. weights moderate with respect to some polynomial, or equivalently to some $v_s$ with $s \geq 0$.

In the sequel $\mathscr{P}^*_{E,\gamma}$ means $\mathscr{P}_{E,\gamma}$ or $\mathscr{P}^0_{E,\gamma}$. The following lemma follows by easy calculations and we leave the proof to the reader (see also [139]). Observe that due to the equality $\mathscr{P}_{E,1} = \mathscr{P}_{E,\gamma} = \mathscr{P}^0_{E,\gamma}$, $0 < \gamma < 1$, it is sufficient to consider $\gamma \geq 1$.

**Lemma 2.2.11.** ([11, Lemma 2.2]) *Consider* $\gamma > 0$. *Then* $\mathscr{P}^*_{E,\gamma}(\mathbb{R}^d)$ *is a group under the pointwise multiplication and with the identity* $m \equiv 1$.

Given a function $f$ defined on $\mathbb{R}^{2d}$ we denote its restrictions to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$ as follows:

$$f_1(x) \coloneqq f(x,0), \qquad f_2(\omega) \coloneqq f(0,\omega), \qquad \forall x,\omega \in \mathbb{R}^d. \tag{2.10}$$

The families $\mathscr{P}^*_{E,\gamma}$ turn out to be closed under restrictions and tensor products in the sense of the following lemma. The proof is omitted, since it follows from definitions and properties of the Euclidean norm.

**Lemma 2.2.12.** ([11, Lemma 2.3]) *Consider* $\gamma > 0$*:*

(*i*) *if* $m \in \mathscr{P}^*_{E,\gamma}(\mathbb{R}^{2d})$, *then* $m_1, m_2 \in \mathscr{P}^*_{E,\gamma}(\mathbb{R}^d)$*;*

(*ii*) *if* $m, w \in \mathscr{P}^*_{E,\gamma}(\mathbb{R}^d)$, *then* $m \otimes w \in \mathscr{P}^*_{E,\gamma}(\mathbb{R}^{2d})$.

Next we exhibit a lemma which will play a key role in Theorem 3.3.5.

**Lemma 2.2.13.** ([11, Lemma 2.4]) *Consider* $\gamma \geq 1$, $r, s \geq 0$, $\tau \in [0,1]$ *and*

$$t \geq \begin{cases} r + s\tau^{1/\gamma} & \text{if} \quad 1/2 \leq \tau \leq 1, \\ r + s(1+\tau^2)^{1/2\gamma} & \text{if} \quad 0 \leq \tau < 1/2. \end{cases} \tag{2.11}$$

*Then for every* $x, \omega, y, \eta \in \mathbb{R}^d$ *the following estimate holds true:*

$$\frac{w^\gamma_{r+s}(x,\omega)}{w^\gamma_r(y,\eta)} \leq w^\gamma_s \otimes w^\gamma_t \Big( \big((1-\tau)x + \tau y, \tau\omega + (1-\tau)\eta\big), \big(\omega - \eta, y - x\big) \Big). \tag{2.12}$$

*Proof.* We first recall that given $0 < p \leq q < \infty$ the following holds true:

$$\|z\|_q \coloneqq \left( \sum_{i=1}^d |z_i|^q \right)^{\frac{1}{q}} \leq \left( \sum_{i=1}^d |z_i|^p \right)^{\frac{1}{p}} =: \|z\|_p, \quad z = (z_1, \ldots, z_d) \in \mathbb{R}^d. \tag{2.13}$$

In fact, consider $z$ such that $\|z\|_p = 1$. Hence $|z_i|^p \leq 1 \Rightarrow |z_i| \leq 1$ for $i = 1, \ldots, d$. Thus $|z_i|^q \leq |z_i|^p$ and $\sum_{i=1}^d |z_i|^q \leq \sum_{i=1}^d |z_i|^p = 1$. Eventually consider $u \in \mathbb{R}^d \setminus \{0\}$, then $\left\| u / \|u\|_p \right\|_q \leq 1$ and (2.13) is proved.

By using the triangular inequality and (2.13) with $q = 1$ and $p = \beta$, we infer that for $0 < \beta \leq 1$

$$\left| \sum_{i=1}^d z_i \right|^\beta \leq \sum_{i=1}^d |z_i|^\beta, \quad z = (z_1, \ldots, z_d) \in \mathbb{R}^d. \tag{2.14}$$

Now, by the triangular inequality and (2.14) with $d = 2$ we obtain

$$|x|^\beta - |y|^\beta \leq |x - y|^\beta, \quad 0 < \beta \leq 1, \;\; x, y \in \mathbb{R}^d. \tag{2.15}$$

Next, we observe that for $z, w \in \mathbb{R}^d$

$$\begin{aligned} &|(\tau z, (1-\tau)w)|^2 = \tau^2 |z|^2 + (1-\tau)^2 |w|^2 = \tau^2 |z|^2 + (\tau^2 + 1 - 2\tau) |w|^2 \\ &= \tau^2(|z|^2 + |w|^2) + (1-2\tau) |w|^2 = \tau^2 |(z,w)|^2 + (1-2\tau) |w|^2 \\ &\leq \begin{cases} \tau^2 |(z,w)|^2 + 0 & \text{if} \quad 1/2 \leq \tau \leq 1, \\ \tau^2 |(z,w)|^2 + 1 |w|^2 + |z|^2 = (1+\tau^2) |(z,w)|^2 & \text{if} \quad 0 \leq \tau < 1/2, \end{cases} \end{aligned}$$

which gives

$$|(\tau z,(1-\tau)w)|^{1/\gamma} \leq \begin{cases} \tau^{1/\gamma}\,|(z,w)|^{1/\gamma} & \text{if} \quad 1/2 \leq \tau \leq 1, \\ (1+\tau^2)^{1/2\gamma}\,|(z,w)|^{1/\gamma} & \text{if} \quad 0 \leq \tau < 1/2. \end{cases} \tag{2.16}$$

We can now prove (2.12):

$$\begin{aligned}
\frac{w^{\gamma}_{r+s}(x,\omega)}{w^{\gamma}_{r}(y,\eta)} &= \exp\left((r+s)\,|(x,\omega)|^{1/\gamma} - r\,|(y,\eta)|^{1/\gamma}\right) \\
&= \exp\left(r\left(|(x,\omega)|^{1/\gamma} - |(y,\eta)|^{1/\gamma}\right) + s\,|(x,\omega)|^{1/\gamma}\right) \\
&\overset{(2.15)}{\leq} \exp\left(r\,|(x,\omega)-(y,\eta)|^{1/\gamma} + s\,|(x,\omega)|^{1/\gamma}\right) \\
&= \exp(r\,|(\omega-\eta,y-x)|^{1/\gamma} + s\,|(x,\omega)|^{1/\gamma} - s\,|(\tau(x-y),(1-\tau)(\omega-\eta))|^{1/\gamma}) \\
&\times \exp(s\,|(\tau(x-y),(1-\tau)(\omega-\eta))|^{1/\gamma}) \\
&\overset{(2.15)}{\leq} \exp(r\,|(\omega-\eta,y-x)|^{1/\gamma} + s\,|(x,\omega)-(\tau(x-y),(1-\tau)(\omega-\eta))|^{1/\gamma} \\
&+ s\,|(\tau(x-y),(1-\tau)(\omega-\eta))|^{1/\gamma}) \\
&= \exp(r\,|(\omega-\eta,y-x)|^{1/\gamma} + s\,|((1-\tau)x+\tau y,\tau\omega+(1-\tau)\eta)|^{1/\gamma} \\
&+ s\,|(\tau x-\tau y,(1-\tau)\omega-(1-\tau)\eta)|^{1/\gamma}) \\
&\overset{(2.16)}{\leq} \begin{cases} \exp((r+s\tau^{1/\gamma})\,|(\omega-\eta,y-x)|^{1/\gamma} \\ \qquad +s\,|((1-\tau)x+\tau y,\tau\omega+(1-\tau)\eta)|^{1/\gamma}) & \text{if} \quad 1/2 \leq \tau \leq 1, \\ \exp((r+s(1+\tau^2)^{1/2\gamma})\,|(\omega-\eta,y-x)|^{1/\gamma} \\ \qquad +s\,|((1-\tau)x+\tau y,\tau\omega+(1-\tau)\eta)|^{1/\gamma}) & \text{if} \quad 0 \leq \tau < 1/2, \end{cases}
\end{aligned}$$

and the claim follows from assumptions (2.11). □

We finish this subsubsection by introducing some polynomial weights which will be used in Theorem 3.3.2 and Lemma 3.3.3.
Let $\tau \in [0,1]$ and $u \geq 0$, then we define the weight of polynomial type

$$m^{\tau}_{u}((x,\omega),(y,\eta)) := (1+|x-\tau\eta|+|\omega+(1-\tau)y|)^{u}, \tag{2.17}$$

where $(x,\omega),(y,\eta) \in \mathbb{R}^{2d}$.

**Remark 2.2.14.** *Let $\tau \in [0,1]$ and $u \geq 0$, then we notice that*

$$m^{\tau}_{u}((x,\omega),(y,\eta)) \lesssim v_u \otimes v_u((x,\omega),(y,\eta)), \quad \forall\,(x,\omega),(y,\eta) \in \mathbb{R}^{2d}.$$

*which will be used in Lemma 3.3.3. Indeed:*

$$\begin{aligned}
m^{\tau}_{u}((x,\omega),(y,\eta)) &= (1+|x-\tau\eta|+|\omega+(1-\tau)y|)^{u} \\
&\lesssim (1+(|x|+|\tau\eta|)^2+(|\omega|+|(1-\tau)y|)^2)^{u/2} \\
&\lesssim (1+|x|^2+\tau^2\,|\eta|^2+|\omega|^2+(1-\tau)^2\,|y|^2)^{u/2} \\
&\lesssim (1+|(x,\omega)|^2+|(y,\eta)|^2)^{u/2} \\
&\leq (1+|(x,\omega)|^2+|(y,\eta)|^2+|(x,\omega)|^2\,|(y,\eta)|^2)^{u/2} \\
&= (1+|(x,\omega)|^2)^{u/2}(1+|(y,\eta)|^2)^{u/2} \\
&= v_u \otimes v_u((x,\omega),(y,\eta)).
\end{aligned}$$

### 2.2.2 Solid QBF spaces on $G$

The definition of a solid quasi-Banach function space $Y$ on $G$ is given. Since It can be useful to describe Wiener Amalgam spaces $W_Q(Y)$, Definition 2.2.22, in terms of sequences, the so-called BUPUs and a particular space of sequences $Y_d$ associated to $Y$ are introduced. We present the space $Y_d$ under specific hypothesis fitting our framework, nevertheless a more general theory is possible, see [120] and [147, Chapter 2]. We recall the $L_x$ and $R_x$ are the left and right translations on $G$, respectively.

**Definition 2.2.15.** *We say that* $(Y, \|\cdot\|_Y)$ *is a* ***function space on*** $G$ *if it is a quasi-normed space consisting of equivalence classes of measurable* $\overline{\mathbb{C}}$*-valued functions on G, where two functions are identified if they coincide a.e..*

*A function space* $(Y, \|\cdot\|_Y)$ *on G is said to be* ***left invariant*** *if* $L_x \colon Y \to Y$ *is well defined and bounded for every* $x \in G$*, similarly we define the* ***right invariance****. We say that Y is* ***bi-invariant*** *if it is both left and right invariant.*

*A function space* $(Y, \|\cdot\|_Y)$ *on G is said* ***solid*** *if given* $g \in Y$ *and* $f \colon G \to \mathbb{C}$ *measurable the following holds true:*

$$|f| \le |g| \quad a.e. \qquad \Rightarrow \qquad f \in Y, \quad \|f\|_Y \le \|g\|_Y\,;$$

*Y is called* ***quasi-Banach function (QBF) space on*** $G$ *if it is complete.*

Without loss of generality, we can assume $\|\cdot\|_Y$ to be a $r$-norm, $0 < r \le 1$, i.e.

$$\|f+g\|_Y^r \le \|f\|_Y^r + \|g\|_Y^r\,, \quad \forall f, g \in Y.$$

This is due to the Aoki-Rolewicz Theorem 2.1.3 and the fact that equivalent quasi-norms induce the same topology, see Lemma 2.1.5 .

**Definition 2.2.16.** *A family* $X = \{x_i\}_{i\in I}$ *in G is called* ***relatively separated*** *if for all compact sets* $K \subseteq G$ *we have*

$$C_{X,K} \coloneqq \sup_{i\in I} \#\{j \in I \,|\, x_i K \cap x_j K \ne \varnothing\} < +\infty. \tag{2.18}$$

*Consider* $X = \{x_i\}_{i\in I}$ *relatively separated family in G,* $Q \subseteq G$ *measurable, relatively compact set of positive measure and* $(Y, \|\cdot\|_Y)$ *solid QBF space on G. Then the* ***discrete sequence space associated to*** $Y$ *is the set*

$$Y_d(X,Q) \coloneqq \left\{ (\lambda_i)_{i\in I} \in \mathbb{C}^I \,|\, \sum_{i\in I} |\lambda_i|\, \chi_{x_i Q} \in Y \right\} \tag{2.19}$$

*endowed with the quasi-norm*

$$\left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)} \coloneqq \left\| \sum_{i\in I} |\lambda_i|\, \chi_{x_i Q} \right\|_Y\,; \tag{2.20}$$

$\mathbb{C}^I$ *is the space of functions from I into* $\mathbb{C}$.

**Lemma 2.2.17.** ([147, Lemma 2.3.10]) *If G is* $\sigma$*-compact, then any relatively separated family X is (at most) countable.*

For the following issue we address the reader to [120, Lemma 2.2] and [147, Lemma 2.3.16].

**Lemma 2.2.18.** *$(Y_d(X,Q), \|\cdot\|_{Y_d(XQ)})$ is a quasi-Banach space. Moreover, if $Y$ is right invariant then $Y_d(X,Q)$ is independent of $Q$ in the sense that another $U \subseteq G$ measurable, relatively compact and with non empty interior yields the same space with an equivalent quasi-norm*

We recall the following result by F. Voigtlaender [147].

**Lemma 2.2.19.** ([147, Lemma 2.3.21]) *Let $v$ be a submultiplicative weight on $G$ and $m$ a weight on $G$ right-moderate w.r.t. $v$. Then for every $0 < p \leq \infty$, for each $Q \subseteq G$ measurable relatively compact set with positive measure and for every relatively separated family $X = \{x_i\}_{i\in I}$ in $G$ we have*

$$(L^p_m(G))_d(X,Q) = \ell^p_{m_X}(I),$$

*where $m_X \colon I \to (0,+\infty), i \mapsto m(x_i)$.*

A generalization of this result was given in [8], see Lemma 5.1.26.

**Definition 2.2.20.** *Let $U \subseteq G$ be a relatively compact, unit neighbourhood. A family $\Psi = \{\psi_i\}_{i\in I}$ of continuous functions on $G$ is called a* ***bounded uniform partition of unity of size** $U$ **($U$-BUPU)*** *if*

- *(i)* $0 \leq \psi_i(x) \leq 1$ *for all $x \in G$ and every $i \in I$;*
- *(ii) there exists $X = \{x_i\}_{i\in I}$* ***$U$-localizing family for*** *$\Psi$, i.e., $X$ is a relatively separated family in $G$ such that*
$$\operatorname{supp}\psi_i \subseteq x_i U \qquad \forall i \in I;$$
- *(iii)* $\sum_{i\in I} \psi_i \equiv 1$.

**Lemma 2.2.21.** *Given any relatively compact unit neighbourhood $U$ in $G$, there always exists a family $\Psi$ which is a $U$-BUPU with some $U$-localizing family $X$. Moreover, since $G$ is $\sigma$-compact, the indexes' set is (at most) countable.*

For the above lemma we refer to [52, Theorem 2] and [147, Lemma 2.3.212].

### 2.2.3 Wiener Amalgam spaces with global component $L^\infty(G)$

We introduce the Wiener Amalgam spaces not in their full generality, but restrict ourselves to cases which ensure "good" properties.

**Definition 2.2.22.** *Consider $Q \subseteq G$ measurable, relatively compact, unit neighbourhood and $f \colon G \to \mathbb{C}$ measurable. We call* ***maximal function of** $f$ **with respect to** $Q$* *the following application*

$$\mathsf{M}_Q f \colon G \to [0,+\infty], \quad x \mapsto \operatorname*{ess\,sup}_{y\in xQ} |f(y)|. \tag{2.21}$$

*We fix a solid QBF space $(Y, \|\cdot\|_Y)$ on $G$ and define the* ***Wiener Amalgam space with window** $Q$**, local component** $L^\infty = L^\infty(G)$ **and global component** $Y$* *as*

$$W_Q(Y) := W_Q(L^\infty, Y) := \{f \in L^\infty_{loc}(G) \mid \mathsf{M}_Q f \in Y\} \tag{2.22}$$

*and endow it with*

$$\|f\|_{W_Q(Y)} := \|f\|_{W_Q(L^\infty,Y)} := \|\mathsf{M}_Q f\|_Y. \tag{2.23}$$

It was proven in [147, Lemma 2.3.4] that the maximal function $\mathsf{M}_Q f$ is measurable.

**Lemma 2.2.23.** *The Wiener Amalgam space $(W_Q(Y), \|\cdot\|_{W_Q(Y)})$ is a solid QBF space on $G$, in particular, $\|\cdot\|_{W_Q(Y)}$ is a $r$-norm, $0 < r \leq 1$, if $\|\cdot\|_Y$ is.*

For each $f \in L^\infty_{loc}(G)$ we have

$$|f(x)| \leq \mathsf{M}_Q f(x) \quad \text{a.e.,} \tag{2.24}$$

which together with the solidity of $Y$ gives the continuous embedding

$$W_Q(L^\infty, Y) \hookrightarrow Y. \tag{2.25}$$

In general the definition of $W_Q(Y)$ may depend on the chosen subset $Q$. However, we shall require some further properties in order to make the Wiener space independent of it. We collect some of the results of [147, Lemma 2.3.16, Theorem 2.3.17] in the following lemma (which holds under milder assumptions).

**Lemma 2.2.24.** *Under the hypothesis presented so far, if the solid QBF space $Y$ on $G$ is right invariant, then the following equivalent facts hold true:*

- $(i)$ *The Wiener Amalgam space $W_Q(L^\infty, Y)$ is right invariant for each measurable, relatively compact, unit neighbourhood $Q \subseteq G$;*
- $(ii)$ *The Wiener Amalgam space $W_Q(L^\infty, Y)$ is independent of the choice of the measurable, relatively compact, unit neighbourhood $Q \subseteq G$, in the sense that different choices yield the same set with equivalent quasi-norms. The equivalence constants depend only on the two sets $Q, Q' \subseteq G$ and on $Y$.*

*If these conditions are fulfilled, $\Psi = \{\psi_i\}_{i\in I}$ is a $U$-BUPU for some localizing family $X = \{x_i\}_{i\in I}$ and $U \subseteq G$ relatively compact unit neighbourhood, then*

$$\|f\|_{W_Q(L^\infty,Y)} \underset{X,Q,Y}{\asymp} \left\|(\|\psi_i \cdot f\|_{L^\infty})_{i\in I}\right\|_{Y_d(X,Q)} \tag{2.26}$$

*for every $f \in W_Q(L^\infty, Y)$ and the constants involved in the above equivalence depend only on $X$, $Q$ and $Y$.*

We remark that the right invariance of $Y$ is sufficient for conditions $(i)$ or $(ii)$ but not necessary; the existence of an $U$-BUPU $\Psi$ is always guaranteed. When one of the above conditions is satisfied, we suppress the index $Q$ in the Wiener space and simply write $W(L^\infty, Y)$ or $W(Y)$.

By considering $Qx$ instead of $xQ$ in the definition of the maximal function, we obtain the "right-sided" version of the Wiener spaces. So that we set the ***right-sided maximal function*** to be

$$\mathsf{M}^R_Q f \colon G \to [0, +\infty], \quad x \mapsto \operatorname*{ess\,sup}_{y\in Qx} |f(y)| \tag{2.27}$$

and define the ***right-sided Wiener Amalgam space*** $W^R_Q(Y)$ similarly as before. Analogous considerations hold for $W^R_Q(Y)$, with the proper cautions about Lemma 2.2.24. In particular, the independence of $W^R_Q(Y)$ from $Q$ is guaranteed if $Y$ is left invariant, see [147, Lemma 2.3.29].

### 2.2.4 An example of solid QBF spaces: $L^{p,q}_m$ spaces

In showing the well-known examples of weighted mixed-norm Lebesgue spaces $L^{p,q}_m$, $0 < p, q \leq \infty$, we prove a number of results which are usually taken al folklore. See in particular: Lemma 2.2.26 for the quasi-norm on $L^{p,q}_m$ being a $\min\{1, p, q\}$-norm, Proposition 2.2.27 for Young's inequality for $L^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})$ and subsequent consequences. The mentioned results are taken from [8].

**Example 2.2.25.** *Let $(X, \mathscr{A}, \mu)$ be a measure space. For $0 < p \leq \infty$, the Lebesgue space $L^p(X) \coloneqq L^p(X, \mathscr{A}, \mu)$ is the collection of equivalence classes of measurable functions $f \colon X \to \overline{\mathbb{C}}$, where two functions coincide if they are equal almost everywhere (a.e.) w.r.t. $\mu$, such that*

$$\|f\|_{L^p} \coloneqq \left( \int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} < +\infty \quad \text{if} \quad 0 < p < \infty,$$
$$\|f\|_{L^\infty} \coloneqq \operatorname*{ess\,sup}_{x \in X} |f(x)| < +\infty \quad \text{if} \quad p = \infty.$$

*For $p \geq 1$ the above application is a norm, for $0 < p < 1$ it is easy to verify that it is a p-norm. Hence $\|\cdot\|_{L^p}$, $0 < p \leq \infty$, is a p-norm and $L^p(X)$, endowed with such a quasi-norm, is known to be a quasi-Banach space. Moreover, if $X$ is a topological group with Haar measure $\mu$, then $(L^p(X), \|\cdot\|_{L^p})$ is a solid QBF space.*
*If $(Y, \mathscr{B}, \nu)$ is another measure space and $0 < q \leq \infty$, then the Lebesgue mixed-norm space $L^{p,q}(X \times Y)$ consists of equivalence classes of measurable equal a.e. functions $f \colon X \times Y \to \overline{\mathbb{C}}$, such that*

$$\|f\|_{L^{p,q}} \coloneqq \left( \int_Y \left( \int_X |f(x,y)|^p \, d\mu(x) \right)^{\frac{q}{p}} d\nu(y) \right)^{\frac{1}{q}} < +\infty \quad \text{if} \quad 0 < p, q < \infty,$$

*analogous definitions when at least one between p and q is $\infty$. $\|\cdot\|_{L^{p,q}}$ is a quasi-norm and $L^{p,q}(X \times Y)$ is complete w.r.t. it. Similarly to the single index case, if $X \times Y$ is a product topological group with Haar measure the product measure $\mu \times \nu$, then $(L^{p,q}(X \times Y), \|\cdot\|_{L^{p,q}_m})$ is a solid QBF space.*

The fact of $\|\cdot\|_{L^{p,q}}$, presented in the above example, being a $\min\{1, p, q\}$-norm seems to be folklore, for no available proof is known to the author. For this reason, we present a proof which was published in [8, Lemma 3.5]. For sake of generality we introduce the *weighted version* of Lebesgue mixed-norm spaces. Let $m \colon X \times Y \to (0, +\infty)$ be a measurable function. Then

$$L^{p,q}_m(X \times Y) \coloneqq \left\{ f \colon X \times Y \to \overline{\mathbb{C}} \text{ measurable} \,|\, f \cdot m \in L^{p,q}(X \times Y) \right\} / \sim,$$

where $\sim$ denotes the equivalence relation where $f \sim g$ if and only if $f = g$ a.e. w.r.t. $\mu \times \nu$. The quasi-norm is the natural one:

$$\|f\|_{L^{p,q}_m} \coloneqq \|f \cdot m\|_{L^{p,q}}.$$

Of course $(L^{p,q}_m, \|\cdot\|_{L^{p,q}_m})$ is a quasi-Banach space. If $X \times Y$ is a product of topological groups, then it is a solid QBF space.

**Lemma 2.2.26.** ([8, Lemma 3.5]) *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be measure spaces, consider $0 < p, q \leq \infty$ and $m \colon X \times Y \to (0, +\infty)$ measurable. Then $\|\cdot\|_{L^{p,q}_m}$ is a $\min\{1, p, q\}$-norm.*

*Proof.* We tackle the unweighted case, the weighted one follows immediately. We recall that for $0 < p \leq \infty$ the application $\|\cdot\|_{L^p}$, see Example 2.2.25, is an $\min\{1, p\}$-norm. Therefore it is a

$\min\{1,p,q\}$-norm also. Let us consider $f,g \in L^{p,q}(X\times Y)$ and define $r := \min\{1,p,q\}$, using the fact that $\|\cdot\|_{L^p(X)}$ is an $r$-norm and $q/r \geq 1$:

$$\begin{aligned}
\|f+g\|_{L^{p,q}}^r &= \left(\int_Y \left(\int_X |f(x,y)+g(x,y)|^p \, d\mu(x)\right)^{\frac{q}{p}} d\nu(y)\right)^{\frac{r}{q}} \\
&= \left(\int_Y \left(\int_X |f(x,y)+g(x,y)|^p \, d\mu(x)\right)^{\frac{r}{p}\frac{q}{r}} d\nu(y)\right)^{\frac{r}{q}} \\
&\leq \left(\int_Y \left(\left(\int_X |f(x,y)|^p \, d\mu(x)\right)^{\frac{r}{p}} + \left(\int_X |g(x,y)|^p \, d\mu(x)\right)^{\frac{r}{p}}\right)^{\frac{q}{r}} d\nu(y)\right)^{\frac{r}{q}} \\
&\leq \left(\int_Y \left(\left(\int_X |f(x,y)|^p \, d\mu(x)\right)^{\frac{r}{p}}\right)^{\frac{q}{r}} d\nu(y)\right)^{\frac{r}{q}} \\
&+ \left(\int_Y \left(\left(\int_X |g(x,y)|^p \, d\mu(x)\right)^{\frac{r}{p}}\right)^{\frac{q}{r}} d\nu(y)\right)^{\frac{r}{q}} \\
&= \|f\|_{L^{p,q}}^r + \|g\|_{L^{p,q}}^r .
\end{aligned}$$

The proof is concluded. □

Also Young's inequality for $L^{p,q}(\mathcal{G}\times\widehat{\mathcal{G}})$, $\mathcal{G}$ LCA group, seems to be folklore. For this reason a proof was provided by E. Cordero and the author in [8].

**Proposition 2.2.27.** ([8, Proposition 4.1]) *Consider* $1 \leq p_i, q_i, r_i \leq \infty$, $i=1,2$, *such that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i = 1,2. \tag{2.28}$$

*If* $F \in L^{p_1,p_2}(\mathcal{G}\times\widehat{\mathcal{G}})$ *and* $H \in L^{q_1,q_2}(\mathcal{G}\times\widehat{\mathcal{G}})$, *then* $F * H \in L^{r_1,r_2}(\mathcal{G}\times\widehat{\mathcal{G}})$ *with*

$$\|F*H\|_{L^{r_1,r_2}} \leq \|F\|_{L^{p_1,p_2}} \|H\|_{L^{q_1,q_2}}. \tag{2.29}$$

*Proof.* We follow the pattern of [12, Part II, Theorem 1, b)]. It suffices to prove the claim for $F,H \geq 0$. Given a measurable function $W\colon \mathcal{G}\times\widehat{\mathcal{G}} \to \mathbb{C}$ and $1 \leq s \leq \infty$, we define the (measurable) function on $\widehat{\mathcal{G}}$

$$\|W\|_{(s)}(\xi) := \begin{cases} \left(\int_{\mathcal{G}} |W(x,\xi)|^s \, dx\right)^{\frac{1}{s}} & \text{if} \quad s<\infty, \\ \operatorname{ess\,sup}_{x\in\mathcal{G}} |W(x,\xi)| & \text{if} \quad s=\infty. \end{cases} \tag{2.30}$$

We show the case $r_1 < \infty$, the case $r_1 = \infty$ is done similarly. In the following we shall use

Minkowski's integral inequality (see [130, Appendix A.1]):

$$\begin{aligned}
\|F * H\|_{(r_1)}(\xi) &= \left(\int_{\mathcal{G}} \left[\int_{\mathcal{G}\times\widehat{\mathcal{G}}} F((x,\xi)-(u,\omega))H(u,\omega)\,du d\omega\right]^{r_1} dx\right)^{\frac{1}{r_1}}\\
&= \left(\int_{\mathcal{G}} \left[\int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} F((x,\xi)-(u,\omega))H(u,\omega)\,du\right) d\omega\right]^{r_1} dx\right)^{\frac{1}{r_1}}\\
&\le \int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} \left[\int_{\mathcal{G}} F((x,\xi)-(u,\omega))H(u,\omega)\,du\right]^{r_1} dx\right)^{\frac{1}{r_1}} d\omega\\
&= \int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} [[F(\cdot,\xi-\omega) * H(\cdot,\omega)](x)]^{r_1}\, dx\right)^{\frac{1}{r_1}} d\omega\\
&= \int_{\widehat{\mathcal{G}}} \|F(\cdot,\xi-\omega) * H(\cdot,\omega)\|_{L^{r_1}(\mathcal{G})}\, d\omega =: I.
\end{aligned}$$

Using Young's inequality (see [96, Theorem 20.18]) with indexes $p_1, q_1, r_1$ as in (2.28) we majorize as

$$\begin{aligned}
I &\le \int_{\widehat{\mathcal{G}}} \|F(\cdot,\xi-\omega)\|_{L^{p_1}(\mathcal{G})} \|H(\cdot,\omega)\|_{L^{q_1}(\mathcal{G})}\, d\omega\\
&= \int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} F(x,\xi-\omega)^{p_1}\, dx\right)^{\frac{1}{p_1}} \left(\int_{\mathcal{G}} H(x,\omega)^{q_1}\, dx\right)^{\frac{1}{q_1}} d\omega\\
&= \int_{\widehat{\mathcal{G}}} \|F\|_{(p_1)}(\xi-\omega)\, \|H\|_{(q_1)}(\omega)\, d\omega\\
&= \left(\|F\|_{(p_1)} * \|H\|_{(q_1)}\right)(\xi).
\end{aligned}$$

Using Young's inequality with indices $p_2, q_2, r_2$ in (2.28) we obtain the desire result. Namely,

$$\begin{aligned}
\|F * H\|_{L^{r_1,r_2}(\mathcal{G}\times\widehat{\mathcal{G}})} &= \left(\int_{\widehat{\mathcal{G}}} \left[\|F * H\|_{(r_1)}(\xi)\right]^{r_2} d\xi\right)^{\frac{1}{r_2}}\\
&\le \left(\int_{\widehat{\mathcal{G}}} \left[\left(\|F\|_{(p_1)} * \|H\|_{(q_1)}\right)(\xi)\right]^{r_2} d\xi\right)^{\frac{1}{r_2}}\\
&= \left\|\|F\|_{(p_1)} * \|H\|_{(q_1)}\right\|_{L^{r_2}(\widehat{\mathcal{G}})}\\
&\le \left\|\|F\|_{(p_1)}\right\|_{L^{p_2}(\widehat{\mathcal{G}})} \left\|\|H\|_{(q_1)}\right\|_{L^{q_2}(\widehat{\mathcal{G}})}\\
&= \|F\|_{L^{p_1,p_2}(\mathcal{G}\times\widehat{\mathcal{G}})} \|H\|_{L^{q_1,q_2}(\mathcal{G}\times\widehat{\mathcal{G}})}.
\end{aligned}$$

This concludes the proof. □

A straightforward consequence is the weighted Young's inequality below.

**Corollary 2.2.28.** ([8, Corollary 4.2]) *Consider* $1 \le p_i, q_i, r_i \le \infty$, $i = 1, 2$, *such that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i = 1, 2. \tag{2.31}$$

*Consider* $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$. *If* $F \in L^{p_1,p_2}_m(\mathcal{G}\times\widehat{\mathcal{G}})$ *and* $H \in L^{q_1,q_2}_v(\mathcal{G}\times\widehat{\mathcal{G}})$, *then* $F*H \in L^{r_1,r_2}_m(\mathcal{G}\times\widehat{\mathcal{G}})$ *with*

$$\|F * H\|_{L^{r_1,r_2}_m} \le \|F\|_{L^{p_1,p_2}_m} \|H\|_{L^{q_1,q_2}_v}. \tag{2.32}$$

Note that Proposition 2.2.27 can be easily generalized to $N$ indices, $N \geq 2$, as in [12, Part II, Theorem 1, b)]:

**Proposition 2.2.29.** ([8, Proposition 4.3]) *Consider $N \in \mathbb{N}$ and let $\mathcal{G}_i$ be a LCA, $\sigma$-finite group with Haar measure $dx_i$, $i = 1, \dots, N$. Consider $1 \leq p_i, q_i, r_i \leq \infty$, $i = 1, \dots, N$, such that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i = 1, \dots, N.$$

*If $F \in L^{p_1,\dots,p_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)$ and $H \in L^{q_1,\dots,q_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)$, then $F * H \in L^{r_1,\dots,r_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)$ with*

$$\|F * H\|_{L^{r_1,\dots,r_N}(\mathcal{G}_1\times\cdots\times\mathcal{G}_N)} \leq \|F\|_{L^{p_1,\dots,p_N}(\mathcal{G}_1\times\cdots\times\mathcal{G}_N)} \|H\|_{L^{q_1,\dots,q_N}(\mathcal{G}_1\times\cdots\times\mathcal{G}_N)},$$

*where the product LCA $\sigma$-finite group $\mathcal{G}_1 \times \cdots \times \mathcal{G}_N$ is endowed with the product Haar measure $dx_1 \dots dx_N$.*

#### 2.2.4.1 Discrete weighted mixed-norm spaces $\ell^{p,q}_m(\mathbb{Z}^{2d})$

By taking $X = \mathbb{Z}^d = Y$ endowed with the counting measure $\mu^{\#}$, we recover the well-know weighted spaces of sequences $\ell^p_m(\mathbb{Z}^d)$ and $\ell^{p,q}_m(\mathbb{Z}^{2d})$. We recall here some properties which will be needed in next chapters.

**Definition 2.2.30.** *Consider two sequences of complex numbers $a = (a_k)_{k\in\mathbb{Z}^d}, b = (b_k)_{k\in\mathbb{Z}^d} \subseteq \mathbb{C}$. We define the* ***convolution sequence*** *of $a$ and $b$ as*

$$a * b := \left(\sum_{n\in\mathbb{Z}^d} a_{k-n} b_n\right)_{k\in\mathbb{Z}^d} \subseteq \overline{\mathbb{C}}. \tag{2.33}$$

*We define the* ***point-wise product sequence*** *of $a$ and $b$ as $a \cdot b := (a_k b_k)_{k\in\mathbb{Z}^d}$.*

We present some properties we need in the sequel, we refer to [74, 75] for proofs.

**Theorem 2.2.31.** (Inclusion relations)
*Let $m$ be any weight on $\mathbb{Z}^d$. Consider $0 < p_1 \leq p_2 \leq \infty$. Then we have the following continuous inclusion:*

$$\ell^{p_1}_m(\mathbb{Z}^d) \hookrightarrow \ell^{p_2}_m(\mathbb{Z}^d), \tag{2.34}$$

*i.e. there exists $C > 0$ such that for all $a \in \ell^{p_1}_m(\mathbb{Z}^d)$*

$$\|a\|_{\ell^{p_2}_m} \leq C \|a\|_{\ell^{p_1}_m}.$$

**Theorem 2.2.32.** (Young's convolution inequality)
*Consider $m \in \mathcal{M}_v(\mathbb{Z}^d)$. Take $0 < p, q, r \leq \infty$ such that:*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \quad \text{for} \quad 1 \leq r \leq \infty, \tag{2.35}$$

$$p = q = r \quad \text{for} \quad 0 < r < 1. \tag{2.36}$$

*Then we have the following continuous inclusion:*

$$\ell^p_m(\mathbb{Z}^d) * \ell^q_v(\mathbb{Z}^d) \hookrightarrow \ell^r_m(\mathbb{Z}^d), \tag{2.37}$$

*i.e. there exists $C > 0$ such that for every $a \in \ell^p_m(\mathbb{Z}^d)$ and $b \in \ell^q_v(\mathbb{Z}^d)$*

$$\|a * b\|_{\ell^r_m} \leq C \|a\|_{\ell^p_m} \|b\|_{\ell^q_v}$$

*and $C$ in independent of $p$, $q$, $r$, $a$, $b$. Moreover if $m \equiv 1 \equiv v$, then $C = 1$.*

**Theorem 2.2.33.** (Hölder's inequality)
*Let $\nu$ be any weight on $\mathbb{Z}^d$. Take $0 < p, q, r \leq \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{2.38}$$

*Then we have the following continuous inclusion:*

$$\ell^p_\nu(\mathbb{Z}^d) \cdot \ell^q_{1/\nu}(\mathbb{Z}^d) \hookrightarrow \ell^r(\mathbb{Z}^d), \tag{2.39}$$

*i.e. there exists $C > 0$ such that for every $a \in \ell^p_\nu(\mathbb{Z}^d)$ and $b \in \ell^q_{1/\nu}(\mathbb{Z}^d)$*

$$\|a \cdot b\|_{\ell^r} \leq C \|a\|_{\ell^p_\nu} \|b\|_{\ell^q_{1/\nu}}.$$

**Lemma 2.2.34.** *If $0 < p_1, p_2 \leq \infty$, with*

$$0 \leq s_2 \leq s_1, \quad \frac{1}{p_2} + \frac{s_2}{d} < \frac{1}{p_1} + \frac{s_1}{d},$$

*then*

$$\ell^{p_2}_{\langle\cdot\rangle^{s_2}}(\mathbb{Z}^d) \hookrightarrow \ell^{p_1}_{\langle\cdot\rangle^{s_1}}(\mathbb{Z}^d). \tag{2.40}$$

### 2.2.5 Coorbit theory for solid QBF spaces on $G$

We are now able to state the coorbit theory in [147, Assumption 2.4.1] in the following items **A**–**G** and **H** –**J**. Although the structure could appear quite cumbersome, lot of elements shall simplify due to our subsequent specific choices in Chapter 5.

**A**. We assume $G$ to be a LCH, $\sigma$-compact group. We consider $\rho\colon G \to \mathcal{U}(\mathcal{H})$ a strongly continuous, unitary, irreducible representation of $G$ for some nontrivial complex Hilbert space $\mathcal{H}$. $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$ (see e.g. [69, 152] ).

**B**. Given $f, g \in \mathcal{H}$, we define the ***(generalized) wavelet transform induced by*** $\rho$, or voice transform, ***of*** $f$ ***w.r.t.*** $g$ as

$$W^\rho_g f\colon G \to \mathbb{C}, \quad x \mapsto \langle f, \rho(x)g\rangle_{\mathcal{H}}, \tag{2.41}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, also denoted by $\langle\cdot,\cdot\rangle$, is the inner product on $\mathcal{H}$ supposed antilinear in the second component. $W^\rho_g f$ is always a continuous and bounded function on $G$, see [152]. We assume the ***representation*** $\rho$ to be ***integrable***, i.e. there exists $g \in \mathcal{H} \smallsetminus \{0\}$ such that $W^\rho_g g \in L^1(G)$; this implies that $\rho$ is also ***square-integrable***: there exists $g \in \mathcal{H} \smallsetminus \{0\}$ such that $W^\rho_g g \in L^2(G)$. Such a $g$ is said to be ***admissible***.

**C**. $(Y, \|\cdot\|_Y)$ will be supposed to be a solid QBF space on $G$ with $\|\cdot\|_Y$, or some equivalent quasi-norm, $r$-norm with $0 < r \leq 1$.

**D**. The Wiener Amalgam space $W_Q(L^\infty, Y)$ is assumed right invariant for each measurable, relatively compact, unit neighbourhood $Q \subseteq G$. We consider a submultiplicative weight $w\colon G \to (0, +\infty)$ such that for some (and hence each) measurable, relatively compact, unit neighbourhood $Q \subseteq G$

$$w(x) \underset{Q}{\gtrsim} |||R_x|||_{W_Q(Y) \to W_Q(Y)} \tag{2.42}$$

and

$$w(x) \underset{Q}{\gtrsim} \Delta(x^{-1})\, |||R_{x^{-1}}|||_{W_Q(Y)\to W_Q(Y)}\,, \tag{2.43}$$

where $\Delta(x)$ is the modular function on $G$. We also require the weight $w$ to be bounded from below, i.e. there exists $c > 0$ such that $w(x) \geq c$ for every $x \in G$.

If the condition on $W_Q(Y)$ in **D** is satisfied, then the Wiener space is independent of $Q$, so that we can omit the lower index. Moreover, this is ensured if $Y$ is right invariant (Lemma 2.2.24).

**E**. We fix a submultiplicative weight $v\colon G \to (0, +\infty)$, which will be called ***control weight for*** $Y$, such that

$$v \geq w, \qquad v \geq w_{\vee,r}, \tag{2.44}$$

where $w$ is defined in **D** and

$$w_{\vee,r}(x) = w(x^{-1})\, \left[\Delta(x^{-1})\right]^{1/r}. \tag{2.45}$$

**F**. The ***class of good vectors*** is defined to be

$$\mathbb{G}_v \coloneqq \left\{g \in \mathcal{H} \,|\, W_g^\rho g \in L_v^1(G)\right\} \tag{2.46}$$

and supposed nontrivial, $\{0\} \subsetneq \mathbb{G}_v$.

**G**. The ***class of analyzing vectors*** is defined as

$$\mathbb{A}_v^r \coloneqq \left\{g \in \mathcal{H} \,|\, W_g^\rho g \in W^R(L^\infty, W(L^\infty, L_v^r))\right\} \tag{2.47}$$

and supposed nontrivial, $\{0\} \subsetneq \mathbb{A}_v^r$.

**Remark 2.2.35.** (*i*) *Observe that, since $v$ is submultiplicative, $L_v^r(G)$ is bi-invariant. This implies that $W(L_v^r)$ is independent of the window $Q$ and it is left invariant, hence also $W^R(W(L_v^r))$ is independent of the window subset. Concretely, this allows us to work with the same $Q$:*

$$\left\|W_g^\rho g\right\|_{W^R(W(L_v^r))} \asymp \left\|\mathsf{M}_Q \mathsf{M}_Q^R W_g^\rho g\right\|_{L_v^r}, \tag{2.48}$$

*(see e.g. Lemma 5.1.10);*

(*ii*) *From the continuous embeddings for $0 < r \leq 1$*

$$W^R(L^\infty, W(L^\infty, L_v^r)) \hookrightarrow W(L^\infty, L_v^r) \hookrightarrow W(L^\infty, L_v^1) \hookrightarrow L_v^1, \tag{2.49}$$

*see [147, p. 113], follows the inclusion $\mathbb{A}_v^r \subseteq \mathbb{G}_v$.*

**H**. For a fixed $g \in \mathbb{G}_v \setminus \{0\}$, the ***space of test vectors*** is the set

$$\mathcal{T}_v \coloneqq \left\{f \in \mathcal{H} \,|\, W_g^\rho f \in L_v^1(G)\right\} \tag{2.50}$$

endowed with the norm

$$\|f\|_{\mathcal{T}_v} \coloneqq \left\|W_g^\rho f\right\|_{L_v^1}. \tag{2.51}$$

$(\mathcal{T}_v, \|\cdot\|_{\mathcal{T}_v})$ is a $\rho$-invariant Banach space which embeds continuously and with density into $\mathcal{H}$ and it is independent from the choice of the window vector $g \in \mathbb{G}_v \setminus \{0\}$, see [147, Lemma 2.4.7]. Recall that often the notation $\mathcal{H}_v^1$ is used in place of $\mathcal{T}_v$, see e.g. [58, 59, 60, 119].

**I**. We call ***reservoir*** the Banach space

$$\mathcal{R}_v \coloneqq \mathcal{T}_v^{\urcorner} \coloneqq \left\{ f\colon \mathcal{H}_v^1 \to \mathbb{C} \,|\, \text{antilinear and continuous} \right\}. \tag{2.52}$$

**J**. We can extend the wavelet transform to $f \in \mathcal{R}_v$ and $g \in \mathcal{T}_v$:

$$W_g^\rho f\colon G \to \mathbb{C}, \quad x \mapsto {}_{\mathcal{R}_v}\langle f, \rho(x)g\rangle_{\mathcal{T}_v}, \tag{2.53}$$

where ${}_{\mathcal{R}_v}\langle\cdot,\cdot\rangle_{\mathcal{T}_v}$ is the duality between $\mathcal{R}_v$ and $\mathcal{T}_v$ that will be denoted simply by $\langle\cdot,\cdot\rangle$. We have that $W_g^\rho f \in C(G) \cap L^\infty_{1/v}(G)$.

**K**. For a fixed vector window $g \in \mathbb{A}_v^r \setminus \{0\}$, the ***coorbit space on*** $G$ ***with respect to*** $Y$ is defined as

$$\mathsf{Co}(Y) \coloneqq \left\{ f \in \mathcal{R}_v \,|\, W_g^\rho f \in W(L^\infty, Y) \right\} \tag{2.54}$$

endowed with the quasi-norm

$$\|f\|_{\mathsf{Co}(Y)} \coloneqq \left\| W_g^\rho f \right\|_{W(L^\infty, Y)}. \tag{2.55}$$

**Theorem 2.2.36.** ([147, Theorem 2.4.9]) *Let* $\mathsf{Co}(Y)$ *be a coorbit space constructed accordingly to items* **A**–**J** *above. Then* $\mathsf{Co}(Y)$ *is independent of* $g \in \mathbb{A}_v^r \setminus \{0\}$, *in the sense that different windows yield equivalent quasi-norms. Moreover,* $(\mathsf{Co}(Y), \|\cdot\|_{\mathsf{Co}(Y)})$ *is a quasi-Banach space continuously embedded into* $\mathcal{R}_v$ *and* $\|\cdot\|_{\mathsf{Co}(Y)}$ *is a r-norm,* $0 < r \leq 1$, *if* $\|\cdot\|_Y$ *is.*

In the following theorem we collect [147, Theorem 2.4.19, Remark 2.4.20].

**Theorem 2.2.37.** *For every* $g \in \mathbb{A}_v^r \setminus \{0\}$ *there exists* $U_0 \subseteq G$ *relatively compact unit neighbourhood such that for each* $U_0$*-BUPU* $\Psi = \{\psi_i\}_{i\in I}$ *with localizing family* $X = \{x_i\}_{i\in I}$ *the following hold true:*

*(i) for each* $i \in I$ *there exists a continuous linear functional*

$$\lambda_i\colon \mathcal{R}_v \to \mathbb{C}$$

*such that* $(\lambda_i(f))_{i\in I} \in Y_d(X)$ *for every* $f \in \mathcal{R}_v$ *and*

$$f = \sum_{i\in I} \lambda_i(f)\rho(x_i)g,, \qquad \forall\, f \in \mathsf{Co}(Y), \tag{2.56}$$

*where the sum converges unconditionally in the w-∗-topology of* $\mathcal{R}_v$. *If the finite sequences are dense in* $Y_d(X)$, *then the series converges unconditionally in* $\mathsf{Co}(Y)$;

*(ii) for all* $\lambda = (\lambda_i)_{i\in I} \in Y_d(X)$ *the series*

$$\mathsf{S}_g^X(\lambda) \coloneqq \sum_{i\in I} \lambda_i \rho(x_i) g \tag{2.57}$$

*is an element of* $\mathsf{Co}(Y)$. *The above sum converges unconditionally in the w-∗-topology of* $\mathcal{R}_v$ *(pointwise). If the finite sequences are dense in* $Y_d(X)$, *then the series converges unconditionally in* $\mathsf{Co}(Y)$ *and there exists* $C > 0$ *such that*

$$\left\| \mathsf{S}_g^X(\lambda) \right\|_{\mathsf{Co}(Y)} \leq C \left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X)}, \qquad \forall\, \lambda \in Y_d(X); \tag{2.58}$$

(*iii*) *for* $f \in \mathcal{R}_v$ *we have*

$$f \in \mathsf{Co}(Y) \Leftrightarrow (\lambda_i(f))_{i\in I} \in Y_d(X) \tag{2.59}$$

*and for every* $f \in \mathsf{Co}(Y)$

$$\|f\|_{\mathsf{Co}(Y)} \asymp \left\|(\lambda_i(f))_{i\in I}\right\|_{Y_d(X)}. \tag{2.60}$$

**Remark 2.2.38.** *Let us remark the main differences with the coorbit theory in Banach setting developed by H. G. Feichtinger and K. Gröchenig [58]:*

(*i*) *in [58] a solid Banach function space* $Y$ *on* $G$ *is considered and supposed continuously embedded in* $L^1_{loc}(G)$*. In particular, we observe how the condition* $Y \hookrightarrow L^1_{loc}(G)$ *is restrictive, in fact even if one would allow* $Y$ *to be quasi-Banach, all the spaces* $L^p(\mathbb{R}^d)$ *with* $0 < p < 1$ *would be excluded;*

(*ii*) *the window space considered in [58] is larger than the one presented so far, namely it is sufficient a non-zero* $g \in \mathcal{A}_v := \mathbb{G}_v$ *and hence the coorbit space is defined as*

$$\mathsf{Co}_{\mathrm{FG}}(Y) := \left\{f \in \mathcal{R}_v \,|\, W^\rho_g f \in Y\right\}, \tag{2.61}$$

*with obvious norm. Hence* $\mathsf{Co}_{\mathrm{FG}}(Y)$ *is a Banach space independent of the chosen window* $g \in \mathcal{A}_v \smallsetminus \{0\}$*.*

It is a natural question whether the two constructions coincide. In the Banach case the answer is positive, see [60, Theorem 8.3] and [119, Theorem 6.1].

**Theorem 2.2.39.** *Consider a solid Banach function space* $Y$ *such that it is bi-invariant and continuously embedded in* $L^1_{loc}(G)$*. Then*

$$\mathsf{Co}_{\mathrm{FG}}(Y) = \mathsf{Co}(Y)$$

*with equivalent norms.*

## 2.3 Time-frequency analysis tools

For a systematic and detailed treatment of time-frequency analysis, and proofs of what follows in the section, we address the reader to [35, 82].

### 2.3.1 Fundamental operators and time-frequency distributions

We present the fundamental operators of time-frequency analysis, e.g. translation and modulation ones. We then study their relations with the Fourier transform and define the short-time Fourier transform (STFT), one of the most common used time-frequency representations, and its main continuity properties. We also recall another time-frequency representation, the cross-$\tau$-Wigner distribution.

**Definition 2.3.1.** *Fix* $x, \omega \in \mathbb{R}^d$*. Let* $f$ *be a function defined on* $\mathbb{R}^d$*.*
*The* ***translation operator*** $T_x$ *is defined as:*

$$T_x f(t) := f(t-x). \tag{2.62}$$

*The* ***modulation operator*** $M_\omega$ *is defined as:*

$$M_\omega f(t) \coloneqq e^{2\pi i\omega t} f(t). \tag{2.63}$$

*The following composition of the previous two operators is called* ***time-frequency shift (TFS) operator****:*

$$\pi(x,\omega) \coloneqq M_\omega T_x. \tag{2.64}$$

Usually we denote a point in the time-frequency space as $z = (x,\omega) \in \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$. The operators $T_x$ and $M_\omega$ satisfy the so called ***commutation relations***:

$$M_\omega T_x = e^{2\pi i x\omega} T_x M_\omega. \tag{2.65}$$

Indeed, if $f$ is any function defined on $\mathbb{R}^d$, we can write:

$$\pi(x,\omega)f(t) = e^{2\pi i\omega t} f(t-x) = e^{2\pi i\omega x} e^{2\pi i\omega(t-x)} f(t-x) = e^{2\pi i x\omega} T_x M_\omega f(t).$$

**Remark 2.3.2.** *The time-frequency shift* $\pi(z)$ *is well defined on the equivalence classes in* $L^p(\mathbb{R}^d)$ *for* $1 \le p \le \infty$ *and* $0 < p < 1$.

**Definition 2.3.3.** *Let* $f$ *be a function on* $\mathbb{R}^d$*. The* ***involution operator*** *is defined as:*

$$f^*(x) \coloneqq \overline{f(-x)}, \qquad \forall x \in \mathbb{R}^d.$$

In order to obtain information about a *local frequency spectrum* of a signal $f$, we restrict $f$ to an interval centred at the instant $x$ object of interest and then take the Fourier transform of this restriction. Such a localization in time is made by multiplying $f$ with a smooth cut-off function $g$, called the ***window function***. We shall work mainly with $g \in \mathcal{S}(\mathbb{R}^d)$. This is the idea under the construction of the short-time Fourier transform.

**Definition 2.3.4.** *Fix* $g \in L^2(\mathbb{R}^d) \setminus \{0\}$*, the window function. The* ***short-time Fourier transform (STFT) of a signal*** $f \in L^2(\mathbb{R}^d)$ ***with respect to*** $g$ *is defined by:*

$$V_g f(x,\omega) \coloneqq \langle f, M_\omega T_x g\rangle, \qquad \forall x,\omega \in \mathbb{R}^d, \tag{2.66}$$

*where* $\langle\cdot,\cdot\rangle$ *is the inner product on* $L^2(\mathbb{R}^d)$.

**Lemma 2.3.5.** *Consider* $f, g \in L^2(\mathbb{R}^d)$*. Then* $V_g f(x,\omega)$ *is uniformly continuous and bounded on* $\mathbb{R}^{2d}$*. Moreover, the following estimate holds true:*

$$\|V_g f\|_{L^\infty(\mathbb{R}^{2d})} \le \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{2.67}$$

**Proposition 2.3.6.** *Consider* $f, g \in L^2(\mathbb{R}^d)$*. Then the following equalities hold true:*

$$\begin{aligned} V_g f(x,\omega) &= \mathcal{F}(f T_x \bar{g})(\omega) \\ &= \langle \hat{f}, T_\omega M_{-x} \hat{g}\rangle \\ &= e^{-2\pi i x\omega}(f * (M_\omega g)^*)(x) \\ &= e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega,-x), \end{aligned} \tag{2.68}$$

*where equation* (2.68) *is called the* ***fundamental identity of time-frequency analysis***.

**Theorem 2.3.7.** (Orthogonality relations for the STFT)
*Consider* $f_i, g_i \in L^2(\mathbb{R}^d)$, $i = 1, 2$. *Then* $V_{g_i} f_i \in L^2(\mathbb{R}^{2d})$ *and:*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)}. \tag{2.69}$$

**Corollary 2.3.8.** *Consider* $f, g \in L^2(\mathbb{R}^d)$. *Then:*

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{2.70}$$

**Remark 2.3.9.** *For a fixed window* $g$*, the linearity of* $V_g \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ *comes straightforward from* (2.66)*. From Corollary 2.3.8 above boundedness follows. Moreover, the operator norm is exactly*

$$\|V_g\|_{B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{2d}))} = \|g\|_{L^2(\mathbb{R}^d)}.$$

*Hence choosing a window function* $g \in L^2(\mathbb{R}^d)$ *with* $\|g\|_{L^2(\mathbb{R}^d)} = 1$ *the operator* $V_g \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ *is an isometry which is not onto because of Lemma 2.3.5.*

Let us introduce the notion of ***vector-valued integrals***, here understood in a ***weak sense***. Sometimes, we shall refer to them also simply as ***formal integrals***.

Consider $B$ a Banach space over $\mathbb{C}$ and suppose it is reflexive, i.e. $B \overset{\mathscr{J}}{\cong} B''$, where the isomorphism is the (conjugate) evaluation map

$$\begin{aligned} \mathscr{J} \colon B &\to B'' \\ b &\mapsto \mathscr{J}_b(\cdot) \end{aligned}$$

defined for $b' \in B'$ as:

$$\mathscr{J}_b(b') := \overline{b'(b)}.$$

Note that we're considering $B', B''$ as the sets of the antilinear and continuous functionals on $B, B'$ respectively. Let $\langle \cdot, \cdot \rangle$ be the duality between $B'$ and $B$, i.e. $\langle h, b \rangle := h(b)$ for $h \in B', b \in B$. Consider $\varphi \colon \mathbb{R}^q \to B$ and assume that for every $h \in B'$

$$\ell_\varphi(h) := \int_{\mathbb{R}^q} \overline{\langle h, \varphi(x) \rangle}\, dx \tag{2.71}$$

is absolutely convergent. Suppose that $\ell_\varphi \in B'' \cong B$. Then there exists a unique $f_\varphi \in B$ such that $\ell_\varphi(\cdot) = \mathscr{J}_{f_\varphi}(\cdot) = \overline{\langle \cdot, f_\varphi \rangle}$. For sake of simplicity we *formally* define:

$$\int_{\mathbb{R}^q} \varphi(x)\, dx := f_\varphi \in B. \tag{2.72}$$

Take $q = 2d$, $B = L^2(\mathbb{R}^d)$ and fix non zero functions $F \in L^2(\mathbb{R}^{2d}), \gamma \in L^2(\mathbb{R}^d)$. Define

$$\varphi \colon \mathbb{R}^{2d} \to L^2(\mathbb{R}^d), (x, \omega) \mapsto F(x, \omega) M_\omega T_x \gamma(\cdot).$$

Observe that actually taking pointwise values of a *function* $F$ which belongs to $L^2(\mathbb{R}^{2d})$ makes no sense. Hence we could choose a representative or work with an equivalence class of functions $\varphi = [\varphi_F]$. In any case we will work with an $F$ which admits a continuous representative, therefore no concern is needed. Remember that an Hilbert space, such as $L^2(\mathbb{R}^d)$, is always reflexive. Moreover it is isomorphic to its dual space via the Riesz-Fréchet Theorem, where we meant the suitable version in which the set of continuous antilinear functionals is considered as dual space. Hence in (2.71) we can read the duality product $\langle\cdot,\cdot\rangle$ as the usual inner product on $L^2(\mathbb{R}^d)$ which is antilinear in the second argument. We want to check that $\ell_\varphi$ is an element of the bidual space of $L^2(\mathbb{R}^d)$. $\ell_\varphi$ is antilinear by construction, take $h \in L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^d)'$ and write

$$\begin{aligned}
\ell_\varphi(h) &= \int_{\mathbb{R}^{2d}} \overline{\langle h, \varphi(x,\omega)\rangle}\, dxd\omega \\
&= \int_{\mathbb{R}^{2d}} \overline{\langle h, F(x,\omega)M_\omega T_x\gamma\rangle}\, dxd\omega \\
&= \int_{\mathbb{R}^{2d}} F(x,\omega)\overline{\langle h, M_\omega T_x\gamma\rangle}\, dxd\omega \\
&= \int_{\mathbb{R}^{2d}} F(x,\omega)\overline{V_\gamma h(x,\omega)}\, dxd\omega \\
&= \langle F, V_\gamma h\rangle_{L^2(\mathbb{R}^{2d})}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and Corollary 2.3.8 we get for every $h \in L^2(\mathbb{R}^d)$:

$$|\ell_\varphi(h)| \leq \|F\|_{L^2(\mathbb{R}^{2d})} \|V_\gamma h\|_{L^2(\mathbb{R}^{2d})} = \|F\|_{L^2(\mathbb{R}^{2d})} \|\gamma\|_{L^2(\mathbb{R}^d)} \|h\|_{L^2(\mathbb{R}^d)}.$$

Then $\ell_\varphi \in L^2(\mathbb{R}^d)''$ and there exists a unique $f_\varphi \in L^2(\mathbb{R}^d)$ such that $\ell_\varphi(\cdot) = \mathscr{J}_{f_\varphi}(\cdot) = \overline{\langle\cdot, f_\varphi\rangle}$ where the duality product between $L^2(\mathbb{R}^d)'$ and $L^2(\mathbb{R}^d)$ can be read as the inner product on $L^2(\mathbb{R}^d)$. Hence defining *formally*

$$\int_{\mathbb{R}^{2d}} F(x,\omega)M_\omega T_x\gamma\, dxd\omega := f_\varphi \in L^2(\mathbb{R}^d),$$

we have that

$$\ell_\varphi(h) = \langle f_\varphi, h\rangle. \tag{2.73}$$

Then the statement of the following theorem is now clear.

**Theorem 2.3.10.** (Inversion formula for the STFT)
*Consider $g, \gamma \in L^2(\mathbb{R}^d)$ and suppose that $\langle g,\gamma\rangle \neq 0$. Then for all $f \in L^2(\mathbb{R}^d)$:*

$$f = \frac{1}{\langle\gamma, g\rangle}\int_{\mathbb{R}^{2d}} V_g f(x,\omega)M_\omega T_x\gamma\, dxd\omega. \tag{2.74}$$

Thanks to the inversion formula (2.74) we are able to find the adjoint of $V_g\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$, where the window is kept fixed. Fix $g \in L^2(\mathbb{R}^d)$ and define the linear operator $A_g\colon L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d), F \mapsto A_g F$, where:

$$A_g F := \int_{\mathbb{R}^{2d}} F(x,\omega)M_\omega T_x g\, dxd\omega. \tag{2.75}$$

It is easy to prove that $A_g \in B(L^2(\mathbb{R}^{2d}), L^2(\mathbb{R}^d))$. Consider $h \in L^2(\mathbb{R}^d)$, $F \in L^2(\mathbb{R}^{2d})$, then by definition of vector-valued integral:

$$\langle A_g F, h\rangle_{L^2(\mathbb{R}^d)} = \langle F, V_g h\rangle_{L^2(\mathbb{R}^{2d})}.$$

Hence $V_g^* = A_g$, which allows us to rewrite the inversion formula (2.74) as follows:

$$f = \frac{1}{\langle\gamma, g\rangle} V_\gamma^* V_g f. \tag{2.76}$$

**Definition 2.3.11.** *Let $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ and consider $T: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ linear and continuous. Then the* **Gabor matrix of $T$ w.r.t.** *$g$ is the (continuous) matrix*

$$G_T(z,w) := G_T^g(z,w) := \langle T\pi(w)g, \pi(z)g\rangle, \qquad \forall z,w \in \mathbb{R}^{2d}. \tag{2.77}$$

Reading the inner product in (2.66) as the duality between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$, which is antilinear in the second argument and linear in the first one, we are allowed to extend the short-time Fourier transform to tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to *smooth window functions* $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

**Definition 2.3.12.** *Fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, the* window function. *The* ***short-time-Fourier transform (STFT) of a tempered distribution*** *$f \in \mathcal{S}'(\mathbb{R}^d)$* ***with respect to*** *$g$ is defined by:*

$$V_g f(x,\omega) := \langle f, M_\omega T_x g\rangle, \qquad \forall x,\omega \in \mathbb{R}^d, \tag{2.78}$$

*where $\langle\cdot,\cdot\rangle$ is the sesquilinear duality between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$.*

We recall some important properties of the STFT of a tempered distribution contained in [82, Theorems 11.2.3 and 11.2.5].

**Theorem 2.3.13.** *Consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. Then the STFT $V_g f$ is a complex-valued continuous function defined on $\mathbb{R}^{2d}$ and there exist constants $C > 0, N \in \mathbb{N}_0$ such that:*

$$|V_g f(x,\omega)| \le C(1+|x|+|\omega|)^N, \qquad \forall\, x,\omega \in \mathbb{R}^d, \tag{2.79}$$

*hence $V_g f$ has at most polynomial growth.*

**Theorem 2.3.14.** *Consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. Then the following are equivalent:*

(*i*) *$f \in \mathcal{S}(\mathbb{R}^d)$;*

(*ii*) *$V_g f \in \mathcal{S}(\mathbb{R}^{2d})$;*

(*iii*) *for all $n \in \mathbb{N}_0$, there exists $C_n > 0$ such that:*

$$|V_g f(x,\omega)| \le C_n(1+|x|+|\omega|)^{-n}, \qquad \forall\, x,\omega \in \mathbb{R}^d. \tag{2.80}$$

We briefly recall another important time-frequency representation, the cross-$\tau$-Wigner distribution, where $\tau \in [0,1]$.

**Definition 2.3.15.** *Consider $f,g \in L^2(\mathbb{R}^d)$ and $\tau \in [0,1]$. The* ***cross-$\tau$-Wigner distribution of $f$ and*** *$g$ is defined by:*

$$W_\tau(f,g)(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\, dt. \tag{2.81}$$

*Taking $g = f$ we get the so-called* ***$\tau$-Wigner distribution of*** *$f$:*

$$W_\tau f(x,\omega) := W_\tau(f,f)(x,\omega).$$

*If $\tau = 1/2$, we call $W_{1/2}(f,g)$* ***cross-$\tau$-Wigner distribution of*** *$f$* ***and*** *$g$ and adopt the notation $W(f,g) := W_{1/2}(f,g)$. When $\tau = 0$, we call $W_0(f,g)$* ***cross-Rihaczek distribution of*** *$f$* ***and*** *$g$ and denote it by $R(f,g) := W_0(f,g)$.*

**Definition 2.3.16.** *Let $f$ be a function defined on $\mathbb{R}^d$. We define the* **reflection** *or* **parity operator** *as:*

$$\mathcal{I}f(t) := f(-t), \qquad \forall t \in \mathbb{R}^d. \tag{2.82}$$

*For $\tau \in (0,1)$ we define the operator*

$$\mathcal{A}_\tau f(t) := f\left(\frac{\tau-1}{\tau}t\right), \qquad \forall t \in \mathbb{R}^d. \tag{2.83}$$

Clearly, $\mathcal{A}_{1/2} = \mathcal{I}$.
For the following results we refer to [35].

**Lemma 2.3.17.** *Let $\tau \in [0,1]$. Then:*

(*i*) *if $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $W_\tau(f,g) \in \mathcal{S}(\mathbb{R}^{2d})$;*

(*ii*) *if $f, g \in L^2(\mathbb{R}^d)$, then $W_\tau(f,g) \in L^2(\mathbb{R}^{2d})$.*

**Lemma 2.3.18.** *Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{S}(\mathbb{R}^d)$.*

(*i*) *If $\tau \in (0,1)$, then*

$$W_\tau(f,g)(x,\omega) = \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} V_{\mathcal{A}_\tau g} f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\omega\right), \quad \forall\,(x,\omega) \in \mathbb{R}^{2d};$$

(*ii*) *if $\tau = 0$, then*

$$W_0(f,g)(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{g}(\omega)} = R(f,g)(x,\omega), \quad \forall\,(x,\omega) \in \mathbb{R}^{2d};$$

(*iii*) *if $\tau = 1$, then*

$$W_1(f,g)(x,\omega) = e^{2\pi i x\omega}\overline{g(x)}\hat{f}(\omega) = \overline{R(g,f)}(x,\omega), \quad \forall\,(x,\omega) \in \mathbb{R}^{2d}.$$

**Proposition 2.3.19.** *Consider $f, g \in L^2(\mathbb{R}^d)$ and $\tau \in [0,1]$. The following hold true:*

(*i*) *for $\tau \in (0,1)$ the function $W_\tau(f,g)$ is uniformly continuous and bounded on $\mathbb{R}^{2d}$ and*

$$\|W(f,g)\|_{L^\infty(\mathbb{R}^{2d})} \leq \frac{1}{\tau^d}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)};$$

(*ii*) *we have:*

$$W_\tau(f,g) = \overline{W_{1-\tau}(g,f)}.$$

*In particular $Wf$ is real-valued;*

(*iii*) *we have:*

$$W_\tau(\hat{f},\hat{g})(x,\omega) = W_{1-\tau}(f,g)(-\omega,x);$$

(*iv*) ***Moyal's formula**: for $f_i, g_i \in L^2(\mathbb{R}^d), i = 1,2,$*

$$\langle W_\tau(f_1,g_1), W_\tau(f_2,g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)}\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^d)}. \tag{2.84}$$

### 2.3.2 Representations of operators

For the following result we address the reader to [82, Theorem 14.3.5] and [35, (4.37) and (4.38)]. Although $\tau$-pseudo-differential operators $\mathrm{Op}_\tau(\sigma)$ will be introduced later in (2.206), we state the theorem now for sake of clarity.

**Theorem 2.3.20.** *Let us consider $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ linear and continuous. Then there exist $K_T, \eta_T \in \mathcal{S}'(\mathbb{R}^{2d})$ and, for every $\tau \in [0,1]$, $\mathrm{a}^T_\tau \in \mathcal{S}'(\mathbb{R}^{2d})$ such that the operator $T$ has the following representations:*

*(i) as an integral operator:*

$$\langle Tf, g\rangle = \langle K_T, g \otimes \overline{f}\rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d); \tag{2.85}$$

*(ii) as a $\tau$-pseudo-differential operator:*

$$T = \mathrm{Op}_\tau(\mathrm{a}^T_\tau); \tag{2.86}$$

*(iii) as a continuous superposition of time-frequency shifts:*

$$T = \int_{\mathbb{R}^{2d}} \eta_T(z)\pi(z)\, dz, \tag{2.87}$$

*in the sense that for every $f, g \in \mathcal{S}(\mathbb{R}^d)$*

$$\langle Tf, g\rangle = \langle \eta_T, V_f g\rangle.$$

**Definition 2.3.21.** *Let $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear and continuous operators and consider the tempered distributions $K_T, \eta_T, \mathrm{a}^T_\tau \in \mathcal{S}'(\mathbb{R}^{2d})$ coming from Theorem 2.3.20.*
*We call $K_T$ the* ***(integral) kernel of the operator*** *$T$ and the following formal expression is called* ***integral representation of*** *$T$:*

$$Tf(x) = \int_{\mathbb{R}^d} K_T(x,y) f(y)\, dy. \tag{2.88}$$

*We call $\eta_T$ the* ***spreading function of the operator*** *$T$ and the formal expression* (2.87) *is called* ***spreading representation of*** *$T$.*
*We call $\mathrm{a}^T_\tau$ the* ***$\tau$-symbol of*** *$T$.*

The kernel $K_T$ and the spreading function $\eta_T$ of $T$ are related by the following formula

$$\eta_T(x,\omega) = \int_{\mathbb{R}^d} K_T(y, y-x) e^{-2\pi i \omega y}\, dy. \tag{2.89}$$

**Remark 2.3.22.** *If the notation used for the operator $T$ is particularly cumbersome, we shall adopt the following equivalent notations*

$$K(T) \coloneqq K_T, \qquad \eta(T) \coloneqq \eta_T. \tag{2.90}$$

### 2.3.3 TFA tools on $\mathcal{G}$ LCA group

The definitions given above on $\mathbb{R}^d$ are here extended to any $\mathcal{G}$ LCA group. We adopt the space of special test functions $\mathcal{S}_\mathcal{C}(\mathcal{G})$, introduced in [88] and defined below, and perform some explicit computations in Lemma 2.3.30 taken from [8]. The definition of $\mathcal{S}_\mathcal{C}(\mathcal{G})$ is based on the following result.

**Theorem 2.3.23** (Structure theorem)**.** ([96, Theorem 24.30]) *Let $\mathcal{G}$ be a LCA group. Then there exist $d \in \mathbb{N}_0$, $\mathcal{G}_0$ LCA group containing a compact open subgroup $\mathcal{K}$, such that $\mathcal{G}$ is isomorphic as topological group to $\mathbb{R}^d \times \mathcal{G}_0$:*

$$\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0.$$

Consequently, being the dual group of the Cartesian product the Cartesian product of the dual groups and $\widehat{\mathbb{R}^d} \cong \mathbb{R}^d$, we have the decomposition

$$\widehat{\mathcal{G}} \cong \mathbb{R}^d \times \widehat{\mathcal{G}}_0,$$

where the dual group $\widehat{\mathcal{G}}_0$ contains the compact open subgroup $\mathcal{K}^\perp$, see e.g. [81, Lemma 6.2.3]. We recall that

$$\mathcal{K}^\perp := \{\xi \in \widehat{\mathcal{G}} \mid \langle \xi, x\rangle = 1 \quad \forall x \in \mathcal{K}\}.$$

**Definition 2.3.24.** *For $x \in \mathcal{G}, \xi \in \widehat{\mathcal{G}}$ and a function $f\colon \mathcal{G} \to \mathbb{C}$ we define the* ***translation operator*** *$T_x$, the* ***modulation operator*** *$M_\xi$ and the* ***time-frequency shift (TFS)*** *$\pi(x,\xi)$ as*

$$T_x f(y) := f(y-x), \quad M_\xi f(y) := \langle \xi, y\rangle f(y), \quad \pi(x,\xi) := M_\xi T_x. \tag{2.91}$$

*For $f, g \in L^2(\mathcal{G})$, the* ***short-time Fourier transform (STFT) of $f$ with respect to*** *$g$ is given by*

$$V_g f(x,\xi) := \langle f, \pi(x,\xi)g\rangle = \int_\mathcal{G} f(y)\overline{\pi(x,\xi)g(y)}\,dy, \qquad \forall (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}, \tag{2.92}$$

*where $\langle \cdot,\cdot \rangle$ is the inner product on $L^2(\mathcal{G})$.*

Once again, $T_x$ and $M_\xi$ fulfil the commutation relations

$$M_\xi T_x = \langle \xi, x\rangle T_x M_\xi. \tag{2.93}$$

The following definitions come form [88, Section 2] and they rely on the Structure theorem 2.3.23.

**Definition 2.3.25.** *We define the* ***generalized Gaussian on*** *$\mathcal{G}$ as*

$$\varphi(x_1,x_2) := e^{-\pi x_1^2}\chi_\mathcal{K}(x_2) =: \varphi_1(x_1)\varphi_2(x_2), \quad \forall (x_1,x_2) \in \mathbb{R}^d \times \mathcal{G}_0, \tag{2.94}$$

*and the* ***set of special test functions***

$$\mathcal{S}_\mathcal{C}(\mathcal{G}) := \operatorname{span}\left\{\pi(\mathbf{x})\varphi \mid \mathbf{x} = (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}\right\} \subseteq L^2(\mathcal{G}). \tag{2.95}$$

**Definition 2.3.26.** *Given $f, g \in L^2(\mathcal{G})$, we define the* ***cross-Rihaczek distribution*** *of $f$ and $g$ by*

$$R(f,g)(x,\xi) := f(x)\overline{\hat{g}(\xi)\langle \xi, x\rangle}, \qquad \forall (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}, \tag{2.96}$$

*$\hat{g}$ being the Fourier transform of $g$. When $f = g$, $R(f,f)$ is called the* ***Rihaczek distribution of*** *$f$.*

The following result comes from [8].

**Lemma 2.3.27.** *Let $\varphi$ be as in (2.94). Then for $x = (x_1, x_2) \in \mathbb{R}^d \times \mathcal{G}_0$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^d \times \widehat{\mathcal{G}}_0$, the Rihaczek distribution of $\varphi$ is*

$$R(\varphi,\varphi)(x,\xi) = c(\mathcal{K})\overline{\langle \xi, x\rangle} e^{-\pi(x_1^2+\xi_1^2)} \otimes \chi_{\mathcal{K}\times\mathcal{K}^\perp}(x_2,\xi_2), \tag{2.97}$$

*where $c(\mathcal{K}) > 0$ is a constant depending on the compact subgroup $\mathcal{K}$.*

*Proof.* It is sufficient to compute as follows:

$$\begin{aligned} R(\varphi,\varphi)(x,\xi) &= R(\varphi_1,\varphi_1)(x_1,\xi_1)R(\varphi_2,\varphi_2)(x_2,\xi_2) \\ &= e^{-2\pi i \xi_1 x_1} e^{-\pi(x_1^2+\xi_1^2)} \chi_{\mathcal{K}}(x_2) c(\mathcal{K}) \chi_{\mathcal{K}^\perp}(\xi_2) \overline{\langle \xi_2, x_2\rangle} \\ &= c(\mathcal{K}) e^{-2\pi i \xi_1 x_1} e^{-\pi(x_1^2+\xi_1^2)} \overline{\langle \xi_2, x_2\rangle} \chi_{\mathcal{K}\times\mathcal{K}^\perp}(x_2,\xi_2) \\ &= c(\mathcal{K})\overline{\langle \xi, x\rangle} e^{-\pi(x_1^2+\xi_1^2)} \otimes \chi_{\mathcal{K}\times\mathcal{K}^\perp}(x_2,\xi_2), \end{aligned}$$

for the constant $c(\mathcal{K})$ coming from the factor $R(\varphi_2,\varphi_2)$ see [88]. □

Hence $R(\varphi,\varphi)(x,\xi)$ is up to a positive constant and a "chirp" a Gaussian on $\mathbb{R}^{2d} \times (\mathcal{G}_0 \times \widehat{\mathcal{G}}_0)$, where we fixed $\mathcal{K} \times \mathcal{K}^\perp$ as compact open subgroup of the not Euclidean component.

**Definition 2.3.28.** *We denote by $\mathcal{J}$ is the topological isomorphism*

$$\mathcal{J} \colon \mathcal{G} \times \widehat{\mathcal{G}} \to \widehat{\mathcal{G}} \times \mathcal{G},\ (x,\xi) \mapsto (-\xi, x). \tag{2.98}$$

**Remark 2.3.29.** *(i) Clearly, $\mathcal{J}^{-1}(\xi,x) = (x,-\xi)$;*

*(ii) If we take $\mathcal{G} = \mathbb{R}^d$, then we have $\mathcal{J} = -J$, $J$ being the operator defined in (2.212).*

We recall the following covariance property [88, Lemma 4.2 (i)]: for $\mathbf{x} = (x,\xi)$, $\mathbf{y} = (y,\eta) \in \mathcal{G} \times \widehat{\mathcal{G}}$, $f, g \in \mathcal{S}_\mathcal{C}(\mathcal{G})$,

$$R(\pi(\mathbf{x})f, \pi(\mathbf{y})g) = \langle \eta, x-y\rangle M_{\mathcal{J}(\mathbf{y}-\mathbf{x})} T_{(x,\eta)} R(f,g). \tag{2.99}$$

In what follows we shall need also the following identity:

$$V_\varphi \varphi(x,\xi) = c(\mathcal{K}) e^{-\frac{\pi}{2}(x_1^2+\xi_1^2)} \otimes \chi_{\mathcal{K}\times\mathcal{K}^\perp}(x_2,\xi_2), \tag{2.100}$$

see [88] for calculations. Using a similar argument as in the estimate [88, formula (12)], one can show that $R(f,g)$ and $V_g f$ are in $L^p_m(\mathcal{G}\times\widehat{\mathcal{G}})$, $0 < p \le \infty$, for arbitrary moderate weight functions and any $f, g \in \mathcal{S}_\mathcal{C}(\mathcal{G})$. Similarly, every function in $\mathcal{S}_\mathcal{C}(\mathcal{G})$ belongs to $L^p_m(\mathcal{G})$, $0 < p \le \infty$. Recall that for any $f, g \in L^2(\mathcal{G})$ [88, formula (8)]

$$V_{M_\eta T_y g} M_\omega T_u f(x,\xi) = \overline{\langle \xi-\omega, u\rangle} \langle \eta, x-u\rangle T_{(u-y,\omega-\eta)} V_g f(x,\xi). \tag{2.101}$$

The previous formula, jointly with (2.99), allows us to write explicitly every STFT and cross-Rihaczek distribution of elements in $\mathcal{S}_\mathcal{C}(\mathcal{G})$.

**Lemma 2.3.30.** ([8, Lemma 2.1]) *Consider $f, g \in \mathcal{S}_\mathcal{C}(\mathcal{G})$, hence*

$$f = \sum_{k=1}^n a_k \pi(\mathbf{u}_k)\varphi, \qquad g = \sum_{j=1}^m b_j \pi(\mathbf{y}_j)\varphi,$$

*for some* $n, m \in \mathbb{N}$, $a_k, b_j \in \mathbb{C}$ *and* $\mathbf{u}_k = (u_k, \omega_k), \mathbf{y}_j = (y_j, \eta_j) \in \mathcal{G} \times \widehat{\mathcal{G}}$. *Then for every* $(x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$*:*

$$V_g f(x,\xi) = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \overline{b_j \langle \xi - \omega_k, u_k\rangle} \langle \eta_j, x - u_k\rangle T_{\mathbf{u}_k - \mathbf{y}_j} V_\varphi \varphi(x,\xi), \tag{2.102}$$

$$R(f,g)(x,\xi) = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j \langle \eta_j, u_k - y_j\rangle M_{\mathcal{J}(\mathbf{y}_j - \mathbf{u}_k)} T_{(u_k,\eta_j)} R(\varphi,\varphi)(x,\xi). \tag{2.103}$$

*Proof.* We write $\mathbf{x} = (x, \xi)$. The first claim follows from (2.101) after the following rephrasing:

$$\begin{aligned} V_g f(\mathbf{x}) &= \langle f, \pi(\mathbf{x}) g\rangle = \langle \sum_{k=1}^{n} a_k \pi(\mathbf{u}_k)\varphi, \pi(\mathbf{x}) \sum_{j=1}^{m} b_j \pi(\mathbf{y}_j)\varphi\rangle \\ &= \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \overline{b_j} \langle \pi(\mathbf{u}_k)\varphi, \pi(\mathbf{x})\pi(\mathbf{y}_j)\varphi\rangle = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \overline{b_j} \left(V_{\pi(\mathbf{y}_j)\varphi} \pi(\mathbf{u}_k)\varphi\right)(\mathbf{x}). \end{aligned}$$

For the second issue we write

$$\begin{aligned} R(f,g)(\mathbf{x}) &= \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j \pi(\mathbf{u}_k)\varphi(x) \widehat{\pi(\mathbf{y}_j)\varphi}(\xi) \overline{\langle \xi, x\rangle} \\ &= \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j R(\pi(\mathbf{u}_k)\varphi, \pi(\mathbf{y}_j)\varphi)(\mathbf{x}) \end{aligned}$$

and use (2.99). □

#### 2.3.3.1 The case $\mathcal{G} = \mathbb{Z}_N$

When dealing with the cyclic group $\mathbb{Z}_N$, $N \in \mathbb{N}$, we make the following identification:

$$\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N). \tag{2.104}$$

Namely, every complex $N$-tuple $(z_0, \ldots, z_{N-1}) \in \mathbb{C}^N$ is identified with the unique function $f\colon \mathbb{Z}_N \to \mathbb{C}^d$ such that $f(t) = z_t$ for every $t = 0, \ldots, N-1$. So that the Euclidean product on $\mathbb{C}^N$ coincides with the standard inner product on $\ell^2(\mathbb{Z}_N)$. Notice that we take the indexes ranging from 0 to $N-1$. Moreover, the argument of $f \in \mathbb{C}^N$, i.e. of $f\colon \mathbb{Z}_N \to \mathbb{C}^N$, is always taken modulus $N$, even if not explicitly stated. Hence, e.g., we shall write $f(N)$ meaning $f(0)$, etc. . We shall denote by $\mathbf{1} \in \mathbb{C}^N$ the constant function equal to 1. It is useful to observe the $\mathbb{Z}_N$ is self-dual, i.e.

$$\widehat{\mathbb{Z}_N} \cong \mathbb{Z}_N.$$

In the finite discrete case translation and modulation operators take the following form

$$T_k f(t) \coloneqq f(t-k), \quad M_l f(t) \coloneqq e^{\frac{2\pi i l t}{N}} f(t),$$

where $f \in \mathbb{C}^N$, $t = 0, \ldots, N-1$, $k, l \in \mathbb{Z}$.
There is an exact analogue of the objects described in Definition 2.3.21 in the finite discrete case, see [64]. In fact, if $T\colon \mathbb{C}^N \to \mathbb{C}^N$ is a linear operator then its kernel $K_T$ is just its matrix representation and we can define its spreading function as

$$\eta_T(u,v) \coloneqq \sum_{k=0}^{N-1} K_T(k, k-u) e^{\frac{-2\pi i k v}{N}}. \tag{2.105}$$

So that $T$ can be seen as a finite superposition of TF-shifts, which in the finite dimensional case are an orthonormal basis, so every matrix can be uniquely described by its spreading function:

$$T = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \eta_T(k,l)\pi(k,l).$$

Of course, $\pi(k,l) := M_l T_k$ and we define the STFT of a signal $f \in \mathbb{C}^N$ w.r.t. the window $g \in \mathbb{C}^N$ as the *matrix* in $\mathbb{C}^{N\times N}$

$$V_g f(u,v) := \langle f, \pi(u,v)g\rangle = \sum_{k=0}^{N-1} f(k)\overline{g(k-u)}e^{\frac{-2\pi i k v}{N}},$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\ell^2(\mathbb{C}^N)$, i.e. the Euclidean product on $\mathbb{C}^N$.

**Definition 2.3.31.** *The* ***discrete Fourier transform (DFT) on*** $\mathbb{C}^N$ *is the linear operator represented by the following* $N \times N$ *complex matrix*

$$(\mathcal{F}_N)_{k,l} := e^{\frac{-2\pi i k l}{N}} \tag{2.106}$$

*which inverse if given by*

$$(\mathcal{F}_N^{-1})_{k,l} = \frac{1}{N} e^{\frac{2\pi i k l}{N}}.$$

*We shall denote by* $\hat{f}$ *the vector* $\mathcal{F}_N f$, $f \in \mathbb{C}^N$.
*The* ***discrete two-dimensional Fourier transform of a matrix*** $a \in \mathbb{C}^N \times \mathbb{C}^N$ *and its inverse are defined as*

$$\mathsf{F}_2 a(u,v) := \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a(k,l) e^{\frac{-2\pi i u k}{N}} e^{\frac{-2\pi i v l}{N}},$$
$$\mathsf{F}_2^{-1} a(u,v) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a(k,l) e^{\frac{2\pi i u k}{N}} e^{\frac{2\pi i v l}{N}}.$$

The action of $\mathsf{F}_2$ on the (pointwise) product of $a$ and $b$ in $\mathbb{C}^{N\times N} \cong \mathbb{C}^N \times \mathbb{C}^N$ is well-known and we mention it for sake of completeness:

$$\mathsf{F}_2(a\cdot b) = \frac{1}{N^2}\left(\mathsf{F}_2 a * \mathsf{F}_2 b\right), \tag{2.107}$$

where the (two-dimensional discrete) convolution on the right-hand side is defined similarly to (2.221).

**Definition 2.3.32.** *The* ***Kronecker delta function*** $\delta \in \mathbb{C}^N$ *is defined as*

$$\delta(u) = \begin{cases} 1 & \textit{for} \quad u = 0, \\ 0 & \textit{for} \quad u = 1,\dots,N-1. \end{cases} \tag{2.108}$$

We recall also the following identity which is due to the subsequent (2.113) and the normalization chosen for the Fourier transform :

$$\mathcal{F}_N\left(\frac{1}{N}\mathbf{1}\right)(u) = \delta(u).$$

In some of the subsequent computations we shall need the function introduced in the following definition. Let us consider two numbers $\alpha, \beta$ such that

$$\alpha, \beta \in \mathbb{N}, \qquad A := \frac{N}{\alpha} \in \mathbb{N}, \quad B := \frac{N}{\beta} \in \mathbb{N}. \tag{2.109}$$

We shall see that they are the core of rectangular lattices defined in (2.138).

**Definition 2.3.33.** *Let $\alpha, \beta, A, B \in \mathbb{N}$ as in* (2.109)*. We call **impulse train**, or **Dirac comb**, the function so defined:*

$$\begin{aligned} Ш_{(\alpha,\beta)}(u, v) &:= \sum_{p=0}^{A-1} \sum_{q=1}^{B-1} \delta(u - \alpha p)\delta(v - \beta q) \\ &= \chi_{\alpha\mathbb{Z}_N}(u) \cdot \chi_{\beta\mathbb{Z}_N}(v) \\ &= \frac{1}{\alpha\beta} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \delta(u - \alpha k)\delta(v - \beta l), \end{aligned} \tag{2.110}$$

*for $u, v = 0, \dots, N - 1$.*

For sake of the reader, we recall and present a proof of the ***Poisson summation formula*** (2.111) and its two-dimensional analogue in the following lemma, see [78] and [110, Theorem 3.2.1].

**Lemma 2.3.34.** *Let $\alpha, \beta, A, B \in \mathbb{N}$ as in* (2.109)*. Then:*

$$\mathcal{F}_N \chi_{\alpha\mathbb{Z}_N} = A\chi_{A\mathbb{Z}_N}, \tag{2.111}$$

$$\mathsf{F}_2 Ш_{(\alpha,\beta)} = AB\, Ш_{(A,B)}. \tag{2.112}$$

*Proof.* Before showing the computations, we recall the following well-known identity for $u, v = 0, \dots, N - 1$:

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i u k}{N}} e^{\frac{-2\pi i v k}{N}} = \begin{cases} N & \text{if} \quad u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{2.113}$$

Hence

$$\begin{aligned} \mathsf{F}_2 Ш_{(\alpha,\beta)}(u, v) &= \frac{1}{\alpha\beta} \sum_{k,l=0}^{N-1} \sum_{p,q=0}^{N-1} \delta(k - \alpha p)\delta(l - \beta q) e^{\frac{-2\pi i k u}{N}} e^{\frac{-2\pi i l v}{N}} \\ &= \frac{1}{\alpha\beta} \sum_{p=0}^{N-1} e^{\frac{-2\pi i \alpha p u}{N}} \sum_{q=0}^{N-1} e^{\frac{-2\pi i \beta q v}{N}}. \end{aligned}$$

Due to (2.113), we see that

$$\sum_{p=0}^{N-1} e^{\frac{-2\pi i \alpha p u}{N}} = \begin{cases} N & \text{if} \quad \alpha u = 0 \mod N \\ 0 & \text{otherwise} \end{cases}$$

and $\alpha u = 0 \mod N$ is equivalent to $u = Al$ for $l = 0, \dots, \alpha - 1$. Arguing similarly for the second summation we have

$$\begin{aligned} \mathsf{F}_2 Ш_{(\alpha,\beta)}(u,v) &= \frac{1}{\alpha\beta} N \sum_{l=0}^{\alpha-1} \delta(u - Al) N \sum_{k=0}^{\beta-1} \delta(v - Bk) \\ &= AB\, Ш_{(A,B)}(u,v). \end{aligned}$$

This concludes the proof. □

**Definition 2.3.35.** *The* ***discrete symplectic Fourier transform of a matrix*** $a \in \mathbb{C}^{N\times N}$ *is defined as*

$$\mathsf{F}_s a(u,v) := \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a(k,l) e^{\frac{2\pi i (lu - kv)}{N}}, \tag{2.114}$$

*with* $u, v = 0, \dots, N-1$.

Hence the relation between $\mathsf{F}_2$ and $\mathsf{F}_s$ is as follows:

$$\mathsf{F}_s a(u,v) = \frac{1}{N} \mathsf{F}_2(a^T)(-u,v) = \frac{1}{N} \mathsf{F}_2 a(v,-u), \tag{2.115}$$

$a^T$ being the transpose of $a$. Recall that given two vectors $f, g \in \mathbb{C}^N$, the tensor product $f \otimes g \in \mathbb{C}^{N\times N}$ is the matrix

$$f \otimes g(u,v) = f(u)g(v), \qquad u, v = 0, \dots, N-1.$$

Eventually we mention that

$$\mathsf{F}_s(f \otimes \hat{g}) = g \otimes \hat{f}. \tag{2.116}$$

## 2.4 Frames in Hilbert spaces

In this section we present frame theory which can be seen as a generalisation of basis theory in Hilbert spaces. Moreover, frames became very popular because they resulted useful in applications. Frame theory allows us the passage from a continuous representation of a signal as in (2.74) to a discrete one. For further details about frame theory and Gabor frames, we suggest [21, 35, 82].

