A curiosity about $(-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[Ne]}$

A CURIOSITY ABOUT $(-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[Ne]}$.

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Abstract. Let α be an irrational real number; the behavior of the sum $S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$ depends on the continued fraction expansion of $\alpha/2$. Since the continued fraction expansion of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible. The partial quotients of the continued fraction expansion of e grow slowly and thus $S_N(2e) = O(\frac{\log(N)^2}{\log\log(N)^2})$, again best possible. The partial quotients of the continued fraction expansion of $e/2$ behave similarly as those of e. Surprisingly enough $S_N(e) = O(\frac{\log(N)}{\log\log(N)})$.

1. Introduction

Let α be an irrational real number; we are interested in the asymptotic behavior of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

The origin of this question seems to go back to [12], where it is remarked that $S_N(\sqrt{2}) = O(\log N)$. More accurate estimates for $S_N(\sqrt{2})$ are available in [5] and were already implicit in [9], where the authors gave an unexpected explicit formula[1] for $S_N(\sqrt{2})$ in terms of the continued fraction expansion[2] $\sqrt{2} = [1; \overline{2}]$.

The behavior of $S_N(\alpha)$ is closely related to the uniform distribution mod 1 of the sequence $(n\alpha/2)_{n\in\mathbb{N}}$. Indeed, $[n\alpha]$ is even if and only if the fractional part $\{n\alpha/2\}$ is in $[0, 1/2)$. Thus,

$$\begin{aligned} S_N(\alpha) &= |\{n = 1, \ldots, N \mid [n\alpha] \text{ even }\}| - |\{n = 1, \ldots, N \mid [n\alpha] \text{ odd }\}| \\ &= 2|\{n = 1, \ldots, N \mid \{n\alpha/2\} \in [0, 1/2)\}| - N \\ &= 2D_N(\alpha/2, 1/2). \end{aligned} \tag{1.1}$$

Here D_N is the local discrepancy:

$$D_N(\alpha, x) = |\{n = 1, \ldots, N \mid \{n\alpha\} \in [0, x)\}| - Nx$$

for $\alpha \in \mathbb{R}$ and $x \in [0, 1]$. A lazy way to bound $D_N(\alpha)$ is to put in the picture the global discrepancy[3]

$$D_N(\alpha) := \sup_{0\le x<y\le 1} \Big| |\{n = 1, \ldots, N \mid \{n\alpha\} \in [x, y)\}| - N(y - x) \Big|.$$

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[1] which can be viewed as an equality between non absolutely convergent Fourier series.

[2] Here and below, $\overline{a_1, \ldots, a_k}$ means $a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots$

[3] Note that some authors, as [3], divide by N in the definition of D_N.

Thus $|D_N(\alpha,1/2)| \le D_N(\alpha)$. For an irrational α, the sequence $(n\alpha)$ is uniform distribution mod 1 by a well known theorem attributed ([3], p.21) independently to Bohl, Sierpiński and Weyl. This means that $D_N(\alpha)=o(N)$. More precise estimates for D_N depend on the diophantine approximation properties of α. We recall that the irrationality exponent $\mu(\alpha)$ of an irrational $\alpha\in\mathbb{R}$ is the infimum (possibly $+\infty$) of the set of positive real numbers μ such that for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that for all $p,\ q\in\mathbb{Z}$ with $q>0$ we have

$$\left|\alpha-\frac{p}{q}\right|>\frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

It is well known that $\mu(\alpha)\ge 2$ with equality for almost all α. It is also well known that μ is invariant by integral Möbius transformations $\alpha\mapsto\frac{a\alpha+b}{c\alpha+d}$ $(a,\ b,\ c,\ d\in\mathbb{Z},\ ad-bc\ne 0)$.

From[4] [3, Theorem 3.2, p.123] $D_N(\alpha)=O_\gamma(N^\gamma)$ for any $\gamma>1-\frac{1}{\mu(\alpha)-1}$. In particular, if $\mu(\alpha)=2$ we have $D_N(\alpha)=O_\gamma(N^\gamma)$ for any $\gamma>0$. A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients (and hence irrationality measure 2). In this case we have (see [3, Theorem 3.4, p.125]) $D_N(\alpha)=O(\log N)$.

