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An Analytic Approach to the Hodge Conjecture

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Abstract

The Dirac-Dolbeault operator for a compact Kähler manifold is a special case of a Dirac operator. The Green function for the Dirac Laplacian over a Riemannian manifold with boundary allows to express the values of the sections of the Dirac bundle in terms of the values on the boundary, extending the mean value theorem of harmonic analysis. Utilizing this representation and the Nash-Moser generalized inverse function theorem we prove the existence of complex submanifolds of a complex projective manifold satisfying globally a certain partial differential equation under a certain injectivity assumption. Next, we show the existence of complex submanifolds whose fundamental classes span the rational Hodge classes, proving the Hodge conjecture for complex projective manifolds.

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