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Fourier sums with exponential weights on $(-1, 1)$: L^1 and L^∞ cases[☆]

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Abstract

We study the behavior of the Fourier sums in orthonormal polynomial systems, related to exponential weights on $(-1, 1)$, in weighted L^1 and uniform metrics.

Keywords: Orthogonal polynomials; Fourier series; Approximation by polynomials; Exponential weights

1. Introduction and main results

Supposing that

$$v^\lambda(x) = (1 - x^2)^\lambda, \qquad w(x) = e^{-(1-x^2)^{-\alpha}},$$

we consider the weight function

$$\sigma(x) = (1 - x^2)^\lambda e^{-(1-x^2)^{-\alpha}} = v^\lambda(x)w(x), \quad \alpha > 0,\ \lambda \geq 0, \tag{1}$$

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for $x \in (-1, 1)$, and the corresponding sequence $\{p_m(\sigma)\}_{m\in\mathbb{N}}$ of orthonormal polynomials with positive leading coefficients $\gamma_m(\sigma)$. If $f \in L^1_\sigma$, i.e. $\int_{-1}^{1} |f\sigma| < \infty$, we can define the mth Fourier sum

$$S_m(\sigma, f) = \sum_{k=0}^{m-1} c_k p_k(\sigma), \quad c_k = \int_{-1}^{1} p_k(\sigma) f\sigma,$$

and investigate under which conditions the function f can be represented by a Fourier series in some suitable function spaces.

Then, letting the weight in (1) be w, we consider function spaces related to the weight function

$$u(x) = (1-x^2)^\mu e^{-\frac{1}{2}(1-x^2)^{-\alpha}} = v^\mu(x)\sqrt{w(x)}, \quad \alpha > 0,\ \mu \geq 0. \tag{2}$$

If we consider the case $1 < p < \infty$, denoting by L^p_u the collection of all measurable functions f, with $\|f\|_{L^p_u} = \|fu\|_p = \left(\int_{-1}^{1} |fu|^p\right)^{1/p}$, then for our aims the following inequality is crucial:

$$\|S_m(\sigma, f)u\|_p \leq \mathcal{C}\|fu\|_p, \tag{3}$$

where $\mathcal{C}$ is a positive constant independent of f and m. Unfortunately, excluding the case $p = 2$ and $u = \sqrt{\sigma}$, inequality (3) does not seem to be true.

To overcome this problem, recently in [7] the authors have proposed approximating a function $f \in L^p_u$ by means of the sequence

$$\{\chi_\theta S_m(\sigma, \chi_\theta f)\}_{m\in\mathbb{N}}, \tag{4}$$

where χ_θ is the characteristic function of the subset of the Mhaskar–Rahmanov–Saff interval $[-a_{\theta m}, a_{\theta m}]$, $a_m = a_m(\sqrt{\sigma})$, $\theta \in (0, 1)$ is fixed, and $1 - a_m \sim m^{-1/(\alpha+1/2)}$. A bound of the form (3) has been proved for this sequence, under suitable assumptions on the weights σ and u. Then the convergence of the sequence (4) to the function f in the L^p_u-metric for $1 < p < \infty$, which has the order of the best polynomial approximation, was also shown.

One of the main tools used for proving the results in [7] was the boundedness of the Hilbert transform in weighted L^p-spaces. Since this cannot hold for $p = 1$ or $p = \infty$, these cases are still open problems. Therefore, to complete the paper [7], here we show the convergence of the sequence in (4), in weighted L^1 and uniform metrics.

Then, letting u be the weight in (2) and $p = 1$ or $p = \infty$, we are going to consider the function spaces

$$L^1_u = \left\{f : fu \in L^1(-1, 1)\right\}$$

and

$$L^\infty_u = C_u = \left\{f \in C^0(-1, 1) : \lim_{x\to\pm 1} f(x)u(x) = 0\right\},$$

with the norms

$$\|f\|_{L^1_u} := \|fu\|_1 = \int_{-1}^{1} |f(x)u(x)|\,dx$$

and

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x\in(-1,1)} |f(x)u(x)|,$$

respectively.

To state our main results, we need some notation. In the sequel, $\mathcal{C}$ will stand for a positive constant that could assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \le \mathcal{C}$. Moreover, we denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by $E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|_p$ the error of the best polynomial approximation in L^p_u, $p \in \{1, \infty\}$.

The following theorem concerns the behavior of the operator $\chi_\theta S_m(\sigma, \chi_\theta f) : C_u \to C_u$.

Theorem 1.1. *Suppose that* $\sigma(x) = (1-x^2)^\lambda e^{-(1-x^2)^{-\alpha}}$ *and* $u(x) = (1-x^2)^\mu e^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, *with* $\alpha > 0$, *and* $\lambda, \mu \ge 0$, *and let* $\theta \in (0,1)$ *be fixed. For every* $f \in C_u$, *we have*

$$\|\chi_\theta S_m(\sigma, \chi_\theta f)\, u\|_\infty \le \mathcal{C}_\theta (\log m) \|\chi_\theta f u\|_\infty, \tag{5}$$

with $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/2}(1/\theta)\right)$ *independent of* m *and* f, *if and only if*

$$\frac{1}{4} \le \mu - \frac{\lambda}{2} \le \frac{3}{4}. \tag{6}$$

Moreover, under the assumption (6), *we get*

$$\|[f - \chi_\theta S_m(\sigma, \chi_\theta f)]\, u\|_\infty \le \mathcal{C}_\theta \left\{(\log m) E_M(f)_{u,\infty} + e^{-cM^{\frac{2\alpha}{2\alpha+1}}} \|fu\|_\infty\right\}, \tag{7}$$

where $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)\frac{m}{2}\right\rfloor$ *and* $c \neq c(m, f, \theta)$.

To complete Theorem 1.1, we remark that the "truncation of the function" seems to be essential, since in (5) and (7) the parameter θ cannot assume the value 1 (see the proof in Section 2).

Moreover, the following remark could be useful. Denote by $S_m(v^\lambda, f)$ the mth Fourier sum with respect to the Jacobi weight $v^\lambda(x) = (1-x^2)^\lambda$, $\lambda \ge 0$, and suppose that $f \in L^\infty_{v^\mu}$, where $v^\mu(x) = (1-x^2)^\mu$, $\mu \ge 0$. In [4] (see also [5, p. 276]) the authors proved that the inequality

$$\|S_m(v^\lambda, f) v^\mu\|_\infty \le \mathcal{C}(\log m)\|f v^\mu\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m, f),$$

holds true if and only if condition (6) and $0 \le \mu \le \lambda + 1$ are fulfilled. Then, the behavior of the sequence $\{\chi_\theta S_m(\sigma, f_\theta)\}_{m\in\mathbb{N}}$ in C_u can be deduced from that of the sequence $\left\{S_m(v^\lambda, f)\right\}_{m\in\mathbb{N}}$ in C_{v^μ}.

In analogy with Theorem 1.1, the next statement shows the behavior of $\chi_\theta S_m(\sigma, \chi_\theta f) : L^1_u \to L^1_u$.

Theorem 1.2. *Let* $\sigma = v^\lambda w$ *and* $u = v^\mu \sqrt{w}$ *be the weights defined above, with* $\alpha > 0$, $\lambda, \mu \ge 0$, *and* $\theta \in (0,1)$ *fixed. For any* $f \in L^1_u$, *the inequality*

$$\|\chi_\theta S_m(\sigma, \chi_\theta f)\, u\|_1 \le \mathcal{C}_\theta (\log m) \|\chi_\theta f u\|_1 \tag{8}$$

holds, with $\mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f)$ *and* $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/2}(1/\theta)\right)$, *if and only if*

$$\frac{v^\mu}{\sqrt{v^\lambda \varphi}} \in L^1, \qquad \frac{1}{v^\mu}\sqrt{\frac{v^\lambda}{\varphi}} \in L^\infty, \tag{9}$$

where $\varphi(x) = \sqrt{1-x^2}$.