This last estimate cannot be improved. Indeed the global discrepancy D_N of every infinite sequence $(u_n)_n$ is[5] $\Omega(\log N)$ (see [13, Theorem 1, p.45]).

Nevertheless, we can construct irrational α such that $|D_N(\alpha,1/2)|$ is as small as we wish. Our first result is:

Theorem 1.1. *Let $\delta\colon\mathbb{N}\to\mathbb{R}^+$ be a function which tends to infinity. Then there exists an irrational number α such that*

$$D_N(\alpha,1/2)=O(\delta(N)).$$

Equivalently, we can find an irrational α such that[6] $S_n(\alpha)=O(\delta(N))$.

By [2, Theorem 8, p.237], for any irrational α there exists a positive constant $A=A(\alpha)$ such that $|\sum_{n=1}^N f(n\alpha)|\ge AN$, where $f(t)=\{t\}-1/2$. By Theorem (1.1) we cannot replace in this statement $\{t\}-1/2$ with $(-1)^{[t]}$, even taking instead of N any function $\delta(N)$ which tends to infinity. See also [10] for a related question.

We then show that for some *classical number* the local discrepancy $D_N(\alpha,1/2)$ can be substantially smaller than $D_N(\alpha)$ and even $o(\log N)$.

Theorem 1.2.

$$\overline{\lim_{N\to+\infty}}\, D_N(e/2)\left(\frac{\log\log N}{\log N}\right)^2=\frac{1}{8} \tag{1.2}$$

and

$$\overline{\lim_{N\to+\infty}}\, |D_N(e/2,1/2)|\frac{\log\log N}{\log N}=\frac{3}{2}. \tag{1.3}$$

[4]The authors state this result in terms of the *type* of α which is equal to $\mu(\alpha)-1$.

[5]Here Ω is the Landau symbol: if $f,\ g$ are two functions with $g>0$ then $f=\Omega(g)$. means $f\ne o(g)$

[6]Note that for any irrational α we have $\overline{\lim}\,|S_n(\alpha)|=+\infty$ ([7, Theorem 1]).

Let's come back to the sum in the title. The question of providing good bound for $S_N(e)$ goes back to H. Pépin [7], who, in the nice self-contained treatement of this matter [8], already get $S_N(e) = O((\log N)^2)$. Equations (1.1) and (1.2) show that $S_N(e)$ is smaller than what one would expect:

$$S_N(e) = (-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[ne]} = O(\log(N)/\log\log(N)). \tag{1.4}$$

Note that

$$S_N(2e) = (-1)^{[2e]} + (-1)^{[4e]} + \cdots + (-1)^{[2Ne]} = O((\log(N)/\log\log(N))^2)$$

is best possible, by (1.1) and by (1.6) of Theorem 1.3 below.

Bounds for $S_N(\alpha)$ are useful to study the convergence of sums of the shape $\sum_n (-1)^{[n\alpha]} u_n$. Let $\Delta u_n = u_{n+1} - u_n$. By partial sommation (as in the solution to [6] proposed by R. Tauraso [14]) we see that such a sum converges if $S_N(\alpha) u_N \to 0$ and $\sum_N S_N(\alpha) \Delta u_N$ converges. By (1.4) both conditions are satisfied when $\alpha = e$ and $u_n = \frac{\log\log(n+1)}{\log(n+1)^2}$. To get more precise and general results, it might be suitable to make a second partial summation, since the arithmetic mean of $S_N(\alpha)$ behave more regularly.

The gain of the factor $\frac{\log\log N}{\log N}$ in (1.3) heavily depends on the particular structure of the continued expansion of $e/2$. Let us give a short explication. Both estimates (1.2) and (1.3) for the global and local discrepancy of $(ne/2)$ depend on the partial quotients $\{a_n\}_{n\geq 1}$ of the continued fraction expansion of $e/2$. This sequence is unbounded. But in the estimate (1.3) only the a_n with $n \not\equiv 2 \mod 3$ come in. The corresponding sequence is now bounded. This phenomenon does not occurr if we replace $e/2$ by e, as the following theorem shows.

