Long Time Decay Estimates in Real Hardy Spaces for the Double Dispersion Equation

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Abstract We study the Cauchy problem for the linear generalized double dispersion equation and derive long time decay estimates for the solution in L^p spaces and in real Hardy spaces.

Keywords Double dispersion equation · Decay estimates · Hardy spaces · Fourier multipliers

1 Introduction

In this note, we extend the results recently obtained by the authors [2] for the Cauchy problem for the linear generalized double dispersion equation

$$\begin{cases} u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u - d\Delta u_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

with $a = b = d = 1$, to the general case of parameters $a > 0$, $b > 0$ and $d > 0$. By using a Mikhlin-Hörmander type multiplier theorem, which provides $\mathcal{H}^p$ boundedness of parameter-dependent operators, we are able to estimate the solution in real Hardy spaces $\mathcal{H}^p$ with $p \leq 2$ (we recall that $\mathcal{H}^p = L^p$ for $p > 1$). Our main result is the following.

Theorem 1 *Let $n \geq 1$, $p \in (0, 2]$, $q_0, q_1 \in (0, p]$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Let $\theta = \theta(n, p) = n(1/p - 1/2)$. Assume that $u_0 \in \mathcal{H}^{q_0}$ with $(1-\Delta)^{\frac{\theta+k+|\alpha|}{2}} u_0 \in \mathcal{H}^p$,*

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and $u_1 \in \mathcal{H}^{q_1}$ *with* $(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}} u_1 \in \mathcal{H}^p$. *Moreover, assume that*

$$n\left(\frac{1}{q_1}-\frac{1}{p}\right) \geq 1,$$

if $k = |\alpha| = 0$. *Then the solution to* (1) *verifies the estimate*

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{\mathcal{H}^p} \leq C(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q_0}-\frac{1}{p}\right)-\theta+k+|\alpha|\right)}\|u_0\|_{\mathcal{H}^{q_0}} \tag{2}$$

$$+ Ce^{-ct}\|(1-\Delta)^{\frac{\theta+k+|\alpha|}{2}} u_0\|_{\mathcal{H}^p} \tag{3}$$

$$+ C(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q_1}-\frac{1}{p}\right)-\theta-1+k+|\alpha|\right)}\|u_1\|_{\mathcal{H}^{q_1}} \tag{4}$$

$$+ Ce^{-ct}\|(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}} u_1\|_{\mathcal{H}^p}, \tag{5}$$

for any $t \geq 0$ *and for some* $C, c > 0$, *independent of the initial data.*

The statement of Theorem 1 is the same of [2, Theorem 1.2], but its proof need suitable modifications when the parameters a, b, d fail to fullfill a condition which is always verified when $a = b = d = 1$. Namely, when a zone of the phase space appears, where the two characteristic roots of the full symbol of (1) are real-valued, and not complex valued. Since this zone only appears at intermediate frequencies, the dissipation remains noneffective and the decay estimates are independent on the specific values assigned to the constants $a, b, d > 0$. We address the interested reader to [7] for a classification of effective and noneffective structural dissipation for damped evolution equations. Decay estimates for evolution equations with effective structural dissipation are obtained in L^p spaces in [3–6, 8, 10] and in real Hardy spaces in [9].

Even if problem (1) is interesting by itself from a theoretical mathematical point of view, it is originated by a real world physical problem.
A presentation of the model is provided in [19]: in some problems of nonlinear wave propagation in waveguides, in case of energy exchange between the surface of nonlinear elastic rod in material (e.g., the Murnaghan material) and an external medium, the following double dispersion equation (DDE)

$$u_{tt} - \Delta u = \frac{1}{4}(6\Delta u^2 + a\Delta u_{tt} - b\Delta^2 u) \tag{6}$$

and the general cubic DDE (CDDE)

$$u_{tt} - \Delta u = \frac{1}{4}(c\Delta u^3 + 6\Delta u^2 + a\Delta u_{tt} - b\Delta^2 u + d\Delta u_t) \tag{7}$$

can be derived from Hamilton Principle. Here $u(t,x)$ is proportional to strain $\frac{\partial U}{\partial x}$, where $U(t,x)$ is the longitudinal displacement, $a > 0$, $b > 0$, and $d \neq 0$ are some constants depending on the Young modulus, the shear modulus μ, density of waveguide ρ and the Poisson coefficient ν. Equations (6) and (7) were studied in literature: the travelling wave solutions, depending upon the phase variable $z = x \pm ct$ were studied by Samsonov in [16, 17], the strain solutions of equations (6) and (7) were analyzed in [12, 18]. Equation (7) is a special case of the following generalized double dispersion equation