**Definition 2.4.1.** *Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ *be an Hilbert space. A sequence* $\{x_n\}_n \subseteq \mathcal{H}$ *is a* **frame** *if there exist constants* $0 < A \leq B$ *such that:*

$$A \|x\|_{\mathcal{H}}^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle_{\mathcal{H}}|^2 \leq B \|x\|_{\mathcal{H}}^2, \qquad \forall\, x \in \mathcal{H}. \tag{2.117}$$

*The constants A and B are called* **lower** *and* **upper frame bound** *respectively. We call* **optimal lower (upper) frame bound** *the largest (smallest) possible lower (upper) frame bound.*

**Remark 2.4.2.** *By frame inequality* (2.117) *we have that* $\|(\langle\cdot, x_n\rangle_{\mathcal{H}})_n\|_{\ell^2}$ *is an equivalent norm for* $\mathcal{H}$. *Moreover, if* $A = 1 = B$, *then* $\|\cdot\|_{\mathcal{H}} = \|(\langle\cdot, x_n\rangle_{\mathcal{H}})_n\|_{\ell^2}$.

**Definition 2.4.3.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be an Hilbert space and consider a frame $\{x_n\}_n \subseteq \mathcal{H}$.*

- $(i)$ *If $A = B$, then $\{x_n\}_n$ is called an* **A-tight frame**, *or simply* **tight frame**;
- $(ii)$ *if $A = 1 = B$, then $\{x_n\}_n$ is called a* **Parseval frame**;
- $(iii)$ *if $\{x_n\}_n$ ceases to be a frame whenever any single element is removed from the sequence, then it is called an* **exact frame**.

**Remark 2.4.4.** $(i)$ *We observe that in a frame the zero vector and repetition of elements are allowed;*

- $(ii)$ *if $\{x_n\}_n$ is a frame for $\mathcal{H}$, then $\sum_{n=1}^{\infty} |\langle x, x_n\rangle_{\mathcal{H}}|^2$ is absolutely convergent. Hence the series is unconditionally convergent. This proves that $\{x_{\sigma(n)}\}_n$ is still a frame for any permutation $\sigma$ of $\mathbb{N}$;*
- $(iii)$ *every orthonormal basis (o.n.b.) $\{e_n\}_n$ is a Parseval and exact frame;*
- $(iv)$ *if $\{x_n\}_n$ is a frame for $\mathcal{H}$, then it is complete. Indeed consider $x \in (\{x_n\}_n)^{\perp}$, then by frame inequality (2.117) we have:*

$$A\,\|x\|_{\mathcal{H}}^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle_{\mathcal{H}}|^2 = 0.$$

*This implies $x = 0_{\mathcal{H}}$, hence $\overline{\operatorname{span}\{x_n\}_n} = \mathcal{H}$.*

**Definition 2.4.5.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be an Hilbert space and consider a sequence $\{x_n\}_n \subseteq \mathcal{H}$. $\{x_n\}_n$ is a* **Bessel sequence** *if*

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle_{\mathcal{H}}|^2 < +\infty, \qquad \forall\, x \in \mathcal{H}. \tag{2.118}$$

**Remark 2.4.6.** *A frame $\{x_n\}_n$ for an Hilbert space $\mathcal{H}$ is always a Bessel sequence.*

**Proposition 2.4.7.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be an Hilbert space and consider a Bessel sequence $\{x_n\}_n \subseteq \mathcal{H}$. Then*

$$\begin{aligned} \mathcal{C}\colon \mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ x &\longmapsto (\langle x, x_n\rangle_{\mathcal{H}})_n \end{aligned} \tag{2.119}$$

*is well defined and it is a linear and continuous operator. The square of the operator norm*

$$B \coloneqq \|\mathcal{C}\|_{\mathrm{Op}}^2 > 0 \tag{2.120}$$

*is called* **Bessel constant for the Bessel sequence** $\{x_n\}_n$.

**Definition 2.4.8.** *The operator $\mathcal{C}\colon \mathcal{H} \to \ell^2$ defined in* (2.119) *is called* **coefficient**, *or* **analysis**, **operator associated to the Bessel sequence** $\{x_n\}_n$.

**Proposition 2.4.9.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be an Hilbert space and consider a Bessel sequence $\{x_n\}_n \subseteq \mathcal{H}$ with Bessel constant $B$. Then:*

- $(i)$ *if $c = (c_n)_n \in \ell^2(\mathbb{N})$, then the series $\sum_{n=1}^{\infty} c_n x_n$ is unconditionally convergent in $\mathcal{H}$;*

*(ii)*

$$\mathcal{D}\colon \ell^2(\mathbb{N}) \longrightarrow \mathcal{H} \qquad (c_n)_n \longmapsto \sum_{n=1}^{\infty} c_n x_n \tag{2.121}$$

*is well defined and it is a linear and continuous operator;*

*(iii)* $\mathcal{D}^* = \mathcal{C}$ *and* $\|\mathcal{D}\|_{\mathrm{Op}} = \|\mathcal{C}\|_{\mathrm{Op}} = \sqrt{B}$;

*(iv) if* $\{x_n\}_n$ *is also a frame, then* $\mathcal{C}$ *is injective and* $\mathcal{D}$ *is surjective.*

**Definition 2.4.10.** *The operator* $\mathcal{D}\colon \ell^2 \to \mathcal{H}$ *defined in* (2.121) *is called* ***reconstruction****, or* ***synthesis****,* ***operator associated to the Bessel sequence*** $\{x_n\}_n$.

*(i) The operator* $S \coloneqq \mathcal{D}\mathcal{C}\colon \mathcal{H} \to \mathcal{H}$ *is called* ***frame operator****.*

*(ii) The operator* $G \coloneqq \mathcal{C}\mathcal{D}\colon \ell^2 \to \ell^2$ *is called* ***Gram operator*** *or* ***Gram matrix****.*

**Theorem 2.4.11.** (Reproducing formulae for a frame)
*Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ *be an Hilbert space and consider a frame* $\{x_n\}_n \subseteq \mathcal{H}$ *with frame bounds* $0 < A \leq B$. *Then:*

*(i) the frame operator* $S\colon \mathcal{H} \to \mathcal{H}$ *is a topological isomorphism, self-adjoint and positive, with*

$$AI \leq S \leq BI,$$

*where* $I$ *is the identity on* $\mathcal{H}$*;*

*(ii)* $S^{-1}$ *is a topological isomorphism, self-adjoint and positive, with*

$$B^{-1}I \leq S^{-1} \leq A^{-1}I;$$

*(iii)* $\{S^{-1}x_n\}_n$ *is a frame for* $H$ *with frame bounds* $0 < B^{-1} \leq A^{-1}$*;*

*(iv) for each* $x \in \mathcal{H}$*, the following* ***reproducing formulae*** *hold true:*

$$x = \sum_{n=1}^{\infty} \langle x, S^{-1}x_n\rangle_{\mathcal{H}} x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle_{\mathcal{H}} S^{-1}x_n, \tag{2.122}$$

*and the above series converge unconditionally in* $\mathcal{H}$*;*

*(v) if the frame is A-tight, then* $S = AI, S^{-1} = A^{-1}I$*, and for every* $x \in \mathcal{H}$

$$x = \frac{1}{A} \sum_{n=1}^{\infty} \langle x, x_n\rangle_{\mathcal{H}} x_n.$$

**Definition 2.4.12.** *Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ *be an Hilbert space and consider a frame* $\{x_n\}_n \subseteq \mathcal{H}$ *with frame operator* $S$*. Then the frame* $\{S^{-1}x_n\}_n$ *is called* ***canonical dual frame****.*

We leave as an exercise for the reader to prove that the frame operator for the canonical dual frame $\{S^{-1}x_n\}_n$ is $S^{-1}$.

**Definition 2.4.13.** *Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ *be an Hilbert space and consider a frame* $\{x_n\}_n \subseteq \mathcal{H}$*. A sequence* $\{y_n\}_n \subseteq \mathcal{H}$ *such that*

$$x = \sum_{n=1}^{\infty} \langle x, y_n\rangle_{\mathcal{H}} x_n \qquad \forall\, x \in \mathcal{H}$$

*with unconditional convergence is called* ***alternative dual for*** $\{x_n\}_n$*. If* $\{y_n\}_n$ *is also a frame, then is called* ***alternative dual frame****.*

### 2.4.1 Gabor frames on Euclidean space

We restrict our attention to the Hilbert space $L^2(\mathbb{R}^d)$ and introduce a specific type of frames, the Gabor ones. They are the most commonly used and studied due to their rather simple construction and being handy for applications.

**Definition 2.4.14.** *We call* ***lattice on*** $\mathbb{R}^{2d}$ *a set of the following type:*

$$\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d,$$

*where* $\alpha, \beta > 0$*. The numbers* $\alpha$ *and* $\beta$ *are called* ***lattice parameters****.*
*Fix a non-zero window function* $g \in L^2(\mathbb{R}^d)$ *and lattice parameters* $\alpha, \beta > 0$*. We call* ***Gabor system*** *the following set:*

$$\mathcal{G}(g, \alpha, \beta) := \left\{ T_{\alpha m} M_{\beta n} g \quad | \quad m, n \in \mathbb{Z}^d \right\}.$$

*If a Gabor system* $\mathcal{G}(g, \alpha, \beta)$ *is a frame for* $L^2(\mathbb{R}^d)$*, then it is called* ***Gabor frame****.*

Due to the commutation relations (2.65), the frame operator $S$ associated to a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ has the form

$$Sf = \sum_{m,n\in\mathbb{Z}^d} \langle f, T_{\alpha m} M_{\beta n} g\rangle T_{\alpha m} M_{\beta n} g = \sum_{m,n\in\mathbb{Z}^d} V_g f(\alpha m, \beta n) M_{\beta n} T_{\alpha m} g.$$

If necessary, we write $S_{g,g}^{\alpha,\beta}$ or $S_{g,g}$ instead of $S$ in order to emphasize the dependence from the window function and the lattice parameters. Anyway, the reason for this notation will be clarified later.

**Remark 2.4.15.** *A Gabor frame can be equivalently defined as*

$$\mathcal{G}(g, \alpha, \beta) := \left\{ M_{\beta n} T_{\alpha m} g \quad | \quad m, n \in \mathbb{Z}^d \right\}$$

*because of commutation relations* (2.65)*.*

Gabor frames are named after the electrical engineer and physicist, most notable for inventing holography, Dennis Gabor (1900-1979). In his paper [73] Gabor conjectured that the Gabor system $\mathcal{G}(\varphi, 1, 1)$, where $\varphi(t) = e^{-\pi t^2}$ is the 1-dimensional Gaussian, was a basis for $L^2(\mathbb{R})$. Indeed, he claimed that every function $f \in L^2(\mathbb{R})$ could be represented as

$$f = \sum_{m,n\in\mathbb{Z}} c_{m,n}(f) M_n T_m \varphi,$$

for some scalars $c_{m,n}(f)$. His conjecture was false, but the previous expansion makes sense using frame theory.

**Proposition 2.4.16.** *Consider* $\mathcal{G}(g, \alpha, \beta)$ *a Gabor frame for* $L^2(\mathbb{R}^d)$ *with frame operator* $S_{g,g}$*. Then*

$$\gamma := S_{g,g}^{-1} g \tag{2.123}$$

*is such that* $\mathcal{G}(\gamma, \alpha, \beta)$ *is the canonical dual frame of* $\mathcal{G}(g, \alpha, \beta)$*. Consequently, every* $f \in L^2(\mathbb{R}^d)$ *has the following frame expansions:*

$$f = \sum_{m,n\in\mathbb{Z}^d} \langle f, T_{\alpha m} M_{\beta n} g\rangle T_{\alpha m} M_{\beta n} \gamma = \sum_{m,n\in\mathbb{Z}^d} \langle f, T_{\alpha m} M_{\beta n} \gamma\rangle T_{\alpha m} M_{\beta n} g, \tag{2.124}$$

*with unconditional convergence in $L^2(\mathbb{R}^d)$. Further, the following norm equivalences hold:*

$$A\,\|f\|_{L^2}^2 \le \sum_{m,n\in\mathbb{Z}^d} |\langle f, T_{\alpha m}M_{\beta n}g\rangle|^2 \le B\,\|f\|_{L^2}^2\,, \tag{2.125}$$

$$B^{-1}\,\|f\|_{L^2}^2 \le \sum_{m,n\in\mathbb{Z}^d} |\langle f, T_{\alpha m}M_{\beta n}\gamma\rangle|^2 \le A^{-1}\,\|f\|_{L^2}^2\,, \tag{2.126}$$

*where $A, B$ are the frame bounds for $\mathcal{G}(g,\alpha,\beta)$.*

The window defined in (2.123) is called ***canonical dual window***.
Consider $g, \gamma \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$. We introduce the following type of operator on $L^2(\mathbb{R}^d)$:

$$S_{g,\gamma}^{\alpha,\beta} f := \sum_{m,n\in\mathbb{Z}^d} \langle f, T_{\alpha m}M_{\beta n}g\rangle T_{\alpha m}M_{\beta n}\gamma, \tag{2.127}$$

defined whenever the series makes sense. Then (2.124) can be rephrased as

$$S_{g,\gamma}^{\alpha,\beta} = I,$$

where $I$ is the identity on $L^2(\mathbb{R}^d)$. Eventually we observe that, if we set $\Lambda := \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, then the operator can be written as

$$S_{g,\gamma}^{\alpha,\beta} f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma.$$

Sometimes we write $\mathcal{G}(g,\Lambda)$ instead of $\mathcal{G}(g,\alpha,\beta)$.

**Corollary 2.4.17.** *Consider $\mathcal{G}(g,\alpha,\beta)$ a Gabor frame for $L^2(\mathbb{R}^d)$ with frame operator $S_{g,g}$. Consider the canonical dual window $\gamma = S_{g,g}^{-1}g$. Then:*

$$S_{g,g}^{-1} = S_{\gamma,\gamma}.$$

**Remark 2.4.18.** *Proposition 2.4.16 provides a discrete time-frequency representation of signals. If $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$, then* (2.124) *is a discrete version of the inversion formula for the STFT. Moreover,* (2.124) *provides a Gabor expansions of $f$ with the canonical set of coefficients given by $c_{m,n} = \langle f, T_{\alpha m}M_{\beta n}\gamma\rangle$. The series expansion* (2.124) *can be rephrased in terms of the STFT as*

$$f = \sum_{m,n\in\mathbb{Z}^d} V_g f(\alpha m, \beta n) M_{\beta n} T_{\alpha m}\gamma,$$

*which is a reconstruction of the signal $f$ from samples of its STFT.*

If we focus on functions $g$ compactly supported in an interval of length $1/\beta$, then there exist Gabor frames $\mathcal{G}(g,\alpha,\beta)$ for $L^2(\mathbb{R})$ where we can also take $g$ smooth, if we choose the lattice parameters $\alpha, \beta$ properly. This was done first by I. Daubechies, A. Grossmann, Y. Meyer in [38], and they were called painless nonorthogonal expansions, since they were easy to construct.

**Theorem 2.4.19.** (Painless Nonorthogonal Expansions)
*Consider lattice parameters $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R})$.*

(*i*) *If $\operatorname{supp}(g) \subseteq [0, \beta^{-1}]$, then $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R})$ if and only if there exist constants $A, B > 0$ such that*

$$A\beta \le \sum_{k\in\mathbb{Z}} |g(x-\alpha k)|^2 \le B\beta \qquad a.e.. \tag{2.128}$$

*In this case, $A$ and $B$ are the frame bounds for $\mathcal{G}(g,\alpha,\beta)$.*

(*ii*) *If* $0 < \alpha\beta < 1$*, then there exists* $g$ *such that* $\operatorname{supp}(g) \subseteq [0, \beta^{-1}]$ *that satisfies* (2.128) *and can be as smooth as we like, even infinitely differentiable.*

(*iii*) *If* $\alpha\beta = 1$*, then any function that is supported in* $[0, \beta^{-1}]$ *and satisfies* (2.128) *is discontinuous.*

(*iv*) *If* $\alpha\beta > 1$ *and* $g$ *is supported in* $[0, \beta^{-1}]$*, then* (2.128) *is not satisfied and* $\mathcal{G}(g, \alpha, \beta)$ *is incomplete in* $L^2(\mathbb{R})$*.*

We end up this section recalling an important result concerning Gabor systems $\mathcal{G}(\varphi, \alpha, \beta)$ where $\varphi(t) = e^{-\pi t^2}$ is the Gaussian function. The following important result is proved independently by Y. Lyubarskii [111] and by K. Seip and R. Wallstén [123, 124] using complex analysis methods.

**Theorem 2.4.20.** *Consider the 1-dimension Gaussian function* $\varphi(t) = e^{-\pi t^2}$*. Then the system* $\mathcal{G}(\varphi, \alpha, \beta)$ *is a frame for* $L^2(\mathbb{R})$ *if and only if* $\alpha\beta < 1$*.*

The generalisation to the $d$-dimensional case can be found in [41, Proposition 10].

**Theorem 2.4.21.** *Consider* $\varphi(t) = 2^{d/4} e^{-\pi t^2}$*,* $t \in \mathbb{R}^d$*. Then the system* $\mathcal{G}(\varphi, \alpha, \beta)$ *is a frame for* $L^2(\mathbb{R}^d)$ *if and only if* $\alpha\beta < 1$*.*

### 2.4.2 Gabor frames on $\mathcal{G}$ LCA group

In what follows we generalize the notion of Gabor frame to any LCA group $\mathcal{G}$. Not every LCA group $\mathcal{G}$ admits a lattice, see definition below, such as $p$-adic groups $\mathbb{Q}_p$ ,hence we use a particular construction due to K. Gröchenig and T. Strohmer in [88]. Having Gabor frames for $L^2(\mathcal{G})$ shall enable us to generalize the convolution relations for modulation spaces firstly given in [9, Proposition 3.1], see Proposition 5.2.14 from [8].

**Definition 2.4.22.** *A* ***lattice*** *in* $\mathcal{G}$ *is a discrete subgroup* $\Lambda$ *such that the quotient group* $\mathcal{G}/\Lambda$ *is compact.*

If $\mathcal{G}$ admits a lattice, there is a relatively compact $U \subseteq \mathcal{G}$, called ***fundamental domain*** for $\Lambda$, such that

$$\mathcal{G} = \bigcup_{w \in \Lambda} (w + U)\,, \qquad (w + U) \cap (u + U) = \varnothing \quad \text{for} \quad w \neq u.$$

**Definition 2.4.23.** *Let* $D \subseteq \mathcal{G}_0$ *a collection of coset representatives of* $\mathcal{G}_0/\mathcal{K}$ *and* $A \in GL(\mathbb{R}^d)$*. We define* $U := A[0,1)^d \times \mathcal{K}$*. The discrete set* $\Lambda := A\mathbb{Z}^d \times D$ *is called* ***quasi-lattice*** *with* ***fundamental domain*** $U$*.*

Observe that we have the following partition

$$\mathcal{G} = \bigcup_{w \in \Lambda} (w + U)\,.$$

**Remark 2.4.24.** *According to the above definition and the Structure Theorem 2.3.23, a quasi-lattice on the phase-space* $\mathcal{G} \times \widehat{\mathcal{G}}$ *is of the type:*

$$\Lambda := \Lambda_1 \times \Lambda_2 := \big(A_1\mathbb{Z}^d \times D_1\big) \times \big(A_2\mathbb{Z}^d \times D_2\big) \cong A_{1,2}\mathbb{Z}^{2d} \times D_{1,2} \tag{2.129}$$

*with fundamental domain*

$$U := U_1 \times U_2 := \big(A_1[0,1)^d \times \mathcal{K}\big) \times \big(A_2[0,1)^d \times \mathcal{K}^{\perp}\big) \cong A_{1,2}[0,1)^{2d} \times \big(\mathcal{K} \times \mathcal{K}^{\perp}\big)\,, \tag{2.130}$$

*where* $D_2 \subseteq \widehat{\mathcal{G}}_0$ *is a set of coset representatives of* $\widehat{\mathcal{G}}_0/\mathcal{K}^\perp$ *and*

$$A_{1,2} := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad D_{1,2} := D_1 \times D_2. \tag{2.131}$$

We shall denote elements of a quasi-lattice $\Lambda$ in $\mathcal{G} \times \widehat{\mathcal{G}}$ as

$$\mathbf{w} = (w, \mu) = ((w_1, w_2), (\mu_1, \mu_2)) \in \Lambda = \Lambda_1 \times \Lambda_2 \subseteq \mathcal{G} \times \widehat{\mathcal{G}}.$$

**Definition 2.4.25.** *Given a quasi-lattice* $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ *and windows* $g, h \in L^2(\mathcal{G})$, *the* ***Gabor system generated by*** $g$ *is*

$$\{\pi(\mathbf{w})g \mid \mathbf{w} \in \Lambda\} = \{\pi(\mathbf{w})g\}_{\mathbf{w}\in\Lambda}.$$

*The coefficient or* ***analysis operator*** *is given by*

$$\mathcal{C}_c^\Lambda := \mathcal{C}_g \colon L^2(\mathcal{G}) \to \ell^2(\Lambda), f \mapsto (\langle f, \pi(\mathbf{w})g\rangle)_{\mathbf{w}\in\Lambda}. \tag{2.132}$$

*Its adjoint is called* ***reconstruction*** *or* ***synthesis operator*** *and has the form*

$$\mathcal{C}_g^{\Lambda *} := \mathcal{C}_g^* \colon \ell^2(\Lambda) \to L^2(\mathcal{G}), (c_{\mathbf{w}})_{\mathbf{w}\in\Lambda} \mapsto \sum_{\mathbf{w}\in\Lambda} c_{\mathbf{w}} \pi(\mathbf{w})g. \tag{2.133}$$

*The* ***Gabor frame operator*** $S_{h,g}$ *is given by*

$$S_{h,g}^\Lambda := S_{h,g} f = \mathcal{C}_h^* \mathcal{C}_g f = \sum_{\mathbf{w}\in\Lambda} \langle f, \pi(\mathbf{w})g\rangle \pi(\mathbf{w})h. \tag{2.134}$$

*We say that* $\{\pi(\mathbf{w})g\}_{\mathbf{w}\in\Lambda}$ *is a* ***Gabor frame for*** $L^2(\mathcal{G})$ *if there exist* $A, B > 0$ *such that*

$$A \|f\|_{L^2}^2 \leq \sum_{\mathbf{w}\in\Lambda} |\langle f, \pi(\mathbf{w})g\rangle|^2 \leq B \|f\|_{L^2}^2, \quad \forall f \in L^2(\mathcal{G}); \tag{2.135}$$

*this is equivalent to saying that* $S_{g,g}$ *is invertible on* $L^2(\mathcal{G})$. *If* $A = B$ *the frame is called* ***tight***. *Moreover, if* $h \in L^2(\mathcal{G})$ *is such that*

$$S_{h,g} = S_{g,h} = I_{L^2}, \tag{2.136}$$

*then* $h$ *is named* ***dual window*** *for the frame* $\{\pi(\mathbf{w})g\}_{\mathbf{w}\in\Lambda}$.

We note that Theorem 2.7 in [88] is still valid for the case of the Gaussian $\varphi$ and considering a Gabor frame not tight. Namely,

**Theorem 2.4.26.** *Let* $\Lambda := \alpha\mathbb{Z}^{2d} \times D_{1,2}$, $\alpha \in (0,1)$, *be a quasi-lattice in* $\mathcal{G} \times \widehat{\mathcal{G}}$. *Consider the Gaussian* $\varphi$ *in* (2.94). *Then*

$$\{\pi(\mathbf{w})\varphi \mid \mathbf{w} \in \Lambda\} \tag{2.137}$$

*is a Gabor frame for* $L^2(\mathcal{G})$.

#### 2.4.2.1 Gabor frames on $\mathcal{G} = \mathbb{Z}_N$

We now pick the group $\mathcal{G} = \mathbb{Z}_N$. Recall that $\widehat{\mathbb{Z}_N} \cong \mathbb{Z}_N$ and also the identification $\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$ (2.104).

**Definition 2.4.27.** *A* ***rectangular lattice in*** $\mathbb{Z}_N \times \mathbb{Z}_N$ *is a set of the following type:*

$$\Lambda := \alpha\mathbb{Z}_N \times \beta\mathbb{Z}_N, \quad \alpha, \beta \in \mathbb{N}, \quad A := \frac{N}{\alpha} \in \mathbb{N}, \quad B := \frac{N}{\beta} \in \mathbb{N}. \tag{2.138}$$

Notice that, since $\alpha, \beta$ are divisors of $N$, $\Lambda$ is indeed a (discrete) subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ such that the quotient is compact. Therefore in this case we no need quasi-lattices.

**Definition 2.4.28.** *The* ***Gabor system generated by a window*** $g \in \mathbb{C}^N$ ***and lattice*** $\Lambda$ *as in* (2.138) *is the set*

$$\begin{aligned} \mathcal{G}(g, \alpha, \beta) &:= \{\pi(k, l)g \, , \, (k, l) \in \Lambda\} \\ &= \{\pi(\alpha k, \beta l)g \, , \, k = 0, \dots, A-1, \, l = 0, \dots, B-1\}. \end{aligned}$$

*We say that* $\mathcal{G}(g, \alpha, \beta)$ *is a* ***Gabor frame for*** $\mathbb{C}^N$ *if*

$$C_1 \|f\|_2^2 \le \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} |\langle f, \pi(\alpha k, \beta l)g\rangle|^2 \le C_2 \|f\|_2^2 \, , \qquad \forall f \in \mathbb{C}^N, \tag{2.139}$$

*for some* $C_1, C_2 > 0$.

We point out that in the above equation

$$\langle f, \pi(\alpha k, \beta l)g\rangle = \sum_{u=0}^{N-1} f(u)\overline{\pi(\alpha k, \beta l)g(u)}$$

and $\|\cdot\|_2$ is the induced norm.

**Remark 2.4.29.** *Since we are in finite-dimension, to ask* $\mathcal{G}(g, \alpha, \beta)$ *to be a frame for* $\mathbb{C}^N$ *is equivalent to ask that it spans* $\mathbb{C}^N$ *[21], where the bounds* $C_1, C_2$ *describe the numerical properties of the transform and the quantity* $\sqrt{C_2/C_1}$ *is the condition number of the analysis, see [5].*

## 2.5 Modulation spaces

The core of this section are function spaces which norm aims to measure the time-frequency concentration of functions or tempered distribution, namely the class of *modulation spaces*. These normed spaces were introduced by H. G. Feichtinger in early 1980's.

We want to investigate how to measure the time-frequency concentration of a function or a distribution. We define a function space whose elements share the same decay properties in the time-frequency plane. To reach this goal, we impose a norm on the SFTF and thus we get the so-called modulation spaces.

In order to remain within the setup of Schwartz functions and tempered distributions, we will present the theory of modulation spaces under the following assumptions:

**Assumptions 2.5.1.** *Every weight on* $\mathbb{R}^d$, $\mathbb{R}^{2d}$, *or on any of their subgroups, is assumed to have at most polynomial growth at infinity (see* (2.8)*).*

For sake of clarity, we recall that this means there exist $s_0, C > 0$ such that

$$m(z) \leq C\langle z\rangle^{s_0} = Cv_{s_0}(z), \tag{2.140}$$

for $z$ in $\mathbb{R}^d$, $\mathbb{R}^{2d}$, etc. .

Assumptions 2.5.1, even if not explicitly stated, are made through all the present thesis work except for: Section 2.8, Section 3.3 of Chapter 3. For more details and the case of other weight classes, namely exponential ones, we refer to [82, Chapter 11].

We shall first define modulation spaces on $\mathbb{R}^d$ in the wider quasi-Banach setting, whereas for modulation spaces on $\mathcal{G}$ LCA group in this section we stick to the Banach case. The quasi-Banach case on $\mathcal{G}$ shall be one of the main result of Chapter 5, see [8] by E. Cordero and the author. On $\mathbb{R}^d$, we prove some new convolution relations due to E. Cordero, F. Nicola and the author [9], see Proposition 2.5.19. This result will be extended to any LCA group in Chapter 5 [8], see Proposition 5.2.14. See Proposition 2.5.20, 2.5.21 and Corollary 2.5.22 are about inclusion relations and an equivalent semi-discrete quasi-norm on $M^{p,q}_m(\mathbb{R}^d)$, see [7]. Proposition 2.5.23 was presented in [3].

### 2.5.1 $M^{p,q}_m$ on Euclidean space, $0 < p, q \leq \infty$

**Definition 2.5.2.** *Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a weight $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $0 < p, q \leq \infty$. We define the* ***modulation space*** *as the following set of tempered distributions:*

$$M^{p,q}_m(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \quad \middle| \quad V_g f \in L^{p,q}_m(\mathbb{R}^{2d}) \right\}. \tag{2.141}$$

For shortness, we write $M^p_m(\mathbb{R}^d)$ in place of $M^{p,p}_m(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^d)$ if $m \equiv 1$. Due to Remark 2.2.8 it's legit to consider $v = v_s$, even if $v_s$ is not submultiplicative for $s \geq 0$.

**Remark 2.5.3.** *Roughly speaking, $M^{p,q}(\mathbb{R}^d)$ contains all those generalised functions such that they behave locally like elements of $\mathcal{F}L^q(\mathbb{R}^d)$ and "decay" like elements of $L^p(\mathbb{R}^d)$ at infinity.*

The most famous modulation spaces are those $M^{p,q}_m(\mathbb{R}^d)$ with $1 \leq p, q \leq \infty$, invented by H. G. Feichtinger in [56]. In that paper he proved they are Banach spaces, whose norm does not depend on the window $g$, in the sense that different window functions in $\mathcal{S}(\mathbb{R}^d)$ yield equivalent norms. Moreover, the window class $\mathcal{S}(\mathbb{R}^d)$ can be extended to the modulation space $M^{1,1}_v(\mathbb{R}^d)$ (so-called weighted Feichtinger algebra). The modulation spaces $M^{p,q}_m(\mathbb{R}^d)$, with $0 < p < 1$ or $0 < q < 1$, were introduced almost twenty years later by Y.V. Galperin and S. Samarah in [75] and then studied in [104, 119, 149] (see also references therein). In this framework, it appears that the largest natural class of windows universally admissible for all spaces $M^{p,q}_m(\mathbb{R}^d)$, $0 < p, q \leq \infty$, (with $m$ having at most polynomial growth) is the Schwartz class $\mathcal{S}(\mathbb{R}^d)$.

**Proposition 2.5.4.** *Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a weight $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $0 < p, q \leq \infty$. Then the application*

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m}, \qquad \forall\, f \in M^{p,q}_m(\mathbb{R}^d),$$

*is a quasi-norm on $M^{p,q}_m(\mathbb{R}^d)$, it is a norm if $1 \leq p, q \leq \infty$.*

From now on we omit the explicit choice of the window function according to the equivalence above.

**Theorem 2.5.5.** *Fix a weight $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $0 < p, q \leq \infty$.*
*Then $(M^{p,q}_m(\mathbb{R}^d), \|\cdot\|_{M^{p,q}_m})$ is a quasi-Banach space. It is a Banach space if $1 \leq p, q \leq \infty$.*

For the following important result we refer to [35, Theorem 2.4.17] or [75, Theorem 3.4].

**Theorem 2.5.6.** *Consider* $m_1, m_2 \in \mathcal{M}_v(\mathbb{R}^{2d})$ *with* $m_2 \lesssim m_1$. *Take* $0 < p_1 \leq p_2 \leq \infty$ *and* $0 < q_1 \leq q_2 \leq \infty$. *Then we have the following continuous inclusion:*

$$M_m^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_m^{p_2,q_2}(\mathbb{R}^d). \tag{2.142}$$

As a consequence, we can extend the result [82, Corollary 11.1.10] which is implicitly used in Theorem 2.5.18.

**Corollary 2.5.7.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. *Take* $0 < p, q \leq \infty$. *Then*

$$M_m^{p,q}(\mathbb{R}^d) \subseteq M_{1/v}^{\infty}(\mathbb{R}^d). \tag{2.143}$$

**Proposition 2.5.8.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ *and* $1 \leq p, q < \infty$. *Then*

$$(M_m^{p,q})'(\mathbb{R}^d) = M_{1/m}^{p',q'}(\mathbb{R}^d),$$

*where* $p'$ *and* $q'$ *are the conjugate exponents and*

$$_{M_{1/m}^{p',q'}}\langle f, g\rangle_{M_m^{p,q}} := \int_{\mathbb{R}^{2d}} V_h f(z)\overline{V_h g(z)}\, dz, \tag{2.144}$$

*for every* $h \in M_v^1(\mathbb{R}^d) \smallsetminus \{0\}$, $f \in M_{1/m}^{p',q'}(\mathbb{R}^d)$ *and* $g \in M_m^{p,q}(\mathbb{R}^d)$.

We shall simply write

$$\langle f, g\rangle = {}_{M_{1/m}^{p',q'}}\langle f, g\rangle_{M_m^{p,q}}.$$

As a consequence, we have a Hölder's inequality for modulation spaces.

**Corollary 2.5.9.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $1 \leq p, q \leq \infty$ *and let* $p'$ *and* $q'$ *be the conjugate exponents. Then*

$$|\langle f, g\rangle| \leq \|f\|_{M_w^{p,q}} \|g\|_{M_{1/w}^{p',q'}}, \quad f \in M_w^{p,q}(\mathbb{R}^d), g \in M_{1/w}^{p',q'}(\mathbb{R}^d). \tag{2.145}$$

*for every* $f \in M_{1/m}^{p',q'}(\mathbb{R}^d)$ *and* $g \in M_m^{p,q}(\mathbb{R}^d)$.

The duality properties for modulation spaces with indices $p < 1$ or $q < 1$ where studied in [105] and completed in [149, Proposition 6.4, page 163].

**Proposition 2.5.10.** *Let* $s \in \mathbb{R}$ *and* $0 < p, q < \infty$. *If* $p \geq 1$ *we denote by* $p'$ *the conjugate exponent of* $p$, *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1;$$

*if* $0 < p < 1$ *we set* $p' := \infty$. *Similarly for* $q$. *Then*

$$(M_{1\otimes v_s}^{p,q}(\mathbb{R}^d))' = M_{1\otimes v_{-s}}^{p',q'}(\mathbb{R}^d). \tag{2.146}$$

**Proposition 2.5.11.** ([82, Proposition 11.3.2, Theorem 11.3.7])
*Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. *Assume* $g \in M_v^1(\mathbb{R}^d) \smallsetminus \{0\}$, $1 \leq p, q \leq \infty$. *Then every* $f \in M_m^{p,q}(\mathbb{R}^d)$ *can be written as vector-valued integral in a weak sense as follows:*

$$f = \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x g\, dx d\omega, \tag{2.147}$$

*and the equality holds in* $M_m^{p,q}(\mathbb{R}^d)$.

**Remark 2.5.12.** *The above vector-valued integral has to be interpreted as follows: if* $f \in M_m^{p,q}(\mathbb{R}^d)$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^d)$*, then*

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^{2d}} V_g f(x,\omega) \langle M_\omega T_x g, \varphi \rangle \, dx d\omega = \int_{\mathbb{R}^{2d}} V_g f(x,\omega) \overline{V_g \varphi(x,\omega)} \, dx d\omega.$$

In the next proposition we need the subsequent polynomial type weights:

$$\tau_s(x,\omega) := \langle \omega \rangle^s = (1+|\omega|^2)^{s/2}, \qquad \forall x, \omega \in \mathbb{R}^d;$$
$$\mu_s(x,\omega) := \langle x \rangle^s = (1+|x|^2)^{s/2}, \qquad \forall x, \omega \in \mathbb{R}^d.$$

**Proposition 2.5.13.** *Fix two indices* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$*. Among the modulation spaces the following function spaces occur:*

(*i*) $L^2$*-spaces:*

$$M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d);$$

(*ii*) *weighted* $L^2$*-spaces:*

$$M^2_{\mu_s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) \mid f(x)\langle x \rangle^s \in L^2(\mathbb{R}^d) \};$$

(*iii*) *Sobolev spaces:*

$$M^2_{\tau_s}(\mathbb{R}^d) = H^s(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) \mid \hat{f}(\omega)\langle \omega \rangle^s \in L^2(\mathbb{R}^d) \};$$

(*iv*) *Shubin-Sobolev spaces:*

$$M^2_{v_s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = Q_s(\mathbb{R}^d); \tag{2.148}$$

(*v*) *spaces* $L^p_{v_s}$*:*

$$L^p_{v_s}(\mathbb{R}^d) \subseteq M^{p,\infty}_{v_s \otimes 1}(\mathbb{R}^d);$$

(*vi*) *the Schwartz class:*

$$\bigcap_{s \geq 0} M^{p,q}_{v_s}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d);$$

(*vii*) *space of tempered distributions:*

$$\bigcap_{s \geq 0} M^{p,q}_{1/v_s}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d).$$

Roughly speaking a weight in $\omega$ regulates the smoothness of $f \in M_m^{p,q}$, whereas a weight in $x$ regulates the decay of $f \in M_m^{p,q}$.

**Remark 2.5.14.** *We remark that*

$$\mathcal{S}(\mathbb{R}^d) \subsetneq \bigcap_{p>0} M^p(\mathbb{R}^d).$$

*The idea in order to prove the previous strict inclusion is the following. The problem is transported to the discrete case: if* $f \in \mathcal{S}(\mathbb{R}^d)$*, then* $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$ *and* $(V_g f(\lambda))_{\lambda \in \Lambda} \in \ell^\infty_{v_s}(\Lambda)$ *for every* $s > 0$*, where* $\Lambda \subseteq \mathbb{R}^{2d}$ *is a suitable lattice. We have the following inclusion*

$$\bigcap_{s>0} \ell^\infty_{v_s}(\Lambda) \subseteq \bigcap_{p>0} \ell^p(\Lambda)$$

*which can be proved to be strict by constructing a suitable sequence which is in* $\bigcap_{p>0} \ell^p$ *but is not rapidly decreasing, i.e. is not an element of* $\bigcap_{s>0} \ell^\infty_{v_s}$*.*

For proofs of the following density result we refer to [82, Proposition 11.3.4] for the Banach case, to [75, Remark 14] for the quasi-Banach one.

**Theorem 2.5.15.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ *(having at most polynomial growth) and* $0 < p, q < \infty$*. Then* $\mathcal{S}(\mathbb{R}^d)$ *is a dense subspace of* $M^{p,q}_m(\mathbb{R}^d)$*.*

Modulation spaces provide a natural setting for time-frequency analysis, thanks to discrete equivalent norms produced by means of Gabor frames. The key result will be Theorem 2.5.18 (see [82, Corollary 12.2.6] for $1 \leq p, q \leq \infty$, and [75, Theorem 3.7] for $0 < p, q < 1$). In order to state it, we briefly introduce the analysis and synthesis operators on modulation spaces.
Consider $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. Fix $\alpha, \beta > 0$, then we denote the restriction of $m$ to the lattice $\Lambda \coloneqq \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ as follows:

$$m_\Lambda(k, n) \coloneqq m(\alpha k, \beta n), \qquad \forall k, n \in \mathbb{Z}^d,$$

similarly for $v$.
The next proposition summarizes definition and well-posedness of the ***analysis***, or ***coefficient***, ***operator*** $\mathcal{C}^{\alpha,\beta}_g$ for modulation spaces. For the Banach case we refer to [82, Theorem 12.2.3], for the quasi-Banach case see [75, Theorem 3.5].

**Proposition 2.5.16.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$*. Fix* $g \in \mathcal{S}(\mathbb{R}^d)$*,* $\alpha, \beta > 0$ *and* $0 < p, q \leq \infty$*. Then the analysis operator*

$$\begin{aligned} \mathcal{C}^{\alpha,\beta}_g \colon M^{p,q}_m(\mathbb{R}^d) &\longrightarrow \ell^{p,q}_{m_\Lambda}(\mathbb{Z}^{2d}) \\ f &\longmapsto (\langle f, \pi(\alpha k, \beta n) g\rangle)_{k,n\in\mathbb{Z}^d} \end{aligned} \tag{2.149}$$

*is well defined and bounded.*

We establish an analogue result for the ***synthesis***, or ***reconstruction***, ***operator*** which summarises [82, Theorem 12.2.4] and [75, Theorem 3.6].

**Proposition 2.5.17.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$*. Fix* $g \in \mathcal{S}(\mathbb{R}^d)$*,* $\alpha, \beta > 0$ *and* $0 < p, q \leq \infty$*. Then the synthesis operator*

$$\begin{aligned} \mathcal{D}^{\alpha,\beta}_g \colon \ell^{p,q}_{m_\Lambda}(\mathbb{Z}^{2d}) &\longrightarrow M^{p,q}_m(\mathbb{R}^d) \\ (c_{k,n})_{k,n\in\mathbb{Z}^d} &\longmapsto \sum_{k,n\in\mathbb{Z}^d}^{\infty} c_{k,n}\pi(\alpha k, \beta n) g \end{aligned} \tag{2.150}$$

*is well defined and bounded. Moreover, if* $p, q < \infty$ *then* $\mathcal{D}^{\alpha,\beta}_g c$ *converges unconditionally in* $M^{p,q}_m(\mathbb{R}^d)$*, otherwise* $\mathcal{D}^{\alpha,\beta}_g c$ *converges weak-∗ in* $M^\infty_{1/v}(\mathbb{R}^d)$*.*

Then we define

$$S^{\alpha,\beta}_{g,\gamma} \coloneqq \mathcal{D}^{\alpha,\beta}_\gamma \mathcal{C}^{\alpha,\beta}_g, \tag{2.151}$$

which can be seen defined on $L^2(\mathbb{R}^d)$ or $M^{p,q}_m(\mathbb{R}^d)$, according to the context.

**Theorem 2.5.18.** *Consider* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$*. Fix* $\alpha, \beta > 0$*,* $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ *such that* $S^{\alpha,\beta}_{g,\gamma} = I$ *on* $L^2(\mathbb{R}^d)$*. Define* $\Lambda \coloneqq \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$*. Then*

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda) g, \quad \forall\, f \in M^{p,q}_m(\mathbb{R}^d), \tag{2.152}$$

*with unconditional convergence in* $M^{p,q}_m(\mathbb{R}^d)$ *if* $0 < p, q < \infty$ *and with weak-$*$ convergence in* $M^\infty_{1/v}(\mathbb{R}^d)$ *otherwise. Furthermore, there are constants* $0 < A \le B$ *such that, for all* $f \in M^{p,q}_m(\mathbb{R}^d)$*,*

$$A\|f\|_{M^{p,q}_m} \le \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}|\langle f,\pi(\alpha k,\beta n)g\rangle|^p m(\alpha k,\beta n)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \le B\|f\|_{M^{p,q}_m}, \tag{2.153}$$

*independently of* $p, q$*, and* $m$*. Similar inequalities hold with* $g$ *replaced by* $\gamma$*.*

Then the above theorem can be summarised as:

$$\|f\|_{M^{p,q}_m(\mathbb{R}^d)} \asymp \|(\langle f,\pi(\lambda)g\rangle)_\lambda\|_{\ell^{p,q}_{\tilde{m}}(\Lambda)} = \|(V_g f(\lambda))_\lambda\|_{\ell^{p,q}_{\tilde{m}}(\Lambda)}. \tag{2.154}$$

We present new convolution relations for modulations spaces proved in [9]. Let us recall that, for the Banach cases, convolution relations were studied in [28] and [136, 137]. The approach used in [9] is general, the techniques use Gabor frames via the equivalence (2.153), plus Hölder's and Young's inequalities for sequences.

**Proposition 2.5.19.** ([9, Proposition 3.1])
*Let* $\nu$ *be weight on* $\mathbb{R}^d$ *and consider* $0 < p, q, r, t, u, \gamma \le \infty$ *such that:*

$$\frac{1}{u} + \frac{1}{t} = \frac{1}{\gamma}, \tag{2.155}$$

*and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \qquad \textit{for } 1 \le r \le \infty, \tag{2.156}$$

*whereas*

$$p = q = r \qquad \textit{for } 0 < r < 1. \tag{2.157}$$

*Fix* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ *and define the restrictions to* $\mathbb{R}^d \times \{0\}$*,* $m_1$ *and* $v_1$*, and to* $\{0\} \times \mathbb{R}^d$*,* $m_2$ *and* $v_2$*, as in* (2.10)*. Then we have the following continuous inclusion:*

$$M^{p,u}_{m_1\otimes\nu}(\mathbb{R}^d) * M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d) \hookrightarrow M^{r,\gamma}_m(\mathbb{R}^d), \tag{2.158}$$

*i.e. for any* $f \in M^{p,u}_{m_1\otimes\nu}(\mathbb{R}^d)$ *and* $h \in M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d)$

$$\|f * h\|_{M^{r,\gamma}_m} \lesssim \|f\|_{M^{p,u}_{m_1\otimes\nu}}\|h\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}}.$$

*Proof.* We use the key idea in [28, Proposition 2.4] to measure the modulation space norm with respect to the Gaussian windows $g_0(x) = e^{-\pi x^2}$ and $g(x) = 2^{-d/2}e^{-\pi x^2/2} = (g_0 * g_0)(x) \in \mathcal{S}(\mathbb{R}^d)$. It is useful noting that $M_\omega(g_0^*) = (M_\omega g_0)^*$, similarly for $g$. A straightforward computation shows

$$V_g f(x,\omega) = e^{-2\pi i x\cdot\omega}(f * (M_\omega g)^*)(x),$$

where actually $g$ can be any window in $L^2(\mathbb{R}^d)$. Consider now $f \in M^{p,u}_{m_1\otimes\nu}(\mathbb{R}^d)$ and $h \in M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d)$, using the identity $M_\omega(g_0^* * g_0^*) = M_\omega g_0^* * M_\omega g_0^*$ we can write the STFT of $f * h$ as follows:

$$\begin{aligned} V_g(f * h)(x,\omega) &= e^{-2\pi i x\cdot\omega}\big((f * h) * (M_\omega g)^*\big)(x) \\ &= e^{-2\pi i x\cdot\omega}\big((f * M_\omega g_0^*) * (h * M_\omega g_0^*)\big)(x). \end{aligned}$$

In the following, we first use the norm equivalence (2.153), written in terms of the STFT as $\|\cdot\|_{M^{r,\gamma}_m} \asymp \|(V_g \cdot (\lambda))_{\lambda\in\Lambda}\|_{\ell^{r,\gamma}_m(\Lambda)}$, where $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. Then we majorize $m$ by

$$m(\alpha k, \beta n) \lesssim m(\alpha k, 0)v(0, \beta n) = m_1(\alpha k)v_2(\beta n),$$

and finally use Young's convolution inequality for sequences (Theorem 2.2.32) in the $k$-variable and Hölder's one (Theorem 2.2.33) in the $n$-variable. The indices $p, q, r, \gamma, t, u$ fulfil the equalities in the assumptions. We show in details the case when $r, \gamma, t, u < \infty$:

$$\begin{aligned}
\|f * h\|_{M^{r,\gamma}_m} &\asymp \|((V_g(f * h))(\alpha k, \beta n)m(\alpha k, \beta n))_{k,n}\|_{\ell^{r,\gamma}_m(\mathbb{Z}^{2d})}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}|(f * M_{\beta n}g_0^*) * (h * M_{\beta n}g_0^*)(\alpha k)|^r m_1(\alpha k)^r\right)^{\gamma/r} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&= \left(\sum_{n\in\mathbb{Z}^d}\|(f * M_{\beta n}g_0^*) * (h * M_{\beta n}g_0^*)\|^\gamma_{\ell^r_{m_1}(\alpha\mathbb{Z}^d)} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\|f * M_{\beta n}g_0^*\|^\gamma_{\ell^p_{m_1}(\alpha\mathbb{Z}^d)}\,\|h * M_{\beta n}g_0^*\|^\gamma_{\ell^q_{v_1}(\alpha\mathbb{Z}^d)}\, v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\|f * M_{\beta n}g_0^*\|^u_{\ell^p_{m_1}(\alpha\mathbb{Z}^d)}\nu(\beta n)^u\right)^{\frac{1}{u}}\left(\sum_{n\in\mathbb{Z}^d}\|h * M_{\beta n}g_0^*\|^t_{\ell^q_{v_1}(\alpha\mathbb{Z}^d)}\frac{v_2(\beta n)^t}{\nu(\beta n)^t}\right)^{\frac{1}{t}}\\
&= \|((V_{g_0}f)(\lambda))_\lambda\|_{\ell^{p,u}_{m_1\otimes\nu}(\Lambda)}\,\|((V_{g_0}h)(\lambda))_\lambda\|_{\ell^{q,t}_{v_1\otimes v_2\nu^{-1}}(\Lambda)}\\
&\asymp \|f\|_{M^{p,u}_{m_1\otimes\nu}}\,\|h\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}},
\end{aligned}$$

where we wrote $m_1 \otimes \nu$ and $v_1 \otimes v_2\nu^{-1}$ instead of $(m_1 \otimes \nu)_\Lambda$ and $(v_1 \otimes v_2\nu^{-1})_\Lambda$, and so on. This concludes the proof. The cases when one among the indexes $r, \gamma, t, u$ is equal to $\infty$ are done similarly. □

We need to introduce an alternative definition of modulation spaces we shall use in the sequel. For $k \in \mathbb{Z}^d$, we denote by $\mathcal{Q}_k$ the unit closed cube centred at $k$. The family $\{\mathcal{Q}_k\}_{k\in\mathbb{Z}^d}$ is a covering of $\mathbb{R}^d$. We define $|\xi|_\infty := \max_{i=1,\dots,d}|\xi_i|$, for $\xi \in \mathbb{R}^d$. Consider now a smooth function $\rho : \mathbb{R}^d \to [0, 1]$ satisfying $\rho(\xi) = 1$ for $|\xi|_\infty \le 1/2$ and $\rho(\xi) = 0$ for $|\xi|_\infty \ge 3/4$. Define

$$\rho_k(\xi) := T_k\rho(\xi) = \rho(\xi - k), \quad \forall k \in \mathbb{Z}^d, \tag{2.159}$$

that is, $\rho_k$ is the translation of $\rho$ at $k$. By the assumption on $\rho$, we infer that $\rho_k(\xi) = 1$ for $\xi \in \mathcal{Q}_k$ and

$$\sum_{k\in\mathbb{Z}^d}\rho_k(\xi) \ge 1, \quad \forall\, \xi \in \mathbb{R}^d.$$

Denote by

$$\sigma_k(\xi) := \frac{\rho_k(\xi)}{\sum_{l\in\mathbb{Z}^d}\rho_l(\xi)}, \quad \forall\xi \in \mathbb{R}^d,\ k \in \mathbb{Z}^d. \tag{2.160}$$

Observe that $\sigma_k(\xi) = \sigma_0(\xi - k) \in \mathcal{D}(\mathbb{R}^d)$ and the sequence $\{\sigma_k\}_{k\in\mathbb{Z}^d}$ is a smooth partition of unity

$$\sum_{k\in\mathbb{Z}^d}\sigma_k(\xi) = 1, \quad \forall\xi \in \mathbb{R}^d.$$

For $k \in \mathbb{Z}^d$, we define the ***frequency-uniform decomposition operator*** by

$$\Box_k := \mathcal{F}^{-1}\sigma_k\mathcal{F}. \tag{2.161}$$

The previous operators allow to introduce an alternative quasi-norm on the weighted modulation spaces $M^{p,q}_{h\otimes w}(\mathbb{R}^d)$ inspired by [148] as follows.

**Proposition 2.5.20.** ([7, Proposition 2.2]) *For $0 < p, q \le \infty$, $h, w \in \mathcal{M}_v(\mathbb{R}^d)$ have*

$$\|f\|_{M^{p,q}_{h\otimes w}(\mathbb{R}^d)} \asymp \left(\sum_{k\in\mathbb{Z}^d} \|\Box_k f\|^q_{L^p_h} w(k)^q\right)^{\frac{1}{q}}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \tag{2.162}$$

*with obvious modification for $q = \infty$.*

*Proof.* The case $p, q \ge 1$ is well known, see for example [35, Proposition 2.3.25]. The cases $0 < p < 1$ or $0 < q < 1$ are an easy modification of that proof. Namely, let us point out the main changes. If $0 < p \le 1$, we consider

$$\Box_k f = \mathcal{F}^{-1}\sigma_k\mathcal{F}f = \mathcal{F}^{-1}\sigma_k T_\xi\bar{\hat{\phi}}\mathcal{F}f, \quad \text{for} \quad \xi \in \mathcal{Q}_k,$$

since $T_\xi\bar{\hat{\phi}} = 1$ in supp $\sigma_k$ for $\xi \in \mathcal{Q}_k$. Using Young's inequality for distributions compactly supported in the frequencies (see [104, Lemma 2.6], which holds also for $L^p_h$, $0 < p \le 1$, with $h$ being $v$-moderate), for $\xi \in \mathcal{Q}_k$, we obtain

$$\|\Box_k f\|_{L^p_h} \lesssim \|\mathcal{F}^{-1}\sigma_k\|_{L^p_v}\|\mathcal{F}^{-1}T_\xi\bar{\hat{\phi}}\mathcal{F}f\|_{L^p_h} \lesssim \|\mathcal{F}^{-1}T_\xi\bar{\hat{\phi}}\mathcal{F}f\|_{L^p_h}.$$

The rest of the proof is analogous to the Banach case and we leave the details to the interested reader. □

An useful embedding is contained in what follows.

**Proposition 2.5.21.** ([7, Proposition 2.3]) *Given $0 < p_1, p_2, q_1, q_2 \le \infty$, with $m, s_1, s_2$ in $\mathbb{R}$, one has*

$$M^{p_1,q_1}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^{s_1}}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^{s_2}}(\mathbb{R}^d) \tag{2.163}$$

*if and only if*

$$p_1 \le p_2 \tag{2.164}$$

*and*

$$q_1 \le q_2,\ s_1 \ge s_2 \qquad \text{or} \qquad q_1 > q_2,\quad \frac{s_1}{d} + \frac{1}{q_1} > \frac{s_2}{d} + \frac{1}{q_2}. \tag{2.165}$$

*Proof.* The Banach case when $m = 0$ was originally shown by H. G. Feichtinger in [56]. We use similar arguments as in that proof. The discrete modulation norm defined in (2.162) is given by

$$\|f\|_{M^{p,q}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^s}} \asymp \left(\sum_{k\in\mathbb{Z}^d} \|\Box_k f\|^q_{L^p_{\langle\cdot\rangle^m}} \langle k\rangle^{sq}\right)^{\frac{1}{q}}.$$

The necessity of (2.164) follows from the fact that $\mathcal{F}L^{p_1}$ is locally contained in $\mathcal{F}L^{p_2}$ if and only if $p_1 \le p_2$ (with strict inclusion if $p_1 < p_2$), cf. [16, 71, 104, 145]. The set of conditions in (2.165) in turn describes the inclusions between weighted $\ell^q$ spaces: $\ell^{q_1}_{\langle\cdot\rangle^{s_1}} \subseteq \ell^{q_2}_{\langle\cdot\rangle^{s_2}}$ if and only if the indices' relations in (2.165) are satisfied, cf. for instance [91, Lemma 2.10]. This concludes the proof. □

**Corollary 2.5.22.** ([7, Corollary 2.4]) *For* $0 < q_1 \leq q_2 \leq \infty$*,* $d \in \mathbb{N}$*,* $m, s, r \in \mathbb{R}$*,* $r > s + d(1/q_1 - 1/q_2)$*, we have the following continuous embeddings:*

$$M^{\infty,q_1}_{\langle\cdot\rangle^m \otimes \langle\cdot\rangle^r}(\mathbb{R}^d) \hookrightarrow M^{\infty,q_2}_{\langle\cdot\rangle^m \otimes \langle\cdot\rangle^r}(\mathbb{R}^d) \hookrightarrow M^{\infty,q_1}_{\langle\cdot\rangle^m \otimes \langle\cdot\rangle^s}(\mathbb{R}^d). \tag{2.166}$$

*Proof.* The first embedding is a straightforward application of the inclusion relations in (2.142). The second one follows by the embedding in Proposition 2.5.21. □

Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. We denote by $\mathcal{M}^{p,q}_m(\mathbb{R}^d)$ the closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M^{p,q}_m$-norm:

$$\mathcal{M}^{p,q}_m(\mathbb{R}^d) \coloneqq \overline{\mathcal{S}(\mathbb{R}^d)}^{\|\cdot\|_{M^{p,q}_m}}. \tag{2.167}$$

Observe that $\mathcal{M}^{p,q}_m(\mathbb{R}^d) = M^{p,q}_m(\mathbb{R}^d)$, whenever the indices $p$ and $q$ are finite. Notice that these spaces enjoy the duality property

$$(\mathcal{M}^{p,q}_m)' = \mathcal{M}^{p',q'}_{1/m} \qquad \text{with} \qquad 1 \leq p, q \leq \infty.$$

**Proposition 2.5.23.** ([3, Proposition 2.2]) *Consider* $1 \leq p, q \leq \infty$*, with* $p', q'$ *being conjugate exponents of* $p, q$*, respectively.*

- *(i) For* $1 \leq p, q \leq \infty$*,* $f \in \mathcal{M}^{p,q}(\mathbb{R}^d)$*,* $h \in \mathcal{M}^{p',q'}(\mathbb{R}^d)$*, we have that* $f * h \in C_0(\mathbb{R}^d)$*;*
- *(ii) For* $1 < p, q < \infty$*,* $f \in M^{p,q}(\mathbb{R}^d)$*,* $h \in M^{p',q'}(\mathbb{R}^d)$*, we have that* $f * h \in C_0(\mathbb{R}^d)$*;*
- *(iii) If either* $f \in M^{\infty,1}(\mathbb{R}^d)$ *and* $h \in M^{1,\infty}(\mathbb{R}^d)$ *or* $f \in M^1(\mathbb{R}^d)$ *and* $h \in M^\infty(\mathbb{R}^d)$*, then* $f * h \in C_b(\mathbb{R}^d)$*.*

*Proof.* These results are well known, see [53] and [54]. For sake of clarity we provide a direct proof.

*(i)* Using the density of $\mathcal{S}(\mathbb{R}^d)$ in both spaces we can find sequences $\{f_n\}_n, \{h_n\}_n \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f_n - f\|_{M^{p,q}} \to 0$ and $\|h_n - h\|_{M^{p',q'}} \to 0$, now $f_n * h_n \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ so that, using

$$|f * h(t)| = |\langle f, \overline{T_t \mathcal{I}(h)}\rangle| \leq \|f\|_{M^{p,q}} \|\overline{T_t \mathcal{I}(h)}\|_{M^{p',q'}} = \|f\|_{M^{p,q}} \|h\|_{M^{p',q'}}, \quad \forall t \in \mathbb{R}^d,$$

$$\begin{aligned}\|f_n * h_n - f * h\|_{L^\infty} &\leq \|f_n * (h_n - h)\|_{L^\infty} + \|(f_n - f) * h\|_{L^\infty} \\ &\leq \|f_n\|_{M^{p,q}} \|h_n - h\|_{M^{p',q'}} + \|f_n - f\|_{M^{p,q}} \|h\|_{M^{p',q'}}.\end{aligned}$$

Hence $f * h \in C_0(\mathbb{R}^d)$. Item *(ii)* is obtained by the same argument as in *(i)*.

*(iii)* Using the convolution relations of Proposition 2.5.19 we infer

$$M^{\infty,1}(\mathbb{R}^d) * M^{1,\infty}(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d) \text{ and } M^1(\mathbb{R}^d) * M^\infty(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d).$$

It follows immediately from the definition of the modulation space $M^{\infty,1}(\mathbb{R}^d)$ that

$$M^{\infty,1}(\mathbb{R}^d) \subseteq (\mathcal{F}L^1(\mathbb{R}^d))_{loc} \cap L^\infty(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d) \tag{2.168}$$

and we are done. □

**Remark 2.5.24.** *We observe that the convolution relations*

$$M^1(\mathbb{R}^d) * M^\infty(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$$

*where already shown in [65, Lemma 8].*

### 2.5.2 $M^{p,q}_m$ on $\mathcal{G}$ LCA group, $1 \le p, q \le \infty$

Since several descriptions are available for modulation spaces on $\mathcal{G}$ LCA group, we chose the following which can be found e.g. in [88]. We shall summarize briefly their main properties which corresponds to the ones for $M^{p,q}_m(\mathbb{R}^d)$.

**Definition 2.5.25.** *Let $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, consider $1 \le p, q \le \infty$ and let $\varphi$ be as in (2.94). For $f \in \mathcal{S}_\mathcal{C}(\mathcal{G})$ we define the application*

$$\|f\|_{M^{p,q}_m} := \|V_\varphi f\|_{L^{p,q}_m} = \left( \int_{\widehat{\mathcal{G}}} \left( \int_{\mathcal{G}} |V_\varphi f(x,\xi)|^p \, m(x,\xi)^p \, dx \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{q}}, \tag{2.169}$$

*with obvious modifications if at least one between p and q is $\infty$. Then we define the* ***modulation space*** *$M^{p,q}_m(\mathcal{G})$ to be the completion $\mathcal{S}_\mathcal{C}(\mathcal{G})$ w.r.t. $\|\cdot\|_{M^{p,q}_m}$ if $p, q < \infty$:*

$$M^{p,q}_m(\mathcal{G}) := \overline{\mathcal{S}_\mathcal{C}(\mathcal{G})}^{\|\cdot\|_{M^{p,q}_m}},$$

*if at least one between p and q is $\infty$ we take the w-$*$ closure.*

As usual, we adopt the notation $M^{p,q} := M^{p,q}_1$ and $M^p_m := M^{p,p}_m$.

**Theorem 2.5.26.** *Let $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $1 \le p, q \le \infty$. Then $M^{p,q}_m(\mathcal{G})$ endowed with $\|\cdot\|_{M^{p,q}_m}$ defined in (2.169) is a Banach space. Moreover, it is independent of the window function in $M^1_v(\mathcal{G}) \smallsetminus \{0\}$, in the sense that given any $g \in M^1_v(\mathcal{G}) \smallsetminus \{0\}$ the application $\|V_g \cdot\|_{L^{p,q}_m}$ is a norm on $M^{p,q}_m(\mathcal{G})$ equivalent to the one defined in (2.169).*

**Remark 2.5.27.** (*i*) *We want to highlight the following inclusion*

$$\mathcal{S}_\mathcal{C}(\mathcal{G}) \subseteq M^1_v(\mathcal{G}).$$

*So that every nonzero function in $\mathcal{S}_\mathcal{C}(\mathcal{G})$ is a suitable window for $M^{p,q}_m(\mathcal{G})$;*

(*ii*) *Taking $\mathcal{G} = \mathbb{R}^d$ we recover the very same modulation spaces which were introduced in Definition 2.5.2.*

**Proposition 2.5.28.** (*i*) *Let $1 \le p, q < \infty$ and consider the conjugate exponents $p'$ and $q'$. Then*

$$(M^{p,q}_m(\mathcal{G}))' = M^{p',q'}_{1/m}(\mathcal{G});$$

(*ii*) *$M^1_m(\mathcal{G})$ and $M^{\infty,1}_m(\mathcal{G})$ are the dual spaces of*

$$\overline{\mathcal{S}_\mathcal{C}(\mathcal{G})}^{\|\cdot\|_{M^{\infty}_{1/m}}} \qquad \text{and} \qquad \overline{\mathcal{S}_\mathcal{C}(\mathcal{G})}^{\|\cdot\|_{M^{1,\infty}_{1/m}}},$$

*respectively;*

(*iii*) *If $f \in M^{\infty,1}_m(\mathcal{G})$ and $g \in M^{1,\infty}_{1/m}$, then $\langle f, g\rangle := \int_{\mathcal{G}\times\widehat{\mathcal{G}}} V_\varphi f(\mathbf{x}) \overline{V_\varphi g(\mathbf{x})} \, d\mathbf{x}$ is well-defined and $|\langle f, g\rangle| \lesssim \|f\|_{M^{\infty,1}_m} \|g\|_{M^{1,\infty}_{1/m}}$.*

#### 2.5.2.1 The Feichtinger algebra $\mathcal{S}_0(\mathcal{G})$

The Feichtinger algebra $\mathcal{S}_0(\mathcal{G})$ [49, 50, 51] has numerous equivalent descriptions, we address the reader to [101] for an exhaustive tour.

**Definition 2.5.29.** *We call* ***Feichtinger algebra over*** $\mathcal{G}$ *the Banach space*

$$\mathcal{S}_0(\mathcal{G}) \coloneqq M^1(\mathcal{G}).$$

*Let $v$ be a submultiplicative weight on $\mathcal{G}\times\widehat{\mathcal{G}}$, then the* ***weighted Feichtinger algebra over*** $\mathcal{G}$ *is*

$$M^1_v(\mathcal{G}).$$

We shall tackle mainly the unweighted case. Of course, if we endow $\mathcal{S}_0(\mathcal{G})$ with the norm

$$\|\cdot\|_{\mathcal{S}_0} \coloneqq \|\cdot\|_{M^1}$$

we have a Banach space which dual is given by $\mathcal{S}_0'(\mathcal{G}) = M^\infty(\mathcal{G})$, which is also called ***set of mild distributions***. In the case of $\mathcal{S}_0(\mathcal{G}) = M^1(\mathcal{G})$, we can pick any window in $\mathcal{S}_0(\mathcal{G})$ itself in order to compute the norm. Namely, given any $g \in \mathcal{S}_0(\mathcal{G}) \setminus \{0\}$ the application $\|V_g\cdot\|_{L^1}$ is a norm on the Feichtinger algebra and different window in $\mathcal{S}_0(\mathcal{G})$ yeld equivalent norms.
An equivalent description of $\mathcal{S}_0(\mathcal{G})$, e.g., is the following:

$$\mathcal{S}_0(\mathcal{G}) = \{f \in L^2(\mathcal{G}) \,|\, \exists g \in L^2(\mathcal{G}) \setminus \{0\} \,:\, V_g f \in L^1(\mathcal{G}\times\widehat{\mathcal{G}})\}.$$

In the following proposition we list just some of the main properties of the Feichtinger algebra.

**Proposition 2.5.30.** *Let us consider $f \in L^2(\mathcal{G})$ such that $V_g f \in L^1(\mathcal{G}\times\widehat{\mathcal{G}})$ for some $g \in L^2(\mathcal{G}) \setminus \{0\}$, i.e. $f \in \mathcal{S}_0(\mathcal{G})$. Then:*

*(i)* $\mathcal{F}[\mathcal{S}_0(\mathcal{G})] = \mathcal{S}_0(\widehat{\mathcal{G}})$ *and* $\|V_g f\|_{L^1} = \left\|V_{\hat{g}}\hat{f}\right\|_{L^1}$*;*

*(ii)* $\pi(\mathbf{x})f \in \mathcal{S}_0(\mathcal{G})$ *for every* $\mathbf{x} \in \mathcal{G}\times\widehat{\mathcal{G}}$*;*

*(iii)* $\mathcal{S}_0(\mathcal{G}) \subseteq C_0(\mathcal{G})$*;*

*(iv)* $\overline{f}, \mathcal{I}f, f^* \in \mathcal{S}_0(\mathcal{G})$*;*

*(v) if* $f \neq 0$*, then* $g \in \mathcal{S}_0(\mathcal{G})$*;*

**Proposition 2.5.31.** *$\mathcal{S}_0(\mathcal{G})$ is a Banach algebra under convolution and pointwise multiplication, i.e. if $f, g \in \mathcal{S}_0(\mathcal{G})$, then $f * g, f\cdot g \in \mathcal{S}_0(\mathcal{G})$ with norm estimates*

$$\|f * g\|_{\mathcal{S}_0} \lesssim \|f\|_{\mathcal{S}_0}\|g\|_{\mathcal{S}_0}, \qquad \|f\cdot g\|_{\mathcal{S}_0} \lesssim \|f\|_{\mathcal{S}_0}\|g\|_{\mathcal{S}_0}.$$

## 2.6 Some specific function spaces

In the present section we shall recall the following function spaces which will be used in the upcoming chapters: Wiener amalgam spaces $W(L^p, L^q_m)(\mathbb{R}^d)$ with indexes $0 < p, q \leq \infty$; Besov spaces $B^{p,q}_s(\mathbb{R}^d)$ [145]; the class of smooth symbols $S^m(\mathbb{R}^{2d})$ [127]; weak $L^{r,\infty}$ spaces [145]. Original results, by E. Cordero and the author, here presented and both published in [7] are: Lemma 2.6.8, which generalizes a characterization of Hörmander's class $S^0_{0,0}$ proved in [87, Lemma 6.1], and the subsequent Lemma 2.6.9.

### 2.6.1 Wiener Amalgam spaces with local component $L^p(\mathbb{R}^d)$ $0 < p \leq \infty$

For more about Wiener amalgam spaces we address the reader, e.g., to [53, 54, 55, 72, 75, 93, 119, 120].

**Definition 2.6.1.** *Consider $p, q \in (0, \infty]$, $m \in \mathcal{M}_v(\mathbb{R}^d)$ and the compact set $Q := [0,1]^d$. The **Wiener amalgam space with local component** $L^p(\mathbb{R}^d)$ **and global component** $L^q_m(\mathbb{R}^d)$, denoted by $W(L^p, L^q_m)(\mathbb{R}^d)$, consists of equivalence classes of functions $f \colon \mathbb{R}^d \to \overline{\mathbb{C}}$ equal a.e. such that $f \in L^p_{loc}(\mathbb{R}^d)$ and for which the **control function**:*

$$F_f^Q(k) := \|f \cdot T_k\chi_Q\|_{L^p} \in \ell^q_m(\mathbb{Z}^d), \qquad \forall k \in \mathbb{Z}^d. \tag{2.170}$$

*The quasi-norm on $W(L^p, L^q_m)(\mathbb{R}^d)$ is given by*

$$\begin{aligned} \|f\|_{W(L^p, L^q_m)} &:= \left\| F_f^Q(k) \right\|_{\ell^q_m} \\ &= \left\| \|f \cdot T_k\chi_Q\|_{L^p} \right\|_{\ell^q_m} \\ &= \left( \sum_{k \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^d} |f(t)|^p \chi_Q(t-k)\, dt \right)^{\frac{q}{p}} m^q(k) \right)^{\frac{1}{q}}, \end{aligned} \tag{2.171}$$

*with suitable adjustments for the cases $p, q = \infty$.*

This special definition allows us to grasp the sense of the amalgam: we first view $f$ "locally" through translations $T_k\chi_Q$ of the sharp cutoff function $\chi_Q$, and measure those local pieces in the $L^p$-norm, then we measure the global behaviour of those local pieces according to the $\ell^q_m$-norm. The "window" through which we view $f$ locally need not be a unit $d$-dimensional cube, cf. [54, 75, 93, 120]. In the sequel we shall use the following properties:

$(i)$ Inclusion relations: For $0 < p_1 \leq p_2 \leq \infty$, $0 < q_2 \leq q_1 \leq \infty$, we have

$$W(L^{p_2}, L^{q_2}_m)(\mathbb{R}^d) \hookrightarrow W(L^{p_1}, L^{q_1}_m)(\mathbb{R}^d). \tag{2.172}$$

$(ii)$ Convolution relations (for the quasi-Banach case see [75, Lemma 2.9]): Consider $m_i \in \mathcal{M}_v$, $0 < p_i, q_i \leq \infty$, $i \in \{1,2,3\}$, and $p_3 \geq 1$. Assume that $L^{p_1} * L^{p_2} \hookrightarrow L^{p_3}$ and $\ell^{q_1}_{m_1} * \ell^{q_2}_{m_2} \hookrightarrow \ell^{q_3}_{m_3}$, then

$$W(L^{p_1}, L^{q_1}_{m_1}) * W(L^{p_2}, L^{q_2}_{m_2}) \hookrightarrow W(L^{p_3}, L^{q_3}_{m_1}). \tag{2.173}$$

$(iii)$ For $m \in \mathcal{M}_v$, $0 < p \leq \infty$, we have

$$L^p_m = W(L^p, L^p_m). \tag{2.174}$$

**Proposition 2.6.2** (Multiplication relations)**.** ([9, Proposition 2.3]) *Consider $m, w \in \mathcal{M}_v$, $0 < p_i, q_i \leq \infty$, $i = \{1,2,3\}$. Assume $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$, then*

$$W(L^{p_1}, L^{q_1}_m) \cdot W(L^{p_2}, L^{q_2}_{w/m}) \hookrightarrow W(L^{p_3}, L^{q_3}_w). \tag{2.175}$$

*Proof.* The result is well known for $1 \leq p_i, q_i \leq \infty$, cf. [53, 93]. Here we show that the same proof works for quasi-Banach spaces. Indeed, since the standard Hölder inequality holds for Lebesgue exponents in $(0, \infty]$, for $f_1 \in W(L^{p_1}, L^{q_1}_m)$, $f_2 \in W(L^{p_2}, L^{q_2}_{w/m})$ we have

$$\|f_1 f_2 T_k\chi_Q\|_{L^{p_3}} = \|(f_1 T_k\chi_Q)(f_2 T_k\chi_Q)\|_{L^{p_3}} \leq \|f_1 T_k\chi_Q\|_{L^{p_1}} \| \|f_2 T_k\chi_Q\|_{L^{p_2}}.$$

Defining $a_k = \|f_1 T_k \chi_Q\|_{p_1}$ and $b_k = \|f_2 T_k \chi_Q\|_{p_2}$ and using Hölder's inequality for sequences $\ell^{q_1}\ell^{q_2} \hookrightarrow \ell^{q_3}$, for $1/q_1 + 1/q_2 = 1/q_3$ $(0 < q_i \leq \infty,\ i = 1,2,3)$, we obtain

$$\|a_k b_k w(k)\|_{\ell^{q_3}} = \|(a_k m(k))(b_k w(k)/m(k))\|_{\ell^{q_3}} \leq \|a_k m(k)\|_{\ell^{q_1}} \|b_k w(k)/m(k)\|_{\ell^{q_2}}.$$

This completes the proof. □

Concerning the theorem below, we address to [75, Theorem 3.3] (see also [82, Theorem 12.2.1] for $p \geq 1$).

**Theorem 2.6.3.** *Assume that $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. For $0 < p < 1$ let $g$ be a non-zero window in $M^r_v(\mathbb{R}^d)$, $r \leq p$. For $1 \leq p \leq \infty$, the function $g$ can be chosen in the larger space $M^1_v(\mathbb{R}^d)$. If $f \in M^p_m(\mathbb{R}^d)$, $0 < p \leq \infty$, then $V_g f \in W(L^\infty, L^p_m)$ and there exists $C > 0$, independent of $f$, such that*

$$\|V_g f\|_{W(L^\infty, L^p_m)} \leq C \|V_g f\|_{L^p_m}.$$

### 2.6.2 Besov spaces

**Definition 2.6.4.** *Consider $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^d)$ and set $\psi_j(\omega) := \psi(2^{-j}\omega)$ for $\omega \in \mathbb{R}^d$ and $j \in \mathbb{N}$. Suppose that:*

$$\begin{aligned} \operatorname{supp} \psi_0 &\subseteq \{\omega \in \mathbb{R}^d : |\omega| \leq 2\}; \\ \operatorname{supp} \psi &\subseteq \{\omega \in \mathbb{R}^d : 1/2 \leq |\omega| \leq 2\}; \\ \psi_0(\omega) + \sum_{j=1}^{\infty} \psi(2^{-j}\omega) &= 1, \quad \forall \omega \in \mathbb{R}^d. \end{aligned}$$

*Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then the* ***Besov space*** *$B^{p,q}_s(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the quasi-norm*

$$\|f\|_{B^{p,q}_s} := \left( \sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\psi_j \mathcal{F} f)\|^q_{L^p} \right)^{1/q} \tag{2.176}$$

*(with usual modifications when $q = \infty$) is finite.*

Besov spaces are generalizations of both Hölder-Zygmund and Sobolev spaces, see e.g. [145]. Precisely, we recapture the Sobolev spaces when $p = q = 2$, $s \in \mathbb{R}$: $B^{2,2}_s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. For $s > 0$, $B^{\infty,\infty}_s(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$, the Hölder-Zygmund classes, whose definition is as follows. For $s > 0$, we can write $s = n + \epsilon$, with $n \in \mathbb{N}_0$ and $\epsilon < 1$. Then $\mathcal{C}^s(\mathbb{R}^d)$ is the space of functions $f \in \mathcal{C}^n(\mathbb{R}^d)$ such that for each multi-index $\alpha \in \mathbb{N}^d_0$, with $|\alpha| = n$, the derivative $\partial^\alpha f$ satisfies the Hölder condition $|\partial^\alpha f(x) - \partial^\alpha f(y)| \leq K|x - y|^\epsilon$, for a suitable $K > 0$.

Inclusion relations between modulation and Besov spaces $B^{\infty,q}_s$ were first obtained for $1 \leq q \leq \infty$ (the Banach setting) in [137, Theorem 2.10] and then for $0 < q \leq \infty$ in [148]: for $0 < q \leq \infty$, set $\theta(q) = \min\{0, 1/q - 1\}$, then

$$B^{\infty,q}_{s+d/q}(\mathbb{R}^d) \hookrightarrow M^{\infty,q}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^d) \hookrightarrow B^{\infty,q}_{s+d\theta(q)}(\mathbb{R}^d), \quad s \in \mathbb{R}. \tag{2.177}$$

### 2.6.3 A class of smooth symbols introduced by Sjöstrand

In [127] J. Sjöstrand continued his study on pseudo-differential operators with rough symbols and he also considered the symbol class object of study of [7], see Chapter 4. Namely, the class $S^m(\mathbb{R}^{2d})$ defined below.

**Definition 2.6.5.** *For $m \in \mathbb{R}$, we define the* ***class of smooth symbols*** *$S^m(\mathbb{R}^{2d})$ as*

$$S^m(\mathbb{R}^{2d}) := \{\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : |\partial^\alpha \sigma(z)| \le C_\alpha \langle z\rangle^m, \quad \alpha \in \mathbb{N}_0^{2d},\ z \in \mathbb{R}^{2d}\}, \tag{2.178}$$

*where $\langle z\rangle$ has been defined in* (2.6).

**Remark 2.6.6.** (*i*) *Notice that this is a special instance of the class $S(w)$ introduced in [127, Formula* (3.2)*];*

(*ii*) *$S^m(\mathbb{R}^{2d})$ contains the so called* ***Shubin classes*** *$\Gamma^m_\rho$, $0 < \rho \le 1$ defined as [125]:*

$$\Gamma^m_\rho(\mathbb{R}^{2d}) = \{\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \mid |\partial^\alpha \sigma(z)| \le C_\alpha \langle z\rangle^{m-\rho|\alpha|},\ \alpha \in \mathbb{N}_0^{2d},\ z \in \mathbb{R}^{2d}\} \tag{2.179}$$

*and can be seen as their limit case for $\rho = 0$;*

(*iii*) *or $m = 0$ we recover the* ***Hörmander class*** *$S^0_{0,0}(\mathbb{R}^{2d})$.*

For any fixed $m \in \mathbb{R}$, the class $S^m(\mathbb{R}^{2d})$ in (2.178) is a Fréchet space when endowed with the sequence of norms $\{|\cdot|_{N,m}\}_{N\in\mathbb{N}_0}$,

$$|\sigma|_{N,m} := \sup_{|\alpha|\le N}\ \sup_{z\in\mathbb{R}^{2d}} |\partial^\alpha\sigma(z)|\langle z\rangle^{-m}, \quad N \in \mathbb{N}_0. \tag{2.180}$$

For $n \in \mathbb{N}_0$, $m \in \mathbb{R} \setminus \{0\}$, we define by $\mathcal{C}^n_m(\mathbb{R}^{2d})$ the space of functions having $n$ derivatives and satisfying (2.180) for $N = n$, whereas $\mathcal{C}^n(\mathbb{R}^{2d})$ is the space of functions with $n$ bounded derivatives. Clearly we have the equalities

$$S^m(\mathbb{R}^{2d}) = \bigcap_{n\in\mathbb{N}_0} \mathcal{C}^n_m(\mathbb{R}^{2d}),\ m \in \mathbb{R} \setminus \{0\}, \quad S^0(\mathbb{R}^{2d}) = \bigcap_{n\in\mathbb{N}_0} \mathcal{C}^n(\mathbb{R}^{2d}).$$

A characterization of the class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$ with modulation spaces was announced by Toft in [138, Remark 3.1] and proved in [87, Lemma 6.1].

**Lemma 2.6.7.** *We have the equalities*

$$\bigcap_{n\in\mathbb{N}_0} \mathcal{C}^n(\mathbb{R}^d) = \bigcap_{s\ge 0} M^\infty_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^d) = \bigcap_{s\ge 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^d). \tag{2.181}$$

*Hence $S^0(\mathbb{R}^{2d}) = \bigcap_{s\ge0} M^\infty_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d}) = \bigcap_{s\ge0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$.*

In what follows we extend the previous outcome to all the classes $S^m(\mathbb{R}^{2d})$, $m \in \mathbb{R}$.

**Lemma 2.6.8.** ([7, Lemma 2.2]) *For $m \in \mathbb{R}$, $0 < q \le \infty$, $n \in \mathbb{N}_0$, $s \in (0, +\infty)$, we have the equalities of Fréchet spaces*

$$S^m(\mathbb{R}^{2d}) = \bigcap_{n\in\mathbb{N}_0} \mathcal{C}^n_m(\mathbb{R}^{2d}) = \bigcap_{n\in\mathbb{N}_0} M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}(\mathbb{R}^{2d}) = \bigcap_{s\ge0} M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}(\mathbb{R}^d) \tag{2.182}$$

*with equivalent families of quasi-norms*

$$\{|\cdot|_{n,m}\}_{n\in\mathbb{N}_0}, \quad \{\|\cdot\|_{M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}}\}_{n\in\mathbb{N}_0}, \quad \{\|\cdot\|_{M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}}\}_{s\ge0}. \tag{2.183}$$

*In particular, for every $n \in \mathbb{N}_0$,*

$$\|f\|_{M^\infty_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}} \le C(n,m)|f|_{n,m}. \tag{2.184}$$

*Proof.* The equality $S^m(\mathbb{R}^{2d}) = \bigcap_{n\in\mathbb{N}_0} M^{\infty,1}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}(\mathbb{R}^{2d})$ was proved in [97, Remark 2.18]. The embeddings in (2.166) then give the equalities in (2.182) with the equivalent families of quasi-norms in (2.183).

Let us show the estimate (2.184). For $f \in \mathcal{C}^n_m(\mathbb{R}^d)$ ($\mathcal{C}^n(\mathbb{R}^d)$ if $m = 0$) and any multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq n$, we consider the function $\partial^\alpha(fT_x\bar{g})$. Taking its Fourier transform we get

$$\mathcal{F}(\partial^\alpha(fT_x\bar{g}))(\omega) = (2\pi i\omega)^\alpha \mathcal{F}(fT_x\bar{g})(\omega) = (2\pi i\omega)^\alpha V_g f(x,\omega). \tag{2.185}$$

In what follows we use the boundedness of $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$, Peetre's inequality $\langle x\rangle^{-m} \leq 2^{-m}\langle x-t\rangle^{|m|}\langle t\rangle^{-m}$, and Leibniz' formula:

$$\begin{aligned}
\langle x\rangle^{-m}\|\mathcal{F}(\partial^\alpha(fT_x\bar{g}))\|_{L^\infty} &\leq \langle x\rangle^{-m}\|\partial^\alpha(fT_x\bar{g})\|_{L^1} \\
&= \left\| \langle x\rangle^{-m} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f\, T_x \partial^{\alpha-\beta}\bar{g} \right\|_{L^1} \\
&\leq 2^{-m} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \|(\partial^\beta f)\langle\cdot\rangle^{-m}\|_{L^\infty} \|(\partial^{\alpha-\beta}\bar{g})\langle\cdot\rangle^{|m|}\|_{L^1} \\
&\leq 2^{-m} \sup_{|\beta|\leq n} \|(\partial^\beta f)\langle\cdot\rangle^{-m}\|_{L^\infty} M_\alpha \\
&\times \max_{\beta\leq\alpha} \binom{\alpha}{\beta} \|(\partial^{\alpha-\beta}\bar{g})\langle\cdot\rangle^{|m|}\|_{L^1} \\
&= C_{\alpha,g,m}|f|_{n,m},
\end{aligned}$$

where $C_{g,\alpha,m} = 2^{-m}M_\alpha \max_{\beta\leq\alpha}\binom{\alpha}{\beta}\|(\partial^{\alpha-\beta}\bar{g})\langle\cdot\rangle^{|m|}\|_{L^1}$ with $M_\alpha = \#\{\beta \in \mathbb{N}_0^d, \beta \leq \alpha\}$. The estimate above and formula (2.185) yield

$$\sup_{x\in\mathbb{R}^d} |V_g f(x,\omega)|\langle x\rangle^{-m} \leq C_{g,\alpha,m}|f|_{n,m}|\omega^\alpha|^{-1}, \quad |\omega| \neq 0, \quad \forall|\alpha| \leq n. \tag{2.186}$$

Now if $f \in \bigcap_{n\in\mathbb{N}_0} \mathcal{C}^n_m(\mathbb{R}^d)$ then for every $\alpha \in \mathbb{N}_0^d$ there exists $C = C_\alpha > 0$ such that the estimate in (2.186) holds true. Since $\langle\omega\rangle^n \leq \sum_{|\alpha|\leq n} c_\alpha|\omega^\alpha|$ for suitable $c_\alpha \geq 0$, we obtain

$$\sup_{x,\omega\in\mathbb{R}^d} |V_g f(x,\omega)|\langle x\rangle^{-m}\langle\omega\rangle^n \leq C|f|_{n,m}, \quad \forall n \in \mathbb{N}_0,$$

for a suitable $C = C(n,m) > 0$ that is (2.184). □

In particular, for $m = 0$ we recapture the outcome of Lemma 2.6.7.

For the case $m = 0$ we can characterize the Hörmander class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$ by Hölder-Zygmund classes $\mathcal{C}^s(\mathbb{R}^{2d}) = B^{\infty,\infty}_s(\mathbb{R}^{2d})$ and by Besov spaces.

**Lemma 2.6.9.** ([7, Lemma 2.3]) *For $0 < q \leq \infty$, we have the equalities*

$$S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{s\geq0} \mathcal{C}^s(\mathbb{R}^{2d}) = \bigcap_{s\geq0} B^{\infty,q}_s(\mathbb{R}^{2d}) = \bigcap_{s\geq0} M^{\infty,q}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d}), \tag{2.187}$$

*with equivalent families of quasi-norms*

$$\{\|\cdot\|_{B^{\infty,\infty}_s}\}_{s\geq0}, \quad \{\|\cdot\|_{B^{\infty,q}_s}\}_{s\geq0}, \quad \{\|\cdot\|_{M^{\infty,q}_{1\otimes\langle\cdot\rangle^s}}\}_{s\geq0}. \tag{2.188}$$

*Proof.* It is a straightforward consequence of Lemma 2.6.8 and the inclusion relations in (2.177). □

### 2.6.4 Weak $L^{r,\infty}$ spaces

We address the reader to [145].

**Definition 2.6.10.** *For $r \in [1,\infty)$, the* **weak** $L^r$ **space** $L^{r,\infty}(\mathbb{R}^d)$ *is the space of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that*

$$\|f\|_{L^{r,\infty}} := \sup_{\alpha>0} \alpha\lambda_f(\alpha)^{\frac{1}{r}} < +\infty, \tag{2.189}$$

*where*

$$\lambda_f(\alpha) := \mu(\{t \in \mathbb{R}^d : |f(t)| > \alpha\}), \qquad \alpha > 0,$$

*$\mu$ being the Lebesgue measure.*

Note that the quantity in (2.189) is a quasi-norm.

For convenience, we write $L^{\infty,\infty}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$. Observe that weak $L^r$ spaces are special instances of *Lorentz spaces* and $L^r(\mathbb{R}^d) \subseteq L^{r,\infty}(\mathbb{R}^d)$, $1 \le r \le \infty$.

**Lemma 2.6.11.** ([3, Lemma 2.8]) *For $r \in [1,\infty)$, $\lambda > 0$ and $\varphi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}^d$, we consider the rescaled Gaussians $\varphi_\lambda(t) := e^{-\pi\lambda t^2}$. Then we have*

$$\|\varphi_\lambda\|_{L^{r,\infty}(\mathbb{R}^d)} = \frac{\left(\frac{d}{2r}\right)^{\frac{d}{2r}}}{\Gamma(\frac{d}{2}+1)\lambda^{\frac{d}{2r}}} e^{-\frac{d}{2r}}. \tag{2.190}$$

*Hence,*

$$\|\varphi_\lambda\|_{L^{r,\infty}(\mathbb{R}^d)} = C(d,r)\lambda^{-\frac{d}{2r}}, \tag{2.191}$$

*with $C(d,r) = e^{-\frac{d}{2r}} \left(\frac{d}{2r}\right)^{\frac{d}{2r}} \Gamma(\frac{d}{2}+1)^{-1}$.*

*Proof.* Observe that for $\alpha \ge 1$ we have $\{t : |\varphi_\lambda(t)| > \alpha\} = \varnothing$. For $0 < \alpha < 1$, $\{t : |\varphi_\lambda(t)| > \alpha\} = \{t : |t| < \pi^{-1/2}\lambda^{-1/2}(\log(1/\alpha))^{1/2}\}$. The Lebesgue measure of the set is given by

$$A_\lambda := \mu(\{t : |t| < \pi^{-1/2}\lambda^{-1/2}(\log(1/\alpha))^{1/2}\}) = \frac{\log(1/\alpha)^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)\lambda^{\frac{d}{2}}}.$$

Now, using the definition of the quasi-norm in (2.189),

$$\begin{aligned}
\|\varphi_\lambda\|_{L^{r,\infty}(\mathbb{R}^d)} &= \sup_{\alpha>0} \alpha\mu(\{t : |\varphi_\lambda(t)| > \alpha\})^{\frac{1}{r}} \\
&= \sup_{0<\alpha<1} \alpha A_\lambda^{\frac{1}{r}} \\
&= \frac{1}{\Gamma(\frac{d}{2}+1)\lambda^{\frac{d}{2r}}} \sup_{0<\alpha<1} \alpha\left(\log(1/\alpha)\right)^{\frac{d}{2r}}.
\end{aligned}$$

An easy computation shows that the function $y(\alpha) := \alpha\left(\log(1/\alpha)\right)^{\frac{d}{2r}}$ on $(0,1)$ admits the maximum point $t_M := e^{-\frac{d}{2r}}$ and the maximum is $y(t_M) = (d/(2r))^{2/(2r)}e^{-2/(2r)}$, so that we obtain the claim. ☐

We observe that in the $L^{r,\infty}$ spaces the rescaled Gaussians behave like in the usual $L^r$ spaces, meaning $\|\varphi_\lambda\|_r \asymp \|\varphi_\lambda\|_{L^{r,\infty}} \asymp \lambda^{-d/(2r)}$.