Moreover, conditions (9) *imply*

$$\|[f - \chi_\theta S_m(\sigma, \chi_\theta f)]\, u\|_1 \le \mathcal{C}_\theta \left\{(\log m) E_M(f)_{u,1} + \mathrm{e}^{-cM^{\frac{2\alpha}{2\alpha+1}}} \|fu\|_1\right\}, \tag{10}$$

where $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) \frac{m}{2} \right\rfloor$ *and* $c \neq c(m, f, \theta)$.

In different contexts, estimates for the weighted L^1-norm of $\chi_\theta S_m(\sigma, \chi_\theta f)$ without the factor $\log m$ are required. Of course we need some further assumptions on the function f.

Theorem 1.3. *With the notation of Theorem 1.2, if* (9) *holds, we have*

$$\|\chi_\theta S_m(\sigma, \chi_\theta f)\, u\|_1 \le \mathcal{C}_\theta \left\| \chi_\theta f u \left(1 + \log^+ |fu| + \log \frac{\mathrm{e}}{1 - \cdot^2}\right)\right\|_1 \tag{11}$$

for any function f *such that the norm on the right-hand side is bounded, with* $\mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f)$ *and* $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/2}(1/\theta)\right)$, *where* $\log^+ y = \begin{cases} \log y & \text{for } y > 1 \\ 0 & \text{for } y \le 1 \end{cases}$.

2. Proofs

We first recall some known results which will be used in the proofs.

Following Levin and Lubinsky in [1, p. 5], we will say that the weight $\varrho(x) = \mathrm{e}^{-Q(x)}$, $|x| < 1$, belongs to the class $\hat{\mathcal{W}}$ and write $\varrho \in \hat{\mathcal{W}}$ if and only if the function $Q : (-1, 1) \in \mathbb{R}$ is an even function that is twice continuously differentiable, and satisfies the following properties:

(i) $Q'(x) \ge 0$ $Q''(x) \ge 0$ for $x \in (0, 1)$;
(ii) $\lim_{x \to 1^-} Q(x) = +\infty$;
(iii) the function

$$T(x) = 1 + \frac{x Q''(x)}{Q'(x)}$$

is increasing in $[0, 1)$ with $T(0) > 1$ and

$$T(x) \sim \frac{Q'(x)}{Q(x)}$$

for x close enough to 1.

One can prove that the weights σ and u in (1) and (2) belong to the class $\hat{\mathcal{W}}$, and the related functions T satisfy $T(x) \sim (1 - x^2)^{-1}$ for x close enough to 1 (see [7, Prop. 2.3]).

The related Mhaskar–Rahmanov–Saff number $a_m = a_m(\varrho)$ is implicitly defined as the positive root of the equation

$$m = \frac{2}{\pi} \int_0^1 a_m t\, Q'(a_m t) \frac{\mathrm{d}t}{\sqrt{1 - t^2}}, \tag{12}$$

and the equivalence (see [3])

$$Q'(a_m) \sim m\sqrt{T(a_m)} \tag{13}$$

can lead to an approximation of a_m. For instance, as regards the weights σ and u, we have

$$1 - a_m(\sigma) \sim 1 - a_m(u) \sim m^{-1/\left(\alpha+\frac{1}{2}\right)}. \tag{14}$$

As regards the number a_m, the following restricted range inequalities hold. Suppose that $\varrho \in \hat{\mathcal{W}}$ and $L > 0$. For any polynomial $P_m \in \mathbb{P}_m$, with $1 \le p \le \infty$ and $s > 1$, we have

$$\|P_m\varrho\|_p \le \mathcal{C}\|P_m\varrho\|_{L^p[-a_m(1-L\delta_m),a_m(1-L\delta_m)]}, \tag{15}$$

and

$$\|P_m\varrho\|_{L^p\{x\ge a_{sm}\}} \le \mathcal{C}\mathrm{e}^{-cmT(a_m)^{-1/2}}\|P_m\varrho\|_{L^p[-a_m,a_m]}, \tag{16}$$

where $a_m = a_m(\varrho)$ and $\delta_m := (mT(a_m))^{-2/3}$, and $\mathcal{C}$ and c are positive constants independent of P_m (see [1, Th. 1.7, p. 12] and [3, Lemma 2.3]).

Let us now recall some properties of the orthonormal polynomials. Let σ be the weight in (1), with $a_m = a_m\left(\sqrt{\sigma}\right)$ and $T(a_m) \sim (1-a_m)^{-1} \sim m^{\frac{1}{\alpha+1/2}}$, and let $\{p_m(\sigma)\}_{m\in\mathbb{N}}$ be the corresponding orthonormal system. Then the equivalences

$$\sup_{x\in(-1,1)} \left|p_m(\sigma,x)\sqrt{\sigma(x)}\sqrt[4]{|a_m^2-x^2|}\right| \sim 1 \tag{17}$$

and

$$\sup_{x\in(-1,1)} \left|p_m(\sigma,x)\sqrt{\sigma(x)}\right| \sim (mT(a_m))^{1/6}, \tag{18}$$

have been proved in [1, formulae (1.38) and (1.39), p. 10] (see also [2, p. 22]).

Suppose that $\theta \in (0,1)$; for any $x \in [-a_{\theta m}, a_{\theta m}]$ we have (see [7])

$$(a_m^2 - x^2) \le (1-x^2) \le \left(1 + \frac{c}{\log(1/\theta)}\right)(a_m^2 - x^2), \tag{19}$$

where c is a positive constant independent of θ and m. Hence, by (17) and (19), we deduce the inequality

$$|p_m(\sigma,x)|\sqrt{\sigma(x)\varphi(x)} \le \mathcal{C}_\theta, \quad |x| \le a_{\theta m}, \tag{20}$$

where

$$\mathcal{C}_\theta = \mathcal{C}\left(1 + \frac{1}{\log(1/\theta)}\right)^{1/4} \tag{21}$$

with $\mathcal{C}$ independent of m and θ.

Moreover, consider the weight $\varphi^2\sigma$, which belongs to the same class of σ (see for instance [7]). From (13) we deduce that its Mhaskar–Rahmanov–Saff number $\bar{a}_m = a_m\left(\sqrt{\varphi^2\sigma}\right)$ satisfies

$$T(\bar{a}_m)^{-1} \sim 1 - \bar{a}_m \sim 1 - a_m,$$

where $a_m = a_m\left(\sqrt{\sigma}\right)$. Then, inequalities analogous to those in (19) hold with a_m replaced by $\bar{a}_m$. Namely, supposing that $\theta \in (0, 1)$, for any $x \in [-a_{\theta m}, a_{\theta m}]$, we have

$$(\bar{a}_m^2 - x^2) \le (1 - x^2) \le \left(1 + \frac{c}{\log(1/\theta)}\right)(\bar{a}_m^2 - x^2). \tag{22}$$

Therefore we get

$$\left|p_m(\varphi^2\sigma, x)\right| \sqrt{\sigma(x)\varphi^3(x)} \le \mathcal{C}_\theta, \quad x \in [-a_{\theta m}, a_{\theta m}], \tag{23}$$

with $\mathcal{C}_\theta$ as in (21).