Theorem 1.3.

$$\overline{\lim_{N\to+\infty}} D_N(e) \left(\frac{\log\log N}{\log N}\right)^2 = \frac{1}{4} \tag{1.5}$$

and

$$\overline{\lim_{N\to+\infty}} |D_N(e, 1/2)| \left(\frac{\log\log N}{\log N}\right)^2 = \frac{1}{4}. \tag{1.6}$$

Relations (1.3) and (1.6) show that the order of growth of $\alpha \mapsto D_N(\alpha, 1/2)$ is *not* invariant with respect to Möbius transformations, contrary to what happen for the global discrepancy.

Although our theorems are straightforward applications of known results ([1] and [11]), it seems that they deserve to be remarked.

2. Computations

Proof of of Theorem 1.1. The proof is an easy application of [11, Example, p.1497]. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function taking odd values and which increases to infinity sufficiently fast. We choose

$$\alpha = \alpha_f = [0; 1, 1, f(1), 1, 1, f(2), \ldots].$$

Let a_j and q_m be the partial quotients and the denominators of the convergents of α. For $N \in \mathbb{N}$ we define $m(N) \in \mathbb{N}$ by the property

$$q_{m(N)} \leq N < q_{m(N)} + 1.$$

Put

$$a_j^+ = \begin{cases} a_j, & \text{if } q_{j-1} \text{ is even and } j \text{ is odd;} \\ 0, & \text{otherwise} \end{cases}$$

and

$$a_j^- = \begin{cases} a_j, & \text{if } q_{j-1} \text{ and } j \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Define the following sums:

$$S_m^+ = \frac{1}{4} \sum_{\substack{2|j \leq m \\ 2\nmid q_j}} a_{j+1} = \frac{1}{4} \sum_{j=0}^{m} a_{j+1}^+, \qquad S_m^- = \frac{1}{4} \sum_{\substack{2\nmid j \leq m \\ 2\nmid q_j}} a_{j+1} = \frac{1}{4} \sum_{j=0}^{m} a_{j+1}^-.$$

Then from [11, Example, p.1497] we have (as in the deduction of Corollary 1.2 from Theorem 1.1 in [1]):

$$\overline{\lim_{N \to +\infty}} D_N(e/2, 1/2)/S_{m(N)}^+ = - \underline{\lim_{N \to +\infty}} D_N(e/2, 1/2)/S_{m(N)}^- = 1. \tag{2.1}$$

From the usual recursive definition of q_m we easily see that[7] q_{j-1} is even iff $j \equiv 0 \mod 3$. Thus

$$\{a_j^+\}_{j\geq 1} = \{\overline{1, 0, 0}\}, \quad \{a_j^-\}_{j\geq 1} = \{\overline{0, 1, 0}\}$$

and

$$S_m^+ \sim S_m^- \sim \frac{1}{4} \sum_{k=1}^{[m/3]} 1 \sim \frac{m}{12}. \tag{2.2}$$

Moreover, from the recursive definition of q_m we have

$$q_m \geq \prod_{j=1}^{[m/3]} f(j).$$

Thus, if f grows sufficiently fast, for $N \in \mathbb{N}$ we have $q_{12[\delta(N)]} \geq N$ and, by definition, $m(N) \leq [12\delta(N)]$. By (2.1) and (2.2) we have $D_N(\alpha, 1/2) = O(\delta(N))$ as desired.

□

[7] To check this property we can of course reduce modulo 2 all the partial coefficients, thus reduce ourselves to compute the well-known convergents of the golden ratio.

Proof of of Theorem 1.2. To prove (1.2) we follow the proof of [1, Theorem 3.2(2), p.286] taking now (*cf* (2.3)) $a_1, a_2, \ldots$ be the partial quotients of the continued fraction expansion[8] of $e/2$

$$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, 1, 3, 7, 3, 1, 7, 1, 3, \ldots]\,. \tag{2.3}$$

We easily find

$$\sum_{i=0}^{m} a_{i+1} \sim 2 \sum_{k=1}^{[m/6]} (2k-1) \sim \frac{1}{18} m^2$$

and

$$\sum_{i=0}^{m} \log a_{i+1} \sim 2 \sum_{k=1}^{[m/6]} \log(2k-1) \sim \frac{1}{3} m \log m.$$

Thus

$$\overline{\lim_{N\to+\infty}}\, D_N(e/2) \left(\frac{\log\log N}{\log N}\right)^2 = \frac{\frac{1}{18}}{4(\frac{1}{3})^2} = \frac{1}{8}.$$