## 2.7 Main operators

The operators presented in this section shall be object of study of next chapters. We shall see: localization operator $A_a^{\psi_1,\psi_2}$ both on $\mathbb{R}^d$ and $\mathcal{G}$, Gabor multipliers $G_a^{g_1,g_2}$ on $\mathbb{R}^d$ and $\mathsf{G}_a^{g_1,g_2}$ on $\mathbb{Z}_N$, pseudo-differential operators $\mathrm{Op}_\tau(\sigma)$ and Born-Jordan operators $\mathrm{Op}_{BJ}(\sigma)$ on $\mathbb{R}^d$, Kohn-Nirenberg operators $\mathrm{Op}_0(\sigma)$ on $\mathcal{G}$ LCA group, Fourier multiplier $T_m$ on $\mathbb{R}^d$ and linear time invariant filters $H$ on $\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$ (recall (2.104)).

### 2.7.1 Localization operators on $\mathbb{R}^d$

**Definition 2.7.1.** *Consider a* ***symbol*** $a \in \mathcal{S}'(\mathbb{R}^{2d})$ *and non-zero* ***windows*** $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$*. Then the* ***localization operator***

$$A_a^{\psi_1,\psi_2} \colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

*is defined by the formal integral*

$$A_a^{\psi_1,\psi_2} f(t) := \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\psi_1} f(x,\omega) M_\omega T_x \psi_2(t)\, dx d\omega, \tag{2.192}$$

*or we can give the definition in a weak sense as follows:*

$$\langle A_a^{\psi_1,\psi_2} f, g\rangle := \langle a, \overline{V_{\psi_1} f} V_{\psi_2} g\rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \tag{2.193}$$

*where* $\langle\cdot,\cdot\rangle$ *in the left-hand side is the sesquilinear duality between* $\mathcal{S}'(\mathbb{R}^d)$ *and* $\mathcal{S}(\mathbb{R}^d)$ *and in the right-hand side between* $\mathcal{S}'(\mathbb{R}^{2d})$ *and* $\mathcal{S}(\mathbb{R}^{2d})$*.*

Localization operators are also called ***STFT multiplier*** because of (2.192).

**Remark 2.7.2.** $A_a^{\psi_1,\psi_2}$ *is well defined as mapping* $A_a^{\psi_1,\psi_2} \colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$*. In fact, for any* $f, g, \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ *we have*

$$V_{\psi_1} f, V_{\psi_2} g \in \mathcal{S}(\mathbb{R}^{2d})$$

*by Theorem 2.3.14. Moreover*

$$V_{\psi_1} f \in \mathcal{S}(\mathbb{R}^{2d}) \quad \Rightarrow \quad \overline{V_{\psi_1} f} \in \mathcal{S}(\mathbb{R}^{2d}).$$

*Since the Schwartz class is closed under pointwise multiplication, we get*

$$\overline{V_{\psi_1} f} V_{\psi_2} g \in \mathcal{S}(\mathbb{R}^{2d})$$

*and thus* $\langle a, \overline{V_{\psi_1} f} V_{\psi_2} g\rangle$ *makes sense since* $a \in \mathcal{S}'(\mathbb{R}^{2d})$*.*
*The linearity of* $A_a^{\psi_1,\psi_2}$ *comes straightforward from* (2.193)*. Indeed*

$$V_{\psi_1} \cdot (x,\omega) = \langle \cdot, M_\omega T_x \psi_1\rangle$$

*is linear, the duality between* $\mathcal{S}'(\mathbb{R}^{2d})$ *and* $\mathcal{S}(\mathbb{R}^{2d})$ *is antilinear in the second argument, hence the complex conjugation over* $V_{\psi_1} f$ *makes of* $A_a^{\psi_1,\psi_2}$ *a linear operator.*

A localization operator $A_a^{\psi_1,\psi_2}$ is defined whenever the vector-valued integral (2.192) makes sense, hence we could chose other sets as spaces of symbols and windows respectively. In (2.193) the brackets $\langle\cdot,\cdot\rangle$ would express the duality on a suitable pair of dual spaces $(B', B)$ and extend the inner product on $L^2(\mathbb{R}^{2d})$. For example, later we shall pick the Gelfand-Shilov space $B = \mathcal{S}^{(1)}(\mathbb{R}^d)$.

**Remark 2.7.3.** *$A_a^{\psi_1,\psi_2}$ defined as in* (2.192) *is well defined and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ (endowed with the w-$*$ topology). In particular if one considers a symbol $a$ in the Lebesgue space $L^q(\mathbb{R}^{2d})$ ($1 \le q < \infty$) and window functions $\psi_1, \psi_2$ in the Feichtinger algebra $M^1(\mathbb{R}^d)$, then the localization operator $A_a^{\psi_1,\psi_2}$ is in the Schatten class $\mathcal{J}^q$ (see [28]). In this case, the localization operator $A_a^{\psi_1,\psi_2}$ is a bounded and compact operator on $L^2(\mathbb{R}^d)$.*

We report the following important characterization of $A_a^{\psi_1,\psi_2} \in \mathcal{J}^p(L^2(\mathbb{R}^d))$, [29, Theorem 1].

**Theorem 2.7.4.** *Consider $1 \le p \le \infty$.*

(*i*) *The mapping*

$$M^{p,\infty}(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d) \to \mathcal{J}^p(L^2(\mathbb{R}^d)), (a, \psi_1, \psi_2) \mapsto A_a^{\psi_1,\psi_2}$$

*is bounded with norm estimate*

$$\left\| A_a^{\psi_1,\psi_2} \right\|_{\mathcal{J}^p} \le B \left\| a \right\|_{M^{p,\infty}} \left\| \psi_1 \right\|_{M^1} \left\| \psi_2 \right\|_{M^1},$$

*for a suitable constant $B > 0$;*

(*ii*) *Assume that $A_a^{\psi_1,\psi_2} \in \mathcal{J}^p(L^2(\mathbb{R}^d))$ for all windows $\psi_1, \psi_2 \in M^1(\mathbb{R}^d)$ and that there exists a constant $B > 0$ depending only on the symbol $a$ such that*

$$\left\| A_a^{\psi_1,\psi_2} \right\|_{\mathcal{J}^p} \le B \left\| \psi_1 \right\|_{M^1} \left\| \psi_2 \right\|_{M^1}, \quad \forall \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d).$$

*Then $a \in M^{p,\infty}(\mathbb{R}^{2d})$.*

**Remark 2.7.5.** *If $\psi_1(t) = e^{-\pi t^2} = \psi_2(t)$, then $A_a = A_a^{\psi_1,\psi_2}$ is the classical anti-Wick operator and the mapping $a \mapsto A_a^{\psi_1,\psi_2}$ is a quantization rule in quantum mechanics [14, 39, 125, 152].*

The terminology *localization operator* appears for the first time in 1988, in a paper by I. Daubechies [37]. Writing $A_a^{\psi_1,\psi_2} f$ as the integral (2.192) makes this choice clearer. Think of $f$ as a signal, an element of $L^2(\mathbb{R}^d)$ is suitable, then we analyse the signal via its STFT. For sake of simplicity consider the symbol $a$ of type $\chi_\Omega$, where $\Omega \subseteq \mathbb{R}^{2d}$ is a compact subset of the time-frequency plane. The product $aV_{\psi_1} f$ is the restriction of the analysed signal to a compact subset in the phase space. Eventually we obtain the modified signal $A_a^{\psi_1,\psi_2} f$ multiplying by $M_\omega T_x \psi_2$ and integrating. According to this perspective, $\psi_1, \psi_2$ are called ***analysis*** and ***synthesis window*** respectively.
In [9] even rough symbols are considered (see Remark 3.2.3), no assumptions on the geometry or support of the generalised function $a$ are made. Besides, the related localization operator $A_a^{\psi_1,\psi_2}$ is not necessarily a self-adjoint operator. It is easy to check that the adjoint of a localization operator is given by

$$(A_a^{\psi_1,\psi_2})^* = A_{\bar{a}}^{\psi_2,\psi_1};$$

hence the self-adjointness property forces the choice $\psi_1 = \psi_2$ and the symbol $a$ real valued, as for the case $A_{\chi_\Omega}^{\psi,\psi}$ mentioned above. The framework of [9] can allow the use of two different windows $\psi_1$ and $\psi_2$ to analyse and synthesize the signal $f$, respectively. Moreover, the symbol $a$ can be a complex-valued function.

### 2.7.2 Localization operators on $\mathcal{G}$ LCA group

We address the reader to Wong's book [152] for a detailed treatment of localization operators on locally compact Hausdorff groups and point out the recent works [108, 109]. Since the equivalent of the Schwartz class on $\mathcal{G}$, i.e. the Schwartz-Bruhat class, is quite cumbersome to deal with, we adopt the more handy Feichtinger algebra $\mathcal{S}_0(\mathcal{G})$.

**Definition 2.7.6.** *Consider windows $\psi_1, \psi_2 \in \mathcal{S}_0(\mathcal{G})$ and symbol $a \in \mathcal{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})$. Then the* ***localization operator with symbol*** $a$ ***and windows*** $\psi_1, \psi_2$ *in $\mathcal{S}_0(\mathcal{G})$ is the linear and continuous operators*

$$A_a^{\psi_1,\psi_2} \colon \mathcal{S}_0(\mathcal{G}) \to \mathcal{S}_0'(\mathcal{G})$$

*formally defined by*

$$A_a^{\psi_1,\psi_2} f(x) = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(u,\omega) V_{\psi_1} f(u,\omega) M_\omega T_u \psi_2(x)\, du d\omega. \tag{2.194}$$

*Equivalently, its weak definition is*

$$\langle A_a^{\psi_1,\psi_2} f, g\rangle = \langle a, \overline{V_{\psi_1} f} V_{\psi_2} g\rangle, \qquad \forall\, f,g \in \mathcal{S}_0(\mathcal{G}). \tag{2.195}$$

**Remark 2.7.7.** (*i*) *It is straightforward computation to check that $A_a^{\psi_1,\psi_2} \colon \mathcal{S}_0(\mathcal{G}) \to \mathcal{S}_0'(\mathcal{G})$ is well defined, linear and continuous as claimed (cf. [101, Theorem 5.3]);*

(*ii*) *Concretely, we shall mainly consider windows $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ rather than in the whole Feichtinger algebra. Notice that if $a \in L^p(\mathcal{G}\times\widehat{\mathcal{G}})$, for any $1 \le p \le \infty$, then $A_a^{\psi_1,\psi_2} \in B(L^2(\mathcal{G}))$, cf. [152, Proposition 12.1, 12.2, 12.3].*

### 2.7.3 Gabor multipliers

We address the reader to [66, 122] for more about Gabor multipliers.

**Definition 2.7.8.** *Let $\alpha, \beta > 0$ and consider the lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, then the* ***Gabor multiplier with windows*** $g_1, g_2 \in L^2(\mathbb{R}^d)$ ***and symbol*** $a$ *can formally be defined as*

$$G_a^{g_1,g_2} f := \sum_{k,n\in\mathbb{Z}^{2d}} a(\alpha k, \beta n) V_{g_1} f(\alpha k, \beta n) T_{\alpha k} M_{\beta n} g_2, \quad \forall f \in L^2(\mathbb{R}^d), \tag{2.196}$$

Observe that a Gabor multiplier is the *discrete* version of a localization operator; in fact it can be obtained from (2.192) by replacing the Lebesgue measure $dx d\omega$ with the discrete measure $\nu = \sum_{k,n\in\mathbb{Z}^d} \delta_{\alpha k, \beta n}$; the integration with respect to $\nu$ becomes the summation

$$\int_{\mathbb{R}^{2d}} F(x,\omega) d\nu(x,\omega) = \sum_{k,n\in\mathbb{Z}^d} F(\alpha k, \beta n).$$

Note that this is a particular instance of a continuous frame multiplier, a (discrete) frame multiplier and their relation, see [2, 4, 6].

#### 2.7.3.1 Gabor multipliers on $\mathcal{G} = \mathbb{Z}_N$

Recall the identification $\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$ (2.104).

**Definition 2.7.9.** *Given a rectangular lattice $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ as in (2.138), windows $g_1, g_2 \in \mathbb{C}^N$,* ***mask*** *or* ***lower symbol*** *$a \in \mathbb{C}^{N\times N}$, we define the* ***(finite) Gabor multiplier*** *applied to $f \in \mathbb{C}^N$ as follows:*

$$\mathsf{G}_{a,\Lambda}^{g_1,g_2} f := \sum_{k=0}^{A-1}\sum_{l=0}^{B-1} a(\alpha k, \beta l) V_{g_1} f(\alpha k, \beta l)\pi(\alpha k, \beta l) g_2. \tag{2.197}$$

Whenever clear, we shall write $\mathsf{G}_a^{g_1,g_2}$ in place of $\mathsf{G}_{a,\Lambda}^{g_1,g_2}$.

**Remark 2.7.10.** *If we pick $\alpha = 1 = \beta$, the the relative finite Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$ is just the localization operator (or STFT multiplier) $A_a^{g_1,g_2}$ on $\mathbb{Z}_N$.*

It is straightforward to obtain the matrix representation of $\mathsf{G}_{a,\Lambda}^{g_1,g_2}$:

$$K(\mathsf{G}_{a,\Lambda}^{g_1,g_2})(u,v) = \sum_{k=0}^{A-1}\sum_{l=0}^{B-1} a(\alpha k, \beta l)\overline{g_1(v-\alpha k)} g_2(u-\alpha k) e^{\frac{2\pi i \beta l(u-v)}{N}}. \tag{2.198}$$

Let us introduce the notation

$$\mathsf{S} := \mathsf{F}_s a, \tag{2.199}$$

where $a \in \mathbb{C}^{N\times N}$ is the symbol of a Gabor multiplier and $\mathsf{F}_s$ the discrete symplectic Fourier transform (2.114). In [46] many results for the interrelation of spreading function and Gabor multiplier are shown. Here we give the related finite dimensional result, like the following:

**Proposition 2.7.11.** ([3, Proposition 6.3]) *The spreading function of a (finite) Gabor multiplier $\mathsf{G}_{a,\Lambda}^{g_1,g_2}$ is given by*

$$\eta(\mathsf{G}_{a,\Lambda}^{g_1,g_2})(u,v) = \frac{N}{\alpha\beta}\sum_{l=0}^{\alpha-1}\sum_{k=0}^{\beta-1} \mathsf{S}(u+Bk, v-Al) V_{g_1} g_2(u,v). \tag{2.200}$$

*Proof.* A direct computation gives

$$\begin{aligned}
\eta(\mathsf{G}_{a,\Lambda}^{g_1,g_2})(u,v) &= \sum_{t=0}^{N-1} K(\mathsf{G}_{a,\Lambda}^{g_1,g_2})(t,t-u) e^{\frac{-2\pi i t v}{N}} \\
&= \sum_{t=0}^{N-1}\sum_{k=0}^{A-1}\sum_{l=0}^{B-1} a(\alpha k, \beta l)\overline{g_1(t-u-\alpha k)} g_2(t-\alpha k) e^{\frac{2\pi i \beta l u}{N}} e^{\frac{-2\pi i t v}{N}} \\
&= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1} a(\alpha k, \beta l) e^{\frac{2\pi i \beta l u}{N}} \\
&\quad \times \sum_{t=0}^{N-1} \overline{g_1(t-u-\alpha k)} g_2(t-\alpha k) e^{\frac{-2\pi i t v}{N}}.
\end{aligned} \tag{2.201}$$

Performing the substitution $t' = t - \alpha k$ in (2.201) gives

$$\begin{aligned}
\sum_{t=0}^{N-1} \overline{g_1(t-u-\alpha k)} g_2(t-\alpha k) e^{\frac{-2\pi i t v}{N}} &= \sum_{t'=0}^{N-1} g_2(t')\overline{g_1(t'-u)} e^{\frac{-2\pi i (t'+\alpha k) v}{N}} \\
&= \sum_{t'=0}^{N-1} g_2(t')\overline{g_1(t'-u)} e^{\frac{-2\pi i v}{N} - \frac{-2\pi i \alpha k v}{N}} \\
&= V_{g_1} g_2(u,v) e^{-\frac{-2\pi i \alpha k v}{N}}.
\end{aligned}$$

Hence, recalling the definition of $\text{Ш}_{(\alpha,\beta)}$, $\mathsf{F}_s$, $\mathsf{F}_2$, and using Lemma 2.3.34 together with (2.107), we get

$$\begin{aligned}
\eta(\mathsf{G}_{a,\Lambda}^{g_1,g_2})(u,v) &= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1} a(\alpha k,\beta l)e^{\frac{2\pi i\beta lu}{N}}e^{-\frac{-2\pi i\alpha kv}{N}}V_{g_1}g_2(u,v)\\
&= N\mathsf{F}_s\left(a\cdot \text{Ш}_{(\alpha,\beta)}\right)(u,v)V_{g_1}g_2(u,v)\\
&= \mathsf{F}_2\left(a^T\cdot \text{Ш}^T_{(\alpha,\beta)}\right)(-u,v)V_{g_1}g_2(u,v)\\
&= \mathsf{F}_2\left(a^T\cdot \text{Ш}_{(\beta,\alpha)}\right)(-u,v)V_{g_1}g_2(u,v)\\
&= \frac{1}{N^2}\left(\mathsf{F}_2 a^T * \mathsf{F}_2\text{Ш}_{(\beta,\alpha)}\right)(-u,v)V_{g_1}g_2(u,v)\\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\frac{1}{N}\mathsf{F}_2 a^T(-u-k,v-l)\mathsf{F}_2\text{Ш}_{(\beta,\alpha)}(k,l)V_{g_1}g_2(u,v)\\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\mathsf{F}_s a(u+k,v-l)AB\text{Ш}_{(B,A)}(k,l)V_{g_1}g_2(u,v)\\
&= \frac{AB}{N}\sum_{l=0}^{\alpha-1}\sum_{k=0}^{\beta-1}\mathsf{F}_s a(u+Bk,v-Al)V_{g_1}g_2(u,v).
\end{aligned}$$

This concludes the proof. □

We shall frequently denote the ***periodization of*** $\mathsf{S}$ by $\mathsf{S}_\mathsf{P}^{BA}$:

$$\mathsf{S}_\mathsf{P}^{BA}(u,v) := \sum_{l=0}^{\alpha-1}\sum_{k=0}^{\beta-1}\mathsf{S}(u+Bk,v-Al), \tag{2.202}$$

the periodicity is meant in the sense that

$$\mathsf{S}_\mathsf{P}^{BA}(u,v) = \mathsf{S}_\mathsf{P}^{BA}(u+Bk,v+Al) \tag{2.203}$$

for $u,v=0,\dots,N-1$ and $k=0,\dots,\beta-1$, $l=0,\dots,\alpha-1$.
So that (2.200) can be written as

$$\eta(\mathsf{G}_{a,\Lambda}^{g_1,g_2})(u,v) = \frac{N}{\alpha\beta}\mathsf{S}_\mathsf{P}^{BA}(u,v)V_{g_1}g_2(u,v). \tag{2.204}$$

The factor $N/\alpha\beta$ is also called *redundancy*. In the finite dimensional case the interpretation of this number is straightforward, because one uses $A\cdot B$ to represent a vector in $\mathbb{R}^N$. This leads to an oversampling of

$$\frac{AB}{N} = \frac{N}{\alpha}\frac{N}{\beta}\frac{1}{N} = \frac{N}{\alpha\beta}.$$

By using the convolution theorem for $\mathsf{F}_s$, cf. (2.107) and (2.115) and see [64, Theorem 4.3], and Lemma 2.3.34 we get

$$\mathsf{F}_s\left(a\cdot \text{Ш}_{(\alpha,\beta)}\right)(u,v) = \frac{1}{\alpha\beta}\sum_{l=0}^{\alpha-1}\sum_{k=0}^{\beta-1}\mathsf{S}(u+Bk,v-Al).$$

Therefore

$$\mathsf{S}_\mathsf{P}^{BA}(u,v) = \alpha\beta\,\mathsf{F}_s\left(a\cdot \text{Ш}_{(\alpha,\beta)}\right)(u,v). \tag{2.205}$$

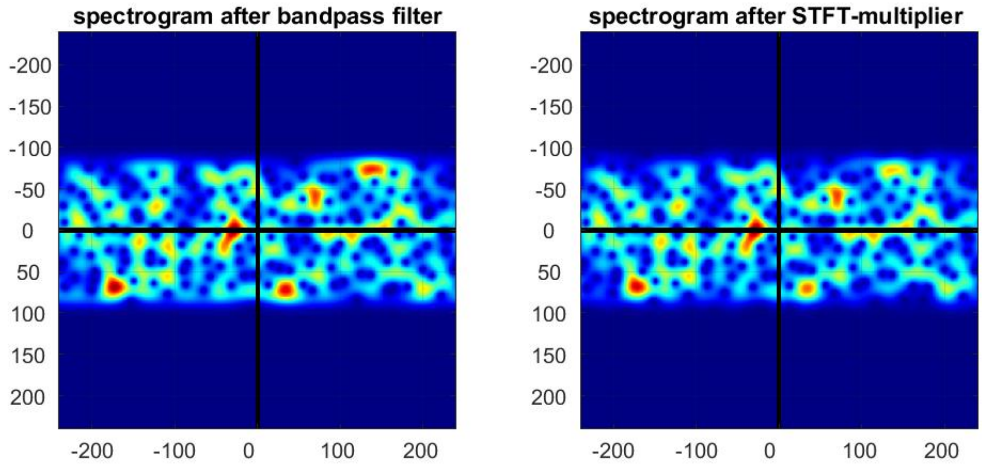



Figure 2.1: The left part shows the (Gaussian) spectrogram of the output of an "ideal band-pass filter" with cut-off frequency $R = 80$, applied to a random signal in $\mathbb{C}^N$, $N = 480$. The right hand side represents the spectrogram of the output of the corresponding STFT multiplier $\mathsf{G}_a^{g_1,g_2}$. Here we use the number of samples as normalization on time and frequency scale.

**Example 2.7.12.** *As example, we consider a low pass filter as it is often implemented in practice. We choose the frequency response $\hat{h} \in \mathbb{C}^N$ equal to the characteristic function, which is 1 on $[-R, R]$ and zero elsewhere. The resulting convolution operator $H$ is compared to the filter generated by a Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$ with symbol $a = \mathbf{1} \otimes \hat{h}$. As analysis and synthesis window for $\mathsf{G}_a^{g_1,g_2}$ we choose the Gaussian window normalized by the factor $1/N$, which is the redundancy since we take $\alpha = \beta = 1$. Both operations are applied to a random vector $f_0$.*

*A graphical comparison of the LTI filter approach and of the Gabor multiplier one is shown in Figure 2.1.*

### 2.7.4 $\mathrm{Op}_\tau(\sigma)$ operators

**Definition 2.7.13.** *Consider a tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\tau \in [0, 1]$. Then the **$\tau$-quantization** $\mathrm{Op}_\tau(\sigma)$ **of** $\sigma$ is the continuous mapping*

$$\mathrm{Op}_\tau(\sigma)\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

*($\mathcal{S}'(\mathbb{R}^d)$ endowed with the w-$*$ topology) defined by the formal integral as:*

$$\mathrm{Op}_\tau(\sigma)f(t) := \int_{\mathbb{R}^{2d}} e^{2\pi i(t-x)\omega}\sigma((1-\tau)t + \tau x, \omega)f(x)\,dxd\omega, \tag{2.206}$$

*or defined weakly by*

$$\langle \mathrm{Op}_\tau(\sigma)f, g\rangle = \langle \sigma, W_\tau(g,f)\rangle, \qquad \forall f, g, \in \mathcal{S}(\mathbb{R}^d), \tag{2.207}$$

*where* $\langle\cdot,\cdot\rangle$ *in the left-hand side is the sesquilinear duality between* $\mathcal{S}'(\mathbb{R}^d)$ *and* $\mathcal{S}(\mathbb{R}^d)$ *and in the right-hand side between* $\mathcal{S}'(\mathbb{R}^{2d})$ *and* $\mathcal{S}(\mathbb{R}^{2d})$.

Naturally, $\sigma$ is the $\tau$-symbol of the operator $\mathrm{Op}_\tau(\sigma)$, see Definition 2.3.21. When $\tau = 0$, we call $\mathrm{Op}_0(\sigma)$ the ***Kohn-Nirenberg quantization of*** $\sigma$ and $\sigma$ is called ***Kohn-Nirenberg symbol of*** $\mathrm{Op}_0(\sigma)$.
When $\tau = 1/2$, we call $\mathrm{Op}_{1/2}(\sigma)$ ***Weyl transform (or quantization) of*** $\sigma$, the distribution $\sigma$ is called then ***Weyl symbol of the operator***. For Weyl operators we adopt also the following alternative notation:

$$L_\sigma \coloneqq \mathrm{Op}_{1/2}(\sigma).$$

**Remark 2.7.14.** $\mathrm{Op}_\tau(\sigma)$ *is well defined as mapping* $\mathrm{Op}_\tau(\sigma)\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. *In fact, for any* $f, g \in \mathcal{S}(\mathbb{R}^d)$ *we have* $W_\tau(g,f) \in \mathcal{S}(\mathbb{R}^{2d})$ *by Lemma 2.3.17. Thus* $\langle \sigma, W(g,f)\rangle$ *makes sense since* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$. *The linearity of* $\mathrm{Op}_\tau(\sigma)$ *follows immediately from the weak definition of* $\mathrm{Op}_\tau(\sigma)$ *because of the relation between Wigner distribution and STFT, Lemma 2.3.18. Then* $W_\tau(g,\cdot)$ *is antilinear and* $\langle \sigma, W_\tau(g,\cdot)\rangle$ *is linear since* $\langle\cdot,\cdot\rangle$ *is the sesquilinear duality between* $\mathcal{S}'(\mathbb{R}^{2d})$ *and* $\mathcal{S}(\mathbb{R}^{2d})$.

**Remark 2.7.15.** (*i*) *The Schwartz Kernel Theorem, stated for* $\mathcal{S}(\mathbb{R}^d)$ *and* $\mathcal{S}'(\mathbb{R}^d)$ *as in [82, Theorem 14.3.4], implies that if* $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *is a continuous operator (where* $\mathcal{S}'(\mathbb{R}^d)$ *is endowed with the w-$*$ topology), then there exists* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *such that*

$$T = L_\sigma.$$

*See in particular [82, Theorem 14.3.5];*

(*ii*) *It is always possible to write* $L_\sigma$ *as the* $\tau$*-quantization of some symbol* $\sigma_\tau$, $\tau \neq 1/2$. *In fact, we have from [35, (4.37) and (4.38)] that*

$$L_\sigma = \mathrm{Op}_\tau(\sigma_\tau) \quad \Leftrightarrow \quad \sigma_\tau = \frac{2^d}{|1-2d|^d} e^{-\pi i(1-2\tau)x\omega} * \sigma(x,\omega). \tag{2.208}$$

*Hence the Schwartz Kernel Theorem can be expressed in term of any* $\tau$*-quantization,* $\tau \in [0,1]$. *Namely, given* $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *linear and continuos and* $\tau \in [0,1]$, *then there exists* $\sigma_\tau \in \mathcal{S}'(\mathbb{R}^{2d})$ *such that*

$$T = \mathrm{Op}_\tau(\sigma_\tau).$$

*This fact has been reported in Theorem 2.3.20.*

As highlighted in Remark 2.7.3, $A_a^{\psi_1,\psi_2}$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ whenever $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$. Hence there exists $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that $A_a^{\psi_1,\psi_2} = L_\sigma$. A calculation in [17, 68, 125] reveals that the Weyl symbol associated to $A_a^{\psi_1,\psi_2}$ is

$$\sigma = a * W(\psi_2, \psi_1), \tag{2.209}$$

then

$$A_a^{\psi_1,\psi_2} = L_{a*W(\psi_2,\psi_1)}. \tag{2.210}$$

In [11, Proposition 2.16], in the Gelfand-Shilov setting, every $\tau$-symbol of $A_a^{\psi_1,\psi_2}$ was explicitly calculated. We state the following result without proof since the one of subsequent Proposition 2.8.16 applies almost verbatim.

**Proposition 2.7.16.** *Consider* $a \in \mathcal{S}'(\mathbb{R}^{2d})$*,* $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ *and* $\tau \in [0,1]$*. Then*

$$A_a^{\psi_1,\psi_2} = \mathrm{Op}_\tau(a * W_\tau(\psi_2, \psi_1)).$$

The crucial relation between the action of any $\tau$-quantization $\mathrm{Op}_\tau(\sigma)$ on time-frequency shifts and the short-time Fourier transform of its symbol is contained in [32, Lemma 4.1].

**Lemma 2.7.17.** *Consider* $\tau \in [0,1]$*,* $g \in \mathcal{S}(\mathbb{R}^d)$*,* $\Phi_\tau := W_\tau(g,g) \in \mathcal{S}(\mathbb{R}^{2d})$*. If* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$*, then*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g, \pi(w)g\rangle| = |V_{\Phi_\tau}\sigma\left(\mathcal{T}_\tau(w,z), J(w-z)\right)|, \quad \forall\, z,w \in \mathbb{R}^{2d}, \tag{2.211}$$

*where* $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{R}^{2d}$ *and* $\mathcal{T}_\tau$ *and* $J$ *are defined as follows:*

$$\mathcal{T}_\tau(w,z) := ((1-\tau)w_1 + \tau z_1, \tau w_2 + (1-\tau)z_2), \qquad J(z) := (z_2, -z_1). \tag{2.212}$$

Notice that $J = -\mathcal{J}$ in the case $\mathcal{G} = \mathbb{R}^d$, $\mathcal{J}$ defied as in (2.98). The following lemma can be viewed as a form of the inversion formula (2.74). We present the proof later for the same result stated in the Gelfand-Shilov setting, as was published in [11, Lemma 3.2].

**Lemma 2.7.18.** *Let* $\tau \in [0,1]$ *and* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$*. If* $g \in \mathcal{S}(\mathbb{R}^d)$ *with* $\|g\|_{L^2} = 1$ *and* $f \in \mathcal{S}(\mathbb{R}^d)$*, then*

$$\mathrm{Op}_\tau(\sigma)f = \int_{\mathbb{R}^{2d}} V_g f(z)\, \mathrm{Op}_\tau(\sigma)(\pi(z)g)\, dz, \tag{2.213}$$

*in the sense that*

$$\langle \mathrm{Op}_\tau(\sigma)f, \varphi\rangle = \int_{\mathbb{R}^{2d}} V_g f(z)\, \langle \mathrm{Op}_\tau(\sigma)(\pi(z)g), \varphi\rangle\, dz, \qquad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

We first recall Schatten class results for the Weyl calculus in terms of modulation spaces, initially proved for $1 \le p \le \infty$ in [17, Theorem 4.5], for $0 < p < 1$ we refer to [142, Theorem 3.4].

**Theorem 2.7.19.** *If the Weyl symbol* $\sigma \in M^{p,1}(\mathbb{R}^{2d})$ *for some* $0 < p < \infty$*, then the operator* $L_\sigma$ *belongs to the Schatten class* $\mathcal{J}^p$ *with*

$$\|L_\sigma\|_{\mathcal{J}^p} \le \|\sigma\|_{M^{p,1}}.$$

*In particular,* $L_\sigma$ *is a compact operator on* $L^2(\mathbb{R}^d)$*.*

### 2.7.5 Born-Jordan operators

We suggest [26, 40] to the interested reader.

**Definition 2.7.20.** *Let* $f, g \in \mathcal{S}(\mathbb{R}^d)$*. We call* ***Born-Jordan distribution of*** $f$ ***and*** $g$ *the function given by*

$$W_{BJ}(f,g) := \int_0^1 W_\tau(f,g)\, d\tau. \tag{2.214}$$

The ***Born-Jordan operator with symbol*** $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ is then defined as

$$\langle \mathrm{Op}_{BJ}(\sigma)f, g\rangle := \langle \sigma, W_{BJ}(g,f)\rangle, \quad \forall f,g \in \mathcal{S}(\mathbb{R}^d). \tag{2.215}$$

### 2.7.6 Pseudo-differential operators on $\mathcal{G}$ LCA group

**Definition 2.7.21.** *Let $\sigma \in \mathcal{S}_0'(\mathcal{G} \times \widehat{\mathcal{G}})$. Then the* ***pseudo-differential operator with Kohn-Nirenberg symbol*** *$\sigma$ is the linear and continuous operator*

$$\mathrm{Op}_0(\sigma)\colon \mathcal{S}_0(\mathcal{G}) \to \mathcal{S}_0'(\mathcal{G})$$

*defined by the formal integral*

$$\mathrm{Op}_0(\sigma)f(x) = \int_{\widehat{\mathcal{G}}} \sigma(x,\xi)\hat{f}(\xi)\langle \xi, x\rangle \, d\xi, \quad \forall x \in \mathcal{G}. \tag{2.216}$$

*Equivalently, we can define it weakly by*

$$\langle \mathrm{Op}_0(\sigma)f, g\rangle = \langle \sigma, R(g,f)\rangle, \qquad \forall f, g \in \mathcal{S}_0(\mathcal{G}), \tag{2.217}$$

*where $\langle \cdot,\cdot\rangle = {}_{\mathcal{S}_0'}\langle \cdot,\cdot\rangle_{\mathcal{S}_0}$ and $R(g,f)$ is the cross-Rihaczek distribution of $f$ and $g$ as in* (2.96).

To see that $\mathrm{Op}_0(\sigma)$ in the above definition is actually well-defined, linear and continuous from $\mathcal{S}_0(\mathcal{G})$ into $\mathcal{S}_0'(\mathcal{G})$ see, e.g., [101, Corollary 4.2, Theorem 5.3].

### 2.7.7 Fourier multiplier

Fourier multipliers [13] are well known in both partial differential equations and signal analysis. They can be viewed as a special instance of Kohn-Nirenberg operators with symbol which depends only on the frequency variables $\omega \in \mathbb{R}^d$.

**Definition 2.7.22.** *Let $m \in \mathcal{S}'(\mathbb{R}^d)$. The* ***Fourier multiplier with multiplier*** *$m \in \mathcal{S}'(\mathbb{R}^d)$ is the linear and continuous operator*

$$T_m\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

*defined by*

$$T_m f(t) := \mathcal{F}^{-1}(m\mathcal{F}f)(t) = \left(\mathcal{F}^{-1}m * f\right)(t), \quad \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{2.218}$$

The function

$$h = \mathcal{F}^{-1}m \tag{2.219}$$

is called the ***impulse response*** or ***transfer function*** in signal processing [116].
Such operator is a well-defined linear mapping from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Boundedness properties of Fourier multipliers $T_m : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ are studied in the classical paper by Hörmander [98]. The most important examples of Fourier multipliers can be obtained by taking $p = q = 2$. Then $T_m$ is bounded if and only if the multiplier $m \in L^\infty(\mathbb{R}^d)$ and $\|T_m\|_{B(L^2)} = \|m\|_{L^\infty}$. For $p = q = 1$ and $p = q = \infty$ the only bounded Fourier multipliers are Fourier transforms of bounded measures. For the cases $p = q \in (1,\infty) \setminus \{2\}$ only sufficient conditions on $m$ are known. The assumptions $m \in L^\infty$ is necessary, though. The main result by Hörmander in [98, Theorem 1.11] (see also its generalization to locally compact groups [1]) states:

**Theorem 2.7.23.** *If $1 < p \le 2 \le q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ with*

$$1/q = 1/r + 1/p, \tag{2.220}$$

*then $T_m$ is bounded $T_m : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$.*

Here $L^{r,\infty}(\mathbb{R}^d)$ is the weak $L^r$-space, see (2.189). For example, every $m$ on $\mathbb{R}^d$ with $|m(\omega)| \le C|\omega|^{-d/r}$, $C > 0$, satisfies $m \in L^{r,\infty}(\mathbb{R}^d)$. For simplicity, we defined $L^{\infty,\infty}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$, so that, inserting $r = \infty$ in Theorem 2.7.23 we recapture the boundedness of the multiplier $T_m$ on $L^2(\mathbb{R}^d)$.

### 2.7.8 Linear time invariant filters on $\mathcal{G} = \mathbb{Z}_N$

Here we apply the identification $\mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$, see (2.104). In the finite discrete setting, i.e. $\mathcal{G} = \mathbb{Z}_N$, Fourier multiplier are also named linear time invariant (LTI) filters [115].

**Definition 2.7.24.** *A* ***Fourier multiplier on*** $\mathbb{Z}_N$*, or* ***linear time invariant (LTI) filter on*** $\mathbb{C}^N$*, or* ***convolution operator***

$$H\colon \mathbb{C}^N \to \mathbb{C}^N$$

*is the linear operator uniquely determined by the so called* ***impulse response*** $h \in \mathbb{C}^N$ *via* ***circular convolution***

$$Hf(u) \coloneqq h * f(u) \coloneqq \sum_{k=0}^{N-1} h(u-k)f(k), \quad \forall f \in \mathbb{C}^N,\, u = 0, \dots, N-1, \tag{2.221}$$

*where* $u - k$ *is considered modulus* $N$*.*

Clearly, $h = H\delta$ where $\delta$ is as in (2.108) and

$$Hf(u) = h * f(u) = \left(\mathcal{F}_N^{-1}\mathcal{F}_N h * \mathcal{F}_N^{-1}\mathcal{F}_N f\right)(u) = \mathcal{F}_N^{-1}\left(\hat{h}\cdot\hat{f}\right)(u),$$

see (2.219), $\hat{h}$ is also called ***frequency response***. It is straight forward to see that a LTI filter $H$ on $\mathbb{C}^N$ has matrix representation

$$K_H(u,v) = h(u-v), \qquad u, v = 0, \dots, N-1. \tag{2.222}$$

We can define the associated discrete spreading function $\eta_H \in \mathbb{C}^{N\times N}$ as

$$\eta_H(u,v) = h \otimes \delta(u,v). \tag{2.223}$$

## 2.8 Gelfand-Shilov setting

We now revisit sections 2.3, 2.5 and 2.7 in the framework of Gelfand-Shilov spaces $\mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ and $\Sigma^\gamma_\tau(\mathbb{R}^d)$. For some references about the Gelfand-Shilov setting we address, e.g. to [76, 118, 141, 143]. We introduce the Gelfand-Shilov spaces, then define the STFT for ultra-distributions and give the definition of ultra-modulation spaces, eventually localization and $\tau$-pseudo-differential operators are shown in the present setting.

In this section, we drop Assumptions 2.5.1 about polynomial growth of the weights involved. In fact, we shall consider the weights $w^\gamma_k$ defined in (2.9).

Proposition 2.8.13 was published by N. Teofanov and the author in [11], it extends the convolutions for modulation spaces presented in Proposition 2.5.19, [9]. Also Lemma 2.8.15 and Proposition 2.8.16 appeared for the first time in [11].

### 2.8.1 Ggelfand-Shilov spaces and their duals

Let $h, \gamma, \tau > 0$ be fixed. Then $S^\gamma_{\tau;h}(\mathbb{R}^d)$ is the Banach space of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{S^\gamma_{\tau;h}} \coloneqq \sup_{p,q\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d} \frac{|x^p\partial^q f(x)|}{h^{|p|+|q|}|p|!^\tau\,|q|!^\gamma} < +\infty, \tag{2.224}$$

endowed with the norm (2.224).

**Definition 2.8.1.** *Let $\gamma,\tau > 0$. The* **Gelfand-Shilov spaces** $\mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ **and** $\Sigma^\gamma_\tau(\mathbb{R}^d)$ *are defined as unions and intersections of $S^\gamma_{\tau;h}(\mathbb{R}^d)$ with respective inductive and projective limit topologies:*

$$\mathcal{S}^\gamma_\tau(\mathbb{R}^d) \coloneqq \bigcup_{h>0} S^\gamma_{\tau;h}(\mathbb{R}^d) \quad \textit{and} \quad \Sigma^\gamma_\tau(\mathbb{R}^d) \coloneqq \bigcap_{h>0} S^\gamma_{\tau;h}(\mathbb{R}^d).$$

Note that $\Sigma^\gamma_\tau(\mathbb{R}^d) \neq \{0\}$ if and only if $\tau+\gamma \geq 1$ and $(\tau,\gamma) \neq (1/2,1/2)$, and $\mathcal{S}^\gamma_\tau(\mathbb{R}^d) \neq \{0\}$ if and only if $\tau+\gamma\geq 1$, see [76, 118]. For every $\tau,\gamma,\varepsilon > 0$ we have

$$\Sigma^\gamma_\tau(\mathbb{R}^d) \hookrightarrow \mathcal{S}^\gamma_\tau(\mathbb{R}^d) \hookrightarrow \Sigma^{\gamma+\varepsilon}_{\tau+\varepsilon}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d). \tag{2.225}$$

If $\tau+\gamma \geq 1$, then the last two inclusions in (2.225) are dense, and if in addition $(\tau,\gamma) \neq (1/2,1/2)$ then the first inclusion in (2.225) is dense. Moreover, for $\gamma < 1$ the elements of $\mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ can be extended to entire functions on $\mathbb{C}^d$ satisfying suitable exponential bounds, [76].
In the sequel we will also use the following notations:

$$\mathcal{S}^{(\gamma)}(\mathbb{R}^d) \coloneqq \Sigma^\gamma_\gamma(\mathbb{R}^d), \qquad \mathcal{S}^{\{\gamma\}}(\mathbb{R}^d) \coloneqq \mathcal{S}^\gamma_\gamma(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}^*(\mathbb{R}^d),$$

where $*$ stands for $(\gamma)$ or $\{\gamma\}$.

**Definition 2.8.2.** *The* **Gelfand-Shilov distribution spaces** $(\mathcal{S}^\gamma_\tau)'(\mathbb{R}^d)$ *and* $(\Sigma^\gamma_\tau)'(\mathbb{R}^d)$ *are the projective and inductive limit respectively of $(S^\gamma_{\tau;h})'(\mathbb{R}^d)$, the topological dual of $S^\gamma_{\tau;h}(\mathbb{R}^d)$:*

$$(\mathcal{S}^\gamma_\tau)'(\mathbb{R}^d) \coloneqq \bigcap_{h>0} (S^\gamma_{\tau;h})'(\mathbb{R}^d) \quad \textit{and} \quad (\Sigma^\gamma_\tau)'(\mathbb{R}^d) \coloneqq \bigcup_{h>0} (S^\gamma_{\tau;h})'(\mathbb{R}^d).$$

It follows that $\mathcal{S}'(\mathbb{R}^d) \hookrightarrow (\mathcal{S}^\gamma_\tau)'(\mathbb{R}^d)$ when $\tau+\gamma\geq 1$, and if in addition $(\tau,\gamma) \neq (1/2,1/2)$, then $(\mathcal{S}^\gamma_\tau)'(\mathbb{R}^d) \hookrightarrow (\Sigma^\gamma_\tau)'(\mathbb{R}^d)$.

The Gelfand-Shilov spaces enjoy beautiful symmetric characterizations which also involve the Fourier transform of their elements. The following result has been reinvented several times, in similar or analogous terms, see [22, 89, 102, 114].

**Theorem 2.8.3.** *Let $\gamma,\tau \geq 1/2$. The following conditions are equivalent:*

*(i)* $f \in \mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ *(resp.* $f \in \Sigma^\gamma_\tau(\mathbb{R}^d)$*);*

*(ii) There exist (resp. for every) constants $A, B > 0$ such that*

$$\|x^p f(x)\|_{L^\infty} \lesssim A^{|p|}|p|!^\tau \quad \textit{and} \quad \|\omega^q \hat{f}(\omega)\|_{L^\infty} \lesssim B^{|q|}|q|!^\gamma, \quad \forall\, p,q \in \mathbb{N}^d_0;$$

*(iii) There exist (resp. for every) constants $A, B > 0$ such that*

$$\|x^p f(x)\|_{L^\infty} \lesssim A^{|p|}|p|!^\tau \quad \textit{and} \quad \|\partial^q f(x)\|_{L^\infty} \lesssim B^{|q|}|q|!^\gamma, \quad \forall\, p,q \in \mathbb{N}^d_0;$$

*(iv) There exist (resp. for every) constants $h, k > 0$ such that*

$$\|f(x)e^{h|x|^{1/\tau}}\|_{L^\infty} < +\infty \quad \textit{and} \quad \|\hat{f}(\omega)e^{k|\omega|^{1/\gamma}}\|_{L^\infty} < +\infty;$$

*(v) There exist (resp. for every) constants $h, B > 0$ such that*

$$\|(\partial^q f)(x)e^{h|x|^{1/\tau}}\|_{L^\infty} \lesssim B^{|q|}|q|!^\gamma, \quad \forall\, q \in \mathbb{N}^d_0. \tag{2.226}$$

Moreover, we could consider any $L^p$-norm, $1 \leq p < \infty$ instead of $L^\infty$-norm in Theorem 2.8.3, cf. [102].

By using Theorem 2.8.3 it can be shown that the Fourier transform is a topological isomorphism between $\mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ and $\mathcal{S}^\tau_\gamma(\mathbb{R}^d)$, $\gamma, \tau \geq 1/2$ ($\mathcal{F}(\mathcal{S}^\gamma_\tau)(\mathbb{R}^d) = \mathcal{S}^\tau_\gamma(\mathbb{R}^d)$), which extends to a continuous linear transform from $(\mathcal{S}^\gamma_\tau)'(\mathbb{R}^d)$ onto $(\mathcal{S}^\tau_\gamma)'(\mathbb{R}^d)$. Similar considerations hold for partial Fourier transforms with respect to some choice of variables. In particular, if $\gamma = \tau$ and $\gamma \geq 1/2$ then $\mathcal{F}(\mathcal{S}^\gamma_\gamma)(\mathbb{R}^d) = \mathcal{S}^\gamma_\gamma(\mathbb{R}^d)$, and if moreover $\gamma > 1/2$, then $\mathcal{F}(\Sigma^\gamma_\gamma)(\mathbb{R}^d) = \Sigma^\gamma_\gamma(\mathbb{R}^d)$, and similarly for their distribution spaces. Due to this fact, corresponding dual spaces are referred to as ***tempered ultra-distributions*** (of Roumieu and Beurling type respectively), see [118].

The combination of global regularity with suitable decay properties at infinity (cf. (2.226)) which is built in the very definition of $\mathcal{S}^\gamma_\tau(\mathbb{R}^d)$ and $\Sigma^\gamma_\tau(\mathbb{R}^d)$, makes them suitable for the study of different problems in mathematical physics, [76, 80, 114]. We refer to [33, 34, 134, 135] for the study of localization operators in the context of Gelfand-Shilov spaces. See also [139, 142, 143] for related studies.

### 2.8.2 Time-frequency distribution and operators

As done for the framework of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and its dual $\mathcal{S}'(\mathbb{R}^d)$ in Section 2.3, we can defined the STFT of $f \in \mathcal{S}^{(1)'}(\mathbb{R}^d)$ w.r.t. $g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$:

$$V_g f(x,\omega) := \langle f, \pi(x,\omega)g\rangle, \qquad x,\omega \in \mathbb{R}^d,$$

where the dual pair $\langle\cdot,\cdot\rangle$ is the one between $\mathcal{S}^{(1)'}(\mathbb{R}^d)$ and $\mathcal{S}^{(1)}(\mathbb{R}^d)$. Since $\mathcal{S}^{(1)}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$, the definition of $W_\tau(f,g)$ makes sense for $f, g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ too. In particular we recall the following results.

**Lemma 2.8.4.** *Let $g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$.*

*(i) If $\tau \in (0,1)$, then*

$$W_\tau(f,g)(x,\omega) = \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} V_{\mathcal{A}_\tau g} f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\omega\right), \quad \forall\,(x,\omega) \in \mathbb{R}^{2d}; \tag{2.227}$$

*(ii) if $\tau = 0$, then*

$$W_0(f,g)(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{g}(\omega)} = R(f,g)(x,\omega), \quad \forall\,(x,\omega) \in \mathbb{R}^{2d};$$

*(iii) if $\tau = 1$, then*

$$W_1(f,g)(x,\omega) = e^{2\pi i x\omega}\overline{g(x)}\hat{f}(\omega) = \overline{R(g,f)}(x,\omega), \quad \forall\,(x,\omega) \in \mathbb{R}^{2d}.$$

**Theorem 2.8.5.** *Let $\mathcal{S}^*(\mathbb{R}^d)$ denote $\mathcal{S}^{\{\gamma\}}(\mathbb{R}^d)$, $\gamma \geq 1/2$, or $\mathcal{S}^{(\gamma)}(\mathbb{R}^d)$, $\gamma > 1/2$. Moreover, let $g \in \mathcal{S}^*(\mathbb{R}^d) \setminus \{0\}$ and $\tau \in [0,1]$. Then the following are true:*

*(i) if $f \in \mathcal{S}^*(\mathbb{R}^d)$, then $W_\tau(f,g), V_g f \in \mathcal{S}^*(\mathbb{R}^{2d})$;*

*(ii) if $f \in (\mathcal{S}^*)'(\mathbb{R}^d)$ and $W_\tau(f,g) \in \mathcal{S}^*(\mathbb{R}^{2d})$ or $V_g f \in \mathcal{S}^*(\mathbb{R}^{2d})$, then $f \in \mathcal{S}^*(\mathbb{R}^d)$.*

*Proof.* The proof for the STFT and $W_{1/2}$ can be found in several sources, see e.g. [89, 132, 139]. The case $\tau \in [0,1]$, $\tau \neq 1/2$, can be proved in a similar fashion and is left for the reader as an exercise. □

### 2.8.3 Ultra-modulation spaces

We use the terminology ultra-modulation spaces in order to emphasize that such spaces may contain ultra-distributions, contrary to the most usual situation when members of modulation spaces are tempered distributions. However, ultra-modulation spaces belong to the family of modulation spaces introduced in [56]. We refer to e.g. [141, 143] for a general approach to the broad class of modulation spaces.
Recall that the weight class $\mathscr{P}_E(\mathbb{R}^{2d})$ was introduced in Subsubsection 2.2.1.1.

**Definition 2.8.6.** *Fix a non-zero window $g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$, a weight $m \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $0 < p, q \leq \infty$. The* ***ultra-modulation space*** *$M^{p,q}_m(\mathbb{R}^d)$ consists of all tempered ultra-distributions $f \in \mathcal{S}^{(1)'}(\mathbb{R}^d)$ such that the quasi-norm*

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} \tag{2.228}$$

*(obvious modifications with $p = \infty$ or $q = \infty$) is finite.*

We collect properties for ultra-modulation spaces them in the following theorem in the same manner of [141, 142], see references therein also.

**Theorem 2.8.7.** *Consider $0 < p, p_1, p_2, q, q_1, q_2 \leq \infty$ and weights $m, m_1, m_2 \in \mathscr{P}_E(\mathbb{R}^{2d})$. Let $\|\cdot\|_{M^{p,q}_m}$ be given by (2.228) for a fixed $g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus \{0\}$. Then:*

- *(i)* $\left(M^{p,q}_m(\mathbb{R}^d), \|\cdot\|_{M^{p,q}_m}\right)$ *is a quasi-Banach, if $p, q \geq 1$ it is a Banach space too;*
- *(ii) if $\tilde{g} \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus \{0\}$, $\tilde{g} \neq g$, then it induces a quasi-norm equivalent to $\|\cdot\|_{M^{p,q}_m}$;*
- *(iii) if $p_1 \leq p_2$, $q_1 \leq q_2$ and $m_2 \lesssim m_1$, then:*

$$\mathcal{S}^{(1)}(\mathbb{R}^d) \hookrightarrow M^{p_1,q_1}_{m_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{m_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}^{(1)'}(\mathbb{R}^d);$$

- *(iv) if $p, q < \infty$, then :*

$$\left(M^{p,q}_m(\mathbb{R}^d)\right)' \cong M^{p',q'}_{1/m}(\mathbb{R}^d),$$

*where*

$$p' := \begin{cases} \infty & \text{if} \quad 0 < p \leq 1 \\ \frac{p}{p-1} & \text{if} \quad 1 < p < \infty \end{cases}$$

*and similarly for $q'$.*

**Remark 2.8.8.** *Point (ii) of the previous theorem tell us that the definition of $M^{p,q}_m(\mathbb{R}^d)$ is independent of the choice of the window. Moreover, it can be shown that the class for window functions can be extended from $\mathcal{S}^{(1)}(\mathbb{R}^d)$ to $M^r_v(\mathbb{R}^d)$, where $r \leq p, q$ and $v \in \mathscr{P}_E(\mathbb{R}^{2d})$ is submultiplicative and such that $m$ is $v$-moderate, [142].*
*We refer to [23] for the density of $\mathcal{S}^{(1)}(\mathbb{R}^d)$ in $M^{p,q}_m(\mathbb{R}^d)$.*

The following proposition is proved in e.g. [133, Theorem 4.1], [139, Theorem 3.9].

**Proposition 2.8.9.** *Consider $\gamma \geq 1$ and $0 < p, q \leq \infty$. Then*

$$\mathcal{S}^{(\gamma)}(\mathbb{R}^d) = \bigcap_{k \geq 0} M^{p,q}_{w^\gamma_k}(\mathbb{R}^d), \qquad \mathcal{S}^{(\gamma)'}(\mathbb{R}^d) = \bigcup_{k \geq 0} M^{p,q}_{1/w^\gamma_k}(\mathbb{R}^d).$$

In some situations it is convenient to consider ultra-modulation spaces as subspaces of $\mathcal{S}^{\{1/2\}'}(\mathbb{R}^d)$ (taking the window $g$ in $\mathcal{S}^{\{1/2\}}(\mathbb{R}^d)$), see for example [23, 142]. However, for our purposes it is sufficient to consider the weights in $\mathscr{P}_E(\mathbb{R}^{2d})$, and then $M^{p,q}_m(\mathbb{R}^d)$ is a subspace of $\mathcal{S}^{(1)'}(\mathbb{R}^d)$. We address the reader to [142, Proposition 1.1] and references quoted there for more details.

We restate [33, Proposition 2.6] in a simplified case suitable to our purposes.

**Proposition 2.8.10.** *Assume* $1 \leq p, q \leq \infty$, $m \in \mathscr{P}_E(\mathbb{R}^{2d})$ *and* $g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ *such that* $\|g\|_{L^2} = 1$. *Then for every* $f \in M^{p,q}_m(\mathbb{R}^d)$ *the following inversion formula holds true:*

$$f = \int_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x g \, dx d\omega, \tag{2.229}$$

*where the equality holds in* $M^{p,q}_m(\mathbb{R}^d)$.

The embeddings between modulation spaces are studied by many authors. We recall the recent contribution [90, Theorem 4.11], which is convenient for our purposes and which will be used in Lemma 3.3.3.

**Theorem 2.8.11.** *Let* $0 < p_j, q_j \leq \infty$, $s_j, t_j \in \mathbb{R}$ *for* $j = 1, 2$ *and consider the polynomial weights* $v_{t_j}$, $v_{s_j}$ *defined as in* (2.7). *Then*

$$M^{p_1,q_1}_{v_{t_1}\otimes v_{s_1}}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{v_{t_2}\otimes v_{s_2}}(\mathbb{R}^d)$$

*if the following two conditions hold true:*

(*i*) $(p_1, p_2, t_1, t_2)$ *satisfies one of the following conditions:*

$$\begin{aligned} (\mathcal{C}_1) \qquad & \frac{1}{p_2} \leq \frac{1}{p_1}, \qquad t_2 \leq t_1, \\ (\mathcal{C}_2) \qquad & \frac{1}{p_2} > \frac{1}{p_1}, \qquad \frac{1}{p_2} + \frac{t_2}{d} < \frac{1}{p_1} + \frac{t_1}{d}; \end{aligned}$$

(*ii*) $(q_1, q_2, s_1, s_2)$ *satisfies one of the conditions* $(\mathcal{C}_1)$ *or* $(\mathcal{C}_2)$ *with* $p_j$ *and* $t_j$ *replaced by* $q_j$ *and* $s_j$ *respectively.*

Discrete equivalent norms produced by means of Gabor frames make of ultra-modulation spaces a natural framework for time-frequency analysis. We address the reader to [75, 82, 141, 142].

**Theorem 2.8.12.** *Consider* $m, v \in \mathscr{P}_E(\mathbb{R}^{2d})$ *such that* $v$ *is submultiplicative and* $m$ *is* $v$-*moderate. Take* $\Lambda := \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, *for some* $\alpha, \beta > 0$, *and* $g, h \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ *such that* $S_{g,h} = I$ *on* $L^2(\mathbb{R}^d)$. *Then*

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda) h = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) h\rangle \pi(\lambda) g, \quad \forall\, f \in M^{p,q}_m(\mathbb{R}^d),$$

*with unconditional convergence in* $M^{p,q}_m(\mathbb{R}^d)$ *if* $0 < p, q < \infty$ *and with weak-*$*$ *convergence in* $M^\infty_{1/v}(\mathbb{R}^d)$ *otherwise. Moreover, there exist* $0 < A \leq B$ *such that, for every* $f \in M^{p,q}_m(\mathbb{R}^d)$,

$$A\|f\|_{M^{p,q}_m} \leq \left( \sum_{n\in\mathbb{Z}^d} \left( \sum_{k\in\mathbb{Z}^d} |\langle f, \pi(\alpha k, \beta n) g\rangle|^p m(\alpha k, \beta n)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq B\|f\|_{M^{p,q}_m},$$

*independently of $p,q,$ and $m$. Equivalently:*

$$\|f\|_{M^{p,q}_m(\mathbb{R}^d)} \asymp \|(\langle f,\pi(\lambda)g\rangle)_\lambda\|_{\ell^{p,q}_m(\Lambda)} = \|(V_g f(\lambda))_\lambda\|_{\ell^{p,q}_m(\Lambda)}. \tag{2.230}$$

*Similar inequalities hold with $g$ replaced by $h$.*

Now we are able to prove the convolution relations for ultra-modulations spaces which will be used to prove the main results of Section 3.3 in Chapter 3. For the Banach cases with weight of at most polynomial growth at infinity, convolution relations were studied in e.g [28, 136, 137]. We modify the technique used in [9] to the Gelfand-Shilov framework presented so far. The essential tool is the equivalence between continuous and discrete norm (2.230).

**Proposition 2.8.13.** ([11, Proposition 2.24]) *Let there be given $0<p,q,r,t,u,\gamma\leq\infty$ such that*

$$\frac{1}{u}+\frac{1}{t}=\frac{1}{\gamma},$$

*and*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}, \qquad \text{for } 1\leq r\leq\infty$$

*whereas*

$$p=q=r, \qquad \text{for } 0<r<1.$$

*Consider $m,v,\nu\in\mathscr{P}_E(\mathbb{R}^{2d})$ such that $m$ is $v$-moderate. Then*

$$M^{p,u}_{m_1\otimes\nu}(\mathbb{R}^d)*M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d)\hookrightarrow M^{r,\gamma}_m(\mathbb{R}^d),$$

*where $m_1$, $v_1$, $v_2$ are defined as in* (2.10).

*Proof.* First observe that due to Lemma 2.2.11 and Lemma 2.2.12 it follows that the ultra-modulation spaces which came into play are well defined.

The main tool is the idea contained in [28, Proposition 2.4]. We take the ultra-modulation norm with respect to the Gaussian windows $g_0(x):=e^{-\pi x^2}\in\mathcal{S}^{\{1/2\}}(\mathbb{R}^d)$ and $g(x):=2^{-d/2}e^{-\pi x^2/2}=(g_0*g_0)(x)\in\mathcal{S}^{\{1/2\}}(\mathbb{R}^d)$.

Since the involution operator $g^*(x)=\overline{g(-x)}$ and the modulation operator $M_\omega$ commute, by a direct computation we have

$$M_\omega(g_0^**g_0^*)=M_\omega g_0^**M_\omega g_0^*$$

and

$$V_g f(x,\omega)=e^{-2\pi i x\omega}(f*M_\omega g^*)(x).$$

Thus, by using the associativity and commutativity of the convolution product, we obtain

$$V_g(f*h)(x,\omega)=e^{-2\pi i x\omega}\big((f*h)*M_\omega g^*\big)(x)=e^{-2\pi i x\omega}\big((f*M_\omega g_0^*)*(h*M_\omega g_0^*)\big)(x)\,.$$

We use the norm equivalence (2.230) for a suitable $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$, and then the $v$-moderateness in order to majorize $m$:

$$m(\alpha k,\beta n)\lesssim m(\alpha k,0)v(0,\beta n)=m_1(\alpha k)v_2(\beta n).$$

Eventually Young's convolution inequality for sequences is used in the $k$-variable and Hölder's one in the $n$-variable. Indeed both inequalities can be used since $p,q,r,\ \gamma,t,u$ fulfil the assumptions of the proposition. We write in details the case when $r,\gamma,t,u<\infty$ and leave to the reader the remaining cases, when one among the indices $r, \gamma, t, u$ is equal to $\infty$, which can be done analogously.

$$\begin{aligned}
\|f*h\|_{M^{r,\gamma}_m} &\asymp \|((V_g(f*h))(\alpha k,\beta n)m(\alpha k,\beta n))_{k,n}\|_{\ell^{r,\gamma}(\mathbb{Z}^{2d})}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}|(f*M_{\beta n}g_0^*)*(h*M_{\beta n}g_0^*)(\alpha k)|^r m_1(\alpha k)^r\right)^{\gamma/r} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&= \left(\sum_{n\in\mathbb{Z}^d}\|(f*M_{\beta n}g_0^*)*(h*M_{\beta n}g_0^*)\|^\gamma_{\ell^r_{m_1}(\alpha\mathbb{Z}^d)} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\|f*M_{\beta n}g_0^*\|^\gamma_{\ell^p_{m_1}(\alpha\mathbb{Z}^d)}\,\|h*M_{\beta n}g_0^*\|^\gamma_{\ell^q_{v_1}(\alpha\mathbb{Z}^d)}\,v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\|f*M_{\beta n}g_0^*\|^u_{\ell^p_{m_1}(\alpha\mathbb{Z}^d)}\nu(\beta n)^u\right)^{\frac{1}{u}}\left(\sum_{n\in\mathbb{Z}^d}\|h*M_{\beta n}g_0^*\|^t_{\ell^q_{v_1}(\alpha\mathbb{Z}^d)}\frac{v_2(\beta n)^t}{\nu(\beta n)^t}\right)^{\frac{1}{t}}\\
&= \|((V_{g_0}f)(\lambda))_\lambda\|_{\ell^{p,u}_{m_1\otimes\nu}(\Lambda)}\,\|((V_{g_0}h)(\lambda))_\lambda\|_{\ell^{q,t}_{v_1\otimes v_2\nu^{-1}}(\Lambda)}\\
&\asymp \|f\|_{M^{p,u}_{m_1\otimes\nu}}\,\|h\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}}.
\end{aligned}$$

Here we wrote $m_1\otimes\nu$ in place of $(m_1\otimes\nu)_\Lambda$, similarly for the other weights appearing in the lower indexes.This concludes the proof. □

### 2.8.4 Localization operators and $\tau$-quantization

Localization operators $A_a^{\psi_1,\psi_2}$ and $\tau$-pseudo-differential operators $\mathrm{Op}_\tau(\sigma)$ can be defined also in the setting of Gelfand-Shilov spaces. Namely, in the definition given in (2.193) and (2.207) substitute the dual pair $(\mathcal{S}',\mathcal{S})$ with $(\mathcal{S}^{(1)'},\mathcal{S}^{(1)})$.
The proof of the following lemma is omitted, since it follows by a slight modification of the proof of [32, Lemma 4.1].

**Lemma 2.8.14.** *Consider $\tau\in[0,1]$, $g\in\mathcal{S}^{(1)}(\mathbb{R}^d)$, $\Phi_\tau := W_\tau(g,g)\in\mathcal{S}^{(1)}(\mathbb{R}^{2d})$. If $\sigma\in\mathcal{S}^{(1)'}(\mathbb{R}^{2d})$, then*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g,\pi(w)g\rangle| = |V_{\Phi_\tau}\sigma\left(\mathcal{T}_\tau(w,z),J(w-z)\right)|,\quad \forall\, z,w\in\mathbb{R}^{2d}, \tag{2.231}$$

*where $z=(z_1,z_2), w=(w_1,w_2)\in\mathbb{R}^{2d}$ and $\mathcal{T}_\tau$ and $J$ are defined as in* (2.212).

The following lemma can be viewed as a form of the inversion formula (2.229).

**Lemma 2.8.15.** ([11, Lemma 3.2]) *Let $\tau\in[0,1]$ and $\sigma\in\mathcal{S}^{(1)'}(\mathbb{R}^{2d})$. If $g\in\mathcal{S}^{(1)}(\mathbb{R}^d)$ with $\|g\|_{L^2}=1$ and $f\in\mathcal{S}^{(1)}(\mathbb{R}^d)$, then*

$$\mathrm{Op}_\tau(\sigma)f = \int_{\mathbb{R}^{2d}} V_gf(z)\,\mathrm{Op}_\tau(\sigma)(\pi(z)g)\,dz, \tag{2.232}$$

*in the sense that*

$$\langle \mathrm{Op}_\tau(\sigma)f,\varphi\rangle = \int_{\mathbb{R}^{2d}} V_gf(z)\,\langle \mathrm{Op}_\tau(\sigma)(\pi(z)g),\varphi\rangle\,dz,\qquad \forall\,\varphi\in\mathcal{S}^{(1)}(\mathbb{R}^d).$$

*Proof.* Consider $\tau \in (0,1)$ and recast the $\tau$-Winger distribution $W_\tau(\varphi, f)$ using the operator $\mathcal{A}_\tau f(t) = f\left(\frac{\tau-1}{\tau}t\right)$ introduced in (2.83):

$$\begin{aligned}
W_\tau(\varphi, f)(x,\omega) &= \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} V_{\mathcal{A}_\tau f}\varphi\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\omega\right)\\
&= \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} \langle \varphi, M_{\frac{1}{\tau}\omega} T_{\frac{1}{1-\tau}x} \mathcal{A}_\tau f\rangle\\
&= \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} \langle \left(\frac{\tau}{1-\tau}\right)^d \mathcal{A}_{1-\tau} T_{-\frac{1}{1-\tau}x} M_{-\frac{1}{\tau}\omega}\varphi, f\rangle\\
&= \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} \int_{\mathbb{R}^{2d}} \overline{V_g f(z)}\, \langle \left(\frac{\tau}{1-\tau}\right)^d \mathcal{A}_{1-\tau} T_{-\frac{1}{1-\tau}x} M_{-\frac{1}{\tau}\omega}\varphi, \pi(z)g\rangle\, dz\\
&= \int_{\mathbb{R}^{2d}} \overline{V_g f(z)}\, \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} \langle \varphi, M_{\frac{1}{\tau}\omega} T_{\frac{1}{1-\tau}x} \mathcal{A}_\tau \pi(z) g\rangle\, dz\\
&= \int_{\mathbb{R}^{2d}} \overline{V_g f(z)}\, \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau}\omega x} V_{\mathcal{A}_\tau \pi(z) g}\varphi\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\omega\right) dz\\
&= \int_{\mathbb{R}^{2d}} \overline{V_g f(z)}\, W_\tau(\varphi, \pi(z)g)(x,\omega)\, dz.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \mathrm{Op}_\tau(\sigma) f, \varphi\rangle = \langle \sigma, W_\tau(\varphi, f)\rangle &= \langle \sigma, \int_{\mathbb{R}^{2d}} \overline{V_g f(z)}\, W_\tau(\varphi, \pi(z)g)(x,\omega)\, dz\rangle\\
&= \int_{\mathbb{R}^{2d}} V_g f(z)\, \langle \sigma, W_\tau(\varphi, \pi(z)g)(x,\omega)\rangle\, dz = \int_{\mathbb{R}^{2d}} V_g f(z)\, \langle \mathrm{Op}_\tau(\sigma)(\pi(z)g), \varphi\rangle\, dz
\end{aligned}$$

and (2.232) holds true when $\tau \in (0,1)$.

For the cases $\tau = 0, 1$ we need the operator $J$ defined in (2.212) and the following equalities which come from easy computations (cf. [82]):

$$V_g f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega, -x),\ \mathcal{F}T_x = M_{-x}\mathcal{F},\ \mathcal{F}M_\omega = T_\omega \mathcal{F},\ T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x.$$

Therefore (2.232) is proved for $\tau = 0, 1$ in the following manner. We put $z = (x,\omega)$ and let $\sigma$ acts on functions of variables $(y, \eta)$:

$$\begin{aligned}
\langle \mathrm{Op}_0(\sigma) f, \varphi\rangle &= \langle \sigma, e^{-2\pi i y\eta} \varphi(y) \overline{\hat{f}(\eta)}\rangle\\
&= \langle \sigma, e^{-2\pi i y\eta} \varphi(y) \int_{\mathbb{R}^{2d}} \overline{V_{\hat{g}} \hat{f}(z') \pi(z') \hat{g}(\eta)}\, dz'\rangle\\
&= \langle \sigma, \int_{\mathbb{R}^{2d}} \overline{V_g f(z)} e^{-2\pi i y\eta} \varphi(y) \overline{e^{2\pi i x\omega} \pi(Jz) \hat{g}(\eta)}\, dz\rangle\\
&= \int_{\mathbb{R}^{2d}} V_g f(z) \langle \sigma, e^{-2\pi i y\eta} \varphi(y) \overline{\widehat{\pi(z) g}(\eta)}\rangle\, dz\\
&= \int_{\mathbb{R}^{2d}} V_g f(z) \langle \sigma, W_0(\varphi, \pi(z)g)\rangle\, dz\\
&= \int_{\mathbb{R}^{2d}} V_g f(z) \langle \mathrm{Op}_0(\sigma) \pi(z) g, \varphi\rangle\, dz.
\end{aligned}$$

The case $\tau = 1$, i.e.

$$\langle \mathrm{Op}_1(\sigma) f, \varphi \rangle = \int_{\mathbb{R}^{2d}} V_g f(z) \langle \mathrm{Op}_1(\sigma)\pi(z) g, \varphi\rangle \, dz,$$

can be proved in the same manner. The details are left to the reader. □

**Proposition 2.8.16.** ([11, Propositio 2.16]) *Consider* $a \in \mathcal{S}'(\mathbb{R}^{2d})$, $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ *and* $\tau \in [0,1]$. *Then*

$$A_a^{\psi_1,\psi_2} = \mathrm{Op}_\tau(a * W_\tau(\psi_2, \psi_1)).$$

*Proof.* For any $\tau \in [0,1]$, let us define

$$A_{a,\tau}^{\psi_1,\psi_2} := \mathrm{Op}_\tau(a * W_\tau(\psi_2, \psi_1)),$$

where $\psi_1, \psi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ and $a \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$. We show that $A_a^{\psi_1,\psi_2} = A_{a,\tau}^{\psi_1,\psi_2}$, for every $\tau \in [0,1]$, using the Schwartz kernel theorem for $\mathcal{S}^{(1)}(\mathbb{R}^d)$ and $\mathcal{S}^{(1)'}(\mathbb{R}^d)$. From the weak definition of $A_a^{\psi_1,\psi_2}$ it follows that

$$\langle A_a^{\psi_1,\psi_2} f, g\rangle = \langle K(A_a^{\psi_1,\psi_2}), g \otimes \overline{f}\rangle,$$

where the kernel $K(A_a^{\psi_1,\psi_2})$ of the operator $A_a^{\psi_1,\psi_2}$ is given by

$$K(A_a^{\psi_1,\psi_2})(t,y) = \int_{\mathbb{R}^{2d}} a(x,\omega) \overline{M_\omega T_x \psi_1}(y) M_\omega T_x \psi_2(t) dx d\omega. \tag{2.233}$$

It remains to calculate the kernel of $A_{a,\tau}^{\psi_1,\psi_2}$. By the commutation relation $T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x$, and the covariance property of $\tau$-Wigner transform:

$$W_\tau(T_x M_\omega f, T_x M_\omega g)(p,q) = W_\tau(f,g)(p-x, q-\omega),$$

we calculate $a * W_\tau(\varphi_2, \varphi_1)$ and obtain

$$\begin{aligned} a * W_\tau(\psi_2,\psi_1)(p,q) &= \int_{\mathbb{R}^{2d}} a(x,\omega) W_\tau(T_x M_\omega \varphi_2, T_x M_\omega \varphi_1)(p,q) dx d\omega \\ &= \int_{\mathbb{R}^{2d}} a(x,\omega) \left( \int_{\mathbb{R}^d} M_\omega T_x \psi_2(p+\tau s) \overline{M_\omega T_x \psi_1}(p-(1-\tau)s) e^{-2\pi i q s} ds \right) dx d\omega, \end{aligned}$$

Now by using a suitable interpretation of the oscillatory integrals in the distributional sense, and appropriate change of variables (cf. [135]) we get

$$\begin{aligned} &\langle A_{a,\tau}^{\psi_1,\psi_2} f, g\rangle = \langle a * W_\tau(\psi_2,\psi_1), W_\tau(g,f)\rangle \\ &= \int_{\mathbb{R}^{2d}} a(x,\omega) \int_{\mathbb{R}^{2d}} \Big( \int_{\mathbb{R}^{2d}} M_\omega T_x \psi_2(p+\tau s) \overline{M_\omega T_x \psi_1}(p-(1-\tau)s) \\ &\times e^{-2\pi i q(s-r)} \overline{g}(p+\tau r) f(p-(1-\tau)r) ds dr \Big) dp dq dx d\omega \\ &= \int_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} a(x,\omega) M_\omega T_x \psi_2(t) \overline{M_\omega T_x \psi_1}(y) dx d\omega \overline{g}(t) f(y) dt dy \\ &= \langle K(A_a^{\psi_1,\psi_2}), g \otimes \overline{f}\rangle, \end{aligned}$$

where $K(A_a^{\psi_1,\psi_2})$ is given by (2.233). By the uniqueness of the kernel we conclude that

$$A_a^{\psi_1,\psi_2} = A_{a,\tau}^{\psi_1,\psi_2}$$

and the proof is finished. □

# Chapter 3

# Eigenfunctions of localization operators on $\mathbb{R}^d$

The core of this chapter are some new results of decay and smoothness for eigenfunctions of localization operators on modulation and ultra-modulation spaces, presented by E. Cordero, F. Nicola and the author in [9] and by N. Teofanov and the author in [11]. The main results state that, roughly speaking, if $f \in L^2(\mathbb{R}^d)$ is an eigenfunction of $A_a^{\psi_1,\psi_2}$ with *suitable* symbol $a$, then one of the following may occur:

$$f \in \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d), \qquad f \in \mathcal{S}(\mathbb{R}^d), \qquad f \in \mathcal{S}^{(\gamma)}(\mathbb{R}^d),$$

see Theorem 3.2.1, 3.2.9 and 3.3.6, respectively.

In order to study $A_a^{\psi_1,\psi_2}$, we investigate first Weyl operators $L_\sigma$ and got results concerning boundedness on modulation spaces and $L^2$-eigenfunctions, this is done in Section 3.1. Section 3.2 reports the main results of [9], see in particular Theorem 3.2.1 and 3.2.9. Eventually, in Section 3.3 results of similar flavour in the Gelfand-Shilov setting are presented from [11].

We recall that the necessary backgrounds can be found in Chapter 2 and warn the reader that in Section 3.3 of the current chapter Assumptions 2.5.1 are dropped, i.e. we shall consider not only weights of polynomial growth but of (sub-)exponential growth as well.

## 3.1 Preliminary results on Weyl operators

The target of this section is Theorem 3.1.2 ([9, Theorem 3.3]) which will tell us when, according to the symbol $\sigma$, it is possible to extend the Weyl operator $L_\sigma$ to modulation spaces and which ones are allowed. The proof presented here is independent and alternative to the one present in [9].
We then derive some consequences about eigenfunctions for $L_\sigma$ which we require only to be in $L^2(\mathbb{R}^d)$.

The proof of the following criterion is contained in [82, Lemma 6.2.1], it will be a useful tool in the sequel.

**Lemma 3.1.1.** (Schur's boundedness test) (*i*) *Consider an infinite matrix* $a = (a_{k,n})_{k,n\in\mathbb{Z}} \subseteq \mathbb{C}$ *and* $1 \le p \le \infty$. *Suppose that:*

$$\sup_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |a_{k,n}| \le C_1 < +\infty, \qquad \sup_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |a_{k,n}| \le C_2 < +\infty.$$

*The linear operator*

$$A\colon \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}), c \mapsto ((Ac)_k)_{k\in\mathbb{Z}},$$

*defined by the matrix-vector multiplication*

$$(Ac)_k := \sum_{n\in\mathbb{Z}} a_{k,n} c_n,$$

*is well defined and bounded. Moreover the following estimate holds:*

$$\|A\|_{\mathrm{Op}} \le C_1^{1/p'} C_2^{1/p};$$

(*ii*) *Consider a measurable function* $K\colon \mathbb{R}^{2d} \to \mathbb{C}$ *and* $1 \le p \le \infty$. *Suppose that:*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,\omega)|\, d\omega \le C_1 < +\infty, \qquad \operatorname*{ess\,sup}_{\omega\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,\omega)|\, dx \le C_2 < +\infty.$$

*The linear operator*

$$A\colon L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), f \mapsto Af,$$

*defined as* ***integral operator with kernel*** $K$ *by*

$$Af(x) := \int_{\mathbb{R}^d} K(x,\omega) f(\omega)\, d\omega,$$

*is well defined and bounded. Moreover the following estimate holds:*

$$\|A\|_{\mathrm{Op}} \le C_1^{1/p'} C_2^{1/p}.$$

**Theorem 3.1.2.** ([9, Theorem 3.3])

(*i*) *Consider* $0 < p, q, \gamma \le \infty$ *such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\gamma}. \tag{3.1}$$

*If* $\sigma \in M^{p,\min\{1,\gamma\}}(\mathbb{R}^{2d})$, *then the Weyl operator*

$$L_\sigma\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

*extends uniquely to a bounded linear operator from* $M^q(\mathbb{R}^d)$ *to* $M^\gamma(\mathbb{R}^d)$;

(*ii*) *Consider* $s, r \ge 0$, $t \ge r + s$, *and a symbol* $\sigma \in M^{\infty,1}_{v_s\otimes v_t}(\mathbb{R}^{2d})$. *Then the Weyl operator*

$$L_\sigma\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

*extends uniquely to a bounded linear operator from* $M^2_{v_r}(\mathbb{R}^d)$ *into* $M^2_{v_{r+s}}(\mathbb{R}^d)$.

*Proof.* $(i)$ Assume $\gamma \geq 1$, then, by (3.1), $p \geq \gamma \geq 1$ and $q \geq \gamma \geq 1$. The claim was proved by Toft in [136, Theorem 4.3]. The case $\gamma < 1$, $p, q \in (0, \infty]$ was again proved by Toft in [142, Theorem 3.1]. For sake of completeness we provide an alternative and independent proof.
Assume first $\gamma \geq 1$, then, by (3.1), $p \geq \gamma \geq 1$ and $q \geq \gamma \geq 1$ and the claim was proved by Toft in [136, Theorem 4.3].
Consider now the case $\gamma < 1$ and $p, q < \infty$. We set

$$G_{\mu,\lambda} := \langle L_\sigma(\pi(\lambda)g), \pi(\mu)g\rangle, \quad \lambda, \mu \in \Lambda, \tag{3.2}$$

where $\Lambda = \alpha\mathbb{Z}^{2d}$ is a lattice in $\mathbb{R}^{2d}$, $g \in \mathcal{S}(\mathbb{R}^d)$ and $\langle\cdot,\cdot\rangle$ is the sesquilinear duality between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. Notice that, by Lemma 2.7.17 with $\tau = 1/2$,

$$|G_{\mu,\lambda}| = \left|V_\Phi\sigma\left(\frac{\mu+\lambda}{2}, J(\mu-\lambda)\right)\right|, \quad \lambda, \mu \in \Lambda, \tag{3.3}$$

where $\Phi := \Phi_{1/2} := W(g, g)$ and $J(z_1, z_2) := (z_2, -z_1)$ for $z_1, z_2 \in \mathbb{R}^d$. We want to show that the operator defined by the matrix $(G_{\lambda,\mu})_{\lambda,\mu}$ is well defined and continuous, namely:

$$G\colon \ell^q(\Lambda) \to \ell^\gamma(\Lambda), c = (c_\lambda)_\lambda \mapsto (Gc_\mu)_\mu,$$

where

$$Gc_\mu := \sum_{\lambda\in\Lambda} G_{\mu,\lambda}c_\lambda.$$

We choose the Gaussian window $g(t) = 2^{d/4}e^{-\pi t^2}$, $t \in \mathbb{R}^d$. It is a straightforward calculation to show that the related Wigner distribution is the rescaled Gaussian $\Phi = W(g, g)(x, \omega) = 2^d e^{-2\pi(x^2+\omega^2)}$. Now, the Gabor system $\mathcal{G}(\Phi, \Lambda \times \Lambda)$ is a frame for $L^2(\mathbb{R}^{2d})$, whenever $\Lambda = \alpha\mathbb{Z}^{2d}$, for any $\alpha > 0$ satisfying $\alpha^2 < 1/2$, as shown by M. de Gosson in [41, Proposition 10] (take $\hbar = 1/(4\pi)$). Hence we choose $\alpha < 1/\sqrt{2}$. Since $\gamma < 1$, from Theorem 2.2.31 we have the continuous inclusion

$$\ell^\gamma(\Lambda) \hookrightarrow \ell^1(\Lambda),$$

therefore for any $a \in \ell^\gamma(\Lambda)$ we have

$$\|a\|_{\ell^1} \lesssim \|a\|_{\ell^\gamma}. \tag{3.4}$$

We can now compute as follows:

$$
\begin{aligned}
\|Gc\|_{\ell^\gamma} &= \left(\sum_{\mu\in\Lambda} |Gc_\mu|^\gamma\right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\mu\in\Lambda} \left|\sum_{\lambda\in\Lambda} G_{\mu,\lambda} c_\lambda\right|^\gamma\right)^{\frac{1}{\gamma}} \\
&\leq \left(\sum_{\mu\in\Lambda} \underbrace{\left(\sum_{\lambda\in\Lambda} |G_{\mu,\lambda}|\,|c_\lambda|\right)}_{=\|(G_{\mu,\lambda}c_\lambda)_\lambda\|_{\ell^1}}^\gamma\right)^{\frac{1}{\gamma}} \\
\text{using (3.4)} &\lesssim \left(\sum_{\mu\in\Lambda} \left(\left(\sum_{\lambda\in\Lambda} |G_{\mu,\lambda}|^\gamma |c_\lambda|^\gamma\right)^{\frac{1}{\gamma}}\right)^\gamma\right)^{\frac{1}{\gamma}} \\
\text{using (3.3)} &= \left(\sum_{\mu\in\Lambda}\sum_{\lambda\in\Lambda} \left|V_\Phi\sigma\left(\frac{\lambda+\mu}{2}, J(\mu-\lambda)\right)\right|^\gamma |c_\lambda|^\gamma\right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} \left|V_\Phi\sigma\left(\frac{\lambda+\mu}{2}, J(\mu-\lambda)\right)\right|^\gamma |c_\lambda|^\gamma\right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\lambda'\in\Lambda}\sum_{\mu\in\Lambda} \left|V_\Phi\sigma\left(\frac{2\mu+J(\lambda')}{2}, \lambda'\right)\right|^\gamma |c_{\mu+J(\lambda')}|^\gamma\right)^{\frac{1}{\gamma}} \qquad (3.5) \\
&\leq \left(\sum_{\lambda'\in\Lambda} \left(\sum_{\mu\in\Lambda} \left|V_\Phi\sigma\left(\frac{2\mu+J(\lambda')}{2}, \lambda\right)\right|^p\right)^{\frac{\gamma}{p}} \left(\sum_{\mu\in\Lambda} |c_{\mu+J(\lambda')}|^q\right)^{\frac{\gamma}{q}}\right)^{\frac{1}{\gamma}},
\end{aligned}
$$

where: in last inequality we applied Hölder's inequality with

$$\frac{1}{\frac{p}{\gamma}} + \frac{1}{\frac{q}{\gamma}} = 1,$$

in (3.5) we set

$$\lambda' := J(\mu-\lambda) = J(\mu) - J(\lambda) \in \Lambda$$

which implies

$$J(\lambda') = J^2(\mu) - J^2(\lambda) = -\mu + \lambda \quad \Rightarrow \quad \lambda = J(\lambda') + \mu.$$

Hence, performing the change

$$\mu + J(\lambda') =: \lambda'' \in \Lambda$$

in the last sum we get $\|c\|_{\ell^q}$. Let us observe that the following inclusions occur:

$$\Lambda \subsetneq \frac{1}{2}\Lambda, \qquad \frac{3}{2}\Lambda \subsetneq \frac{1}{2}\Lambda.$$

Then the change of variables

$$\mu + J(\lambda')/2 =: \mu' \in \Lambda + \frac{1}{2}\Lambda = \frac{3}{2}\Lambda$$

allows us to majorize as follows

$$\begin{aligned}
\|Gc\|_{\ell^\gamma} &\lesssim \|c\|_{\ell^q} \left( \sum_{\lambda' \in \Lambda} \left( \sum_{\mu \in \Lambda} \left| V_\Phi \sigma \left( \frac{2\mu + J(\lambda')}{2}, \lambda' \right) \right|^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}} \\
&= \|c\|_{\ell^q} \left( \sum_{\lambda' \in \Lambda} \left( \sum_{\mu' \in \frac{3}{2}\Lambda} |V_\Phi \sigma (\mu', \lambda')|^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}} \\
&\leq \|c\|_{\ell^q} \left( \sum_{\lambda' \in \frac{1}{2}\Lambda} \left( \sum_{\mu' \in \frac{1}{2}\Lambda} |V_\Phi \sigma (\mu', \lambda')|^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}} \\
&= \|c\|_{\ell^q} \left( \sum_{\lambda' \in \frac{1}{2}\Lambda} \left( \sum_{\mu' \in \frac{1}{2}\Lambda} |\langle \sigma, \pi(\mu', \lambda')\Phi \rangle|^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}}.
\end{aligned}$$

We apply the the previous argument, from [41, Proposition 10] we have that $\mathcal{G}(\Phi, \frac{1}{2}\Lambda \times \frac{1}{2}\Lambda)$ is a frame for $L^2(\mathbb{R}^{2d})$ if an only if

$$\left(\frac{\alpha}{2}\right)^2 < \frac{1}{2} \Leftrightarrow \alpha < \sqrt{2}.$$

Since we chose $\alpha < 1/\sqrt{2}$, $\mathcal{G}(\Phi, \frac{1}{2}\Lambda \times \frac{1}{2}\Lambda)$ is a frame; by assumption $\sigma \in M^{p,\gamma}(\mathbb{R}^{2d})$ and, using the characterization in Theorem 2.5.18, this is equivalent to saying

$$\|\sigma\|_{M^{p,\gamma}} \asymp \left( \sum_{\lambda' \in \frac{1}{2}\Lambda} \left( \sum_{\mu' \in \frac{1}{2}\Lambda} |\langle \sigma, \pi(\mu', \lambda')\Phi \rangle|^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}}. \tag{3.6}$$

Therefore by (3.6) we get continuity for the linear operator $G$:

$$\|Gc\|_{\ell^\gamma} \lesssim \|c\|_{\ell^q} \left( \sum_{\lambda' \in \frac{1}{2}\Lambda} \left( \sum_{\mu' \in \frac{1}{2}\Lambda} |\langle \sigma, \pi(\mu', \lambda')\Phi \rangle|^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}} \lesssim \|c\|_{\ell^q} \|\sigma\|_{M^{p,\gamma}}.$$

As shown in [86], $\mathcal{G}(g, \Lambda) = \mathcal{G}(g, \alpha, \alpha)$ is a frame if $\alpha < 1$ and it admits a dual window $h \in \mathcal{S}(\mathbb{R}^d)$. Then, thanks to Theorem 2.5.18, we can expand $f \in \mathcal{S}(\mathbb{R}^d) \subseteq M^q(\mathbb{R}^d)$ by means of the Gabor atoms

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)h \rangle \pi(\lambda)g$$

and write

$$L_\sigma f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)h \rangle L_\sigma(\pi(\lambda)g). \tag{3.7}$$

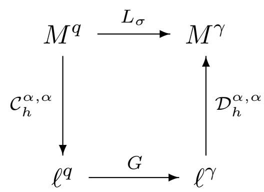
Expanding now the function $L_\sigma f$ and using (3.7) we get

$$\begin{aligned}
L_\sigma f &= \sum_{\mu\in\Lambda} \langle L_\sigma f, \pi(\mu)g\rangle \pi(\mu)h \\
&= \sum_{\mu\in\Lambda} \langle \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle L_\sigma(\pi(\lambda)g), \pi(\mu)g\rangle \pi(\mu)h \\
&= \sum_{\mu\in\Lambda}\sum_{\lambda\in\Lambda} \underbrace{\langle f, \pi(\lambda)h\rangle}_{=\mathcal{C}_h^{\alpha,\alpha}(f)_\lambda} \underbrace{\langle L_\sigma(\pi(\lambda)g), \pi(\mu)g\rangle}_{=G_{\mu,\lambda}} \pi(\mu)h.
\end{aligned}$$

Recall the continuity properties of the coefficient operator $\mathcal{C}_h^{\alpha,\alpha}$ from $M^{p,q}$ into $\ell^{p,q}(\Lambda)$ and of the synthesis operator $\mathcal{D}_h^{\alpha,\alpha}$ from $\ell^{p,q}(\Lambda)$ into $M^{p,q}$, $0 < p, q \le \infty$ (Proposition 2.5.16 and Proposition 2.5.17), then we have just shown that $L_\sigma$ (considered as linear operator with dense domain $\mathcal{S}(\mathbb{R}^d) \subseteq M^q(\mathbb{R}^d)$) can be decomposed as

$$L_\sigma = \mathcal{D}_h^{\alpha,\alpha} \circ G \circ \mathcal{C}_h^{\alpha,\alpha}$$

and the following diagram is commutative:




Since we proved the continuity of the operator $G$ from $\ell^q$ into $\ell^\gamma$, $L_\sigma$ is a continuous and linear operator with dense domain, hence it admits a unique continuous linear extension.
The cases when $\gamma < 1$ and $p = \infty$ or $q = \infty$ can be treated similarly.