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u - d\Delta u_t = \Delta f(u). \tag{8}$$

The double dispersion equation and its generalized form have attracted lots of researchers' interests and many interesting results have been established: the global existence and asymptotic decay of solution to the problem (8) are proved in [2] for $a = b = d = 1$ and nonsmooth $f(u)$. As customary, the proof is based on the contraction mapping principle and makes use of the sharp decay estimates for the linearized problem. However, in this case the oscillations coming from the wave part of the equation produces two issues when one works in L^p spaces with $p \in (1, 2)$: a loss of regularity which is known from the theory of damped wave equations, and a loss of decay rate, which is known from the theory of strongly damped wave equations.

The double dispersion equation has been well investigated in recent times, in particular see [1, 15, 19, 20].

2 Notation

We denote by $\mathscr{F}$ the Fourier transform with respect to the space variable x,

$$\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-ix\xi}\, dx,$$

and we write $\hat{\varphi}(\xi) = \mathscr{F} f(\xi)$, and $\hat{\varphi}(t,\xi) = (\mathscr{F}\varphi(t,\cdot))(\xi)$.

Differential operators are denoted by $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the length of α.

With the symbol Δ we denote the Laplace operator as $\Delta = \sum_{i=1}^n \partial_{x_i}^2$. Fractional powers $s > 0$ of $-\Delta$ and $1 - \Delta$ are intended as defined by their action

$$(-\Delta)^s \varphi = \mathscr{F}^{-1}(|\xi|^{2s}\hat{\varphi}), \qquad (1-\Delta)^s\varphi = \mathscr{F}^{-1}(\langle\xi\rangle^{2s}\hat{\varphi}),$$

where

$$\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

Similarly, we define the Riesz potentials (see also the Appendix) for $s > 0$:

$$I_s\varphi = \mathscr{F}^{-1}(|\xi|^{-s}\hat{\varphi}), \qquad (1-\Delta)^{-s}\varphi = \mathscr{F}^{-1}(\langle\xi\rangle^{-2s}\hat{\varphi}).$$

By $W^{m,p}$, $p \in [1,\infty]$ we denote the usual Sobolev space of L^p functions with derivatives up to the order m in L^p, recalling that $W^{m,p} = (1-\Delta)^{-\frac{m}{2}} L^p$ if $p > 1$. Moreover we use the following.

Definition 1 Let $f, g \colon \Omega \to \mathbb{R}$ be two functions. We use the notation $f \lesssim g$ (resp. $f \gtrsim g$) if there exists a constant $C > 0$ such that $f(y) \leq Cg(y)$ (resp. $f(y) \geq Cg(y)$) for all $y \in \Omega$.

The definition of real Hardy spaces $\mathcal{H}^p$ and some of their properties are collected in the Appendix.

3 Fundamental Solution and Decay Estimates

Applying to (1) Fourier transform w.r.t. x, we get

$$\begin{cases} \hat{u}_{tt} + |\xi|^2\hat{u} + a|\xi|^2\hat{u}_{tt} + b|\xi|^4\hat{u} + d|\xi|^2\hat{u}_t = 0, \quad t \geq 0,\ \xi \in \mathbb{R}^n, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi),\ \hat{u}_t(0,\xi) = \hat{u}_1(\xi). \end{cases} \tag{9}$$

Solving the characteristic equation

$$(1 + a|\xi|^2)\lambda^2 + d|\xi|^2\lambda + (|\xi|^2 + b|\xi|^4) = 0, \tag{10}$$

we have the characteristic roots:

$$\lambda_\pm = \frac{-d|\xi|^2 \pm |\xi|\sqrt{-4ab|\xi|^4 + (d^2 - 4a - 4b)|\xi|^2 - 4}}{2(1 + a|\xi|^2)}. \tag{11}$$

