The indivisibles: a travel in time and space from Archimedes to Cavalieri / Les indivisibles: un voyage dans le temps et l'espace d'Archimède à Cavalieri.

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Abstract: This paper presents a teaching experiment that brings together the history of mathematics and mathematics laboratory of three-dimensional Euclidean geometry, with the use of artefacts and physical experiences. It has been realized with a class group of 24 high school students (12^{th} grade), who were encouraged to become time traveller historians and mathematicians, investigating analogies and differences between Archimedes' and Cavalieri's methods to estimate volumes. The project had a double goal: from a research point of view, it pointed at evaluating the effectiveness of an historical inspired activity to update students' common culture about mathematics, while from a didactical point of view, the aim of this experience was exploiting the feeling of personal discovery that epitomizes hands-on activities as a pivot to promote a critical attitude towards Euclidean geometry as well as to endorse a historical approach to calculus.

Keywords: geometry, calculus, indivisibles, history of mathematics, Cavalieri

Résumé : Cet article présente une expérience pédagogique associant le laboratoire d'histoire des mathématiques et de la géométrie euclidienne tridimensionnelle à l'utilisation d'objets et d'expériences physiques. Il a été réalisé avec un groupe de classe de 24 lycéens (12e année), qui ont été encouragés à devenir des historiens et des mathématiciens voyageant dans le temps, en recherchant des analogies et des différences entre les méthodes d'Archimedes et de Cavalieri pour estimer les volumes. Le projet avait un double objectif: du point de vue de la recherche, il visait à évaluer l'efficacité d'une activité inspirée par l'histoire pour actualiser la culture commune des élèves en mathématiques, alors que d'un point de vue didactique, le but de cette expérience était de sentiment de découverte personnelle qui incarne les activités pratiques comme un pivot pour promouvoir une attitude critique à l'égard de la géométrie euclidienne ainsi que pour approuver une approche historique du calcul

Mots clés : geométrie, calculs, indivisibles, histoire des mathématiques, Cavalieri

Introduction and theoretical framework

Rethinking history of mathematics may prove to be a turning point to create meaningful classroom activities: indeed, it can be a formidable source of ideas for constructing students' mathematical skills and develop their sensitivity and interest in the evolution of mathematical concepts as well as the related symbolism and lexis.

As Marie-Anne Pech points out (Pech, 2013), mathematics consents a double journey: it makes us travel in time because our current knowledge was built by the humanity of yesterday and in space because this knowledge has to be understood in its context, taking the situated cultural approach into account.

Understanding knowledge in its context implies that "the concepts of 'mathematician' and 'scientific community' have to be differentiated according to the location and the historical period, [in fact] mathematicians are social beings and the development of mathematics is a process of interaction between mathematicians, hence, obviously there is always a social element in the history of mathematics" (Bos & Mehrtens, 1977, p.9).

In particular, we can describe European mathematicians of the XVIth and XVIIth century as pioneers who embodied the atmosphere of unearthing that characterized science in that period: "the mathematician, like an explorer, must find his way through fog and wilderness and retrieve the elusive gems. Mathematics, for them, is a science of discovery: it is about the uncovering of secret and hidden gems of knowledge. Its goals have little in common with traditional Euclidean geometry and much in common with the aims and purposes of the newly emerging experimental sciences." (Alexander, 2012, p.9).

Furthermore, Pech reminds us that, in drawing at the history of mathematics as a source for significant educational tasks, we have to consider two different aspects: the perception of history as a tool to motivate students, to humanize mathematics and to deepen the learning process, and the idea of history as an objective in itself to learn what mathematics is, to grasp its meaning, to show its constant evolutions in time and space and develop metamathematical reflections.

In this frame of thought, this teaching experiment aimed at "seeing history not only as a window from which to draw a better knowledge of the nature of mathematics but as a means of transforming the teaching of the subject itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance the students' development of mathematical thinking" (Furinghetti and Radford, 2008, p.626).