*(ii)* Let $g \in \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$ and consider $f \in \mathcal{S}(\mathbb{R}^d) \subseteq M^2_{v_r}(\mathbb{R}^d)$. From (2.213) with $\tau = 1/2$ we have

$$L_\sigma f = \int_{\mathbb{R}^{2d}} V_g f(z) L_\sigma(\pi(z)g)\, dz, \tag{3.8}$$

in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the distribution $L_\sigma f$ acts as follows:

$$\langle L_\sigma f, \varphi\rangle = \int_{\mathbb{R}^{2d}} V_g f(z)\, \langle L_\sigma(\pi(z)g), \varphi\rangle\, dz.$$

We can express the STFT of the tempered distribution $L_\sigma f$ as

$$V_g(L_\sigma f)(w) = \langle L_\sigma f, \pi(w)g\rangle = \int_{\mathbb{R}^{2d}} V_g f(z)\, \langle L_\sigma \pi(z)g, \pi(w)g\rangle\, dz,$$

where $w \in \mathbb{R}^{2d}$. The desired result thus follows if we can prove that the map

$$M(\sigma)\colon L^2_{v_r}(\mathbb{R}^{2d}) \to L^2_{v_{r+s}}(\mathbb{R}^{2d}), F \mapsto M(\sigma)F$$

defined by

$$M(\sigma)F(w) := \int_{\mathbb{R}^{2d}} F(z)\, \langle L_\sigma \pi(z)g, \pi(w)g\rangle\, dz$$

is continuous. Setting $\Phi = W(g,g)$ and using (2.211), we see that it is sufficient to prove that the integral operator with integral kernel

$$\left| V_\Phi \sigma \left( \frac{z+w}{2}, J(w-z) \right) \right| \langle z \rangle^{-r} \langle w \rangle^{r+s}$$

is bounded on $L^2(\mathbb{R}^{2d})$. This follows from Shur's test, Lemma 3.1.1. Indeed, by assumption $\sigma \in M^{\infty,1}_{v_s \otimes v_t}(\mathbb{R}^{2d})$, so that performing a change of variables

$$\int_{\mathbb{R}^{2d}} \sup_{z \in \mathbb{R}^{2d}} \left| V_\Phi \sigma \left( \frac{z+w}{2}, J(w-z) \right) \right| \langle z+w \rangle^s \langle w-z \rangle^t dw < +\infty;$$

we used sup instead of ess sup since every function involved is continuous. Thus

$$\sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left| V_\Phi \sigma \left( \frac{z+w}{2}, J(w-z) \right) \right| \langle z+w \rangle^s \langle w-z \rangle^t \, dw < +\infty$$

and similarly

$$\sup_{w \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left| V_\Phi \sigma \left( \frac{z+w}{2}, J(w-z) \right) \right| \langle z+w \rangle^s \langle w-z \rangle^t \, dz < +\infty.$$

Hence it is sufficient to prove that for some positive constant $C > 0$ we have

$$\langle z+w \rangle^{-s} \langle w-z \rangle^{-t} \langle z \rangle^{-r} \langle w \rangle^{r+s} \le C, \quad \forall\, w, z \in \mathbb{R}^{2d}. \tag{3.9}$$

Let us prove the estimate (3.9). Setting $x = z + w$, $y = w - z$, the inequality (3.9) can be rephrased as

$$\langle x \rangle^{-s} \langle y \rangle^{-t} \langle x-y \rangle^{-r} \langle x+y \rangle^{r+s} \le C, \quad \forall\, x, y \in \mathbb{R}^{2d}. \tag{3.10}$$

For $|x| < 2|y|$, observe that $|x+y| < 3|y|$ and since $t \ge r+s$ we get the estimate (3.10). For $|x| \ge 2|y|$, we use $\langle x+y \rangle \asymp \langle x-y \rangle \asymp \langle x \rangle$ and (3.10) immediately follows. □

The following corollary is a easy consequence due to Proposition 2.5.13. We recall that a description of the Shubin-Sobolev spaces $Q_s$ is given in (2.148).

**Corollary 3.1.3.** ([9, Corollary 3.4])
*If $s, r \ge 0$, $t \ge r+s$, and the symbol $\sigma \in M^{\infty,1}_{v_s \otimes v_t}(\mathbb{R}^{2d})$, then the pseudo-differential operator $L_\sigma$ extends uniquely to a bounded operator from $Q_r(\mathbb{R}^d)$ into $Q_{r+s}(\mathbb{R}^d)$ .*

An application of the previous theorem concerns the study of eigenfunctions' properties for Weyl operators.

**Proposition 3.1.4.** ([9, Proposition 3.5])
*Consider a Weyl symbol $\sigma \in M^{p,\gamma}$ for some $0 < p < \infty$ and every $\gamma > 0$. Any eigenfunction $f \in L^2(\mathbb{R}^d)$ of $L_\sigma$ is in $\cap_{\gamma > 0} M^\gamma(\mathbb{R}^d)$.*

*Proof.* We use Theorem 3.1.2 with $q = 2$: if the symbol $\sigma$ is in $M^{p,\gamma}(\mathbb{R}^{2d})$, for every $\gamma > 0$, then the Weyl operator acts continuously from $M^2(\mathbb{R}^d)$ into $M^{\gamma_1}(\mathbb{R}^d)$, where $\gamma_1$ is such that

$$\frac{1}{p} + \frac{1}{2} = \frac{1}{\gamma_1}.$$

Since $p < \infty$, we have
$$1/\gamma_1 = 1/p + 1/2 > 0 + 1/2 \Rightarrow 0 < \gamma_1 < 2.$$
Since $\sigma \in M^{p,\min\{1,\gamma_1\}}(\mathbb{R}^{2d})$, by Theorem 3.1.2 part $(i)$ $L_\sigma$ admits a unique continuous linear extension
$$L_\sigma \colon M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d) \to M^{\gamma_1}(\mathbb{R}^d).$$
If $f \in M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ is an eigenfunction of $L_\sigma$ with eigenvalue $\lambda \neq 0$, we have
$$f = \frac{1}{\lambda} L_\sigma f \in M^{\gamma_1}(\mathbb{R}^d).$$
Starting with $f \in M^{\gamma_1}(\mathbb{R}^d)$, we repeat the same argument, obtaining that the eigenfunction $f$ is in the smaller modulation space $M^{\gamma_2}(\mathbb{R}^d)$, with $\gamma_2$ solution of
$$\frac{1}{p} + \frac{1}{\gamma_1} = \frac{1}{\gamma_2}.$$
We observe that $0 < \gamma_2 < \gamma_1$ since $p < \infty$. Indeed since $\sigma \in M^{p,\min\{1,\gamma_2\}}(\mathbb{R}^{2d})$, by Theorem 3.1.2 part $(i)$ $L_\sigma$ admits a unique continuous linear extension
$$L_\sigma \colon M^{\gamma_1}(\mathbb{R}^d) \to M^{\gamma_2}(\mathbb{R}^d).$$
If $f \in M^{\gamma_1}(\mathbb{R}^d)$ is an eigenfunction of $L_\sigma$ with eigenvalue $\lambda \neq 0$, we have
$$f = \frac{1}{\lambda} L_\sigma f \in M^{\gamma_2}(\mathbb{R}^d).$$
Continuing this way we construct a decreasing sequence of indices $\gamma_n > 0$ which explicit expression is
$$\gamma_n = \frac{2}{2n/p + 1}, \qquad n \in \mathbb{N}_0.$$
The proof is a simple induction argument which is left to the reader. Hence
$$f \in \bigcap_{n \in \mathbb{N}_0} M^{\gamma_n}(\mathbb{R}^d).$$
Moreover $\lim_{n\to\infty} \gamma_n = 0$ and the claim follows from the inclusion relations for modulation spaces Theorem 2.5.6. □

A boot-strap argument similar to the previous one allows us to prove the following regularity result for $L^2$-eigenfunctions of Weyl operators.

**Proposition 3.1.5.** ([9, Proposition 3.6])
*Consider a Weyl symbol $\sigma \in M^{\infty,1}_{v_s \otimes v_t}(\mathbb{R}^{2d})$ for some $s > 0$ and every $t > 0$. Any eigenfunction $f \in L^2(\mathbb{R}^d)$ of $L_\sigma$ is in $\mathcal{S}(\mathbb{R}^d)$.*

*Proof.* We use Theorem 3.1.2 part $(ii)$ with $r = 0$. If the symbol $\sigma$ is in $M^{\infty,1}_{v_s\otimes v_t}(\mathbb{R}^{2d})$, for some $s > 0$ and every $t > 0$, then the Weyl operator $L_\sigma$ acts continuously from $M^2_{v_0}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ into $M^2_{v_s}(\mathbb{R}^d)$. Starting now with the eigenfunction $f$ in $M^2_{v_s}(\mathbb{R}^d)$ and repeating the same argument with $t \geq s$ we obtain that the eigenfunction is in $M^2_{v_{s+s}}(\mathbb{R}^d)$. Proceeding this way we infer that $f \in \cap_{n\in\mathbb{N}_0} M^2_{v_{ns}}(\mathbb{R}^d)$. The inclusion relations for modulation spaces and the property
$$\cap_{r>0} M^2_{v_r}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d),$$
see $(vi)$ Proposition 2.5.13, prove the claim. □

## 3.2 Decay and smoothness results

Eventually we state and prove the main results about decay and smoothness for eigenfunctions of localization operators which are supposed to be in $L^2(\mathbb{R}^d)$. The essential meaning of the main decay's statement, Theorem 3.2.1 ([9, Theorem 3.7]), will be made explicit via Parseval Gabor frames. We present regularity properties for $L^2$-eigenfunctions of some $A_a^{\psi_1,\psi_2}$ which is a simple consequence of Proposition 3.1.5. Eventually we tackle the case of $A_a^{\psi_1,\psi_2}$ with symbol in $L^q_m(\mathbb{R}^{2d})$, $1 \leq q < \infty$.

**Theorem 3.2.1.** ([9, Theorem 3.7])
*Consider a symbol $a \in M^{p,\infty}(\mathbb{R}^{2d})$, for some $0 < p < \infty$, and non-zero windows $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$. Any eigenfunction $f \in L^2(\mathbb{R}^d)$ of $A_a^{\psi_1,\psi_2}$ satisfies $f \in \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d)$.*

*Proof.* Since the windows $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$, the cross-Wigner distribution is in $\mathcal{S}(\mathbb{R}^{2d}) \subseteq M^{q,\gamma}(\mathbb{R}^{2d})$, for every $0 < q, \gamma \leq \infty$. We next apply the convolution relations for modulation spaces (2.158) with $u = \infty$, $t = \gamma$ and $v = m = \nu \equiv 1$.
If $p \geq 1$, choose $q = 1$, so that $r = p$, and

$$\sigma := a * W(\psi_2, \psi_1) \in M^{p,\infty}(\mathbb{R}^{2d}) * M^{1,\gamma}(\mathbb{R}^{2d}) \hookrightarrow M^{p,\gamma}(\mathbb{R}^{2d}), \quad \forall \gamma > 0.$$

If $0 < p < 1$, choose $p = q = r$ so that

$$\sigma := a * W(\psi_2, \psi_1) \in M^{p,\infty}(\mathbb{R}^{2d}) * M^{p,\gamma}(\mathbb{R}^{2d}) \hookrightarrow M^{p,\gamma}(\mathbb{R}^{2d}), \quad \forall \gamma > 0.$$

In both cases we obtain that $\sigma \in M^{p,\gamma}(\mathbb{R}^{2d})$, for every $\gamma > 0$. Hence the claim immediately follows by Proposition 3.1.4 and the realization of the localization operator as a Weyl one: $A_a^{\psi_1,\psi_2} = L_\sigma$, where $\sigma = a * W(\psi_2, \psi_1)$. □

**Remark 3.2.2.** *Notice that $f \in \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d)$ does not imply $f \in \mathcal{S}(\mathbb{R}^d)$, as pointed out in Remark 2.5.14.*

**Remark 3.2.3.** *We want to emphasize the vast class of symbols which are included in assumptions of Theorem 3.2.1. For example, a rough symbol as the Dirac delta $\delta$ is allowed. Indeed, consider $g \in \mathcal{S}(\mathbb{R}^{2d})$ and compute:*

$$\begin{aligned} V_g\delta(z,w) &= \langle \delta, M_w T_z g \rangle \\ &= \overline{e^{2\pi i w t} g(t-z)}\big|_{t=0} \\ &= \overline{g(0-z)}. \end{aligned}$$

*Hence $\delta \in M^{1,\infty}(\mathbb{R}^{2d})$ and Theorem 3.2.1 holds true for any $L^2$-eigenfunction of $A_\delta^{\psi_1,\psi_2}$, with $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$.*

As a consequence of Theorem 3.2.1, the eigenfunctions of a localization operator $A_a^{\psi_1,\psi_2}$ are extremely concentrated on the time-frequency space, having very few Gabor coefficients large whereas all the others are negligible.

**Definition 3.2.4.** *Consider a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ for $L^2(\mathbb{R}^d)$. For $N \in \mathbb{N}$, we define the* ***set of all linear combinations of Gabor atoms consisting of at most** $N$ **terms*** *as*

$$\Sigma_N := \left\{ p = \sum_{k,n \in F} c_{k,n} \pi(\alpha k, \beta n) g \mid c_{k,n} \in \mathbb{C},\ F \subseteq \mathbb{Z}^d \times \mathbb{Z}^d,\ \#F \leq N \right\}. \tag{3.11}$$

*Given a function $f \in L^2(\mathbb{R}^d)$, the* ***$N$-term approximation error in*** *$L^2(\mathbb{R}^d)$ (with respect to the Gabor frame $\mathcal{G}(g, \alpha, \beta)$) is*

$$\sigma_N(f) := \inf_{p \in \Sigma_N} \|f - p\|_{L^2}. \tag{3.12}$$

**Remark 3.2.5.** *Note that $\Sigma_N$ is not a linear subspace since $\Sigma_N + \Sigma_N = \Sigma_{2N}$. That is why the approximation of a signal $f$ by elements of $\Sigma_N$ is often referred to as non-linear approximation. Namely, $\sigma_N(f)$ is the error produced when $f$ is approximated optimally by a linear combination of $N$ Gabor atoms.*

Assume $f \in M^p(\mathbb{R}^d)$ for some $0 < p < 2$. Thus, in particular, $f \in L^2(\mathbb{R}^d)$ since $M^p(\mathbb{R}^d) \subseteq M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ for $0 < p < 2$. Consider now a *Parseval* Gabor frame $\mathcal{G}(g, \Lambda)$ for $L^2(\mathbb{R}^d)$, where $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and $g \in \mathcal{S}(\mathbb{R}^d)$. The series of Gabor coefficients in

$$\|f\|_{L^2}^2 = \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2$$

is absolutely convergent, hence also unconditionally convergent. Thus we can rearrange the Gabor coefficients $|\langle f, \pi(\lambda)g \rangle|$ in a decreasing order. Precisely, set $c_{k,n} := \langle f, \pi(\alpha k, \beta n)g \rangle$, $k, n \in \mathbb{Z}^d$, and let $\iota : \mathbb{N} \to \mathbb{Z}^d \times \mathbb{Z}^d$ be any bijection satisfying

$$|c_{\iota(1)}| \geq |c_{\iota(2)}| \geq \cdots \geq |c_{\iota(m)}| \geq |c_{\iota(m+1)}| \geq \cdots .$$

Such a bijection always exists due to the convergence of the series. The sequence $(\tilde{c}_m)_{m \in \mathbb{N}} := (|c_{\iota(m)}|)_{m \in \mathbb{N}}$ is called the ***non-increasing rearrangement of*** $(c_{k,n})_{k,n}$ above. With this notations, the best approximation of $f$ in $\Sigma_N$ is

$$p_{opt} := \sum_{m=1}^{N} c_{\iota(m)} \pi(\iota(m))g$$

and the the $N$-term approximation error becomes

$$\sigma_N(f) = \inf_{p \in \Sigma_N} \|f - p\|_{L^2} = \|f - p_{opt}\|_{L^2} = \left( \sum_{m=N+1}^{\infty} |c_{\iota(m)}|^2 \right)^{\frac{1}{2}}.$$

We observe that $\iota$, the non-increasing rearrangement $(\tilde{c}_m)_{m \in \mathbb{N}}$ and $p_{opt}$ are not unique. Anyway different best approximations $p_{opt}$ and $p'_{opt}$ both realize the $N$-term approximation error.
By abuse of notation, given $a = (a_m)_m$, $a_m \geq 0$ for every $m$, a non-increasing sequence ( $a_1 \geq a_2 \geq \cdots \geq a_m \geq a_{m+1} \geq \cdots$ ) we write

$$\sigma_N(a) := \left( \sum_{m=N+1}^{\infty} a_m^2 \right)^{\frac{1}{2}}.$$

The key tool is now the following lemma from [82], see also [129] and [43].

**Lemma 3.2.6.** ([82, Lemma 12.4.1])
*Let $a = (a_m)_m$, $a_m \geq 0$ for every $m$, be a non-increasing sequence and consider $0 < p < 2$. Set*

$$\gamma := \frac{1}{p} - \frac{1}{2} > 0. \tag{3.13}$$

*Then there exists a constant $C = C(p) > 0$, such that*

$$\frac{1}{C}\|a\|_{\ell^p} \leq \left( \sum_{N=1}^{\infty} (N^\gamma \sigma_{N-1}(a))^p \frac{1}{N} \right)^{\frac{1}{p}} \leq C\|a\|_{\ell^p}. \tag{3.14}$$

**Proposition 3.2.7.** ([9, Proposition 3.8])
*Assume $f \in M^p(\mathbb{R}^d)$ for some $0 < p < 2$. Consider a Parseval Gabor frame $\mathcal{G}(g, \Lambda)$ for $L^2(\mathbb{R}^d)$, where $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha, \beta > 0$, and $g \in \mathcal{S}(\mathbb{R}^d)$. Then, there exists $C = C(p) > 0$ such that the $N$-term approximation error with respect to the Parseval Gabor frame $\mathcal{G}(g, \Lambda)$ satisfies*

$$\sigma_N(f) \leq C\|f\|_{M^p} N^{-\gamma}, \tag{3.15}$$

*where $\gamma > 0$ is defined as in* (3.13).

*Proof.* The sequence of Gabor coefficients of $f$, given by $(\langle f, \pi(\alpha k, \beta n)g\rangle)_{k,n\in\mathbb{Z}^d}$, is in $\ell^p(\Lambda)$ by Theorem 2.5.18, with

$$\|f\|_{M^p} \asymp \|(\langle f, \pi(\alpha k, \beta n)g\rangle)_{k,n}\|_{\ell^p(\Lambda)}$$

and the sequence $(|\langle f, \pi(\alpha k, \beta n)g\rangle|)_{k,n}$ can be rearranged in a non-increasing one $(a_m)_{m\in\mathbb{N}}$, as explained above. Applying Lemma 3.2.6 to such a sequence, from the right-hand side inequality in (3.14) we infer (3.15). □

**Corollary 3.2.8.** ([9, Corollary 3.9])
*Consider a Parseval Gabor frame $\mathcal{G}(g, \Lambda)$ for $L^2(\mathbb{R}^d)$, where $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha, \beta > 0$, and $g \in \mathcal{S}(\mathbb{R}^d)$. Under the assumptions of Theorem 6.2.1, any $f$ eigenfunction of $A_a^{\psi_1,\psi_2}$ is highly compressed onto a few Gabor atoms $\pi(\lambda)g$, in the sense that its $N$-term approximation error (with respect to $\mathcal{G}(g, \Lambda)$) satisfies the following property: for every $r > 0$ there exists $C = C(r) > 0$ such that*

$$\sigma_N(f) \leq CN^{-r}. \tag{3.16}$$

*Proof.* By Theorem 3.2.1, the eigenfunction fulfils $f \in M^p(\mathbb{R}^d)$, for every $p > 0$. Hence the assumptions of Proposition 3.2.7 are satisfied for every $0 < p < 2$. The claim follows by choosing $r := \gamma = 1/p - 1/2$, as defined in (3.13). □

We next consider the case of localization operators with symbols $a \in M^\infty_{v_s\otimes 1}(\mathbb{R}^{2d})$, $s > 0$. In this case $L^2$-eigenfunctions reveal to be Schwartz functions, as shown below.

**Theorem 3.2.9.** ([9, Theorem 3.10])
*Consider a symbol $a \in M^\infty_{v_s\otimes 1}(\mathbb{R}^{2d})$, for some $s > 0$, and non-zero windows $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ is an eigenfunction of $A_a^{\psi_1,\psi_2}$, then $f \in \mathcal{S}(\mathbb{R}^d)$.*

*Proof.* The assumption $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ implies

$$W(\psi_2, \psi_1) \in \mathcal{S}(\mathbb{R}^{2d}) \subseteq M^{1,1}_{v_r\otimes v_t}(\mathbb{R}^{2d}),$$

for every $r, t > 0$. We next apply the convolution relations for modulation spaces (2.158), obtaining that $A_a^{\psi_1,\psi_2} = L_\sigma$ with $\sigma := a * W(\psi_2, \psi_1) \in M^{\infty,1}_{v_s\otimes v_t}(\mathbb{R}^{2d})$, for some $s > 0$ and every $t > 0$. Hence the claim immediately follows by Proposition 3.1.5. □

**Remark 3.2.10.** *From Proposition 2.5.13 and inclusion relations between modulation spaces Theorem 2.5.6 we have that the previous theorem holds true also for symbols of the type $a \in L^p_{v_s}(\mathbb{R}^{2d})$, for some $0 < p \leq \infty$ and $s > 0$.*

**Remark 3.2.11.** *The nice properties of eigenfunctions for localization operators studied so far seem to depend on the fact that such operators are not only compact but belong to the Schatten class $\mathcal{J}^p$, $0 < p < \infty$ (cf. [29, Theorem 1]). C. Fernández and A. Galbis in [67, Theorem 3.15] characterize compact localization operators. Namely, fix $g_0 \in \mathcal{S}(\mathbb{R}^d)$ and $a \in M^\infty(\mathbb{R}^{2d})$, then the following conditions are equivalent:*

(*i*) *The localization operator* $A_a^{\psi_1,\psi_2}$ *is compact on* $L^2(\mathbb{R}^d)$ *for every* $\psi_1, \psi_2$ *in* $\mathcal{S}(\mathbb{R}^d)$;

(*ii*) *For every* $R > 0$,

$$\lim_{|x|\to\infty} \sup_{|\omega|\le R} |V_{g_0} a(x,\omega)| = 0. \tag{3.17}$$

*It seems that for symbols satisfying condition* (3.17) *the techniques developed above do not work anymore. It would be very interesting to know whether for compact operators that are not in the Schatten class* $\mathcal{J}^p$, $0 < p < \infty$, *the* $L^2$ *eigenfunctions do gain any additional smoothness and regularity.*

We now consider localization operators with symbols in weighted Lebesgue spaces. Let us recall that any localization operator $A_a^{\psi_1,\psi_2}$ with windows in $\mathcal{S}(\mathbb{R}^d)$ and symbol $a \in L^q(\mathbb{R}^{2d})$, with $1 \le q < \infty$, is a compact operator, cf. [152, Proposition 13.3]. The case of weighted Lebesgue spaces and, more generally, Potential Sobolev spaces was treated in [17]: let us stress that any localization operator $A_a^{\psi_1,\psi_2}$ with Schwartz windows and symbol $a$ in $L^q_m(\mathbb{R}^{2d})$, with $1 \le q < \infty$ is a compact operator on $L^2(\mathbb{R}^d)$.

**Theorem 3.2.12.** ([9, Theorem 4.1]) *Let* $m \in \mathcal{M}_v$, $m(z) \ge 1$ *for every* $z \in \mathbb{R}^{2d}$, $a \in L^q_m(\mathbb{R}^{2d})$, $1 \le q < \infty$, *and non-zero windows* $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$. *Any eigenfunction* $f \in L^2(\mathbb{R}^d)$ *of* $A_a^{\psi_1,\psi_2}$ *satisfies* $f \in \bigcap_{p>0} M^p_m(\mathbb{R}^d)$.

*Proof.* By assumption and using (2.174), we start with a symbol $a$ in $L^q_m(\mathbb{R}^{2d}) = W(L^q, L^q_m)(\mathbb{R}^{2d})$. Consider the eigenvector $f \in L^2(\mathbb{R}^d)$ and the window $\psi_1 \in \mathcal{S}(\mathbb{R}^d)$. Then by Theorem 2.6.3 the STFT $V_{\psi_1} f$ is in the Wiener amalgam space $W(L^\infty, L^2)(\mathbb{R}^{2d})$. Proposition 2.6.2 yields that $aV_{\psi_1} f \in W(L^q, L^{p_1}_m)(\mathbb{R}^{2d})$, with

$$\frac{1}{q} + \frac{1}{2} = \frac{1}{p_1},$$

so that the index $p_1$ satisfies $p_1 < \min\{q, 2\}$. Consider now a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$. Using the weak definition of $A_a^{\psi_1,\psi_2}$

$$\begin{aligned} V_g(A_a^{\psi_1,\psi_2} f)(w) = \langle A_a^{\psi_1,\psi_2} f, \pi(w)g\rangle &= \int_{\mathbb{R}^{2d}} (aV_{\psi_1} f)(z)\langle \pi(z)\psi_2, \pi(w)g\rangle\, dz \\ &= \int_{\mathbb{R}^{2d}} (aV_{\psi_1} f)(z)\langle \psi_2, \pi(-z)\pi(w)g\rangle\, dz \\ &= \int_{\mathbb{R}^{2d}} (aV_{\psi_1} f)(z) e^{-2\pi i z_1 w_2}\langle \psi_2, \pi(w-z)g\rangle\, dz \end{aligned}$$

so that,

$$|V_g(A_a^{\psi_1,\psi_2} f)(w)| \le \int_{\mathbb{R}^{2d}} |(aV_{\psi_1} f)(z)|\, |V_g\psi_2(w-z)| dz = |aV_{\psi_1} f| * |V_g\psi_2|(w). \tag{3.18}$$

We estimate

$$|V_g(A_a^{\psi_1,\psi_2} f)(w)| \le |aV_{\psi_1} f| * |V_g\psi_2|(w) \in W(L^q, L^{p_1}_m)(\mathbb{R}^{2d}) * W(L^\infty, L^1_v)(\mathbb{R}^{2d}).$$

Observing that $W(L^\infty, L^1_v)(\mathbb{R}^{2d}) \hookrightarrow W(L^{q'}, L^1_v)(\mathbb{R}^{2d})$ and applying the convolution relations (2.173) we infer $|V_g(A_a^{\psi_1,\psi_2} f)| \in W(L^\infty, L^{p_1}_m) \hookrightarrow L^{p_1}_m$. This proves that $A_a^{\psi_1,\psi_2} f \in M^{p_1}_m(\mathbb{R}^d)$. Recalling the assumption $A_a^{\psi_1,\psi_2} f = \lambda f$, $\lambda \ne 0$, we infer $f \in M^{p_1}_m(\mathbb{R}^d)$.

We now repeat the previous argument starting with $f \in M^{p_1}_m(\mathbb{R}^d)$. By Theorem 2.6.3 the STFT $V_{\psi_1} f \in W(L^\infty, L^{p_1}_m)(\mathbb{R}^{2d})$ and $aV_{\psi_1} f \in W(L^q, L^{p_2}_{m^2}) \hookrightarrow W(L^q, L^{p_2}_m)$, (since $m^2 \geq m$), with

$$\frac{1}{q} + \frac{1}{p_1} = \frac{1}{p_2},$$

so that $p_2 < p_1$. Arguing as above we infer $|V_g(A^{\psi_1,\psi_2}_a f)(w)| \in W(L^\infty, L^{p_2}_m) \hookrightarrow L^{p_2}_m$. Thus, the eigenfunction $f$ belongs to the smaller space $M^{p_2}_m$.
Continuing this way we construct a strictly decreasing sequence of indices $p_n > 0$ and such that

$$\lim_{n\to\infty} p_n = 0.$$

By induction and using the same argument as above one immediately obtains that if $f \in M^{p_n}_m(\mathbb{R}^d)$ then $f \in M^{p_{n+1}}_m(\mathbb{R}^d)$. This concludes the proof. □

## 3.3 Gelfand-Shilov setting

This last section presents the main results achieved in [11] about localization operators $A^{\psi_1,\psi_2}_a$ in the framework of ultra-modulation spaces. Once more, we stress that Assumptions 2.5.1 are dropped in the current section.

In the following item, we show how the $\tau$-quantization $\mathrm{Op}_\tau(\sigma)$, $\tau \in [0,1]$, can be extended between ultra-modulation spaces under suitable assumptions on the weights. We remark that the following theorem is contained in more general [142, Theorem 3.1]. A more elementary proof of the same claim when Lebesgue parameters are greater than or equal to 1 is given in [140, Theorem A.2]. In contrast to [140, 142], different arguments were used in [11] and are presented below. Namely, the Schur test in combination with Lemmas 2.8.14 and 2.8.15 is used. We note that [9, Theorem 3.3] (Theorem 3.1.2) is a particular case of Theorem 3.3.1 when restricted to polynomial weights and the duality between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, taking $\tau = 1/2$.

**Theorem 3.3.1.** ([11, Theorem 3.3]) *Consider $\tau \in [0,1]$, $m_0 \in \mathscr{P}_E(\mathbb{R}^{4d})$ and $m_1, m_2 \in \mathscr{P}_E(\mathbb{R}^{2d})$ such that*

$$\frac{m_2(x,\omega)}{m_1(y,\eta)} \lesssim m_0((1-\tau)x + \tau y, \tau\omega + (1-\tau)\eta, \omega - \eta, y - x), \quad \forall\, x,\omega,y,\eta \in \mathbb{R}^d. \tag{3.19}$$

*Fix a symbol $\sigma \in M^{\infty,1}_{m_0}(\mathbb{R}^{2d})$. Then the pseudo-differential operator $\mathrm{Op}_\tau(\sigma)$, from $\mathcal{S}^{(1)}(\mathbb{R}^d)$ to $\mathcal{S}^{(1)'}(\mathbb{R}^d)$, extends uniquely to a bounded and linear operator from $M^p_{m_1}(\mathbb{R}^d)$ into $M^p_{m_2}(\mathbb{R}^d)$ for every $1 \leq p < \infty$.*

*Proof.* Let $g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$ and consider $f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \subseteq M^p_{m_1}(\mathbb{R}^d)$. Due to the normalization chosen $\|g\|_{L^2} = \|\hat{g}\|_{L^2}$ and we recall the inversion formula (2.229) which can be seen as a pointwise equality between smooth functions in this case (see [82, Proposition 11.2.4]): $f = \int_{\mathbb{R}^{2d}} V_g f(z)\pi(z)g\, dz$.

By Lemma 2.8.15 we have

$$\begin{aligned} V_g(\mathrm{Op}_\tau(\sigma)f)(w) &= \langle \mathrm{Op}_\tau(\sigma)f, \pi(w)g\rangle \\ &= \int_{\mathbb{R}^{2d}} V_g f(z)\, \langle \mathrm{Op}_\tau(\sigma)\pi(z)g, \pi(w)g\rangle\, dz. \end{aligned} \tag{3.20}$$

In the next step we prove that the map $M_\tau(\sigma) : F \mapsto M_\tau(\sigma)F$, defined by

$$M_\tau(\sigma)F(w) := \int_{\mathbb{R}^{2d}} F(z) \langle \mathrm{Op}_\tau(\sigma)\pi(z)g, \pi(w)g\rangle \, dz$$

is continuous from $L^p_{m_1}(\mathbb{R}^{2d})$ to $L^p_{m_2}(\mathbb{R}^{2d})$.

Using (2.231), we see that it is equivalent to prove that the integral operator with kernel

$$K_\tau(z,w) := |V_{\Phi_\tau}\sigma\left(\mathcal{T}_\tau(w,z), J(w-z)\right)| \frac{1}{m_1(z)} m_2(w),$$

where $\mathcal{T}_\tau$ ans $J$ are defined in (2.212), is bounded on $L^p(\mathbb{R}^{2d})$. We do this using the Schur test, see Lemma 3.1.1. First we majorize $K_\tau$ with another integral kernel $Q_\tau$ using the condition (3.19) with $w = (x,\omega) \in \mathbb{R}^{2d}$ and $z = (y,\eta) \in \mathbb{R}^{2d}$:

$$\begin{aligned} K_\tau(z,w) &= \frac{m_2(w) m_0(\mathcal{T}_\tau(w,z), J(w-z))}{m_1(z) m_0(\mathcal{T}_\tau(w,z), J(w-z))} |V_{\Phi_\tau}\sigma(\mathcal{T}_\tau(w,z), J(w-z))| \\ &\lesssim |V_{\Phi_\tau}\sigma(\mathcal{T}_\tau(w,z), J(w-z))| \, m_0(\mathcal{T}_\tau(w,z), J(w-z)) \\ &=: Q_\tau(z,w). \end{aligned}$$

We now show that $Q_\tau$ satisfies the Schur conditions. By appropriate change of variables ($w' \equiv w'_z(w) := J(w-z)$, where $z$ is fixed) we obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |Q_\tau(z,w)| \, dw &= \int_{\mathbb{R}^{2d}} \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} |V_\Phi \sigma\left(z,w'\right)| \, m_0\left(z,w'\right) \, dw' \\ &= \|\sigma\|_{M^{\infty,1}_{m_0}} < +\infty. \end{aligned}$$

Furthermore, by the change of variables $w' \equiv w'_w(z) := J(w-z)$ for every $w$ fixed, we obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{w\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |Q_\tau(z,w)| \, dz &= \int_{\mathbb{R}^{2d}} \operatorname*{ess\,sup}_{w\in\mathbb{R}^{2d}} |V_{\Phi_\tau} \sigma\left(w,w'\right)| \, m_0\left(w,w'\right) \, dw' \\ &= \|\sigma\|_{M^{\infty,1}_{m_0}} < +\infty. \end{aligned}$$

Since $K_\tau \lesssim Q_\tau$, it follows that

$$\operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |K_\tau(z,w)| \, dw < +\infty \quad \text{and} \quad \operatorname*{ess\,sup}_{w\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |K_\tau(z,w)| \, dz < +\infty.$$

Hence from the Schur test it follows that $M_\tau(\sigma)$ is continuous, and due to (3.20) we notice that

$$V_g \circ \mathrm{Op}_\tau(\sigma) f = M_\tau(\sigma) \circ V_g f,$$

where the right hand-side is continuous and takes elements of $\mathcal{S}^{(1)}(\mathbb{R}^d) \subseteq M^p_{m_1}(\mathbb{R}^d)$ into $L^p_{m_2}(\mathbb{R}^{2d})$. Therefore $\mathrm{Op}_\tau(\sigma)$ is linear, continuous and densely defined. This concludes the proof. □

Schatten class properties for various classes of pseudo-differential operators in the framework of time-frequency analysis are studied by many authors, let us mention just [82, 29, 113, 142]. However, for our purposes it is convenient to recall [106, Theorem 1.2] about Schatten class property for pseudo-differential operators $\mathrm{Op}_\tau(\sigma)$ with symbols in modulation spaces.

**Theorem 3.3.2.** *Let $\tau \in [0,1]$, $0 < p < 2$, $d \in \mathbb{N}$ and*

$$u > \frac{2d}{p} - d. \tag{3.21}$$

*Consider $\sigma \in M^2_{m^\tau_u}(\mathbb{R}^{2d})$, where $m^\tau_u$ is defined as in* (2.17). *Then*

$$\mathrm{Op}_\tau(\sigma) \in \mathcal{J}^p(L^2(\mathbb{R}^d)).$$

**Lemma 3.3.3.** ([11, Lemma 3.5]) *Let $\tau \in [0,1]$, $\gamma \geq 1$ and $d \in \mathbb{N}$. Fix*

$$u, s, t > 0, \qquad l > u + d, \qquad j \geq u.$$

*Then*

$$M^{\infty,1}_{w^\gamma_s \otimes w^\gamma_t}(\mathbb{R}^{2d}) \hookrightarrow M^{\infty,1}_{v_l \otimes v_j}(\mathbb{R}^{2d}) \hookrightarrow M^2_{v_u \otimes v_u}(\mathbb{R}^{2d}) \hookrightarrow M^2_{m^\tau_u}(\mathbb{R}^{2d}).$$

*Proof.* The first inclusion is due to the inclusion relations between ultra-modulation spaces since $v_l \otimes v_j \lesssim w^\gamma_s \otimes w^\gamma_t$. The last inclusion follows similarly since $m^\tau_u \lesssim v_u \otimes v_u$, as it is shown in Remark 2.2.14.

For the second inclusion we use Theorem 2.8.11: $(\infty, 2, l, u)$ fulfils the condition $(\mathcal{C}_2)$ and $(1, 2, j, u)$ fulfils the condition $(\mathcal{C}_1)$. This concludes the proof. □

On account of the following corollary all the operators considered in Theorem 3.3.5 are compact on $L^2(\mathbb{R}^d)$.

**Corollary 3.3.4.** ([11, Corollary 3.6]) *Let $\tau \in [0,1]$, $\gamma \geq 1$ and $s,t > 0$. Consider $\sigma \in M^{\infty,1}_{w^\gamma_s \otimes w^\gamma_t}(\mathbb{R}^{2d})$. Then $\mathrm{Op}_\tau(\sigma)$ is compact on $L^2(\mathbb{R}^d)$.*

*Proof.* The claim follows by Lemma 3.3.3 with $u$ satisfying (3.21), after choosing any $0 < p < 2$, in addition with Theorem 3.3.2. □ □

Now we prove the decay property of the eigenfunctions of $\mathrm{Op}_\tau(\sigma)$ when the symbol belongs to certain weighted ultra-modulation spaces. This result improves Theorem 3.2.9 ([9, Theorem 3.10]), in the sense that we show how faster decay of the symbol implies stronger regularity and decay properties for the eigenfunctions of the corresponding operator. More precisely, Theorem 3.2.9 deals with polynomial decay, whereas Theorem 3.3.5 allows to consider sub-exponential decay as well.

**Theorem 3.3.5.** ([11, Theorem 3.7]) *Fix $\tau \in [0,1]$, $\gamma \geq 1$ and $s > 0$. Consider a symbol $\sigma \in M^{\infty,1}_{w^\gamma_s \otimes w^\gamma_t}(\mathbb{R}^{2d})$ for every $t \geq 0$. Any $f \in L^2(\mathbb{R}^d)$ eigenfunction of $\mathrm{Op}_\tau(\sigma)$ belongs to $\mathcal{S}^{(\gamma)}(\mathbb{R}^d)$.*

*Proof.* We first observe that $\sigma \in M^{\infty,1}_{w^\gamma_s \otimes w^\gamma_t}(\mathbb{R}^{2d})$ for every $t \geq 0$ is equivalent to require that $t$ fulfils (2.11) due to the inclusion relations. By (2.12) from Lemma 2.2.13 it follows that

$$\frac{w^\gamma_{r'+s'}(x,\omega)}{w^\gamma_{r'}(y,\eta)} \leq w^\gamma_{s'} \otimes w^\gamma_{t'}\Big(\big((1-\tau)x + \tau y, \tau\omega + (1-\tau)\eta\big), \big(\omega - \eta, y - x\big)\Big),$$

for every $x, \omega, y, \eta \in \mathbb{R}^d$, where $s', r' \geq 0$ and $t'$ which fulfils (2.11). We consider first the case $1/2 \leq \tau \leq 1$ and fix $s' = s > 0$.

Take $r' = 0$, $t \geq s\tau^{1/\gamma}$, and apply Theorem 3.3.1 with $p = 2$, $m_0 = w^\gamma_s \otimes w^\gamma_t$, $m_1 = w^\gamma_0$ and $m_2 = w^\gamma_s$ which satisfy (3.19). Thus $\mathrm{Op}_\tau(\sigma)$ extends to a continuous operator from $M^2_{w^\gamma_0}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ to $M^2_{w^\gamma_s}(\mathbb{R}^d)$. Starting with $f \in L^2(\mathbb{R}^d)$ we get $f = \lambda^{-1}\,\mathrm{Op}_\tau(\sigma) f \in M^2_{w^\gamma_s}(\mathbb{R}^d)$.

Now, take $r' = s$, $t \geq s + s\tau^{1/\gamma}$, and apply Theorem 3.3.1 with $p = 2$, $m_0 = w_s^\gamma \otimes w_t^\gamma$, $m_1 = w_s^\gamma$ and $m_2 = w_{2s}^\gamma$ which satisfy (3.19). Thus $\mathrm{Op}_\tau(\sigma)$ restricts to a continuous operator from $M^2_{w_s^\gamma}(\mathbb{R}^d)$ to $M^2_{w_{2s}^\gamma}(\mathbb{R}^d)$, so starting with $f \in M^2_{w_s^\gamma}(\mathbb{R}^d)$ we get $f = \lambda^{-1}\mathrm{Op}_\tau(\sigma)f \in M^2_{w_{2s}^\gamma}(\mathbb{R}^d)$.

Repeating the same argument, and using the inclusion relations between ultra-modulation spaces we obtain:

$$f \in \bigcap_{n\in\mathbb{N}_0} M^2_{w_{ns}^\gamma}(\mathbb{R}^d) = \bigcap_{k\geq 0} M^2_{w_k^\gamma}(\mathbb{R}^d) = \mathcal{S}^{(\gamma)}(\mathbb{R}^d).$$

The case $0 \leq \tau < 1/2$ is done similarly. This concludes the proof. □

We conclude the chapter with the analogue of Theorem 3.2.9. Note that by Corollary 3.3.4 it follows that the localization operators $A_a^{\psi_1,\psi_2}$ in the following statement are compact on $L^2(\mathbb{R}^d)$.

**Theorem 3.3.6.** ([11, Theorem 3.8]) *Consider $\gamma \geq 1$, $s > 0$, $a \in M^\infty_{w_s^\gamma\otimes 1}(\mathbb{R}^{2d})$ and $\psi_1, \psi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$. Any $f \in L^2(\mathbb{R}^d)$ eigenfunction of $A_a^{\psi_1,\psi_2}$ belongs to $\mathcal{S}^{(\gamma)}(\mathbb{R}^d)$.*

*Proof.* Since $\psi_1, \psi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ it follows that $W(\psi_2, \psi_1) \in \mathcal{S}^{(1)}(\mathbb{R}^{2d}) \subseteq M^1_{w_r^\gamma\otimes w_t^\gamma}(\mathbb{R}^{2d})$, for every $r, t \geq 0$. It is easy to check that $w_s^\gamma \otimes w_t^\gamma$ is $w_r^\gamma \otimes w_t^\gamma$-moderate for every $t \geq 0$ and every $r \geq s$, i.e.

$$w_s^\gamma \otimes w_t^\gamma((x,\omega) + (y,\eta)) \leq w_r^\gamma \otimes w_t^\gamma(x,\omega) w_s^\gamma \otimes w_t^\gamma(y,\eta), \quad x,\omega,y,\eta \in \mathbb{R}^d.$$

We write $A_a^{\psi_1,\psi_2} = \mathrm{Op}_{1/2}(\sigma)$ with $\sigma = a * W(\psi_2,\psi_1)$, and then apply Proposition 2.8.13 in order to infer $\sigma \in M^{\infty,1}_{w_s^\gamma\otimes w_t^\gamma}(\mathbb{R}^{2d})$ for every $t \geq s/2^{1/\gamma}$:

$$M^\infty_{w_s^\gamma\otimes 1}(\mathbb{R}^{2d}) * M^1_{w_r^\gamma\otimes w_t^\gamma}(\mathbb{R}^{2d}) \hookrightarrow M^{\infty,1}_{w_s^\gamma\otimes w_t^\gamma}(\mathbb{R}^{2d}).$$

The claim now follows by Theorem 3.3.5. □

# Chapter 4

# Characterization of smooth symbol classes by Gabor matrix decay

The present chapter illustrates a characterization of the symbol class $S^m(\mathbb{R}^{2d})$ (2.178), introduced by J. Sjöstrand in [127], by mean of the Gabor matrix of $\mathrm{Op}_\tau(\sigma)$. The results reported in what follows are due to E. Cordero and the author [7]. In particular, the first part of the main result can be roughly summarized as follows.

*Fix $m \in \mathbb{R}$. The following properties are equivalent:*

*(i)* $\sigma \in S^m\left(\mathbb{R}^{2d}\right)$*;*

*(ii)* $\sigma \in \mathcal{S}'\left(\mathbb{R}^{2d}\right)$ *and for every $s \geq 0$, $0 < q \leq \infty$, there exists a function $H_\tau \in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$, with*

$$\|H_\tau\|_{L^q_{\langle\cdot\rangle^s}} \leq C, \quad \forall \tau \in [0,1];$$

*such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,g, \pi(u)\,g\rangle| \leq H_\tau(u-z)\langle \mathcal{T}_\tau(z,u)\rangle^m, \qquad \forall u,z \in \mathbb{R}^{2d}.$$

Above, $\mathcal{T}_\tau$ is the transformation defined in (2.212).
We provide also a discrete version of the above item (*ii*), see Theorem 4.2.3. For the Hörmander class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$, the Gabor matrix characterization for Weyl operators was shown by K. Gröchenig and Z. Rzeszotnik in [87, Theorem 6.2] (see also [121]) in the case $q = \infty$. So this result can be viewed as an extension to any $0 < q \leq \infty$ and $\tau \in [0,1]$. The key tool in order to prove such generalization is Lemma 2.6.8, which proof is contained in Chapter 2, and which gives the characterization

$$S^m(\mathbb{R}^{2d}) = \bigcap_{s\geq 0} M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s},$$

where $0 < q \leq \infty$ and the weights $\langle\cdot\rangle^{-m}, \langle\cdot\rangle^s$ were introduced in (2.7).

Section 4.1 shows some estimates for the STFT of $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ w.r.t. $W_\tau(g,g) \in \mathcal{S}(\mathbb{R}^{2d})$ uniform in $\tau \in [0,1]$. In particular, a proof alternative to the one published in [7] is provided for Proposition 4.1.1.
The main result mentioned above is achieved by proving a characterization of similar flavour for the modulation spaces

$$M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d}),$$

where $m, s \in \mathbb{R}$ and $0 < q \leq \infty$. See Theorem 4.2.1 in Section 4.2. As a consequence, we are able to infer boundedness properties for $\tau$-pseudo-differential operators on modulation spaces along with a uniform bound in $\tau \in [0,1]$ for the norm on $\mathrm{Op}_\tau(\sigma)$, see Subsection 4.2.1. In Subsection 4.2.2 we give an estimate for the Gabor matrix of a Born-Jordan operator, Theorem 4.2.10, and obtain as straightforward consequence some continuity results for $\mathrm{Op}_{BJ}$ between modulation spaces, Corollary 4.2.12.

We remind to the reader that the necessary preliminaries and definitions can be found in Chapter 2, in particular the definitions of $\mathcal{C}^s$, $\mathcal{C}^n$ and $\mathcal{C}^n_m$ can be found in Subsection 2.6.3. In this chapter Assumptions 2.5.1 hold, i.e. every weight is supposed of at most polynomial growth.

## 4.1 Uniform estimates in $\tau$

In this section, we focus on the quasi-norm of $M^{p,q}_w(\mathbb{R}^{2d})$ computed w.r.t. the window $W_\tau(g,g) \in \mathcal{S}(\mathbb{R}^{2d})$ and prove that it does not depend on $\tau \in [0,1]$, in the sense of (4.2). After stating and proving Proposition 4.1.1, we exhibit a number of unpublished lemmas in order to give an alternative proof of (4.2).

Let us first represent the Gabor matrix as a kernel of an integral operator. Consider a linear and bounded operator $T$ from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. The inversion formula (2.147) for $g \in M^1_v(\mathbb{R}^d)$, $\|g\|_{L^2} = 1$ is simply $V_g^* V_g = I$. The operator $T$ can be written as

$$T = V_g^* V_g T V_g^* V_g. \tag{4.1}$$

The linear transformation $V_g T V_g^*$ is an integral operator with kernel given by the Gabor matrix of $T$ which was defined in (2.77):

$$G_T(u,z) = \langle T\pi(z)g, \pi(u)g\rangle, \quad \forall u, z \in \mathbb{R}^{2d}.$$

By definition and the inversion formula, $V_g$ is bounded from $M^{p,q}_w(\mathbb{R}^d)$ to $L^{p,q}_w(\mathbb{R}^{2d})$ and $V_g^*$ from $L^{p,q}_w(\mathbb{R}^{2d})$ to $M^{p,q}_w(\mathbb{R}^d)$. Hence the continuity properties of $T$ on modulation spaces can be obtained by the corresponding ones of the operator $V_g T V_g^*$ on mixed-norm $L^{p,q}_w$ spaces. These issues will be studied in Proposition 4.2.7 and Corollary 4.2.12 and can be achieved by studying the Gabor matrix decay of $T$.

**Proposition 4.1.1.** ([7, Proposition 3.1]) *Consider $0 < p, q \leq \infty$, $\tau \in [0,1]$, $w \in \mathcal{M}_v(\mathbb{R}^{4d})$ of at most polynomial growth, $G \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and define $\Phi_\tau := W_\tau(g,g)$. Then there exist $A = A(v,g,G) > 0$, $B = B(v,g,G) > 0$ such that*

$$A \|V_G \sigma\|_{L^{p,q}_w} \leq \|V_{\Phi_\tau} \sigma\|_{L^{p,q}_w} \leq B \|V_G \sigma\|_{L^{p,q}_w}, \tag{4.2}$$

*for every $\tau \in [0,1]$ and $\sigma \in M^{p,q}_w(\mathbb{R}^{2d})$.*

*Proof.* By Proposition 2.2 and Remark 2.3 in [42], the mapping

$$(\tau, f, g) \mapsto W_\tau(f,g)$$

is continuous from $\mathbb{R} \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{2d})$ and locally uniformly bounded. Since $\Phi_\tau$ for $\tau \in [0,1]$ belongs to a bounded set in $\mathcal{S}(\mathbb{R}^{2d})$, the result follows immediately from [82, Theorem 11.3.7] for $p, q \geq 1$ and [75, Theorem 3.1] for $0 < p, q \leq \infty$. □

For sake of completeness, we give an alternative and independent proof of previous Proposition 4.1.1. In order to do so, we need some technical lemmas.

We begin with a generalization of [27, Lemma 3.2].

**Lemma 4.1.2.** *Consider $\tau \in [0,1]$ and define*

$$\psi(t) := e^{-\pi t^2}, \qquad \Psi(x,\omega) := e^{-\pi(x^2+\omega^2)}, \qquad \Psi_\tau := W_\tau(\psi,\psi), \tag{4.3}$$

*for $t, x, \omega \in \mathbb{R}^d$. Then for $v$ submultiplicative weight on $\mathbb{R}^{4d}$ there exists a constant $C > 0$ such that*

$$\|V_\Psi \Psi_\tau\|_{L^1_v} \leq C, \qquad \forall \tau \in [0,1]. \tag{4.4}$$

*Consequently*

$$\|\Psi_\tau\|_{M^1_v} \leq C, \qquad \forall \tau \in [0,1]. \tag{4.5}$$

*Proof.* We first observe that any submultiplicative weight function $v$ on $\mathbb{R}^{4d}$ can grow at most exponentially, i.e., there exist $C > 0$ and $b > 0$ such that

$$v(z) \leq Ce^{b|z|}, \qquad \forall z \in \mathbb{R}^{4d}, \tag{4.6}$$

see, e.g. [35, Lemma 2.1.4]. Following the proof in [27] and using (4.6), we can majorize in the following manner:

$$\begin{aligned}
\|V_\Psi \Psi_\tau\|_{L^1_v} &\leq C_1 \int_{\mathbb{R}^{2d}} e^{-\pi \frac{z_1^2+z_2^2}{2\tau^2-2\tau+5}} e^{b|z|} I_1 \, dz_1 dz_2 \\
&\leq C_1 \int_{\mathbb{R}^{2d}} e^{-\pi \frac{z_1^2+z_2^2}{2\tau^2-2\tau+5}} e^{b(|z_1|+|z_2|)} I_1 \, dz_1 dz_2,
\end{aligned}$$

where $C > 0$ depends only on the weight $v$ and $I_1$ is an integral over $\mathbb{R}^{2d}$ which can be controlled from above by

$$I_1 \leq \tilde{C}_2 e^{\pi \frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} + \frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|} e^{\pi \frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} + \frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1|},$$

for some $a > 0$ and $\tilde{C}_2 > 0$ independent of $\tau$. Hence setting $C_2 := C_1 \tilde{C}_2$ we have

$$\begin{aligned}
\|V_\Psi \Psi_\tau\|_{L^1_v} &\leq C_2 \int_{\mathbb{R}^d} e^{-\pi \frac{3z_1^2}{2\tau^2-2\tau+5} + \pi \frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} + \frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1| + b|z_1|} \, dz_1 \\
&\quad \times \int_{\mathbb{R}^d} e^{-\pi \frac{3z_2^2}{2\tau^2-2\tau+5} + \pi \frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} + \frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2| + b|z_2|} \, dz_2 \\
&= C_2 \Bigg( \underbrace{\int_{\mathbb{R}^d} e^{-\pi \frac{3z_1^2}{2\tau^2-2\tau+5} + \pi \frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} + \frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1| + b|z_1|} \, dz_1}_{=:I_2} \Bigg)^2 .
\end{aligned}$$

The integral $I_2$ can be controlled as

$$I_2 = \int_{\mathbb{R}^d} e^{-\pi \frac{3(2\tau^2-2\tau+2)-(1-2\tau)^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} z_1^2 + \left(a\frac{|1-2\tau|}{2\tau^2-2\tau+2} + b\right)|z_1|} \, dz_1 \leq \int_{\mathbb{R}^d} e^{-\pi C_3 z_1^2 + \overbrace{(aC_4 + b)}^{=:C_5 \geq 0}|z_1|} \, dz_1,$$

being

$$C_3 = \min_{\tau\in[0,1]} \frac{3(2\tau^2-2\tau+2)-(1-2\tau)^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)} = \frac{1}{2}, \quad C_4 = \max_{\tau\in[0,1]} \frac{|1-2\tau|}{2\tau^2-2\tau+2} = \frac{1}{2}.$$

Since $e^{-\pi\frac{C_3}{2}z_1^2+C_5|z_1|} \to 0$ as $|z_1| \to +\infty$, for $\varepsilon > 0$ fixed there exists $R > 0$ such that $e^{-\pi\frac{C_3}{2}z_1^2+C_5|z_1|} < \varepsilon$ for every $z_1 \notin B_R(0)$. Therefore

$$\begin{aligned} I_2 &\leq \int_{\mathbb{R}^d} e^{-\pi C_3 z_1^2+C_5|z_1|}\,dz_1 = \int_{B_R(0)} e^{-\pi C_3 z_1^2+C_5|z_1|}\,dz_1 + \int_{\mathbb{R}^d\setminus B_R(0)} e^{-\pi C_3 z_1^2+C_5|z_1|}\,dz_1 \\ &\leq e^{C_5 R}\int_{B_R(0)} e^{-\pi C_3 z_1^2}\,dz_1 + \varepsilon\int_{\mathbb{R}^d\setminus B_R(0)} e^{-\pi\frac{C_3}{2}z_1^2}\,dz_1 < +\infty, \end{aligned}$$

Hence there exists $C > 0$ such that $\|V_\Psi \Psi_\tau\|_{L^1_v} \leq C$, uniformly w.r.t. $\tau \in [0,1]$. The equivalence

$$\|\Psi_\tau\|_{M^1_v} \asymp \|V_\Psi \Psi_\tau\|_{L^1_v}$$

concludes the proof. □

**Corollary 4.1.3.** *Fix $G \in \mathcal{S}(\mathbb{R}^{2d})$ and consider an even, submultiplicative weight $v$ on $\mathbb{R}^{4d}$. Let $\Psi_\tau$ be the function defined in (4.3). Then there exists a constant $C > 0$ such that*

$$\|V_{\Psi_\tau} G\|_{L^1_v} \leq C, \qquad \forall \tau \in [0,1]. \tag{4.7}$$

*Proof.* The claim is a straightforward consequence of Lemma 4.1.2, the switching property of the STFT (see, e.g., [35, Lemma 1.2.3])

$$V_{\Psi_\tau} G(z,\zeta) = e^{-2\pi z\zeta} V_G \Psi_\tau(-z,-\zeta), \qquad \forall (z,\zeta) \in \mathbb{R}^{4d}.$$

and the even property of the weight $v$. □

In the following lemma we summarize [27, Lemma 2.5, Lemma 2.6, Corollary 2.7]. For $\tau \in (0,1)$, define the matrix

$$\mathcal{B}_\tau := \begin{bmatrix} 0_d & \sqrt{\frac{1-\tau}{\tau}} I_d \\ -\sqrt{\frac{\tau}{1-\tau}} I_d & 0_d \end{bmatrix}. \tag{4.8}$$

**Lemma 4.1.4.** *Let $f,g \in \mathcal{S}(\mathbb{R}^d)$, $\tau \in [0,1]$ and define $\Phi_\tau := W_\tau(g,g)$. Consider $z = (z_1,z_2), \zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^{2d}$. If $\tau \in (0,1)$, then*

$$\begin{aligned} V_{\Phi_\tau} W_\tau(f,f)(z,\zeta) &= e^{-2\pi z_2\zeta_2} V_g f(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1)\overline{V_g f(z_1 + (1-\tau)\zeta_2, z_2 - \tau\zeta_1)} \\ &= e^{-2\pi z_2\zeta_2} V_g f(z + \sqrt{\tau(1-\tau)}\mathcal{B}_\tau^T\zeta)\overline{V_g f(z + \sqrt{\tau(1-\tau)}\mathcal{B}_\tau\zeta)};, \end{aligned} \tag{4.9}$$

*where $\mathcal{B}_\tau^T$ stands for the transpose of $\mathcal{B}_\tau$.*
*If $\tau = 1$, then*

$$V_{\Phi_1} W_1(f,f)(z,\zeta) = e^{-2\pi z_2\zeta_2} V_g f(z_1 - \zeta_2, z_2)\overline{V_g f(z_1, z_2 - \zeta_1)}; \tag{4.10}$$

*If $\tau = 0$, then*

$$V_{\Phi_0} W_0(f,f)(z,\zeta) = e^{-2\pi z_2\zeta_2} V_g f(z_1, z_2 + \zeta_1)\overline{V_g f(z_1 + \zeta_2, z_2)}. \tag{4.11}$$

**Remark 4.1.5.** *Notice that, for $\zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^{2d}$ and $\tau \in (0,1)$, we have*

$$|\sqrt{\tau(1-\tau)}\mathcal{B}_\tau\zeta|^2 = |-\tau\zeta_1|^2 + |(1-\tau)\zeta_2|^2 \leq |\zeta_1|^2 + |\zeta_2|^2 = |\zeta|^2. \tag{4.12}$$

*Therefore*

$$\langle\sqrt{\tau(1-\tau)}\mathcal{B}_\tau\zeta\rangle^s \leq \langle\zeta\rangle^s, \qquad \forall \tau \in (0,1), \forall \zeta \in \mathbb{R}^{2d}, \forall s \geq 0. \tag{4.13}$$

**Lemma 4.1.6.** *Consider $f, g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $\tau \in [0,1]$, $v$ a submultiplicative weight on $\mathbb{R}^{4d}$ of at most polynomial growth, and define $\Phi_\tau := W_\tau(g,g)$. Then there exists a constant $C > 0$ such that*

$$\|V_{\Phi_\tau} W_\tau(f,f)\|_{L^1_v} \leq C, \qquad \forall \tau \in [0,1]. \tag{4.14}$$

*Proof.* We divide the proof in three cases: $\tau \in (0,1)$, $\tau = 1$ and $\tau = 0$.

For $\tau \in (0,1)$ we apply (4.9), the change of variable $y = z + \sqrt{\tau(1-\tau)}\mathcal{B}_\tau \zeta$, so that $z + \sqrt{\tau(1-\tau)}\mathcal{B}_\tau^T \zeta = y - J\zeta$ (where $J$ is as in (2.212)), the submultiplicativity of $v$ as well as the (at most) polynomial growth, and finally Remark 4.1.5:

$$\begin{aligned}
\|V_{\Phi_\tau} W_\tau(f,f)\|_{L^1_v} &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, v(y - \sqrt{\tau(1-\tau)}\mathcal{B}_\tau \zeta, \zeta)\, dy d\zeta \\
&\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, v(y,0) v(-\sqrt{\tau(1-\tau)}\mathcal{B}_\tau \zeta, 0) v(0,\zeta)\, dy d\zeta \\
&\lesssim \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, \langle y \rangle^s \langle -\sqrt{\tau(1-\tau)}\mathcal{B}_\tau \zeta \rangle^s \langle \zeta \rangle^s\, dy d\zeta \\
&\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, \langle y \rangle^s \langle \zeta \rangle^s \langle \zeta \rangle^s\, dy d\zeta \\
&= \int_{\mathbb{R}^{2d}} (|V_g f| * |V_g f \langle \cdot \rangle^s|)(J\zeta) \langle \zeta \rangle^{2s}\, d\zeta \\
&= \| |V_g f| * |V_g f \langle \cdot \rangle^s| \|_{L^1_{\langle \cdot \rangle^{2s}}} < +\infty.
\end{aligned}$$

The convergence is due to the fact that $f, g \in \mathcal{S}(\mathbb{R}^d)$, therefore $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$.

For $\tau = 1$ we apply (4.10) and the change of variable $y_1 = z_1, y_2 = z_2 - \zeta_1$; arguing as in the previous stage we obtain the result. In detail,

$$\begin{aligned}
\|V_{\Phi_1} W_1(f,f)\|_{L^1_v} &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, v(y_1, y_2 + \zeta_1, \zeta_1, \zeta_2)\, dy d\zeta \\
&\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, v(y,0) v(0,0,\zeta_1,0) v(0,\zeta)\, dy d\zeta \\
&\lesssim \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, \langle y \rangle^s \langle \zeta_1 \rangle^s \langle \zeta \rangle^s\, dy d\zeta \\
&\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g f(y - J\zeta) V_g f(y)|\, \langle y \rangle^s \langle \zeta \rangle^s \langle \zeta \rangle^s\, dy d\zeta \\
&= \int_{\mathbb{R}^{2d}} (|V_g f| * |V_g f \langle \cdot \rangle^s|)(J\zeta) \langle \zeta \rangle^{2s}\, d\zeta \\
&= \| |V_g f| * |V_g f \langle \cdot \rangle^s| \|_{L^1_{\langle \cdot \rangle^{2s}}} < +\infty.
\end{aligned}$$

The case $\tau = 0$ follows the same argument as before via (4.11).

In each case we found the same upper bound which does not depend on $\tau \in [0,1]$. The proof is concluded. ☐

**Corollary 4.1.7.** *Consider $\tau \in [0,1]$, $v$ a submultiplicative weight on $\mathbb{R}^{4d}$ of at most polynomial growth, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and define $\Phi_\tau := W_\tau(g,g)$. Then there exists a constant $C > 0$ such that*

$$\|\Phi_\tau\|_{M^1_v} \leq C, \qquad \forall \tau \in [0,1]. \tag{4.15}$$

*Proof.* Fix a window $G \in \mathcal{S}(\mathbb{R}^{2d}) \smallsetminus \{0\}$ and consider the functions $\psi$ and $\Psi_\tau$ defined in (4.3). Using the change-window property of the STFT (see, e.g., [35, 1.2.29]), Moyal's formula for $\tau$-Wigner distributions (see (2.84)) and Young's inequality for mixed-normed spaces (see Proposition 2.2.27 or, e.g., [35, Theorem 2.2.3]),

$$\|V_G\Phi_\tau\|_{L^1_v} \leq \frac{1}{|\langle \Psi_\tau, \Psi_\tau\rangle|} \|V_{\Psi_\tau}\Phi_\tau * V_G\Psi_\tau\|_{L^1_v} \leq \|\psi\|_{L^2}^{-4} \|V_{\Psi_\tau}\Phi_\tau\|_{L^1_v} \|V_G\Psi_\tau\|_{L^1_v}.$$

The desired result follows now by Lemma 4.1.2, 4.1.6 and the fact that

$$\|\Phi_\tau\|_{M^1_v} \asymp \|V_G\Phi_\tau\|_{L^1_v},$$

where the constants involved do not depend on $\tau \in [0,1]$. This concludes the proof. □

**Corollary 4.1.8.** *Consider $\tau \in [0,1]$, $v$ an even submultiplicative weight on $\mathbb{R}^{4d}$ of at most polynomial growth, $G \in \mathcal{S}(\mathbb{R}^{4d}) \smallsetminus \{0\}$, $g \in \mathcal{S}(\mathbb{R}^d) \smallsetminus \{0\}$ and define $\Phi_\tau \coloneqq W_\tau(g,g)$. Then there exists a constant $C > 0$ such that*

$$\|V_{\Phi_\tau}G\|_{L^1_v} \leq C, \qquad \forall \tau \in [0,1]. \tag{4.16}$$

*Proof.* The proof is a straightforward consequence of Corollary 4.1.7 and the switching property of the STFT, cf. the proof of Corollary 4.1.3. □

*Alternative proof for Proposition 4.1.1.* Let $\Psi_\tau = W_\tau(\psi,\psi)$, and $\psi$ be as in (4.3). Using the change-window property of the STFT (see, e.g., [35, 1.2.29]), Moyal's formula for $\tau$-Wigner distributions (2.84) and Young's inequality for mixed-normed spaces (see, e.g., [35, Theorem 2.2.3]), and Corollary 4.1.8:

$$\begin{aligned}
\|V_{\Phi_\tau}\sigma\|_{L^{p,q}_w} &\leq \frac{1}{|\langle \Psi_\tau, \Psi_\tau\rangle|} \||V_{\Psi_\tau}\sigma| * |V_{\Phi_\tau}\Psi_\tau|\|_{L^{p,q}_w} \leq \|\psi\|_{L^2}^{-4} \|V_{\Psi_\tau}\sigma\|_{L^{p,q}_w} \|V_{\Phi_\tau}\Psi_\tau\|_{L^1_v} \\
&\leq C\,\|\psi\|_{L^2}^{-4} \|V_{\Psi_\tau}\sigma\|_{L^{p,q}_w} \\
&\leq C\,\|\psi\|_{L^2}^{-4} \frac{1}{|\langle G, G\rangle|} \||V_G\sigma| * |V_{\Psi_\tau}G|\|_{L^{p,q}_w} \\
&\leq C\,\|\psi\|_{L^2}^{-4} \|G\|_{L^2}^{-2} \|V_G\sigma\|_{L^{p,q}_w} \|V_{\Psi_\tau}G\|_{L^1_v} \\
&\leq \tilde{C}\,\|\psi\|_{L^2}^{-4} \|G\|_{L^2}^{-2} \|V_G\sigma\|_{L^{p,q}_w},
\end{aligned}$$

with $\tilde{C} > 0$ and independent of $\tau$. Similarly,

$$\|V_G\sigma\|_{L^{p,q}_w} \leq \tilde{C}\,\|\psi\|_{L^2}^{-4} \|g\|_{L^2}^{-4} \|V_{\Phi_\tau}\sigma\|_{L^{p,q}_w}.$$

The choice

$$A \coloneqq \tilde{C}^{-1}\,\|\psi\|_{L^2}^{4} \|g\|_{L^2}^{4}, \qquad B \coloneqq \tilde{C}\,\|\psi\|_{L^2}^{-4} \|G\|_{L^2}^{-2}$$

let us conclude the proof. □

In what follows, [32, Lemma 4.1] is needed. It was reported in Lemma 2.7.17.

## 4.2 Main results

In order to obtain the announced characterization of $S^m(\mathbb{R}^{2d})$, we first characterize $\tau$-pseudo-differential operators with symbols in $M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$. After that we shall describe the class $S^m(\mathbb{R}^{2d})$ in terms of the Gabor matrix of $\mathrm{Op}_\tau(\sigma)$ and provide some continuity results for such operators. In the last Subsection 4.2.2, Born-Jordan operators shall be considered.

**Theorem 4.2.1.** ([7, Theorem 3.2]) *Consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ such that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. For $\tau \in [0,1]$, let $\mathcal{T}_\tau$ be the transformation defined in (2.212). For any $s,m \in \mathbb{R}$, $0 < q \leq \infty$, the following properties are equivalent:*

*(i)* $\sigma \in M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$;

*(ii)* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a function* $H_\tau \in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ *satisfying*

$$\|H_\tau\|_{L^q_{\langle\cdot\rangle^s}} \leq C, \quad \forall \tau \in [0,1], \tag{4.17}$$

*such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,g, \pi(u)\,g\rangle| \leq H_\tau(u-z)\langle \mathcal{T}_\tau(z,u)\rangle^m, \qquad \forall u,z \in \mathbb{R}^{2d}; \tag{4.18}$$

*(iii)* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a sequence* $h_\tau \in \ell^q_{\langle\cdot\rangle^s}(\Lambda)$, *with* $\|h_\tau\|_{\ell^q_{\langle\cdot\rangle^s}} \leq C$ *for every* $\tau \in [0,1]$, *such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\,\pi(\mu)\,g, \pi(\lambda)\,g\rangle| \leq h_\tau(\lambda-\mu)\langle \mathcal{T}_\tau(\mu,\lambda)\rangle^m, \qquad \forall \lambda,\mu \in \Lambda. \tag{4.19}$$

*Proof.* The proof follows the pattern of the corresponding one for Weyl operators with symbols in weighted Sjöstrand's classes [83, Theorem 3.2].
$(i) \Rightarrow (ii)$ This implication comes easily from the characterization (2.211). In details, observing that $\langle Ju\rangle = \langle u\rangle$,

$$\begin{aligned}|\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,g, \pi(u)\,g\rangle| &= |V_{\Phi_\tau}\sigma\,(\mathcal{T}_\tau(z,u)\,, J\,(u-z))| \\ &\leq \sup_{w\in\mathbb{R}^{2d}} \left(|V_{\Phi_\tau}\sigma(w, J\,(u-z)\,|\langle w\rangle^{-m}\right)\langle \mathcal{T}_\tau(z,u)\rangle^m \\ &= H_\tau(u-z)\langle \mathcal{T}_\tau(z,u)\rangle^m,\end{aligned}$$

where

$$H_\tau(u) := \sup_{w\in\mathbb{R}^{2d}} \left(|V_{\Phi_\tau}\sigma(w,Ju)|\langle w\rangle^{-m}\right).$$

For $0 < q < \infty$,

$$\|H_\tau\|_{L^q_{\langle\cdot\rangle^s}} = \left(\int_{\mathbb{R}^{2d}} \left[\sup_{w\in\mathbb{R}^{2d}} \left(|V_{\Phi_\tau}\sigma(w,Ju)|\langle w\rangle^{-m}\right)\right]^q \langle u\rangle^{qs} du\right)^{\frac{1}{q}} \asymp \|\sigma\|_{M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}},$$

Hence by Proposition 4.1.1 we obtain the estimate (4.17). The case $q = \infty$ is analogous.
$(ii) \Rightarrow (i)$ Consider the change of variables $y = \mathcal{T}_\tau(z,u)$ and $t = J(u-z)$, so that

$$\begin{cases} z(y,t) &= y - U_\tau J^{-1}t \\ u(y,t) &= y + (I_{2d} - U_\tau)J^{-1}t \end{cases}, \qquad U_\tau z := \begin{bmatrix} \tau I_d & 0 \\ 0 & (1-\tau)I_d \end{bmatrix} z = \mathcal{T}_\tau(0,z) \tag{4.20}$$

and $u(y,t)-z(y,t)=J^{-1}t$. For $0<q<\infty$, using (2.211) and (4.18),

$$\begin{aligned}\|\sigma\|_{M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^{s}}} &\asymp \left(\int_{\mathbb{R}^{2d}}\left(\sup_{y\in\mathbb{R}^{2d}}|V_{\Phi_\tau}\sigma(y,t)|\langle y\rangle^{-m}\right)^q\langle t\rangle^{qs}dt\right)^{\frac{1}{q}}\\ &=\left(\int_{\mathbb{R}^{2d}}\left(\sup_{y\in\mathbb{R}^{2d}}|\langle \mathrm{Op}_\tau(\sigma)\pi(z(y,t))g,\pi(u(y,t))g\rangle|\langle\mathcal{T}_\tau(z,u)\rangle^{-m}\right)^q\langle t\rangle^{qs}dt\right)^{\frac{1}{q}}\\ &\leq\left(\int_{\mathbb{R}^{2d}}\left|H_\tau(J^{-1}t)\right|^q\langle t\rangle^{qs}dt\right)^{\frac{1}{q}}\\ &\leq C,\end{aligned}$$

where we used (4.17). The case $q=\infty$ is analogous.
$(ii)\Leftrightarrow(iii)$ The argument requires that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then the equivalence can be proved similarly to [30, Theorem 3.1] and [83, Theorem 3.2]. □

**Remark 4.2.2.** *Under the assumptions of the previous theorem, let us consider the following statements:*

$(ii)'$ $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$ *and*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g,\pi(u)g\rangle|\leq H_\tau(u-z)\langle\mathcal{T}_\tau(z,u)\rangle^m,\qquad\forall u,z\in\mathbb{R}^{2d},$$

*for some* $\tau\in[0,1]$ *and* $H_\tau\in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$*;*

$(iii)'$ $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$ *and*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(\mu)g,\pi(\lambda)g\rangle|\leq h_\tau(\lambda-\mu)\langle\mathcal{T}_\tau(\mu,\lambda)\rangle^m,\qquad\forall\lambda,\mu\in\Lambda,$$

*for some* $\tau\in[0,1]$ *and* $h_\tau\in\ell^q_{\langle\cdot\rangle^s}(\Lambda)$.

*Then the proof of Theorem 4.2.1 shows that the* $(ii)'$ *and* $(iii)'$ *imply* $(i)$, *i.e.* $\sigma\in M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$.

We possess all the instruments for the characterization of $S^m(\mathbb{R}^{2d})$.

**Theorem 4.2.3.** ([7, Theorem 1.1]) *Consider* $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ *and a lattice* $\Lambda$ *such that* $\mathcal{G}(g,\Lambda)$ *is a Gabor frame for* $L^2(\mathbb{R}^d)$. *Fix* $m\in\mathbb{R}$. *The following properties are equivalent:*

$(i)$ $\sigma\in S^m(\mathbb{R}^{2d})$*;*

$(ii)$ $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$ *and for every* $s\geq0$, $0<q\leq\infty$, *there exists a function* $H_\tau\in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$, *satisfying* (4.17)

$$\|H_\tau\|_{L^q_{\langle\cdot\rangle^s}}\leq C,\quad\forall\tau\in[0,1],$$

*such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g,\pi(u)g\rangle|\leq H_\tau(u-z)\langle\mathcal{T}_\tau(z,u)\rangle^m,\qquad\forall u,z\in\mathbb{R}^{2d};\tag{4.21}$$

$(iii)$ $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$ *and for every* $s\geq0$, $0<q\leq\infty$, *there exists a sequence* $h_\tau\in\ell^q_{\langle\cdot\rangle^s}(\Lambda)$, *with* $\|h_\tau\|_{\ell^q_{\langle\cdot\rangle^s}}\leq C$ *for every* $\tau\in[0,1]$, *such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(\mu)g,\pi(\lambda)g\rangle|\leq h_\tau(\lambda-\mu)\langle\mathcal{T}_\tau(\mu,\lambda)\rangle^m,\qquad\forall\lambda,\mu\in\Lambda.\tag{4.22}$$

*Proof.* The proof is a direct application of the characterization of the classes $S^m(\mathbb{R}^{2d})$ presented in (2.182) and Theorem 4.2.1. □

**Remark 4.2.4.** *Observations similar to the ones in Remark 4.2.2 can be made for the above theorem.*

For $m = 0$, $\tau = 1/2$ and $q = \infty$, we recapture the characterization for the Hörmander class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$ shown by K. Gröchenig and Z. Rzeszotnik in [87, Theorem 6.2] (see also [121]).

The following issue is an improvement of [24, Theorem 2.4] and relies on the new characterization of $S^m(\mathbb{R}^{2d})$ proved in Lemma 2.6.8.

**Proposition 4.2.5.** ([7, Proposition 3.3]) *Consider $g \in \mathcal{S}(\mathbb{R}^d) \smallsetminus \{0\}$, $m \in \mathbb{R}$ and $\sigma \in S^m\left(\mathbb{R}^{2d}\right)$. For any $n \in \mathbb{N}_0$ there exists $C = C(n) > 0$, which does not depend on $\sigma$ or $\tau$, such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g, \pi(u)g\rangle| \le C|\sigma|_{n,m}\frac{\langle \mathcal{T}_\tau(z,u)\rangle^m}{\langle u-z\rangle^n}, \qquad \forall \tau \in [0,1],\ \forall u, z \in \mathbb{R}^{2d}. \tag{4.23}$$

*Proof.* Using the characterization of the Hörmander classes $S^m(\mathbb{R}^{2d})$ in (2.182) we infer that $\sigma \in M^\infty_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}(\mathbb{R}^{2d})$ and, for any $n \in \mathbb{N}_0$, the norm estimate in (2.184) says that there exists $C = C(n,m)$ such that

$$\|\sigma\|_{M^\infty_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}} \le C(n,m)|\sigma|_{n,m}, \tag{4.24}$$

where $C(n,m) > 0$ is independent of $\sigma$. For $z, w \in \mathbb{R}^{2d}$ we use Lemma 2.7.17 and the norm estimate in (4.24) which yield

$$\begin{aligned}|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g, \pi(u)g\rangle| &= |V_{\Phi_\tau}\sigma(\mathcal{T}_\tau(z,u), J(u-z))|\\ &\le C|\sigma|_{n,m}\frac{\langle \mathcal{T}_\tau(z,u)\rangle^m}{\langle u-z\rangle^n},\end{aligned}$$

that is the desired result. □

For $s \in [0,+\infty) \smallsetminus \mathbb{N}_0$, the estimate reads as follows.

**Proposition 4.2.6.** ([7, Proposition 3.4]) *Consider $g \in \mathcal{S}(\mathbb{R}^d) \smallsetminus \{0\}$, $\tau \in [0,1]$, $m \in \mathbb{R}$ and $\sigma \in S^m\left(\mathbb{R}^{2d}\right)$. For any $s \in [0,+\infty) \smallsetminus \mathbb{N}_0$ there exists $C = C(s,m) > 0$, which does not depend on $\sigma$ or $\tau$, such that*

$$|\langle \mathrm{Op}_\tau(\sigma)\pi(z)g, \pi(u)g\rangle| \le C|\sigma|_{n+1,m}\frac{\langle \mathcal{T}_\tau(z,u)\rangle^m}{\langle u-z\rangle^s}, \qquad \forall u, z \in \mathbb{R}^{2d}, \tag{4.25}$$

*where $n = [s]$ is the integer part of $s$.*

*Proof.* The result is attained by the the same argument as Proposition 4.2.5 and the inclusion relations between modulation spaces in Theorem 2.5.6. □

### 4.2.1 Boundedness results

The characterization of the class $S^m(\mathbb{R}^{2d})$ in Lemma 2.6.8 and Theorem 4.2.1 are the key tool for boundedness properties of $\tau$-pseudo-differential operators on weighted modulation spaces.

**Proposition 4.2.7.** ([7, Proposition 3.5]) *Consider $\tau \in [0,1]$, $m \in \mathbb{R}$, $\sigma \in S^m(\mathbb{R}^{2d})$, $0 < p,q \leq \infty$. Then $\mathrm{Op}_\tau(\sigma)$, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, extends uniquely to a bounded operator*

$$\mathrm{Op}_\tau(\sigma)\colon M^{p,q}_{\langle\cdot\rangle^{r+m}}(\mathbb{R}^d) \to M^{p,q}_{\langle\cdot\rangle^r}(\mathbb{R}^d),$$

*for every $r \in \mathbb{R}$.*

*Proof.* Choose $g \in \mathcal{S}(\mathbb{R}^d)$ and a lattice $\Lambda$ such that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Define $t := \min\{1,p,q\}$ and choose $s > (2d + |r|)/t$. Using the equivalent discrete quasi-norm for the modulation space (2.153), the estimate in (4.19) and Young's convolution inequality in [74, Theorem 3.1], we obtain the result. Namely,

$$\begin{aligned}\|\mathrm{Op}_\tau(\sigma)f\|_{M^{p,q}_{\langle\cdot\rangle^r}} &\asymp \|V_g(\mathrm{Op}_\tau(\sigma)f)\|_{\ell^{p,q}_{\langle\cdot\rangle^r}(\Lambda)} \leq \left\| h_\tau * |V_g f|\,\langle\cdot\rangle^{|m|}\right\|_{\ell^{p,q}_{\langle\cdot\rangle^r}(\Lambda)}\\ &\leq \|h_\tau\|_{\ell^t_{\langle\cdot\rangle^s}(\Lambda)} \|V_g f\langle\cdot\rangle^m\|_{\ell^{p,q}_{\langle\cdot\rangle^r}(\Lambda)} \leq C\,\|f\|_{M^{p,q}_{\langle\cdot\rangle^{r+m}}}.\end{aligned}$$

Alternatively, since $\sigma \in S^m = \bigcap_{s\geq 0} M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ by Lemma 2.6.8, one can use [142, Theorem 3.1] with $p = \infty$ and $q \leq 1$ small enough to yield the claim. □

**Remark 4.2.8.** (*i*) *For $\sigma \in S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$ and we recapture the continuity of*

$$\mathrm{Op}_\tau(\sigma)\colon M^{p,q}_{\langle\cdot\rangle^r}(\mathbb{R}^d) \to M^{p,q}_{\langle\cdot\rangle^r}(\mathbb{R}^d).$$

*This was already shown in [137] for $p,q \geq 1$, for the quasi-Banach cases see [142];*

(*ii*) *For $p = q = 2$ we have the continuity between the Shubin-Sobolev spaces $Q_{r+m}(\mathbb{R}^d)$ and $Q_r(\mathbb{R}^d)$.*

**Corollary 4.2.9.** ([7, Corollary 3.7]) *Consider $m,r \in \mathbb{R}$, $\sigma \in S^m(\mathbb{R}^{2d})$, $0 < p,q \leq \infty$. Let $\|\mathrm{Op}_\tau(\sigma)\|$ denote the norm of $\mathrm{Op}_\tau(\sigma)$ in $B(M^{p,q}_{\langle\cdot\rangle^{r+m}}(\mathbb{R}^d), M^{p,q}_{\langle\cdot\rangle^r}(\mathbb{R}^d))$. Then there exists a constant $C > 0$ such that*

$$\|\mathrm{Op}_\tau(\sigma)\| \leq C, \qquad \forall \tau \in [0,1]. \tag{4.26}$$

*Proof.* The claim is evident from proof of Proposition 4.2.7. □

### 4.2.2 Born-Jordan operators

We recall that Born-Jordan operators $\mathrm{Op}_{BJ}(\sigma)$ were defined in (2.215).

**Theorem 4.2.10.** ([7, Theorem 3.8]) *Consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. For $m \in \mathbb{R}$ consider $\sigma \in S^m(\mathbb{R}^{2d})$. Then for every $s \geq 0$, $0 < q \leq \infty$, $\tau \in [0,1]$ there exists a function $H_\tau \in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ which satisfies (4.17) and such that*

$$|\langle \mathrm{Op}_{BJ}(\sigma)\,\pi(z)\,g, \pi(u)\,g\rangle| \leq \langle z\rangle^m \int_0^1 H_\tau(u-z)\,d\tau, \qquad \forall u,z \in \mathbb{R}^{2d}. \tag{4.27}$$

*Proof.* For $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, $\mathrm{Op}_{BJ}(\sigma)$ is linear and continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, see [42]. For $z, u \in \mathbb{R}^{2d}$, $\sigma \in S^m(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^d)$ we compute

$$\begin{aligned}\langle \mathrm{Op}_{BJ}(\sigma)\pi(z)g, \pi(u)g\rangle &= \langle \sigma, W_{BJ}(\pi(u)g, \pi(z)g)\rangle \\ &= \int_{\mathbb{R}^{2d}} \sigma(y) \int_0^1 \overline{W_\tau(\pi(u)g, \pi(z)g)(y)}\, d\tau dy =: I.\end{aligned}$$

From [42, Proposition 2.2, Remark 2.3] we have that the mapping

$$\mathbb{R} \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d}), \quad (t, \varphi, \psi) \mapsto W_t(\varphi, \psi)$$

is continuous and locally uniformly bounded. Thus $W_{BJ}(\varphi, \psi) \in \mathcal{S}(\mathbb{R}^{2d})$ and the integral $I$ is absolutely convergent, so that

$$I = \int_0^1 \int_{\mathbb{R}^{2d}} \sigma(y)\overline{W_\tau(\pi(u)g, \pi(z)g)(y)}\, dyd\tau = \int_0^1 \langle \mathrm{Op}_\tau(\sigma)\, \pi(z)\, g, \pi(u)\, g\rangle\; d\tau.$$

By Peetre's inequality:

$$\begin{aligned}\langle \mathcal{T}_\tau(z,u)\rangle^m &= \langle z_1 + \tau(u_1 - z_1), z_2 + (1-\tau)(u_2 - z_2)\rangle^m \\ &\lesssim \langle z\rangle^m \langle u - z\rangle^{|m|},\end{aligned}$$

for every $u = (u_1, u_2)$, $z = (z_1, z_2) \in \mathbb{R}^{2d}$. Hence, using Theorem 4.2.3,

$$|I| \le \int_0^1 |\langle \mathrm{Op}_\tau(\sigma)\, \pi(z)\, g, \pi(u)\, g\rangle|\; d\tau \lesssim \int_0^1 H_\tau(u-z)\, \langle u - z\rangle^{|m|}\, d\tau\, \langle z\rangle^m.$$

Then the function $H_\tau(z)\, \langle z\rangle^{|m|}$ satisfies condition (4.17). □

**Remark 4.2.11.** *(i) For $q \ge 1$, we can define $H(z) := \int_0^1 H_\tau(z) d\tau$. Using Minkowski's integral inequality we infer $H \in L^q_{\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ and the estimate (4.27) becomes*

$$|\langle \mathrm{Op}_{BJ}(\sigma)\, \pi(z)\, g, \pi(u)\, g\rangle| \le H(u - z)\, \langle z\rangle^m, \qquad \forall u, z \in \mathbb{R}^{2d}.$$

*Notice that for $0 < q < 1$ Minkowski's integral inequality is not true in general.*
*(ii) Arguing as in Theorem 4.2.10, we may discretize the Gabor matrix decay in (4.27) as follows: consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a lattice $\Lambda$ in $\mathbb{R}^{2d}$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. If $\sigma \in S^m(\mathbb{R}^{2d})$ then for every $s \ge 0$, $0 < q \le \infty$, there exists a sequence $h_\tau \in \ell^q_{\langle\cdot\rangle^s}(\Lambda)$ with $\|h_\tau\|_{\ell^q_{\langle\cdot\rangle^s}} \le C$ for every $\tau \in [0,1]$ such that*

$$|\langle \mathrm{Op}_{BJ}(\sigma)\, \pi(\mu)\, g, \pi(\lambda)\, g\rangle| \le \langle \mu\rangle^m \int_0^1 h_\tau(\lambda - \mu) d\tau, \qquad \forall \lambda, \mu \in \Lambda.$$

**Corollary 4.2.12.** ([7, Corollary 3.10]) *Consider $m \in \mathbb{R}$, $\sigma \in S^m(\mathbb{R}^{2d})$, $0 < p, q \le \infty$. Then $\mathrm{Op}_{BJ}(\sigma)$, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, extends uniquely to a bounded operator*

$$\mathrm{Op}_{BJ}(\sigma)\colon M^{p,q}_{\langle\cdot\rangle^{r+m}}(\mathbb{R}^d) \to M^{p,q}_{\langle\cdot\rangle^r}(\mathbb{R}^d),$$

*for every $r \in \mathbb{R}$.*

*Proof.* The proof is similar to the one of Proposition 4.2.7, using the decay for Gabor matrix of $\mathrm{Op}_{BJ}(\sigma)$ found in Theorem 4.2.10, with $h_\tau$ replaced by $\int_0^1 h_\tau(\cdot)d\tau$. Then, for $t \geq 1$ we use Minkowski's inequality to write

$$\left\| \int_0^1 h_\tau(\cdot)d\tau \right\|_{\ell^t_{\langle \cdot \rangle^s}} \leq \int_0^1 \|h_\tau\|_{\ell^t_{\langle \cdot \rangle^s}} d\tau \leq C.$$

For $t < 1$ we use the inclusion relations (2.40) and majorize

$$\left\| \int_0^1 h_\tau(\cdot)d\tau \right\|_{\ell^t_{\langle \cdot \rangle^s}} \lesssim \left\| \int_0^1 h_\tau(\cdot)d\tau \right\|_{\ell^1_{\langle \cdot \rangle^{\tilde{s}}}},$$

with $\tilde{s} \geq 0$ such that $1/t + s/(2d) < 1 + \tilde{s}/(2d)$, that is

$$\tilde{s} > \frac{2d}{t}(1-t),$$

and we proceed as above. $\square$

# Chapter 5

# Quasi-Banach modulation spaces and localization operators on locally compact abelian groups

The first important achievement presented in this chapter is a new definition, due to E. Cordero and the author [8], of modulation spaces $M_m^{p,q}$ over a LCA group $\mathcal{G}$ with indexes $0 < p, q \leq \infty$. Concretely, we shall set

$$M_m^{p,q}(\mathcal{G}) \coloneqq \{f \in \mathcal{S}_0'(\mathcal{G}) \,|\, V_g f \in W(L^\infty, L_m^{p,q})(\mathcal{G} \times \widehat{\mathcal{G}})\},$$

where $\mathcal{S}_0'(\mathcal{G})$ is the dual of the Feichtinger algebra, $g$ is a suitable window and $W(L^\infty, L_m^{p,q})(\mathcal{G}\times\widehat{\mathcal{G}})$ the Wiener amalgam space on $\mathcal{G} \times \widehat{\mathcal{G}}$ with local component $L^\infty$ and global one $L_m^{p,q}$. The quasi-norm is the natural one:

$$\begin{aligned}\|f\|_{M_m^{p,q}} &\coloneqq \|V_g f\|_{W(L_m^{p,q})} \\ &\coloneqq \left( \int_{\widehat{\mathcal{G}}} \left( \int_{\mathcal{G}} \operatorname*{ess\,sup}_{(u,\omega)\in Q+(x,\xi)} |V_g f(u,\omega)|^p \, m(x,\xi)^p \, dx \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{q}},\end{aligned}$$

where $Q \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ is a suitable unit neighbourhood; the modifications when $p = \infty$ or $q = \infty$ are obvious.
We shall see that such definition recovers all the already known modulation spaces, i.e. it coincides with

(*i*) $M_m^{p,q}(\mathbb{R}^d)$ with $0 < p, q \leq \infty$;

(*ii*) $M_m^{p,q}(\mathcal{G})$ with $1 \leq p, q \leq \infty$, any $\mathcal{G}$ LCA group.

The novelty of the definition here presented relies in the fact that it allows us to deal with the quasi-Banach case, i.e. $p < 1$ or $q < 1$, on every $\mathcal{G}$ LCA group, not only the Euclidean space $\mathbb{R}^d$. Moreover, we prove that if $\mathcal{G}$ is discrete or compact, then we can consider the "usual" $L_m^{p,q}$-quasi-norm of the STFT instead of the Wiener one. Namely, if $\mathcal{G}$ is discrete or compact, then

$$M_m^{p,q}(\mathcal{G}) = \{f \in \mathcal{S}_0'(\mathcal{G}) \,|\, V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})\}.$$

If the above equality holds true for every LCA group is still an open problem.