If we consider $\xi_- < |\xi| < \xi_+$ and $d > \bar{d}$, where explicity

$$\xi_\pm = \sqrt{\frac{(d^2 - 4a - 4b) \pm \sqrt{(d^2 - 4a - 4b)^2 - 64ab}}{8ab}}, \tag{12}$$

$$\bar{d} = \sqrt{4a + 4b + 8\sqrt{ab}} = 2\sqrt{a} + 2\sqrt{b}, \tag{13}$$

then the characteristic roots are real and distinct. In this zone, it holds

$$\hat{u}(t,\xi) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\hat{u}_0 + \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{u}_1. \tag{14}$$

The presence of this zone is neglected in [2], due to the choice $a = b = d = 1$, which implies that $\bar{d} = 4 > 1 = d$. However, out of this zone, namely at low frequencies $|\xi| < \xi_-$ and at high frequencies $|\xi| > \xi_+$, the analysis is qualitatively equivalent to the study carried on in [2]. For this reason, we omit the study of these two zones and only study the "new" intermediate zone $|\xi| \in (\xi_-, \xi_+)$.

More precisely, we will fix $\varepsilon > 0$ in the proof, sufficiently small, and we will study the region

$$\xi_- + \varepsilon \le |\xi| \le \xi_+ - \varepsilon.$$

For the sake of brevity, we also omit the study of the two transition regions $|\xi| \in (\xi_- - \varepsilon, \xi_- + \varepsilon)$ and $|\xi| \in (\xi_+ - \varepsilon, \xi_+ + \varepsilon)$ (near the surfaces $|\xi| = \xi_\pm$, at which $\lambda_- = \lambda_+$).

For the sake of brevity, in the following we deal with $u_0 = 0$. We denote

$$u(t, \cdot) = G(t, \cdot) * u_1 = \mathscr{F}^{-1}(\hat{G}(t, \xi)\hat{u}_1). \tag{15}$$

In order to prove $\mathcal{H}^p$ estimates with $p \in (0, 2)$, the derivatives of $\hat{G}(t, \xi)$ come into play.

Theorem 2 *Let $n \ge 1$, $p \in (0, 2)$, $q \in (0, p]$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Assume that $\varphi \in \mathcal{H}^q$ with φ supported in $\{\xi \in \mathbb{R}^n : |\xi| \in [\xi_- + \varepsilon, \xi_+ - \varepsilon]\}$. Then we have the estimate*

$$\|\partial_t^k \partial_x^\alpha G(t, \cdot) * \varphi\|_{\mathcal{H}^p} \lesssim e^{-ct} \|\varphi\|_{\mathcal{H}^p}, \tag{16}$$

for any $t \ge 0$ and for some $c > 0$.

Proof We consider the Fourier multiplier (see Definition 3)

$$m(t, \xi) = \langle \xi \rangle^{-\theta - k - |\alpha|} (i\xi)^\alpha \partial_t^k \hat{G}(t, \xi),$$

and we prove that the operator T_m is $\mathcal{H}^p$-bounded, with

$$\|m(t, \cdot)\|_{\mathcal{M}(\mathcal{H}^p)} \lesssim e^{-ct}, \tag{17}$$

for some $c > 0$. Let us fix $\varepsilon > 0$, sufficiently small. For any

$$\xi_- + \varepsilon \le |\xi| \le \xi_+ - \varepsilon,$$

it holds $\lambda_+ - \lambda_- \gtrsim c_\varepsilon > 0$.
We notice that we may estimate

$$|\partial_\xi^\gamma (\lambda_+ - \lambda_-)(\xi)| \lesssim |\xi|^{-|\gamma|}. \tag{18}$$

Taking into account of (18), writing

$$\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t}}{\lambda_+ - \lambda_-}(1 - e^{(\lambda_- - \lambda_+)t}),$$

together with

$$\partial_{\xi_j} e^{\lambda_+ t} = t e^{\lambda_+ t} \partial_{\xi_j} \lambda_+,$$
$$\partial_{\xi_j} e^{(\lambda_- - \lambda_+)t} = t e^{(\lambda_- - \lambda_+)t} \partial_{\xi_j} (\lambda_- - \lambda_+),$$

we may estimate

$$|\partial_\xi^\gamma \partial_t^k \hat{G}(t,\xi)| \lesssim |\xi|^{k-|\gamma|} (1+t)^{|\gamma|} e^{\lambda_+ t},$$