Let us denote by x_k, $k = 1, \dots, m$, the zeros of $p_m(\sigma)$, located as

$$-a_m(1 - c\delta_m) < x_1 < x_2 \cdots < x_m < a_m(1 - c\delta_m),$$

with $c > 0$ and $\delta_m \sim m^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)}$. Then the formula

$$\Delta x_k \, |p_m(\sigma, x)| \sqrt{\sigma(x)} \sim \frac{|x - x_k|}{\left|a_m^2 - x_k^2\right|^{1/4}} \tag{24}$$

holds for any $x \in (-1, 1)$, where x_k is a node closest to x and $\Delta x_k = x_{k+1} - x_k$ (see [1, formula (12.7), p. 134]). As a consequence, if $x \in [-a_{\theta m}, a_{\theta m}] \cap I_k$, where

$$I_k = \left[x_k + \frac{\Delta x_k}{8}, x_{k+1} - \frac{\Delta x_k}{8}\right],$$

we have

$$|p_m(\sigma, x)| \sqrt{\sigma(x)\varphi(x)} \sim 1, \quad x \in I_k \cap [-a_{\theta m}, a_{\theta m}], \tag{25}$$

since $\varphi(x_k) = \sqrt{1 - x_k^2} \sim \sqrt{a_m^2 - x_k^2}$ for $|x_k| \le a_{\theta m}$.

The following Bernstein inequality has been proved in [6] (see also [11] for more general weights).

Theorem 2.1. *Suppose that $u(x) = (1 - x^2)^\mu \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, with $\mu \ge 0$ and $\alpha > 0$. Then, for any $P_m \in \mathbb{P}_m$, we get*

$$\sup_{x \in (-1,1)} \left|P_m'(x)\varphi(x)u(x)\right| \le \mathcal{C} m \|P_m u\|_\infty, \tag{26}$$

where $\varphi(x) = \sqrt{1 - x^2}$ and $\mathcal{C}$ is independent of m and P_m.

If f belongs to L^1_u, its mth Fourier sum $S_m(\sigma, f)$ is defined in the usual way as

$$S_m(\sigma, f, x) = \sum_{k=0}^{m-1} c_k(\sigma, f) p_k(\sigma, x) = \int_{-1}^{1} K_m(\sigma, x, t) f(t)\sigma(t)\, \mathrm{d}t,$$

where $c_k(\sigma, f) = \int_{-1}^{1} p_k(\sigma, t) f(t)\sigma(t)\,\mathrm{d}t$ is the kth Fourier coefficient of f in the system $\{p_m(\sigma)\}_{m\in\mathbb{N}}$ and

$$K_m(\sigma, x, t) = \sum_{k=0}^{m-1} p_k(\sigma, x) p_k(\sigma, t)$$

$$= \frac{\gamma_{m-1}(\sigma)}{\gamma_m(\sigma)} \frac{p_m(\sigma, x) p_{m-1}(\sigma, t) - p_{m-1}(\sigma, x) p_m(\sigma, t)}{x - t} \tag{27}$$

is the Christoffel–Darboux kernel. By using the Pollard formula, this kernel can be written as follows:

$$K_m(\sigma, x, t) = -\alpha_m p_m(\sigma, x) p_m(\sigma, t)$$

$$+ \beta_m \frac{p_m(\sigma, x) p_{m-1}(\varphi^2\sigma, t)\varphi^2(t) - p_{m-1}(\varphi^2\sigma, x)\varphi^2(x) p_m(\sigma, t)}{x - t} \tag{28}$$

where $\varphi^2(t) = 1 - t^2$,

$$\alpha_m = \left(1 + \frac{\gamma_{m+1}(\varphi^2\sigma)\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)^2}\right)^{-1} \frac{\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)}$$

and

$$\beta_m = \left(1 + \frac{\gamma_{m+1}(\varphi^2\sigma)\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)^2}\right)^{-1} \frac{\gamma_{m+1}(\varphi^2\sigma)\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)^2}.$$

In [7, Prop. 2.2] it has been shown that $\gamma_m(\sigma)/\gamma_{m\pm1}(\varphi^2\sigma) \sim 1$ and then $\alpha_m \sim 1 \sim \beta_m$.

In order to prove the error estimates (7) and (10), we need the following lemma.

Lemma 2.2. *Let σ and u be the weights in (1) and (2), with arbitrary parameters $\alpha > 0$, $\lambda, \mu \geq 0$. Then, for any $f \in C_u$, we have*

$$\|S_m(\sigma, f)u\|_\infty \leq \mathcal{C} m^\nu \|fu\|_\infty, \tag{29}$$

and, for any $f \in L^1_u$, we get

$$\|S_m(\sigma, f)u\|_1 \leq \mathcal{C} m^\nu \|fu\|_1, \tag{30}$$

for some $\nu > 0$, where $\mathcal{C}$ is independent of m and f in both cases.

Proof. Let us first prove (29). We first observe that, since $u = v^{\mu-\lambda/2}\sqrt{\sigma}$, for any $P_m \in \mathbb{P}_m$, with $1 \leq p \leq \infty$, we have (see [7, Prop. 2.1])

$$\|P_m u\|_p \leq \mathcal{C} \begin{cases} \|P_m u\|_{L^p[-a_{sm}, a_{sm}]}, \ s > 1, & \text{if } \mu - \lambda/2 < 0 \\ \|P_m u\|_{L^p[-a_m, a_m]}, & \text{otherwise} \end{cases} \tag{31}$$

where $a_m = a_m(\sqrt{\sigma})$.

Then, by inequality (31), we have

$$\|S_m(\sigma, f)u\|_\infty = \sup_{x\in[-a_{sm},a_{sm}]} \left| u(x) \int_{-1}^{1} K_m(\sigma, x, t) f(t)\sigma(t)\,\mathrm{d}t \right|$$
$$\leq \|fu\|_\infty \sup_{x\in[-a_{sm},a_{sm}]} \left\{ \int_{|x-t|\geq\frac{\varphi(x)}{m}} + \int_{|x-t|\leq\frac{\varphi(x)}{m}} \right\} \left| K_m(\sigma, x, t)\frac{\sigma(t)}{u(t)} u(x)\right| \mathrm{d}t$$
$$\leq \|fu\|_\infty \sup_{x\in[-a_{sm},a_{sm}]} \{I_1 + I_2\}, \tag{32}$$

where $s > 1$.