Technically speaking, the new definition of $M_m^{p,q}(\mathcal{G})$ was inspired by the idea of H. G. Feichtinger and K. Gröchenig in [58]: to view modulation spaces on $\mathcal{G}$ as particular coorbit spaces over the Heisenberg group $\mathcal{G}\times\widehat{\mathcal{G}}\times\mathbb{T}$. However, the coorbit theory proposed by H. G. Feichtinger and K. Gröchenig in their works [58, 59, 60] is not suitable for the quasi-Banach case. The right construction is provided by the new coorbit theory started by H. Rauhut in [119] and developed by F. Voigtlaender in his Ph.D. thesis [147], see also the more recent contribution [146]. The reader can find a short survey of the mentioned coorbit theory for quasi-Banach spaces in Chapter 2, Section 2.2.

The second most important contribution shown in this chapter is a number of boundedness results for cross-Rihaczek distribution $R(f,g)$ (2.96) and pseudo-differential operators with Kohn-Nirenberg symbol $\mathrm{Op}_0(\sigma)$ (2.216). Concerning the mapping $R$ between modulation spaces, we address the reader to Proposition 5.2.2, which borrows techniques from[31, Theorem 3.1] and [25, Theorem 4]. As a consequence, we have the boundedness result for $\mathrm{Op}_0(\sigma)$ between modulation spaces, which generalizes [31, Theorem 5.1]. A result similar to Theorem 3.1.2 is obtained for Kohn-Nirenberg operators as well, although the techniques are way much different due to the high level of technicalities. Indeed, we first need to prove Gabor frame expansions for $M_m^{p,q}(\mathcal{G})$, which are done by mean of quasi-lattices on groups [88], then we work on a quotient group derived form $\mathcal{G}\times\widehat{\mathcal{G}}$ instead of the whole phase-space. See Theorem 5.2.17 and its proof carefully. Once this result is established, we easily infer Proposition 5.2.18 about $L^2$-eigenfunctions of $\mathrm{Op}_0(\sigma)$. We report the statement for sake of clarity:

*Consider a symbol $\sigma$ on the phase space such that for some $0<p<\infty$ we have $\sigma\in\bigcap_{\gamma>0}M^{p,\gamma}(\mathcal{G}\times\widehat{\mathcal{G}})$. If $\lambda\in\sigma_P(\mathrm{Op}_0(\sigma))\smallsetminus\{0\}$, then any eigenfunction $f\in L^2(\mathcal{G})$ with eigenvalue $\lambda$ satisfies $f\in\bigcap_{\gamma>0}M^{\gamma}(\mathcal{G})$.*

Moreover, once we have proved Theorem 5.2.11 and 5.2.12, i.e. continuity for coefficient $\mathcal{C}_g$ and synthesis $\mathcal{C}_g^*$ operators on modulation spaces, we get new convolution relations for $M_m^{p,q}(\mathcal{G})$ in Proposition 5.2.14. This is the natural generalization of what is proved for the Euclidean case in [9], see Proposition 2.5.19.

The third and last major contribution this chapter presents concerns localization operators $A_a^{\psi_1,\psi_2}$ in the same fashion of [9], see Theorem 3.2.1. Namely, using the representation of $A_a^{\psi_1,\psi_2}$ as Kohn-Nirenberg operator

$$A_a^{\psi_1,\psi_2}=\mathrm{Op}_0(a*R(\psi_2,\psi_1)),$$

see Proposition 5.3.2, the main result of [8] Theorem 5.3.3 is obtained:

*Let $0<p<\infty$ and $a\in M^{p,\infty}(\mathcal{G}\times\widehat{\mathcal{G}})$. Consider $\psi_1,\psi_2\in\mathcal{S}_{\mathcal{C}}(\mathcal{G})\smallsetminus\{0\}$. Suppose that $\sigma_P(A_a^{\psi_1,\psi_2})\smallsetminus\{0\}\neq\varnothing$ and $\lambda\in\sigma_P(A_a^{\psi_1,\psi_2})\smallsetminus\{0\}$. Then any eigenfunction $f\in L^2(\mathcal{G})$ with eigenvalue $\lambda$ satisfies*

$$f\in\bigcap_{\gamma>0}M^{\gamma}(\mathcal{G}).$$

The chapter is structured as follows. Section 5.1 is devoted to the construction of modulation spaces over $\mathcal{G}$ and the study of their main expected properties. There we make a specific choice for every item from **A** to **J** presented in Chapter 2 Subsection 2.2.5. Subsection 5.2.1 deals with continuity for the Rihaczek distribution between modulation spaces and a first result for Kohn-Nirenberg operators $\mathrm{Op}_0(\sigma)$. In Subsection 5.2.2 we put to work quasi-lattices in order to have frames expansion in the new modulation spaces. The new convolutions relations for $M^{p,q}_m(\mathcal{G})$ Proposition 5.2.14 are presented here. Subsection 5.2.3 contains the result for $L^2$-eigenfunction of $\mathrm{Op}_0(\sigma)$. Eventually, Section 5.3 is devoted to the main result Theorem 5.3.3 about $L^2$-eigenfunction of localization operators $A_a^{\psi_1,\psi_2}$ over $\mathcal{G}$ LCA group.

We recall that Assumptions 2.2.2 on $\mathcal{G}$ hold in this chapter even if not explicitly stated.

## 5.1 Quasi-Banach modulation spaces on LCA groups

The following concepts are taken for granted and can be found in Chapter 2: left $L_x$ and right $R_x$ translations, relatively separated families, a discrete space $Y_d$ associated to $Y$, BUPUs, maximal functions $\mathsf{M}_Q f$, Wiener amalgam spaces $W_Q(Y) = W_Q(L^\infty, Y)$. Definition 2.2.3 contains the hypothesis on weights and the class $\mathcal{M}_v$ used in what follows. Note that the coorbit space construction is listed in items $\mathbf{A} - \mathbf{J}$ (unitary representation $\rho$, wavelet transform $W_g^\rho f$, assumptions on weights, sets $\mathbb{G}_v$, $\mathbb{A}_v^r$, $\mathcal{T}_v$, $\mathcal{R}_v$) in Subsection 2.2.5 of Chapter 2. Each of these items will be revisited in this section under specific choices, see list $\mathbf{A}'$ –$\mathbf{J}'$ below.

Relying on the theory Chapter 2 Subsection 2.2.5, we are able to give a definition of modulation spaces on LCA groups which covers Feichtinger's orginal one [56] and deals with the quasi-Banach case. The subsequent construction of $M^{p,q}_m(\mathcal{G})$ was suggested for the Banach case in [58, p. 67], although the coorbit theory applied here is different.

Since the group $\mathbb{H}_{\mathcal{G}}$ defined below is noncommutative, we adopt the multiplicative notation for its operation.

**Definition 5.1.1.** *Let $\mathbb{T}$ be the torus with the complex multiplication. We define the* ***Heisenberg-type group associated to*** *$\mathcal{G}$, Heisenberg group for short, as*

$$\mathbb{H}_{\mathcal{G}} := \mathcal{G} \times \widehat{\mathcal{G}} \times \mathbb{T}, \tag{5.1}$$

*endowed with the product topology and the following operation:*

$$(x, \xi, \tau)(x', \xi', \tau') = (x + x', \xi + \xi', \tau\tau'\langle \xi', x\rangle)\,, \tag{5.2}$$

*for $(x, \xi, \tau), (x', \xi', \tau') \in \mathbb{H}_{\mathcal{G}}$.*

The group $\mathbb{H}_{\mathcal{G}}$ is also called ***Mackey obstruction group*** of $\mathcal{G} \times \widehat{\mathcal{G}}$, see [20, Section 4], in particular Example 4.6 therein.

**Lemma 5.1.2.** ([8, Lemma 3.2]) *The topological product space $\mathbb{H}_{\mathcal{G}}$ with the operation in (5.2) is a topological LCH, $\sigma$-compact, noncommutative, unimodular group with Haar measure the product measure $dxd\xi d\tau$, $dx$ and $d\xi$ being dual Haar measures on $\mathcal{G}$ and $\widehat{\mathcal{G}}$ and $d\tau(\mathbb{T}) = 1$.*

*Proof.* Hausdorff property, local compactness, $\sigma$-compactness and noncommutativity are trivial. For the proof that $\mathbb{H}_{\mathcal{G}}$ is a topological unimodular group we refer to Theorem 3 in [100], for the bi-invariance of $dxd\xi d\tau$ see [100, p. 12] or, alternatively, [20, Lemma 4.3]. □

The identity in $\mathbb{H}_{\mathcal{G}}$ is $(e, \hat{e}, 1)$ and the inverse of an element $(x, \xi, \tau)$ is

$$(x, \xi, \tau)^{-1} = (-x, -\xi, \overline{\tau}\langle \xi, x\rangle).$$

**Lemma 5.1.3.** ([8, Lemma 3.3]) *The mapping*

$$\varrho\colon \mathbb{H}_{\mathcal{G}} \to \mathcal{U}(L^2(\mathcal{G})), (x, \xi, \tau) \mapsto \tau M_\xi T_x \tag{5.3}$$

*is a unitary, strongly continuous, irreducible, integrable representation of $\mathbb{H}_{\mathcal{G}}$ on $L^2(\mathcal{G})$. We call $\varrho$* ***Schrödinger representation****.*

*Proof.* Well-posedness of $\varrho$ is trivial, from the commutations relations (2.93) it is straightforward to see that $\varrho$ is a group homomorphism. Observe that

$$\pi\colon \mathcal{G}\times\widehat{\mathcal{G}} \to \mathcal{U}(L^2(\mathcal{G})), (x,\xi)\mapsto M_\xi T_x$$

is a *projective representation* in the terminology of [20, Definition 4.1]. In fact, (i) $\pi(e,\hat{e}) = I_{L^2}$; (ii) from the commutation relations (2.93) we obtain

$$\pi\left((x,\xi)+(x',\xi')\right) = \overline{\langle\xi',x\rangle}\pi(x,\xi)\pi(x',\xi'),$$

where $\langle\cdot,\cdot\rangle$ is continuous on $\mathcal{G}\times\widehat{\mathcal{G}}$; (iii) the continuity of the STFT guarantees the required measurability. To verify that $\varrho$ is strongly continuous, one can proceed as in the Euclidean case, see e.g. [35]. The result then follows from [20, Lemma 4.4 (ii)].
The fact that $\varrho$ is irreducible was proved in [100], see page 14 before §5. For the integrability, consider the Gaussian $\varphi\in L^2(\mathcal{G})$ in (2.94) and observe that the torus is compact and $\left|W_\varphi^\varrho\varphi\right| = |V_\varphi\varphi|$ (see (2.41) for the definition of $W_\varphi^\varrho\varphi$). Then from (2.100) we have $V_\varphi\varphi\in L^1(\mathcal{G}\times\widehat{\mathcal{G}})$ and $W_\varphi^\varrho\varphi\in L^1(\mathbb{H}_\mathcal{G})$. This concludes the proof. □

**Definition 5.1.4.** *We define the extension of $m\in\mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$ as*

$$\tilde{m}\colon \mathbb{H}_\mathcal{G}\to(0,+\infty), (x,\xi,\tau)\mapsto m(x,\xi). \tag{5.4}$$

*For $0<p,q\le\infty$, the space $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ consists of those equivalence classes of measurable complex-valued functions on $\mathbb{H}_\mathcal{G}$, where two functions are identified if they coincide a.e., for which the following application is finite*

$$\|F\|_{L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})} := \|F\|_{L_{\tilde{m}}^{p,q}} := \left(\int_{\widehat{\mathcal{G}}\times\mathbb{T}}\left(\int_\mathcal{G}|F(x,\xi,\tau)|^p\, m(x,\xi)^p\,dx\right)^{\frac{q}{p}} d\xi d\tau\right)^{\frac{1}{q}}, \tag{5.5}$$

*obvious modifications for $p=\infty$ or $q=\infty$.*

$(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}), \|\cdot\|_{L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})})$ is a solid QBF space on $\mathbb{H}_\mathcal{G}$. If $m$ is moderate with respect to a submultiplicative weight $v$ on $\mathcal{G}\times\widehat{\mathcal{G}}$, then $\tilde{m}$ is left- and right-moderate w.r.t. $\tilde{v}$ on $\mathbb{H}_\mathcal{G}$, $\tilde{v}$ as in (5.4). Therefore $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ is left and right invariant, see Definition 2.2.15.

**Lemma 5.1.5.** *Consider $0<p,q\le\infty$. Then $\|\cdot\|_{L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})}$ is an $r$-norm on $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ with $r:=\min\{1,p,q\}$.*

*Proof.* This is just Lemma 2.2.26 with $X=\mathcal{G}$ and $Y=\widehat{\mathcal{G}}\times\mathbb{T}$. □

**Lemma 5.1.6.** ([8, Lemma 3.6]) *Consider $m\in\mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$ and $0<p,q\le\infty$. Then there exists $C=C(m,v)>0$ such that for any $F\in L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ and $(x,\xi,\tau)\in\mathbb{H}_\mathcal{G}$*

$$\left\|R_{(x,\xi,\tau)}F\right\|_{L_{\tilde{m}}^{p,q}} \le Cv(-x,-\xi)\,\|F\|_{L_{\tilde{m}}^{p,q}}, \quad \left\|L_{(x,\xi,\tau)}F\right\|_{L_{\tilde{m}}^{p,q}} \le Cv(x,\xi)\,\|F\|_{L_{\tilde{m}}^{p,q}}. \tag{5.6}$$

*Proof.* The claim is a straightforward calculation which follows by the bi-invariance of the Haar measure on $\mathbb{H}_\mathcal{G}$. For $p,q\ne\infty$,

$$\begin{aligned}
\left\|R_{(x,\xi,\tau)}F\right\|_{L_{\tilde{m}}^{p,q}}^q &= \int_{\widehat{\mathcal{G}}\times\mathbb{T}}\left(\int_\mathcal{G}|F((u,\omega,t)(x,\xi,\tau))|^p\,\tilde{m}(u,\omega,t)^p\,du\right)^{\frac{q}{p}} d\omega dt\\
&\underset{m,v}{\lesssim} \int_{\widehat{\mathcal{G}}\times\mathbb{T}}\left(\int_\mathcal{G}|F(u',\omega',t')|^p\,\tilde{m}(u',\omega',t')^p\tilde{v}((x,\xi,\tau)^{-1})^p\,du'\right)^{\frac{q}{p}} d\omega'dt'\\
&= v(-x,-\xi)^q\,\|F\|_{L_{\tilde{m}}^{p,q}}^q.
\end{aligned}$$

Left translations are treated similarly, as well as the cases $p=\infty$ or $q=\infty$. □

Due to the symmetry of $v$ (Definition 2.2.3), the first inequality in (5.6) reads as

$$\left\|R_{(x,\xi,\tau)}F\right\|_{L^{p,q}_{\tilde{m}}} \leq Cv(x,\xi)\,\|F\|_{L^{p,q}_{\tilde{m}}}\,.$$

**Lemma 5.1.7.** ([8, Lemma 3.7]) *Let $0 < p, q \leq \infty$. Fix $V_{\mathcal{G}} \subseteq \mathcal{G}$ and $V_{\widehat{\mathcal{G}}} \subseteq \widehat{\mathcal{G}}$ open, relatively compact, neighbourhoods of $e \in \mathcal{G}$ and $\hat{e} \in \widehat{\mathcal{G}}$, respectively. Define*

$$V := V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T}. \tag{5.7}$$

*Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. Then there exists $C = C(m,v) > 0$ such that for every $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$*

$$|||R_{(x,\xi,\tau)}|||_{W_V(L^{p,q}_{\tilde{m}}) \to W_V(L^{p,q}_{\tilde{m}})} \leq Cv(-x,-\xi). \tag{5.8}$$

*Proof.* $V$ is a open, relatively compact, unit neighbourhood and the set

$$V_{1,2} := V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \tag{5.9}$$

is also open, relatively compact, unit neighbourhood in $\mathcal{G} \times \widehat{\mathcal{G}}$. For $F \in L^\infty_{loc}(\mathbb{H}_{\mathcal{G}})$

$$\mathsf{M}_V[R_{(x,\xi,\tau)}F] = \mathsf{M}_{V(x,\xi,\tau)}F, \tag{5.10}$$

see [147, Lemma 2.3.18, 1.]. For any $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$

$$V(x,\xi,\tau) = (V_{\mathcal{G}} + x) \times \left(V_{\widehat{\mathcal{G}}} + \xi\right) \times \bigcup_{u \in V_{\mathcal{G}}} \mathbb{T}\tau\langle \xi, u\rangle = (x,\xi,\tau)V.$$

If $F \in W_V(L^{p,q}_{\tilde{m}})$ and $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$, from what just observed we obtain:

$$\mathsf{M}_V[R_{(x,\xi,\tau)}F](u,\omega,t) = \operatorname*{ess\,sup}_{(y,\eta,s) \in (u,\omega,t)(x,\xi,\tau)V} |F(y,\eta,s)| = R_{(x,\xi,\tau)}[\mathsf{M}_V F(u,\omega,t)].$$

Eventually by using (5.6)

$$\begin{aligned}\left\|R_{(x,\xi,\tau)}F\right\|_{W_V(L^{p,q}_{\tilde{m}})} &= \left\|\mathsf{M}_V[R_{(x,\xi,\tau)}F]\right\|_{L^{p,q}_{\tilde{m}}} = \left\|R_{(x,\xi,\tau)}[\mathsf{M}_V F]\right\|_{L^{p,q}_{\tilde{m}}} \\ &\leq Cv(-x,-\xi)\,\|\mathsf{M}_V F\|_{L^{p,q}_{\tilde{m}}} = Cv(-x,-\xi)\,\|F\|_{W_V(L^{p,q}_{\tilde{m}})}\,,\end{aligned}$$

for some $C = C(m,v) > 0$. This concludes the proof. □

As already highlighted, inequality (5.8) can be equivalently written with $v(x,\xi)$ in place of $v(-x,-\xi)$. Observe that the constant $C$ involved in (5.6) and (5.8) is the one coming from the $v$-moderateness condition: $m((x,\xi)+(u,\omega)) \leq Cv(x,\xi)m(u,\omega)$.

**Corollary 5.1.8.** ([8, Corollary 3.8]) *Let $0 < p, q \leq \infty$. Consider $Q \subseteq \mathbb{H}_{\mathcal{G}}$ measurable, relatively compact, unit neighbourhood and $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. Then there exists $C_Q = C(Q,m,v) > 0$ such that for every $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$*

$$|||R_{(x,\xi,\tau)}|||_{W_Q(L^{p,q}_{\tilde{m}}) \to W_Q(L^{p,q}_{\tilde{m}})} \leq C_Q v(-x,-\xi). \tag{5.11}$$

*Proof.* The claim follows from the independence of the Wiener Amalgam space $W(L^{p,q}_{\tilde{m}})$ from the window subset (Lemma 2.2.24) together with Lemma 5.1.7. □

**Remark 5.1.9.** ([8, Remark 3.9]) *Consider the (generalized) wavelet transform induced by the Schrödinger representation $\varrho$ in* (2.41) *taking $G = \mathbb{H}_{\mathcal{G}}$ and $f, g \in \mathcal{H} = L^2(\mathcal{G})$:*

$$W_g^{\varrho} f \colon \mathbb{H}_{\mathcal{G}} \to \mathbb{C}, (x, \xi, \tau) \mapsto \langle f, \tau M_\xi T_x g\rangle_{L^2(\mathcal{G})}. \tag{5.12}$$

*This is a continuous and bounded function. It is straightforward to see that*

$$W_g^{\varrho} f(x, \xi, \tau) = \langle f, \tau M_\xi T_x g\rangle = \overline{\tau} V_g f(x, \xi), \quad \forall\, (x, \xi, \tau) \in \mathbb{H}_{\mathcal{G}}, \tag{5.13}$$

*which implies*

$$\left|W_g^{\varrho} f(x, \xi, \tau)\right| = |V_g f(x, \xi)|, \quad \forall\, (x, \xi, \tau) \in \mathbb{H}_{\mathcal{G}}. \tag{5.14}$$

*Therefore for $f, g \in L^2(\mathcal{G})$, being $\mathbb{T}$ compact,*

$$W_g^{\varrho} f \in L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}}) \quad \Leftrightarrow \quad V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}}) \tag{5.15}$$

*and*

$$W_g^{\varrho} f \in W(L^\infty(\mathbb{H}_{\mathcal{G}}), L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})) \quad \Leftrightarrow \quad V_g f \in W(L^\infty(\mathcal{G} \times \widehat{\mathcal{G}}), L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})). \tag{5.16}$$

We are now able to revisit steps **A** − **J**, in Subsection 2.2.5 of Chapter 2, as follows.

**A′**. For $G = \mathbb{H}_{\mathcal{G}}$ the Heisenberg group associated to $\mathcal{G}$, $\mathcal{H} = L^2(\mathcal{G})$ and $\rho = \varrho \colon \mathbb{H}_{\mathcal{G}} \to L^2(\mathcal{G})$ the Schrödinger representation, the requirements of **A** are fulfilled due to Lemma 5.1.2 and 5.1.3.

**B′**. $W_g^{\varrho} f$ was described in (5.12) and the integrability of $\varrho$ was proved in Lemma 5.1.3 as well as that every element of $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ is admissible for $\varrho$.

**C′**. Take $Y = L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})$ (Definition 5.1.4) and $r = \min\{1, p, q\}$ (Lemma 5.1.5).

**D′**. The right invariance for each measurable, relatively compact, unit neighbourhood $Q \subseteq \mathbb{H}_{\mathcal{G}}$ of $W_Q(L^\infty, L_{\tilde{m}}^{p,q})$ is guaranteed by the right invariance of $L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})$, Lemma 5.1.6 and Lemma 2.2.24. Since $\mathbb{H}_{\mathcal{G}}$ is unimodular, (2.42) and (2.43) can be summarized as

$$w(x, \xi, \tau) \underset{Q}{\gtrsim} \left|\left|\left| R_{(x,\xi,\tau)^{\pm 1}} \right|\right|\right|_{W_Q(L_{\tilde{m}}^{p,q}) \to W_Q(L_{\tilde{m}}^{p,q})}, \tag{5.17}$$

for some (hence every) measurable, relatively compact, unit neighbourhood $Q \subseteq \mathbb{H}_{\mathcal{G}}$. Therefore, on account of (5.11) and the definition of $\mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, we can take $w = \tilde{v}$ the extension of $v$ defined as in (5.4).

**E′**. We take $\tilde{v}$ as control weight for $L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})$, see **E**.

**F′**. The class of good vectors we are considering is

$$\mathbb{G}_{\tilde{v}} \coloneqq \left\{g \in L^2(\mathcal{G}) \,|\, W_g^{\varrho} g \in L_{\tilde{v}}^1\right\}. \tag{5.18}$$

We shall prove that it is nontrivial.

**G′**. Our class of analysing vectors is

$$\mathbb{A}_{\tilde{v}}^r \coloneqq \left\{g \in L^2(\mathcal{G}) \,|\, W_g^{\varrho} g \in W^R(L^\infty, W(L^\infty, L_{\tilde{v}}^r))\right\}. \tag{5.19}$$

It is due to [147, Lemma 2.4.9] that $\mathbb{A}^r_{\tilde{v}}$ is a vector space, as observed in the proof of [147, Theorem 2.4.9], and that

$$W^{\varrho}_h g \in W^R(L^\infty, W(L^\infty, L^r_{\tilde{v}})) \tag{5.20}$$

for every $g, h \in \mathbb{A}^r_{\tilde{v}}$.

**Lemma 5.1.10.** ([8, Lemma 3.10]) *Let us define*

$$\mathscr{A}_{\tilde{v}} \coloneqq \mathscr{A}_{\tilde{v}}(\mathcal{G}) \coloneqq \bigcap_{0<r\leq 1} \mathbb{A}^r_{\tilde{v}}. \tag{5.21}$$

*The following inclusions hold true:*

$$\mathcal{S}_{\mathcal{C}}(\mathcal{G}) \subseteq \mathscr{A}_{\tilde{v}} \subseteq \mathbb{G}_{\tilde{v}}. \tag{5.22}$$

*Proof.* The only inclusion to be shown is the first one, the second one was already mentioned in Remark 2.2.35 (ii). Fix $0 < r \leq 1$. First, we show that the Gaussian $\varphi \in L^2(\mathcal{G})$ in (2.94) belongs to $\mathbb{A}^r_{\tilde{v}}$. From (2.100):

$$W^{\varrho}_{\varphi}\varphi(x,\xi,\tau) = \overline{\tau} c(\mathcal{K}) e^{-\frac{\pi}{2}(x_1^2+\xi_1^2)} \otimes \chi_{\mathcal{K}\times\mathcal{K}^\perp}(x_2,\xi_2) = \overline{\tau} V_\varphi \varphi(x,\xi),$$

for some $c(\mathcal{K}) > 0$. Take $V \subseteq \mathbb{H}_{\mathcal{G}}$ as in (5.7) and observe that if $F \in L^\infty_{loc}(\mathbb{H}_{\mathcal{G}})$

$$\mathsf{M}_V[\mathsf{M}^R_V F](x,\xi,\tau) = \operatorname*{ess\,sup}_{(u,\omega,t)\in(x,\xi,\tau)V} \left| \operatorname*{ess\,sup}_{(y,\eta,s)\in V(u,\omega,t)} |F(y,\eta,s)| \right| \leq \operatorname*{ess\,sup}_{(y,\eta,s)\in V(x,\xi,\tau)V} |F(y,\eta,s)|\,.$$

If $F = W^{\varrho}_{\varphi}\varphi$, adopting notation of (5.9), we get

$$\begin{aligned} \mathsf{M}_V[\mathsf{M}^R_V W^{\varrho}_{\varphi}\varphi](x,\xi,\tau) &\leq \operatorname*{ess\,sup}_{(y,\eta,s)\in V(x,\xi,\tau)V} |\overline{s} V_\varphi\varphi(y,\eta)| = \operatorname*{ess\,sup}_{(y,\eta)\in V_{1,2}+(x,\xi)+V_{1,2}} |V_\varphi\varphi(y,\eta)| \\ &= \operatorname*{ess\,sup}_{(y,\eta)\in(x,\xi)+2V_{1,2}} |V_\varphi\varphi(y,\eta)| = \mathsf{M}_{2V_{1,2}} V_\varphi\varphi(x,\xi), \end{aligned}$$

where $2V_{1,2} \coloneqq V_{1,2} + V_{1,2}$ is a open, relatively compact, unit neighbourhood in $\mathcal{G} \times \widehat{\mathcal{G}}$. From the solidity of $L^r_{\tilde{v}}$,

$$\left\|W^{\varrho}_{\varphi}\varphi\right\|_{W^R(W(L^r_{\tilde{v}}))} \asymp \left\|W^{\varrho}_{\varphi}\varphi\right\|_{W^R_V(W_V(L^r_{\tilde{v}}))} \leq \left\|\mathsf{M}_{2V_{1,2}} V_\varphi\varphi\right\|_{L^r_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})} \tag{5.23}$$

and we shall prove the right-hand side to be finite. Due to the arbitrariness of $V_{\mathcal{G}}$ and $V_{\widehat{\mathcal{G}}}$, we can assume that

$$V_{1,2} = V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \cong (E_1 \times D_1) \times (E_2 \times D_2) \cong (E_1 \times E_2) \times (D_1 \times D_2)\,, \tag{5.24}$$

where $E_1, E_2 \subseteq \mathbb{R}^d$, $D_1 \subseteq \mathcal{G}_0$ and $D_2 \subseteq \widehat{\mathcal{G}}$ are open, relatively compact, unit neighbourhoods. As done previously,

$$\begin{aligned} E_{1,2} &\coloneqq E_1 \times E_2 \subseteq \mathbb{R}^{2d}, \quad D_{1,2} \coloneqq D_1 \times D_2 \subseteq \mathcal{G}_0 \times \widehat{\mathcal{G}}_0, \\ 2E_{1,2} &\coloneqq E_{1,2} + E_{1,2}, \quad 2D_{1,2} \coloneqq D_{1,2} + D_{1,2}. \end{aligned}$$

Hence

$$\begin{aligned}\mathsf{M}_{2V_{1,2}}V_\varphi\varphi(x,\xi)&=c(\mathcal{K})\operatorname*{ess\,sup}_{\substack{((y_1,\eta_1),(y_2,\eta_2))\in\\((x_1,\xi_1),(x_2,\xi_2))+2E_{1,2}\times 2D_{1,2}}}\left|e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)}\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)\right|\\&=c(\mathcal{K})\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}}\left|e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)}\right|\operatorname*{ess\,sup}_{(y_2,\eta_2)\in(x_2,\xi_2)+2D_{1,2}}\left|\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)\right|.\end{aligned}$$

Since $v(x,\xi)$ is submultiplicative, using the structure theorem we can majorize as follows:

$$v(x,\xi)=v((x_1,\xi_1),(x_2,\xi_2))\le v((x_1,\xi_1),(e_0,\hat{e}_0))v((0,0),(x_2,\xi_2)),$$

where $x=(x_1,x_2)\in\mathbb{R}^d\times\mathcal{G}_0$, $\xi=(\xi_1,\xi_2)\in\mathbb{R}^d\times\widehat{\mathcal{G}}_0$. Let us define

$$v_1(x_1,\xi_1)\coloneqq v((x_1,\xi_1),(e_0,\hat{e}_0)),\quad v_2(x_2,\xi_2)\coloneqq v((0,0),(x_2,\xi_2)),$$

$(x_1,\xi_1)\in\mathbb{R}^{2d}$ and $(x_2,\xi_2)\in\mathcal{G}_0\times\widehat{\mathcal{G}}_0$, which are still submultiplicative. Hence

$$\begin{aligned}\left\|\mathsf{M}_{2V_{1,2}}V_\varphi\varphi\right\|^r_{L^r_v(\mathcal{G}\times\widehat{\mathcal{G}})}\le&\,c(\mathcal{K})^r\underbrace{\int_{\mathbb{R}^{2d}}\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}}\left|e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)}\right|^r v_1(x_1,\xi_1)^r\,dx_1d\xi_1}_{=:I_1}\\&\times\underbrace{\int_{\mathcal{G}_0\times\widehat{\mathcal{G}}_0}\operatorname*{ess\,sup}_{(y_2,\eta_2)\in(x_2,\xi_2)+2D_{1,2}}\left|\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)\right|v_2(x_2,\xi_2)^r\,dx_2d\xi_2}_{=:I_2}.\end{aligned}$$

For $N>2d$ and considering the weight $\langle\cdot\rangle\coloneqq(1+|\cdot|^2)^{1/2}$, we can write

$$\begin{aligned}I_1&=\int_{\mathbb{R}^{2d}}\frac{\langle(x_1,\xi_1)\rangle^N}{\langle(x_1,\xi_1)\rangle^N}\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}}\left|e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)}\right|^r v_1(x_1,\xi_1)^r\,dx_1d\xi_1\\&\le\int_{\mathbb{R}^{2d}}\frac{1}{\langle(x_1,\xi_1)\rangle^N}\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}}\left[e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)}v_1(y_1,\eta_1)^r\langle(y_1,\eta_1)\rangle^N\right]dx_1d\xi_1\\&\le\int_{\mathbb{R}^{2d}}\frac{1}{\langle(x_1,\xi_1)\rangle^N}\operatorname*{ess\,sup}_{(y_1,\eta_1)\in\mathbb{R}^{2d}}\left[e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)}v_1(y_1,\eta_1)^r\langle(y_1,\eta_1)\rangle^N\right]dx_1d\xi_1.\end{aligned}$$

In fact,

$$\begin{aligned}&\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}}e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)}\langle(x_1,\xi_1)\rangle^N v_1(x_1,\xi_1)^r\\&\qquad\le\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}}\left[e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)}v_1(y_1,\eta_1)^r\langle(y_1,\eta_1)\rangle^N\right]\\&\qquad\le\|e^{\frac{-r\pi}{2}|\cdot|^2}v_1^r(\cdot)\langle\cdot\rangle^N\|_{L^\infty(\mathbb{R}^{2d})}<+\infty\end{aligned}$$

because $v_1$ is submultiplicative so it can grow at most exponentially [35, Lemma 2.1.4]. Hence $I_1<+\infty$.

We now study the integral $I_2$. Observe that the integrand is not equal to zero if and only if $(\mathcal{K}\times\mathcal{K}^\perp)\cap((x_2,\xi_2)+2D_{1,2})\ne\varnothing$, which means that there exist $k\in\mathcal{K}\times\mathcal{K}^\perp$ and $h\in 2D_{1,2}$, all depending on $(x_2,\xi_2)$, such that $k=(x_2,\xi_2)+h$ if and only if $(x_2,\xi_2)=k-h$, which implies $(x_2,\xi_2)\in\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}$. Equivalently, $(x_2,\xi_2)\notin\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}$ if and only if $\operatorname*{ess\,sup}_{(y_2,\eta_2)\in(x_2,\xi_2)+2D_{1,2}}|\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)|=0$, that implies

$$\operatorname*{ess\,sup}_{(y_2,\eta_2)\in(x_2,\xi_2)+2D_{1,2}}|\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)|\le\chi_{\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}}(x_2,\xi_2).$$

Note that $\mathcal{K}\times\mathcal{K}^{\perp}-2D_{1,2}$ is relatively compact, hence of finite measure. The local boundedness of the submultiplicative weight $v_2$, shown in [147, Theorem 2.2.22], ensures that the integral on $\mathcal{G}_0\times\widehat{\mathcal{G}}_0$ is finite.
So far we have shown $\varphi\in\mathbb{A}^r_{\tilde{v}}$. We now consider $f=\sum_{k=1}^n a_k\pi(u_k,\omega_k)\varphi\in\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ and apply (5.23), Lemma 2.3.30 and left/right invariance of $W_{2V_{1,2}}(L^r_v(\mathcal{G}\times\widehat{\mathcal{G}}))$:

$$\begin{aligned}\left\|W^{\varrho}_f f\right\|_{W^R_V(W_V(L^r_{\tilde{v}}))} &\le \left\|\mathsf{M}_{2V_{1,2}}V_f f\right\|_{L^r_v(\mathcal{G}\times\widehat{\mathcal{G}})} = \|V_f f\|_{W_{2V_{1,2}}(L^r_v(\mathcal{G}\times\widehat{\mathcal{G}}))}\\ &= \left\|\sum_{k,j=1}^n a_k\overline{a_j\langle\xi-\omega_k,u_k\rangle}\langle\omega_j,x-u_k\rangle T_{(u_k,\omega_k)-(u_j,\omega_j)}V_{\varphi}\varphi(x,\xi)\right\|_{W_{2V_{1,2}}(L^r_v(\mathcal{G}\times\widehat{\mathcal{G}}))}\\ &\underset{n,r}{\lesssim}\sum_{k,j=1}^n|a_k a_j|\left\|T_{(u_k,\omega_k)-(u_j,\omega_j)}V_{\varphi}\varphi(x,\xi)\right\|_{W_{2V_{1,2}}(L^r_v(\mathcal{G}\times\widehat{\mathcal{G}}))}<+\infty.\end{aligned}$$

This concludes the proof. □

Of course, $\mathscr{A}_{\tilde{v}}$ is a vector space. We shall use the extended notation $\mathscr{A}_{\tilde{v}}(\mathcal{G})$ only when confusion may occur. It is also clear that writing $\mathscr{A}_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})$ we mean the weight $v$ to be defined on $(\mathcal{G}\times\widehat{\mathcal{G}})\times(\widehat{\mathcal{G}}\times\mathcal{G})$, as done in the subsequent Corollary 5.1.11.

**Corollary 5.1.11.** ([8, Corollary 3.11]) *Let $f,g\in\mathcal{S}_{\mathcal{C}}(\mathcal{G})$, then $R(f,g)\in\mathscr{A}_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})$.*

*Proof.* The proof follows the same arguments in Lemma 5.1.10, together with (2.97) and Lemma 2.3.30. □

**H′**. For a fixed $g\in\mathbb{G}_{\tilde{v}}\smallsetminus\{0\}$, the space of test vectors is

$$\mathcal{T}_{\tilde{v}}\coloneqq\left\{f\in L^2(\mathcal{G})\,|\,W^{\varrho}_g f\in L^1_{\tilde{v}}(\mathbb{H}_{\mathcal{G}})\right\} \tag{5.25}$$

endowed with the norm

$$\|f\|_{\mathcal{T}_{\tilde{v}}}\coloneqq\left\|W^{\varrho}_g f\right\|_{L^1_{\tilde{v}}}. \tag{5.26}$$

$(\mathcal{T}_{\tilde{v}},\|\cdot\|_{\mathcal{T}_{\tilde{v}}})$ is a $\varrho$-invariant Banach space which embeds continuously into $L^2(\mathcal{G})$ and it is independent from the choice of the window vector $g\in\mathbb{G}_{\tilde{v}}\smallsetminus\{0\}$, see [147, Lemma 2.4.7].

**Lemma 5.1.12.** ([8, Lemma 3.12]) *For any $g\in\mathcal{S}_{\mathcal{C}}(\mathcal{G})\smallsetminus\{0\}$, the following equality holds true*

$$\mathbb{G}_{\tilde{v}}=\mathcal{T}_{\tilde{v}}=\left\{f\in L^2(\mathcal{G})\,|\,V_g f\in L^1_v(\mathcal{G}\times\widehat{\mathcal{G}})\right\}. \tag{5.27}$$

*Proof.* The second equality is just Remark 5.1.9, for the first one the proof follows the pattern of [15, Proposition 3.6]. From [147, Lemma 2.4.7]: $\mathbb{G}_{\tilde{v}}\subseteq\mathcal{T}_{\tilde{v}}$. Being the Duflo-Moore operator ([47, Theorem 3]) the identity, the orthogonality relations for $f,h\in L^2(\mathcal{G})$ and $g,\gamma\in\mathbb{G}_{\tilde{v}}$ are

$$\langle W^{\varrho}_g f,W^{\varrho}_{\gamma}h\rangle_{L^2(\mathbb{H}_{\mathcal{G}})}=\langle\gamma,g\rangle_{L^2(\mathcal{G})}\langle f,h\rangle_{L^2(\mathcal{G})},$$

see [147, Theorem 2.4.3]. Fix $f \in \mathcal{T}_{\tilde{v}}$, take $\gamma = g \neq 0$, $h = \varrho(x,\xi,\tau)f$ and using Fubini's Theorem, symmetry and submultiplicativity of $\tilde{v}$ we compute

$$\begin{aligned}
&\left\|W_f^\varrho f\right\|_{L^1_{\tilde{v}}} = \int_{\mathbb{H}_\mathcal{G}} |\langle f, \varrho(x,\xi,\tau)f\rangle|\, \tilde{v}(x,\xi,\tau)\, dx d\xi d\tau \\
&= \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \left|\langle W_g^\varrho f, W_g^\varrho[\varrho(x,\xi,\tau)f]\rangle\right| \tilde{v}(x,\xi,\tau)\, dx d\xi d\tau \\
&\leq \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f(y,\eta,s) W_g^\varrho[\varrho(x,\xi,\tau)f](y,\eta,s)\right| dy d\eta ds\, \tilde{v}(x,\xi,\tau)\, dx d\xi d\tau \\
&= \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f(y,\eta,s)\right| \left(\int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho[\varrho(x,\xi,\tau)f](y,\eta,s)\right| \tilde{v}(x,\xi,\tau)\, dx d\xi d\tau\right) dy d\eta ds.
\end{aligned}$$

Observe

$$W_g^\varrho[\varrho(x,\xi,\tau)f](y,\eta,s) = \langle \varrho(x,\xi,\tau)f, \varrho(y,\eta,s)g\rangle = W_g^\varrho f\left((x,\xi,\tau)^{-1}(y,\eta,s)\right),$$

so that

$$\begin{aligned}
\left\|W_f^\varrho f\right\|_{L^1_{\tilde{v}}} &\leq \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f(y,\eta,s)\right| \\
&\quad \times \left(\int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f\left((x,\xi,\tau)^{-1}(y,\eta,s)\right)\right| \tilde{v}(x,\xi,\tau)\, dx d\xi d\tau\right) dy d\eta ds \\
&\leq \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f(y,\eta,s)\right| \\
&\quad \times \left(\int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f(x',\xi',\tau')\right| \tilde{v}(x',\xi',\tau')\tilde{v}(y,\eta,s)\, dx' d\xi' d\tau'\right) dy d\eta ds \\
&= \frac{1}{\|g\|_{L^2}^2} \left(\int_{\mathbb{H}_\mathcal{G}} \left|W_g^\varrho f(x',\xi',\tau')\right| \tilde{v}(x',\xi',\tau')\, dx' d\xi' d\tau'\right)^2 \\
&= \frac{1}{\|g\|_{L^2}^2} \left\|W_g^\varrho f\right\|_{L^1_{\tilde{v}}}^2 < +\infty.
\end{aligned}$$

Hence $f \in \mathbb{G}_{\tilde{v}}$ and the proof in concluded. □

**Lemma 5.1.13.** ([8, Lemma 3.13]) *$\mathcal{S}_\mathcal{C}(\mathcal{G})$ is dense in $(\mathcal{T}_{\tilde{v}}, \|\cdot\|_{\mathcal{T}_{\tilde{v}}})$.*

*Proof.* In Lemma 5.1.10 we have shown that the Gaussian $\varphi$ in (2.94) belongs to $\mathbb{G}_{\tilde{v}}$. Then from [147, Lemma 2.4.7, 5.] we have that

$$\mathcal{S}_\mathcal{C}^\varrho(\mathcal{G}) \coloneqq \operatorname{span}\{\varrho(x,\xi,\tau)\varphi \mid (x,\xi,\tau) \in \mathbb{H}_\mathcal{G}\}$$

is dense in $(\mathcal{T}_{\tilde{v}}, \|\cdot\|_{\mathcal{T}_{\tilde{v}}})$. The claim follows from the trivial fact that $\mathcal{S}_\mathcal{C}^\varrho(\mathcal{G}) = \mathcal{S}_\mathcal{C}(\mathcal{G})$. □

**I′**. The reservoir is the Banach space

$$\mathcal{R}_{\tilde{v}} \coloneqq \mathcal{T}_{\tilde{v}}^\urcorner \coloneqq \{f \colon \mathcal{T}_{\tilde{v}} \to \mathbb{C} \mid \text{antilinear and continuous}\}. \tag{5.28}$$

**Remark 5.1.14.** ([8, Remark 3.14]) *Recall the definition of $\mathcal{S}_0(\mathcal{G})$ given in Definition 2.5.29. If $v \equiv 1$, then*

$$\mathcal{T}_1 = \mathbb{G}_1 = \mathcal{S}_0(\mathcal{G}), \quad \mathcal{R}_1 = \mathcal{S}_0'(\mathcal{G}). \tag{5.29}$$

*If $v$ is not constant, then*

$$\mathcal{T}_{\tilde{v}} = \mathbb{G}_{\tilde{v}} \hookrightarrow \mathcal{S}_0(\mathcal{G}), \qquad \mathcal{R}_{\tilde{v}} \hookleftarrow \mathcal{S}_0'(\mathcal{G}). \tag{5.30}$$

**Corollary 5.1.15.** ([8, Corollary 3.15]) *The following inclusion holds true:*

$$\mathscr{A}_{\tilde{v}} \subseteq C_0(\mathcal{G}), \tag{5.31}$$

*the latter being the space of continuous complex-valued functions on $\mathcal{G}$ which vanish at infinity.*

*Proof.* Combining Lemma 5.1.10 and Remark 5.1.14 we have $\mathscr{A}_{\tilde{v}} \subseteq \mathbb{G}_{\tilde{v}} = \mathcal{T}_{\tilde{v}} \subseteq S_0(\mathcal{G})$. We conclude using the fact that $S_0(\mathcal{G}) \subseteq C_0(\mathcal{G})$, see e.g. [101, Theorem 4.1]. □

**J′**. We extend the wavelet transform to $f \in \mathcal{R}_{\tilde{v}}$ and $g \in \mathcal{T}_{\tilde{v}}$:

$$W^{\varrho}_g f\colon \mathbb{H}_{\mathcal{G}} \to \mathbb{C}, (x, \xi, \tau) \mapsto {}_{\mathcal{R}_{\tilde{v}}}\langle f, \tau M_\xi T_x g\rangle_{\mathcal{T}_{\tilde{v}}}. \tag{5.32}$$

From now on we shall simply write $\langle \cdot,\cdot \rangle$. Observe $W^{\varrho}_g f \in C(\mathbb{H}_{\mathcal{G}}) \cap L^\infty_{1/\tilde{v}}(\mathbb{H}_{\mathcal{G}})$.

**Remark 5.1.16.** ([8, Remark 3.16]) *The class $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ defined in* (2.95) *actually depends on the compact open subgroup $\mathcal{K}$ in $\mathcal{G}_0$, where $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$. Then we might write $\mathcal{S}^{\mathcal{K}}_{\mathcal{C}}$ in place of $\mathcal{S}_{\mathcal{C}}$. Observe that if $\mathcal{K}'$ is a compact open subgroup different from $\mathcal{K}$ Lemma 5.1.10 is still valid. More generally, if $\mathbb{K}$ is the class of all compact open subgroups in $\mathcal{G}_0$:*

$$\mathbb{S}_{\mathcal{C}}(\mathcal{G}) \coloneqq \bigcup_{\mathcal{K}\in\mathbb{K}} \mathcal{S}^{\mathcal{K}}_{\mathcal{C}}(\mathcal{G}) \subseteq \mathscr{A}_{\tilde{v}} \subseteq \mathbb{G}_{\tilde{v}}. \tag{5.33}$$

*Therefore, coorbit spaces (defined in the subsequent* (5.34)*) are independent of the window $g \in \mathbb{S}_{\mathcal{C}}(\mathcal{G})$. Concretely, this gives us the freedom to chose the subgroup $\mathcal{K}$ which fits better to our purposes, as done in the proof of Lemma 5.1.38. Arguing similarly, we could replace $e^{-\pi x_1^2}$ in* (2.94) *with any $e^{-ax_1^2}$, $a > 0$. This fact will be used in Proposition 5.2.14.*
*From now on, for sake of simplicity, we shall only use the notation $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ with the convention that $\mathcal{K}$ and the coefficient of the Gaussian on $\mathbb{R}^d$ can be chosen freely, so that we shall never explicitly use the symbol $\mathbb{S}_{\mathcal{C}}(\mathcal{G})$.*

**K′**. The coorbit space on $\mathbb{H}_{\mathcal{G}}$ with respect to $L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})$, $0 < p, q \leq \infty$, is, for some fixed non-zero window $g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$,

$$\mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) \coloneqq \mathsf{Co}(L^{p,q}_{\tilde{m}}) \coloneqq \left\{ f \in \mathcal{R}_{\tilde{v}} \,|\, W^{\varrho}_g f \in W(L^\infty, L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) \right\} \tag{5.34}$$

endowed with the quasi-norm

$$\|f\|_{\mathsf{Co}(L^{p,q}_{\tilde{m}})} \coloneqq \left\| W^{\varrho}_g f \right\|_{W(L^\infty, L^{p,q}_{\tilde{m}})}. \tag{5.35}$$

We stress that $\mathsf{Co}(L^{p,q}_{\tilde{m}})$ is independent of the window $g$ and $(\mathsf{Co}(L^{p,q}_{\tilde{m}}), \|\cdot\|_{\mathsf{Co}(L^{p,q}_{\tilde{m}})})$ is a quasi-Banach space continuously embedded into $\mathcal{R}_{\tilde{v}}$. Moreover, $\|\cdot\|_{\mathsf{Co}(L^{p,q}_{\tilde{m}})}$ is a $r$-norm, with $r = \min\{1, p, q\}$. Notice that

$$\mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) = \left\{ f \in \mathcal{R}_{\tilde{v}} \,|\, V_g f \in W(L^\infty, L^{p,q}_m(\mathcal{G} \times \widehat{\mathcal{G}})) \right\}.$$

**Remark 5.1.17.** ([8, Remark 3.17]) *It is clear from the general coorbit theory presented in Chapter 2, that the set $\mathscr{A}_{\tilde{v}}$ defined in* (5.21) *is the maximal window space for all the coorbit spaces $\mathsf{Co}(L^{p,q}_{\tilde{m}})$, $0 < p, q \leq \infty$. For sake of simplicity we shall mainly work with window functions in the smaller class $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ and adopt the whole space $\mathscr{A}_{\tilde{v}}$ only when necessary, as done in Section 5.2.*

The coorbit spaces are independent of the reservoir, in the sense shown below.

**Proposition 5.1.18.** ([8, Proposition 3.18]) *Fix a non-zero window $g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$, then*

$$\mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) = \left\{ f \in \mathcal{S}_0'(\mathcal{G}) \,|\, W^{\varrho}_g f \in W(L^{\infty}, L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) \right\}, \tag{5.36}$$

*in the sense that the restriction map*

$$\left\{ f \in \mathcal{S}_0'(\mathcal{G}) \,|\, W^{\varrho}_g f \in W(L^{p,q}_{\tilde{m}}) \right\} \to \ \mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})), f \mapsto f|_{\mathcal{T}_{\tilde{v}}}$$

*is a bijection.*

*Proof.* If $v \equiv 1$ the claim is trivial since $\mathcal{T}_1 = \mathcal{S}_0$ and $\mathcal{R}_1 = \mathcal{S}_0'$, with equal norms, see Remark 5.1.14. If $v$ is not constant, then $v \gtrsim 1$ (since $v$ is bounded from below), and the thesis follows from what observed in Remark 5.1.14 and [147, Theorem 2.4.9, 3.]. □

**Definition 5.1.19.** *Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $0 < p, q \leq \infty$. The **modulation space** $M^{p,q}_m(\mathcal{G})$ is defined as*

$$M^{p,q}_m(\mathcal{G}) \coloneqq \ \mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})), \tag{5.37}$$

*endowed with the quasi-norm*

$$\|\cdot\|_{M^{p,q}_m} \coloneqq \|\cdot\|_{\ \mathsf{Co}(L^{p,q}_{\tilde{m}})}\,. \tag{5.38}$$

We adopt the notations $M^p_m = M^{p,p}_m$ and $M^{p,q} = M^{p,q}_1$.

**Theorem 5.1.20.** ([8, Theorem 3.20]) *For $0 < p, q \leq \infty$, the modulation spaces $(M^{p,q}_m(\mathcal{G}), \|\cdot\|_{M^{p,q}_m})$ are quasi-Banach spaces continuously embedded into $\mathcal{R}_{\tilde{v}}$ which do not depend on the window function $g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}) \smallsetminus \{0\}$, in the sense that different windows yield equivalent quasi-norms.*

*Proof.* Since $(M^{p,q}_m(\mathcal{G}), \|\cdot\|_{M^{p,q}_m}) = (\ \mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})), \|\cdot\|_{\ \mathsf{Co}(L^{p,q}_{\tilde{m}})})$, the claim follows from the coorbit spaces theory, Lemma 5.1.10 and [147, Theorem 2.4.9]. □

**Remark 5.1.21.** ([8, Remark 3.21]) *If $g, h \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}) \smallsetminus \{0\}$ (or $\mathscr{A}_{\tilde{v}} \smallsetminus \{0\}$) and $f \in M^{p,q}_m(\mathcal{G})$, then from the proof in [147, Theorem 2.4.9] we see that*

$$\|W^{\varrho}_h f\|_{W_Q(L^{p,q}_{\tilde{m}})} \underset{Q,v,r}{\lesssim} \frac{\left\|W^{\varrho}_g h\right\|_{W_Q(L^r_{\tilde{v}})}}{\|g\|^2_{L^2}} \left\|W^{\varrho}_g f\right\|_{W_Q(L^{p,q}_{\tilde{m}})} = \frac{\|h\|_{M^r_v(\mathcal{G})}}{\|g\|^2_{L^2}} \left\|W^{\varrho}_g f\right\|_{W_Q(L^{p,q}_{\tilde{m}})}, \tag{5.39}$$

*where $r = \min\{1, p, q\}$ as in $\mathbf{C}'$; actually we could replace $r$ with any $r'$ such that $0 < r' \leq r$. In the Banach case we have $r = 1$ and recapture [82, (11.33)], after taking into account Theorem 5.1.33 and Remark 5.1.37.*

In order to prove the expected inclusion relations between modulation spaces, we need particular types of relatively separated families, BUPUs and discrete spaces. The proofs of some subsequent lemmas are omitted because well known or trivial.

**Lemma 5.1.22.** ([8, Lemma 3.22]) *Let $Q, Q' \subseteq \mathbb{H}_{\mathcal{G}}$ be relatively compact, unit neighbourhoods and $\mathfrak{F} = \{(x_l, \xi_l, \tau_l)\}_{l \in L} \subseteq \mathbb{H}_{\mathcal{G}}$ relatively separated family, consider $0 < p, q \leq \infty$ and $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. Then*

$$(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{F}, Q) = (L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{F}, Q')$$

*with equivalent quasi-norms. Moreover, the equivalence constants depend only on $Q$, $Q'$, $m$ and $v$:*

$$\left\|(\lambda_l)_{l \in L}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{F},Q)} \underset{Q,Q',m,v}{\asymp} \left\|(\lambda_l)_{l \in L}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{F},Q')}\,.$$

*In particular, they do not depend on $\mathfrak{F}$ or $p$ and $q$.*

*Proof.* From (5.6) we have that for every $0 < p, q \leq \infty$ and $(x, \xi, \tau) \in \mathbb{H}_\mathcal{G}$

$$|||R_{(x,\xi,\tau)}|||_{L^{p,q}_{\tilde{m}} \to L^{p,q}_{\tilde{m}}} \leq Cv(x, \xi),$$

where $C = C(m, v) > 0$ is the constant of $v$-moderateness for $m$. Since $L^{p,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G})$ is right invariant, the proof goes like the one of [147, Lemma 2.3.16] applying the additional majorization above. $\square$

**Lemma 5.1.23.** ([8, Lemma 3.23]) *Let $Q, U \subseteq \mathbb{H}_\mathcal{G}$ be relatively compact, unit neighbourhoods, $\Delta = \{\delta_l\}_{l \in L}$ $U$-BUPU on $\mathbb{H}_\mathcal{G}$ with $U$-localizing family $\mathfrak{F} = \{(x_l, \xi_l, \tau_l)\}_{l \in L} \subseteq \mathbb{H}_\mathcal{G}$, consider $0 < p, q \leq \infty$ and $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. Then*

$$\|f\|_{W_Q(L^{p,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))} \underset{Q,U,\mathfrak{F},m,v}{\asymp} \left\|(\|\delta_l \cdot f\|_{L^\infty})_{l \in L}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{F},Q)}.$$

*In particular, the equivalence constants do not depend on $p$ and $q$.*

*Proof.* The result come from the proof [147, Theorem 2.3.17], see (2.26), together with Lemma 5.1.22. $\square$

**Lemma 5.1.24.** ([8, Lemma 3.24]) *Consider $X = \{x_i\}_{i \in I} \subseteq \mathcal{G}$, $\Xi = \{\xi_j\}_{j \in J} \subseteq \widehat{\mathcal{G}}$ and $T = \{\tau_z\}_{z \in Z} \subseteq \mathbb{T}$ relatively separated families. Then $\mathfrak{X} \coloneqq X \times \Xi \times T$ is a relatively separated family in $\mathbb{H}_\mathcal{G}$.*

We remark that if the group is $\sigma$-compact, then any relatively separated family is (at most) countable, Lemma 2.2.17.

**Lemma 5.1.25.** ([8, Lemma 3.25]) *Let $U \subseteq \mathcal{G}$ and $D \subseteq \widehat{\mathcal{G}}$ be relatively compact, unit neighbourhoods. Consider $\Psi = \{\psi_i\}_{i \in I}$ $U$-BUPU with localizing family $X = \{x_i\}_{i \in I}$ and $\Gamma = \{\gamma_j\}_{j \in J}$ $D$-BUPU with localizing family $\Xi = \{\xi_j\}_{j \in J}$. Then*

$$\Psi \otimes \Gamma \otimes \mathbb{I} \coloneqq \{\psi_i \otimes \gamma_j \otimes \chi_\mathbb{T},\ (i, j) \in I \times J\} \tag{5.40}$$

*is a $U \times D \times \mathbb{T}$-BUPU in $\mathbb{H}_\mathcal{G}$ with localizing family $\mathfrak{X} \coloneqq X \times \Xi \times \{1\}$.*

The following is a generalization of Lemma 2.2.19 and we follow the pattern of its proof. Although we present it for the Heisenberg group $\mathbb{H}_\mathcal{G} \cong \mathcal{G} \times \left(\widehat{\mathcal{G}} \times \mathbb{T}\right)$, it can be easily adapted to any product group $G_1 \times G_2$, $G_1$ and $G_2$ even not abelian. A similar result for $1 \leq p = q \leq \infty$ had been stated in [54, Remark 4, p. 518] without proof.

**Lemma 5.1.26.** ([8, Lemma 3.26]) *Consider $X = \{x_i\}_{i \in I} \subseteq \mathcal{G}$ and $\Xi = \{\xi_j\}_{j \in J} \subseteq \widehat{\mathcal{G}}$ relatively separated families, $\mathfrak{X}$ as in Lemma 5.1.25, and $V = V_\mathcal{G} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T}$ as in* (5.7)*. For $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $0 < p, q \leq \infty$,*

$$(L^{p,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X}, V) = \ell^{p,q}_{m_\mathbf{X}}(I \times J), \tag{5.41}$$

*where*

$$m_\mathbf{X} \colon I \times J \to (0, +\infty), (i, j) \mapsto m(x_i, \xi_j), \tag{5.42}$$

*with equivalence of the relative quasi-norms depending on $X$, $\Xi$, $V_\mathcal{G}$, $V_{\widehat{\mathcal{G}}}$, $v$, $p$ and $q$.*

*Proof.* The proof is divided into four cases.
*Case* $p, q < \infty$. Consider a sequence $(\lambda_i)_{i\in I} \in \mathbb{C}^I$. For every $x \in \mathcal{G}$, we define $I_x$ the subset of indexes

$$I_x = \{i \in I \,|\, \chi_{x_i+V_\mathcal{G}}(x) \neq \varnothing\} \subseteq \{i \in I \,|\, \left(x_i + \overline{V}_\mathcal{G}\right) \cap (x + \{e\}) \neq \varnothing\}. \tag{5.43}$$

From [147, Lemma 2.3.10], we have

$$\#\{i \in I \,|\, (x_i + V_\mathcal{G}) \cap (x + \{e\}) \neq \varnothing\} \leq C_{X,\overline{V}_\mathcal{G}} < +\infty, \qquad \forall x \in \mathcal{G}, \tag{5.44}$$

$C_{X,\overline{V}_\mathcal{G}} \in \mathbb{N}$ as in (2.18). Whence $\#I_x \leq C_{X,\overline{V}_\mathcal{G}}$ and

$$\begin{aligned}\left(\sum_{i\in I} |\lambda_i|\, \chi_{x_i+V_\mathcal{G}}(x)\right)^p &\leq (\#I_x \cdot \max\{|\lambda_i| \mid i \in I_x\})^p \leq C^p_{X,\overline{V}_\mathcal{G}} \max\{|\lambda_i|^p \mid i \in I_x\} \\ &\leq C^p_{X,\overline{V}_\mathcal{G}} \sum_{i\in I_x} |\lambda_i|^p = C^p_{X,\overline{V}_\mathcal{G}} \sum_{i\in I} |\lambda_i|^p\, \chi_{x_i+V_\mathcal{G}}(x).\end{aligned}$$

Vice versa

$$\begin{aligned}\left(\sum_{i\in I} |\lambda_i|\, \chi_{x_i+V_\mathcal{G}}(x)\right)^p &\geq (\max\{|\lambda_i| \mid i \in I_x\})^p = \max\{|\lambda_i|^p \mid i \in I_x\} \\ &\geq C^{-p}_{X,\overline{V}_\mathcal{G}} \sum_{i\in I_x} |\lambda_i|^p = C^{-p}_{X,\overline{V}_\mathcal{G}} \sum_{i\in I} |\lambda_i|^p\, \chi_{x_i+V_\mathcal{G}}(x).\end{aligned}$$

Hence we have shown the equivalence

$$\left(\sum_{i\in I} |\lambda_i|\, \chi_{x_i+V_\mathcal{G}}(x)\right)^p \asymp \sum_{i\in I} |\lambda_i|^p\, \chi_{x_i+V_\mathcal{G}}(x). \tag{5.45}$$

Analogous equivalences hold for every relatively separated family and sequence on the corresponding set of indexes, which under our hypothesis are always countable. Due to the chosen $V$,

$$\chi_{(x_i,\xi_j,1)V}(x,\xi,\tau) = \chi_{x_i+V_\mathcal{G}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) \qquad \forall\, (x,\xi,\tau) \in \mathbb{H}_\mathcal{G}.$$

Taking a sequence $(\lambda_{ij})_{i\in I,j\in J} \in \mathbb{C}^{I\times J}$ and using twice the equivalence (5.45), we compute

$$\begin{aligned}
\left\|(\lambda_{ij})_{i,j}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X},V)} &= \left(\int_{\widehat{\mathcal{G}}\times\mathbb{T}}\left(\int_{\mathcal{G}}\left(\sum_{i\in I,j\in J}|\lambda_{ij}|\,\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^p m(x,\xi)^p\,dx\right)^{\frac{q}{p}} d\xi d\tau\right)^{\frac{1}{q}} \\
&\asymp \left(\int_{\widehat{\mathcal{G}}}\left(\int_{\mathcal{G}}\sum_{i\in I,j\in J}|\lambda_{ij}|^p\,\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)m(x,\xi)^p dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}} \\
&= \left(\int_{\widehat{\mathcal{G}}}\left(\sum_{j\in J}\sum_{i\in I}|\lambda_{ij}|^p\int_{\mathcal{G}} m(x,\xi)^p\chi_{x_i+V_{\mathcal{G}}}(x)dx\,\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}} \\
&\asymp \left(\int_{\widehat{\mathcal{G}}}\sum_{j\in J}\left(\sum_{i\in I}|\lambda_{ij}|^p\int_{\mathcal{G}} m(x,\xi)^p\chi_{x_i+V_{\mathcal{G}}}(x)\,dx\right)^{\frac{q}{p}}\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\,d\xi\right)^{\frac{1}{q}} \\
&= \left(\sum_{j\in J}\int_{V_{\widehat{\mathcal{G}}}}\left(\sum_{i\in I}|\lambda_{ij}|^p\int_{V_{\mathcal{G}}} m(x+x_i,\xi+\xi_j)^p dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}}.
\end{aligned}$$

The monotone convergence theorem justifies the interchanges of integration with summation performed. From [147, Corollary 2.2.23] we have

$$\left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}} v\right)^{-1} m((x,\xi)+(u,\omega)) \le m(u,\omega) \le \left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}} v\right) m((x,\xi)+(u,\omega)), \tag{5.46}$$

for every $(u,\omega)\in\mathcal{G}\times\widehat{\mathcal{G}}$ and $(x,\xi)\in\overline{V}_{1,2}$, with $V_{1,2}$ defined in (5.9). Therefore, if $\xi\in V_{\widehat{\mathcal{G}}}$, we have

$$\int_{V_{\mathcal{G}}} m(x+x_i,\xi+\xi_j)^p\,dx \underset{v,\overline{V}_{1,2}}{\asymp} \int_{V_{\mathcal{G}}} m(x_i,\xi_j)^p\,dx = m(x_i,\xi_j)^p dx(V_{\mathcal{G}}). \tag{5.47}$$

Using the equivalences above,

$$\begin{aligned}
\left\|(\lambda_{ij})_{i,j}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X},V)} &\asymp \left(\sum_{j\in J}\int_{V_{\widehat{\mathcal{G}}}}\left(\sum_{i\in I}|\lambda_{ij}|^p\int_{V_{\mathcal{G}}} m(x+x_i,\xi+\xi_j)^p dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}} \\
&\asymp \left(\sum_{j\in J}\left(\sum_{i\in I}|\lambda_{ij}|^p\, m(x_i,\xi_j)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{p,q}_{m_{\mathbf{X}}}(I\times J)}.
\end{aligned}$$

*Case* $p=q=\infty$. For $(x,\xi)\in\mathcal{G}\times\widehat{\mathcal{G}}$ we define

$$\mathbf{I}_{(x,\xi)} = \{(i,j)\in I\times J \,|\, \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)\neq\varnothing\}.$$

Arguing as for (5.43) and (5.44), we have that there exists $N = N(\mathbf{X},V_{1,2}) = C_{\mathbf{X},\overline{V}_{1,2}} \in \mathbb{N}$ (see (2.18)) such that $\#\mathbf{I}_{(x,\xi)} \le N$, where $\mathbf{X} \coloneqq X\times\Xi$ and $V_{1,2}$ as in (5.9). Using (5.46), for

$(\lambda_{ij})_{i\in I,j\in J} \in \mathbb{C}^{I\times J}$,

$$\begin{aligned}\sum_{i\in I,j\in J} |\lambda_{ij}|\, \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi) m(x,\xi) &= \sum_{(i,j)\in \mathbf{I}_{(x,\xi)}} |\lambda_{ij}|\, m(x_i+u_i(x), \xi_j+\omega_j(\xi)) \\ &\asymp \sum_{(i,j)\in \mathbf{I}_{(x,\xi)}} |\lambda_{ij}|\, m(x_i,\xi_j) \\ &= \sum_{i\in I,j\in J} |\lambda_{ij}|\, m(x_i,\xi_j)\chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi),\end{aligned} \tag{5.48}$$

where $(u_i(x), \omega_j(\xi)) \in V_{1,2}$ for every $(i,j) \in \mathbf{I}_{(x,\xi)}$. Consider now $(\lambda_{ij})_{i\in I,j\in J} \in \ell^\infty_{m_{\mathbf{X}}}(I\times J)$. Then

$$\begin{aligned}\left\|(\lambda_{ij})_{i,j}\right\|_{(L^\infty_{\widetilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} &= \left\|(\lambda_{ij})_{i,j}\right\|_{(L^\infty_{m}(\mathcal{G}\times\widehat{\mathcal{G}}))_d(\mathbf{X},V_{1,2})} \\ &= \left\|\sum_{i\in I,j\in J} |\lambda_{ij}|\, \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} \\ &\asymp \left\|\sum_{i\in I,j\in J} |\lambda_{ij}|\, m(x_i,\xi_j)\chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} \\ &\le \left\|\sum_{i\in I,j\in J} \sup_{l,s} |\lambda_{ls}|\, m(x_l,\xi_s)\chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} \\ &\le \left\|(\lambda_{ij})_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)} \left\|N\chi_{\mathcal{G}\times\widehat{\mathcal{G}}}\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} = N\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)}.\end{aligned}$$

Vice versa, if $(\lambda_{ij})_{i\in I,j\in J} \in (L^\infty_{\widetilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)$,

$$\begin{aligned}\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)} &= \sup_{i\in I,j\in J} |\lambda_{ij}|\, m_{\mathbf{X}}(i,j) = \sup_{i\in I,j\in J} |\lambda_{i,j}|\, \chi_{(x_i,\xi_j)+V_{1,2}}(x_i,\xi_j)m(x_i,\xi_j) \\ &\le \sup_{i\in I,j\in J} \left\||\lambda_{ij}|\, \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} \\ &\le \sup_{i\in I,j\in J} \left\|\sum_{l\in I,s\in J} |\lambda_{ls}|\, \chi_{(x_l,\xi_s)+V_{1,2}}(x,\xi)m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} \\ &= \left\|\sum_{l\in I,s\in J} |\lambda_{ls}|\, \chi_{(x_l,\xi_s)+V_{1,2}}(x,\xi)m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})} \\ &= \left\|(\lambda_{ij})_{i,j}\right\|_{(L^\infty_{\widetilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)}.\end{aligned}$$

*Case $p=\infty$ and $q<\infty$.* We show the equivalence

$$\begin{aligned}&\operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I,j\in J} |\lambda_{ij}|\, m_{\mathbf{X}}(i,j)\chi_{x_i+V_\mathcal{G}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) \\ &\qquad \underset{\Xi,V_{\widehat{\mathcal{G}}}}{\asymp} \sum_{j\in J} \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, m_{\mathbf{X}}(i,j)\chi_{x_i+V_\mathcal{G}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi).\end{aligned} \tag{5.49}$$

In fact, arguing as in (5.43) and (5.44), for $\xi \in \widehat{\mathcal{G}}$ fixed and $J_\xi := \{j \in J \mid \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) \neq \varnothing\}$, there exists $M = M(\Xi, V_{\widehat{\mathcal{G}}}) \in \mathbb{N}$ such that $\#J_\xi \leq M$. Therefore,

$$\begin{aligned}
\operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I, j\in J} |\lambda_{ij}|\, m_{\mathbf{X}}(i,j)\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) &= \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{j\in J_\xi}\sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j) \\
&\leq \sum_{j\in J_\xi} \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j) \\
&= \sum_{j\in J} \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j\in J} \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) &= \sum_{j\in J_\xi} \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j) \\
&\leq M \max\{\operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j) \mid j \in J_\xi\} \\
&\leq M \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{j\in J_\xi}\sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j) \\
&= M \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{j\in J}\sum_{i\in I} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi).
\end{aligned}$$

Finally, using the previous cases, the equivalences in (5.48) and (5.49), we can write

$$\begin{aligned}
\left\|(\lambda_{ij})_{i,j}\right\|_{(L^{\infty,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X},V)} &= \left(\int_{\widehat{\mathcal{G}}} \left(\operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I, j\in J} |\lambda_{ij}|\, \chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) m(x,\xi)\right)^q d\xi\right)^{\frac{1}{q}} \\
&\asymp \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j\in J} \operatorname*{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} |\lambda_{ij}|\, m_{\mathbf{X}}(i,j)\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^q d\xi\right)^{\frac{1}{q}} \\
&= \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j\in J} \left\|\sum_{i\in I} |\lambda_{ij}|\, m_{\mathbf{X}}(i,j)\chi_{x_i+V_{\mathcal{G}}}(\cdot)\right\|_{L^\infty(\mathcal{G})} \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^q d\xi\right)^{\frac{1}{q}} \\
&= \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j\in J} \left\|(\lambda_{ij} m_{\mathbf{X}}(i,j))_{i\in I}\right\|_{(L^\infty(\mathcal{G}))_d(X,V_{\mathcal{G}})} \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^q d\xi\right)^{\frac{1}{q}} \\
&\asymp \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j\in J} \left\|(\lambda_{ij} m_{\mathbf{X}}(i,j))_{i\in I}\right\|_{\ell^\infty(I)} \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^q d\xi\right)^{\frac{1}{q}} \\
&= \left\|\left(\left\|(\lambda_{ij} m_{\mathbf{X}}(i,j))_{i\in I}\right\|_{\ell^\infty(I)}\right)_{j\in J}\right\|_{(L^q(\widehat{\mathcal{G}}))_d(\Xi,V_{\widehat{\mathcal{G}}})} \\
&\asymp \left\|\left(\left\|(\lambda_{ij} m_{\mathbf{X}}(i,j))_{i\in I}\right\|_{\ell^\infty(I)}\right)_{j\in J}\right\|_{\ell^q(J)} = \left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{\infty,q}_{m_{\mathbf{X}}}(I\times J)}.
\end{aligned}$$

*Case $p<\infty$ and $q=\infty$.* Similarly to what has been done before,

$$\begin{aligned}
\left\|(\lambda_{ij})_{i,j}\right\|_{(L^{p,\infty}_{\tilde m}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} &= \operatorname*{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\left(\int_\mathcal{G}\left(\sum_{i\in I,j\in J}|\lambda_{ij}|\,\chi_{x_i+V_\mathcal{G}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^p m(x,\xi)^p dx\right)^{\frac{1}{p}}\\
&\asymp \operatorname*{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\left(\int_\mathcal{G}\sum_{i\in I,j\in J}|\lambda_{ij}|^p\,\chi_{x_i+V_\mathcal{G}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)m(x,\xi)^p dx\right)^{\frac{1}{p}}\\
&\asymp \operatorname*{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\left(\sum_{i\in I,j\in J}|\lambda_{ij}|^p\,m(x_i,\xi_j)^p\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^{\frac{1}{p}}\\
&\asymp \operatorname*{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\sum_{j\in J}\left(\sum_{i\in I}|\lambda_{ij}|^p\,m(x_i,\xi_j)^p\right)^{\frac{1}{p}}\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\\
&= \operatorname*{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\sum_{j\in J}\left\|(\lambda_{ij}m_\mathbf{X}(i,j))_{i\in I}\right\|_{\ell^p(I)}\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\\
&= \left\|\left(\left\|(\lambda_{ij}m_\mathbf{X}(i,j))_{i\in I}\right\|_{\ell^p(I)}\right)_{j\in J}\right\|_{(L^\infty(\widehat{\mathcal{G}}))_d(\Xi,V_{\widehat{\mathcal{G}}})}\\
&\asymp \left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{p,\infty}_{m_\mathbf{X}}(I\times J)}.
\end{aligned}$$

The proof is concluded. ∎

**Remark 5.1.27.** ([8, Remark 3.27]) *We want to state explicitly the equivalence constants involved in the previous lemma. We distinguish four cases, as done in the proof.*
Case $p,q<\infty$. *We have*

$$A^{-1}B\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{p,q}_{m_\mathbf{X}}(I\times J)} \le \left\|(\lambda_{ij})_{i,j}\right\|_{(L^{p,q}_{\tilde m}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \le AB\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{p,q}_{m_\mathbf{X}}(I\times J)},$$

*where*

$$\begin{aligned}
A &:= A(X,\Xi,V_\mathcal{G},V_{\widehat{\mathcal{G}}},v,p) := C_{X,\overline{V}_\mathcal{G}}C^{\frac{1}{p}+1}_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}\left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}} v\right),\\
B &:= B(V_\mathcal{G},V_{\widehat{\mathcal{G}}},p,q) := dx(V_\mathcal{G})^{\frac{1}{p}}\,d\xi(V_{\widehat{\mathcal{G}}})^{\frac{1}{q}}.
\end{aligned}$$

Case $p=q=\infty$. *The equivalence is*

$$\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{\infty}_{m_\mathbf{X}}(I\times J)} \le \left\|(\lambda_{ij})_{i,j}\right\|_{(L^{\infty}_{\tilde m}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \le \left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}} v\right)C_{\mathbf{X},\overline{V}_{1,2}}\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{\infty}_{m_\mathbf{X}}(I\times J)}.$$

Case $p=\infty$ and $q<\infty$. *We got*

$$D\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{\infty,q}_{m_\mathbf{X}}(I\times J)} \le \left\|(\lambda_{ij})_{i,j}\right\|_{(L^{\infty,q}_{\tilde m}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \le E\left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{\infty,q}_{m_\mathbf{X}}(I\times J)},$$

*where*

$$D := D(\Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, q) := C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}^{-2} \left( \sup_{\overline{V}_{1,2} \cup -\overline{V}_{1,2}} v \right)^{-1} d\xi(V_{\widehat{\mathcal{G}}})^{\frac{1}{q}},$$

$$E := B(X, \Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, q) := C_{X,\overline{V}_{\mathcal{G}}} C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}} \left( \sup_{\overline{V}_{1,2} \cup -\overline{V}_{1,2}} v \right) d\xi(V_{\widehat{\mathcal{G}}})^{\frac{1}{q}}.$$

Case $p < \infty$ and $q = \infty$. *The last equivalence is given by*

$$L \left\| (\lambda_{ij})_{i,j} \right\|_{\ell^{p,\infty}_{m_{\mathbf{X}}}(I \times J)} \leq \left\| (\lambda_{ij})_{i,j} \right\|_{(L^{p,\infty}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X}, V)} \leq M \left\| (\lambda_{ij})_{i,j} \right\|_{\ell^{p,\infty}_{m_{\mathbf{X}}}(I \times J)},$$

*where*

$$L := L(X, \Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, p) := C_{X,\overline{V}_{\mathcal{G}}}^{-1} C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}^{-1} \left( \sup_{\overline{V}_{1,2} \cup -\overline{V}_{1,2}} v \right)^{-1} dx(V_{\mathcal{G}})^{\frac{1}{p}},$$

$$M := M(X, \Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, p) := C_{X,\overline{V}_{\mathcal{G}}} C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}^{2} \left( \sup_{\overline{V}_{1,2} \cup -\overline{V}_{1,2}} v \right) dx(V_{\mathcal{G}})^{\frac{1}{p}}.$$

*We recall that the definition of the constants* $C_{X,\overline{V}_{\mathcal{G}}}, C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}, C_{\mathbf{X},\overline{V}_{1,2}}$ *is given in* (2.18).

On account of the constants shown in the previous remark, we have the following corollary.

**Corollary 5.1.28.** ([8, Corollary 3.28]) *Fix* $0 < \delta \leq \infty$ *and take* $p, q$ *such that* $0 < \delta \leq p, q \leq \infty$. *Under the same assumptions of Lemma 5.1.26, there are two constants*

$$C_1 := C_1(X, \Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, \delta) > 0 \quad \text{and} \quad C_1 := C_1(X, \Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, \delta) > 0$$

*such that*

$$C_1 \left\| (\lambda_{ij})_{i,j} \right\|_{\ell^{p,q}_{m_{\mathbf{X}}}(I \times J)} \leq \left\| (\lambda_{ij})_{i,j} \right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X}, V)} \leq C_2 \left\| (\lambda_{ij})_{i,j} \right\|_{\ell^{p,q}_{m_{\mathbf{X}}}(I \times J)}$$

*for every sequence* $(\lambda_{ij})_{i,j}$ *in* $(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X}, V) = \ell^{p,q}_{m_{\mathbf{X}}}(I \times J)$.

*Proof.* We notice that if $b \geq 1$, then $b^{\frac{1}{p}}$ is a strictly decreasing function of $p \in (0, \infty)$ and $b^{\frac{1}{p}} \geq 1$. Likewise $b^{-\frac{1}{p}}$ is strictly increasing and $0 < b^{-\frac{1}{p}} \leq 1$. The claim follows now from Remark 5.1.27. $\square$

**Remark 5.1.29.** *Although in Lemma 5.1.26 we considered* $V = V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T}$ *with* $V_{\mathcal{G}}$ *and* $V_{\widehat{\mathcal{G}}}$ *open, this last assumption can be relaxed into measurability. Even in this case the above lemma and the subsequent Corollary 5.1.30 hold true.*

**Corollary 5.1.30.** ([8, Corollary 3.30]) *Consider* $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ *and* $m_1, m_2 \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *such that* $m_2 \lesssim m_1$. *Let* $V$, $X$, $\Xi$ *and* $\mathfrak{X}$ *be as in Lemma 5.1.26. Then*

$$\left(L^{p_1,q_1}_{\tilde{m}_1}(\mathbb{H}_{\mathcal{G}})\right)_d(\mathfrak{X}, V) \hookrightarrow \left(L^{p_2,q_2}_{\tilde{m}_2}(\mathbb{H}_{\mathcal{G}})\right)_d(\mathfrak{X}, V). \tag{5.50}$$

*Proof.* It is a straightforward consequence of Lemma 5.1.26 and the continuous inclusions

$$\ell^{p_1,q_1}_{m_{1,\mathbf{X}}}(I \times J) \hookrightarrow \ell^{p_2,q_2}_{m_{2,\mathbf{X}}}(I \times J), \tag{5.51}$$

since $m_{2,\mathbf{X}} \lesssim m_{1,\mathbf{X}}$. $\square$

**Proposition 5.1.31.** ([8, Proposition 3.31]) *Consider* $0 < p_1 \le p_2 \le \infty$*,* $0 < q_1 \le q_2 \le \infty$ *and* $m_1, m_2 \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *such that* $m_2 \lesssim m_1$*. Then we have the following continuous inclusions:*

$$M_{m_1}^{p_1,q_1}(\mathcal{G}) \hookrightarrow M_{m_2}^{p_1,q_2}(\mathcal{G}). \tag{5.52}$$

*Proof.* Under the hypothesis of Lemma 5.1.25, it is always possible to find a BUPU on $\mathbb{H}_\mathcal{G}$ of the type (5.40), see [147, Lemma 2.3.12]. For such a BUPU

$$\Psi \otimes \Gamma \otimes \mathbb{I} = \{\psi_i \otimes \gamma_j \otimes \chi_\mathbb{T},\ (i,j) \in I \times J\},$$

the corresponding localizing family $\mathfrak{X} = X \times \Xi \times \{1\}$ fulfils the requirements of Corollary 5.1.30. To get the desired result we use the equivalence of quasi-norms shown in (2.26):

$$\begin{aligned}
\|f\|_{M_{m_2}^{p_2,q_2}} &\asymp \left\|W_g^\varrho f\right\|_{W(L_{\tilde{m}_2}^{p_2,q_2})} \asymp \left\|\left(\left\|(\psi_i \otimes \gamma_j \otimes \chi_\mathbb{T}) \cdot W_g^\varrho f\right\|_{L^\infty}\right)_{i,j}\right\|_{(L_{\tilde{m}_2}^{p_2,q_2}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \\
&\lesssim \left\|\left(\left\|(\psi_i \otimes \gamma_j \otimes \chi_\mathbb{T}) \cdot W_g^\varrho f\right\|_{L^\infty}\right)_{i,j}\right\|_{(L_{\tilde{m}_1}^{p_1,q_1}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \\
&\asymp \left\|W_g^\varrho f\right\|_{W(L_{\tilde{m}_1}^{p_1,q_1})} \asymp \|f\|_{M_{m_1}^{p_1,q_1}}.
\end{aligned}$$

This concludes the proof. □

If $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$, from the submultiplicativity and symmetry of $v$ we have $1/m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$. This remark is implicitly used in the following issue.

**Proposition 5.1.32.** ([8, Proposition 3.32]) *If* $1 \le p, q < \infty$*, then* $(M_m^{p,q}(\mathcal{G}))' = M_{1/m}^{p',q'}(\mathcal{G})$ *under the duality*

$$\langle f, h\rangle = \langle V_g f, V_g h\rangle_{L^2(\mathcal{G}\times\widehat{\mathcal{G}})}, \tag{5.53}$$

*for all* $f \in M_m^{p,q}(\mathcal{G})$*,* $h \in M_{1/m}^{p',q'}(\mathcal{G})$ *and some* $g \in \mathcal{S}_\mathcal{C}(\mathcal{G}) \setminus \{0\}$*.*

*Proof.* For $1 \le p, q \le \infty$, $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ is a solid bi-invariant Banach function space continuously embedded into $L_{loc}^1(\mathbb{H}_\mathcal{G})$. Therefore, from Theorem 2.2.39 combined with Remark 5.1.9, we have

$$M_m^{p,q}(\mathcal{G}) = \mathsf{Co}(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) = \mathsf{Co}_{\mathrm{FG}}(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) = \{f \in \mathcal{R}_{\tilde{v}} \mid V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})\} \tag{5.54}$$

with

$$\|V_g f\|_{W(L_m^{p,q})} \asymp \|V_g f\|_{L_m^{p,q}}. \tag{5.55}$$

The proof then goes as in [82, Theorem 11.3.6], after noticing that we can identify $(L_m^1)'$ with $L_{1/m}^\infty$ since under our assumptions $\mathcal{G} \times \widehat{\mathcal{G}}$ is $\sigma$-finite, similarly for mixed-norm cases. □

**Theorem 5.1.33.** ([8, Theorem 3.33])

(*i*) *If* $0 < p, q < \infty$*, then* $\mathcal{S}_\mathcal{C}(\mathcal{G})$ *is quasi-norm-dense in* $M_m^{p,q}(\mathcal{G})$*;*

(*ii*) *If* $1 \le p, q \le \infty$ *and at least one between* $p$ *and* $q$ *is equal to* $\infty$*, then* $\mathcal{S}_\mathcal{C}(\mathcal{G})$ *is w-*∗*-dense in* $M_m^{p,q}(\mathcal{G})$*.*

*Proof.* For any $0 \le p, q \le \infty$, $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ is a subspace of $M^{p,q}_m(\mathcal{G})$, cf. the computations in the proof of Lemma 5.1.10 and the inclusions in (5.52).

($i$) Let $\varphi$ be as in (2.94) and consider the relatively compact unit neighbourhood $U_0$ coming from Theorem 2.2.37. Without loss of generality we can assume $U_0 = V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T} = V$ as in (5.7), see the proofs of [147, Theorem 2.4.19] and [147, Lemma 2.4.17]. Then there exists a $U_0$-BUPU with localizing family $\mathfrak{X} = \{(x_i, \xi_j, 1)\}_{(i,j)\in I\times J}$ such that any $f \in M^{p,q}_m(\mathcal{G})$ can be written as

$$f = \sum_{i\in I, j\in J} \lambda_{ij}(f)\varrho(x_i, \xi_j, 1)\varphi = \sum_{i\in I, j\in J} \lambda_{ij}(f)\pi(x_i, \xi_j)\varphi, \tag{5.56}$$

with unconditional convergence in $M^{p,q}_m(\mathcal{G})$ since the finite sequences are dense in $\ell^{p,q}_{m\mathbf{X}}(I \times J) = (L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X}, V)$, $p, q < \infty$.

($ii$) We show the case $p = q = \infty$, the remaining ones are analogous. From Proposition 5.1.32, $M^{\infty}_m(\mathcal{G})$ can be seen as the dual of $M^1_{1/m}(\mathcal{G})$. Therefore, with $\varphi$ the Gaussian in (2.94),

$${}^{\perp}\mathcal{S}_{\mathcal{C}}(\mathcal{G}) = \{f \in M^1_{1/m} \mid \langle V_\varphi f, V_\varphi h\rangle = 0, \quad \forall\, h \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})\}. \tag{5.57}$$

For fixed $(u, \omega) \in \mathcal{G} \times \widehat{\mathcal{G}}$ consider $h = \pi(u, \omega)\varphi \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$. From Lemma 2.3.30

$$V_\varphi h(x, \xi) = \overline{\langle \xi - \omega, u\rangle} T_{(u,\omega)} V_\varphi \varphi(x, \xi). \tag{5.58}$$

In particular, from (2.100), $V_\varphi h(u, \omega) \neq 0$ and it is continuous. Therefore if $f \in {}^{\perp}\mathcal{S}_{\mathcal{C}}(\mathcal{G})$

$$\langle V_\varphi f, V_\varphi h\rangle = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} V_\varphi f(x, \xi)\overline{V_\varphi h(x, \xi)}\, dx d\xi = 0 \quad \Rightarrow \quad V_\varphi f \overline{V_\varphi h} = 0 \,\text{a.e.},$$

but since $V_\varphi f \overline{V_\varphi h}$ is continuous this implies $V_\varphi f(x, \xi)\overline{V_\varphi h(x, \xi)} = 0$ for every $(x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$. Necessarily $V_\varphi f$ vanishes on a neighbourhood of $(u, \omega)$. On account of the arbitrariness of the point $(u, \omega) \in \mathcal{G} \times \widehat{\mathcal{G}}$, we have $V_\varphi f \equiv 0$ which also means $W^{\varrho}_\varphi f \equiv 0$. Since the application

$$W^{\varrho}_\varphi \colon \mathcal{R}_{\tilde{v}} \to C(\mathbb{H}_{\mathcal{G}}) \cap L^{\infty}_{1/\tilde{v}}(\mathbb{H}_{\mathcal{G}})$$

is injective, see [147, Lemma 2.4.8], we infer $f = 0$. Therefore ${}^{\perp}\mathcal{S}_{\mathcal{C}}(\mathcal{G}) = \{0\}$ and

$$\overline{\mathcal{S}_{\mathcal{C}}(\mathcal{G})}^{w-*} = \left({}^{\perp}\mathcal{S}_{\mathcal{C}}(\mathcal{G})\right)^{\perp} = (\{0\})^{\perp} = M^{\infty}_m(\mathcal{G}).$$

This concludes the proof. □

**Lemma 5.1.34.** ([8, Lemma 3.34]) *For every , $0 < p, q \le \infty$ and $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$*

$$\mathscr{A}_{\tilde{v}}(\mathcal{G}) \subseteq M^{p,q}_m(\mathcal{G}).$$

*Proof.* We just need to show that for every $0 < r \le 1$ the inclusion

$$\mathscr{A}_{\tilde{v}}(\mathcal{G}) \subseteq M^r_v(\mathcal{G}) \tag{5.59}$$

holds true, then the claim follows from the inclusion relations for modulation spaces. From (5.20) and the inclusion relations in [147, p. 113], if $g \in \mathscr{A}_{\tilde{v}} \subseteq \mathbb{A}^r_{\tilde{v}}$ and $\varphi$ is the Gaussian as in (2.94), we get that

$$W^{\varrho}_\varphi g \in W^R(L^\infty, W(L^\infty, L^r_{\tilde{v}})) \hookrightarrow W(L^\infty, L^r_{\tilde{v}}).$$

Hence $g \in M^r_v(\mathcal{G})$. □

**Corollary 5.1.35.** ([8, Corollary 3.35]) *If $0 < p, q < \infty$, then $\mathscr{A}_{\tilde{v}}$ is quasi-norm-dense in $M^{p,q}_m(\mathcal{G})$.*

*Proof.* The claim follows from the above theorem, the previous lemma and the inclusion $\mathcal{S}_{\mathcal{C}} \subseteq \mathscr{A}_{\tilde{v}}$. □

**Corollary 5.1.36.** ([8, Corollary 3.36]) *For every $f \in \mathcal{S}_0'(\mathcal{G})$ there exists a net $(f_\alpha)_{\alpha\in A} \subseteq \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ such that*

$$\lim_{\alpha\in A} \langle f_\alpha, h\rangle_{L^2(\mathcal{G})} = {}_{\mathcal{S}_0'}\langle f, h\rangle_{\mathcal{S}_0}, \qquad \forall\, h \in \mathcal{S}_0(\mathcal{G}). \tag{5.60}$$

*Proof.* From Lemma 5.1.13 we have that $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ is norm-dense in $\mathcal{T}_1 = \mathcal{S}_0(\mathcal{G})$. From [101, Proposition 6.15] there exists a bounded net $(f_\beta)_{\beta\in B} \subseteq \mathcal{S}_0(\mathcal{G})$ such that

$$\lim_{\beta\in B} \langle f_\beta, h\rangle_{L^2(\mathcal{G})} = {}_{\mathcal{S}_0'}\langle f, h\rangle_{\mathcal{S}_0}, \qquad \forall\, h \in \mathcal{S}_0(\mathcal{G}). \tag{5.61}$$

This concludes the proof. □

**Remark 5.1.37.** ([8, Remark 3.37])

(*i*) *From Theorem 5.1.33 and relations* (5.54) *and* (5.55) *it follows that the modulation spaces introduced in Definition 5.1.19 coincide with the classical ones in [56, 88]. This implies that*

$$M^1_m(\mathcal{G}) \cong \Big(clos_{M^\infty_{1/m}}(\mathcal{S}_{\mathcal{C}}(\mathcal{G}))\Big)', \tag{5.62}$$

*the dual of the closure of $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ with respect to the norm on $M^\infty_{1/m}(\mathcal{G})$. If $f \in M^{\infty,1}_m(\mathcal{G})$ and $g \in M^{1,\infty}_{1/m}(\mathcal{G})$, then for $\varphi$ as in* (2.94)

$$\Big|\langle V_\varphi f, V_\varphi g\rangle_{L^2(\mathcal{G}\times\widehat{\mathcal{G}})}\Big| \lesssim \|f\|_{M^{\infty,1}_m} \|g\|_{M^{1,\infty}_{1/m}}. \tag{5.63}$$

*See [88, Proposition 2.2];*

(*ii*) *The theory for $\mathcal{G} = \mathbb{R}^d$ developed in [75] is recovered for every $0 < p, q \le \infty$. In fact, it was observed in [119, Section 8] that from [75, Lemma 3.2] follows the equality*

$$\mathsf{Co}(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathbb{R}^d})) = \{f \in \mathcal{S}' \mid V_g f \in L^{p,q}_m(\mathbb{R}^{2d})\} \qquad 0 < p, q \le \infty,$$

*with equivalent quasi-norms.*

For a general LCA group $\mathcal{G}$ it is an open problem whether a construction of the type

$$\{f \in \mathcal{R}_{\tilde{v}} \mid V_g f \in L^{p,q}_m(\mathcal{G}\times\widehat{\mathcal{G}})\},$$

with obvious quasi-norm, could make sense or not when at least one between $p$ and $q$ is smaller than 1. However, we are able to answer affirmatively if $\mathcal{G}$ is discrete or compact, see the lemma and corollary below.