where we used that $(1 + t^{|\gamma|}) \lesssim (1+t)^{|\gamma|}$. Therefore,

$$|\partial_\xi^\gamma m(t,\xi)| \lesssim |\xi|^{-|\gamma|} (1+t)^{|\gamma|} e^{-ct},$$

where

$$c = \min_{|\xi| \in [\xi_-, \xi_+]} (-\lambda_+) > 0.$$

The minimum is nonnegative, since λ_+ is nonpositive. We remark that

$$\lambda_+ \to -\frac{d|\xi|_\pm^2}{2(1 + a|\xi|_\pm^2)} \quad \text{as } |\xi| \to \xi_\pm.$$

By applying Theorem 3 in the Appendix, with $a = 0$ and $A = 1 + t$, we obtain

$$\|m(t,\cdot)\|_{\mathcal{M}(\mathcal{H}^p)} \lesssim (1+t)^\theta \, e^{-ct}.$$

Therefore, we obtain (17) with a different c. This completes the proof. □

Remark 1 The polynomial decay rate of formula (4) comes from the multiplier estimate at low frequencies (as it happens for damped waves in the whole space $\mathbb{R}^n$, in general), whereas the regularity of the initial data $(1-\Delta)^{\frac{\theta+k+|\alpha|-j}{2}} u_j \in \mathcal{H}^p$, $j = 0, 1$, comes from the multiplier estimate at high frequencies (see [2] for the proof). In the intermediate frequencies, on the one hand we derive an exponential decay, on the other hand, no regularity issue comes into play.

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Appendix

We recall how the Hardy spaces $\mathcal{H}^p(\mathbb{R}^n)$ are presented by Fefferman and Stein [11]. We use the notation $\mathcal{H}^p$ instead of the classical notation H^p to avoid possible confusion with the Sobolev space $W^{p,2}$.

Fix, once for all, a radial nonnegative function $\phi \in C_c^\infty(\mathbb{R}^n)$ supported in the unit ball with integral equal to 1. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define the *maximal function* $M_\phi u$ by

$$M_\phi u(x) = \sup_{0<t<\infty} |(u * \phi_t)(x)|,$$

where $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2 Let $0 < p < \infty$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathcal{H}^p(\mathbb{R}^n)$ if and only if $M_\phi u \in L^p(\mathbb{R}^n)$, i.e.,

$$\|u\|_{\mathcal{H}^p} = \|M_\phi u\|_{L^p} < \infty.$$

For $p = \infty$, we set $\mathcal{H}^\infty(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

The spaces $\mathcal{H}^p(\mathbb{R}^n)$ are independent of the choice of ϕ. For $p = 1$, $\|u\|_{\mathcal{H}^1}$ is a norm and $\mathcal{H}^1(\mathbb{R}^n)$ is a normed space densely contained in $L^1(\mathbb{R}^n)$. For $p > 1$, $\|u\|_{\mathcal{H}^p}$ is a norm equivalent to the usual L^p norm and we denote $\mathcal{H}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, by abusing notation. For $0 < p \leq 1$, the space $\mathcal{H}^p(\mathbb{R}^n)$ is a complete metric space with the distance

$$d(u, v) = \|u - v\|_{\mathcal{H}^p}^p, \qquad u, v \in \mathcal{H}^p(\mathbb{R}^n).$$

Although $\mathcal{H}^p(\mathbb{R}^n)$ is not locally convex for $0 < p < 1$ and $\|u\|_{\mathcal{H}^p}$ is not truly a norm, we will still refer to $\|u\|_{\mathcal{H}^p}$ as the "norm" of u, as it is customary.

The property $f \in \mathcal{H}^p$ can be characterized by appropriate singular integrals in a way that has some analogy with the earlier maximal characterization [14, Theorem C]: a function $f \in L^2$ belongs to $\mathcal{H}^p$ when $p \in (0, 1]$, if and only if $f \in L^p$ and $R_\alpha f \in L^p$, for $|\alpha| \leq k$, where $k = 1 + [(n-1)(1/p - 1)]$, and $R_\alpha f$ denotes the Riesz transform of f, defined via the Fourier transform by

$$\widehat{R_\alpha f}(\xi) = (i\xi|\xi|^{-1})^\alpha \hat{f}(\xi).$$

Moreover,

$$\|f\|_{\mathcal{H}^p} \approx \sum_{|\alpha|\leq k} \|R_\alpha f\|_{L^p}.$$

Another number fixes the order of the moment conditions which the functions in Hardy spaces shall verify. Indeed,

$$\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0, \qquad \forall\, |\alpha| \leq [n(1/p - 1)]$$

for any function $f \in \mathcal{H}^p \cap \mathcal{C}_c^\infty$.