To prove (1.3) we apply again the formula in [11, Example, p.1497]. Let a_j and q_m be the partial quotients and the denominators of the convergents of (2.3). Let $m(N)$, $a_j^{\pm}$ and $S_m^{\pm}$ be as in the the proof of Theorem 1.1. Then

$$\overline{\lim_{N\to+\infty}}\, D_N(e/2, 1/2)/S^{+}_{m(N)} = -\underline{\lim_{N\to+\infty}}\, D_N(e/2, 1/2)/S^{-}_{m(N)} = 1. \tag{2.4}$$

From (2.3) and from the usual recursive definition of q_m we see (*cf* note[7]) that q_{j-1} is even iff $j \equiv 2 \mod 3$. Thus

$$\{a_j^{+}\}_{j\ge1} = \{2, \overline{0, 3, 0, 0, 0, 3}\}, \qquad \{a_j^{-}\}_{j\ge1} = \{2, \overline{0, 0, 1, 0, 1, 0}\}$$

and

$$S_m^{+} \sim \frac{1}{4} \sum_{k=1}^{[m/6]} (3+3) \sim \frac{1}{4} m, \qquad S_m^{-} \sim \frac{1}{4} \sum_{k=1}^{[m/6]} (1+1) \sim \frac{1}{12} m.$$

Moreover (*cf* (2.3))

$$\log q_m \sim \sum_{i=1}^{m} \log a_i \sim 2 \sum_{k=1}^{[m/6]} \log(2k-1) \sim 2\frac{m}{6} \log m = \frac{1}{3} m \log m$$

which, by definition of $m(N)$, easily implies $m(N) \sim 3\frac{\log N}{\log\log N}$. Replacing these estimates in (2.6) we get

$$\overline{\lim_{N\to+\infty}}\, D_N(e/2, 1/2)/(\tfrac{3}{4}\tfrac{\log N}{\log\log N}) = -\underline{\lim_{N\to+\infty}}\, D_N(e/2, 1/2)/(\tfrac{3}{12}\tfrac{\log N}{\log\log N}) = 1.$$

Equation (1.3) follows.

□

[8] which can be easily computed from the well-known Euler continued fraction of e, *e.g.* by known algorithms [4].

Proof of of Theorem 1.3. Equation (1.5) is a special case of [1, Theorem 3.2(2), p.286]. The deduction of (1.6) follows the same lines as that of (1.3). Let a_j and q_m be the partial quotients and the denominators of the convergents of the continued fraction expansion of e

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots] \tag{2.5}$$

Let $m(N)$, $a_j^\pm$ and $S_m^\pm$ as in the the proof of Theorem 1.1. From [11, Example, p.1497]:

$$\overline{\lim_{N\to+\infty}} D_N(e, 1/2)/S^+_{m(N)} = - \underline{\lim_{N\to+\infty}} D_N(e, 1/2)/S^-_{m(N)} = 1. \tag{2.6}$$

From (2.5) and from the usual recursive definition of q_m we easily see that q_{j-1} is even iff $j \equiv 0, 4 \mod 6$. Thus (*cf* (2.5))

$$\{a_j^+\}_{j\geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, 0, 1, 0, 1, 0, 12, 0, \dots\};$$
$$\{a_j^-\}_{j\geq 1} = \{0, 2, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 10, 0, 0, 0, 0, \dots\}$$

and

$$S_m^+ \sim \frac{1}{4}\sum_{k=1}^{[m/6]}(1+1+4k) \sim \frac{1}{72}m^2, \qquad S_m^- \sim \frac{1}{4}\sum_{k=1}^{[m/6]}(4k-2) \sim \frac{1}{72}m^2.$$

Moreover (*cf* again (2.5))

$$\log q_m \sim \sum_{i=1}^{m} \log a_i \sim \sum_{k=1}^{[m/3]} \log(2k) \sim \frac{1}{3} m \log m$$

which implies $m(N) \sim 3\frac{\log N}{\log\log N}$. Replacing these estimates in (2.6) we get

$$\overline{\lim_{N\to+\infty}} D_N(e, 1/2)/(\tfrac{1}{8}(\tfrac{\log N}{\log\log N})^2) = - \underline{\lim_{N\to+\infty}} D_N(e, 1/2)/(\tfrac{1}{8}(\tfrac{\log N}{\log\log N})^2) = 1.$$

Equation (1.6) follows.

□

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