**Lemma 5.1.38.** ([8, Lemma 3.38]) *Let $0 < p, q \le \infty$. Suppose $\mathcal{G}$ is discrete or compact. Then there exists $C > 0$ such that for every $f \in M^{p,q}_m(\mathcal{G})$*

$$\left\|W^\varrho_g f\right\|_{W(L^{p,q}_{\tilde{m}})} \le C \left\|W^\varrho_g f\right\|_{L^{p,q}_{\tilde{m}}}, \tag{5.64}$$

*for some $g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}) \setminus \{0\}$.*

*Proof.* If we prove for some suitable unit neighbourhood $Q \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ that there exists $C > 0$ such that

$$\|V_g f\|_{W_Q(L^{p,q}_m)} \leq C \|V_g f\|_{L^{p,q}_m}, \tag{5.65}$$

then (5.64) holds true, see Remark 5.1.9. Moreover, as shown in Proposition 5.1.18, we can consider the modulation spaces as subsets of $\mathcal{S}_0'(\mathcal{G})$.
*Case $\mathcal{G}$ discrete.* $\widehat{\mathcal{G}}$ is compact and the structure theorem reads as $\mathcal{G} = \mathcal{G}_0$ and $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}_0$. In the definition of the Gaussian function (2.94) we take, Remark 5.1.16, the open and compact subgroup $\mathcal{K} = \{e\}$, therefore

$$\varphi(x) \coloneqq \chi_{\{e\}}(x) =: \delta_e(x).$$

We also choose $Q \coloneqq \{e\} \times \widehat{\mathcal{G}}$, which is a measurable, relatively compact, unit neighbourhood. Fix $f \in M^{p,q}_m(\mathcal{G})$, from [101, Proposition 6.15], we have that there exists a bounded net $(f_\alpha)_{\alpha \in A} \subseteq \mathcal{S}_0(\mathcal{G})$ such that

$$\lim_{\alpha \in A} \langle f_\alpha, h\rangle_{L^2(\mathcal{G})} = {}_{\mathcal{S}_0'}\langle f, h\rangle_{\mathcal{S}_0}, \qquad \forall\, h \in \mathcal{S}_0(\mathcal{G}). \tag{5.66}$$

Recall that $\mathcal{S}_\mathcal{C}(\mathcal{G}) \subseteq \mathcal{S}_0(\mathcal{G})$, then adopting the widow function $\varphi$, we compute

$$\begin{aligned}
V_\varphi f(x,\xi) &= \langle f, \pi(x,\xi)\delta_e\rangle = \lim_{\alpha \in A} \langle f_\alpha, \pi(x,\xi)\delta_e\rangle = \lim_{\alpha \in A} \sum_{u \in \mathcal{G}} f_\alpha(u) \overline{\langle \xi, u\rangle \delta_x(u)} \\
&= \lim_{\alpha \in A} f_\alpha(x) \overline{\langle \xi, x\rangle} = \overline{\langle \xi, x\rangle} \lim_{\alpha \in A} f_\alpha(x), \\
\mathsf{M}_Q V_\varphi f(x,\xi) &= \operatorname*{ess\,sup}_{(y,\eta) \in (x,\xi) + \{e\} \times \widehat{\mathcal{G}}} \left| \overline{\langle \eta, y\rangle} \lim_{\alpha \in A} f_\alpha(y) \right| = \operatorname*{ess\,sup}_{(y,\eta) \in \{x\} \times \widehat{\mathcal{G}}} \left| \lim_{\alpha \in A} f_\alpha(y) \right| \\
&= \left| \lim_{\alpha \in A} f_\alpha(x) \right| = |V_\varphi f(x,\xi)|.
\end{aligned}$$

Therefore

$$\|V_\varphi f\|_{W_Q(L^{p,q}_m)} = \|\mathsf{M}_Q V_\varphi f\|_{L^{p,q}_m} = \|V_\varphi f\|_{L^{p,q}_m}.$$

*Case $\mathcal{G}$ compact.* The argument is identical to the previous one, take $\mathcal{K} = \mathcal{G}$ and $Q \coloneqq \mathcal{G} \times \{\hat{e}\}$. □

**Corollary 5.1.39.** ([8, Corollary 3.39]) *Suppose $\mathcal{G}$ is discrete or compact. Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $0 < p, q \leq \infty$. Then*

$$M^{p,q}_m(\mathcal{G}) = \{f \in \mathcal{S}_0'(\mathcal{G}) \mid V_g f \in L^{p,q}_m(\mathcal{G} \times \widehat{\mathcal{G}})\}$$

*and*

$$\|f\|_{M^{p,q}_m} \asymp \|V_g f\|_{L^{p,q}_m}, \tag{5.67}$$

*for some $g \in \mathcal{S}_\mathcal{C}(\mathcal{G}) \smallsetminus \{0\}$.*

*Proof.* We consider $M^{p,q}_m(\mathcal{G})$ as a subspace of $\mathcal{S}_0'(\mathcal{G})$ instead of $\mathcal{R}_{\tilde{v}}$ (Proposition 5.1.18). The claim then follows from the continuous embedding $W(L^{p,q}_{\tilde{m}}) \hookrightarrow L^{p,q}_{\tilde{m}}$, Lemma 5.1.38 and Remark 5.1.9. □

## 5.2 Continuity of the Rihaczek and Kohn-Nirenberg operators

In this section we investigate the continuity of the Rihaczek distribution (2.96) on modulation spaces and infer boundedness results for the Kohn-Nirenberg operators, defined in (2.216).

### 5.2.1 Boundedness results

We first study the boundedness of the Rihaczek distribution on modulation spaces. The techniques are mainly borrowed from [31, Theorem 3.1] and [25, Theorem 4] for the Wigner distribution on $\mathbb{R}^d$.
From now on we shall mainly work with $\mathcal{S}_0(\mathcal{G})$ and $\mathcal{S}_0'(\mathcal{G})$ instead of $\mathcal{T}_{\tilde{v}}$ and $\mathcal{R}_{\tilde{v}}$ (Proposition 5.1.18).

We need to extend [88, formula (51)] to wider classes of functions. Namely,

**Lemma 5.2.1.** ([8, Lemma 4.4]) *Consider $\psi \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ and $f, g \in \mathcal{S}_0'(\mathcal{G})$. Then*

$$V_{R(\psi,\psi)}R(g,f)((x,\xi),(\omega,u)) = \overline{\langle \xi, u\rangle} V_\psi g(x, \xi+\omega)\overline{V_\psi f(x+u,\xi)}, \tag{5.68}$$

*with $x, u \in \mathcal{G}$ and $\xi, \omega \in \widehat{\mathcal{G}}$.*

*Proof.* For $f, g, \psi \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ formula (5.68) is proved in [88, formula (51)]. Consider now $f, g \in \mathcal{S}_0'(\mathcal{G})$. From Corollary 5.1.36 there exist nets $\{f_\alpha\}_{\alpha\in A}$ and $\{g_\alpha\}_{\alpha \in A}$ in $\mathcal{S}_0(\mathcal{G})$ which converge pointwisely to $f$ and $g$ in $\mathcal{S}_0'(\mathcal{G})$. Therefore for every $x, u \in \mathcal{G}$ and $\xi \in \widehat{\mathcal{G}}$,

$$\lim_{\alpha\in A} V_\psi f_\alpha(x+u,\xi) = \lim_{\alpha \in A}\langle f_\alpha, \pi(x+u,\xi)\psi\rangle = \langle f, \pi(x+u,\xi)\psi\rangle = V_\psi f(x+u,\xi),$$

and similarly for $V_\psi g$. For the left-hand side of (5.68), observe that

$$R(f_\alpha, g_\alpha)(x,\xi) = \overline{\langle \xi, x\rangle}\mathcal{F}_2(f_\alpha \otimes \overline{g}_\alpha)(x,\xi).$$

The partial Fourier transform $\mathcal{F}_2$ is a topological isomorphism from $\mathcal{S}_0(\mathcal{G}\times\mathcal{G})$ onto $\mathcal{S}_0(\mathcal{G}\times\widehat{\mathcal{G}})$ and from $\mathcal{S}_0'(\mathcal{G}\times\mathcal{G})$ onto $\mathcal{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})$. Write $\mathbf{x} = (x,\xi)$ and $\boldsymbol{\omega} = (\omega, u)$,

$$\begin{aligned}\lim_{\alpha\in A} V_{R(\psi,\psi)}R(f_\alpha, g_\alpha)(\mathbf{x},\boldsymbol{\omega}) &= \lim_{\alpha\in A}\langle \overline{\langle\cdot,\cdot\rangle}\mathcal{F}_2(f_\alpha\otimes \overline{g}_\alpha), \pi(\mathbf{x},\boldsymbol{\omega})R(\psi,\psi)\rangle \\ &= \langle \overline{\langle\cdot,\cdot\rangle}\mathcal{F}_2(f\otimes\overline{g}), \pi(\mathbf{x},\boldsymbol{\omega})R(\psi,\psi)\rangle \\ &= V_{R(\psi,\psi)}R(f,g)(\mathbf{x},\boldsymbol{\omega}),\end{aligned}$$

being $R(\psi,\psi) \in \mathcal{S}_0(\mathcal{G}\times\widehat{\mathcal{G}})$. This concludes the proof. □

**Proposition 5.2.2.** ([8, Proposition 4.5]) *Consider $p, q, p_i, q_i \in (0,\infty]$, $i = 1, 2$, such that*

$$p_i, q_i \le q, \quad i = 1,2; \tag{5.69}$$

$$\min\left\{\frac{1}{p_1}+\frac{1}{p_2}, \frac{1}{q_1}+\frac{1}{q_2}\right\} \ge \frac{1}{p}+\frac{1}{q}. \tag{5.70}$$

*Let $v$ be a even submultiplicative weight bounded from below on $\mathcal{G}\times\widehat{\mathcal{G}}$, and $\mathcal{J}$ the isomorphism in (2.98). For $g \in M^{p_1,q_1}_v(\mathcal{G})$ and $f \in M^{p_2,q_2}_v(\mathcal{G})$, we have $R(g,f) \in M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, with*

$$\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}} \lesssim \|g\|_{M^{p_1,q_1}_v}\|f\|_{M^{p_2,q_2}_v}. \tag{5.71}$$

*Proof.* Consider $\psi \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$, $f \in M_v^{p_2,q_2}(\mathcal{G})$, $g \in M_v^{p_1,q_1}(\mathcal{G})$. By Lemma 5.2.1 the STFT of the Rihaczek distribution is given by

$$V_{R(\psi,\psi)}R(g,f)((x,\xi),(\omega,u)) = \overline{\langle \xi, u\rangle} V_\psi g(x,\xi+\omega)\overline{V_\psi f(x+u,\xi)}, \tag{5.72}$$

with $x,u \in \mathcal{G}$ and $\xi,\omega \in \widehat{\mathcal{G}}$. Corollary 5.1.11 shows that $R(\psi,\psi) \in \mathscr{A}_{\widetilde{1\otimes v\circ\mathcal{J}^{-1}}}(\mathcal{G}\times\widehat{\mathcal{G}})$. Consider $V_{\mathcal{G}} \subseteq \mathcal{G}$ and $V_{\widehat{\mathcal{G}}} \subseteq \widehat{\mathcal{G}}$ open, relatively compact, unit neighbourhoods. According to the notation in (5.9) we define

$$V_{1,2} = V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}}, \quad V_{2,1} \coloneqq V_{\widehat{\mathcal{G}}} \times V_{\mathcal{G}}, \quad O \coloneqq V_{1,2} \times V_{2,1} \times \mathbb{T}. \tag{5.73}$$

Set

$$H_g((x,\xi),(\omega,u),\tau) \coloneqq V_\psi g(x,\xi+\omega) \quad \text{and} \quad H_f((x,\xi),(\omega,u),\tau) \coloneqq \overline{V_\psi f(x+u,\xi)},$$

which are functions on the Heisenberg group associated to $\mathcal{G}\times\widehat{\mathcal{G}}$. Notice

$$\mathsf{M}_O[\overline{\tau} V_{R(\psi,\psi)}R(g,f)] = \mathsf{M}_O[H_g \cdot H_f] \le \mathsf{M}_O[H_g]\cdot \mathsf{M}_O[H_f].$$

We compute

$$\begin{aligned}
\mathsf{M}_O[H_g]((x,\xi),(\omega,u),\tau) &= \operatorname*{ess\,sup}_{\substack{((y,\eta),(\nu,z),s)\in\\ ((x,\xi),(\omega,u),\tau)O}} |V_\psi g(y,\eta+\nu)| \\
&= \operatorname*{ess\,sup}_{\nu\in\omega+V_{\widehat{\mathcal{G}}}} \operatorname*{ess\,sup}_{(y,\eta)\in(x,\xi)+V_{1,2}} \left|T_{(e,-\nu)}V_\psi g(y,\eta)\right| \\
&= \operatorname*{ess\,sup}_{\nu\in\omega+V_{\widehat{\mathcal{G}}}} \left(\mathsf{M}_{V_{1,2}}[T_{(e,-\nu)}V_\psi g](x,\xi)\right) \\
&= \operatorname*{ess\,sup}_{\nu\in\omega+V_{\widehat{\mathcal{G}}}} \left(T_{(e,-\nu)}[\mathsf{M}_{V_{1,2}}V_\psi g(x,\xi)]\right) \\
&= \operatorname*{ess\,sup}_{\nu\in\omega+V_{\widehat{\mathcal{G}}}} \left(\mathsf{M}_{V_{1,2}}V_\psi g(x,\xi+\nu)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathsf{M}_O[H_f]((x,\xi),(\omega,u),\tau) &= \operatorname*{ess\,sup}_{z\in u+V_{\mathcal{G}}} \left(T_{(-z,\hat{e})}[\mathsf{M}_{V_{1,2}}V_\psi f(x,\xi)]\right) \\
&= \operatorname*{ess\,sup}_{z\in u+V_{\mathcal{G}}} \left(\mathsf{M}_{V_{1,2}}V_\psi f(x+z,\xi)\right).
\end{aligned}$$

By the modulation spaces independence of the window in $\mathscr{A}_{\widetilde{1\otimes v\circ\mathcal{J}^{-1}}}(\mathcal{G}\times\widehat{\mathcal{G}})$, we can write

$$\begin{aligned}
\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}} &\asymp \Bigg(\int_{\widehat{\mathcal{G}}\times\mathcal{G}\times\mathbb{T}}\Bigg(\int_{\mathcal{G}\times\widehat{\mathcal{G}}}\big|\mathsf{M}_O[\overline{\tau}V_{R(\psi,\psi)}R(g,f)]((x,\xi),(\omega,u),\tau)\big|^p\,dxd\xi\Bigg)^{\frac{q}{p}} \\
&\qquad\qquad \times v^q\circ\mathcal{J}^{-1}(\omega,u)\,d\omega dud\tau\Bigg)^{\frac{1}{q}} \\
&\le \Bigg(\int_{\widehat{\mathcal{G}}\times\mathcal{G}}\Bigg(\int_{\mathcal{G}\times\widehat{\mathcal{G}}}\operatorname*{ess\,sup}_{\nu\in\omega+V_{\widehat{\mathcal{G}}}}\big(\mathsf{M}_{V_{1,2}}V_\psi g(x,\xi+\nu)\big)^p \\
&\qquad \times\operatorname*{ess\,sup}_{z\in u+V_{\mathcal{G}}}\big(\mathsf{M}_{V_{1,2}}V_\psi f(x+z,\xi)\big)^p\,dxd\xi\Bigg)^{\frac{q}{p}} v^q\circ\mathcal{J}^{-1}(\omega,u)\,d\omega du\Bigg)^{\frac{1}{q}} \\
&= \Bigg(\int_{\widehat{\mathcal{G}}\times\mathcal{G}}\Bigg(\operatorname*{ess\,sup}_{(\nu,z)\in(\omega,u)+V_{2,1}}\int_{\mathcal{G}\times\widehat{\mathcal{G}}}\mathsf{M}_{V_{1,2}}V_\psi g(x,\xi+\nu)^p\mathsf{M}_{V_{1,2}}V_\psi f(x+z,\xi)^p\,dxd\xi\Bigg)^{\frac{q}{p}} \\
&\qquad \times v^q\circ\mathcal{J}^{-1}(\omega,u)\,d\omega du\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

The inner integral can be rephrased using the left-right invariance of Haar measure and the involution $h^*(\cdot) \coloneqq \overline{h(-\cdot)}$ as follows:

$$\begin{aligned}
&\int_{\mathcal{G}\times\widehat{\mathcal{G}}}\mathsf{M}_{V_{1,2}}V_\psi g(x,\xi+\nu)^p\mathsf{M}_{V_{1,2}}V_\psi f(x+z,\xi)^p\,dxd\xi \\
&\quad= \int_{\mathcal{G}\times\widehat{\mathcal{G}}}\mathsf{M}_{V_{1,2}}V_\psi g(x',\xi')^p\mathsf{M}_{V_{1,2}}V_\psi f((x',\xi')+(z,-\nu))^p\,dx'd\xi' \\
&\quad= \int_{\mathcal{G}\times\widehat{\mathcal{G}}}(\mathsf{M}_{V_{1,2}}V_\psi g)^*(x'',\xi'')^p\mathsf{M}_{V_{1,2}}V_\psi f((z,-\nu)-(x'',\xi''))^p\,dx''d\xi'' \\
&\quad= (\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}*(\mathsf{M}_{V_{1,2}}V_\psi f)^p(z,-\nu) \\
&\quad= (\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}*(\mathsf{M}_{V_{1,2}}V_\psi f)^p\circ\mathcal{J}^{-1}(\nu,z).
\end{aligned}$$

Whence, using [147, Lemma 2.3.23], we majorize

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\nu,z)\in(\omega,u)+V_{2,1}}(\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}*(\mathsf{M}_{V_{1,2}}V_\psi f)^p\circ\mathcal{J}^{-1}(\nu,z) \\
&\qquad= \operatorname*{ess\,sup}_{(z',\nu')\in\mathcal{J}^{-1}(\omega,u)+\mathcal{J}^{-1}V_{2,1}}(\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}*(\mathsf{M}_{V_{1,2}}V_\psi f)^p(z',\nu') \\
&\qquad= \mathsf{M}_{\mathcal{J}^{-1}V_{2,1}}[(\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}*(\mathsf{M}_{V_{1,2}}V_\psi f)^p](\mathcal{J}^{-1}(\omega,u)) \\
&\qquad\le [\mathsf{M}_{\mathcal{J}^{-1}V_{2,1}}[(\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}]*(\mathsf{M}_{V_{1,2}}V_\psi f)^p](\mathcal{J}^{-1}(\omega,u)) \\
&\qquad= [\mathsf{M}_{\mathcal{J}^{-1}V_{2,1}}[(\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}]*(\mathsf{M}_{V_{1,2}}V_\psi f)^p]\circ\mathcal{J}^{-1}(\omega,u).
\end{aligned}$$

Setting $U \coloneqq -\mathcal{J}^{-1}V_{2,1}+V_{1,2}$, which is an open, relatively compact, unit neighbourhood, we obtain

$$\begin{aligned}\mathsf{M}_{\mathcal{J}^{-1}V_{2,1}}[(\mathsf{M}_{V_{1,2}}V_\psi g)^{*\,p}](u,\omega) &= \operatorname*{ess\,sup}_{(y,\eta)\in(u,\omega)+\mathcal{J}^{-1}V_{2,1}} \ \operatorname*{ess\,sup}_{(x,\xi)\in-(y,\eta)+V_{1,2}} |V_\psi g(x,\xi)|^p \\ &\le \operatorname*{ess\,sup}_{(y,\eta)\in(u,\omega)+\mathcal{J}^{-1}V_{2,1}} \ \operatorname*{ess\,sup}_{\substack{(x,\xi)\in\\ -(u,\omega)-\mathcal{J}^{-1}V_{2,1}+V_{1,2}}} |V_\psi g(x,\xi)|^p \\ &= \operatorname*{ess\,sup}_{\substack{(x,\xi)\in\\ -(u,\omega)-\mathcal{J}^{-1}V_{2,1}+V_{1,2}}} |V_\psi g(x,\xi)|^p \\ &= (\mathsf{M}_U V_\psi g(-u,-\omega))^p = ([\mathsf{M}_U V_\psi g(u,\omega)]^*)^p\,.\end{aligned}$$

Observe that for positive functions $h, l$ on $\mathcal{G}\times\widehat{\mathcal{G}}$ and $v$ a submultiplicative weight we can write

$$((h*l)v)(x,\xi) \le (hv*lv)(x,\xi), \qquad (x,\xi)\in\mathcal{G}\times\widehat{\mathcal{G}}, \tag{5.74}$$

moreover $v^p$ is submultiplicative as well. Therefore

$$\begin{aligned}\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}} &\lesssim \Bigg(\int_{\widehat{\mathcal{G}}\times\mathcal{G}} \Big([([\mathsf{M}_U V_\psi g]^*)^p * (\mathsf{M}_{V_{1,2}}V_\psi f)^p]\circ\mathcal{J}^{-1}(\omega,u)\Big)^{\frac{q}{p}} \\ &\qquad\qquad \times v^q\circ\mathcal{J}^{-1}(\omega,u)\,d\omega du\Bigg)^{\frac{1}{q}} \\ &\le \left(\int_{\widehat{\mathcal{G}}\times\mathcal{G}} \big(([\mathsf{M}_U V_\psi g]^*\cdot v)^p * (\mathsf{M}_{V_{1,2}}V_\psi f\cdot v)^p\big)^{\frac{q}{p}}\circ\mathcal{J}^{-1}(\omega,u)\,d\omega du\right)^{\frac{1}{q}} \\ &= \left\|([\mathsf{M}_U V_\psi g]^*\cdot v)^p * (\mathsf{M}_{V_{1,2}}V_\psi f\cdot v)^p\right\|^{1/p}_{L^{q/p}(\mathcal{G}\times\widehat{\mathcal{G}})}.\end{aligned}$$

By Young's convolution inequality and following the same argumenta as in the proofs in [31, Theorem 3.1] and [25, Theorem 4] for the Euclidean case (replacing the Wigner distribution with the Rihaczek) we infer the estimate

$$\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}} \lesssim \|g\|_{M^{p_1,q_1}_v}\,\|f\|_{M^{p_2,q_2}_v}\,, \tag{5.75}$$

with indices satisfying the conditions (5.69) and (5.70). Following the patterns of [25, 31] the same result is obtained when $p=\infty$ or $q=\infty$. □

The boundedness properties of the Rihaczek distributions enter the study of Kohn-Nirenberg pseudo-differential operators $\mathrm{Op}_0(\sigma)$, defined in (2.216) and (2.217), in the same fashion of [88].

The boundedness result for Weyl operators in the Euclidean setting [31, Theorem 5.1] can be written for Kohn-Nirenberg operators on groups as follows.

**Theorem 5.2.3.** ([8, Theorem 4.6]) *Consider $p,q,p_i,q_i\in[1,\infty]$, $i=1,2$, such that:*

$$q\le\min\{p_1',q_1',p_2,q_2\}; \tag{5.76}$$

$$\min\left\{\frac{1}{p_1}+\frac{1}{p_2'},\frac{1}{q_1}+\frac{1}{q_2'}\right\}\ge\frac{1}{p'}+\frac{1}{q'}. \tag{5.77}$$

*Consider $v$ submultiplicative weight even and bounded from below on $\mathcal{G}\times\widehat{\mathcal{G}}$. If $\sigma\in M^{p,q}_{1\otimes\frac{1}{v}\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, then $\mathrm{Op}_0(\sigma)$ is a bounded operator from $M^{p_1,q_1}_v(\mathcal{G})$ into $M^{p_2,q_2}_{1/v}(\mathcal{G})$ with estimate*

$$\|\mathrm{Op}_0(\sigma)f\|_{M^{p_2,q_2}_{1/v}} \lesssim \|\sigma\|_{M^{p,q}_{1\otimes\frac{1}{v}\circ\mathcal{J}^{-1}}}\,\|f\|_{M^{p_1,q_1}_v}\,. \tag{5.78}$$

*Proof.* It follows by duality using Proposition 5.2.2 and the weak definition of $\mathrm{Op}_0(\sigma)$ in (2.217). □

### 5.2.2 Gabor frames on quasi-lattices

The key tool in the boundedness properties of Kohn-Nirenberg operators on quasi-Banach modulation spaces is the Gabor frame theory. For a detailed treatment of frame theory see, e.g., [21]. In what follows we shall recall and prove new properties for Gabor frames on a LCA group. As a byproduct, we generalize the convolution relations for modulation spaces firstly given in [9, Proposition 3.1], see Proposition 5.2.14.

**Lemma 5.2.4.** *$\mathcal{G}_0/\mathcal{K}$ and $\widehat{\mathcal{G}}_0/\mathcal{K}^\perp$ are discrete.*

*Proof.* We show only the case $\mathcal{G}_0/\mathcal{K}$, the remaining one is identical. Let $p_0\colon \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{K}$ be the projection and consider a point $[x] \in \mathcal{G}_0/\mathcal{K}$. Then $p_0^{-1}([x]) = x + \mathcal{K}$ is open since $\mathcal{K}$ is. Therefore, being every point open, the space is discrete. □

**Lemma 5.2.5.** ([8, Lemma 4.7]) *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice as in* (2.129)*. Then $\Lambda$ is a relatively separated family.*

*Proof.* We use Lemma 5.1.24. The fact that $A_{1,2}\mathbb{Z}^{2d}$ is relatively separated in $\mathbb{R}^{2d}$ is trivial. We only have to show that $D_1$ is relatively separated in $\mathcal{G}_0$; $D_2$ is treated similarly. For a fixed compact set $Q_0 \subseteq \mathcal{G}_0$ we have to show that

$$C_{D_1,Q_0} = \sup_{x\in D_1} \#\{y \in D_1 \mid (x + Q_0) \cap (y + Q_0) \neq \varnothing\} < +\infty.$$

Since $Q_0$ is compact and $\mathcal{K}$ is an open subgroup, there exist $q_1, \ldots, q_n \in Q_0$ such that

$$Q_0 \subseteq \bigcup_{i=1}^{n} (q_i + \mathcal{K}) =: Q_0'.$$

For $x, y \in D_1$, if $(x + Q_0)\cap(y + Q_0) \neq \varnothing$ then $(x + Q_0')\cap(y + Q_0') \neq \varnothing$, hence $C_{D_1,Q_0} \leq C_{D_1,Q_0'}$. Assume that $(x + Q_0') \cap (y + Q_0') \neq \varnothing$, then there are $i_0, j_0 \in \{1, \ldots, n\}$ and $k_{i_0}, k_{j_0} \in \mathcal{K}$ such that

$$x + q_{i_0} + k_{i_0} = y + q_{j_0} + k_{j_0} \quad \Leftrightarrow \quad y = x + q_{i_0} - q_{j_0} + k_{i_0} - k_{j_0}.$$

Fix $x \in D_1$, quotienting by $\mathcal{K}$,

$$[y]^\bullet = [x + q_{i_0} - q_{j_0}]^\bullet \quad \Rightarrow \quad \#\{y \in D_1 \mid (x + Q_0') \cap (y + Q_0') \neq \varnothing\} \leq n^2, \tag{5.79}$$

where $[y]^\bullet$ denotes the projection of $y \in \mathcal{G}_0$ onto the quotient $\mathcal{G}_0/\mathcal{K}$. This proves $C_{D_1,Q_0} \leq C_{D_1,Q_0'} < +\infty$. The desired result follows now from Lemma 5.1.24. □

**Corollary 5.2.6.** ([8, Corollary 4.8]) *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice. Then $\Lambda$ is at most countable.*

*Proof.* We use the fact that $\Lambda$ is a relatively separated family and Lemma 2.2.17. □

In the following issue about the existence of a particular BUPU, we use the quasi-lattice $\Lambda$ both as localizing family and as indexes' set. The argument was presented in [57, Remark 2.5].

**Lemma 5.2.7.** ([8, Lemma 4.9]) *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice as in* (2.129) *with fundamental domain $U$ as in* (2.130)*. Then there exist two open, relatively compact, unit neighbourhoods $Q$ and $V_{1,2}$ in $\mathcal{G} \times \widehat{\mathcal{G}}$, where $V_{1,2}$ as in* (5.9)*, such that $Q \subsetneq V_{1,2}$ and there is a $V_{1,2}$-BUPU*

$$\{\psi_w \otimes \gamma_\mu\}_{(w,\mu)\in\Lambda}$$

*with localizing family $\Lambda$ and such that for every $(w, \mu) \in \Lambda$*

$$\psi_w \otimes \gamma_\mu \equiv 1 \qquad on \qquad (w, \mu) + Q.$$

**Remark 5.2.8.** ([8, Remark 4.10]) *Without loss of generality, the unit neighbourhood $Q$ of the previous lemma can be chosen such that*

$$(\{0_{\mathbb{R}^d}\} \times \mathcal{K}) \times \big(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^{\perp}\big) \subsetneq Q. \tag{5.80}$$

*Therefore for every $(w,\mu) \in \Lambda = \Lambda_1 \times \Lambda_2$ we have*

$$\psi_w \equiv 1 \quad on \quad w + (\{0_{\mathbb{R}^d}\} \times \mathcal{K}), \qquad \gamma_\mu \equiv 1 \quad on \quad \mu + \big(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^{\perp}\big).$$

The following is a consequence of Theorem 2.4.26.

**Corollary 5.2.9.** ([8, Corollary 4.13]) *There exists $\alpha \in (0,1)$ such that the Gabor frame $\{\pi(\mathbf{w})\varphi \,|\, \mathbf{w} \in \Lambda\}$ defined in (2.137) admits a dual window $h \in \mathscr{A}_{\tilde{v}}$.*

*Proof.* We first tackle the problem of finding a dual window. The proof is similar to that in [88, Theorem 2.7]. We distinguish three cases.

*Case $\mathcal{G} = \mathbb{R}^d$.* In this case the frame we are considering is

$$\{\pi(w_1,\mu_1)e^{-\pi x_1^2},\, (w_1,\mu_1) \in \alpha\mathbb{Z}^{2d}\}, \quad \alpha \in (0,1). \tag{5.81}$$

We fix $\alpha$ such that $\alpha^{2d} < (d+1)^{-1}$. Then the existence of a dual window $\gamma_0$ for the Gabor frame generated by the first Hermite function $H_0$ (the Gaussian) was proved by K. Gröchenig and Y. Lyubarskii, see [85, 86]. In particular in [86, Remarks 2] was observed that $\gamma_0$ belongs to the Gelfand-Shilov space $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$, cf. [76].

*Case $\mathcal{G} = \mathcal{G}_0$.* In this case the frame that we are dealing with is the orthonormal basis for $L^2(\mathcal{G}_0)$

$$\{\pi(w_2,\mu_2)\chi_{\mathcal{K}}(x_2),\, (w_2,\mu_2) \in D_{1,2}\}. \tag{5.82}$$

Therefore $\chi_{\mathcal{K}}$ is a dual window itself.

*Case $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$.* The frame in this case is the tensor product of the previous ones:

$$\{\pi(\mathbf{w})\varphi,\, \mathbf{w} = ((w_1,w_2),(\mu_1,\mu_2)) \in \Lambda = \big(\alpha\mathbb{Z}^d \times D_1\big) \times \big(\alpha\mathbb{Z}^d \times D_2\big)\}, \tag{5.83}$$

where $\varphi(x_1,x_2) = e^{-\pi x_1^2}\chi_{\mathcal{K}}(x_2) = (\varphi_1 \otimes \varphi_2)(x_1,x_2)$. Recall that the functions of the type $f_1 \otimes f_2$, with $f_1 \in L^2(\mathbb{R}^d)$ and $f_2 \in L^2(\mathcal{G}_0)$, are dense in $L^2(\mathbb{R}^d \times \mathcal{G}_0)$. Let us show that

$$h(x_1,x_2) \coloneqq (\gamma_0 \otimes \chi_{\mathcal{K}})(x_1,x_2) \tag{5.84}$$

is a dual window. In fact,

$$\begin{aligned} &\sum_{\mathbf{w}\in\Lambda} \langle f_1 \otimes f_2, \pi(\mathbf{w})\varphi\rangle \pi(\mathbf{w})\gamma_0 \otimes \chi_{\mathcal{K}} \\ &\quad = \sum_{(w_1,\mu_1)} \langle f_1, \pi(w_1,\mu_1)\varphi_1\rangle \pi(w_1,\mu_1)\gamma_0 \sum_{(w_2,\mu_2)} \langle f_2, \pi(w_2,\mu_2)\varphi_2\rangle \pi(w_2,\mu_2)\chi_{\mathcal{K}} \\ &\quad = f_1 \otimes f_2; \end{aligned}$$

similarly,

$$\sum_{\mathbf{w}\in\Lambda} \langle f_1 \otimes f_2, \pi(\mathbf{w})\gamma_0 \otimes \chi_{\mathcal{K}}\rangle \pi(\mathbf{w})\varphi = f_1 \otimes f_2.$$

The claim follows by density argument.

We now prove that $h \in \mathscr{A}_{\tilde{v}}$ in the general case $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$. Similarly to the wavelet transform of the generalized Gaussian $\varphi$ in (2.94), see (2.100), we obtain

$$W_h^\varrho h(x,\xi,\tau) = \overline{\tau} c(\mathcal{K}) V_{\gamma_0}\gamma_0 \otimes \chi_{\mathcal{K}\times\mathcal{K}^\perp}(x,\xi). \tag{5.85}$$

Since $V_{\gamma_0}\gamma_0 \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d})$, see e.g. [11, Theorem 2.13], calculations similar to the ones performed in Lemma 5.1.10 yield the desired result. □

**Lemma 5.2.10.** ([8, Lemma 4.14]) *Let $\Lambda = \alpha\mathbb{Z}^{2d} \times D_{1,2}$, $\alpha \in \mathbb{R}$, be a quasi-lattice in $\mathcal{G} \times \widehat{\mathcal{G}}$. Consider the function*

$$\varphi^\circ(x) := \varphi^\circ(x_1,x_2) := 2^{-\frac{d}{2}} meas(\mathcal{K}) e^{-\frac{\pi}{2}x_1^2} \otimes \chi_{\mathcal{K}}(x_2) \in \mathscr{A}_{\tilde{v}} \tag{5.86}$$

*for $x = (x_1,x_2) \in \mathbb{R}^d \times \mathcal{G}_0 \cong \mathcal{G}$, where $meas(\mathcal{K})$ is the (finite) measure of the compact open closed subgroup $\mathcal{K}$ in $\mathcal{G}_0$. Then there exist $\alpha \in (0,1)$ and a function $h^\circ \in \mathscr{A}_{\tilde{v}}$ such that*

$$\{\pi(\mathbf{w})\varphi^\circ \mid \mathbf{w} \in \Lambda\} \tag{5.87}$$

*is a Gabor frame for $L^2(\mathcal{G})$ with dual window $h^\circ$.*

*Proof.* The result is obtained using the same arguments as in Theorem 2.4.26 and Corollary 5.2.9, combined with [35, Lemma 3.2.2]. □

**Theorem 5.2.11.** ([8, Theorem 4.15]) *Let $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$ be a quasi-lattice with fundamental domain $U$. Consider $0 < p,q \leq \infty$, $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$ and $g \in \mathscr{A}_{\tilde{v}}$. Then the coefficient operator $\mathcal{C}_g$ admits a unique continuous and linear extension*

$$\mathcal{C}_g \colon M_m^{p,q}(\mathcal{G}) \to \ell_{m_\Lambda}^{p,q}(\Lambda), \tag{5.88}$$

*where $m_\Lambda$ is the restriction of $m$ to $\Lambda$. Moreover, if $0 < \delta \leq \infty$ is such that $0 < \delta \leq \min\{p,q\} \leq \infty$, then there is a constant $C = C(\delta) > 0$, such that*

$$|||\mathcal{C}_g|||_{M_m^{p,q}\to\ell_{m_\Lambda}^{p,q}} \leq C$$

*for all $p,q \geq \delta$. The constant $C = C(\delta)$ may depend on other elements, but not on $p$ and $q$.*

*Proof.* Consider $f \in M_m^{p,q}(\mathcal{G})$. Let $\{\psi_w \otimes \gamma_\mu\}_{(w,\mu)\in\Lambda}$ be the BUPU on $\mathcal{G}\times\widehat{\mathcal{G}}$ constructed in Lemma 5.2.7. Since tensor product of BUPUs is a BUPU (Lemma 5.1.25) it follows that $\{\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}\}_{(w,\mu)\in\Lambda}$ is a $V$-BUPU on $\mathbb{H}_{\mathcal{G}}$, $V$ as in (5.7), with localizing family $\mathfrak{X} = \Lambda \times \{1\}$ and such that

$$(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}})(w,\mu,1) = 1 \qquad \forall\,(w,\mu) \in \Lambda. \tag{5.89}$$

Hence

$$|\langle f, \pi(w,\mu)g\rangle| = \left|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}})(w,\mu,1)\cdot W_g^\varrho f(w,\mu,1)\right| \leq \left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}})\cdot W_g^\varrho f\right\|_{L^\infty}.$$

By Lemma 5.1.26,

$$\begin{aligned}
\|\mathcal{C}_g f\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)} &= \left\|(\langle f, \pi(w,\mu)g\rangle)_{(w,\mu)\in\Lambda}\right\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)} \\
&\leq \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}})\cdot W_g^\varrho f\right\|_{L^\infty}\right)_{(w,\mu)\in\Lambda}\right\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)} \\
&\asymp \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}})\cdot W_g^\varrho f\right\|_{L^\infty}\right)_{(w,\mu)\in\Lambda}\right\|_{(L_{\tilde{m}}^{p,q})_d(\mathfrak{X},V)} \\
&\asymp \left\|W_g^\varrho f\right\|_{W(L_{\tilde{m}}^{p,q})} = \|f\|_{M_m^{p,q}},
\end{aligned}$$

where in the last equivalence we used Lemma 5.1.23, see also (2.26). The last claim comes from Lemma 5.1.23 and Corollary 5.1.28. □

**Theorem 5.2.12.** ([8, Theorem 4.16]) *Let $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$ be a quasi-lattice with fundamental domain $U$. Consider $0 < p, q \le \infty$, $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$ and $g \in \mathscr{A}_{\tilde{v}}$. Then the synthesis operator $\mathcal{C}_g^*$ admits a unique continuous and linear extension*

$$\mathcal{C}_g^* \colon \ell^{p,q}_{m_\Lambda}(\Lambda) \to M^{p,q}_m(\mathcal{G}), \tag{5.90}$$

*where $m_\Lambda$ is the restriction of $m$ to $\Lambda$. If $p, q \neq \infty$, then the series representing $\mathcal{C}_g^*(c)$ converges unconditionally in $M^{p,q}_m(\mathcal{G})$. Otherwise $\mathcal{C}_g^*(c)$ w-$*$-converges in $M^\infty_{1/v}(\mathcal{G})$. Moreover, if $0 < \delta \le \infty$ is such that $0 < \delta \le \min\{p, q\} \le \infty$, then there is a constant $C = C(\delta) > 0$, such that*

$$|||\mathcal{C}_g^*|||_{\ell^{p,q}_{m_\Lambda}\to M^{p,q}_m} \le C$$

*for all $p, q \ge \delta$. The constant $C = C(\delta)$ may depend on other elements, but not on $p$ and $q$.*

*Proof.* The proof follows the pattern displayed in [75]. Let $(x, \xi, \tau) \in \mathbb{H}_\mathcal{G}$ and $c = (c_\mathbf{w})_{\mathbf{w}\in\Lambda} \in \ell^{p,q}_{m_\Lambda}(\Lambda)$, then we write

$$\begin{aligned}\left|W^g_g[\mathcal{C}_g^*(c)](x,\xi,\tau)\right| &= \left|V_g\left[\sum_{\mathbf{w}\in\Lambda} c_\mathbf{w}\pi(\mathbf{w})g\right](x,\xi)\right| = \left|\sum_{\mathbf{w}\in\Lambda} c_\mathbf{w} V_g\pi(\mathbf{w})g(x,\xi)\right|\\ &\le \sum_{\mathbf{w}\in\Lambda}|c_\mathbf{w}|\,|T_\mathbf{w}V_g g(x,\xi)| =: F^g_c(x,\xi,\tau).\end{aligned}$$

Let $\{\psi_w\otimes\gamma_\mu\}_{(w,\mu)\in\Lambda}$ be the $V_{1,2}$-BUPU on $\mathcal{G}\times\widehat{\mathcal{G}}$ constructed in Lemma 5.2.7. Then $\{\psi_w\otimes\gamma_\mu\otimes\chi_\mathbb{T}\}_{(w,\mu)\in\Lambda}$ is a $V$-BUPU, $V$ as in (5.7), on $\mathbb{H}_\mathcal{G}$ with localizing family $\mathfrak{X} = \Lambda\times\{1\}$. Using the norm equivalence in (2.26) and Lemma 5.1.26

$$\begin{aligned}\left\|\mathcal{C}_g^*(c)\right\|_{M^{p,q}_m} \asymp \left\|\mathcal{C}_g^*(c)\right\|_{W_V(L^{p,q}_{\tilde{m}})} &\lesssim \left\|\left(\|(\psi_w\otimes\gamma_\mu\otimes\chi_\mathbb{T})\cdot F^g_c\|_{L^\infty}\right)_{\mathbf{w}\in\Lambda}\right\|_{(L^{p,q}_{\tilde{m}})_d(\mathfrak{X},V)}\\ &\asymp \left\|\left(\|(\psi_w\otimes\gamma_\mu\otimes\chi_\mathbb{T})\cdot F^g_c\|_{L^\infty}\right)_{\mathbf{w}\in\Lambda}\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)}.\end{aligned}$$

We control the latter sequence as follows:

$$\begin{aligned}\|(\psi_w\otimes\gamma_\mu\otimes\chi_\mathbb{T})\cdot F^g_c\|_{L^\infty} &\le \sum_{\mathbf{u}\in\Lambda}|c_\mathbf{u}| \operatorname*{ess\,sup}_{(x,\xi)\in\mathbf{w}+V_{1,2}} |T_\mathbf{u}V_g g(x,\xi)|\\ &= \sum_{\mathbf{w}\in\Lambda}|c_\mathbf{w}|\,\mathsf{M}_{V_{1,2}}V_g g(\mathbf{w}-\mathbf{u})\\ &= \Big((|c_\mathbf{u}|)_\mathbf{u} * (\mathsf{M}_{V_{1,2}}V_g g(\mathbf{u}))_\mathbf{u}\Big)(\mathbf{w}).\end{aligned}$$

We set $t = \min\{1, p\}$ and $s = \min\{1, p, q\}$. Using the convolution relations for the sequences' spaces in [75, Lemma 2.7], we obtain

$$\begin{aligned}\left\|\mathcal{C}_g^*(c)\right\|_{M^{p,q}_m} &\lesssim \left\|\left(\Big((|c_\mathbf{u}|)_\mathbf{u} * (\mathsf{M}_{V_{1,2}}V_g g(\mathbf{u}))_\mathbf{u}\Big)(\mathbf{w})\right)_{\mathbf{w}\in\Lambda}\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)}\\ &\lesssim \|c\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)} \left\|(\mathsf{M}_{V_{1,2}}V_g g(\mathbf{w}))_{\mathbf{w}\in\Lambda.}\right\|_{\ell^{t,s}_{v_\Lambda}(\Lambda)}.\end{aligned}$$

Arguing as in the proof of Theorem 5.2.11 and using Lemma 5.1.26 and (2.26) again

$$\begin{aligned}\left\|(\mathsf{M}_{V_{1,2}}V_g g(\mathbf{w}))_{\mathbf{w}\in\Lambda}\right\|_{\ell^{a,b}_{v_\Lambda}(\Lambda)} &\leq \left\|\left(\left\|(\psi_w\otimes\gamma_\mu\otimes\chi_{\mathbb{T}})\cdot \mathsf{M}_V W^{\varrho}_g g\right\|_{L^\infty}\right)_{\mathbf{w}\in\Lambda}\right\|_{\ell^{t,s}_{v_\Lambda}(\Lambda)}\\ &\asymp \left\|\left(\left\|(\psi_w\otimes\gamma_\mu\otimes\chi_{\mathbb{T}})\cdot \mathsf{M}_V W^{\varrho}_g g\right\|_{L^\infty}\right)_{\mathbf{w}\in\Lambda}\right\|_{(L^{t,s}_{\tilde{v}})_d(\mathfrak{X},V)}\\ &\asymp \left\|\mathsf{M}_V W^{\varrho}_g g\right\|_{W_V(L^{t,s}_{\tilde{v}})} = \left\|\mathsf{M}_V\mathsf{M}_V W^{\varrho}_g g\right\|_{L^{t,s}_{\tilde{v}}}\\ &\leq \left\|\mathsf{M}_{V^2} W^{\varrho}_g g\right\|_{L^{t,s}_{\tilde{v}}} = \left\|W^{\varrho}_g g\right\|_{W_{V^2}(L^{t,s}_{\tilde{v}})},\end{aligned}$$

where we set $V^2 := VV$ (multiplicative notation in $\mathbb{H}_{\mathcal{G}}$). As reported in Remark 2.2.35, for any $0 < r \leq 1$ we have the continuous inclusion

$$W^R(L^\infty, W(L^\infty, L^r_{\tilde{v}})) \hookrightarrow W(L^\infty, L^r_{\tilde{v}}). \tag{5.91}$$

Arguing as in Proposition 5.1.31 and taking $r < \min\{t,s\}$ we obtain

$$W(L^\infty, L^r_{\tilde{v}}) \hookrightarrow W(L^\infty, L^{t,s}_{\tilde{v}}). \tag{5.92}$$

The fact that $g$ is in $\mathscr{A}_{\tilde{v}}$ (defined in (5.21)) implies then

$$\left\|W^{\varrho}_g g\right\|_{W_{V^2}(L^{t,s}_{\tilde{v}})} < +\infty$$

and

$$\left\|\mathcal{C}^*_g(c)\right\|_{M^{p,q}_m} \lesssim \|c\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)}. \tag{5.93}$$

Unconditional convergence for the series defining $\mathcal{C}^*_g(c)$ in $M^{p,q}_m(\mathcal{G})$ if $p,q\neq\infty$, and w-$*$-convergence in $M^{\infty}_{1/v}(\mathcal{G})$ otherwise, is inferred as in [82, Theorem 12.2.4]. The last claim comes from Lemma 5.1.23 and Corollary 5.1.28. □

**Theorem 5.2.13.** ([8, Theorem 4.17]) *Let $0 < p,q \leq \infty$, $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$ and $\varphi$ as in (2.94). Consider $h \in \mathscr{A}_{\tilde{v}}$ such that*

$$S_{h,\varphi} = S_{\varphi,h} = I_{L^2}, \tag{5.94}$$

*for a suitable quasi-lattice $\Lambda = \Lambda_1\times\Lambda_2 \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$. Then*

$$f = \sum_{\mathbf{w}\in\Lambda}\langle f, \pi(\mathbf{w})\varphi\rangle\pi(\mathbf{w})h = \sum_{\mathbf{w}\in\Lambda}\langle f, \pi(\mathbf{w})h\rangle\pi(\mathbf{w})\varphi \tag{5.95}$$

*with unconditional convergence in $M^{p,q}_m(\mathcal{G})$ if $p,q\neq\infty$, and w-$*$-convergence in $M^{\infty}_{1/v}(\mathcal{G})$ otherwise. Moreover, for every $f \in M^{p,q}_m(\mathcal{G})$ we have the following quasi-norm equivalences:*

$$\|f\|_{M^{p,q}_m} \asymp \left(\sum_{\mu\in\Lambda_2}\left(\sum_{w\in\Lambda_1}|V_\varphi f(w,\mu)|^p\, m(w,\mu)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left\|(V_\varphi f(\mathbf{w}))_{\mathbf{w}\in\Lambda}\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)}, \tag{5.96}$$

$$\|f\|_{M^{p,q}_m} \asymp \left(\sum_{\mu\in\Lambda_2}\left(\sum_{w\in\Lambda_1}|V_h f(w,\mu)|^p\, m(w,\mu)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left\|(V_h f(\mathbf{w}))_{\mathbf{w}\in\Lambda}\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)},$$

*and similarly if $p=\infty$ or $q=\infty$.*

*Proof.* The proof is based on the continuity of $\mathcal{C}_\varphi$, $\mathcal{C}_\varphi^*$, $\mathcal{C}_h$ and $\mathcal{C}_h^*$. The pattern is the same of [82, Corollary 12.2.6]. □

Expansions and equivalences analogous to (5.95) and (5.96) hold for $\varphi^\circ$ and $h^\circ$ defined in Lemma 5.2.10.
The following is a generalization of Proposition 2.5.19, which treated treated only the group $\mathbb{R}^d$.

**Proposition 5.2.14.** *Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, define for $x \in \mathcal{G}$ and $\xi \in \widehat{\mathcal{G}}$*

$$m_1(x) := m(x, \hat{e}), \quad v_1(x) := v(x, \hat{e}), \quad v_2(\xi) := v(e, \xi). \tag{5.97}$$

*Let $\nu(\xi) > 0$ be an arbitrary weight function on $\widehat{\mathcal{G}}$ such that*

$$m_1 \otimes \nu, v_1 \otimes v_2\nu^{-1} \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}}). \tag{5.98}$$

*Let $0 < p, q, r, t, u, \gamma \leq \infty$, with*

$$\frac{1}{u} + \frac{1}{t} = \frac{1}{\gamma}, \tag{5.99}$$

*and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \qquad \text{for } 1 \leq r \leq \infty \tag{5.100}$$

*whereas*

$$p = q = r, \qquad \text{for } 0 < r < 1. \tag{5.101}$$

*Then*

$$M^{p,u}_{m_1\otimes\nu}(\mathcal{G}) * M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathcal{G}) \hookrightarrow M^{r,\gamma}_m(\mathcal{G}) \tag{5.102}$$

*with quasi-norm inequality*

$$\|f * g\|_{M^{r,\gamma}_m} \lesssim \|f\|_{M^{p,u}_{m_1\otimes\nu}} \|g\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}}. \tag{5.103}$$

*Proof.* We follow the patter displayed in [9, Proposition 3.1]. A direct computation gives $\varphi * \varphi = \varphi^\circ$, where $\varphi$ is defined in (2.94) and $\varphi^\circ$ in (5.86). Similarly, the following identities can be easily checked:

$$V_h f(x,\xi) = \overline{\langle \xi, x\rangle} \left(f * M_\xi[h^*]\right)(x), \quad M_\xi[\varphi^{\circ *}](x) = \left(M_\xi[\varphi^*] * M_\xi[\varphi^*]\right)(x)$$

(recall the involution $h^*(x) = \overline{h(-x)}$). Using associativity and commutativity of the convolution product we can write

$$V_{\varphi^\circ}(f * g)(x,\xi) = \overline{\langle \xi, x\rangle} \left((f * M_\xi[\varphi^*]) * (g * M_\xi[\varphi^*])\right)(x). \tag{5.104}$$

In what follows we will use the frame expansions in Theorem 5.2.13 with $\varphi^\circ$ in place of $\varphi$, see Lemma 5.2.10. We majorize the weight $m$ by

$$m(\mathbf{w}) = m(w,\mu) \lesssim m(w,\hat{e})v(e,\mu) = m_1(w)v_2(\mu) \qquad \mathbf{w} = (w,\mu) \in \Lambda,$$

use Young's convolution inequality for sequences in the $w$-variable and Hölder's one in the $\mu$-variable. The indices $p, q, r, \gamma, t, u$ fulfil the equalities in the assumptions. We show in details the case when $r, \gamma, t, u < \infty$. The others are similar. Namely,

$$\begin{aligned}
\|f * g\|_{M_m^{r,\gamma}} &\asymp \left\| ((V_{\varphi^\circ}(f * h))(\mathbf{w}) m(\mathbf{w}))_{\mathbf{w}\in\Lambda} \right\|_{\ell^{r,\gamma}(\Lambda)} \\
&= \left( \sum_{\mu\in\Lambda_2} \left( \sum_{w\in\Lambda_1} |V_{\varphi^\circ}(f * g)(w,\mu)|^r \, m(w,\mu)^r \right)^{\frac{\gamma}{r}} \right)^{\frac{1}{\gamma}} \\
&\lesssim \left( \sum_{\mu\in\Lambda_2} \left( \sum_{w\in\Lambda_1} |(f * M_\mu[\varphi^*]) * (g * M_\mu[\varphi^*])(w)|^r \, m_1(w)^r \right)^{\frac{\gamma}{r}} v_2(\mu)^\gamma \right)^{\frac{1}{\gamma}} \\
&= \left( \sum_{\mu\in\Lambda_2} \left\| ((f * M_\mu[\varphi^*]) * (g * M_\mu[\varphi^*])(w))_{w\in\Lambda_1} \right\|^\gamma_{\ell^r_{m_1}(\Lambda_1)} v_2(\mu)^\gamma \right)^{\frac{1}{\gamma}} \\
&\lesssim \left( \sum_{\mu\in\Lambda_1} \left\| ((f * M_\mu[\varphi^*])(w))_{w\in\Lambda_1} \right\|^\gamma_{\ell^p_{m_1}(\Lambda_1)} \left\| ((g * M_\mu[\varphi^*])(w))_{w\in\Lambda_1} \right\|^\gamma_{\ell^q_{v_1}(\Lambda_1)} \right. \\
&\qquad \left. \times\, v_2(\mu)^\gamma \frac{\nu(\mu)^\gamma}{\nu(\mu)^\gamma} \right)^{\frac{1}{\gamma}} \\
&\leq \left( \sum_{\mu\in\Lambda_2} \left\| ((f * M_\mu[\varphi^*])(w))_{w\in\Lambda_1} \right\|^u_{\ell^p_{m_1}(\Lambda_1)} \nu(\mu)^u \right)^{\frac{1}{u}} \\
&\times \left( \sum_{\mu\in\Lambda_2} \left\| ((g * M_\mu[\varphi^*])(w))_{w\in\Lambda_1} \right\|^t_{\ell^q_{m_1}(\Lambda_1)} \frac{v_2(\mu)^t}{\nu(\mu)^t} \right)^{\frac{1}{t}} \\
&= \left\| (V_\varphi f(\mathbf{w}))_{\mathbf{w}\in\Lambda} \right\|_{\ell^{p,u}_{m_1\otimes\nu}(\Lambda)} \left\| (V_\varphi g(\mathbf{w}))_{\mathbf{w}\in\Lambda} \right\|_{\ell^{q,t}_{m_1\otimes v_2\nu^{-1}}(\Lambda)} \\
&\asymp \|f\|_{M^{p,u}_{m_1\otimes\nu}} \|g\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}},
\end{aligned}$$

the last equivalence is (5.96). This concludes the proof. □

Let us introduce the closed and compact subgroups of $\mathcal{G} \times \widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}} \times \mathcal{G}$, respectively:

$$\mathbb{U}(\mathcal{G}) := (\{0_{\mathbb{R}^d}\} \times \mathcal{K}) \times \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp\right), \ \mathbb{U}(\widehat{\mathcal{G}}) := \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp\right) \times (\{0_{\mathbb{R}^d}\} \times \mathcal{K}). \tag{5.105}$$

Given $\mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}$, we will denote its projection on $(\mathcal{G} \times \widehat{\mathcal{G}})/\mathbb{U}(\mathcal{G})$ by

$$\overset{\bullet}{\mathbf{x}} \quad \text{or} \quad [\mathbf{x}]^\bullet,$$

and similarly for the projection of $\boldsymbol{\xi} \in \widehat{\mathcal{G}} \times \mathcal{G}$ onto $(\widehat{\mathcal{G}} \times \mathcal{G})/\mathbb{U}(\widehat{\mathcal{G}})$.
Let $\Lambda = A_{1,2}\mathbb{Z}^{2d} \times D_{1,2} \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ and $\Gamma = A_{3,4}\mathbb{Z}^{2d} \times D_{3,4} \subseteq \widehat{\mathcal{G}} \times \mathcal{G}$ be quasi-lattices, then their projections

$$\mathscr{D}(\mathcal{G}) := \mathscr{D}(\mathcal{G}, A_{1,2}) := \overset{\bullet}{\Lambda} \qquad \text{and} \qquad \mathscr{D}(\widehat{\mathcal{G}}) := \mathscr{D}(\widehat{\mathcal{G}}, A_{3,4}) := \overset{\bullet}{\Gamma} \tag{5.106}$$

are discrete and at most countable LCA groups. Given a distribution $f$ in $\mathcal{R}_{\tilde{v}}$, or $\mathcal{S}_0'$, and a window $g \in \mathscr{A}_{\tilde{v}}$, the function

$$\dot{V}_g f(\dot{\mathbf{x}}) := \sup_{\mathbf{z}\in\mathbb{U}(\mathcal{G})} |V_g f(\mathbf{x}+\mathbf{z})| = \mathsf{M}_{\mathbb{U}(\mathcal{G})} V_g f(\mathbf{x}) \tag{5.107}$$

is well defined on the quotient group $(\mathcal{G}\times\widehat{\mathcal{G}})/\mathbb{U}(\mathcal{G})$. In fact, if $\mathbf{u}$ is such that $\dot{\mathbf{x}} = \dot{\mathbf{u}}$, then there exists $\mathbf{n} \in \mathbb{U}(\mathcal{G})$ such that $\mathbf{u} = \mathbf{x}+\mathbf{n}$. Setting $\mathbf{y} = \mathbf{n}+\mathbf{z} \in \mathbb{U}(\mathcal{G})$ we have

$$\sup_{\mathbf{z}\in\mathbb{U}(\mathcal{G})} |V_g f(\mathbf{u}+\mathbf{z})| = \sup_{\mathbf{z}\in\mathbb{U}(\mathcal{G})} |V_g f(\mathbf{x}+\mathbf{n}+\mathbf{z})| = \sup_{\mathbf{y}\in\mathbb{U}(\mathcal{G})} |V_g f(\mathbf{x}+\mathbf{y})|\,.$$

Similarly, given a weight $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$, the function

$$\dot{m}(\dot{\mathbf{x}}) := \sup_{\mathbf{z}\in\mathbb{U}(\mathcal{G})} m(\mathbf{x}+\mathbf{z}) \tag{5.108}$$

is well defined on the quotient.

**Lemma 5.2.15.** ([8, Lemma 4.19]) *Consider a quasi-lattice $\Lambda$ in $\mathcal{G}\times\widehat{\mathcal{G}}$. Let $g \in \mathscr{A}_{\tilde{v}}$, $0 < p,q \le \infty$, $m \in \mathcal{M}_v(\mathcal{G}\times\widehat{\mathcal{G}})$ and define the mapping*

$$\dot{\mathcal{C}}_g \colon M^{p,q}_m(\mathcal{G}) \to \ell^{p,q}_{\dot{m}}(\mathscr{D}(\mathcal{G})),\ f \mapsto \left(\dot{V}_g f(\dot{\mathbf{w}})\right)_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})}, \tag{5.109}$$

*where the weight $\dot{m}$ is understood to be restricted on $\mathscr{D}(\mathcal{G})$. Then there exists a constant $C > 0$ such that for every $f \in M^{p,q}_m(\mathcal{G})$ we have*

$$\left\|\dot{\mathcal{C}}_g f\right\|_{\ell^{p,q}_{\dot{m}}(\mathscr{D}(\mathcal{G}))} \le C\,\|f\|_{M^{p,q}_m}\,. \tag{5.110}$$

*Proof.* The BUPU $\{\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}},\ \mathbf{w} = (w,\mu) \in \Lambda\}$ coming from Lemma 5.2.7 is such that

$$\psi_w \otimes \gamma_\mu \equiv 1 \qquad \text{on} \qquad \mathbf{w} + \mathbb{U}(\mathcal{G}).$$

Noticing that the projection of $\Lambda$ onto $\mathscr{D}(\mathcal{G})$ is one-to-one we have without ambiguity

$$\dot{V}_g f(\dot{\mathbf{w}}) \le \|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot V_g f\|_{L^\infty} = \left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot W^\varrho_g f\right\|_{L^\infty},$$

where $(w,\mu)$ is the only representative of $\dot{\mathbf{w}}$ in the quasi-lattice. Since $\mathbb{U}(\mathcal{G})$ is compact there exists a constant $C = C(\mathbb{U}(\mathcal{G}), v) > 0$ such that

$$\frac{1}{C} m(\mathbf{x}+\mathbf{z}) \le m(\mathbf{x}) \le C m(\mathbf{x}+\mathbf{z}), \tag{5.111}$$

for every $\mathbf{x} \in \mathcal{G}\times\widehat{\mathcal{G}}$ and $\mathbf{z} \in \mathbb{U}(\mathcal{G})$, see [147, Corollary 2.2.23]. For $\mathbf{x} = \mathbf{w} \in \Lambda$, taking the supremum over $\mathbf{z}$ in $\mathbb{U}(\mathcal{G})$ we can unambiguously write

$$\dot{m}(\dot{\mathbf{w}}) \asymp m(\mathbf{w}). \tag{5.112}$$

All together we have

$$\begin{aligned}\left\|\dot{\mathcal{C}}_g f\right\|_{\ell^{p,q}_{\dot{m}}(\mathscr{D}(\mathcal{G}))} &= \left\|\left(\dot{V}_g f(\dot{\mathbf{w}}) \cdot \dot{m}(\dot{\mathbf{w}})\right)_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})}\right\|_{\ell^{p,q}(\mathscr{D}(\mathcal{G}))} \\ &\lesssim \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot W^\varrho_g f\right\|_{L^\infty} \cdot m(\mathbf{w})\right)_{\mathbf{w}\in\Lambda}\right\|_{\ell^{p,q}(\Lambda)}.\end{aligned}$$

Then we conclude as in the proof of Theorem 5.2.11. □

### 5.2.3 Eigenfunctions of Kohn-Nirenberg operators

We have now all the instruments to study the eigenfunctions for Kohn-Nirenberg operators. Let us first introduce the Gabor matrix of $\mathrm{Op}_0(\sigma)$.

**Definition 5.2.16.** *Consider $g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ and $\sigma \in \mathcal{S}_0'(\mathcal{G} \times \widehat{\mathcal{G}})$. The* ***Gabor matrix of the Kohn-Nirenberg operator*** $\mathrm{Op}_0(\sigma)$ ***(with respect to*** $g$***)*** *is defined by*

$$[M(\sigma)]_{\mathbf{x},\mathbf{y}} := \langle \mathrm{Op}_0(\sigma)\pi(\mathbf{y})g, \pi(\mathbf{x})g\rangle, \qquad \forall \mathbf{x}, \mathbf{y} \in \mathcal{G} \times \widehat{\mathcal{G}}. \tag{5.113}$$

The machinery developed in the previous subsection let us generalize what stated in [9, Thereom 3.3 (i)] (Theorem 3.1.2) for Weyl operators on $\mathbb{R}^d$ and proved separately in [136, Theorem 4.3] and [142, Theorem 3.1]. We will then obtain properties for the eigenfunctions in $L^2(\mathcal{G})$ of $\mathrm{Op}_0(\sigma)$ similar to the ones for Weyl operators on the Euclidean space, cf. [9, Proposition 3.5] (Proposition 3.1.4).

We start with the boundedness properties of Kohn-Nirenberg operators.

**Theorem 5.2.17.** ([8, Theorem 4.21]) *Consider $0 < p, q, \gamma \leq \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\gamma} \tag{5.114}$$

*and a symbol $\sigma \in M^{p,\min\{1,\gamma\}}(\mathcal{G} \times \widehat{\mathcal{G}})$. Then Kohn-Nirenberg operator $\mathrm{Op}_0(\sigma)\colon \mathcal{S}_0(\mathcal{G}) \to \mathcal{S}_0'(\mathcal{G})$ admits a unique linear continuous extension*

$$\mathrm{Op}_0(\sigma)\colon M^q(\mathcal{G}) \to M^\gamma(\mathcal{G}).$$

*Proof.* We distinguish two cases: $\gamma \leq 1$ and $\gamma > 1$.
*Case* $\gamma \leq 1$. Let $\varphi$ be as in (2.94) and consider $h \in \mathscr{A}_{\tilde{v}}$ and a quasi-lattice $\Lambda$ such that $S_{h,\varphi} = S_{\varphi,h} = I_{L^2}$. Write

$$\mathrm{Op}_0(\sigma) = \mathcal{C}_h^* \circ \mathcal{C}_\varphi \circ \mathrm{Op}_0(\sigma) \circ \mathcal{C}_\varphi^* \circ \mathcal{C}_h =: \mathcal{C}_h^* \circ M(\sigma) \circ \mathcal{C}_h. \tag{5.115}$$

We shall prove that the Gabor matrix $M(\sigma)$ is linear and continuous from $\ell^q(\Lambda)$ into $\ell^\gamma(\Lambda)$. It is sufficient to prove that the diagram

$$\begin{array}{ccc} M^q & \xrightarrow{\mathrm{Op}_0(\sigma)} & M^\gamma \\ {\scriptstyle \mathcal{C}_h}\downarrow & & \uparrow{\scriptstyle \mathcal{C}_h^*} \\ \ell^q & \xrightarrow{M(\sigma)} & \ell^\gamma \end{array}$$

is commutative. We show in detail the cases $p < +\infty$ and $q < +\infty$, the others are similar. For $f \in M^q(\mathcal{G})$, using the decomposition in (5.115) and the notation for the Gabor matrix (5.113), we have

$$\begin{aligned} \mathrm{Op}_0(\sigma)f &= \sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda} \langle \mathrm{Op}_0(\sigma)\pi(\mathbf{u})\varphi, \pi(\mathbf{w})\varphi\rangle \langle f, \pi(\mathbf{u})h\rangle \pi(\mathbf{w})h \\ &= \sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda} [M(\sigma)]_{\mathbf{w},\mathbf{u}} \langle f, \pi(\mathbf{u})h\rangle \pi(\mathbf{w})h, \end{aligned}$$

so that

$$M(\sigma)\colon \ell^q(\Lambda) \to \ell^\gamma(\Lambda), \quad (c_{\mathbf{w}})_{\mathbf{w}\in\Lambda} \mapsto \left(\sum_{\mathbf{u}\in\Lambda} [M(\sigma)]_{\mathbf{w},\mathbf{u}}\, c_{\mathbf{u}}\right)_{\mathbf{w}\in\Lambda}.$$

From the weak definition (2.217) and (2.99) we can write each entry of the (discrete) Gabor matrix of $\mathrm{Op}_0(\sigma)$ as follows:

$$\begin{aligned}
[M(\sigma)]_{\mathbf{w},\mathbf{u}} &= \langle \mathrm{Op}_0(\sigma)\pi(\mathbf{u})\varphi, \pi(\mathbf{w})\varphi\rangle \\
&= \langle \sigma, R(\pi(\mathbf{w})\varphi, \pi(\mathbf{u})\varphi)\rangle \\
&= \langle \sigma, \langle \nu, w-u\rangle M_{\mathcal{J}(\mathbf{u}-\mathbf{w})} T_{(w,\nu)} R(\varphi,\varphi)\rangle \\
&= \overline{\langle \nu, w-u\rangle} V_\Phi\sigma\left((w,\nu), \mathcal{J}(\mathbf{u}-\mathbf{w})\right),
\end{aligned}$$

where $\mathbf{w} = (w,\mu)$, $\mathbf{u} = (u,\nu)$ and $\Phi \coloneqq R(\varphi,\varphi) \in \mathscr{A}_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})$. We introduce the mapping

$$\mathsf{T}_0\colon \left(\mathcal{G}\times\widehat{\mathcal{G}}\right) \times \left(\mathcal{G}\times\widehat{\mathcal{G}}\right) \to \mathcal{G}\times\widehat{\mathcal{G}}, \quad ((w,\mu),(u,\nu)) \mapsto (w,\nu) \tag{5.116}$$

and write

$$\left|[M(\sigma)]_{\mathbf{w},\mathbf{u}}\right| = \left|V_\Phi\sigma\left(\mathsf{T}_0(\mathbf{w},\mathbf{u}), \mathcal{J}(\mathbf{u}-\mathbf{w})\right)\right|. \tag{5.117}$$

Since $\gamma \le 1$, we have $\|c\|_{\ell^1} \le \|c\|_{\ell^\gamma}$ and we estimate

$$\begin{aligned}
\|M(\sigma)c\|_{\ell^\gamma(\Lambda)} &= \left(\sum_{\mathbf{w}\in\Lambda}\left|\sum_{\mathbf{u}\in\Lambda}[M(\sigma)]_{\mathbf{w},\mathbf{u}}\, c_{\mathbf{u}}\right|^\gamma\right)^{\frac{1}{\gamma}} \\
&\le \left(\sum_{\mathbf{w}\in\Lambda}\left(\sum_{\mathbf{u}\in\Lambda}\left|[M(\sigma)]_{\mathbf{w},\mathbf{u}}\right| |c_{\mathbf{u}}|\right)^\gamma\right)^{\frac{1}{\gamma}} \\
&\le \left(\sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda}\left|[M(\sigma)]_{\mathbf{w},\mathbf{u}}\right|^\gamma |c_{\mathbf{u}}|^\gamma\right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda}\left|V_\Phi\sigma\left(\mathsf{T}_0(\mathbf{w},\mathbf{u}), \mathcal{J}(\mathbf{u}-\mathbf{w})\right)\right|^\gamma |c_{\mathbf{u}}|^\gamma\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Let us majorize each entry of the matrix as follows:

$$\begin{aligned}
\left|V_\Phi\sigma\left(\mathsf{T}_0(\mathbf{w},\mathbf{u}), \mathcal{J}(\mathbf{u}-\mathbf{w})\right)\right| &\le \sup_{\mathbf{z}\in\mathbb{U}(\mathcal{G}),\boldsymbol{\delta}\in\mathbb{U}(\widehat{\mathcal{G}})} \left|V_\Phi\sigma\left(\mathsf{T}_0(\mathbf{w},\mathbf{u})+\mathbf{z}, \mathcal{J}(\mathbf{u}-\mathbf{w})+\boldsymbol{\delta}\right)\right| \\
&= \overset{\bullet}{V}_\Phi\sigma\left([\mathsf{T}_0(\mathbf{w},\mathbf{u})]^\bullet, [\mathcal{J}(\mathbf{u}-\mathbf{w})]^\bullet\right),
\end{aligned} \tag{5.118}$$

where the function on the quotient group was introduced in (5.107). Fix $\mathbf{w},\mathbf{u}\in\Lambda$ and consider $\mathbf{x} = (x,\xi)$, $\mathbf{y} = (y,\eta)$ such that $\overset{\bullet}{\mathbf{w}} = \overset{\bullet}{\mathbf{x}}$ and $\overset{\bullet}{\mathbf{u}} = \overset{\bullet}{\mathbf{y}}$. Then there exist unique $\mathbf{z} = (z,\zeta) = ((0,z_2),(0,\zeta_2)), \mathbf{n} = (n,\iota) = ((0,n_2),(0,\iota_2)) \in \mathbb{U}(\mathcal{G})$ such that

$$\mathbf{x} = \mathbf{w}+\mathbf{z}, \qquad \mathbf{y} = \mathbf{u}+\mathbf{n}.$$

Therefore

$$\begin{aligned}
\mathsf{T}_0(\mathbf{x},\mathbf{y}) = \mathsf{T}_0(\mathbf{w}+\mathbf{z},\mathbf{u}+\mathbf{n}) &= ((w_1, w_2+z_2),(\nu_1,\nu_2+\iota_2)) \\
&= \mathsf{T}_0(\mathbf{w},\mathbf{u}) + ((0,z_2),(0,\iota_2))
\end{aligned}$$

where $((0,z_2),(0,\iota_2)) \in \mathbb{U}(\mathcal{G})$, so that we have shown

$$\dot{\mathbf{w}} = \dot{\mathbf{x}},\ \dot{\mathbf{u}} = \dot{\mathbf{y}} \qquad \Rightarrow \qquad [\mathsf{T}_0(\mathbf{w},\mathbf{u})]^\bullet = [\mathsf{T}_0(\mathbf{x},\mathbf{y})]^\bullet . \tag{5.119}$$

Similarly,

$$\mathcal{J}(\mathbf{y}-\mathbf{x}) = \mathcal{J}(\mathbf{u}+\mathbf{n}-\mathbf{w}-\mathbf{z}) = \mathcal{J}(\mathbf{u}-\mathbf{w}) + \mathcal{J}(\mathbf{n}-\mathbf{z})$$

and being $\mathcal{J}(\mathbf{n}-\mathbf{z}) \in \mathbb{U}(\widehat{\mathcal{G}})$ we have proved

$$\dot{\mathbf{w}} = \dot{\mathbf{x}},\ \dot{\mathbf{u}} = \dot{\mathbf{y}} \qquad \Rightarrow \qquad [\mathcal{J}(\mathbf{u}-\mathbf{w})]^\bullet = [\mathcal{J}(\mathbf{y}-\mathbf{x})]^\bullet . \tag{5.120}$$

Hence the function in (5.118) depends only on the cosets of $\mathbf{w}$ and $\mathbf{u}$, so that the application

$$\dot{H}(\dot{\mathbf{u}},\dot{\mathbf{w}}) := \dot{V}_\Phi\sigma\left([\mathsf{T}_0(\mathbf{w},\mathbf{u})]^\bullet, [\mathcal{J}(\mathbf{u}-\mathbf{w})]^\bullet\right) \tag{5.121}$$

is well defined. A sequence $c = (c_\mathbf{w})_{\mathbf{w}\in\Lambda}$ on the quasi-lattice $\Lambda$ uniquely determines a sequence on $\mathscr{D}(\mathcal{G}) = \dot{\Lambda}$ simply by

$$\dot{c} := \left(c_{\dot{\mathbf{w}}} := c_\mathbf{w}\right)_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \tag{5.122}$$

with

$$\|c\|_{\ell^q(\Lambda)} = \left\|\dot{c}\right\|_{\ell^q(\mathscr{D}(\mathcal{G}))} .$$

Using Hölder's inequality in the $\dot{\mathbf{u}}$ variable (observe $1/(p/\gamma)+1/(q/\gamma)=1$) and the consideration above:

$$\begin{aligned}
\|M(\sigma)c\|_{\ell^\gamma(\Lambda)} &\leq \left(\sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda} \dot{H}(\dot{\mathbf{u}},\dot{\mathbf{w}})^\gamma |c_\mathbf{u}|^\gamma\right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})}\ \sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \dot{H}(\dot{\mathbf{u}},\dot{\mathbf{w}})^\gamma \left|c_{\dot{\mathbf{u}}}\right|^\gamma\right)^{\frac{1}{\gamma}} \\
&\leq \left(\sum_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \left(\sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \dot{H}(\dot{\mathbf{u}},\dot{\mathbf{w}})^{\gamma\frac{p}{\gamma}}\right)^{\frac{\gamma}{p}} \left(\sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \left|c_{\dot{\mathbf{u}}}\right|^{\gamma\frac{q}{\gamma}}\right)^{\frac{\gamma}{q}}\right)^{\frac{1}{\gamma}} \\
&= \|c\|_{\ell^q(\Lambda)} \left(\sum_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \left(\sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \dot{V}_\Phi\sigma\left([\mathsf{T}_0(\mathbf{w},\mathbf{u})]^\bullet, [\mathcal{J}(\mathbf{u}-\mathbf{w})]^\bullet\right)^p\right)^{\frac{\gamma}{p}}\right)^{\frac{1}{\gamma}} .
\end{aligned}$$

Let us perform the following change of variables:

$$\dot{\boldsymbol{\theta}} := [\mathcal{J}(\mathbf{u}-\mathbf{w})]^\bullet \in \mathscr{D}(\widehat{\mathcal{G}}) = [\mathcal{J}\Lambda]^\bullet . \tag{5.123}$$

Notice that $\mathcal{J}\Lambda \subseteq \widehat{\mathcal{G}}\times\mathcal{G}$ is a quasi-lattice. Then there exists $\boldsymbol{\delta} \in \mathbb{U}(\widehat{\mathcal{G}})$ such that $\boldsymbol{\theta}+\boldsymbol{\delta} = \mathcal{J}(\mathbf{u}-\mathbf{w})$ and

$$\mathbf{u}-\mathbf{w} = \mathcal{J}^{-1}(\boldsymbol{\theta}+\boldsymbol{\delta}) \Rightarrow \mathbf{w} = \mathbf{u} - \mathcal{J}^{-1}(\boldsymbol{\theta}) - \mathcal{J}^{-1}(\boldsymbol{\delta}) \Rightarrow \dot{\mathbf{w}} = [\mathbf{u} - \mathcal{J}^{-1}(\boldsymbol{\theta})]^\bullet,$$

since $-\mathcal{J}^{-1}(\boldsymbol{\delta}) \in \mathbb{U}(\mathcal{G})$. Recalling (5.119) and writing $\boldsymbol{\theta} = (\theta, s) = ((\theta_1, \theta_2), (s_1, s_2)) \in \mathcal{J}\Lambda$, we have

$$\begin{aligned}[\mathsf{T}_0(\mathbf{w}, \mathbf{u})]^\bullet &= [\mathsf{T}_0(\mathbf{u} - \mathcal{J}^{-1}(\boldsymbol{\theta}), \mathbf{u})]^\bullet = [\mathsf{T}_0((u - s, \nu + \theta), (u, \nu))]^\bullet \\ &= [(u - s, \nu)]^\bullet = [\mathbf{u} - (s, \hat{e})]^\bullet.\end{aligned}$$

In the above calculation we can choose as representative of $\dot{\boldsymbol{\theta}}$ the only one in $\mathcal{J}\Lambda$ without loss of generality. In fact, write $\Lambda = (\alpha\mathbb{Z}^d \times D_1) \times (\alpha\mathbb{Z}^d \times D_2)$, $\mathcal{J}\Lambda = (\alpha\mathbb{Z}^d \times -D_2) \times (\alpha\mathbb{Z}^d \times D_1)$, and consider $\boldsymbol{\eta} = (\eta, l) = ((\eta_1, \eta_2), (l_1, l_2))$ such that $\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\eta}}$ and $\boldsymbol{\eta} \notin \mathcal{J}\Lambda$. Being $\mathbb{U}(\widehat{\mathcal{G}}) = (\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp) \times (\{0_{\mathbb{R}^d}\} \times \mathcal{K})$, it necessarily follows that $\theta_1 = \eta_1$ and $s_1 = l_1$ in $\alpha\mathbb{Z}^d$, $[\theta_2]^\bullet = [\eta_2]^\bullet$ in $\widehat{\mathcal{G}}_0/\mathcal{K}^\perp$, $[s_2]^\bullet = [l_2]^\bullet$ in $\mathcal{G}_0/\mathcal{K}$ and $[(l, \hat{e})]^\bullet \in \dot{\Lambda}$.

Eventually we set

$$\dot{\mathbf{z}} := \dot{\mathbf{u}} - [(s, \hat{e})]^\bullet \in \mathscr{D}(\mathcal{G}) = \dot{\Lambda} \tag{5.124}$$

and using Lemma 5.2.15

$$\begin{aligned}&\left(\sum_{\dot{\mathbf{w}} \in \mathscr{D}(\mathcal{G})} \left(\sum_{\dot{\mathbf{u}} \in \mathscr{D}(\mathcal{G})} \dot{V}_\Phi \sigma\left([\mathsf{T}_0(\mathbf{w}, \mathbf{u})]^\bullet, [\mathcal{J}(\mathbf{u} - \mathbf{w})]^\bullet\right)^p\right)^{\frac{\gamma}{p}}\right)^{\frac{1}{\gamma}} \\ &\qquad = \left(\sum_{\dot{\boldsymbol{\theta}} \in \mathscr{D}(\widehat{\mathcal{G}})} \left(\sum_{\dot{\mathbf{z}} \in \mathscr{D}(\mathcal{G})} \dot{V}_\Phi \sigma(\dot{\mathbf{z}}, \dot{\boldsymbol{\theta}})^p\right)^{\frac{\gamma}{p}}\right)^{\frac{1}{\gamma}} = \left\|\dot{\mathcal{C}}_\Phi \sigma\right\|_{\ell^{p,\gamma}(\mathscr{D}(\mathcal{G}) \times \mathscr{D}(\widehat{\mathcal{G}}))} \\ &\qquad \lesssim \|\sigma\|_{M^{p,\gamma}(\mathcal{G} \times \widehat{\mathcal{G}})} < +\infty.\end{aligned}$$

*Case* $\gamma > 1$. Observe that $p \geq \gamma > 1$ and $q \geq \gamma > 1$. Consider first $p \neq \infty$. The desired result is obtained by duality. By Proposition 5.1.32 $M^\gamma(\mathcal{G}) \cong (M^{\gamma'}(\mathcal{G}))'$, we hence show that if $f \in M^q(\mathcal{G})$ then $\mathrm{Op}_0(\sigma)f$ is a continuous linear functional on $M^{\gamma'}(\mathcal{G})$. Let $g \in M^{\gamma'}(\mathcal{G})$, from the weak definition (2.217) and the fact that $M^{p,1}(\mathcal{G} \times \widehat{\mathcal{G}}) \cong (M^{p',\infty}(\mathcal{G} \times \widehat{\mathcal{G}}))'$ we get:

$$|\langle \mathrm{Op}_0(\sigma)f, g\rangle| = |\langle \sigma, R(g, f)\rangle| \leq \|\sigma\|_{M^{p,1}} \|R(g, f)\|_{M^{p',\infty}}.$$

The indexes' conditions in (5.69) and (5.70) become

$$\gamma', q \leq \infty, \tag{5.125}$$

$$\frac{1}{\gamma'} + \frac{1}{q} \geq \frac{1}{p}. \tag{5.126}$$

The first one is trivial, the second follows from the assumption (5.114). Therefore

$$\|R(g, f)\|_{M^{p',\infty}} \lesssim \|g\|_{M^{\gamma'}} \|f\|_{M^q}$$

and the boundedness of $\mathrm{Op}_0(\sigma)$ from $M^q(\mathcal{G})$ into $M^\gamma(\mathcal{G})$ follows.
If $p = \infty$ the argument is similar, we use the duality (5.63) between $M^{\infty,1}$ and $M^{1,\infty}$. □

**Proposition 5.2.18.** ([8, Proposition 4.22]) *Consider a symbol $\sigma$ on the phase space such that for some $0 < p < \infty$*

$$\sigma \in \bigcap_{\gamma > 0} M^{p,\gamma}(\mathcal{G} \times \widehat{\mathcal{G}}). \tag{5.127}$$

*Any eigenfunction $f \in L^2(\mathcal{G})$ of $\mathrm{Op}_0(\sigma)$ satisfies $f \in \bigcap_{\gamma > 0} M^\gamma(\mathcal{G})$.*

*Proof.* We use Theorem 5.2.17 and follow the proof pattern of [9, Proposition 3.5]. □

## 5.3 Localization Operators on Groups

The aim of this section is to infer a result for $L^2$-eigenfunctions of localization operators which extends the one obtained in the Euclidean setting in Theorem 3.2.1 [9]. We recall that the definition of $A^{\psi_1,\psi_2}_a$ was given in (2.194).
Given a function $F$ on $\mathcal{G}\times\mathcal{G}$, we introduce the operator $\mathfrak{T}_b$:

$$\mathfrak{T}_b F(x,u) := F(x,u-x). \tag{5.128}$$

Recall that $\mathcal{F}_2$ stands for the partial Fourier transform with respect to the second variable of measurable functions $\sigma$ defined on $\mathcal{G}\times\widehat{\mathcal{G}}$. We shall consider $\mathcal{F}_2\sigma$ to be defined on $\mathcal{G}\times\mathcal{G}$, instead of $\mathcal{G}\times\widehat{\widehat{\mathcal{G}}}$, due to the Pontryagin's duality. $\mathfrak{T}_b$ and $\mathcal{F}_2$ are automorphisms of $\mathcal{S}_0(\mathcal{G}\times\mathcal{G})$ and $\mathcal{S}_0(\mathcal{G}\times\widehat{\mathcal{G}})$, respectively, which extend to automorphisms of $\mathcal{S}_0'(\mathcal{G}\times\mathcal{G})$ and $\mathcal{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})$ by transposition.

**Lemma 5.3.1.** ([8, Lemma 5.2]) *Consider $\sigma\in\mathcal{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})$ and $f,g\in\mathcal{S}_0(\mathcal{G})$. Then*

$$\langle \mathrm{Op}_0(\sigma)f,g\rangle_{L^2(\mathcal{G})} = \langle k_\sigma, g\otimes\overline{f}\rangle_{L^2(\mathcal{G}\times\mathcal{G})} \tag{5.129}$$

*where the kernel $k_\sigma$ is given by*

$$k_\sigma(x,u) := K(\mathrm{Op}_0(\sigma))(x,u) := \int_{\widehat{\mathcal{G}}} \sigma(x,\xi)\overline{\langle u-x,\xi\rangle}\,d\xi = \mathfrak{T}_b(\mathcal{F}_2\sigma(x,u)). \tag{5.130}$$

*Proof.* The proof carries over from the Euclidean case almost verbatim, see e.g. [35, formula (4.3)]. □

The following issue presents the connection between localization and Kohn-Nirenberg operators on LCA groups, extending the Euclidean case proved in Proposition 2.8.16 [11].

**Proposition 5.3.2.** ([8, Proposition 5.3]) *Consider windows $\psi_1,\psi_2\in\mathcal{S}_0(\mathcal{G})$ and a symbol $a\in\mathcal{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})$. Then we have*

$$A^{\psi_1,\psi_2}_a = \mathrm{Op}_0(a * R(\psi_2,\psi_1)). \tag{5.131}$$

*Proof.* The proof is similar to the Euclidean case. We detail it for sake of clarity. We first compute the kernel $K(A^{\psi_1,\psi_2}_a)$ of $A^{\psi_1,\psi_2}_a$:

$$\begin{aligned}\langle A^{\psi_1,\psi_2}_a f,g\rangle &= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)\left(\int_{\mathcal{G}} f(u)\overline{\pi(x,\xi)\psi_1(u)}\,du\right)\left(\int_{\mathcal{G}}\overline{g(y)}\pi(x,\xi)\psi_2(y)\,dy\right)dxd\xi\\ &= \int_{\mathcal{G}\times\mathcal{G}} f(u)\overline{g(y)}k(y,u)\,dydu,\end{aligned}$$

with

$$K(A^{\psi_1,\psi_2}_a)(y,u) := \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)\overline{\pi(x,\xi)\psi_1(u)}\pi(x,\xi)\psi_2(y)\,dxd\xi.$$

Using Lemma 5.3.1, we set $\mathfrak{T}_b \circ \mathcal{F}_2(\sigma) = K(A_a^{\psi_1,\psi_2})$ and compute $\sigma$ using (2.99) as follows:

$$\begin{aligned}
\mathcal{F}_2^{-1} \circ \mathfrak{T}_b^{-1}(k) &= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)\mathcal{F}_2^{-1} \circ \mathfrak{T}_b^{-1} \left(\pi(x,\xi)\psi_2 \otimes \overline{\pi(x,\xi)\psi_1}(y,u)\right) dxd\xi \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)\mathcal{F}_2^{-1} \left(\pi(x,\xi)\psi_2(y) \cdot \overline{\pi(x,\xi)\psi_1(u+y)}\right) dxd\xi \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)\pi(x,\xi)\psi_2(y) \int_{\mathcal{G}} \overline{\pi(x,\xi)\psi_1(u+y)}\langle\omega,u\rangle \, du \, dxd\xi \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)\pi(x,\xi)\psi_2(y)\overline{\langle\omega,y\rangle\mathcal{F}(\pi(x,\xi)\psi_1)(\omega)} \, dxd\xi \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)R(\pi(x,\xi)\psi_2, \pi(x,\xi)\psi_1)(y,\omega) \, dxd\xi \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} a(x,\xi)R(\psi_2,\psi_1)((y,\omega)-(x,\xi)) \, dxd\xi \\
&= a * R(\psi_2,\psi_1)(y,\omega).
\end{aligned}$$

We then infer the thesis from the kernels' theorem [50, Theorem B3]. □

**Theorem 5.3.3.** ([8, Theorem 5.4]) *Let* $0 < p < \infty$ *and* $a \in M^{p,\infty}(\mathcal{G}\times\widehat{\mathcal{G}})$. *Consider* $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}) \smallsetminus \{0\}$. *Any eigenfunction* $f \in L^2(\mathcal{G})$ *of* $A_a^{\psi_1,\psi_2}$ *satisfies*

$$f \in \bigcap_{\gamma>0} M^\gamma(\mathcal{G}). \tag{5.132}$$

*Proof.* Observe that for $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ we have $R(\psi_2,\psi_1) \in \mathscr{A}_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})$, by Corollary 5.1.11. Therefore $R(\psi_2,\psi_1)$ belongs to every modulation space on the phase space; this is easily seen by using (5.21), the inclusion relations (2.49) and the inclusion between modulation spaces in Proposition 5.1.31. Then the argument is the same as in [9, Theorem 3.7]: we write $A_a^{\psi_1,\psi_2}$ in the Kohn-Nirenberg form (Proposition 5.3.2)

$$A_a^{\psi_1,\psi_2} = \mathrm{Op}_0(a * R(\psi_2,\psi_1)), \tag{5.133}$$

use the convolution relations in Proposition 5.2.14 and infer the thesis applying Proposition 5.2.18. □

# Chapter 6

# Localization operators as Fourier multipliers

In this chapter we investigate under which conditions is it possible to write a localization operator $A_a^{\psi_1,\psi_2}$, or STFT multiplier accordingly to the terminology adopted in [3], with symbol only in the frequencies, i.e. symbols of type $a = 1 \otimes m$ with $m$ defined on $\mathbb{R}^d$, as a Fourier multiplier first on $\mathbb{R}^d$, then in the framework of $\mathbb{Z}_N$. Namely, we study when the equality

$$A_{1\otimes m}^{\psi_1,\psi_2} = T_{m_2} \qquad \text{on} \quad \mathcal{S}(\mathbb{R}^d), \quad M^1(\mathbb{R}^d), \quad \text{or} \quad L^2(\mathbb{R}^d)$$

holds true. Of course, in the finite discrete case the equality above reduces to an equality between $N \times N$ matrices. We recall that a localization operator $A_a^{\psi_1,\psi_2}$ on $\mathbb{Z}_N$ is just a finite Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$ (2.197) and we talk about linear time invariant (LTI) filters $H$ (2.221) rather than Fourier multipliers.
To give a hint of the results in chapter, the equality $A_{1\otimes m}^{\psi_1,\psi_2} = T_{m_2}$ holds true on $\mathcal{S}(\mathbb{R}^d)$ is and only if

$$m_2 = m * \mathcal{F}^{-1}(\mathcal{C}_{\psi_1,\psi_2}),$$

with $m, m_2 \in \mathcal{S}'(\mathbb{R}^d)$, $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ or other suitable function spaces, and $\mathcal{C}_{\psi_1,\psi_2}$ is the window correlation function defined in (6.2) below as

$$\mathcal{C}_{\psi_1,\psi_2}(y) := (\mathcal{I}\psi_2 * \overline{\psi}_1)(y), \tag{6.1}$$

$\mathcal{I}$ being the reflection operator. We address the reader to Theorem 6.2.1.
In particular, if we choose $m = m_2$ the equality $A_{1\otimes m}^{\psi_1,\psi_2} = T_m$ holds for any multiplier $m \in \mathcal{S}(\mathbb{R}^d)$ if and only if

$$\mathcal{C}_{\psi_1,\psi_2} = 1 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

The equality above is very restrictive, so that $A_{1\otimes m}^{\psi_1,\psi_2} = T_{m_2}$ on $\mathcal{S}(\mathbb{R}^d)$ never holds for classical anti-Wick operators, i.e. $\psi_1 = \psi_2 = e^{-\pi t^2}$, whose Gaussian windows provide a smoothing effect. Indeed, Theorem 6.4.3 states that

*If $1 < p \leq 2 \leq q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ with indices satisfying* (6.13)*, then the anti-Wick operator $A_{1\otimes m}^{\psi,\psi}$ is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$.*

Please note the similarity (and differences) to Hörmander's result, Theorem 2.7.23). The previous result holds true for more general STFT multipliers $A_{1\otimes m}^{\psi_1,\psi_2}$ with $\psi_1, \psi_2 \in \mathcal{S}'(\mathbb{R}^d)$ such that the window correlation function satisfies $\mathcal{C}_{\psi_1,\psi_2} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, cf. Theorem 6.4.1 in Section 6.4 below. For $p = 2$, the boundedness of the Fourier multiplier $T_m$ in Theorem 2.7.23 forces the indices' choice: $q = 2$ and $r = \infty$, whereas condition in (6.13)

$$\frac{1}{q} \leq \frac{1}{r} + \frac{1}{p}$$

is more flexible, allowing to choose $q \geq 2$ and $r \leq \infty$. The necessity of condition (6.13) for anti-Wick operators is proved in Theorem 6.4.4. We just mention that the problem of representation and approximation of linear operators by means of Gabor multipliers (and suitable modifications) was studied by M. Dörfler and B. Torrésani in [46], further investigations are contained in [77, 107]. More generally, approximating problems for pseudo-differential operators via STFT multipliers ("wave packets" were exhibited in the work by A. Cordoba and C. Fefferman [36], see also Folland [68] and the Ph.D. thesis [48]). However here the focus is strictly different.