In Theorem 2 we use a variant of the celebrated Mikhlin-Hörmander multiplier theorem for Hardy spaces (see [13]) to obtain the boundedness of operators acting on $\mathcal{H}^p(\mathbb{R}^n)$.

Definition 3 Let m be a bounded function on $\mathbb{R}^n$ and consider the operator T_m defined by

$$T_m f = \mathscr{F}^{-1}\big(m(\xi)\hat{f}(\xi)\big). \tag{19}$$

We say that m is a Fourier multiplier for $\mathcal{H}^p$ if $T_m f \in \mathcal{H}^p$ for all $f \in \mathcal{H}^p$ and

$$\|T_m f\|_{\mathcal{H}^p} \leq C\|f\|_{\mathcal{H}^p}; \tag{20}$$

in other words, if T_m can be extended to a bounded operator from $\mathcal{H}^p$ to $\mathcal{H}^p$.
In this context, $\mathcal{M}(\mathcal{H}^p)$ denotes the set of all the Fourier multipliers for $\mathcal{H}^p$. The norm $\|m\|_{\mathcal{M}(\mathcal{H}^p)}$ is defined to be the operator norm of T_m in $\mathcal{H}^p$, i.e.

$$\|m\|_{\mathcal{M}(\mathcal{H}^p)} = \sup_{f\in\mathcal{H}^p, f\neq 0} \frac{\|T_m f\|_{\mathcal{H}^p}}{\|f\|_{\mathcal{H}^p}}. \tag{21}$$

Theorem 3 *Let $p \in (0,2)$, and $\theta = \theta(n,p) = n(1/p - 1/2)$. Assume that $m \in C^k(\mathbb{R}^n)$, with $m(\xi) = 0$ in a neighborhood of the origin, and $k = \max\{[\theta], [\frac{n}{2}]\} + 1$. If*

$$|\partial_\xi^\gamma m(\xi)| \leq |\xi|^{-a\theta}(A|\xi|^{a-1})^{|\gamma|}, \ |\gamma| \leq k,$$

for some constant $a \geq 0$ and $A \geq 1$, then $m \in \mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))$ and

$$\|m\|_{\mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))} \leq CA^\theta,$$

where $C > 0$ is a constant independent of A.

Theorem 4 *Let $p \in (0,2)$, and $\theta = \theta(n,p) = n(1/p - 1/2)$. Assume that $m \in C^k(\mathbb{R}^n \setminus \{0\})$, with $m(\xi) = 0$ for $|\xi| \geq 1$, and $k = \max\{[\theta], [\frac{n}{2}]\} + 1$. If*

$$|\partial_\xi^\gamma m(\xi)| \leq |\xi|^{a\theta}(A|\xi|^{-a-1})^{|\gamma|}, \ |\gamma| \leq k,$$

for some constant $a \geq 0$ and $A \geq 1$, then $m \in \mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))$ and

$$\|m\|_{\mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))} \leq CA^\theta,$$

where C is a constant independent of A.

Let I_r be the Riesz potential with order $r > 0$, defined by means of $I_r f(\xi) = \mathscr{F}^{-1}(|\xi|^{-r} \hat{f}(\xi))$. If $r \in (0, n)$, then there exists $c_{n,r}$ such that

$$I_r f(x) = c_{n,r} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-r}} \, dy$$

and sufficiently smooth f. Real Hardy spaces have the property that the Hardy-Littlewood-Sobolev theorem for Riesz potential, valid in L^p spaces, with $p > 1$, extends to $\mathcal{H}^p$, with $p \in (0, \infty)$, see [14, Theorem F].

Theorem 5 *Consider $r > 0$ and $0 < p < n/r$. Then, there exists $C = C(r, p) > 0$ such that*

$$\|I_r f\|_{\mathcal{H}^q(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{r}{n}.$$

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