Here, our aim is to obtain a raw estimate for $\|S_m(\sigma, f)u\|_\infty$. Therefore, instead of the Pollard decomposition, we will use the definition in (27) of the Christoffel–Darboux kernel. By (18), we have

$$|K_m(\sigma, x, t)| \frac{\sigma(t)}{u(t)} u(x) = |K_m(\sigma, x, t)| \sqrt{\sigma(x)\sigma(t)} \left(\frac{1-x^2}{1-t^2}\right)^{\mu-\lambda/2}$$
$$\leq \frac{\left[|p_m(\sigma, x)p_{m-1}(\sigma, t)| + |p_{m-1}(\sigma, x)p_m(\sigma, t)|\right]\sqrt{\sigma(x)\sigma(t)}}{|x-t|} \left(\frac{1-x^2}{1-t^2}\right)^{\mu-\lambda/2}$$
$$\leq \frac{\mathcal{C}}{|x-t|}(mT(a_m))^{1/3}(1-a_{sm})^{-|\mu-\lambda/2|}.$$

Hence, with $|x|, |t| \leq a_{sm}$, we get

$$I_1 \leq \mathcal{C}(mT(a_m))^{1/3}(1-a_{sm})^{-|\mu-\lambda/2|} \int_{|x-t|\geq\frac{\varphi(x)}{m}} \frac{\mathrm{d}t}{|x-t|}$$
$$\leq \mathcal{C}(mT(a_m))^{1/3}(1-a_{sm})^{-|\mu-\lambda/2|}\log m. \tag{33}$$

Whereas, for the integrand of the term I_2 in (32), by the mean value theorem, we have

$$|K_m(\sigma, x, t)| \frac{\sigma(t)}{u(t)} u(x)$$
$$= \left\{ \left|p'_m(\sigma, \xi_1)p_{m-1}(\sigma, t)\right| + \left|p'_{m-1}(\sigma, \xi_2)p_m(\sigma, t)\right| \right\} \sqrt{\sigma(x)\sigma(t)} \left(\frac{1-x^2}{1-t^2}\right)^{\mu-\lambda/2}$$

with $\xi_1, \xi_2 \in (x, t)$. We recall that if $t, y \in [x - \varphi(x)/m, x + \varphi(x)/m]$, we have $(1-t^2) \sim (1-y^2) \sim (1-x^2)$ and $\sigma(t) \sim \sigma(y) \sim \sigma(x)$ (see [6]). Then, by using (18), the Bernstein-type inequality (26) and, again, (18), we get

$$|K_m(\sigma, x, t)| \frac{\sigma(t)}{u(t)} u(x)$$
$$\leq \mathcal{C}\,(mT(a_m))^{\frac{1}{6}} \frac{m}{\varphi(x)} \left[\left|p'_m(\sigma, \xi_1)\right| \frac{\varphi(\xi_1)}{m}\sqrt{\sigma(\xi_1)} + \left|p'_{m-1}(\sigma, \xi_2)\right| \frac{\varphi(\xi_2)}{m}\sqrt{\sigma(\xi_2)} \right]$$
$$\leq \mathcal{C}\,(mT(a_m))^{\frac{1}{3}} \frac{m}{\varphi(x)}.$$

Hence we obtain

$$I_2 \le \mathcal{C}\,(mT(a_m))^{1/3}\,\frac{m}{\varphi(x)}\int_{|x-t|\le\frac{\varphi(x)}{m}}\mathrm{d}t$$
$$\le \mathcal{C}\,(mT(a_m))^{1/3}\,. \tag{34}$$

Therefore, by (32)–(34), we deduce (29) since $T(a_m)\sim(1-a_m)^{-1}$ and by (14).

Now, let us prove (30). Setting $g(x)=\operatorname{sgn}\{S_m(\sigma,f,x)\}$ and reversing the integrals, we have

$$\begin{aligned}\|S_m(\sigma,f)u\|_1 &= \int_{-1}^{1} g(x)\int_{-1}^{1}K_m(\sigma,x,t)f(t)\sigma(t)\,\mathrm{d}t\,u(x)\,\mathrm{d}x\\ &= \int_{-1}^{1}\left|f(t)\sigma(t)\int_{-1}^{1}g(x)K_m(\sigma,x,t)u(x)\,\mathrm{d}x\right|\mathrm{d}t\\ &\le \|fu\|_1\sup_{|t|\le1}\left|\frac{\sigma(t)}{u(t)}\int_{-1}^{1}g(x)K_m(\sigma,x,t)u(x)\,\mathrm{d}x\right|\\ &= \|fu\|_1\left\|S_m\left(\sigma,\frac{gu}{\sigma}\right)\frac{\sigma}{u}\right\|_\infty.\end{aligned}$$

Then, by (29), inequality (30) follows. □

From inequality (16), we can deduce the following proposition (see [7]).

Proposition 2.3. *Suppose that $0<a<1$, let u be the weight in (2) and suppose that $1\le p\le\infty$. There exists an integer $M\ge1$ such that, for any function $f\in L^p_u$, we have*

$$\|fu\|_{L^p\{|x|\ge a\}}\le\mathcal{C}\left\{E_M(f)_{u,p}+\mathrm{e}^{-cMT(a_M)^{-1/2}}\|fu\|_p\right\}, \tag{35}$$

where $\mathcal{C}$, c are positive constants independent of f and M.

We remark that, by Proposition 2.3, if $f_\theta=\chi_\theta f$, with χ_θ the characteristic function of $[-a_{\theta m}(\sqrt{\sigma}),a_{\theta m}(\sqrt{\sigma})]$, where σ is the weight in (1), for m sufficiently large we can estimate the L^p_u-distance between f and f_θ by (35) with $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)\frac{m}{2}\right\rfloor$, taking into account (31).

Proof of Theorem 1.1. Let us first prove that conditions (6) imply inequality (5).

In order to estimate $\|\chi_\theta S_m(\sigma,f_\theta)u\|_\infty$, where $f_\theta=\chi_\theta f$, it suffices to consider the quantity $|S_m(\sigma,f_\theta,x)|u(x)$ for $x\in[-a_{\theta m},a_{\theta m}]$. By (28), since $\alpha_m\sim1\sim\beta_m$, we have

$$\begin{aligned}|S_m(\sigma,f_\theta,x)|u(x) \le{}& \mathcal{C}\left\{|p_m(\sigma,x)|u(x)\left|\int_{-a_{\theta m}}^{a_{\theta m}}p_m(\sigma,t)f(t)\sigma(t)\mathrm{d}t\right|\right.\\ &\left.+u(x)\left|\int_{-a_{\theta m}}^{a_{\theta m}}\widetilde{K}(x,t)f(t)\sigma(t)\mathrm{d}t\right|\right\}\\ &=: \mathcal{C}\{A_1+A_2\},\end{aligned} \tag{36}$$

where

$$\widetilde{K}(x,t)=\frac{p_m(\sigma,x)p_{m-1}(\varphi^2\sigma,t)\varphi^2(t)-p_{m-1}(\varphi^2\sigma,x)\varphi^2(x)p_m(\sigma,t)}{x-t}.$$

For the term A_1, by (20), we have

$$\begin{aligned} A_1 \leq & \left| p_m(\sigma, x)\sqrt{\sigma(x)\varphi(x)} \right| v^{\mu-\frac{\lambda}{2}-\frac{1}{4}}(x) \| f_\theta u \|_\infty \\ & \times \int_{-a_{\theta m}}^{a_{\theta m}} \left| p_m(\sigma, t)\sqrt{\sigma(t)\varphi(t)} \right| v^{-\mu+\frac{\lambda}{2}-\frac{1}{4}}(t)\, \mathrm{d}t \\ \leq & \, \mathcal{C}_\theta^2 \| f_\theta u \|_\infty v^{\mu-\frac{\lambda}{2}-\frac{1}{4}}(x) \int_{-a_{\theta m}}^{a_{\theta m}} v^{-\mu+\frac{\lambda}{2}-\frac{1}{4}}(t)\, \mathrm{d}t \\ \leq & \, \mathcal{C}_\theta^2 \| f_\theta u \|_\infty \int_{-a_{\theta m}}^{a_{\theta m}} v^{-\mu+\frac{\lambda}{2}-\frac{1}{4}}(t)\, \mathrm{d}t, \end{aligned}$$

since, by (6), $\mu - \lambda/2 - 1/4 \geq 0$. Moreover, by (6), we have $-\mu + \lambda/2 - 1/4 \geq -1$ and then the integral on the right-hand side is bounded, except for the case $-\mu + \lambda/2 - 1/4 = -1$. Anyway we get

$$A_1 \leq \mathcal{C}_\theta^2 (\log m) \| f_\theta u \|_\infty, \tag{37}$$

where $\mathcal{C}_\theta$ is the constant in (21).