All the results the chapter presents are due to [3], it is structured as follows. Section 6.1 is devoted to the study of the already mentioned window correlation function $\mathcal{C}_{\psi_1,\psi_2}$, also a boundedness result on modulation spaces for $A_a^{\psi_1,\psi_2}$ is obtained. The equality $A_{1\otimes m}^{\psi_1,\psi_2} = T_{m_2}$ in considered in Section 6.2 whereas the case $m = m_2$ is tackled in Section 6.3. The smoothing effect of a STFT multiplier, Theorem 6.4.1, is the main object of Section 6.4, in Subsection 6.4.1 the anti-Wick case is taken into account. Eventually, Section 6.5 deals with the finite discrete setting of $\mathbb{Z}_N$ and it is an extension of [48, Chapter 2].

We recall that all the needed definitions and backgrounds can be found in Chapter 2 and Assumptions 2.5.1 hold in the present chapter. Moreover, we shall assume every weight continuous and even.

## 6.1 Window correlation function

In the following theorem we present an optimal result for $M^{p,q}$-boundedness (and in particular $L^2$-boundedness) of STFT multipliers. We extend Theorem 5.2 in [31] and Theorem 1.1 in [28].

**Theorem 6.1.1.** ([3, Theorem 2.4]) *Consider $s \geq 0$, $p_1, p_2, q_1, q_2 \in [1,\infty]$, with $1/p_1 + 1/p_2 \geq 1$, $1/q_1 + 1/q_2 \geq 1$. If $\psi_1 \in M_{v_s}^{p_1,q_1}(\mathbb{R}^d)$, $\psi_2 \in M_{v_s}^{p_2,q_2}(\mathbb{R}^d)$, and $a \in M^{\infty,1}(\mathbb{R}^{2d})$, then $A_a^{\psi_1,\psi_2}$ is bounded on every $M_{v_s}^{p,q}(\mathbb{R}^d)$, $p,q \in [1,\infty]$. In particular, the operator $A_a^{\psi_1,\psi_2}$ is bounded on the Shubin-Sobolev space $Q_s = M_{v_s}^2$ (In particular, for $s = 0$, $A_a^{\psi_1,\psi_2}$ is bounded on $L^2(\mathbb{R}^d)$).*

*Proof.* If $\psi_1 \in M_{v_s}^{p_1,q_1}(\mathbb{R}^d)$, $\psi_2 \in M_{v_s}^{p_2,q_2}(\mathbb{R}^d)$ with $1/p_1 + 1/p_2 \geq 1$, $1/q_1 + 1/q_2 \geq 1$, by [25, Theorem 4] we infer that their cross-Wigner distribution $W(\psi_2,\psi_1)$ is in $M_{1\otimes v_s}^{1,\infty}(\mathbb{R}^{2d})$. Rewriting the STFT multiplier $A_a^{\psi_1,\psi_2}$ as a Weyl operator $L_\sigma$ with $\sigma = a * W(\psi_2,\psi_1)$, the convolution relations for modulation spaces in Proposition 2.5.19 give

$$\sigma \in M^{\infty,1}(\mathbb{R}^{2d}) * M_{1\otimes v_s}^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M_{1\otimes v_s}^{\infty,1}(\mathbb{R}^{2d}).$$

The result follows by the continuity properties of Weyl operators in [31, Theorem 5.2]. □

For sake of completeness let us recall [28, Corollary 4.2]:

**Proposition 6.1.2.** *If $a \in M^\infty(\mathbb{R}^{2d})$ and $\psi_1, \psi_2 \in M_v^1(\mathbb{R}^d)$, $w \in \mathcal{M}_v$, then $A_a^{\psi_1,\psi_2}$ is bounded on $M_w^{p,q}(\mathbb{R}^d)$ for $1 \leq p,q \leq \infty$. In particular, it is bounded on $L^2(\mathbb{R}^d)$.*

**Definition 6.1.3.** *Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$. The* ***window correlation function of the pair*** *$(\psi_1, \psi_2)$ is defined by*

$$\mathcal{C}_{\psi_1,\psi_2}(y) := (\mathcal{I}\psi_2 * \bar{\psi_1})(y), \quad \forall y \in \mathbb{R}^d. \tag{6.2}$$

*The* ***shifted window correlation function of the pair*** *$(\psi_1, \psi_2)$ is defined for all $t, y \in \mathbb{R}^d$ as:*

$$\mathbb{G}_{\psi_1,\psi_2}(t, y) := \int_{\mathbb{R}^d} \psi_2(t-u)\overline{\psi_1(y-u)}\, du. \tag{6.3}$$

**Remark 6.1.4.** *(i) The window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ is a continuous function and it enjoys several properties depending on the function/distribution space of the two windows $\psi_1, \psi_2$, cf- Proposition 6.1.5;*

*(ii) It is straightforward to show that $\mathbb{G}_{\psi_1,\psi_2} \in L^\infty(\mathbb{R}^{2d})$. Observe that the definition of $\mathbb{G}_{\psi_1,\psi_2}$ also works for windows $\psi_1, \psi_2$ belonging to function/distributions spaces other than $L^2(\mathbb{R}^d)$ (see Proposition 6.1.5). We can rewrite the shifted window correlation function $\mathbb{G}_{\psi_1,\psi_2}$ on $\mathbb{R}^{2d}$ as a* time shift *of the mapping $\mathcal{C}_{\psi_1,\psi_2}$ on $\mathbb{R}^d$ defined in* (6.2). *In fact, a straightforward computation shows that*

$$\mathbb{G}_{\psi_1,\psi_2}(t, y) = \mathcal{C}_{\psi_1,\psi_2}(y-t) = T_t\mathcal{C}_{\psi_1,\psi_2}(y), \quad \forall t, y \in \mathbb{R}^d. \tag{6.4}$$

Let us study the properties of $\mathcal{C}_{\psi_1,\psi_2}$.

**Proposition 6.1.5.** ([3, Proposition 2.6]) *The window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ enjoys the following properties.*

*(i) If $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $\mathcal{C}_{\psi_1,\psi_2} \in \mathcal{S}(\mathbb{R}^d)$.*

*(ii) If either $\psi_1$ is in $\mathcal{S}'(\mathbb{R}^d)$ and $\psi_2 \in \mathcal{S}(\mathbb{R}^d)$ or $\psi_1$ is in $\mathcal{S}(\mathbb{R}^d)$ and $\psi_2 \in \mathcal{S}'(\mathbb{R}^d)$ then $\mathcal{C}_{\psi_1,\psi_2} \in \mathcal{C}(\mathbb{R}^d)$ with at most polynomial growth.*

*(iii) If $\psi_1 \in L^p(\mathbb{R}^d)$, $\psi_2 \in L^{p'}(\mathbb{R}^d)$, with $1 < p < \infty$, $1/p + 1/p' = 1$, then $\mathcal{C}_{\psi_1,\psi_2} \in C_0(\mathbb{R}^d)$. If either $p = 1$ ($p' = \infty$) or $p = \infty$ ($p' = 1$) then $\mathcal{C}_{\psi_1,\psi_2} \in C_b(\mathbb{R}^d)$. The same statements hold if we replace the Lebesgue space $L^p(\mathbb{R}^d)$ (resp. $L^{p'}(\mathbb{R}^d)$) with the modulation space $M^p(\mathbb{R}^d)$ (resp. $M^{p'}(\mathbb{R}^d)$).*

*(iv) If $\psi_1 \in M^{p,u}_{w_1\otimes\nu}(\mathbb{R}^d)$, $\psi_2 \in M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d)$, with $1 \le p, q, r, t, u, \gamma \le \infty$ satisfying*

$$\frac{1}{u} + \frac{1}{t} \ge \frac{1}{\gamma},$$

*and*

$$\frac{1}{p} + \frac{1}{q} \ge 1 + \frac{1}{r},$$

*and the weights as in the assumptions of Proposition 2.5.19, then $\mathcal{C}_{\psi_1,\psi_2}$ is in $\in M^{r,\gamma}_w(\mathbb{R}^d)$, with norm inequality*

$$\|\mathcal{C}_{\psi_1,\psi_2}\|_{M^{r,\gamma}_w} \lesssim \|\psi_1\|_{M^{p,u}_{w_1\otimes\nu}} \|\psi_2\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}}.$$

*Proof.* The proofs of items $(i)$, $(ii)$ follow by the convolution properties for the Schwartz class $\mathcal{S}$, its dual $\mathcal{S}'$ respectively, see, e.g., the textbooks [70, 94]. Item $(iii)$ is a consequence of the convolution properties for $L^p(\mathbb{R}^d)$ spaces which can be found e.g., in [70, 94]. For modulation spaces $M^p$ we use the convolution properties in Proposition 2.5.23.

(*iv*). By assumption all the weights under consideration are even, so that $\mathcal{I}\psi_2 \in M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d)$ whenever $\psi_2 \in M^{q,t}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d)$. Moreover modulation spaces are closed under complex conjugation, hence the result immediately follows by applying the convolution relations in Proposition 2.5.19. □

**Example 6.1.6.** ([3, Example 2.7]) *In what follows we exhibit examples of window correlation functions.*

(*i*) *Consider two $L^2$-normalized Gaussian functions $\psi_1(t) = \psi_2(t) = 2^{d/4}e^{-\pi t^2}$, $t \in \mathbb{R}^d$. In this case, the window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ in* (6.2) *is a Gaussian as well*

$$\mathcal{C}_{\psi_1,\psi_2}(t) = \mathcal{I}(\psi_1 * \mathcal{I}(\hat{\psi_2}))(t) = 2^{d/2}(e^{\pi(\cdot)^2} * e^{\pi(\cdot)^2})(-t) = e^{-\frac{\pi}{2}t^2}, \quad t \in \mathbb{R}^d; \tag{6.5}$$

(*ii*) *Consider $\psi_1 = \chi_{[0,1]^d}$, $\psi_2(t) = 1$, for every $t \in \mathbb{R}^d$. Observe $\psi_1 \in L^1(\mathbb{R}^d)$, $\psi_2 \in L^\infty(\mathbb{R}^d)$. Then the window correlation function becomes*

$$\mathcal{C}_{\psi_1,\psi_2}(t) = \psi_1 * \mathcal{I}(\bar{\psi_2})(-t) = \int_{[0,1]^d} dy = 1, \quad \forall t \in \mathbb{R}^d.$$

## 6.2 Study the equality $A^{\psi_1,\psi_2}_{1\otimes m} = T_{m_2}$

The following issue has been already anticipated at the beginning of the chapter.

**Theorem 6.2.1.** ([3, Theorem 3.1]) *Fix multiplier symbols $m, m_2 \in \mathcal{S}'(\mathbb{R}^d)$ (resp. $m, m_2 \in M^\infty(\mathbb{R}^d)$) and windows $\psi_1, \psi_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$). Then the equality*

$$A^{\psi_1,\psi_2}_{1\otimes m} = T_{m_2} \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d) \ (\text{resp. } M^1(\mathbb{R}^d)) \tag{6.6}$$

*holds if and only if*

$$m_2 = m * \mathcal{F}^{-1}(\mathcal{C}_{\psi_1,\psi_2}) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \ (\text{resp. } M^\infty(\mathbb{R}^d)). \tag{6.7}$$

*The same conclusions hold under the following assumptions:*

(*i*) *The symbols $m, m_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(\psi_1, \psi_2)$ in $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ (resp. $M^\infty(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$);*

(*ii*) *The symbols $m, m_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(\psi_1, \psi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ (resp. $M^1(\mathbb{R}^d) \times M^\infty(\mathbb{R}^d)$).*

*Proof.* Assume $m, m_2 \in \mathcal{S}'(\mathbb{R}^d)$ and $(\psi_1, \psi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. First, we show that the operators $A^{\psi_1,\psi_2}_{1\otimes m}$ and $T_{m_2}$ are well defined and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. For every $f, g \in \mathcal{S}(\mathbb{R}^d)$, the weak definition of STFT multiplier (2.193) and the standard properties of the STFT give the result, since $V_{\psi_1}f \in \mathcal{S}(\mathbb{R}^{2d})$ and $V_{\psi_2}g \in \mathcal{S}(\mathbb{R}^{2d})$ and the mappings $V_{\psi_1}$, $V_{\psi_2}$ are continuous on $\mathcal{S}(\mathbb{R}^d)$, see for example [35, Chapter 1]. For the Fourier multiplier we use the continuity of $\mathcal{F}$ (resp. $\mathcal{F}^{-1}$) on $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^d)$) and of the product $\mathcal{S}(\mathbb{R}^d) \cdot \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

Writing them as integral operators we obtain

$$A^{\psi_1,\psi_2}_{1\otimes m} f(t) = \int_{\mathbb{R}^d} K(A^{\psi_1,\psi_2}_{1\otimes m})(t, y) f(y) dy,$$

with kernel

$$K(A^{\psi_1,\psi_2}_{1\otimes m})(t,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(t-y)\omega} m(\omega)\psi_2(t-x)\overline{\psi_1(y-x)}\,dxd\omega$$

$$= \hat{m}(y-t)\mathbb{G}_{\psi_1,\psi_2}(t,y) = T_t(\hat{m}\mathcal{C}_{\psi_1,\psi_2})(y), \tag{6.8}$$

and

$$T_{m_2}f(t) = \int_{\mathbb{R}^d} K(T_{m_2})(t,y)f(y)dy, \tag{6.9}$$

with kernel

$$K(T_{m_2})(t,y) = \int_{\mathbb{R}^d} e^{2\pi i(t-y)\omega} m_2(\omega)d\omega = \hat{m}_2(y-t) = T_t\hat{m}_2(y). \tag{6.10}$$

By the Schwartz' kernel theorem the operators $A^{\psi_1,\psi_2}_{1\otimes m}$ and $T_{m_2}$ coincide if and only if their kernels $K(A^{\psi_1,\psi_2}_{1\otimes m})$ and $K(T_{m_2})$ coincide in $\mathcal{S}'(\mathbb{R}^{2d})$. Equating the kernels we obtain (6.19).

Consider now case $(i)$: $m, m_2 \in \mathcal{S}(\mathbb{R}^d)$ and $(\psi_1,\psi_2) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. We use similar arguments as above, observing that the STFT $V_{\psi_1}f \in \mathcal{S}'(\mathbb{R}^{2d})$ for every $f \in \mathcal{S}(\mathbb{R}^d)$ (cf. [35, Chapter 1]). The case $(ii)$ is analogous and left to the reader.

Second, assume $m, m_2 \in M^\infty(\mathbb{R}^d)$, $\psi_1,\psi_2 \in M^1(\mathbb{R}^d)$. We use the same arguments as in the first step, simply replacing $\mathcal{S}$ with $M^1$ and its dual $\mathcal{S}'$ with $(M^1)' = M^\infty$. Hence, we obtain that $T_{m_2}$ and the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$ are well-defined linear and bounded operators from $M^1(\mathbb{R}^d)$ into $M^\infty(\mathbb{R}^d)$. Rewriting them as integral operators and using the kernel theorem in the framework of modulation spaces [50, 61] we come up to the result. The cases: $(i)$ $m, m_2 \in M^1(\mathbb{R}^d)$, $\psi_1 \in M^1(\mathbb{R}^d)$ $\psi_2 \in M^\infty(\mathbb{R}^d)$, $(ii)$ $m, m_2 \in M^1(\mathbb{R}^d)$, $\psi_1 \in M^\infty(\mathbb{R}^d)$ $\psi_2 \in M^1(\mathbb{R}^d)$ are similar. □

In this case the symbol $m$ of the STFT multiplier is *smoothed* by the convolution with the Fourier transform of the window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ and the result is a multiplier symbol $m_2$ of $T_{m_2}$ smoother than $m$. For example, if you consider $m \in M^\infty(\mathbb{R}^d)$, $\psi_1,\psi_2 \in M^1(\mathbb{R}^d)$, as explained in Proposition 6.1.5 $(iv)$, then we have

$$m_2 = m * \mathcal{F}^{-1}(\mathcal{C}_{\psi_1,\psi_2}) \in M^\infty(\mathbb{R}^d) * \mathcal{F}^{-1}M^1(\mathbb{R}^d).$$

Using the convolution property in Proposition 2.5.19

$$m_2 \in M^\infty(\mathbb{R}^d) * \mathcal{F}^{-1}M^1(\mathbb{R}^d) = M^\infty(\mathbb{R}^d) * M^1(\mathbb{R}^d) \subseteq M^{\infty,1}(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d) \tag{6.11}$$

and we infer that the multiplier symbol $m_2$ belongs to $\mathcal{C}_b(\mathbb{R}^d)$. Then one can play with the convolution properties for modulation (and other function) spaces to obtain a Fourier multipliers' symbol $m_2$ in different function spaces.

For applications it is often useful to consider windows $\psi_1,\psi_2 \in L^2(\mathbb{R}^d)$ and multiplier $m \in L^\infty(\mathbb{R}^d)$. In this case the multiplier $m_2$ enjoys the smoothing below.

**Lemma 6.2.2.** ([3, Lemma 3.2]) *Assume $\psi_1,\psi_2 \in L^2(\mathbb{R}^d)$, $m \in L^\infty(\mathbb{R}^d)$. Then $m_2$ as in* (6.7) *belongs to $C_b(\mathbb{R}^d)$.*

*Proof.* For $\psi_1,\psi_2 \in L^2(\mathbb{R}^d)$, the window correlation function satisfies $\mathcal{F}^{-1}\mathcal{C}_{\psi_1,\psi_2} \in L^1(\mathbb{R}^d)$, since $\mathcal{I}\psi_2, \bar{\psi_1} \in L^2(\mathbb{R}^d)$ and

$$\mathcal{F}^{-1}(\mathcal{C}_{\psi_1,\psi_2}) \in \mathcal{F}^{-1}(L^2(\mathbb{R}^d) * L^2(\mathbb{R}^d)) = \mathcal{F}^{-1}L^2(\mathbb{R}^d) \cdot \mathcal{F}^{-1}L^2(\mathbb{R}^d)$$
$$= L^2(\mathbb{R}^d) \cdot L^2(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d).$$

Hence, by Proposition 6.1.5 (*iii*) we obtain

$$m_2 \in L^\infty(\mathbb{R}^d) * L^1(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d),$$

as desired. □

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## 6.3 Study the equality $A^{\psi_1,\psi_2}_{1\otimes m} = T_m$

First, we recall that the Hörmander's condition $p \le 2 \le q$ in Theorem 2.7.23 is sharp. More precisely, if there exists a function $F$ such that $\{F > 0\}$ has non-zero measure and for all $m : \mathbb{R}^d \to \mathbb{R}$ with $|m| \le |F|$, $T_m : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded, then $p \le 2 \le q$ (cf. [98, Theorem 1.12]). Moreover, also (2.220) is necessary by the $L^p$ inequalities for potentials (see [130, pag. 119]). We present a direct proof by rescaling arguments of the following necessary condition. $L^{r,\infty}$ denotes the weak $L^r$ spaces (2.189), we use $\|\cdot\|_p := \|\cdot\|_{L^p}$ where $L^p(\mathbb{R}^d)$ is the usual Lebesgue space.

**Proposition 6.3.1.** ([3, Proposition 1.2]) *For $p, q, r \in (1, \infty]$ we assume that the Fourier multiplier $T_m$ satisfies*

$$\|T_m f\|_q \le C\|m\|_{L^{r,\infty}}\|f\|_p, \quad \text{for every } f,\, m \in \mathcal{S}(\mathbb{R}^d), \tag{6.12}$$

*then we must have the indices' relation:*

$$\frac{1}{q} \le \frac{1}{r} + \frac{1}{p}. \tag{6.13}$$

*Proof.* Let us choose the multiplier $m(t) = m_\lambda(t) := \varphi_\lambda(t) = e^{-\pi\lambda t^2}$ and the function $f(t) = \varphi_\lambda(t)$ as well. Observe that $\widehat{\varphi_\lambda}(\xi) = \lambda^{-d/2}e^{-\pi\lambda^{-1}\xi^2}$, so that we compute

$$\begin{aligned} T_{m_\lambda}\varphi_\lambda(t) &= \lambda^{-d/2}\mathcal{F}^{-1}(e^{-\pi\frac{\lambda^2+1}{\lambda}\xi^2})(t) \\ &= (\lambda^2+1)^{-d/2}e^{-\frac{\pi\lambda}{\lambda^2+1}t^2}. \end{aligned}$$

The $L^q$ norm of the function above is given by

$$\|T_{m_\lambda}\varphi_\lambda\|_q \asymp \lambda^{-\frac{d}{2q}}(\lambda^2+1)^{-\frac{d}{q'}},$$

with $q'$ being the conjugate exponent of $q$. We have $\|\varphi_\lambda\|_p \asymp \lambda^{-d/(2p)}$. Assuming now (6.12) in our context

$$\|T_{m_\lambda}\varphi_\lambda\|_q \le C \le \|m_\lambda\|_{L^{r,\infty}}\|\varphi_\lambda\|_p$$

we get

$$\lambda^{-\frac{d}{2q}}(\lambda^2+1)^{-\frac{d}{q'}} \le C\lambda^{-\frac{d}{2r}}\lambda^{-\frac{d}{2p}}.$$

Letting $\lambda \to 0^+$ we obtain the desired estimate (6.13). □

For any symbol $a(x,\omega) = (1\otimes m)(x,\omega) = m(\omega)$, $x,\omega \in \mathbb{R}^d$, the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$ can be formally re-written in terms of the related correlation function. Assume for simplicity that the windows $\psi_1, \psi_2$ and multiplier $m = m(\omega)$ are in $\mathcal{S}(\mathbb{R}^d)$. We start with $f \in \mathcal{S}(\mathbb{R}^d)$; for every fixed $t \in \mathbb{R}^d$, the integrals below are absolutely convergent and we are allowed to use Fubini's

Theorem. Moreover, it is straightforward to see that $A^{\psi_1,\psi_2}_{1\otimes m} f \in \mathcal{S}(\mathbb{R}^d)$. Simple computations give

$$A^{\psi_1,\psi_2}_{1\otimes m} f(t) = \int_{\mathbb{R}^d} e^{2\pi i\omega t} m(\omega) \int_{\mathbb{R}^d} f(y) e^{-2\pi i\omega y} \mathbb{G}_{\psi_1,\psi_2}(t,y)\,dy\,d\omega$$

$$= \int_{\mathbb{R}^d} e^{2\pi i\omega t} m(\omega) \int_{\mathbb{R}^d} f(y) e^{-2\pi i\omega y} T_t \mathcal{C}_{\psi_1,\psi_2}(y)\,dy\,d\omega \tag{6.14}$$

$$= \int_{\mathbb{R}^d} e^{2\pi i\omega t} m(\omega) \mathcal{F}(f T_t \mathcal{C}_{\psi_1,\psi_2})(\omega)\,d\omega. \tag{6.15}$$

Note that, if we assume condition (6.1), then $T_t \mathcal{C}_{\psi_1,\psi_2} = 1$ for every $t \in \mathbb{R}^d$ and $A^{\psi_1,\psi_2}_{1\otimes m} = T_m$, as desired.

The equality (6.15) suggests the introduction of a new time-frequency representation closely related to the STFT.

**Definition 6.3.2.** *For $\psi_1 \in L^1(\mathbb{R}^d), \psi_2 \in L^2(\mathbb{R}^d)$, we define the* ***two-window short-time Fourier transform of a signal*** *$f \in L^2(\mathbb{R}^d)$ by*

$$\int_{\mathbb{R}^d} e^{-2\pi i\omega y} f(y) T_t \mathcal{C}_{\psi_1,\psi_2}(y)\,dy := \langle f, M_\omega T_t \overline{\mathcal{C}}_{\psi_1,\psi_2}\rangle = V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\omega), \quad . \tag{6.16}$$

*for every $(t,\omega) \in \mathbb{R}^{2d}$.*

For $\psi_1 \in L^1(\mathbb{R}^d), \psi_2 \in L^2(\mathbb{R}^d)$, Young's Inequality gives $\overline{\mathcal{C}}_{\psi_1,\psi_2} \in L^2(\mathbb{R}^d)$. Thus, the integral above is absolutely convergent for every $f \in L^2(\mathbb{R}^d)$. The same argument applies if we replace the condition $\psi_1 \in L^1(\mathbb{R}^d), \psi_2 \in L^2(\mathbb{R}^d)$ with the more general one $\psi_1 \in L^p(\mathbb{R}^d), \psi_2 \in L^q(\mathbb{R}^d)$ such that $1/p + 1/q = 3/2$.

Using (6.15), the action of the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$ can be rewritten as

$$A^{\psi_1,\psi_2}_{1\otimes m} f(t) = \int_{\mathbb{R}^d} e^{2\pi i\omega t} m(\omega) V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\omega)\,d\omega = \mathcal{F}_2^{-1}[m V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\cdot)](t), \quad t \in \mathbb{R}^d \tag{6.17}$$

where $\mathcal{F}_2^{-1}$ denotes the partial Fourier transform w.r.t. the second coordinate $\omega$. The formal equality above can be made rigorous by studying the properties of the two-window short-time Fourier transform $V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}}$ and the multiplier symbol $m(\omega)$.

The following issue stems from Theorem 6.2.1 with $m = m_2$.

**Corollary 6.3.3.** ([3, Corollary 4.2]) *Fix a multiplier symbol $m \in \mathcal{S}'(\mathbb{R}^d)$ (resp. $m \in M^\infty(\mathbb{R}^d)$) and windows $\psi_1, \psi_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$). Then the equality*

$$A^{\psi_1,\psi_2}_{1\otimes m} = T_m \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d) \ (\text{resp. } M^1(\mathbb{R}^d)) \tag{6.18}$$

*holds if and only if*

$$\hat{m}\mathcal{C}_{\psi_1,\psi_2} = \hat{m} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \ (\text{resp. } M^\infty(\mathbb{R}^d)). \tag{6.19}$$

*The same conclusions hold under the following assumptions:*

*(i) The symbol $m$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(\psi_1,\psi_2)$ in $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ (resp. $M^\infty(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$);*

*(ii) The symbol $m$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(\psi_1,\psi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ (resp. $M^1(\mathbb{R}^d) \times M^\infty(\mathbb{R}^d)$).*

Straightforward consequences of the result above are the following.

**Corollary 6.3.4.** ([3, Corollary 4.3]) *Consider either* $(\psi_1, \psi_2) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ *or* $(\psi_1, \psi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$*. Then the equality* (6.19) *holds for every symbol* $m \in \mathcal{S}(\mathbb{R}^d)$ *if and only if condition* (6.1) *is satisfied.*

*Proof.* The condition (6.1) immediately follows if we take $m(\omega) = e^{-\pi\omega^2} \in \mathcal{S}(\mathbb{R}^d)$ in the equality (6.19). □

**Corollary 6.3.5.** ([3, Corollary 4.4]) *It is not possible to find* $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ *such that the equality* (6.18) *holds for every multiplier* $m \in \mathcal{S}'(\mathbb{R}^d)$*.*

*Proof.* Taking $m(\omega) = e^{-\pi\omega^2} \in \mathcal{S}(\mathbb{R}^d)$ in the equality (6.19) we obtain condition (6.1). Since $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$, by Proposition 6.1.5 we infer $\mathcal{C}_{\psi_1,\psi_2} \in \mathcal{S}(\mathbb{R}^d)$, thus condition (6.1) is never satisfied. □

Let us try to understand the condition (6.1) better for operators having windows/symbols in modulation spaces.

Notice that under the assumption $\psi_1, \psi_2 \in M^1(\mathbb{R}^d)$ the window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ is in $M^1(\mathbb{R}^d)$ (use Proposition 6.1.5 (*iv*) or the well-known fact that $M^1$ is an algebra under convolution). As a consequence of Theorem 6.2.1, if we want condition (6.19) to be satisfied for every multiplier $m \in M^\infty(\mathbb{R}^d)$, the window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ must satisfy

$$\mathcal{C}_{\psi_1,\psi_2}(t) = 1, \quad t \in \mathbb{R}^d. \tag{6.20}$$

But this is not possible since $\mathcal{C}_{\psi_1,\psi_2} \in M^1(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$.

To overcome this issue, we look for windows in a bigger class that could guarantee condition (6.20). This requires smoother symbols.

**Theorem 6.3.6.** ([3, Theorem 4.5]) *Consider* $p_1, p_2, q_1, q_2 \in [1, \infty]$*, with* $1/p_1 + 1/p_2 \geq 1$*,* $1/q_1 + 1/q_2 \geq 1$*,* $\psi_1 \in M^{p_1,q_1}(\mathbb{R}^d)$*,* $\psi_2 \in M^{p_2,q_2}(\mathbb{R}^d)$*, and* $m \in M^{\infty,1}(\mathbb{R}^d)$*. Then both the Fourier multiplier* $T_m$ *and the STFT multiplier* $A^{\psi_1,\psi_2}_{1\otimes m}$ *are well-defined linear and bounded operators on* $L^2(\mathbb{R}^d)$ *and the equality* (6.18) *holds on* $M^\infty(\mathbb{R}^d)$ *if and only if condition* (6.19) *is satisfied on* $M^\infty(\mathbb{R}^d)$*. As a consequence, if we want* (6.19) *to be fulfilled for every symbol* $m \in M^{\infty,1}(\mathbb{R}^d)$*, the window correlation function* $\mathcal{C}_{\psi_1,\psi_2}$ *must satisfy* (6.20)*.*

*Proof.* We start with $\psi_1, \psi_2 \in M^1(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$, for every $p, q \in [1, \infty]$. Notice that, if the multiplier $m \in M^{\infty,1}(\mathbb{R}^d)$, then the localization symbol $(1 \otimes m)$ is in $M^{\infty,1}(\mathbb{R}^{2d})$, since

$$(1 \otimes m) \in M^{\infty,1}(\mathbb{R}^d) \otimes M^{\infty,1}(\mathbb{R}^d) \subseteq M^{\infty,1}(\mathbb{R}^{2d})$$

and we have $1 \in M^{\infty,1}(\mathbb{R}^d)$. In fact, for any fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$, we work out

$$V_g 1(x, \omega) = \mathcal{F}(T_x \bar{g})(\omega) = M_{-x}\hat{\bar{g}}(\omega), \quad (x, \omega) \in \mathbb{R}^{2d},$$

so that

$$\|1\|_{M^{\infty,1}(\mathbb{R}^d)} \asymp \|V_g 1\|_{L^{\infty,1}(\mathbb{R}^{2d})} = \|\hat{\bar{g}}\|_{L^1(\mathbb{R}^d)} = \|g\|_{\mathcal{F}L^1(\mathbb{R}^d)} < +\infty.$$

Hence by Theorem 6.1.1 the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$ is bounded on any $M^{p,q}(\mathbb{R}^d)$ and in particular on $L^2(\mathbb{R}^d)$. This is also the case for the Fourier multiplier $T_m$ with $m \in M^{\infty,1}(\mathbb{R}^d)$, since the inclusion relation in (2.168) gives in particular $m \in L^\infty(\mathbb{R}^d)$ and hence $T_m \in B(L^2)$ [98]. Using Theorem 6.2.1, such operators coincide whenever condition (6.19) is satisfied.

Next, consider $\psi_1 \in M^{p_1,q_1}(\mathbb{R}^d)$, $\psi_2 \in M^{p_2,q_2}(\mathbb{R}^d)$ satisfying the assumptions. We shall show that the related kernel $K(A^{\psi_1,\psi_2}_{1\otimes m})$ of $A^{\psi_1,\psi_2}_{1\otimes m}$ is in $M^\infty(\mathbb{R}^{2d})$. In fact, Proposition 6.1.5 (*iv*) gives the window correlation function $\mathcal{C}_{\psi_1,\psi_2} \in M^{\infty,1}(\mathbb{R}^d)$. If the multiplier $m \in M^{\infty,1}(\mathbb{R}^d)$, then $\hat{m} \in W(\mathcal{F}L^\infty, L^1)(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^\infty, L^\infty)(\mathbb{R}^d) = M^\infty(\mathbb{R}^d)$ (cf., e.g., [35, Chapter 2]) and the multiplication relations for modulation spaces [35, Prop. 2.4.23]

$$\|\hat{m}\mathcal{C}_{\psi_1,\psi_2}\|_{M^\infty} \lesssim \|\hat{m}\|_{M^\infty}\|\mathcal{C}_{\psi_1,\psi_2}\|_{M^{\infty,1}} \lesssim \|m\|_{M^{\infty,1}}\|\psi_1\|_{M^{p_1,q_1}}\|\psi_2\|_{M^{p_2,q_2}} < \infty.$$

Hence we obtain condition (6.19). □

Thanks to the results above, if the window functions $\psi_1$ and $\psi_2$ are non-smooth, they can satisfy condition (6.1), as in the following issue.

**Example 6.3.7.** ([3, Example 4.6]) *An example of window correlation functions $\mathcal{C}_{\psi_1,\psi_2}$ satisfying* (6.1). *Consider $\psi_2 = 1 \in M^{\infty,1}(\mathbb{R}^d)$ and any $\psi_1 \in M^{1,\infty}(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} \psi_1(y)\,dy = 1. \tag{6.21}$$

*This gives* (6.20). *In particular, observe that* (6.21) *is fulfilled if we consider $\psi_1(t) = e^{-\pi t^2} \in \mathcal{S}(\mathbb{R}^d) \subseteq M^{1,\infty}(\mathbb{R}^d)$. Hence, the operators $A^{\psi_1,\psi_2}_{1\otimes m}$ and $T_m$ coincide for every multiplier $m \in M^{\infty,1}(\mathbb{R}^d)$.*

The realm of modulation spaces seems the only possible environment to get the equality $A^{\psi_1,\psi_2}_{1\otimes m} = T_m$. Also for the standard case of $L^2$-window functions the equality fails, as shown below.

**Theorem 6.3.8.** ([3, Theorem 4.7]) *Consider $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$, and the multiplier $m \in L^\infty(\mathbb{R}^d)$. Then both the Fourier multiplier $T_m$ and the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$ are well-defined linear and bounded operators on $L^2(\mathbb{R}^d)$ and the equality*

$$A^{\psi_1,\psi_2}_{1\otimes m} = T_m \quad \text{on} \quad L^2(\mathbb{R}^d) \tag{6.22}$$

*holds if and only if condition* (6.19) *is satisfied. As a consequence, if we want* (6.19) *to be fulfilled for every multiplier $m \in L^\infty(\mathbb{R}^d)$, the window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ must satisfy* (6.20), *and this is* ***never*** *the case.*

*Proof.* The boundedness of $A^{\psi_1,\psi_2}_{1\otimes m}$ on $L^2(\mathbb{R}^d)$ is shown in [151]. For the Fourier multiplier we recall that $T_m$ is bounded on $L^2(\mathbb{R}^d)$ since $m$ is in $L^\infty(\mathbb{R}^d)$ [98]. Condition (6.19) then follows by Theorem 6.2.1. The window correlation function $\mathcal{C}_{\psi_1,\psi_2}$ never satisfies (6.19) because $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ implies $\mathcal{C}_{\psi_1,\psi_2} \in C_0(\mathbb{R}^d)$, by Proposition 6.1.5 (*iii*). □

A natural question is whether we can consider windows $\psi_1 \in M^p(\mathbb{R}^d)$, $\psi_2 \in M^{p'}(\mathbb{R}^d)$, $1 \le p, p' \le \infty$, $1/p + 1/p' = 1$, and the multiplier $m \in L^\infty(\mathbb{R}^d)$. This is the case explained below.

**Proposition 6.3.9.** ([3, Proposition 4.8]) *If we consider $\psi_1 \in M^p(\mathbb{R}^d)$, $\psi_2 \in M^{p'}(\mathbb{R}^d)$, $1 \le p, p' \le \infty$, $1/p + 1/p' = 1$, and multiplier $m \in L^\infty(\mathbb{R}^d)$, then the result in Theorem 6.2.1 holds true. In particular, the equality in* (6.19) *is fulfilled if and only if condition* (6.20) *is satisfied.*

*Proof.* The Fourier multiplier $T_m$ is obviously well-defined, linear and bounded from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, since $T_m$ is bounded on $L^2(\mathbb{R}^d)$.

We recall, for $f, g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ with $\|\gamma\|_{L^2} = 1$, the switching property of the STFT [35, Lemma 1.2.3] and the change of window in [35, Lemma 1.2.29]. Indeed, for $(x,\omega) \in \mathbb{R}^{2d}$:

$$V_f g(x,\omega) = e^{-2\pi i x\cdot\omega}\overline{V_g f(-x,-\omega)}, \qquad |V_g f(x,\omega)| \leq \left(|V_\gamma f| * V_g\gamma\right)(x,\omega).$$

For the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$, we use its weak definition in (2.193), Hölder's inequality, the mentioned switching property and change of window; for every $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
|\langle A^{\psi_1,\psi_2}_a f, g\rangle| &= |\langle a, \overline{V_{\psi_1} f} V_{\psi_2} g\rangle| \\
&\leq \|a\|_{L^\infty(\mathbb{R}^{2d})}\|\overline{V_{\psi_1} f} V_{\psi_2} g\|_{L^1(\mathbb{R}^{2d})} \\
&\leq \|m\|_{L^\infty(\mathbb{R}^d)}\|V_{\psi_1} f\|_{L^p(\mathbb{R}^{2d})}\|V_{\psi_2} g\|_{L^{p'}(\mathbb{R}^{2d})} \\
&= \|m\|_{L^\infty(\mathbb{R}^d)}\|V_\gamma f * V_{\psi_1}\gamma\|_{L^p(\mathbb{R}^{2d})}\|V_\gamma g * V_{\psi_2}\gamma\|_{L^{p'}(\mathbb{R}^{2d})} \\
&\leq \|m\|_{L^\infty(\mathbb{R}^d)}\|V_\gamma f\|_{L^1(\mathbb{R}^{2d})}\|V_{\psi_1}\gamma\|_{L^p(\mathbb{R}^{2d})}\|V_\gamma g\|_{L^1(\mathbb{R}^{2d})}\|V_{\psi_2}\gamma\|_{L^{p'}(\mathbb{R}^{2d})} \\
&= \|m\|_{L^\infty(\mathbb{R}^d)}\|f\|_{M^1(\mathbb{R}^d)}\|V_\gamma \psi_1\|_{L^p(\mathbb{R}^{2d})}\|g\|_{M^1(\mathbb{R}^d)}\|V_\gamma \psi_2\|_{L^{p'}(\mathbb{R}^{2d})} \\
&= \|m\|_{L^\infty(\mathbb{R}^d)}\|f\|_{M^1(\mathbb{R}^d)}\|\psi_1\|_{M^p(\mathbb{R}^d)}\|\psi_2\|_{M^{p'}(\mathbb{R}^d)}\|g\|_{M^1(\mathbb{R}^d)}.
\end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^d)$, the estimate above gives the continuity of $A^{\psi_1,\psi_2}_{1\otimes m}$ from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. Then, arguing as in the proof of Theorem 6.2.1 we obtain the claim. □

Considering $\psi_2(t) = 1$ for every $t \in \mathbb{R}^d$, hence $\psi_2 \in L^\infty(\mathbb{R}^d) \subseteq M^\infty(\mathbb{R}^d)$, and any $\psi_1 \in M^1(\mathbb{R}^d)$ satisfying (6.21), we provide examples for condition (6.20) being satisfied.

## 6.4 Smoothing effects of STFT multipliers

Thanks to the smoothing effect of the two-window STFT we obtain boundedness results for STFT multipliers which extend the case of Fourier multipliers. The main tool is to use the representation of $A^{\psi_1,\psi_2}_a$ in (6.15), that is

$$A^{\psi_1,\psi_2}_{1\otimes m} f(t) = \int_{\mathbb{R}^d} e^{2\pi i\omega t} m(\omega)\mathcal{F}(f T_t \mathcal{C}_{\psi_1,\psi_2})(\omega)d\omega = \mathcal{F}_2^{-1}[m V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\cdot)].$$

**Theorem 6.4.1.** ([3, Theorem 5.1]) *Assume* $1 < p \leq 2 \leq q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ *such that condition* (6.13) *is satisfied. Consider windows* $\psi_1, \psi_2 \in \mathcal{S}'(\mathbb{R}^d)$ *such that the correlation function satisfies*

$$\mathcal{C}_{\psi_1,\psi_2} \in L^{p'}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{6.23}$$

*Then the STFT operator* $A^{\psi_1,\psi_2}_{1\otimes m}$ *is bounded from* $L^p(\mathbb{R}^d)$ *into* $L^q(\mathbb{R}^d)$.

*Proof.* Consider a function $f$ in $L^p(\mathbb{R}^d)$, $p \leq 2$, then

$$\|f T_t \mathcal{C}_{\psi_1,\psi_2}\|_1 \leq \|f\|_p \|T_t \mathcal{C}_{\psi_1,\psi_2}\|_{p'} = \|f\|_p \|\mathcal{C}_{\psi_1,\psi_2}\|_{p'}, \quad \forall t \in \mathbb{R}^d$$

and

$$\|f T_t \mathcal{C}_{\psi_1,\psi_2}\|_p \leq \|f\|_p \|T_t \mathcal{C}_{\psi_1,\psi_2}\|_\infty \leq \|f\|_p \|\mathcal{C}_{\psi_1,\psi_2}\|_\infty, \quad \forall t \in \mathbb{R}^d.$$

So that by complex interpolation, $f T_t \mathcal{C}_{\psi_1,\psi_2} \in L^s(\mathbb{R}^d)$, for every $1 \leq s \leq p$ (hence $1/s \geq 1/p$) $\forall t \in \mathbb{R}^d$, with

$$\|f T_t \mathcal{C}_{\psi_1,\psi_2}\|_{L^s(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)},$$

for a constant $C > 0$ independent of $t$.

By Theorem 2.7.23, if $m \in L^{r,\infty}(\mathbb{R}^d)$, then the Fourier multiplier

$$T_m f = \mathcal{F}_2^{-1}[m V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\cdot)] = \mathcal{F}_2^{-1}[m\mathcal{F}_2(fT_t\mathcal{C}_{\psi_1,\psi_2})]$$

acts continuously from $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, with $q \geq 2$ satisfying the index condition in (6.13). □

**Remark 6.4.2.** *If $\psi_1 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\psi_2 \in L^2(\mathbb{R}^d)$ (or vice versa) then the window correlation function satisfies condition* (6.23)*. In fact, by Proposition 6.1.5 it follows that $\mathcal{C}_{\psi_1,\psi_2} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subseteq L^{p'}(\mathbb{R}^d)$, for every $2 \leq p' \leq \infty$.*

This shows the *smoothing effect* of the two-window STFT $V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f$. For simplicity, let us consider $f \in L^2(\mathbb{R}^d)$. The Fourier multiplier $T_m$ takes the function $f \in L^2(\mathbb{R}^d)$ and considers its Fourier transform $\hat{f}$ that lives in $L^2(\mathbb{R}^d)$ by Plancherel theorem, but we cannot infer any other further property for $f$. Instead, in the STFT multiplier $A^{\psi_1,\psi_2}_{1\otimes m}$ we replace $\hat{f}$ with the two-window STFT $V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f$. Assuming the condition (6.23), we obtain that $V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f \in C_b(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ and uniformly continuous on $\mathbb{R}^{2d}$ (cf. [35, Proposition 1.2.10, Corollary 1.2.12]), and this implies $V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\cdot) \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for every fixed $t \in \mathbb{R}^d$, so that the related multiplier $\mathcal{F}_2^{-1}[m V_{\overline{\mathcal{C}}_{\psi_1,\psi_2}} f(t,\cdot)]$ can enjoy the smoothing effect above, uniformly with respect to $t \in \mathbb{R}^d$.

### 6.4.1 The anti-Wick case

Thanks to the discussions above, we can state that an anti-Wick operator $A^{\varphi,\varphi}_{1\otimes m}$, with Gaussian windows $\varphi(t) = 2^{d/4} e^{-\pi t^2}$ and multiplier symbol $m \in \mathcal{S}'(\mathbb{R}^d)$, can *never* be written in the Fourier multiplier form. In fact, recalling that the window correlation function in this case is given by $\mathcal{C}_{\varphi,\varphi}(t) = e^{-\frac{\pi}{2}t^2}$, cf. formula (6.5), we infer that condition (6.1) is never satisfied.

Let us better understand the smoothing effects for such operators. Using the expression in (6.15), we can write

$$A^{\varphi,\varphi}_{1\otimes m} f(t) = \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) \mathcal{F}(fT_t(e^{-\frac{\pi}{2}(\cdot)^2}))(\omega) d\omega.$$

The anti-Wick operator in terms of the two-window STFT defined in (6.16) can be written as

$$A^{\varphi,\varphi}_{1\otimes m} f(t) = \mathcal{F}_2^{-1}[m V_{\mathcal{C}_{\varphi,\varphi}} f(t,\cdot)], \quad t \in \mathbb{R}^d.$$

Roughly speaking, here the signal $f$ is first smoothed by multiplying with the shifted Gaussian $T_t(e^{-\frac{\pi}{2}(\cdot)^2})$, that is

$$g_t(y) := f(y) T_t(e^{-\frac{\pi}{2}(\cdot)^2})(y). \tag{6.24}$$

Then, the multiplier $T_m$ is applied to the modified signal $g_t$. In other words,

$$A^{\varphi,\varphi}_{1\otimes m} f(t) = T_m(g_t)(t), \quad f \in L^2(\mathbb{R}^d). \tag{6.25}$$

From the equality above, it is clear the smoothing effect of the anti-Wick operator $A^{\varphi,\varphi}_{1\otimes m}$ with respect to the Fourier multiplier $T_m$, stated in Theorem 6.4.3, that we are going to prove very easily.

**Theorem 6.4.3.** ([3, Theorem 1.3]) *If $1 < p \leq 2 \leq q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ with indices satisfying* (6.13)*, then the anti-Wick operator $A^{\varphi,\varphi}_{1\otimes m}$ is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$.*

*Proof.* Since the window correlation function $\mathcal{C}_{\varphi,\varphi}(t) = e^{-\frac{\pi}{2}t^2}$ is in $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^{p'}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, for any $2 \le p' < \infty$ condition in (6.23) is satisfied and the thesis follows by Theorem 6.4.1. □

We end up this section by showing the necessity of the indices' relation in (6.13).

**Theorem 6.4.4.** ([3, Theorem 5.3]) *If there exists a $C > 0$ such that the anti-Wick operator satisfies*

$$\|A^{\varphi,\varphi}_{1\otimes m} f\|_q \le C\|m\|_{L^{r,\infty}}\|f\|_p, \quad \forall f, m \in \mathcal{S}(\mathbb{R}^d), \tag{6.26}$$

*then condition* (6.13) *holds true.*

*Proof.* We write condition (6.26) for the multipliers $m_\lambda(\xi) = \varphi_\lambda(\xi) = e^{-\pi\lambda\xi^2}$, $\lambda > 0$, and functions $f_\lambda(t) = \varphi_\lambda(t)$ as well. Then we compute the anti-Wick operator $A^{\varphi,\varphi}_{1\otimes m_\lambda} f_\lambda$. A tedious computation shows

$$A^{\varphi,\varphi}_{1\otimes m_\lambda} f_\lambda(t) = c_\lambda e^{-\pi b_\lambda t^2},$$

with

$$c_\lambda := \frac{2^{d/2}}{(6\lambda^2 + 4\lambda + 1)^{d/2}}, \quad b_\lambda := \frac{2\lambda(6\lambda^3 + 10\lambda^2 + 9\lambda + 1)}{(6\lambda^2 + 4\lambda + 1)(2\lambda + 1)^2}.$$

This yields the norm estimate

$$\|A^{\varphi,\varphi}_{1\otimes m_\lambda} f_\lambda\|_q \asymp c_\lambda b_\lambda^{-\frac{d}{2q}} \asymp \frac{(2\lambda+1)^{\frac{d}{q}}}{\lambda^{\frac{d}{2q}}(6\lambda^2 + 4\lambda + 1)^{\frac{d}{2q'}}(6\lambda^3 + 10\lambda^2 + 9\lambda + 1)^{\frac{d}{2q}}}.$$

Letting $\lambda \to 0^+$ we infer the inequality in (6.13). □

## 6.5 Finite discrete setting: representation of LTI filter as Gabor Multiplier

Using intuition and visual comparison as an indication that the implementation of a LTI filter by a Gabor multiplier seems to work quite well, but being aware of continuous results, we are now going to analyse under which conditions it is analytically possible to have equivalence between a LTI filter and a Gabor multiplier. We will see immediately in the first theorem that exactly the most interesting class of perfect filters with characteristic function as frequency response does not qualify as suited candidates.

The result below is a consequence of the general setting in Theorem 6.3.8, but the estimate (6.27) is new.

**Theorem 6.5.1.** ([3, Theorem 6.5]) *Let $T_{m_2} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a LTI filter with frequency response $m_2 = \hat{h} = \chi_\Omega$, $\Omega \subsetneq \mathbb{R}$ interval. Then $T_{m_2}$ can never be represented exactly as Gabor multiplier with symbol $a = 1 \otimes m$, $m \in L^\infty(\mathbb{R})$, and*

$$\|T_{m_2} - G_a^{g_1,g_2}\|_{\mathrm{Op}} \ge \frac{1}{2} \tag{6.27}$$

*for every and $g_1, g_2 \in L^2(\mathbb{R})$.*

*Proof.* The spreading function is a Banach Gelfand Triple isomorphism between $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ and $(\mathcal{S}_0, L^2, \mathcal{S}_0')(\mathbb{R}^2)$, see [63] for notations. Therefore two operators are identical in $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ if and only if their spreading functions are identical. The integral kernel of a Fourier multiplier with symbol $m_2$ was calculated in (6.10) and it is related to the spreading function as follows:

$$\begin{aligned}
\eta(T_{m_2})(x,\omega) &= \int_{\mathbb{R}^d} K(T_{m_2})(y, y-x)e^{-2\pi i\omega y}\,dy \\
&= \int_{\mathbb{R}^d} T_y\hat{m}_2(y-x)e^{-2\pi i\omega y}\,dy \\
&= \int_{\mathbb{R}^d} \hat{m}_2(-x)e^{-2\pi i\omega y}\,dy \\
&= \left(\mathcal{I}\circ\mathcal{F}(m_2)\otimes\delta\right)(x,\omega) \\
&= \left(h\otimes\delta\right)(x,\omega).
\end{aligned}$$

The spreading function of a Gabor multiplier $G_a^{g_1,g_2}$ defined trough a lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ is given by

$$\eta(G_a^{g_1,g_2})(x,\omega) = \mathcal{F}_s(a)(x,\omega)\cdot V_{g_1}g_2(x,\omega) =: \mathscr{A}(x,\omega)\cdot V_{g_1}g_2(x,\omega), \tag{6.28}$$

where $\mathscr{A} = \mathcal{F}_s a = \mathcal{F}^{-1}(m)\otimes\mathcal{F}(1) = \mathcal{F}^{-1}(m)\otimes\delta$ is the $(\frac{1}{\beta},\frac{1}{\alpha})$-periodic symplectic Fourier Transform of the symbol $a = 1\otimes m$ (compare [46]). Therefore a Gabor multiplier is equivalent to a convolution operator if and only if

$$\left(h\otimes\delta\right)(x,\omega) = \left(\mathcal{F}^{-1}(m)\otimes\delta\right)V_{g_1}g_2(x,\omega) \qquad \forall(x,\omega)\in\mathbb{R}^{2d}. \tag{6.29}$$

This gives

$$h(x) = \mathcal{F}^{-1}(m)V_{g_1}g_2(x,0) \quad \Leftrightarrow \quad \hat{h} = m * \mathcal{F}\left(V_{g_1}g_2(\cdot,0)\right). \tag{6.30}$$

Let us calculate

$$\begin{aligned}
\mathcal{F}\left(V_{g_1}g_2(\cdot,0)\right)(\omega) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_2(t)\overline{g_1(t-x)}\,dt e^{-2\pi i\omega x}\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_2(t)\overline{g_1(x')}e^{-2\pi i(x-y)}\,dx'dt \\
&= \mathcal{F}(g_2)(\omega)\mathcal{F}^{-1}(\overline{g_1})(\omega).
\end{aligned}$$

Therefore

$$\hat{h} = m * \mathcal{F}(g_2)\mathcal{F}^{-1}(\overline{g_1}). \tag{6.31}$$

Since the windows $g_1, g_2$ belong to $L^2(\mathbb{R}^d)$, we have $s(\omega) := \mathcal{F}(g_2)\mathcal{F}^{-1}(\overline{g_1}) \in L^1(\mathbb{R}^d)$ and the right-hand side of (6.31) is bounded and uniformly continuous. Since we are assuming $\hat{h}$ to be the characteristic function of an interval $\Omega \subsetneq \mathbb{R}$, we obtained the first assertion of the thesis. About estimate (6.27) we distinguish two cases. If there is $\omega_0 \in \mathbb{R}$ such that $|s(\omega_0)| = 1/2$, being the image of $\hat{h}$ the set $\{0,1\}$, then

$$\left|\hat{h}(\omega_0) - s(\omega_0)\right| \geq \left|\left|\hat{h}(\omega_0)\right| - |s(\omega_0)|\right| \geq \frac{1}{2}$$

which implies

$$\sup_{\omega\in\mathbb{R}}\left|\hat{h}(\omega) - s(\omega)\right| = \|T_{m_2} - G_a^{g_1,g_2}\|_{Op} \geq \frac{1}{2}.$$

If the value $1/2$ is never attained by $|s(\omega)|$ the argument is identical. This concludes the proof. □

In the finite discrete case the problem presents itself in a similar way. In the next theorem we state the necessary conditions on the window functions in order to get perfect equivalence. It can be seen that without subsampling perfect equivalence would in theory be always possible if $\operatorname{supp}(h) \subseteq \operatorname{supp}(\overline{\mathcal{I}g_1} * g_2)$. Numerically, we observe however the same behaviour as we have in the continuous case.

**Theorem 6.5.2.** ([3, Theorem 6.6]) *Le us fix a LTI filter $H\colon \mathbb{C}^N \to \mathbb{C}^N$ with impulse response $h \in \mathbb{C}^N$ and lattice constants $\alpha, \beta \geq 1$.*
*If $H$ can be written as a Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$ with lattice constants $\alpha, \beta$, for some symbol $a \in \mathbb{C}^{N\times N}$ and window functions $g_1, g_2 \in \mathbb{C}^N$, then the following hold for every $\forall u \in \operatorname{supp}(h)$:*

*1)* $V_{g_1}g_2(u,0) = (\overline{\mathcal{I}g_1} * g_2)(u) = \mathcal{C}_{g_1,g_2}(u) \neq 0$;

*2)* $V_{g_1}g_2(u + Bk, lA) = 0,\ \forall k = 0,\dots,\beta-1, \quad \forall l = 1,\dots,\alpha-1$;

*3)* $V_{g_1}g_2(u + Bk, 0) = 0,\ \forall k = 1,\dots,\beta-1 \quad s.t. \quad (u + Bk) \notin \operatorname{supp}(h)$;

*4)* $V_{g_1}g_2(u + Bk, 0) = \frac{h(u+Bk)}{h(u)} V_{g_1}g_2(u,0),\ \forall k = 1,\dots,\beta-1 \quad s.t. \quad (u + Bk) \in \operatorname{supp}(h)$.

*Vice versa, if there are window functions $g_1, g_2 \in \mathbb{C}^N$ fulfilling 1)–4), then there exists a symbol $a \in \mathbb{C}^{N\times N}$ such that $H = \mathsf{G}_a^{g_1,g_2}$.*

*Proof.* Let us assume that $H = \mathsf{G}_a^{g_1,g_2}$ for some $a \in \mathbb{C}^{N\times N}$, $g_1, g_2 \in \mathbb{C}^N$. Two operators are identical if and only if their spreading functions are identical. From (2.223) and (2.204), $H = \mathsf{G}_a^{g_1,g_2}$ if and only if

$$(h \otimes \delta)(u,v) = \frac{N}{\alpha\beta} \mathsf{S}_{\mathsf{P}}^{BA}(u,v) V_{g_1}g_2(u,v) \tag{6.32}$$

This, in turn is equivalent to

$$h(u) = N\,(\alpha\beta)^{-1}\,\mathsf{S}_{\mathsf{P}}^{BA}(u,0) V_{g_1}g_2(u,0), \qquad u = 0,\dots,N-1; \tag{6.33}$$

$$0 = N\,(\alpha\beta)^{-1}\,\mathsf{S}_{\mathsf{P}}^{BA}(u,v) V_{g_1}g_2(u,v), \qquad u,v = 0,\dots,N-1, \quad v \neq 0. \tag{6.34}$$

From equation (6.33) condition 1) follows. In fact, suing the switching property property of the STFT [35, Lemma 1.2.3] $V_{g_1}g_2(u,0) = \overline{V_{g_2}g_1(-u,0)} = \mathcal{C}_{g_1,g_2}(u)$. Note that for $\mathsf{S}_{\mathsf{P}}^{BA}(u,0) = 0$ by equation (6.32) we get $u \notin \operatorname{supp}(h)$. Hence by equation (6.34) together with the periodicity of $\mathsf{S}_{\mathsf{P}}^{BA}$ follows condition 2) follows. The periodicity of $\mathsf{S}_{\mathsf{P}}^{BA}$ in the time domain together with equation (6.33) gives condition 3). Finally by (6.33) and (2.203) we compute

$$\frac{\alpha\beta}{N} \frac{h(u)}{V_{g_1}g_2(u,0)} = \mathsf{S}_{\mathsf{P}}^{BA}(u,0) = \mathsf{S}_{\mathsf{P}}^{BA}(u + Bk, 0) = \frac{\alpha\beta}{N} \frac{h(u+Bk)}{V_{g_1}g_2(u+Bk,0)} \tag{6.35}$$

for $k = 1,...,\beta-1$, $(u + kB) \in \operatorname{supp}(h)$, hence we get condition 4).
On the other hand, let us consider $g_1, g_2 \in \mathbb{C}^N$ fulfilling conditions 1) − 4). Let us define for $u = 0,\dots,N-1$

$$V(u) := \begin{cases} V_{g_1}g_2(u,0) & \text{if} \quad u \in \operatorname{supp}(h) \\ 1 & \text{otherwise} \end{cases} \tag{6.36}$$

and

$$C(u) := \begin{cases} \#\{\{u + B\mathbb{Z}_N\} \cap \operatorname{supp}(h)\} & \text{if} \quad \{u + B\mathbb{Z}_N\} \cap \operatorname{supp}(h) \neq \varnothing \\ 1 & \text{otherwise,} \end{cases} \tag{6.37}$$

we notice that $C(u+Bk) = C(u)$ for any $k = 0, \dots, \beta-1$, since $u + B\mathbb{Z}_N = u + Bk + B\mathbb{Z}_N$. Let us observe that

$$\begin{aligned}\frac{h}{C\cdot V} * \chi_{B\mathbb{Z}_N}(u) &= \sum_{k=0}^{N-1} \frac{h(u-k)}{C(u-k)V(u-k)}\chi_{B\mathbb{Z}_N}(k) = \frac{1}{C(u)}\sum_{k=0}^{N-1}\frac{h(u-k)}{V(u-k)}\chi_{B\mathbb{Z}_N}(k)\\ &= \frac{1}{C}\cdot\left(\frac{h}{V} * \chi_{B\mathbb{Z}_N}\right)(u).\end{aligned}$$

We define

$$\mathsf{S}_{\mathsf{P}}^{BA}(u,v) \coloneqq \frac{\alpha\beta}{N}\left(\frac{h}{C\cdot V} * \chi_{B\mathbb{Z}_N}\right)\otimes \chi_{A\mathbb{Z}_N}(u,v), \tag{6.38}$$

which is periodic in the sense of (2.203) since $C(u+Bk) = C(u)$ for any $k = 0, \dots, \beta-1$ and

$$\begin{aligned}\left(\frac{h}{V} * \chi_{B\mathbb{Z}_N}\right)(u+Bk) &= \frac{1}{C(u)}\sum_{j=0}^{N-1}\frac{h(j)}{V(j)}\chi_{B\mathbb{Z}_N}(u+Bk-j)\\ &= \frac{1}{C(u)}\sum_{j=0}^{N-1}\frac{h(j)}{V(j)}\chi_{B\mathbb{Z}_N-Bk}(u-j)\\ &= \frac{1}{C(u)}\sum_{j=0}^{N-1}\frac{h(j)}{V(j)}\chi_{B\mathbb{Z}_N}(u-j)\\ &= \left(\frac{h}{V} * \chi_{B\mathbb{Z}_N}\right)(u).\end{aligned}$$

In order to verify (6.33), fix $u \in \{0, \dots, N-1\}$ and let us write the partition

$$\{0, \dots, \beta-1\} = S_{in}(u) \cup S_{out}(u),$$

where

$$\begin{aligned}S_{in}(u) &\coloneqq \{k \in \{0, \dots, \beta-1\} \mid u + Bk \in \operatorname{supp}(h)\},\\ S_{out}(u) &\coloneqq \{k \in \{0, \dots, \beta-1\} \mid u + Bk \notin \operatorname{supp}(h)\}.\end{aligned}$$

Therefore if $u \in \operatorname{supp}(h)$ we have $0 \in S_{in}(u) \neq \varnothing$, starting from the right-hand side of (6.33) and using 4) we get

$$\begin{aligned}N(\alpha\beta)^{-1}\mathsf{S}_{\mathsf{P}}^{BA}(u,0)V_{g_1}g_2(u,0) &= \frac{1}{C(u)}\sum_{k=0}^{\beta-1}\frac{h(u+Bk)}{V(u+Bk)}V_{g_1}g_2(u,0)\\ &= \frac{1}{C(u)}\sum_{k\in S_{in}(u)}\frac{h(u+Bk)}{V_{g_1}g_2(u+Bk,0)}V_{g_1}g_2(u,0)\\ &= \frac{1}{C(u)}\sum_{k\in S_{in}(u)}\frac{h(u)}{V_{g_1}g_2(u,0)}V_{g_1}g_2(u,0)\\ &= \frac{1}{C(u)}C(u)h(u).\end{aligned}$$

If $u \notin \operatorname{supp}(h)$ and $S_{in}(u) = \varnothing$, then $\mathsf{S}_{\mathsf{P}}^{BA}(u,0) = 0$ and (6.33) is fulfilled. If $u \notin \operatorname{supp}(h)$ and $S_{in}(u) \neq \varnothing$, then $u + Bj = z \in \operatorname{supp}(h)$ for some $j \in S_{in}(u)$. Hence we can write $u = z - Bj = z + Bs$ for a certain $s \in \{0, \dots, \beta - 1\}$ and from 3) we get $V_{g_1}g_2(u,0) = V_{g_1}g_2(z + Bs, 0) = 0$, which guarantees (6.33).
Equation (6.34) is fulfilled if $v \notin A\mathbb{Z}_N \setminus \{0\}$. Let us fix $v \in A\mathbb{Z}_N \setminus \{0\}$ and distinguish two cases: if $u$ appearing in (6.34) belongs to $\operatorname{supp}(h) + B\mathbb{Z}_N$, then $V_{g_1}g_2(u,v) = 0$ due to 2) and we are done; if $u$ does not belong to $\operatorname{supp}(h) + B\mathbb{Z}_N$, then $\mathsf{S}_{\mathsf{P}}^{BA}(u,v) = 0$ and (6.34) if verified once more.
Eventually, in order to find a symbol $a$ which gives the function $\mathsf{S}_{\mathsf{P}}^{BA}$ defined above, we use (2.205):

$$\alpha\beta \mathsf{F}_s(a \cdot \text{Ш}_{\alpha,\beta})(u,v) = \frac{\alpha\beta}{N}\left(\frac{h}{C \cdot V} * \chi_{B\mathbb{Z}_N}\right) \otimes \chi_{A\mathbb{Z}_N}(u,v). \tag{6.39}$$

Being $\mathsf{F}_s^{-1} = \mathsf{F}_s$ and for (2.111) we derive

$$\begin{aligned} a(u,v)\text{Ш}_{(\alpha,\beta)}(u,v) &= \frac{1}{N}\mathsf{F}_s\left(\left(\frac{h}{C \cdot V} * \chi_{B\mathbb{Z}_N}\right) \otimes \chi_{A\mathbb{Z}_N}\right)(u,v) \\ &= \frac{1}{N}A^{-1}\chi_{\alpha\mathbb{Z}_N}(u)\mathcal{F}_N\left(\frac{h}{C \cdot V} * \chi_{B\mathbb{Z}_N}\right)(v) \\ &= \frac{\alpha}{N^2}\chi_{\alpha\mathbb{Z}_N}(u)\mathcal{F}_N\left(\frac{h}{C \cdot V}\right)(v)\beta\chi_{\beta\mathbb{Z}_N}(v) \\ &= \frac{\alpha\beta}{N^2}\mathcal{F}_N\left(\frac{h}{C \cdot V}\right)(v)\text{Ш}_{(\alpha,\beta)}(u,v). \end{aligned}$$

So that a possible choice for the symbol is

$$a(u,v) = \frac{\alpha\beta}{N^2}\left(\mathbf{1} \otimes \mathcal{F}_N\left(\frac{h}{C \cdot V}\right)\right)(u,v). \tag{6.40}$$

This concludes the proof. □

**Remark 6.5.3.** *Theorem 6.5.2 can be seen as a special result on the reproducing property, compare [99] or [131] equation (4):*

$$f(t) \equiv \sqrt{2\pi}T \sum_{k=-\infty}^{\infty} f(kT)\psi(t - kT).$$

*If* $\operatorname{supp}(h) \subseteq (-B, B)$ *we would get perfect reproduction for* $V_{g_1}g_2(u,0) = 1$ *on* $u \in \operatorname{supp}(h)$ *and* $V_{g_1}g_2(u,v) = 0$ *outside the fundamental region of the adjoint lattice for* $(u,v) \notin (-B,B) \times (-A,A)$*. If* $\operatorname{supp}(h) \subseteq (-B,B)$*, the region* $X$ *with* $\operatorname{supp}(h) \subseteq X \subseteq (-B,B)$ *introduces the freedom to choose* $(\overline{\mathcal{I}g_1} * g_2)(u)$ *having smooth decay on* $X$*. As for an LTI filter* $\eta_H(u,v) = 0 \quad \forall v \neq 0$*, see* (2.223)*, we have this freedom in the frequency domain for* $Y := \{(x,y) : 0 < |y| < A\}$ *irrespective of the choice of* $h$*.*

The conditions given in Theorem 6.5.2 will be central for the remaining part of this section. Therefore a visual outline of them is shown in Figure 6.1. The next theorem can be seen as a special case of the last result, having no subsampling, i.e. $\alpha = \beta = 1$.

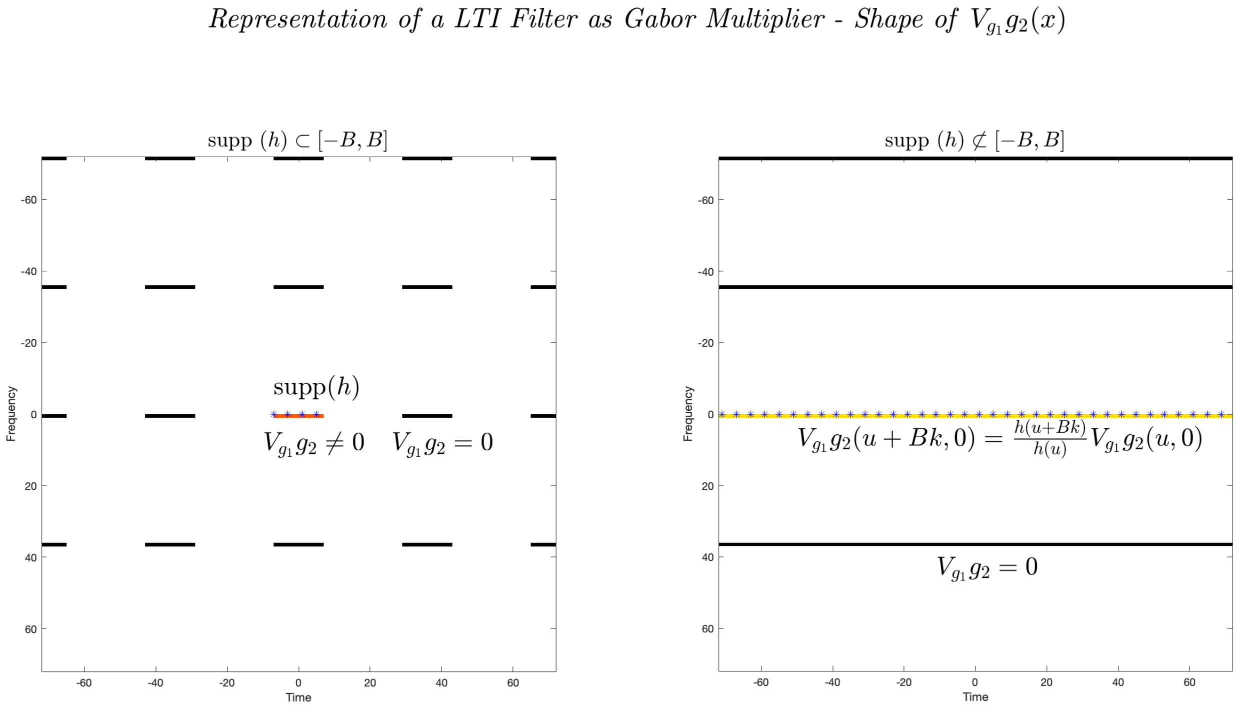


Figure 6.1: This figure gives a visual outline of Theorem 6.5.2 on the representation of a LTI filter by a Gabor multiplier. The conditions on the support of $V_{g_1}g_2$ are shown once for supp$(h) \subseteq [-B,B]$ and once for supp$(h) \not\subseteq [-B,B]$. Black lines indicate the regions where $V_{g_1}g_2$ has to be zero.

**Theorem 6.5.4.** ([3, Theorem 6.8]) *Consider a LTI filter $H\colon \mathbb{C}^N \to \mathbb{C}^N$ with impulse response $h \in \mathbb{C}^N$ and $g_1, g_2 \in \mathbb{C}^N$ with $\mathcal{C}_{g_1,g_2}(u) \neq 0$ for every $u = 0, \ldots, N-1$. Then the $H$ can be represented as Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$ with $\alpha = \beta = 1$ and lower symbol*

$$a = \frac{1}{N^2}\left(\mathbf{1} \otimes \mathcal{F}_N\left(\frac{h}{\mathcal{C}_{g_1,g_2}}\right)\right). \tag{6.41}$$

*Proof.* Let us observe that, since $\alpha = \beta = 1$, we have $\mathsf{S} = \mathsf{S}_{\mathsf{P}}^{BA}$, see (2.199) and (2.205). Taking $a$ as in (6.41), recalling $\overline{\mathcal{I}g_1} * g_2(\cdot) = V_{g_1}g_2(\cdot,0)$ and $\mathcal{F}_N(N^{-1}\mathbf{1})(v) = \delta(v)$, we compute

$$\mathsf{S}(u,v) = \mathsf{F}_s a(u,v) = \frac{1}{N}\mathsf{F}_s\left(\frac{1}{N}\mathbf{1} \otimes \mathcal{F}_N\left(\frac{h(\cdot)}{V_{g_1}g_2(\cdot,0)}\right)\right)(u,v) = \frac{1}{N}\frac{h(u)}{V_{g_1}g_2(u,0)} \cdot \delta(v).$$

Similarly to what done in the proof of Theorem 6.5.2, $H$ and $\mathsf{G}_a^{g_1,g_2}$ coincide if their spreading functions do; on account of the previous computation we get

$$h \otimes \delta(u,v) = N\mathsf{S}(u,v) V_{g_1} g_2(u,v) = \frac{h(u)}{V_{g_1} g_2(u,0)} \delta(v) V_{g_1} g_2(u,v)$$

which is true since $V_{g_1} g_2(u,0) = \overline{\mathcal{I} g_1} * g_2(u) = \mathcal{C}_{g_1,g_2}(u) \neq 0$ for every $u$. This concludes the proof. □

This means, given window functions, for which the convolution (up to $\mathcal{I}$ and a conjugation) is non-zero on the support of the impulse response $h$, a LTI filter $H$ can always be represented exactly as Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$. The error between the LTI filter and the Gabor multiplier is the error introduced through subsampling of the mask $a$. The representation is always possible if we allow for the degenerate case of $g_1 = g_2 = \mathbf{1}$. We should, however, keep in mind that if we want to have a meaningful parameter set for applications this is, after all, a very strong condition on the smoothness of $\hat{h}$. Even if met, for applications, the exact representation is not too well suited due to poor calculation efficiency and bad numerical behaviour for $\overline{\mathcal{I} g_1} * g_2$ close to zero.

Knowing from Theorem 6.5.4 that every LTI filter with bandlimited impulse response $h$ can be represented as Gabor multiplier, we are now turning the focus to the opposite direction, asking whether it is clear that a Gabor multiplier having a mask constant in time is equivalent to a LTI filter. Reading equation (6.41) the other way round, we see implicitly that a Gabor multiplier with time invariant symbol is a convolution operator. The frequency response of this convolution operator, however, is not exactly equal to the frequency mask of the Gabor multiplier but smoothed by a convolution with the Fourier transform of the window functions. In Figure 6.2 a visual representation can be found. Smooth window functions have the advantage of preserving the edges of the frequency mask rather well at the cost of a longer time delay needed in return. Theorem 6.5.5 formalizes this fact.

**Theorem 6.5.5.** ([3, Theorem 6.9]) *Consider a Gabor multiplier $\mathsf{G}_a^{g_1,g_2}$ with no time subsampling, i.e. $\alpha = 1$, windows $g_1, g_2 \in \mathbb{C}^N$ with $g_1$ symmetric and symbol*

$$a = \mathbf{1} \otimes \hat{h} \tag{6.42}$$

*for some $\hat{h} \in \mathbb{C}^N$. Then, it is also a LTI filter with impulse response*

$$\frac{1}{\beta} \sum_{k=0}^{\beta-1} h(\cdot + Bk)(\overline{g_1} * g_2)(\cdot). \tag{6.43}$$

*Proof.* We start from the kernel representation of the Gabor multiplier (2.198) with $\alpha = 1$

$$\begin{aligned} K(\mathsf{G}_a^{g_1,g_2})(u,v) &= \sum_{k=0}^{N-1} \sum_{l=0}^{B-1} a(k,\beta l) \overline{g_1(v-k)} g_2(u-k) e^{\frac{2\pi i \beta l (u-v)}{N}} \\ &= \sum_{k=0}^{N-1} \sum_{l=0}^{B-1} \hat{h}(\beta l) \overline{g_1(v-k)} g_2(u-k) e^{\frac{2\pi i \beta l (u-v)}{N}} \\ &= \sum_{l=0}^{B-1} \hat{h}(\beta l) e^{\frac{2\pi i \beta l (u-v)}{N}} \sum_{k=0}^{N-1} \overline{g_1(v-k)} g_2(u-k). \end{aligned} \tag{6.44}$$

Fixing $v \in \{0, \ldots, N-1\}$, performing the change of variable $t = v - k$ and using the symmetry of $g_1$, we write the second factor as

$$\begin{aligned}
\sum_{k=0}^{N-1} \overline{g_1(v-k)} g_2(u-k) &= \sum_{t=0}^{N-1} \overline{g_1(t)} g_2(u-v+t) \\
&= \sum_{t=0}^{N-1} \overline{g_1(t)} g_2(u-v-t) \\
&= (\overline{g_1} * g_2)(u-v).
\end{aligned}$$

For the first factor in (6.44), using (2.111):

$$\begin{aligned}
\sum_{l=0}^{B-1} \hat{h}(\beta l) e^{\frac{2\pi i \beta l (u-v)}{N}} &= \mathcal{F}_N^{-1}(\hat{h} \cdot \chi_{\beta \mathbb{Z}_N})(u-v) \\
&= \left( \mathcal{F}_N^{-1} \hat{h} * \mathcal{F}_N^{-1} \chi_{\beta \mathbb{Z}_N} \right)(u-v) \\
&= \sum_{k=0}^{N-1} h(u-v-k) \frac{B}{N} \chi_{B\mathbb{Z}_N}(-k) \\
&= \frac{1}{\beta} \sum_{k=0}^{\beta-1} h(u-v+Bk).
\end{aligned}$$

Eventually we get

$$K(\mathsf{G}_a^{g_1,g_2})(u,v) = \frac{1}{\beta} \sum_{k=0}^{\beta-1} h(u-v+Bk)(\overline{g_1} * g_2)(u-v) \tag{6.45}$$

and the result follows by (2.222). □

We observe that the convolution in (6.43) is the restriction of $V_{g_1} g_2$ to the time-axis, since we are considering a symmetric window $g_1$.
It is important to note that the LTI property is only valid in case of no time subsampling. In the case of a common Gabor multiplier with $\alpha > 1$, in contrast, the second sum in equation (6.44) would depend on $u$ and be $\alpha-$periodic, explicitly:

$$\sum_{k=0}^{A-1} \overline{g_1(v-\alpha k)} g_2(u-\alpha k).$$

Therefore as soon as we have time domain subsampling of the signal, the LTI property of the operator is lost even though the mask being constant in time.

As already mentioned, it becomes apparent that an LTI filter can be considered as a special case of a Gabor multiplier with degenerated window functions $g_1 = g_2 = \mathbf{1}$. We want to put emphasis also on the interconnection between sharp frequency cut off of the filter and smoothness of the window functions corresponding to a time delay in filtering. Condition 1) of Theorem 6.5.2 requires the impulse response $h$ to have a faster decay than $\overline{\mathcal{I}g_1} * g_2$. This means that in case we want to have a sharp cut off in the frequency filter $\hat{h}$, which corresponds to a slow decay in $h$, we have to choose a smooth window function which corresponds to a large time lag.

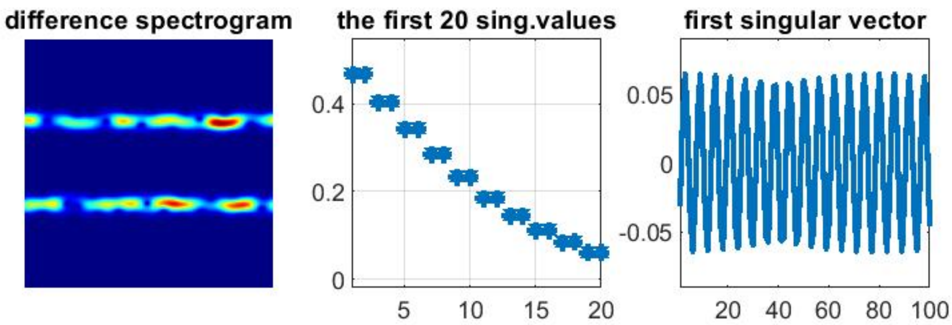


Figure 6.2: The figure shows the effect of implementing a Gabor (STFT) multiplier with mask $a = \mathbf{1} \otimes \chi_\Omega$, $\Omega = [-R, R]$, with $R = 80$ and $N = 480$. The resulting operator is still an LTI operator as long as no subsampling is performed ($\alpha = \beta = 1$), but now looking at the difference of the spectrograms given in the first plot, which is strongly concentrated around the cut-off frequency. The central plot shows the 20 largest singular values of the difference between the implemented STFT multiplier and the perfect low pass filter. In the last plot, we show only a segment of the first singular vector of the difference, to demonstrate the high regular oscillations.

# Chapter 7

# Quantum Harmonic Analysis

This last chapter presents the result by F. Luef and the author contained in [10]. The main aim is to provide a class of operators in quantum harmonic analysis (QHA) which is the counterpart of the Feichtinger algebra $\mathcal{S}_0$ in classical harmonic analysis. For this reason, we shall name such class the set of Feichtinger operators and denote it by $\mathbb{S}_0$. The space $\mathbb{S}_0$, which turns out to be a Banach $*$-algebra, was introduced by H. G. Feichtinger and M. S. Jakobsen in [62] and can be described as follows:

$$\mathbb{S}_0 = \{S\colon \mathcal{S}_0'(\mathbb{R}^d) \to \mathcal{S}_0(\mathbb{R}^d) \mid S \text{ is linear, continuous and } K_S \in \mathcal{S}_0(\mathbb{R}^{2d})\},$$

where $K_S$ is the integral kernel of $S$. $\mathbb{S}_0$ proves to be a valid alternative to the Fréchet space of Schwartz operators $\mathfrak{S}$, introduced in [103], which consists of all the pseudo-differential operators with Weyl symbol in the Schwartz class $\mathcal{S}(\mathbb{R}^{2d})$. Roughly speaking, to work with $\mathbb{S}_0$ in place of $\mathfrak{S}$ is more advantageous as well as to work with $\mathcal{S}_0(\mathbb{R}^d)$ instead of $\mathcal{S}(\mathbb{R}^d)$. We also introduce a weighted version of $\mathbb{S}_0$, $\mathbb{M}^1_s$, and exhibit a characterization for $\mathfrak{S}$ in terms of such weighted classes. Theorem 7.3.6 states the following:

$$\mathfrak{S} = \bigcap_{s\geq 0} \mathbb{M}^1_s.$$

As a consequence, we obtain a result in the spirit of [95]: Corollary 7.3.7 shows that, if the STFT of any $\tau$-symbol $\mathrm{a}^S_\tau$ of an operator $S \in B(L^2)$ is rapidly decaying, then $S$ belongs to the Schwartz class $\mathfrak{S}$.

As preliminaries, in order to show that Feichtinger operators are a suitable environment for QHA, new tools are introduced and new perspectives on the well-known $\tau$-quantization $\mathrm{Op}_\tau$ are shed. E.g., for every $\tau \in [0,1]$ we define the $\tau$-Wigner distribution of an operator $S$ as follows:

$$W_\tau S(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i t\omega} K_S(x+\tau t, x-(1-\tau)t)\, dt.$$

To give a flavour of our results concerning the interplay of $\mathrm{Op}_\tau$ and $W_\tau$, we report the statement of Theorem 7.2.7.

*For every $\tau \in [0,1]$ the following mappings are linear and continuous:*

$$\mathrm{Op}_\tau\colon \mathcal{S}_0'(\mathbb{R}^{2d}) \to \mathbb{S}_0', \qquad W_\tau\colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d}).$$

*Moreover,* $\mathrm{Op}_\tau$ *is the Banach space adjoint of* $W_\tau$*:* $\mathrm{Op}_\tau = W_\tau^*$*, i.e. for every* $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ *and* $S \in \mathbb{S}_0$

$$ {}_{\mathcal{S}_0'}\langle a, W_\tau S\rangle_{\mathcal{S}_0} = {}_{\mathbb{S}_0'}\langle \mathrm{Op}_\tau(a), S\rangle_{\mathbb{S}_0}. $$

The chapter is structured as follows. In Section 7.1 we recall the necessary preliminary tools, e.g. convolutions between functions and operators $a \star S$ or between operators and operators $T \star S$, and introduce some new ones. Namely: the $\tau$-STFT $V^\tau$, the $\tau$-Wigner distribution for operators $W_\tau$, the Fourier-$\tau$-Wigner transform for operators $\mathcal{F}_{W_\tau}$, the $\tau$-spreading representation operator $\mathrm{SR}^\tau$. Section 7.2 introduces Feichtinger operators $\mathbb{S}_0$, recalling the important Outer and Inner Kernel Theorems from [62], and studies them in the framework of QHA. In Subsection 7.2.1 the mappings $\mathcal{F}_{W_\tau}$ and $W_\tau$ are studied on $\mathbb{S}_0$, as well as $\mathrm{Op}_\tau$ and $\mathrm{SR}^\tau$ on $\mathcal{S}_0'(\mathbb{R}^{2d})$. Subsection 7.2.2 extends the convolution $\star$ to elements in $\mathbb{S}_0$ and $\mathbb{S}_0'$, showing that Feichtinger operators are actually a suitable environment for QHA. As a consequence, we are able to extend $\mathcal{F}_{W_\tau}$ and $W_\tau$ to $\mathbb{S}_0'$. In Subsection 7.2.3 we define the $\tau$-Cohen's class representation, with kernel $a$, of an operator $S$ $Q_a^\tau(S)$ and the definition of $Q_S^\tau(f)$ from [108] is recalled. Already known objects will be recovered in the form of $Q_a^\tau(S)$ and we shall observe that $Q_a^\tau(S)$ coincides with the $\tau$-symbol of the mixed-state localization operator $a \star S$. Some interplays between the Gabor matrix of an operator, the $\tau$-Cohen's class, the trace and the $\tau$-Wigner distribution are then exhibited. Eventually, Subsection 7.3 introduces the weighted classes of Feichtinger operators $\mathbb{M}_s^1$ and provides a characterization for Schwartz operators $\mathfrak{S}$ in terms of $\mathbb{M}_s^1$.

## 7.1 Preliminaries

We recall the already known tools of QHA and introduce new one, such as $V_g^\tau f$, $\mathcal{F}_{W_\tau} S$, $W_\tau S$, $\mathrm{SR}^\tau a$. The $\tau$-quantization $\mathrm{Op}_\tau$ and the cross-$\tau$-Wigner distribution $W_\tau(f,g)$ can be found in Chapter 2.
Even if not specified, the parameter $\tau$ always belongs to $[0,1]$.

### 7.1.1 A continuum of (new) time-frequency representations

**Definition 7.1.1.** *Given* $\tau \in [0,1]$*, the* $\tau-$***time-frequency shift (τ-TFS) at*** $(x,\omega) \in \mathbb{R}^{2d}$ *is defined to be*

$$ \pi^\tau(x,\omega) := e^{-2\pi i \tau x\omega} M_\omega T_x = M_{(1-\tau)\omega} T_x M_{\tau\omega}. \tag{7.1} $$

For $\tau = 0$ we recover the usual TFS $\pi^0 = \pi$. The following relations are due to easy computations, hence we left them to the reader:

$$ \begin{aligned} \pi^\tau(x,\omega)\pi^\tau(x',\omega') &= e^{-2\pi i[(1-\tau)x\omega' - \tau x'\omega]}\pi^\tau(x+x',\omega+\omega'), \\ \pi^\tau(x,\omega)\pi^\tau(x',\omega') &= e^{-2\pi i[x\omega' - x'\omega]}\pi^\tau(x',\omega')\pi^\tau(x,\omega), \\ \pi^\tau(x,\omega)^* &= \pi^{1-\tau}(-x,-\omega) = e^{-2\pi i(1-\tau)x\omega}\pi(-x,-\omega). \end{aligned} $$

**Definition 7.1.2.** *For* $f,g \in L^2(\mathbb{R}^d)$ *we define the* $\tau$***-short-time Fourier transform (τ-STFT) of*** $f$ ***w.r.t*** $g$*:*

$$ V_g^\tau f(x,\omega) := \langle f, \pi^\tau(x,\omega)g\rangle, \qquad \forall x,\omega \in \mathbb{R}^d. \tag{7.2} $$

Of course, the $\tau$-STFT can be defined for any suitable dual pair. As can be easily verified, each mapping

$$\pi^\tau : \mathbb{R}^{2d} \to \mathcal{U}(L^2(\mathbb{R}^d)),$$

where $\mathcal{U}(L^2(\mathbb{R}^d))$ denotes the unitary operators on $L^2$, is a projective representation. So that $V^\tau$ is the wavelet transform associated to $\pi^\tau$, hence $V_g^\tau f$ is a continuous function.

**Remark 7.1.3.** *Of course for $\tau = 0$ we come back to the usual STFT $V^0 = V$ and we have*

$$V_g^\tau f(x,\omega) = e^{2\pi i \tau x\omega} V_g f(x,\omega). \tag{7.3}$$

*From the above equation, we notice that $V_g^{\frac{1}{2}} f$ is exactly the* ***cross-ambiguity function of*** *$f$* ***and*** *$g$ $A(f,g)$:*

$$V_g^{\frac{1}{2}} f(x,\omega) = A(f,g)(x,\omega). \tag{7.4}$$

### 7.1.2 Fundamental and new tools of QHA

In this subsection we introduce the fundamentals definitions of quantum harmonic analysis which were introduced by R. Werner in [150]. We shall see in Definition 7.2.18 how to extend the following definitions.

**Definition 7.1.4.** *Consider $z \in \mathbb{R}^{2d}$ and $T \in B(L^2(\mathbb{R}^d))$. The* ***translation of*** *$T$* ***at*** *$z$ is*

$$\alpha_z(T) := \pi(z) T \pi(z)^*. \tag{7.5}$$

*The* ***involution of*** *$T$ is set to be:*

$$\check{T} := \mathcal{I} T \mathcal{I}. \tag{7.6}$$

*Given $a \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{J}^1$, trace class on $L^2(\mathbb{R}^d)$, the* ***convolution between*** *$a$* ***and*** *$S$ is the operator*

$$a \star S := S \star a := \int_{\mathbb{R}^{2d}} a(z) \alpha_z(S)\, dz, \tag{7.7}$$

*where the integral has to be understood in weak sense. The* ***convolution of two operators*** *$S, T \in \mathcal{J}^1$ is the function defined for every $z \in \mathbb{R}^{2d}$ as*

$$S \star T(z) := \operatorname{tr}\left(S \alpha_z(\check{T})\right). \tag{7.8}$$

It is straightforward to check that $\alpha_z \alpha_{z'} = \alpha_{z+z'}$. In this chapter, we reserve the symbol $\otimes$ for rank-one operators. Namely, given $f, g \in L^2(\mathbb{R}^d)$:

$$(f \otimes g)\psi := \langle \psi, g \rangle f, \qquad \forall \psi \in L^2(\mathbb{R}^d). \tag{7.9}$$

Trivially, the kernel of the operator $f \otimes g$ is the tensor product *of functions* $f(x)\overline{g(y)}$:

$$(f \otimes g)\psi(t) = \langle \psi, g \rangle f(t) = \int_{\mathbb{R}^d} f(t)\overline{g(x)}\psi(x)\, dx.$$

So that, when we will need the *functions'* tensor product $f(x)g(y)$, we shall adopt the notation

$$K_{f \otimes \overline{g}}(x,y) = f(x)g(y), \tag{7.10}$$

$K_S$ being the integral kernel of the operator $S$.
We now interpret (2.81) as the cross-$\tau$-Wigner distribution of the rank-one operator $f \otimes g$. Hence, we naturally define the $\tau$-Wigner distribution of an operator as follows.

**Definition 7.1.5.** *Let $S$ be an operator with integral kernel $K_S$ and let $\tau \in [0,1]$. Then the* ***$\tau$-Wigner distribution of** $S$ is*

$$W_\tau S(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i t\omega} K_S(x+\tau t, x-(1-\tau)t)\, dt. \tag{7.11}$$

**Definition 7.1.6.** *For $S \in \mathcal{J}^1$ and $\tau \in [0,1]$, the* ***Fourier-$\tau$-Wigner transform of** $S$ is defined to be:*

$$\mathcal{F}_{W_\tau} S(z) := \operatorname{tr}\left(\pi^\tau(z)^* S\right), \qquad \forall z \in \mathbb{R}^{2d}. \tag{7.12}$$

For $\tau = 1/2$ we recover the usual Fourier-Wigner transform [150].

**Definition 7.1.7.** *We call* ***$\tau$-spreading representation of** $S \in B(L^2)$ an expression of type*

$$S = \int_{\mathbb{R}^{2d}} h(z)\pi^\tau(z)\, dz, \tag{7.13}$$

*where the integral is understood in weak sense. The function $h$ is called* ***$\tau$-spreading function of** $S$.*

We shall see the $\tau$-spreading representation as mapping which assign to a function an operator, hence we give the following definition.

**Definition 7.1.8.** *For $\tau \in [0,1]$ and $h \in L^1(\mathbb{R}^{2d})$ we define the* ***$\tau$-spreading representation operator*** *as the mapping*

$$h \mapsto \mathrm{SR}^\tau h := \int_{\mathbb{R}^{2d}} h(z)\pi^\tau(z)\, dz. \tag{7.14}$$

Let $\mathcal{F}_\sigma$ denote the symplectic Fourier transform, we are know able to collect in the following lemma a number of important relations which involve many of the tools presented so far. The proofs are standard computations and the canonical decompositions of $S$ and $T$ [126] are used, we leave them to the interested reader.

**Lemma 7.1.9.** *Let $f, g, \in L^2(\mathbb{R}^d)$, $S, T \in \mathcal{J}^1$, $a \in L^1(\mathbb{R}^{2d})$ and $\tau \in [0,1]$. Then:*

- *(i)* $\mathcal{F}_\sigma(W_\tau(f\otimes g)) = V_g^\tau f$;
- *(ii)* $\mathcal{F}_{W_\tau}(f\otimes g) = V_g^\tau f$;
- *(iii)* $W_\tau S = \mathcal{F}_\sigma \mathcal{F}_{W_\tau} S$;
- *(iv)* $\mathcal{F}_{W_\tau} S(x,\omega) = e^{-2\pi i(1/2-\tau)x\omega}\mathcal{F}_{W_{1/2}} S(x,\omega)$;
- *(v)* $\mathcal{F}_\sigma(S \star T) = \mathcal{F}_{W_\tau} S \cdot \mathcal{F}_{W_{1-\tau}} T = \mathcal{F}_{W_{1-\tau}} S \cdot \mathcal{F}_{W_\tau} T$;
- *(vi)* $\mathcal{F}_{W_\tau}(a \star S) = \mathcal{F}_\sigma a \cdot \mathcal{F}_{W_\tau} S$;
- *(vii)* $\mathcal{F}_{W_\tau} S$ *is the $\tau$-spreading function of $S$, i.e.* $S = \int_{\mathbb{R}^{2d}} \mathcal{F}_{W_\tau} S(z)\pi^\tau(z)\, dz$.

We notice that if we consider the rank-one operator $S = f \otimes g$, then item *(iii)* and *(ii)* of the previous lemma give

$$W_\tau(f,g) = W_\tau(f\otimes g) = \mathcal{F}_\sigma V_g^\tau f. \tag{7.15}$$

### 7.1.3 $\tau$-quantization of functions

Recall that given an operator $S$, we denote by $\mathrm{a}_\tau^S$ its $\tau$-symbol, i.e. that tempered distribution such that $\mathrm{Op}_\tau\left(\mathrm{a}_\tau^S\right) = S$.

**Remark 7.1.10.** *Under suitable assumptions, for example $a \in L^1(\mathbb{R}^{2d})$, straightforward calculations give*

$$\mathrm{Op}_\tau(a) = \int_{\mathbb{R}^{2d}} \mathcal{F}_\sigma a(z)\pi^\tau(z)\,dz,$$

*and since also $\mathcal{F}_{W_\tau}\mathrm{Op}_\tau(a)$ is the $\tau$-spreading function of $\mathrm{Op}_\tau(a)$ we have*

$$a = \mathcal{F}_\sigma\mathcal{F}_{W_\tau}\mathrm{Op}_\tau(a). \tag{7.16}$$

*Hence for $S \in \mathcal{J}^1$*

$$\mathrm{a}_\tau^S = \mathcal{F}_\sigma\mathcal{F}_{W_\tau} S = W_\tau S. \tag{7.17}$$

Given $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $f, g \in \mathcal{S}_0(\mathbb{R}^d)$, we recall the definition of ***cross-$\tau$-Cohen's class representation of $f$ and $g$, with kernel $a$***:

$$Q_a^\tau(f,g) \coloneqq a * W_\tau(f,g). \tag{7.18}$$

We shall extend this definition in the next section.

## 7.2 Feichtinger operators

In this section we summarize some important results concerning a class of operators studied in [62]. For such operators, introduced below, we adopt the name "Feichtinger operators" for reasons which will appear clear. We stick to the Euclidean setting in which we are interested, although the treatment shown in [62] is far more general.

**Definition 7.2.1.** *The* ***set of Feichtinger operators*** *is defined to be*

$$\begin{aligned}\mathbb{S}_0 \coloneqq \{S\colon \mathcal{S}_0'(\mathbb{R}^d) \to \mathcal{S}_0(\mathbb{R}^d) \mid &\ S \textit{ is linear, continuous and} \\ &\textit{maps norm bounded w-}* \textit{ convergent sequences in } \mathcal{S}_0' \\ &\textit{into norm convergent sequences in } \mathcal{S}_0\}.\end{aligned} \tag{7.19}$$

We adopt the following notation:

$$\mathbb{S}_0' \coloneqq B(\mathcal{S}_0(\mathbb{R}^d), \mathcal{S}_0'(\mathbb{R}^d)) \tag{7.20}$$

and state the so called Outer Kernel Theorem [62, Theorem 1.1].

**Theorem 7.2.2** (Outer Kernel)**.** *The Banach space $\mathbb{S}_0'$ is isomorphic to $\mathcal{S}_0'(\mathbb{R}^{2d})$ via the map $T \mapsto K_T$, where the relation between $T$ and its kernel $K_T$ is given by*

$${}_{\mathcal{S}_0'}\langle Tf, g\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle K_T, K_{g\otimes f}\rangle_{\mathcal{S}_0}, \qquad \forall\, f, g, \in \mathcal{S}_0(\mathbb{R}^d).$$

The following synthesized result goes under the name of Inner Kernel Theorem, we present it in our setting, i.e. $\mathcal{S}_0'$ is meant to be the sent of conjugate-linear continuous functionals on $\mathcal{S}_0$. To this end, we introduce the following notation: given $\sigma, \nu \in \mathcal{S}_0'(\mathbb{R}^d)$, we denote by $\nu\widetilde{\otimes}\overline{\sigma}$ the unique element of $\mathcal{S}_0'(\mathbb{R}^{2d})$ such that

$$_{\mathcal{S}_0'}\langle \nu\widetilde{\otimes}\overline{\sigma}, K_{\psi\otimes\overline{\varphi}}\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle \nu,\psi\rangle_{\mathcal{S}_0}\overline{_{\mathcal{S}_0'}\langle \sigma,\overline{\varphi}\rangle_{\mathcal{S}_0}}, \qquad \forall\, \psi, \varphi \in \mathcal{S}_0(\mathbb{R}^{2d}).$$

We address the reader to [62, Theorem 1.3], Lemma 3.1 and Corollary 3.10 also.

**Theorem 7.2.3** (Inner Kernel)**.** *The space of Feichtinger operators $\mathbb{S}_0$ is a Banach space if endowed with the norm of $B(\mathcal{S}_0', \mathcal{S}_0)$ and it is naturally isomorphic as Banach space to $\mathcal{S}_0(\mathbb{R}^{2d})$ through the map $T \mapsto K_T$, where the relation between $T$ and its kernel $K_T$ is given by*

$$_{\mathcal{S}_0'}\langle \nu, T\sigma\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle \nu\widetilde{\otimes}\overline{\sigma}, , K_T\rangle_{\mathcal{S}_0}, \qquad \forall\, \sigma, \nu, \in \mathcal{S}_0'(\mathbb{R}^d).$$

*Moreover, $\mathbb{S}_0$ is Banach algebra under composition. If $S, T \in \mathbb{S}_0$, then*

$$K_{S\circ T}(y,u) = \int_{\mathbb{R}^d} K_T(y,t)K_S(t,u)\,dt. \tag{7.21}$$

On account of Theorem 7.2.2 and 7.2.3, $\mathbb{S}_0'$ is the (conjugate) topological dual of $\mathbb{S}_0$ and the duality is given by

$$_{\mathbb{S}_0'}\langle T, S\rangle_{\mathbb{S}_0} = {}_{\mathcal{S}_0'}\langle K_T, K_S\rangle_{\mathcal{S}_0}.$$

**Lemma 7.2.4.** *Let $S \in \mathbb{S}_0$, then there are two non-unique sequences $\{f_n\}_n, \{g_n\}_n \subseteq \mathcal{S}_0(\mathbb{R}^d)$ such that*

$$S = \sum_{n=1}^{\infty} f_n \otimes g_n, \qquad \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{S}_0} \|g_n\|_{\mathcal{S}_0} < +\infty, \qquad K_S = \sum_{n=1}^{\infty} K_{f_n\otimes g_n}.$$

*Moreover,*

$$\mathbb{S}_0 \hookrightarrow \mathcal{J}^1$$

*with*

$$\mathrm{tr}(S) = \int_{\mathbb{R}^d} K_S(x,x)\,dx.$$

*Proof.* We just have to prove the continuous inclusion of Feichtinger operators into $\mathcal{J}^1$, all the remaining statements can be found in [62], see in particular Corollary 3.15 and Remark 9. The claim comes after an easy computation:

$$\begin{aligned}\|S\|_{\mathcal{J}^1} = |\mathrm{tr}(A)| &\leq \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} |f_n(x)g_n(x)|\,dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} |f_n(x)g_n(x)|\,dx \\ &\leq \sum_{n=1}^{\infty} \|f_n\|_{L^2} \|g_n\|_{L^2} \lesssim \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{S}_0} \|g_n\|_{\mathcal{S}_0} < \infty.\end{aligned}$$

Since $\mathcal{S}_0(\mathbb{R}^{2d}) = \mathcal{S}_0(\mathbb{R}^d)\hat{\otimes}\mathcal{S}_0(\mathbb{R}^d)$, see e.g. [62, Lemma 2.1], we get

$$\|S\|_{\mathcal{J}^1} \lesssim \|K_S\|_{\mathcal{S}_0} \asymp \|S\|_{\mathbb{S}_0}$$

and the proof is concluded. □

Together with the observations in [62, p. 4], we have

$$\mathbb{S}_0 \hookrightarrow \mathcal{J}^1 \hookrightarrow \mathcal{J}^2 \hookrightarrow B(L^2(\mathbb{R}^d)) \hookrightarrow \mathbb{S}_0'. \tag{7.22}$$

The fact that all Feichtinger operators are trace class implies the validity of Lemma 7.1.9.

### 7.2.1 $\tau$-quantization of operators

The present subsection is devoted to the study of $\mathrm{Op}_\tau$, $W_\tau$, $\mathcal{F}_{W_\tau}$ and $\mathrm{SR}^\tau$ on one of the following spaces: $\mathbb{S}_0$, $\mathbb{S}_0'$, $\mathcal{S}_0(\mathbb{R}^{2d})$ and $\mathcal{S}_0'(\mathbb{R}^{2d})$.
We shall see, e.g. that $\mathrm{Op}_\tau$ can be interpreted as the Banach space adjoint of $W_\tau$, see Theorem 7.2.6 and Theorem 7.2.7. The following remark is the key insight for the mentioned results.

**Remark 7.2.5.** *Let us consider $f, g \in L^2(\mathbb{R}^d)$ such that $f \neq 0$, $a \in L^2(\mathbb{R}^{2d})$ and $\{f_j\}_j$ o.n.b. for $L^2$ with $f_1 = f$. Then we compute as follows:*

$$\begin{aligned}\langle \mathrm{Op}_\tau(a)f, g\rangle &= \langle \mathrm{Op}_\tau(a)f, \sum_{j=1}^{\infty}\langle g, f_j\rangle f_j\rangle = \sum_{j=1}^{\infty}\langle \mathrm{Op}_\tau(a)\left(\langle f_j, g\rangle f\right), f_j\rangle \\ &= \sum_{j=1}^{\infty}\langle \mathrm{Op}_\tau(a)(f\otimes g)f_j, f_j\rangle = \mathrm{tr}\left(\mathrm{Op}_\tau(a)(f\otimes g)\right).\end{aligned}$$

*Taking into account the weak definition of* $\mathrm{Op}_\tau(a)$ *and* (7.11) *we can write*

$$\langle \mathrm{Op}_\tau(a)f, g\rangle = \langle a, W_\tau((f\otimes g)^*)\rangle = \mathrm{tr}\left(\mathrm{Op}_\tau(a)(f\otimes g)\right) = {}_{\mathcal{J}^\infty}\langle \mathrm{Op}_\tau(a), (f\otimes g)^*\rangle_{\mathcal{J}^1}. \tag{7.23}$$

*We can perform computations similar to the ones above for $S \in \mathcal{J}^1$ with canonical decomposition $\sum_{k=1}^{\infty}\lambda_k f_k \otimes g_k$ after extending $\{f_k\}_k$ to an o.n.b., in this case we obtain*

$$\langle a, W_\tau S\rangle = \mathrm{tr}\left(\mathrm{Op}_\tau(a)S^*\right) = {}_{\mathcal{J}^\infty}\langle \mathrm{Op}_\tau(a), S\rangle_{\mathcal{J}^1}. \tag{7.24}$$

**Theorem 7.2.6.** *For every $\tau \in [0, 1]$ the following mappings are linear and continuous:*

$$\mathrm{Op}_\tau\colon L^2(\mathbb{R}^{2d}) \to \mathcal{J}^\infty, \qquad W_\tau\colon \mathcal{J}^1 \to L^2(\mathbb{R}^{2d}).$$

*Moreover,* $\mathrm{Op}_\tau$ *is the Banach space adjoint of* $W_\tau$*:* $\mathrm{Op}_\tau = W_\tau^*$.

*Proof.* The boundedness of $\mathrm{Op}_\tau$ is trivial; the proof of the continuity of $W_\tau$ follows the same pattern shown in the proof of the subsequent Theorem 7.2.7. The last claim is just (7.24). □

**Theorem 7.2.7.** *For every $\tau \in [0, 1]$ the following mappings are linear and continuous:*

$$\mathrm{Op}_\tau\colon \mathcal{S}_0'(\mathbb{R}^{2d}) \to \mathbb{S}_0', \qquad W_\tau\colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d}).$$

*Moreover,* $\mathrm{Op}_\tau$ *is the Banach space adjoint of* $W_\tau$*:* $\mathrm{Op}_\tau = W_\tau^*$*, i.e. for every $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $S \in \mathbb{S}_0$*

$${}_{\mathcal{S}_0'}\langle a, W_\tau S\rangle_{\mathcal{S}_0} = {}_{\mathbb{S}_0'}\langle \mathrm{Op}_\tau(a), S\rangle_{\mathbb{S}_0}. \tag{7.25}$$

*Proof.* The boundedness and linearity of $\mathrm{Op}_\tau$ need no proof; using the formal representation of $\mathrm{Op}_\tau(a)$ we can given the formal expression for its kernel:

$$K_{\mathrm{Op}_\tau(a)}(t, x) = \int_{\mathbb{R}^d} e^{2\pi i(t-x)\omega} a((1-\tau)t + \tau x, \omega)\, d\omega. \tag{7.26}$$

Let us consider first $f, g \in \mathcal{S}_0$, then a standard argument, see e.g. [35, Proposition 1.3.25], gives that

$$W_\tau(f\otimes g) = W_\tau(f, g) \in \mathcal{S}_0(\mathbb{R}^{2d}) \qquad \text{with} \qquad \|W_\tau(f\otimes g)\|_{\mathcal{S}_0} \lesssim \|f\|_{\mathcal{S}_0}\|g\|_{\mathcal{S}_0}.$$

Since Lemma 7.1.9 holds true on $\mathbb{S}_0$, we write $W_\tau = \mathcal{F}_\sigma \mathcal{F}_{W_\tau}$ and use a representation for $S$ of type $\sum_{n=1}^\infty f_n \otimes g_n$ as shown in Lemma 7.2.4. Now we compute:

$$\begin{aligned} \mathcal{F}_{W_\tau} S(z) &= \mathrm{tr}(\pi^\tau(z)^* S) = \mathrm{tr}(\sum_{n=1}^\infty \pi^\tau(z)^*(f_n \otimes g_n)) \\ &= \sum_{n=1}^\infty \langle \pi^\tau(z)^* f_n, g_n\rangle = \sum_{n=1}^\infty V_{g_n}^\tau f_n(z). \end{aligned} \tag{7.27}$$

Taking a suitable window for the norm on $\mathcal{S}_0(\mathbb{R}^{2d})$ [101, Theorem 5.3] we have

$$\|\mathcal{F}_{W_\tau} S\|_{\mathcal{S}_0} \leq \sum_{n=1}^\infty \left\|V_{g_n}^\tau f_n\right\|_{\mathcal{S}_0} = \sum_{n=1}^\infty \|f_n\|_{\mathcal{S}_0} \|g_n\|_{\mathcal{S}_0} < +\infty.$$

Therefore

$$\begin{aligned} \|\mathcal{F}_{W_\tau} S\|_{\mathcal{S}_0} &\leq \inf\{\sum_{n=1}^\infty \|f_n\|_{\mathcal{S}_0} \|g_n\|_{\mathcal{S}_0}, S = \sum_{n=1}^\infty f_n \otimes g_n\} \\ &\leq \inf\{\sum_{n=1}^\infty \|f_n\|_{\mathcal{S}_0} \|g_n\|_{\mathcal{S}_0}, K_S = \sum_{n=1}^\infty K_{f_n \otimes g_n}\} \\ &= \|K_S\|_{\mathcal{S}_0} \asymp \|S\|_{\mathbb{S}_0}. \end{aligned}$$

We proved the boundedness of $\mathcal{F}_{W_\tau}\colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$, the continuity of the symplectic Fourier transform $\mathcal{F}_\sigma\colon \mathcal{S}_0(\mathbb{R}^{2d}) \to \mathcal{S}_0(\mathbb{R}^{2d})$ is well-known, the continuity of $W_\tau\colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$ follows. Concerning the last claim, we proceed as follows:

$$\begin{aligned} {}_{\mathbb{S}_0'}\langle \mathrm{Op}_\tau(a), S\rangle_{\mathbb{S}_0} &= {}_{\mathcal{S}_0'}\langle K_{\mathrm{Op}_\tau(a)}, K_S\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle K_{\mathrm{Op}_\tau(a)}, \sum_{n=1}^\infty K_{f \otimes g_n}\rangle_{\mathcal{S}_0} \\ &= \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle K_{\mathrm{Op}_\tau(a)}, K_{f \otimes g_n}\rangle_{\mathcal{S}_0} = \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle \mathrm{Op}_\tau(a) g_n, f_n\rangle_{\mathcal{S}_0} \\ &= \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle a, W_\tau(f_n \otimes g_n)\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle a, \sum_{n=1}^\infty W_\tau(f_n \otimes g_n)\rangle_{\mathcal{S}_0} \\ &= {}_{\mathcal{S}_0'}\langle a, W_\tau S\rangle_{\mathcal{S}_0}. \end{aligned}$$

the proof is concluded. □

On account of Theorem 7.2.6 and 7.2.7, it seems reasonable to interpret

$$W_\tau S$$

as the $\tau$-quantization of an operator in $\mathbb{S}_0$ or $\mathcal{J}^1$.

**Corollary 7.2.8.** *(i) For every $\tau \in [0,1]$ the mapping $W_\tau\colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$ is a topological isomorphism with inverse given by $\mathrm{Op}_\tau\colon \mathcal{S}_0(\mathbb{R}^{2d}) \to \mathbb{S}_0$;*

*(ii) A linear and continuous operator $S\colon \mathcal{S}_0(\mathbb{R}^d) \to \mathcal{S}_0'(\mathbb{R}^d)$ belongs to $\mathbb{S}_0$ if and only if $W_\tau S \in \mathcal{S}_0(\mathbb{R}^{2d})$ for some (hence every) $\tau \in [0,1]$.*

*Proof.* $(i)$ We observed in (7.17) that $W_\tau S$ is just the $\tau$-symbol $\mathrm{a}_\tau^S$ of $S$, here we take $S \in \mathbb{S}_0$. Therefore

$$\mathrm{Op}_\tau \circ W_\tau S = \mathrm{Op}_\tau(\mathrm{a}_\tau^S) = S.$$

We now show that if we start with $a \in \mathcal{S}_0(\mathbb{R}^{2d})$, then $\mathrm{Op}_\tau(a)$ belongs to $\mathbb{S}_0$. From (7.26), we have that the kernel of $\mathrm{Op}_\tau(a)$ can be written as

$$K_{\mathrm{Op}_\tau(a)}(t,x) = \int_{\mathbb{R}^d} e^{2\pi i(t-x)\omega} a((1-\tau)t+\tau x,\omega)\, d\omega = \Psi_\tau \mathcal{F}_2^{-1} a(t,x),$$

where: $\mathcal{F}_2^{-1}$ is the inverse of the Fourier transform in the second variable; $\Psi_\tau$ is the change of variables induced by the matrix so defined

$$\begin{bmatrix} 1-\tau & \tau \\ 1 & -1 \end{bmatrix}, \qquad \Psi_\tau F(t,x) \coloneqq F((1-\tau)t+\tau x, t-x). \tag{7.28}$$

Being $a$ in the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^{2d})$ we have $\mathcal{F}_2^{-1}a \in \mathcal{S}_0(\mathbb{R}^{2d})$, therefore also $\Psi_\tau \mathcal{F}_2^{-1} a$ is in $\mathcal{S}_0(\mathbb{R}^{2d})$ which means that $\mathrm{Op}_\tau(a)$ is an element of $\mathbb{S}_0$. The fact that $\mathrm{Op}_\tau$ is continuous from $\mathcal{S}_0(\mathbb{R}^{2d})$ into $\mathbb{S}_0$ is clear from the applications $\mathcal{F}_2^{-1}$ and $\Psi_\tau$. Eventually:

$$W_\tau \circ \mathrm{Op}_\tau(a) = \mathrm{a}_\tau^{\mathrm{Op}_\tau(a)} = a.$$

$(ii)$ The claim is a straightforward consequence of $(i)$. □

**Corollary 7.2.9.** $(i)$ *For every $\tau \in [0,1]$ $\mathcal{F}_{W_\tau} : \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$ is a topological isomorphisms with inverse given by the $\tau$-spreading representation*

$$\mathrm{SR}^\tau : \mathcal{S}_0(\mathbb{R}^{2d}) \to \mathbb{S}_0\,, a \mapsto \int_{\mathbb{R}^{2d}} a(z)\pi^\tau(z)\, dz; \tag{7.29}$$

$(ii)$ *Let us define*

$$\mathrm{SR}^\tau : \mathcal{S}_0'(\mathbb{R}^{2d}) \to \mathbb{S}_0'\; a \mapsto \int_{\mathbb{R}^{2d}} a(z)\pi^\tau(z)\, dz, \tag{7.30}$$

*where the integral has to be understood weakly as follows:*

$${}_{\mathcal{S}_0'}\langle (\mathrm{SR}^\tau a) f, g\rangle_{\mathcal{S}_0} \coloneqq {}_{\mathcal{S}_0'}\langle a, V_f^\tau g\rangle_{\mathcal{S}_0}, \qquad a \in \mathcal{S}_0'(\mathbb{R}^{2d}),\ f,g \in \mathcal{S}_0(\mathbb{R}^d).$$

*Then* $\mathrm{SR}^\tau$ *as in* (7.30) *is well-defined, linear, continuous, extends* (7.29) *and it is the Banach space adjoint of $\mathcal{F}_{W_\tau}$ in* $(i)$*:*

$$\mathrm{SR}^\tau = \mathcal{F}_{W_\tau}^*, \tag{7.31}$$

*in the sense that for every $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $S \in \mathbb{S}_0$*

$${}_{\mathcal{S}_0'}\langle a, \mathcal{F}_{W_\tau} S\rangle_{\mathcal{S}_0} = {}_{\mathbb{S}_0'}\langle \mathrm{SR}^\tau a, S\rangle_{\mathbb{S}_0} = {}_{\mathcal{S}_0'}\langle K_{\mathrm{SR}^\tau a}, K_S\rangle_{\mathcal{S}_0};$$

$(iii)$ *Every function $F \in \mathcal{S}_0(\mathbb{R}^{2d})$ admits an expansion of the following type:*

$$F = \sum_{n=1}^{\infty} V_{g_n}^\tau f_n,$$

*for some sequences $\{f_n\}_n, \{g_n\}_n \subseteq \mathcal{S}_0(\mathbb{R}^d)$ such that $\sum_{n=1}^\infty \|f_n\|_{\mathcal{S}_0} \|g_n\|_{\mathcal{S}_0} < +\infty$.*

*Proof.* (*i*) First we notice that if we start with $a \in \mathcal{S}_0(\mathbb{R}^{2d})$, then $\mathrm{SR}^\tau a$ is the Feichtinger operator with kernel

$$K_{\mathrm{SR}^\tau a}(y,u) = \int_{\mathbb{R}^d} a(y-u,\omega)e^{2\pi i y\omega}\,d\omega = \mathcal{F}_2^{-1}[a(y-u,\cdot)](y).$$

Clearly $\mathrm{SR}^\tau$ is continuous from $\mathcal{S}_0(\mathbb{R}^{2d})$ into $\mathbb{S}_0$.
Since we have $W_\tau = \mathcal{F}_\sigma\mathcal{F}_{W_\tau}$ and $\mathcal{F}_\sigma$ is an automorphism of $\mathcal{S}_0(\mathbb{R}^{2d})$, we can write $\mathcal{F}_{W_\tau} = \mathcal{F}_\sigma W_\tau$ and which is an isomorphism due to Corollary 7.2.8. To prove that $\mathrm{SR}^\tau$ is the inverse of $\mathcal{F}_{W_\tau}$ we use (7.27), take $S = \sum_{n=1}^\infty f_n \otimes g_n \in \mathbb{S}_0$ and $\psi, f \in \mathcal{S}_0(\mathbb{R}^d)$:

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle(\mathrm{SR}^\tau \circ \mathcal{F}_{W_\tau}S)\psi,\varphi\rangle_{\mathcal{S}_0} &= \int_{\mathbb{R}^{2d}} \mathcal{F}_{W_\tau}S(z)\,{}_{\mathcal{S}_0'}\langle\pi^\tau(z)\psi,\varphi\rangle_{\mathcal{S}_0}\,dz\\
&= \sum_{n=1}^\infty \int_{\mathbb{R}^{2d}} V_{g_n}^\tau f_n(z)\overline{V_\psi^\tau\varphi(z)}\,dz\\
&= \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle f_n,\varphi\rangle_{\mathcal{S}_0}\overline{{}_{\mathcal{S}_0'}\langle g_n,\psi\rangle_{\mathcal{S}_0}}\\
&= {}_{\mathcal{S}_0'}\langle\sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle\psi,g_n\rangle_{\mathcal{S}_0} f_n,\varphi\rangle_{\mathcal{S}_0}\\
&= {}_{\mathcal{S}_0'}\langle\sum_{n=1}^\infty (f_n\otimes g_n)\psi,\varphi\rangle_{\mathcal{S}_0}\\
&= {}_{\mathcal{S}_0'}\langle S\psi,\varphi\rangle_{\mathcal{S}_0},
\end{aligned}$$

in the third equality we used Moyal's identity. For the composition $\mathcal{F}_{W_\tau} \circ \mathrm{SR}^\tau$, notice that this is the identity on $\mathcal{S}_0(\mathbb{R}^{2d})$ due lo Lemma 7.1.9 (*vii*).
(*ii*) Well-posedness, linearity and continuity of $\mathrm{SR}^\tau$ from $\mathcal{S}_0'(\mathbb{R}^{2d})$ into $\mathbb{S}_0'$ are standard. Trivially (7.30) extends (7.29). To see that $\mathrm{SR}^\tau$ is the Banach space adjoint of $\mathcal{F}_{W_\tau}$ from $\mathbb{S}_0$ into $\mathcal{S}_0(\mathbb{R}^{2d})$, take $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $S \in \mathbb{S}_0$. In the following calculations we use: the already mentioned (7.27), the representation for Feichtinger operators and their kernel given in Lemma 7.2.4, the Outer and Inner Kernel Theorem:

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle a,\mathcal{F}_{W_\tau}S\rangle_{\mathcal{S}_0} &= \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle a,V_{g_n}^\tau f_n\rangle_{\mathcal{S}_0} = \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle(\mathrm{SR}^\tau a)g_n,f_n\rangle_{\mathcal{S}_0}\\
&= \sum_{n=1}^\infty {}_{\mathcal{S}_0'}\langle K_{\mathrm{SR}^\tau a},K_{f_n\otimes g_n}\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle K_{\mathrm{SR}^\tau a},K_S\rangle_{\mathcal{S}_0}\\
&= {}_{\mathbb{S}_0'}\langle\mathrm{SR}^\tau a,S\rangle_{\mathbb{S}_0}.
\end{aligned}$$

(*iii*) The last claim is a direct consequence of the computations in (7.27) and the surjectivity of $\mathcal{F}_{W_\tau}$. □

### 7.2.2 A suitable environment for QHA

In Section 7.1 we introduced convolutions between a function and an operator and two operators. M. Keyl, J. Kiukas and R. Werner [103] showed that such convolutions make sense for wider classes of (generalized) functions and operators. We summarized here the main results; in what follows $\mathfrak{S}$ denotes the set of pseudo-differential operators with Weyl symbol in the Schwartz class $\mathcal{S}(\mathbb{R}^{2d})$ and $\mathfrak{S}'$ those pseudo-differential operators with Weyl symbol in $\mathcal{S}'(\mathbb{R}^{2d})$. On account of the Schwartz Kernel Theorem we can identify $\mathfrak{S}'$ with the linear and continuous operators from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

**Proposition 7.2.10.** *(i) Let us take $S, T \in \mathfrak{S}$, $A \in \mathfrak{S}'$, $b \in \mathcal{S}(\mathbb{R}^{2d})$ and $a \in \mathcal{S}'(\mathbb{R}^{2d})$. Then the following convolutions can be defined and they extend the ones defined in Subsection 7.1.2:*

$$S \star T \in \mathcal{S}(\mathbb{R}^{2d}), \quad S \star A \in \mathcal{S}'(\mathbb{R}^{2d}), \quad b \star S \in \mathfrak{S}, \quad a \star S, b \star A \in \mathfrak{S}';$$

*(ii) The Fourier-Wigner transform can be extended to a topological isomorphism $\mathcal{F}_{W_{1/2}} \colon \mathfrak{S}' \to \mathcal{S}'(\mathbb{R}^{2d})$;*

*(iii) We have $\mathcal{F}_\sigma(S \star T) = \mathcal{F}_{W_{1/2}}S \cdot \mathcal{F}_{W_{1/2}}T$ and $\mathcal{F}_{W_{1/2}}(b \star S) = \mathcal{F}_\sigma b \cdot \mathcal{F}_{W_{1/2}}S$ whenever $S, T$ and $b$ are such that the convolutions are defined as in part (i);*

*(iv) The Weyl symbol of $A \in \mathfrak{S}'$ is given by $\mathcal{F}_\sigma \mathcal{F}_{W_{1/2}} A$.*

The authors of [103] proved that the class of so-called Schwartz operators $\mathfrak{S}$ has the structure of a Fréchet space. Since we believe that Fréchet spaces can be rather cumbersome to work with, in this subsection we show that the Banach space of Feichtinger operators $\mathbb{S}_0$ is a valid alternative to $\mathfrak{S}$. We first need some preliminary results about $\mathcal{S}_0$ and $\mathbb{S}_0$.

**Lemma 7.2.11.** *Given $f \in \mathcal{S}_0'(\mathbb{R}^d)$, there exists a sequence $\{f_n\}_n \subseteq \mathcal{S}_0(\mathbb{R}^d)$ which w-$*$ converges to $f$ and it is bounded by $\|f\|_{\mathcal{S}_0'}$, i.e.*

$$\lim_{n\to+\infty} \langle f_n, g \rangle = {}_{\mathcal{S}_0'}\langle f, g \rangle_{\mathcal{S}_0} \qquad \forall\, g \in \mathcal{S}_0(\mathbb{R}^d), \qquad \sup_n \|f_n\|_{\mathcal{S}_0} \leq \|f\|_{\mathcal{S}_0'}.$$

*Proof.* Let us fix $f \in \mathcal{S}_0'(\mathbb{R}^d) \smallsetminus \{0\}$ and call $R \coloneqq \|f\|_{\mathcal{S}_0'}$. From [101, Proposiiton 6.15], there exists a net $\{f_\alpha\}_{\alpha \in A} \subseteq \mathcal{S}_0(\mathbb{R}^d)$ which w-$*$ converges to $f$ in $\mathcal{S}_0'$ and such that $\|f_\alpha\|_{\mathcal{S}_0'} \leq R$ for every $\alpha \in A$. Calling

$$B_R \coloneqq \left\{ f \in \mathcal{S}_0'(\mathbb{R}^d) \mid \|f\|_{\mathcal{S}_0'} \leq R \right\} \quad \text{and} \quad E_R \coloneqq \mathcal{S}_0(\mathbb{R}^d) \cap B_R,$$

where $\mathcal{S}_0$ is identified with its natural immersion in $\mathcal{S}_0'$, this means that

$$E_R \subseteq B_R \subseteq \overline{E_R}^{w-*}.$$

$\overline{E_R}^{w-*}$ is bounded in $\mathcal{S}_0'(\mathbb{R}^d)$. In fact, if $f_0 \in \overline{E_R}^{w-*}$, then there exists a net $\{f_\alpha\}_{\alpha \in A} \subseteq E_R$ such that it w-$*$ converges to $f_0$, hence

$$\|f_0\|_{\mathcal{S}_0'} \leq \liminf_{\alpha \in A} \|f_\alpha\|_{\mathcal{S}_0'} = \lim_{\alpha \in A} \inf\{\|f_\beta\|_{\mathcal{S}_0'} \mid \alpha \preceq \beta\} \leq \lim_{\alpha \in A} R = R.$$

In particular, this shows that $\overline{E_R}^{w-*} \subseteq B_R$ and we get

$$\overline{E_R}^{w-*} = B_R.$$

Being $\mathcal{S}_0$ separable, from [112, Therem 2.6.23] the relative w-$*$ topology on $B_R$ is induced by a metric, hence the topological w-$*$ closure of $E_R$ equals its sequential w-$*$ closure. Hence there exists a sequence $\{f_n\}_n \subseteq E_R$ which w-$*$ converges to $f$ in $\mathcal{S}_0'(\mathbb{R}^d)$. □

**Remark 7.2.12.** *The above lemma holds also for any LCA second countable group $\mathcal{G}$ replacing $\mathbb{R}^d$, see [45, Theorem 2] for the separability of $\mathcal{S}_0(\mathcal{G})$.*

**Lemma 7.2.13.** *Given $S \in \mathbb{S}_0'$, there exists a sequence $\{S_n\}_n \subseteq \mathbb{S}_0$ such that*

(*i*) $\|S_n\|_{\mathbb{S}_0'} \lesssim \|S\|_{\mathbb{S}_0'}$;

(*ii*) $\lim_{n\to+\infty} \left|{}_{\mathcal{S}_0'}\langle (S-S_n)f, g\rangle_{\mathcal{S}_0}\right| = 0$ *for all* $f, g \in \mathcal{S}_0(\mathbb{R}^d)$.

*Proof.* it is a straightforward application of the Kernel Theorem 7.2.2 and 7.2.2 and of Lemma 7.2.11. □

Convergence as in item ($i$) of the above lemma will be also denoted by

$$S_n \overset{w-*}{\underset{n}{\longrightarrow}} S \quad \text{in} \quad \mathbb{S}_0' \qquad \text{or} \qquad S = w\text{-}*\text{-}\lim_n S_n \qquad \text{in} \quad \mathbb{S}_0'.$$

**Lemma 7.2.14.** *Let $S\colon \mathcal{S}_0 \to \mathcal{S}_0'$ be in $\mathbb{S}_0$. Then the Banach space adjoint $S^*\colon \mathcal{S}_0' \to \mathcal{S}_0$ is in $\mathbb{S}_0$ with kernel*

$$K_{S^*}(y,u) = \overline{K_S(u,y)}. \tag{7.32}$$

*Proof.* We take $f, g \in \mathcal{S}_0(\mathbb{R}^d)$, then

$$\begin{aligned} {}_{\mathcal{S}_0'}\langle Sf, g\rangle_{\mathcal{S}_0} &= \int_{\mathbb{R}^{2d}} K_T(y,u)\overline{g(y)}f(u)\,dydu \\ &= \int_{\mathbb{R}^d} f(u)\overline{\int_{\mathbb{R}^d} \overline{K_S(y,u)}g(y)\,dy}\,du \\ &= {}_{\mathcal{S}_0'}\langle f, S^*g\rangle_{\mathcal{S}_0}. \end{aligned}$$

Hence $S^*g(y) = \int_{\mathbb{R}^d} \overline{K_S(u,y)}g(u)\,du$, this means $K_{S^*}(y,u) = \overline{K_S(u,y)}$ which is an element of $\mathcal{S}_0(\mathbb{R}^{2d})$. □

**Corollary 7.2.15.** *$\mathbb{S}_0$ is a Banach $*$-algebra.*

We notice that $(S^*)\check{} = (\check{S})^*$, so that from now on we shall simply write $\check{S}^*$ when necessary.

**Lemma 7.2.16.** (*i*) *The following applications are surjective isometries:*

($i-a$) *$\alpha_z\colon \mathbb{S}_0 \to \mathbb{S}_0$, for every $z = (x,\omega) \in \mathbb{R}^{2d}$, and*

$$K_{\alpha_z S}(y,u) = e^{2\pi i(y-u)\omega}K_S(y-x, u-x); \tag{7.33}$$

($i-b$) *$\check{}\colon \mathbb{S}_0 \to \mathbb{S}_0$ and*

$$K_{\check{S}}(y,u) = K_S(-y-u); \tag{7.34}$$

($i-c$) *$\alpha_z\colon \mathbb{S}_0' \to \mathbb{S}_0'$, for every $z \in \mathbb{R}^{2d}$;*

($i-d$) *$\check{}\colon \mathbb{S}_0' \to \mathbb{S}_0'$;*

(*ii*) *Let $S, T \in \mathbb{S}_0$ and $b \in \mathcal{S}_0(\mathbb{R}^{2d})$, then*

$$S \star T \in \mathcal{S}_0(\mathbb{R}^{2d}), \qquad b \star S \in \mathbb{S}_0;$$

(*iii*) *The kernel of the mixed-state localization operator* $b\star S$ *is given by*

$$K_{b\star S}(y,u)=\int_{\mathbb{R}^d} b(x,\omega)e^{2\pi i(y-u)\omega}K_S(y-x,u-x)\,dxd\omega; \tag{7.35}$$

*for very* $z=(x,\omega)\in\mathbb{R}^{2d}$ *the kernel of* $S\alpha_z\check{T}$ *is*

$$K_{S\alpha_z\check{T}}(y,u)=\int_{\mathbb{R}^d} e^{2\pi i(y-t)\omega}K_T(x-y,x-t)K_S(t,u)\,dt. \tag{7.36}$$

*Proof.* (*i*) We leave all the direct computations to the interest reader, we just point out that to prove $\alpha_z S,\check{S}\in\mathbb{S}_0$ the result [62, Corollary 3.3] is useful. A continuous and linear operator $S\colon\mathcal{S}_0\to\mathcal{S}_0'$ is a Feichtinger operator iff

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\left|{}_{\mathcal{S}_0'}\langle S\pi(z)g_1,\pi(w)g_2\rangle_{\mathcal{S}_0}\right|\,dzdw$$

is finite for every $g_1,g_2\in\mathcal{S}_0(\mathbb{R}^d)$.
(*ii*) We first tackle the convolution between two Feichtinger operators. On account of item (*i*) and the fact that $\mathbb{S}_0$ is a Banach algebra under composition, we have that for every $z=(x,\omega)\in\mathbb{R}^{2d}$ $S\alpha_z\check{T}$ is in $\mathbb{S}_0$. Then we compute using [62, Corollary 3.15]:

$$\begin{aligned}
S\star T(z)=\operatorname{tr}(S\alpha_z\check{T})&=\int_{\mathbb{R}^d}K_{S\alpha_z\check{T}}(y,y)\,dy=\int_{\mathbb{R}^{2d}}K_{\alpha_z\check{T}}(y,t)K_S(t,y)\,dtdy\\
&=\int_{\mathbb{R}^{2d}}e^{2\pi i(y-t)\omega}K_T(x-y,x-t)K_S(t,y)\,dtdy\\
&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}K_T(x-y,x-t)K_S(t,y)e^{-2\pi it\omega}\,dt\right)e^{2\pi iy\omega}\,dy\\
&=\mathcal{F}_2^{-1}\mathcal{F}_1\left(\Phi T_{(x,x)}K_T\cdot K_S\right)(\omega,\omega),
\end{aligned}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the partial Fourier transforms with respect to the first and second variable, respectively, and $\Phi F(t,y):=F(-y,-t)$. Consider now $f,g,h,l\in\mathcal{S}_0(\mathbb{R}^d)$, it is useful to compute the following where $P$ is the parity operator:

$$\begin{aligned}
\mathcal{F}_2^{-1}\mathcal{F}_1\left(\Phi T_{(x,x)}K_{h\otimes l}\cdot K_{f\otimes g}\right)(\omega,\omega)&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}h(x-y)\overline{l(x-t)}f(t)\overline{g(y)}e^{-2\pi it\omega}\,dt\right)e^{2\pi iy\omega}\,dy\\
&=\int_{\mathbb{R}^d}f(t)e^{-2\pi it\omega}\overline{l(x-t)}\,dt\cdot\int_{\mathbb{R}^d}\overline{g(y)}e^{2\pi iy\omega}h(x-y)\,dy\\
&=V_{\mathcal{I}l}f(-x,\omega)\cdot\overline{V_{\mathcal{I}h}g(-x,\omega)}.
\end{aligned}$$

So that $\mathcal{F}_2^{-1}\mathcal{F}_1\left(\Phi T_{(x,x)}K_{h\otimes l}\cdot K_{f\otimes g}\right)(\omega,\omega)$ is in $\mathcal{S}_0(\mathbb{R}^{2d})$ as a function of $(x,\omega)$. We consider now two representation $S=\sum_{n=1}^\infty f_n\otimes g_n$ and $T=\sum_{n=1}^\infty h_n\otimes l_n$, see Lemma 7.2.4, so that

$$K_S=\sum_{n=1}^\infty K_{f_n\otimes g_n},\qquad K_T=\sum_{n=1}^\infty K_{h_n\otimes l_n}.$$

It follows that we can write

$$\begin{aligned}S\star T(z) &= \mathcal{F}_2^{-1}\mathcal{F}_1\left(\Phi T_{(x,x)}\sum_{M}^{\infty}K_{h_m\otimes l_m}\cdot\sum_{n=1}^{\infty}K_{f_n\otimes g_n}\right)(\omega,\omega)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathcal{F}_2^{-1}\mathcal{F}_1\left(\Phi T_{(x,x)}K_{h_m\otimes l_m}\cdot K_{f_n\otimes g_n}\right)(\omega,\omega)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{\mathcal{I}l_m}f_n(-x,\omega)\cdot\overline{V_{\mathcal{I}h_m}g_n(-x,\omega)}\in\mathcal{S}_0(\mathbb{R}^{2d}),\end{aligned}$$

the convergence is guaranteed by Lemma 7.2.4.
Concerning $b\star S$, the subsequent estimate for every $f,g\in\mathcal{S}_0(\mathbb{R}^d)$ proves that $b\star S\in\mathbb{S}_0'$:

$$\left|{}_{\mathcal{S}_0'}\langle(b\star S)f,g\rangle_{\mathcal{S}_0}\right|\le\int_{\mathbb{R}^{2d}}|b(z)|\left|{}_{\mathcal{S}_0'}\langle S\pi(z)^*f,\pi(z)^*g\rangle_{\mathcal{S}_0}\right|\,dz\lesssim\|b\|_{L^1}\|S\|_{\mathbb{S}_0'}\|f\|_{\mathcal{S}_0}\|g\|_{\mathcal{S}_0}.$$

We exploit [62, Theorem 3.2 (ii)] to show that $b\star S$ is in $\mathbb{S}_0$, take $g_1,g_2\in\mathcal{S}_0(\mathbb{R}^d)$:

$$\begin{aligned}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\left|{}_{\mathcal{S}_0'}\langle(b\star S)\pi(w)g_1,\pi(u)g_2\rangle_{\mathcal{S}_0}\right|\,dwdu &\le\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|b(z)|\\ &\quad\times\left|{}_{\mathcal{S}_0'}\langle S\pi(w-z)g_1,\pi(u-z)g_2\rangle_{\mathcal{S}_0}\right|\,dzdwdu\\ &=\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\left|{}_{\mathcal{S}_0'}\langle S\pi(w')g_1,\pi(u')g_2\rangle_{\mathcal{S}_0}\right|\,dw'du'\cdot\int_{\mathbb{R}^{2d}}|b(z)|\,dz<+\infty.\end{aligned}$$

$(iii)$ We compute explicitly the kernel of the operator given by the convolution $b\star S$, here $z=(x,\omega)\in\mathbb{R}^{2d}$:

$$\begin{aligned}{}_{\mathcal{S}_0'}\langle(b\star S)f,g\rangle_{\mathcal{S}_0} &=\int_{\mathbb{R}^{2d}}b(x,\omega)\int_{\mathbb{R}^{2d}}K_S(y,u)\overline{\pi(-z)g(y)}\pi(-z)f(u)\,dydu\,dxd\omega\\ &=\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}b(x,\omega)e^{2\pi i(y-u)\omega}K_S(y,u)\overline{g(y+x)}f(u+x)\,dxd\omega\,dydu.\end{aligned}$$

The change of variables $y'=y+u, u'=u+x$ gives the desired result. The last claim is just a direct application of (7.33), (7.34) and the Banach algebra property for $\mathbb{S}_0$ [62, Lemma 3.10]. □

**Corollary 7.2.17.** *Let $S,T\in\mathbb{S}_0$ with representations $S=\sum_{n=1}^{\infty}f_n\otimes g_n$ and $T=\sum_{n=1}^{\infty}h_n\otimes l_n$, where $\{f_n\}_n,\{g_n\}_n,\{h_n\}_n,\{l_n\}_n\subseteq\mathcal{S}_0(\mathbb{R}^d)$ with $\sum_{n=1}^{\infty}\|f_n\|_{\mathcal{S}_0}\|g_n\|_{\mathcal{S}_0}<+\infty$, $\sum_{n=1}^{\infty}\|h_n\|_{\mathcal{S}_0}\|l_n\|_{\mathcal{S}_0}<+\infty$. Then, with the notations introduced in the proof of Lemma 7.2.16, for every $z=(x,\omega)\in\mathbb{R}^{2d}$:*

$$\begin{aligned}S\star T(z) &=\mathcal{F}_2^{-1}\mathcal{F}_1\left(\Phi T_{(x,x)}K_T\cdot K_S\right)(\omega,\omega)\\ &=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{\mathcal{I}l_m}f_n(-x,\omega)\cdot\overline{V_{\mathcal{I}h_m}g_n(-x,\omega)}.\end{aligned}\tag{7.37}$$

**Definition 7.2.18.** *Let $A\in\mathbb{S}_0'$, $a\in\mathcal{S}_0'(\mathbb{R}^{2d})$, $S\in\mathbb{S}_0$ and $b\in\mathcal{S}_0(\mathbb{R}^{2d})$. Consider any sequences $\{A_n\}_n\subseteq\mathbb{S}_0$ and $\{a_n\}_n\subseteq\mathcal{S}_0(\mathbb{R}^{2d})$ such that*

$$A_n\xrightarrow[n]{w-*}A\quad in\quad\mathbb{S}_0'\qquad and\qquad a_n\xrightarrow[n]{w-*}a\quad in\quad\mathcal{S}_0'(\mathbb{R}^{2d}).$$

*Then we define the following* ***convolutions****:*

$$S \star A := w\text{-}*\text{-}\lim_n S \star A_n \quad \textit{in} \quad \mathcal{S}_0'(\mathbb{R}^{2d}); \tag{7.38}$$

$$a \star S := S \star a := w\text{-}*\text{-}\lim_n a_n \star S \quad \textit{in} \quad \mathbb{S}_0'; \tag{7.39}$$

$$b \star A := A \star b := w\text{-}*\text{-}\lim_n b \star A_n \quad \textit{in} \quad \mathbb{S}_0'. \tag{7.40}$$

**Remark 7.2.19.** *The reader may find useful to keep in mind the following simple identities, they will be used in the proof of the subsequent proposition. Consider $S \in \mathbb{S}_0, \psi, \varphi, f, g \in \mathcal{S}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}^{2d}$:*

$$\begin{aligned} \alpha_z(\psi \otimes \varphi) &= \pi(z)\psi \otimes \pi(z)\varphi; \\ (\psi \otimes \varphi)(K_{f\otimes\overline{g}}) &= \langle f, \varphi\rangle(\psi \otimes g); \\ (\psi \otimes \varphi) \star \check{S}(z) &= {}_{\mathcal{S}_0'}\langle \pi(z)S\pi(z)^*\psi, \varphi\rangle_{\mathcal{S}_0}. \end{aligned}$$

**Proposition 7.2.20.** *The convolutions defined in Definition 7.2.18:*

$(i)$ *do not depend on the sequences chosen; moreover, taking $A, a, S, b$ as in Definition 7.2.18:*

$${}_{\mathcal{S}_0'}\langle S \star A, b\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle K_A, K_{b\star\check{S}^*}\rangle_{\mathcal{S}_0}; \tag{7.41}$$

$${}_{\mathcal{S}_0'}\langle (a \star S)f, g\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle a, (g \otimes f) \star \check{S}^*\rangle_{\mathcal{S}_0}; \tag{7.42}$$

$${}_{\mathcal{S}_0'}\langle (b \star A)f, g\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle K_A, K_{b^*\star(g\otimes f)}\rangle_{\mathcal{S}_0}, \tag{7.43}$$

*where $b^*(z) := \overline{b(-z)}$;*

$(ii)$ *extend the definitions given in Subsection 7.1.2;*

$(iii)$ *are commutative;*

$(iv)$ *are associative, in particular if $z \in \mathbb{R}^{2d}$, $T, Q \in \mathbb{S}_0$, $\sigma \in \mathcal{S}_0(\mathbb{R}^{2d})$ and $A, a, S, b$ as in Definition 7.2.18 then:*

$$(S \star (T \star b))(z) = ((S \star T) * b)(z); \tag{7.44}$$

$$S \star (T \star Q) = (S \star T) \star Q; \tag{7.45}$$

$$(S \star b) \star \sigma = S \star (b * \sigma); \tag{7.46}$$

$$S \star (T \star a) = (S \star T) * a; \tag{7.47}$$

$$A \star (T \star b) = (A \star T) \star b; \tag{7.48}$$

$$S \star (T \star A) = (S \star T) \star A; \tag{7.49}$$

*in the above identities $*$ denotes the usual convolution between two functions or a function and a distribution.*

*Proof.* $(i)$ If we show (7.41), (7.42) and (7.43), then the rest claimed in $(i)$ is obvious.
We start with(7.41). Let $b \in \mathcal{S}_0(\mathbb{R}^{2d})$ and $z = (x, \omega) \in \mathbb{R}^{2d}$, in the subsequent computations we

use Lemma 7.2.14 and 7.2.16:

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle S\star A, b\rangle_{\mathcal{S}_0} &= \lim_{n\to+\infty} {}_{\mathcal{S}_0'}\langle S\star A_n, b\rangle_{\mathcal{S}_0} = \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} \operatorname{tr}(S\alpha_z\check{A}_n)\overline{b(z)}\,dz\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^d} K_{S\alpha_z\check{A}_n}(y,y)\,dy\overline{b(z)}\,dz\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(y-t)\omega}K_{A_n}(x-y,x-t)K_S(t,y)\,dtdy\,\overline{b(z)}\,dz\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(t'-y')\omega}K_{A_n}(y',t')K_S(x-t',x-y')\,dt'dy'\,\overline{b(z)}\,dz\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_{A_n}(y',t')\overline{\left(\int_{\mathbb{R}^{2d}}\overline{K_S(x-t',x-y')}e^{2\pi i(y'-t')\omega}b(z)\,dz\right)}\,dy'dt'\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_{A_n}(y',t')\overline{\left(\int_{\mathbb{R}^{2d}}\overline{K_{\check{S}}(t'-x,y'-x)}e^{2\pi i(y'-t')\omega}b(z)\,dz\right)}\,dy'dt'\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_{A_n}(y',t')\overline{\left(\int_{\mathbb{R}^{2d}}K_{\check{S}^*}(y'-x,t'-x)e^{2\pi i(y'-t')\omega}b(z)\,dz\right)}\,dy'dt'\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_{A_n}(y',t')\overline{K_{b\star\check{S}^*}(y',t')}\,dy'dt'.
\end{aligned}$$

About (7.42), we take $f,g\in\mathcal{S}_0(\mathbb{R}^d)$ and compute directly keeping in mind Remark 7.2.19:

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle (a\star S)f, g\rangle_{\mathcal{S}_0} &= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} a_n(z)\,{}_{\mathcal{S}_0'}\langle \pi(z)S\pi(z)^*f, g\rangle_{\mathcal{S}_0}\,dz\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} a_n(z)\overline{{}_{\mathcal{S}_0'}\langle \pi(z)S^*\pi(z)^*g, f\rangle_{\mathcal{S}_0}}\,dz\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} a_n(z)\overline{(g\otimes f)\star\check{S}^*(z)}\,dz.
\end{aligned}$$

We tackle then (7.43):

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle (b\star A)f, g\rangle_{\mathcal{S}_0} &= \lim_{n\to+\infty} {}_{\mathcal{S}_0'}\langle K_{b\star A_n}, K_{g\otimes f}\rangle_{\mathcal{S}_0}\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^{2d}} b(x,\omega)e^{2\pi i(y-u)\omega}K_{A_n}(y-x,u-x)\,dxd\omega\Big)\\
&\times\overline{g(y)}f(u)\,dydu\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} K_{A_n}(y',u')\overline{\Big(\int_{\mathbb{R}^{2d}}\overline{b(x,\omega)}e^{-2\pi i(y'-u')\omega}}\\
&\overline{\times\, g(y'+x)\overline{f(u'+x)}\,dxd\omega\Big)}\,dy'du'\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} K_{A_n}(y',u')\overline{\Big(\int_{\mathbb{R}^{2d}} b^*(x',\omega')e^{2\pi i(y'-u')\omega'}}\\
&\overline{\times\, g(y'-x')\overline{f(u'+x')}\,dx'd\omega'\Big)}\,dy'du'\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^{2d}} K_{A_n}(y',u')\overline{K_{b^*\star(g\otimes f)}(y',u')}\,dy'du',
\end{aligned}$$

where for sake of brevity we put $b^*(z) := \overline{b(-z)}$.
$(ii)$ and $(iii)$ are trivial.

$(iv)$ We prove just (7.44), (7.45) and (7.46). The remaining identities can be derived easily by the interested reader.
In order to show (7.44) we compute for $z \in \mathbb{R}^{2d}$:

$$\begin{aligned}
(S \star (T \star b))(z) &= \operatorname{tr}\left(S \circ \alpha_z \left(\left(\int_{\mathbb{R}^{2d}} b(z)\alpha_w T\, dw\right)\check{}\right)\right) \\
&= \operatorname{tr}\left(S \circ \left(\int_{\mathbb{R}^{2d}} b(w)\alpha_z\left((\alpha_w T)\check{}\right) dw\right)\right) \\
&= \operatorname{tr}\left(S \circ \int_{\mathbb{R}^{2d}} b(w)\alpha_z \alpha_{-w}\check{T}\, dw\right) \\
&= \operatorname{tr}\left(S \circ \int_{\mathbb{R}^{2d}} b(-w')\alpha_{w'}\alpha_z\check{T}\, dw'\right) \\
&= \int_{\mathbb{R}^{2d}} b(-w')\operatorname{tr}\left(S\alpha_{w'+z}\check{T}\right) dw',
\end{aligned}$$

where the last equality is due, e.g., to [128, Proposition 2.9]. we can the rephrase the last right-side term as

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} b(z-w'')\operatorname{tr}\left(S\alpha_{w''}\check{T}\right) dw'' &= \int_{\mathbb{R}^{2d}} b(z-w'')(S\star T)(w'')\, dw'' \\
&= ((S\star T) * b)(z).
\end{aligned}$$

About (7.45), the following property for the trace is useful:

$$\int_{\mathbb{R}^{2d}} \operatorname{tr}(S\alpha_w T)\, dw = \operatorname{tr}(S)\operatorname{tr}(T),$$

where $S, T \in \mathcal{J}^1$. Take now $f, g \in \mathcal{S}_0(\mathbb{R}^d)$:

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle (S\star(T\star Q))f, g\rangle_{\mathcal{S}_0} &= \int_{\mathbb{R}^{2d}} \operatorname{tr}(T\alpha_z\check{Q})\, {}_{\mathcal{S}_0'}\langle \alpha_z S f, g\rangle_{\mathcal{S}_0}\, dz \\
&= \int_{\mathbb{R}^{2d}} \operatorname{tr}(Q\alpha_z\check{T})\operatorname{tr}((\alpha_z S)(f\otimes g))\, dz \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \operatorname{tr}(Q(\alpha_z\check{T})\alpha_w((\alpha_z S)(f\otimes g)))\, dwdz \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \operatorname{tr}((f\otimes g)(\alpha_w Q)\alpha_z((\alpha_w\check{T})S))\, dzdw \\
&= \int_{\mathbb{R}^{2d}} \operatorname{tr}(S\alpha_w\check{T})\operatorname{tr}((\alpha_w Q)(f\otimes g))\, dw \\
&= {}_{\mathcal{S}_0'}\langle ((S\star T)\star Q)f, g\rangle_{\mathcal{S}_0}.
\end{aligned}$$

Also the last identity (7.46) comes from a direct computation, for $f, g \in \mathcal{S}_0(\mathbb{R}^d)$:

$$\begin{aligned}
{}_{\mathcal{S}_0'}\langle ((S \star b) \star \sigma) f, g\rangle_{\mathcal{S}_0} &= \int_{\mathbb{R}^{2d}} \sigma(z) {}_{\mathcal{S}_0'}\langle \alpha_z (S \star b) f, g\rangle_{\mathcal{S}_0}\, dz \\
&= \int_{\mathbb{R}^{2d}} \sigma(z) \int_{\mathbb{R}^{2d}} b(w) {}_{\mathcal{S}_0'}\langle (\alpha_w S)\pi(z)^* f, \pi(z)^* g\rangle_{\mathcal{S}_0}\, dwdz \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \sigma(z) b(w) {}_{\mathcal{S}_0'}\langle (\alpha_{w+z} S) f, g\rangle_{\mathcal{S}_0}\, dwdz \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \sigma(z) b(w) \operatorname{tr}((\alpha_{w+z} S)(f \otimes g))\, dwdz \\
&= \int_{\mathbb{R}^{2d}} b(w) \int_{\mathbb{R}^{2d}} \sigma(z' - w) \operatorname{tr}((\alpha_{z'} S)(f \otimes g))\, dz' dw \\
&= \int_{\mathbb{R}^{2d}} ( \int_{\mathbb{R}^{2d}} b(w) \sigma(z' - w)\, dz') \operatorname{tr}((\alpha_{z'} S)(f \otimes g))\, dw \\
&= \int_{\mathbb{R}^{2d}} b * \sigma(z') {}_{\mathcal{S}_0'}\langle (\alpha_{z'} S) f, g\rangle_{\mathcal{S}_0}\, dz' \\
&= {}_{\mathcal{S}_0'}\langle (S \star (b * \sigma)) f, g\rangle_{\mathcal{S}_0}.
\end{aligned}$$

The proof is concluded. □

**Corollary 7.2.21.** *The mappings $\mathcal{F}_{W_\tau}$ and $W_\tau$ defined on $\mathbb{S}_0$ can be extended to topological isomorphisms*

$$\mathcal{F}_{W_\tau} \colon \mathbb{S}_0' \to \mathcal{S}_0'(\mathbb{R}^{2d}) \qquad \text{and} \qquad W_\tau \colon \mathbb{S}_0' \to \mathcal{S}_0'(\mathbb{R}^{2d})$$

*by duality:*

$$(7.50) \qquad {}_{\mathcal{S}_0'}\langle \mathcal{F}_{W_\tau} S, a\rangle_{\mathcal{S}_0} \coloneqq {}_{\mathbb{S}_0'}\langle S, \mathrm{SR}^\tau a\rangle_{\mathbb{S}_0}, \qquad {}_{\mathcal{S}_0'}\langle W_\tau S, a\rangle_{\mathcal{S}_0} \coloneqq {}_{\mathbb{S}_0'}\langle S, \mathrm{Op}_\tau\, a\rangle_{\mathbb{S}_0},$$

*where $S \in \mathbb{S}_0'$ and $a \in \mathcal{S}_0(\mathbb{R}^{2d})$. The inverses are given by*

$$\mathrm{SR}^\tau \colon \mathcal{S}_0'(\mathbb{R}^{2d}) \to \mathbb{S}_0' \qquad \text{and} \qquad \mathrm{Op}_\tau \colon \mathcal{S}_0'(\mathbb{R}^{2d}) \to \mathbb{S}_0',$$

*respectively.*

*Proof.* The definitions in (7.50) rely on the fact that $\mathrm{Op}_\tau = W_\tau^*$ and $\mathrm{SR}^\tau = \mathcal{F}_{W_\tau}^*$, see Theorem 7.2.7 and Corollary 7.2.9. It is straightforward to see that if $S \in \mathbb{S}_0'$, then $\mathcal{F}_{W_\tau} S$ and $W_\tau S$ defined as in (7.50) are in $\mathcal{S}_0'(\mathbb{R}^{2d})$. Also linearity and boundedness of $\mathcal{F}_{W_\tau} \colon \mathbb{S}_0' \to \mathcal{S}_0'(\mathbb{R}^{2d})$ and $W_\tau \colon \mathbb{S}_0' \to \mathcal{S}_0'(\mathbb{R}^{2d})$ are easy to verify as well as the fact that they extend $\mathcal{F}_{W_\tau} \colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$ and $W_\tau \colon \mathbb{S}_0 \to \mathcal{S}_0(\mathbb{R}^{2d})$.

We show that $W_\tau$ is an isomorphisms with inverse $\mathrm{Op}_\tau$, then $\mathcal{F}_{W_\tau}$ is treated in the same way. $W_\tau$ is injective because $\mathrm{Op}_\tau \colon \mathcal{S}_0(\mathbb{R}^{2d}) \to \mathbb{S}_0$ is an isomorphism. Fix now $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$, there exists a sequence $\{a_n\}_n \subseteq \mathcal{S}_0(\mathbb{R}^{2d})$ such that $a_n \overset{w-*}{\underset{n}{\longrightarrow}} a$ in $\mathcal{S}_0'(\mathbb{R}^{2d})$. Since $W_\tau$ is an isomorphism between $\mathbb{S}_0$ and $\mathcal{S}_0(\mathbb{R}^{2d})$, there exists $\{A_n\}_n \subseteq \mathbb{S}_0$ such that $a_n = W_\tau A_n$. We see that there is $A \in \mathbb{S}_0'$ such that $A_n \overset{w-*}{\underset{n}{\longrightarrow}} A$ in $\mathbb{S}_0'$, in fact taking $b \in \mathcal{S}_0(\mathbb{R}^{2d})$

$${}_{\mathcal{S}_0'}\langle a, b\rangle_{\mathcal{S}_0} = \lim_{n \to +\infty} {}_{\mathcal{S}_0'}\langle W_\tau A_n, b\rangle_{\mathcal{S}_0} = \lim_{n \to +\infty} {}_{\mathbb{S}_0'}\langle A_n, \mathrm{Op}_\tau\, b\rangle_{\mathbb{S}_0}.$$

Hence $a = W_\tau A$, which proves that $W_\tau$ is onto. We show now that $W_\tau \circ \mathrm{Op}_\tau$ is the identity on $\mathcal{S}_0'(\mathbb{R}^{2d})$, take $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $b \in \mathcal{S}_0(\mathbb{R}^{2d})$:

$${}_{\mathcal{S}_0'}\langle W_\tau \circ \mathrm{Op}_\tau\, a, b\rangle_{\mathcal{S}_0} = {}_{\mathbb{S}_0'}\langle \mathrm{Op}_\tau\, a, \mathrm{Op}_\tau\, b\rangle_{\mathbb{S}_0} = {}_{\mathcal{S}_0'}\langle a, W_\tau \circ \mathrm{Op}_\tau\, b\rangle_{\mathcal{S}_0} = {}_{\mathcal{S}_0'}\langle a, b\rangle_{\mathcal{S}_0}.$$

The first identity is just (7.50), the second one is (7.25) and the last one is $(i)$ of Corollary 7.2.8. For the other way round, take $S \in \mathbb{S}_0'$ and $T \in \mathbb{S}_0$:

$$_{\mathbb{S}_0'}\langle \mathrm{Op}_\tau \circ W_\tau S, T\rangle_{\mathbb{S}_0} = {}_{\mathcal{S}_0'}\langle W_\tau S, W_\tau T\rangle_{\mathcal{S}_0} = {}_{\mathbb{S}_0'}\langle S, \mathrm{Op}_\tau \circ W_\tau T\rangle_{\mathbb{S}_0} = {}_{\mathbb{S}_0'}\langle S, T\rangle_{\mathbb{S}_0}.$$

The first identity is (7.25), the second one is (7.50) and the last one is $(i)$ of Corollary 7.2.8. □

### 7.2.3 $\tau$-Cohen's class of operators

In the present subsection we define $Q_a^\tau(S)$ and the definition of $Q_S^\tau(f)$ from [108] is recalled. We shall see which already known object can be recovered by $Q_a^\tau(S)$ and observe that it coincides with the $\tau$-symbol of the mixed-state localization operator $a \star S$. Some interplays between the Gabor matrix of an operator $G_T$, the $\tau$-Cohen's class, the trace and the $\tau$-Wigner distribution are exhibited afterwards.

**Definition 7.2.22.** *For $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ we define the* ***$\tau$-Cohen's class representation, with kernel** $a$**, of an operator*** $S \in \mathbb{S}_0$ *as*

$$Q_a^\tau(S) := a * W_\tau S. \tag{7.51}$$

Of course, the rank-1 case $f \otimes g$ we recover the definition given in (7.18). We recall also the definition given in [108] of Cohen's class representation of a function $f \in \mathcal{S}_0(\mathbb{R}^d)$ w.r.t. the operator $S \in \mathbb{S}_0'$ by

$$Q_S f := (f \otimes f) \star \check{S}. \tag{7.52}$$

It can be easily seen that for every $z \in \mathbb{R}^{2d}$

$$Q_S f(z) = (f \otimes f) \star \check{S}(z) = \langle (\alpha_z S) f, f, \rangle.$$

**Remark 7.2.23.** *Consider $a \in \mathcal{S}_0'(\mathbb{R}^{2d})$ and $S \in \mathbb{S}_0$, then we see that the $\tau$-Cohen's class representation of $S$ w.r.t. $a$ is just the $\tau$-symbol of the mixed-state localization operator $a \star S$:*

$$\mathrm{a}_\tau^{a\star S} = W_\tau(a \star S) = a * W_\tau S = Q_a^\tau(S).$$

**Lemma 7.2.24.** *Let $S \in \mathbb{S}_0$ have a representation $\sum_{n=1}^\infty f_n \otimes g_n$, take $f, \varphi, \psi \in \mathcal{S}_0(\mathbb{R}^d)$ and $\{h_n\}_n \subseteq \mathcal{S}_0(\mathbb{R}^d)$ with*

$$\sum_{n=1}^\infty \|h_n\|_{\mathcal{S}_0}^2 < +\infty.$$

*. Then for every $z \in \mathbb{R}^{2d}$:*

$$Q^\tau_{W_{1-\tau}(\check{\psi},\check{\varphi})}(S)(z) = \sum_{n=1}^\infty V_\varphi f_n(z) \overline{V_\psi g_n(z)}; \tag{7.53}$$

$$Q^\tau_{W_{1-\tau}(\check{\varphi},\check{\varphi})}(\sum_{n=1}^\infty h_n \otimes h_n)(z) = \sum_{n=1}^\infty |V_\varphi h_n(z)|^2. \tag{7.54}$$

*Proof.* Clearly, it is sufficient to prove the first identity. We show first that for $f, g \in \mathcal{S}_0(\mathbb{R}^d)$

$$Q_a^\tau(f, g) = (f \otimes g) \star \mathrm{Op}_{1\text{-}\tau}(a). \tag{7.55}$$

In fact, applying $\mathcal{F}_\sigma$ to the right-hand side first we get

$$\mathcal{F}_\sigma((f\otimes g)\star \mathrm{Op}_{1\text{-}\tau}(a)) = \mathcal{F}_{W_\tau}(f\otimes g)\cdot \mathcal{F}_{W_{1-\tau}}\mathrm{Op}_{1\text{-}\tau}(a) = V_g^\tau f\cdot \mathcal{F}_\sigma a.$$

We apply $\mathcal{F}_\sigma$ a second time:

$$(f\otimes g)\star \mathrm{Op}_{1\text{-}\tau}(a) = \mathcal{F}_\sigma V_g^\tau f * \mathcal{F}_\sigma\mathcal{F}_\sigma a = W_\tau(f,g)*a.$$

We can now compute as below:

$$\begin{aligned} Q^\tau_{W_{1-\tau}(\check\psi,\check\varphi)}(S) &= W_{1-\tau}(\check\psi,\check\varphi)*W_\tau(\sum_{n=1}^\infty f_n\otimes g_n) = \sum_{n=1}^\infty W_{1-\tau}(\check\psi,\check\varphi)*W_\tau(f_n,g_n)\\ &= \sum_{n=1}^\infty (f_n\otimes g_n)\star \mathrm{Op}_{1\text{-}\tau}(W_{1-\tau}(\check\psi,\check\varphi)) = \sum_{n=1}^\infty (f_n\otimes g_n)\star(\check\psi\otimes\check\varphi)\\ &= \sum_{n=1}^\infty V_\varphi f_n(z)\overline{V_\psi g_n(z)}, \end{aligned}$$

where the last equality is due to [108] □

We denote by $T\geq 0$ fact $T$ being positive, i.e.

$$\langle Tf,f\rangle \geq 0, \qquad \forall\, f\in L^2(\mathbb{R}^d).$$

An operator $T\in\mathcal{J}^1$ and $T\geq 0$ is also called a quantum state.
Let us take $T\in\mathbb{S}_0'$ and $\varphi\in\mathcal{S}$, then the Gabor matrix of $T$ (w.r.t. $\varphi$) is defined as

$$G_T(z,w) := \langle T\pi(w)\varphi,\pi(z)\varphi\rangle, \qquad z=(x,\omega), w=(u,v)\in\mathbb{R}^{2d}. \tag{7.56}$$

We notice that the Gabor matrix of an operator does not depend on $\tau$, in the sense that

$$G_T(z,w) = \langle T\pi(w)\varphi,\pi(z)\varphi\rangle = \langle T\pi^\tau(w)\varphi,\pi^\tau(z)\varphi\rangle, \qquad \forall\,\tau\in[0,1].$$

**Remark 7.2.25.** *We point out that the diagonal of the Gabor matrix of $T$, w.r.t. $\varphi$, is the Cohen's class representation of $\varphi$ w.r.t. $T$ up to a reflection:*

$$G_T(-z,-z) = Q_T\varphi(z). \tag{7.57}$$

*In fact*

$$\begin{aligned} G_T(-z,-z) &= \langle T\pi(-z)\varphi,\pi(-z)\varphi\rangle = \langle T\pi(z)^*\varphi,\pi(z)^*\varphi\rangle\\ &= \langle(\alpha_z T)\varphi,\varphi,\rangle = Q_T\varphi(z). \end{aligned}$$

Let $F$ and $H$ be functions of $(z,w)\in\mathbb{R}^{4d}$ and let $\Theta$ be a real $4d\times 4d$ matrix. Then the twisted convolution induced by $\Theta$ is defined as

$$F\natural_\Theta H(z,w) := \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} F(z',w')H(z-z',w-w')e^{2\pi i(z,w)\Theta(z',w')}\,dz'dw'. \tag{7.58}$$

**Lemma 7.2.26.** *Let $T,S\in\mathcal{J}^1$, $T,S\geq 0$. Then for every $\tau\in[0,1]$ we have*

$$\mathrm{tr}(TS) = \int_{\mathbb{R}^{2d}} W_\tau T(z)\overline{W_\tau S(z)}\,dz. \tag{7.59}$$

*Proof.* Since $T$ and $S$ are trace-class and positive, they can be described as

$$T=\sum_{n=1}^{\infty}\lambda_n f_n\otimes f_n,\qquad S=\sum_{n=1}^{\infty}\mu_n g_n\otimes g_n$$

for some o.n. sets $\{f_n\}_n$ and $\{g_n\}_n$ in $L^2$ and $\lambda_n,\mu_n\geq 0$. Let $\{e_n\}_n$ be an o.n.b. for $L^2(\mathbb{R}^d)$:

$$\operatorname{tr}(TS)=\sum_{n=1}^{\infty}\langle TSe_n,e_n\rangle=\sum_{i,j}^{\infty}\lambda_j\mu_i\left|\langle f_j,g_i\rangle\right|^2.$$

On the other hand

$$\int_{\mathbb{R}^{2d}}W_\tau T(z)\overline{W_\tau S(z)}\,dz=\sum_{i,j}^{\infty}\lambda_j\mu_i\int_{\mathbb{R}^{2d}}W_\tau f_j(z)\overline{W_\tau g_i(z)}\,dz=\sum_{i,j}^{\infty}\lambda_j\mu_i\left|\langle f_j,g_i\rangle\right|^2,$$

where the last equality is due to Moyal's identity. This concludes the proof. □

**Remark 7.2.27.** *Since we assume $S\geq 0$, $S$ is self-adjoint and for $\tau=1/2$ we have that $W_{1/2}S$ is real-valued. In fact, using the representation given in the proof of Lemma 7.2.26:*

$$W_{1/2}S=\sum_{n=1}^{\infty}\mu_n W_{1/2}g_n$$

*with every $W_{1/2}g_n$ real-valued and $\mu_n\geq 0$. Hence for $\tau=1/2$ we recover [95, lemma 2.7].*

**Lemma 7.2.28.** *Let $T\in\mathcal{J}^1$ and consider $\varphi\in\mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2}=1$. Then*

$$\operatorname{tr}T=\int_{\mathbb{R}^{2d}}\langle(\alpha_z T)\varphi,\varphi\rangle\,dz=\int_{\mathbb{R}^{2d}}Q_T\varphi(z)\,dz=\int_{\mathbb{R}^{2d}}G_T(z,z)\,dz. \tag{7.60}$$

*Proof.* The proof follows from a direct computation using the representations presented in the proof of Lemma 7.2.26 and Moyal's identity involving the function $\varphi$:

$$\langle f_j,g_i\rangle=\langle V_\varphi f_j,V_\varphi g_i\rangle.$$

We leave details to the interested reader. □

**Lemma 7.2.29.** *Let $T\in\mathcal{J}^1$, $T\geq 0$ and consider $\varphi\in\mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2}=1$. Then for every $z\in\mathbb{R}^{2d}$:*

$$Q_T\varphi(z)=\int_{\mathbb{R}^{2d}}W_\tau T(w)\overline{W_\tau\varphi(z+w)}\,dw=W_\tau T*(W_\tau\varphi)^*(z), \tag{7.61}$$

*where $(W_\tau\varphi)^*(w)=\overline{W_\tau\varphi(-w)}$.*

*Proof.* We compute directly

$$\begin{aligned}Q_T\varphi(z)&=\langle\pi(z)T\pi(z)^*\varphi,\varphi\rangle=\operatorname{tr}(T(\pi(z)^*\varphi\otimes\pi(z)^*\varphi))\\&=\int_{\mathbb{R}^{2d}}W_\tau T(w)\overline{W_\tau(\pi(z)^*\varphi\otimes\pi(z)^*\varphi)(w)}\,dw,\end{aligned}$$

the last equation holds because of Lemma 7.2.26. An easy calculation gives

$$W_\tau(\pi(z)^*\varphi\otimes\pi(z)^*\varphi)(w)=W_\tau\varphi(z+w),$$

which is also known as covariance property and concludes the proof. □

**Lemma 7.2.30.** *Let $T \in \mathcal{J}^1$, $T \geq 0$ and consider $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2} = 1$. Then for every $z, w \in \mathbb{R}^{2d}$:*

$$|G_T(z,w)|^2 \leq Q_T\varphi(-z)Q_T\varphi(-w).$$

*Proof.* The claim follows from the Cauchy-Schwarz inequality for the inner product induced by the positive operator $T$ and Remark 7.2.25. □

**Lemma 7.2.31.** *Let $0_d$ and $I_d$ denote the zero and identity $d \times d$ matrices, respectively. Let us define*

$$\Theta := \begin{bmatrix} 0_d & 0_d & 0_d & 0_d \\ I_d & 0_d & 0_d & 0_d \\ 0_d & 0_d & 0_d & 0_d \\ 0_d & 0_d & -I_d & 0_d \end{bmatrix}.$$

*Let $T \in \mathcal{J}^1$ and consider $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2} = 1$. For $z = (x, \omega), w = (u, v) \in \mathbb{R}^{2d}$ we have*

$$\begin{aligned} G_T(z,w) &= G_T \natural_\Theta (G_{\varphi\otimes\varphi})^*(z,w) \\ &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} G_T(z',w')(G_{\varphi\otimes\varphi})^*(z-z', w-w')e^{2\pi i(\omega x' - u'v)}\, dz'dw', \end{aligned} \tag{7.62}$$

*where $z' = (x', \omega'), w' = (u', v') \in \mathbb{R}^{2d}$.*

*Proof.* We apply twice Moyal's identity:

$$\begin{aligned} G_T(z,w) &= \int_{\mathbb{R}^{2d}} V_\varphi[T\pi(w)\varphi](z')\overline{V_\varphi[\pi(z)\varphi](z')}\, dz' \\ &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} V_\varphi[\pi(w)\varphi](w')\overline{V_\varphi[T^*\pi(z')\varphi](w')}\langle \pi(z')\varphi, \pi(z)\varphi\rangle\, dz'dw' \\ &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} G_T(z',w')\langle \pi(w)\varphi, \pi(w')\varphi\rangle\langle \pi(z')\varphi, \pi(z)\varphi\rangle\, dz'dw'. \end{aligned}$$

It is then a direct, although tedious, calculation to show that

$$\langle \pi(z)\varphi, \pi(z')\varphi\rangle\langle \pi(w')\varphi, \pi(w)\varphi\rangle = (G_{\varphi\otimes\varphi})^*(z-z', w-w')e^{2\pi i(\omega x' - u'v)}.$$

The proof is concluded. □

**Lemma 7.2.32.** *Let $T \in \mathcal{J}^1$, $T \geq 0$ and consider $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2} = 1$. Then for any $\tau \in [0,1]$:*

$$W_\tau T(z) = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} e^{-2\pi i[(\omega x' - \omega' x) + (\frac{1}{2} - \frac{3}{4}\tau)x'\omega' + x'v]} G_T\left(\frac{z'}{2} - w, -\frac{z'}{2} - w\right) dwdz', \tag{7.63}$$

*where $z = (x, \omega), z' = (x', \omega'), w = (u, v) \in \mathbb{R}^{2d}$.*

*Proof.* We start rephrasing the $\tau$-Wigner distribution of $T$:

$$W_\tau T(z) = \mathcal{F}_\sigma \mathcal{F}_{W_\tau} T(z) = \int_{\mathbb{R}^{2d}} e^{-2\pi i(\omega x' - \omega' x)}\, \mathrm{tr}(\pi^\tau(z')^* T)\, dz'.$$

Recalling the properties for $\pi^\tau$, see Section 7.1, we see that

$$\begin{aligned}\pi^\tau(z'/2+z'/2) &= e^{2\pi i[(1-\tau)\frac{x'\omega'}{4}-\tau\frac{x'\omega'}{4}]}\pi^\tau(z'/2)\pi^\tau(z'/2)\\ &= e^{\frac{\pi}{2}i(1-2\tau)x'\omega'}\pi^\tau(z'/2)\pi^\tau(z'/2).\end{aligned}$$

Taking the adjoint we get $\pi^\tau(z')^* = e^{-\frac{\pi}{2}i(1-2\tau)x'\omega'}\pi^\tau(z'/2)^*\pi^\tau(z'/2)^*$ and we write using Lemma 7.2.28:

$$\begin{aligned}\mathrm{tr}(\pi^\tau(z')^*T) &= e^{-\frac{\pi}{2}i(1-2\tau)x'\omega'}\,\mathrm{tr}(\pi^\tau(z'/2)^*T\pi^\tau(z'/2)^*)\\ &= e^{-\frac{\pi}{2}i(1-2\tau)x'\omega'}\int_{\mathbb{R}^{2d}}\langle T\pi^\tau(z'/2)^*\pi^\tau(w)^*\varphi,\pi^\tau(z'/2)\pi^\tau(w)^*\varphi\rangle\,dw\\ &= e^{-\frac{\pi}{2}i(1-2\tau)x'\omega'}e^{-\frac{\pi}{2}i(1-\tau)x'\omega'}\\ &\times\int_{\mathbb{R}^{2d}}\langle T\pi^\tau(-z'/2)\pi^\tau(-w)\varphi,\pi^\tau(z'/2)\pi^\tau(-w)\varphi\rangle\,dw\\ &= e^{-\frac{\pi}{2}i(2-3\tau)x'\omega'}\int_{\mathbb{R}^{2d}}\langle T\pi(-z'/2)\pi(-w)\varphi,\pi(z'/2)\pi(-w)\varphi\rangle\,dw\\ &= e^{-\frac{\pi}{2}i(2-3\tau)x'\omega'}\int_{\mathbb{R}^{2d}}e^{-2\pi ix'v}\langle T\pi(-z'/2-w)\varphi,\pi(z'/2-w)\varphi\rangle\,dw.\end{aligned}$$

The proof is concluded. □

## 7.3 A characterization for Schwartz operators

In this section we introduced the weighted version of $\mathbb{S}_0$ and give an alternative description of the class $\mathfrak{S}$. We recall the polynomial weight defined in (2.7):

$$v_s(z) \coloneqq (1+|z|^2)^{\frac{s}{2}},\qquad z\in\mathbb{R}^{2d},$$

where $s\geq 0$. In order to avoid an extremely cumbersome notation, just for the weight functions $v_s$ we shall use in the present chapter the following:

$$v_s\otimes v_s(z,w) \coloneqq K_{v_s\otimes\overline{v_s}} = v_s(z)v_s(w),\qquad \forall z,w\in\mathbb{R}^{2d}.$$

**Definition 7.3.1.** *For $s\geq 0$ we define the* ***weighted class of Feichtinger operators*** *as*

$$\mathbb{M}^1_s \coloneqq \{S\colon \mathcal{S}_0'(\mathbb{R}^d)\to\mathcal{S}_0(\mathbb{R}^d)\,|\,S \text{ is linear, continuous with kernel } K_S\in M^1_{v_s\otimes v_s}(\mathbb{R}^{2d})\}. \tag{7.64}$$

*For $S$ in $\mathbb{M}^1_s$ we define the application*

$$\|S\|_{\mathbb{M}^1_s} \coloneqq \|K_S\|_{M^1_{v_s\otimes v_s}}. \tag{7.65}$$

**Remark 7.3.2.** (*i*) *For $s=0$ we recover the unweighted Feichtinger operators $\mathbb{S}_0$;*

(*ii*) *The application defined in (7.65) is a norm on $\mathbb{M}^1_s$ and it is easy to see that $(\mathbb{M}^1_s,\|\cdot\|_{\mathbb{M}^1_s})$ is a Banach space and the following continuous inclusion holds true for every $s\geq 0$:*

$$\mathbb{M}^1_s\hookrightarrow\mathbb{S}_0. \tag{7.66}$$

**Lemma 7.3.3.** *Let $S\in\mathbb{M}^1_s$, then there exist $\{f_n\}_n,\{g_n\}_n\subseteq M^1_{v_s\otimes v_s}(\mathbb{R}^{2d})$ such that*

$$S=\sum_{n=1}^\infty f_n\otimes g_n,\qquad \sum_{n=1}^\infty\|f_n\|_{M^1_{v_s}}\|g_n\|_{M^1_{v_s}}\leq+\infty,\quad K_S=\sum_{n=1}^\infty K_{f_n\otimes g_n}.$$

*Proof.* The proof follows from the fact that

$$M^1_{v_s\otimes v_s}(\mathbb{R}^{2d}) = M^1_{v_s}(\mathbb{R}^d)\hat{\otimes}M^1_{v_s}(\mathbb{R}^d).$$

See proof of Lemma 7.2.4 also. □

**Theorem 7.3.4.** *For every $\tau \in [0,1]$ the mapping $W_\tau \colon \mathbb{M}^1_s \to M^1_{v_s\otimes v_s}(\mathbb{R}^{2d})$ is a topological isomorphism with inverse given by $\mathrm{Op}_\tau \colon M^1_{v_s\otimes v_s}(\mathbb{R}^{2d}) \to \mathbb{M}^1_s$.*

*Proof.* The proof follows the same patter of Theorem 7.2.7 and Corollary 7.2.8. □

**Corollary 7.3.5.** *An operator $S$ belong to $\mathbb{M}^1_s$ iif for some (hence every) $\tau \in [0,1]$ $W_\tau S \in M^1_{v_s\otimes v_s}(\mathbb{R}^{2d})$.*

**Theorem 7.3.6.** *The following holds true:*

$$\mathfrak{S} = \bigcap_{s\geq 0} \mathbb{M}^1_s. \tag{7.67}$$

*Proof.* On account of Corollary 7.3.5, $S$ belongs to the set on the right-hand side if and only if

$$W_\tau S \in \bigcap_{s\geq 0} M^1_{v_s\otimes v_s}(\mathbb{R}^{2d}) = \mathcal{S}(\mathbb{R}^{2d}).$$

The claim follows since $W_{1/2}S$ is the Weyl symbol of $S$, i.e. $\mathrm{a}^S_{1/2} = W_{1/2}S$. □

We recall that a function $F$ on $\mathbb{R}^{2d}$ is called rapidly decaying if for every multiindex $\alpha, \beta \in \mathbb{N}^d_0$ we have

$$\sup_{x,\omega\in\mathbb{R}^d} \left|x^\alpha \omega^\beta F(x,\omega)\right| < +\infty,$$

where, if $x = (x_1, \ldots, x_d)$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$, $x^\alpha$ stands for $x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d}$.

In [95, Theorem 1.1] is given a sufficient condition for a positive trace-class operator to be in $\mathfrak{S}$. Namely, if $T \in B(L^2)$, $T \geq 0$, is such that $W_\tau T$ exists for some $\tau \in [0,1]$ and it is rapidly decreasing, then $T \in \mathfrak{S}$ and $W_\tau T$ exists for every $\tau \in [0,1]$. In this spirit, we provide the following sufficient condition for a generic $S \in B(L^2)$. Observe that we do not not require $S$ to be positive.

**Corollary 7.3.7.** *Let $S \in B(L^2)$ and assume that form some $\tau \in [0,1]$ $W_\tau S$ exists. Suppose also that, w.r.t. some non-zero window in $L^2(\mathbb{R}^{2d})$, the STFT of $W_\tau S$ is rapidly decaying. Then $W_\tau S$ exists for every $\tau \in [0,1]$ and $S$ is in $\mathfrak{S}$.*

*Proof.* Let us pick $G \in L^2(\mathbb{R}^{2d}) \smallsetminus \{0\}$. If $V_G W_\tau S$ is rapidly decaying then $S \in \mathbb{M}^1_s$ for every $s \geq 0$. The claim follows from Theorem 7.3.6. □

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