Whereas, for the term A_2, we can write

$$\begin{aligned} A_2 \leq & \, \| f_\theta u \|_\infty \left\{ \int_{|x-t| \geq \frac{\varphi(x)}{m}} + \int_{|x-t| \leq \frac{\varphi(x)}{m}} \right\} \left| \widetilde{K}(x,t) \right| \frac{\sigma(t)u(x)}{u(t)}\, \mathrm{d}t \\ =: & \, \| f_\theta u \|_\infty \{ B_1 + B_2 \}, \end{aligned} \tag{38}$$

with $|x|, |t| \leq a_{\theta m}$.

Let us consider B_1. Setting $h_m(x,t) = p_m(\sigma, x) p_{m-1}(\varphi^2\sigma, t)\varphi^2(t)\sqrt{\sigma(x)\sigma(t)}$, by (20) and (23), we get

$$\begin{aligned} B_1 = & \int_{|x-t| \geq \frac{\varphi(x)}{m}} \left| \frac{h_m(x,t) - h_m(t,x)}{x-t} \right| \left(\frac{1-x^2}{1-t^2} \right)^{\mu-\lambda/2} \mathrm{d}t \\ \leq & \, \mathcal{C}_\theta^2 \int_{|x-t| \geq \frac{\varphi(x)}{m}} \frac{1}{|x-t|} \left[\left(\frac{1-x^2}{1-t^2} \right)^{\mu-\lambda/2-1/4} + \left(\frac{1-x^2}{1-t^2} \right)^{\mu-\lambda/2+1/4} \right] \mathrm{d}t. \end{aligned}$$

So, in order to estimate B_1, it is sufficient to consider integrals of the form

$$\int_{|x-t| \leq \phi(x)/m} \frac{1}{|x-t|} \left(\frac{1-x^2}{1-t^2} \right)^{\delta} \mathrm{d}t, \quad 0 \leq \delta \leq 1, \; |x|, |t| \leq a_{\theta m},$$

taking into account (6). These integrals are dominated by $\log m$ and then

$$B_1 \leq \mathcal{C} \log m. \tag{39}$$

Let us now consider the term B_2. Setting $Q(x) = p_{m-1}(\varphi^2\sigma, x)\varphi^2(x)$, we can write

$$\begin{aligned} B_2 \leq & \int_{|x-t| \leq \frac{\varphi(x)}{m}} \left| \frac{p_m(\sigma,x) - p_m(\sigma,t)}{x-t} \right| |Q(t)| \sqrt{\sigma(x)\sigma(t)} \left(\frac{1-x^2}{1-t^2} \right)^{\mu-\lambda/2} \mathrm{d}t \\ & + \int_{|x-t| \leq \frac{\varphi(x)}{m}} \left| \frac{Q(x) - Q(t)}{x-t} \right| |p_m(\sigma,t)| \sqrt{\sigma(x)\sigma(t)} \left(\frac{1-x^2}{1-t^2} \right)^{\mu-\lambda/2} \mathrm{d}t \\ =: & \, B_2' + B_2''. \end{aligned} \tag{40}$$

We are going to estimate only the term B_2'', since B_2' can be handled in a similar way. To this end we recall that, for $y, t \in [x-\varphi(x)/m, x+\varphi(x)/m]$, $|x| \le a_{\theta m}$, we have $1-y^2 \sim 1-t^2 \sim 1-x^2$ and $\sigma(y) \sim \sigma(t) \sim \sigma(x)$ (see [6]). Then, using (20) and the mean value theorem with $\xi \in (x,t)$, setting $\bar{a}_{m-1} = \bar{a}_{m-1}(\sqrt{\varphi^2\sigma})$, we get

$$\begin{aligned}
&\left|\frac{Q(x)-Q(t)}{x-t}\right| |p_m(\sigma,t)|\sqrt{\sigma(x)\sigma(t)}\left(\frac{1-x^2}{1-t^2}\right)^{\mu-\lambda/2} \le \mathcal{C}_\theta \frac{|Q'(\xi)|\sqrt{\sigma(\xi)}}{\sqrt{\varphi(\xi)}}\\
&\le \mathcal{C}_\theta\left[\frac{|p'_{m-1}(\varphi^2\sigma,\xi)|\varphi^2(\xi)\sqrt{\sigma(\xi)}}{\sqrt{\varphi(\xi)}} + \frac{|p_{m-1}(\varphi^2\sigma,\xi)|\sqrt{\sigma(\xi)}}{\sqrt{\varphi(\xi)}}\right]\\
&\le \mathcal{C}_\theta\left[|p'_{m-1}(\varphi^2\sigma,\xi)||\bar{a}_{m-1}^2-\xi^2|^{1/4}\sqrt{\varphi^2(\xi)\sigma(\xi)} + \frac{|p_{m-1}(\varphi^2\sigma,\xi)|\sqrt{\sigma(\xi)\varphi^3(\xi)}}{1-x^2}\right],
\end{aligned}$$

since $1-\xi^2 \sim \bar{a}_{m-1}^2-\xi^2$. By (23), the second addend is dominated by $\mathcal{C}/(1-x^2)$. It remains to estimate the first addend. We observe that there exists a polynomial $q \in \mathbb{P}_m$ such that $q(y) \sim |\bar{a}_{m-1}^2-y^2|^{1/4}$ and $\varphi(y)q'(y) \le \mathcal{C}m|\bar{a}_{m-1}^2-y^2|^{1/4}$ for $y \in [-1+m^{-2}, 1-m^{-2}]$ (see [8]). Therefore we get

$$\begin{aligned}
D(x) &:= |p'_{m-1}(\varphi^2\sigma,\xi)||\bar{a}_{m-1}^2-\xi^2|^{1/4}\sqrt{\varphi^2(\xi)\sigma(\xi)}\\
&\sim |p'_{m-1}(\varphi^2\sigma,\xi)q(\xi)|\sqrt{\varphi^2(\xi)\sigma(\xi)}\\
&\le \mathcal{C}\left|\left(p_{m-1}(\varphi^2\sigma)q\right)'(\xi)\right|\sqrt{\varphi^2(\xi)\sigma(\xi)} + \mathcal{C}|p_{m-1}(\varphi^2\sigma,\xi)|q'(\xi)|\sqrt{\varphi^2(\xi)\sigma(\xi)}\\
&\sim \frac{m}{\varphi(x)}\left\{\left|\left(p_{m-1}(\varphi^2\sigma)q\right)'(\xi)\right|\frac{\varphi(\xi)}{m}\sqrt{\varphi^2(\xi)\sigma(\xi)}\right.\\
&\quad\left. + |p_{m-1}(\varphi^2\sigma,\xi)|q'(\xi)|\frac{\varphi(\xi)}{m}\sqrt{\varphi^2(\xi)\sigma(\xi)}\right\}
\end{aligned}$$

and, using the Bernstein-type inequality (26), the restricted range inequality (15), the properties of q and (23),

$$\begin{aligned}
D(x) &\le \mathcal{C}\frac{m}{\varphi(x)}\max_{y\in(-1,1)}\left|\left(p_{m-1}(\varphi^2\sigma)q\right)(y)|\sqrt{\varphi^2(y)\sigma(y)}\right|\\
&\quad + \mathcal{C}\frac{m}{\varphi(x)}\max_{y\in[-a_{\theta m},a_{\theta m}]}|p_{m-1}(\varphi^2\sigma,y)|\bar{a}_{m-1}^2-y^2|^{1/4}\sqrt{\varphi^2(y)\sigma(y)}\\
&\le \mathcal{C}_\theta\frac{m}{\varphi(x)},
\end{aligned}$$

where $\mathcal{C}_\theta$ is the constant in (21). Hence the integrand of B_2'' is dominated by

$$\mathcal{C}_\theta^2\left(\frac{1}{1-x^2}+\frac{m}{\varphi(x)}\right) \le \mathcal{C}_\theta^2\frac{m}{\varphi(x)},$$

since $|x| \le a_{\theta m} \le 1-m^{-2}$, for m sufficiently large. It follows that

$$B_2'' \le \mathcal{C}_\theta^2\frac{m}{\varphi(x)}\int_{|x-t|\le\frac{\varphi(x)}{m}} \mathrm{d}t \le \mathcal{C}_\theta^2.$$

Taking into account (40), we have

$$B_2 \leq \mathcal{C}_\theta^2. \tag{41}$$

Then, combining (37)–(39) and (41) in (36), and taking the supremum over all $x \in [-a_{\theta m}, a_{\theta m}]$, we obtain (5).

Now, let us prove that inequality (5) implies conditions (6). We note that if (5) holds, with χ_θ the characteristic function of $[-a_{\theta m}, a_{\theta m}]$ and $f_\theta = \chi_\theta f$, we have

$$\|\chi_\theta S_{m+1}(\sigma, f_\theta)\, u\|_\infty \leq \mathcal{C}(\log m)\|f_\theta u\|_\infty$$

and then

$$\left\|\chi_\theta\left[S_{m+1}(\sigma, f) - S_m(\sigma, f_\theta)\right] u\right\|_\infty \leq \mathcal{C}(\log m)\|f_\theta u\|_\infty,$$

i.e.

$$\|\chi_\theta p_m(\sigma) u\|_\infty \left|\int_{-1}^{1} p_m(\sigma, t)\frac{\sigma(t)}{u(t)}\frac{f_\theta(t)u(t)}{\|f_\theta u\|_\infty}\mathrm{d}t\right| \leq \mathcal{C}\log m.$$

It follows that

$$\|\chi_\theta p_m(\sigma)\sqrt{\sigma\varphi}v^{\mu-\lambda/2-1/4}\|_\infty \sup_{\|g\|_\infty=1}\left|\int_{-1}^{1}\chi_\theta p_m(\sigma)\sqrt{\sigma\varphi}v^{-\mu+\lambda/2-1/4}g\right| \leq \mathcal{C}\log m. \tag{42}$$

For the first factor, denoting by x_k the zero of $p_m(\sigma)$ such that $x_k < a_{\theta m} < x_{k+1}$, setting $\bar{x}_k = (x_{k-1} + x_k)/2$, by using (25), we get

$$\begin{aligned}\|\chi_\theta p_m(\sigma)\sqrt{\sigma\varphi}v^{\mu-\lambda/2-1/4}\|_\infty &\geq \left|p_m(\sigma)\sqrt{\sigma\varphi}v^{\mu-\lambda/2-1/4}\right|(\bar{x}_k)\\ &\geq \mathcal{C}v^{\mu-\lambda/2-1/4}(a_{\theta m}) \sim (1-a_{\theta m})^{\mu-\lambda/2-1/4},\end{aligned}$$

since $1 - \bar{x}_k \sim 1 - a_{\theta m}$. Moreover, the second factor in (42) is the norm of the functional $\Gamma : g \in L^\infty \mapsto \mathbb{R}$ defined by $\Gamma(g) = \int_{-1}^{1}\chi_\theta p_m(\sigma)\sqrt{\sigma\varphi}v^{-\mu+\lambda/2-1/4}g$, and then

$$\begin{aligned}&\sup_{\|g\|_\infty=1}\left|\int_{-1}^{1}\chi_\theta(t)p_m(\sigma,t)\sqrt{\sigma(t)\varphi(t)}v^{-\mu+\lambda/2-1/4}(t)g(t)\,\mathrm{d}t\right|\\ &\quad = \int_{-1}^{1}\chi_\theta(t)\left|p_m(\sigma,t)\sqrt{\sigma(t)\varphi(t)}\right|v^{-\mu+\lambda/2-1/4}(t)\,\mathrm{d}t.\end{aligned}$$

It is easy to show that (see [7])

$$\int_{-1}^{1}\chi_\theta(t)\left|p_m(\sigma,t)\sqrt{\sigma(t)\varphi(t)}\right|v^{-\mu+\lambda/2-1/4}(t)\,\mathrm{d}t \geq \mathcal{C}\int_0^{a_{\theta m}}(1-t)^{-\mu+\lambda/2-1/4}(t)\,\mathrm{d}t.$$

Therefore, from (42), it follows that

$$(1-a_{\theta m})^{\mu-\lambda/2-1/4}\int_0^{a_{\theta m}}(1-t)^{-\mu+\lambda/2-1/4}(t)\,\mathrm{d}t \leq \mathcal{C}\log m.$$

Hence, taking into account (14), if one of the assumptions of (6) is not fulfilled, we get a contradiction.

Finally, to prove inequality (7), let $P_M \in \mathbb{P}_M$ be the best polynomial approximation of $f \in C_u$. By inequality (5), Lemma 2.2 and Proposition 2.3, for m sufficiently large, we have

$$\begin{aligned}\|[f-\chi_\theta S_m(\sigma, f_\theta)]\, u\|_\infty &\le \|(f-P_M)\, u\|_\infty + \|\chi_\theta S_m\,(\sigma, f_\theta - \chi_\theta P_M)\, u\|_\infty \\ &\quad + \|S_m\,(\sigma, P_M - \chi_\theta P_M)\, u\|_\infty + \|(P_M - \chi_\theta P_M)\, u\|_\infty \\ &\le \mathcal{C}_\theta (\log m) E_M(f)_{u,\infty} + \mathcal{C}(m^\nu + 1)\, \|(P_M - \chi_\theta P_M)\, u\|_\infty \\ &\le \mathcal{C}_\theta \left\{ (\log m) E_M(f)_{u,\infty} + \mathrm{e}^{-cM^{\frac{2\alpha}{2\alpha+1}}}\, \|P_M u\|_\infty \right\} \\ &\le \mathcal{C}_\theta \left\{ (\log m) E_M(f)_{u,\infty} + \mathrm{e}^{-cM^{\frac{2\alpha}{2\alpha+1}}}\, \|f u\|_\infty \right\},\end{aligned}$$

which was our claim. □

Let us denote by $\mathcal{H}(f)$ the Hilbert transform of a function f, extended to $(-1, 1)$. Namely,

$$\mathcal{H}(f, x) = \int_{-1}^{1} \frac{f(t)}{x-t}\, \mathrm{d}t \qquad x \in (-1, 1),$$

is the Cauchy principal value of this integral. We recall that the formula

$$\int_{-1}^{1} \mathcal{H}(f) g = -\int_{-1}^{1} \mathcal{H}(g) f \tag{43}$$

holds if $f \in L^p$ and $g \in L^q$, $1 < p < \infty$, $1/p + 1/q = 1$. Moreover, if $f \in L^\infty$ and $g \in L \log^+ L$, i.e. $\int_{-1}^{1} |g(x)| \log^+ |g(x)|\, \mathrm{d}x < \infty$, the inversion (43) is still true (see [10]) and

$$\|f \mathcal{H}(g)\|_1 \le \|g(1 + \log^+ |g|)\|_1 \|f\|_\infty. \tag{44}$$

Now, let us prove Theorem 1.3 before Theorem 1.2. In order to do this, we need the following lemma, whose proof will be given in the Appendix.

Lemma 2.4. *Suppose that $v^\gamma(x) = (1 - x^2)^\gamma$, with $0 < \gamma < 1$, and let G be a function such that $\|G\|_\infty < \infty$. Then we have*

$$\left\| g v^\gamma \mathcal{H}(G v^{-\gamma}) \right\|_1 \le \mathcal{C} \|G\|_\infty \left\| g \left(1 + \log^+ |g| + \log \frac{\mathrm{e}}{1 - \cdot^2} \right) \right\|_1 \tag{45}$$

for any function g such that the norm on the right-hand side is bounded, with $\mathcal{C} \ne \mathcal{C}(G, g)$.

Note that an analogy of the previous lemma was proved in [9] with the L^1-norm replaced by the L^p-norm, $p > 1$, while for $p = 1$ we did not find any result in the literature.

Proof of Theorem 1.3. Supposing that $f_\theta = \chi_\theta f$, by (28) we have

$$\begin{aligned}\|\chi_\theta S_m(\sigma, f_\theta) u\|_1 &\le \mathcal{C} \left\{ \int_{-a_{\theta m}}^{a_{\theta m}} \left| p_m(\sigma, x) u(x) \int_{-a_{\theta m}}^{a_{\theta m}} p_m(\sigma, t) f_\theta(t) \sigma(t) \mathrm{d}t \right| \mathrm{d}x \right. \\ &\quad + \int_{-a_{\theta m}}^{a_{\theta m}} \left| p_m(\sigma, x) u(x) \int_{-a_{\theta m}}^{a_{\theta m}} \frac{p_{m-1}(\varphi^2 \sigma, t) \varphi^2(t)}{x-t} f_\theta(t) \sigma(t) \mathrm{d}t \right| \mathrm{d}x \\ &\quad \left. + \int_{-a_{\theta m}}^{a_{\theta m}} \left| p_{m-1}(\varphi^2 \sigma, x) \varphi^2(x) u(x) \int_{-a_{\theta m}}^{a_{\theta m}} \frac{p_m(\sigma, t)}{x-t} f_\theta(t) \sigma(t) \mathrm{d}t \right| \mathrm{d}x \right\} \\ &=: \mathcal{C} \left\{ I_1 + I_2 + I_3 \right\}.\end{aligned} \tag{46}$$

For the term I_1, by (20) and (9), we have

$$I_1 \le \mathcal{C}_\theta^2 \int_{-a_{\theta m}}^{a_{\theta m}} \frac{v^\mu(x)}{\sqrt{v^\lambda(x)\varphi(x)}} \mathrm{d}x \int_{-a_{\theta m}}^{a_{\theta m}} \frac{1}{v^\mu(t)} \sqrt{\frac{v^\lambda(t)}{\varphi(t)}}\, |f(t)u(t)|\, \mathrm{d}t$$
$$\le \mathcal{C}_\theta^2 \|f_\theta u\|_1, \tag{47}$$

where $\mathcal{C}_\theta$ is given by (21).

Consider now the term I_2. Since $\chi_\theta p_m(\sigma)u \in L^\infty(-1,1)$ and $\chi_\theta p_{m-1}(\varphi^2\sigma)\varphi^2 f_\theta \sigma \in L\log^+ L(-1,1)$, the inversion of the integrals is possible (see [10]). Then, by (23), we have

$$I_2 \le \mathcal{C}_\theta \int_{-a_{\theta m}}^{a_{\theta m}} |f_\theta(t)| u(t) v^{\frac{\lambda}{2}+\frac{1}{4}-\mu}(t)\, |\mathcal{H}(\chi_\theta p_m(\sigma)u, t)|\, \mathrm{d}t.$$

Note that conditions (9) imply $-1 < \mu - \lambda/2 - 1/4 \le -1/2$. Then, by (9) and (20), we can use Lemma 2.4 with $\gamma = \lambda/2 - \mu + 1/4$, $G = \chi_\theta p_m(\sigma)\sqrt{\varphi\sigma}$ and $g = f_\theta u$, obtaining

$$I_2 \le \mathcal{C}_\theta^2 \left\| f_\theta u \left(1 + \log^+ |f_\theta u| + \log \frac{\mathrm{e}}{1-\cdot^2}\right)\right\|_1, \tag{48}$$

where $\mathcal{C}_\theta$ is given by (21).

In order to estimate I_3, we proceed in a similar way. We first reverse the integrals and use (20). Then, taking into account that, by (9), $-1/2 < \mu - \lambda/2 + 1/4 \le 0$, we can use Lemma 2.4 with $\gamma = \lambda/2 - \mu - 1/4$, $G = \chi_\theta p_{m-1}(\varphi^2\sigma)\varphi^2\sqrt{\varphi^3\sigma}$ and $g = f_\theta u$. By (23), we get

$$I_3 = \int_{-a_{\theta m}}^{a_{\theta m}} \left| p_m(\sigma,t) f_\theta(t)\sigma(t)\mathcal{H}\left(\chi_\theta p_{m-1}(\varphi^2\sigma)\varphi^2 u, t\right)\right| \mathrm{d}t$$
$$\le \mathcal{C}_\theta \int_{-a_{\theta m}}^{a_{\theta m}} \left| f_\theta u v^{-\mu+\frac{\lambda}{2}-\frac{1}{4}}(x)\mathcal{H}\left(\chi_\theta p_{m-1}(\varphi^2\sigma)\varphi^2 u, t\right)\right| \mathrm{d}t$$
$$\le \mathcal{C}_\theta^2 \left\| f_\theta u \left(1 + \log^+ |f_\theta u| + \log \frac{\mathrm{e}}{1-\cdot^2}\right)\right\|_1. \tag{49}$$

Combining (47), (48) and (49) in (46), inequality (11) follows. □

Proof of Theorem 1.2. In order to prove that assumptions (9) imply inequality (8), we can proceed like in the second part of the proof of Lemma 2.2. Then, setting $g(x) = \operatorname{sgn}\{S_m(\sigma, \chi_\theta f, x)\}$, we have

$$\|\chi_\theta S_m(\sigma, f_\theta)\, u\|_1 \le \mathcal{C}\|f_\theta u\|_1 \left\|\chi_\theta S_m\left(\sigma, \chi_\theta \frac{gu}{\sigma}\right)\frac{\sigma}{u}\right\|_\infty.$$

Now, by Theorem 1.1, conditions (9) imply

$$\left\|\chi_\theta S_m\left(\sigma, \chi_\theta \frac{gu}{\sigma}\right)\frac{\sigma}{u}\right\|_\infty \le \mathcal{C}(\log m)\, \|\chi_\theta g\|_\infty$$

and (8) follows.

We omit the proof that the inequality in (8) implies the condition in (9) and the proof of the estimate in (10), since one can apply arguments analogous to those used in the second part of the proof of Theorem 1.1. □

Appendix

Proof of Lemma 2.4. Let us estimate $|F(t)|$, where $F(t) := v^{\gamma}(t)\mathcal{H}(Gv^{-\gamma}, t)$. In view of the symmetry, we can assume that $-1 < t < 0$. We first consider the case $-1 < t < -1/2$. We can write

$$\mathcal{H}(Gv^{-\gamma}, t) = \int_{-1}^{2t+1} \frac{(Gv^{-\gamma})(x)}{x-t}\,\mathrm{d}x + \int_{2t+1}^{1} \frac{(Gv^{-\gamma})(x)}{x-t}\,\mathrm{d}x =: I_1 + I_2. \tag{50}$$

For I_2 we have

$$\begin{aligned}|I_2| &\le \|G\|_\infty \left[\int_{2t+1}^{\frac{1}{2}} \frac{(1-x^2)^{-\gamma}}{x-t}\,\mathrm{d}x + \int_{\frac{1}{2}}^{1} \frac{(1-x^2)^{-\gamma}}{x-t}\,\mathrm{d}x\right]\\ &\le \mathcal{C}\|G\|_\infty \left[\int_{2t+1}^{\infty} (1+x)^{-\gamma-1}\,\mathrm{d}x + \int_0^1 (1-x)^{-\gamma}\,\mathrm{d}x\right]\\ &\le \mathcal{C}\|G\|_\infty \left[(1+t)^{-\gamma} + 1\right] \le \mathcal{C}\|G\|_\infty v^{-\gamma}(t).\end{aligned} \tag{51}$$

The term I_1 can be rewritten as

$$\begin{aligned}I_1 &= v^{-\gamma}(t)\int_{-1}^{2t+1} \frac{G(x)}{x-t}\,\mathrm{d}x + \int_{-1}^{2t+1} G(x)\frac{v^{-\gamma}(x) - v^{-\gamma}(t)}{x-t}\,\mathrm{d}x\\ &= v^{-\gamma}(t)\mathcal{H}(G,t) - v^{-\gamma}(t)\int_{2t+1}^{1} \frac{G(x)}{x-t}\,\mathrm{d}x + \int_{-1}^{2t+1} G(x)\frac{v^{-\gamma}(x) - v^{-\gamma}(t)}{x-t}\,\mathrm{d}x\\ &=: v^{-\gamma}(t)\mathcal{H}(G,t) + A_1 + A_2.\end{aligned} \tag{52}$$

For A_1 we have

$$\begin{aligned}|A_1| &\le v^{-\gamma}(t)\|G\|_\infty \int_{2t+1}^{1} \frac{\mathrm{d}x}{x-t} \le v^{-\gamma}(t)\|G\|_\infty \log\left|\frac{1-t}{1+t}\right|\\ &\le \mathcal{C}v^{-\gamma}(t)\|G\|_\infty \log\frac{\mathrm{e}}{1-t^2}.\end{aligned} \tag{53}$$

Whereas, for A_2, by the mean value theorem we have

$$\begin{aligned}&\left|\frac{(1-x)^{-\gamma}(1+x)^{-\gamma} - (1-t)^{-\gamma}(1+t)^{-\gamma}}{x-t}\right|\\ &\le \mathcal{C}\left[(1+x)^{-\gamma} + \left|\frac{(1+x)^{-\gamma} - (1+t)^{-\gamma}}{x-t}\right|\right]\end{aligned} \tag{54}$$

whence

$$|A_2| \le \mathcal{C}\|G\|_\infty \left[\int_{-1}^{2t+1} (1+x)^{-\gamma}\mathrm{d}x + \int_{-1}^{2t+1} \left|\frac{(1+x)^{-\gamma} - (1+t)^{-\gamma}}{x-t}\right|\mathrm{d}x\right].$$

Then, setting $1 + x = (1 + t)u$, we obtain

$$|A_2| \le \mathcal{C}\|G\|_\infty \left[1 + (1+t)^{-\gamma}\int_0^2 \left|\frac{u^{-\gamma}-1}{u-1}\right|\mathrm{d}u\right]$$

$$\leq \mathcal{C}\|G\|_\infty \left[1 + (1+t)^{-\gamma} \int_0^2 u^{-\gamma} |1-u|^{-1+\gamma} \, du\right]$$
$$\leq \mathcal{C}\|G\|_\infty v^{-\gamma}(t). \tag{55}$$

Combining (50)–(53) and (55), we get

$$\left|\mathcal{H}(Gv^{-\gamma}, t)\right| \leq \mathcal{C} v^{-\gamma}(t) \left[\|G\|_\infty + |\mathcal{H}(G,t)| + \|G\|_\infty \log \frac{e}{1-t^2}\right] \tag{56}$$

for $-1 < t < -1/2$.

Let us now consider the case $-1/2 < t < 0$. We can write

$$\begin{aligned}\mathcal{H}(Gv^{-\gamma}, t) &= v^{-\gamma}(t)\mathcal{H}(G,t) + \int_{-1}^{1} \frac{v^{-\gamma}(x) - v^{-\gamma}(t)}{x-t} G(x) \, dx \\ &= v^{-\gamma}(t)\mathcal{H}(G,t) + \left\{\int_{-1}^{-\frac{1}{2}} + \int_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1}\right\} \frac{v^{-\gamma}(x) - v^{-\gamma}(t)}{x-t} G(x) \, dx \\ &= v^{-\gamma}(t)\mathcal{H}(G,t) + \{B_1 + B_2 + B_3\}.\end{aligned} \tag{57}$$

For B_2, by the mean value theorem, we have

$$|B_2| \leq \|G\|_\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{v^{-\gamma}(x) - v^{-\gamma}(t)}{x-t}\right| dx \leq \mathcal{C}. \tag{58}$$

For B_1, by (54), we get

$$\begin{aligned}|B_1| &\leq \mathcal{C}\|G\|_\infty \left[\int_{-1}^{-\frac{1}{2}} (1+x)^{-\gamma} \, dx + \int_{-1}^{-\frac{1}{2}} \left|\frac{(1+x)^{-\gamma} - (1+t)^{-\gamma}}{x-t}\right| dx\right] \\ &\leq \mathcal{C}\|G\|_\infty \left[1 + (1+t)^{-\gamma} \int_0^{\frac{1}{2(1+t)}} u^{-\gamma} |1-u|^{-1+\gamma} du\right] \\ &\leq \mathcal{C}\|G\|_\infty \left[1 + (1+t)^{-\gamma} \int_0^{1} u^{-\gamma} (1-u)^{-1+\gamma} du\right] \\ &\leq \mathcal{C}\|G\|_\infty v^{-\gamma}(t),\end{aligned} \tag{59}$$

since $1 + t \geq 1/2$ and $0 < \gamma < 1$. The term B_3 can be estimated similarly to B_1. Then, by (57)–(59), we get

$$\left|v^{\gamma}(t)\mathcal{H}(G,t)\right| \leq \mathcal{C}\left(\|G\|_\infty + |\mathcal{H}(G,t)|\right)$$

for $-1/2 < t < 1$. Combining this last inequality with (56), we get

$$|F(t)| \leq \mathcal{C}\left[\|G\|_\infty + |\mathcal{H}(G,t)| + \|G\|_\infty \log \frac{e}{1-t^2}\right],$$

whence, supposing that $g \in L \log^+ L$, by (44), we obtain

$$\|gF\|_1 \le \mathcal{C}\left[\|G\|_\infty\|g\|_1 + \|g\mathcal{H}(G)\|_1 + \|G\|_\infty \int_{-1}^{1} |g(t)| \log \frac{\mathrm{e}}{1-t^2}\,\mathrm{d}t\right]$$
$$\le \mathcal{C}\|G\|_\infty \left[\|g\|_1 + \|g(1+\log^+|g|)\|_1 + \int_{-1}^{1} |g(t)| \log \frac{\mathrm{e}}{1-t^2}\,\mathrm{d}t\right],$$

i.e. (45). □

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