The inspiration came from the idea of exploiting the feeling of personal discovery that epitomizes hands-on activities to promote a critical attitude towards Euclidean geometry and to endorse a historical approach to calculus, embracing what Thomas (2015) writes in his note on Rashed Roshdi's 2011 work "D'Al-Khwarizmi à Descartes - Etudes sur l'histoire des mathématiques classiques" about "breaking the chronological boundaries inherited from political history (ancient, medieval, classical, modern mathematics), and reflect on the place of the History of Sciences, between epistemology and social sciences" (translation from the CIEAEM 70 2° announcement).

The project had a double purpose: my research interests laid in evaluating the effectiveness of a historically inspired activity to update students' common culture about mathematics. From a didactical point of view the goal of this activity was to assess if students, following a historical pattern that placed them in the position of past mathematicians, and creating with their own hands an artefact connected to a specific mathematical concept, could become more aware of the underlying mathematical meanings and be more prompted to take them in. In the long run (not included in this paper), I also aimed at laying the groundwork to observe if, getting acquainted to Cavalieri's idea that "a plane is composed of straight lines like a cloth of threads and a volume is composed of flat areas like a book of pages" (Cavalieri, 1647), could support their future understanding of the modern integration theory.

Method and activity

The activity – strictly connected to the Italian national curriculum - is conceived as a didactical transposition of Cavalieri's work, focused on the use of indivisibles to derive the formula of the volume of the sphere. The original procedure applies Cavalieri's principle (i.e. the equivalence of the volumes resulting from the equivalence of corresponding flat sections) to compare the volume of the solid delimited by a hemisphere and its circumscribed cylinder (the *bowl* in Fig.1a) to that of the cone of equal height and radius. The educational path stems from the proof given in the early 1600s by Luca Valerio - commonly known as the *bowl (scodella) of Galileo*, (Fig.1b) because Galileo reports it in his 1638 book "Discorsi e dimostrazioni matematiche intorno a due nuove scienze"– with an alternate history completion.

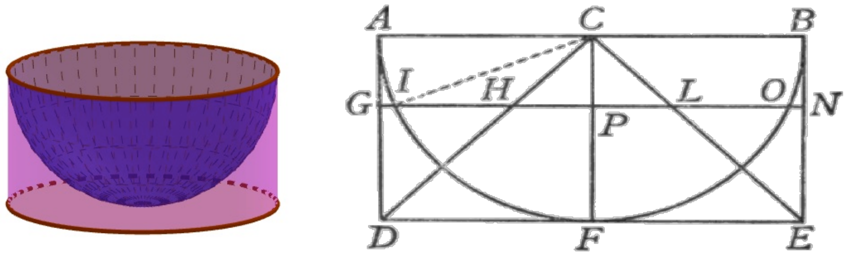

Figure 1: Galileo's bowl (a) and proof sketch (b)

Travelling back in time, the classical proof of the indivisibles equivalence is traded with its physical verification, obtained by applying Archimedes' mechanical equilibrium principle. Students are made aware that this is not the first encounter between indivisibles and levers: thanks to the rediscovery in 1906 of Archimedes' celebrated Palimpsest, containing The Method of Mechanical Theorems, we know in fact that the method of indivisibles was already used in the III century BC. Nevertheless, Archimedes did not consider it a mathematically rigorous method, therefore he used indivisibles, combined with the mechanical method, to discover the relations between areas and volumes and then he proved the same results by exhaustion.

The teaching experiment described in this paper was carried out with a class of 24 students attending the 12th grade of Liceo Scientifico (17-18 y.o.), results were gathered by the author, who was the classroom teacher, through field notes collected during the observation of the classroom activity and the following collective discussion and assessment of students' reports.

The experience is framed within the practice of the mathematics laboratory, introduced by UMI CIIM in 2001 as "not intended as opposed to a classroom, but rather as a methodology", and exploits "history of mathematics, […] as a possible and effective laboratory tool" (Bartolini Bussi, 2010 p. 42, translated by the author). In this context, students are prompted to become apprentice mathematicians and time traveller historians and are encouraged to underline and appreciate analogies and differences between Archimedes' and Cavalieri's methods to evaluate volumes.

From the procedural point of view, students are first made aware that, being independent of the postulates of Euclidean space, Cavalieri's principle is a kind of postulate itself, which provides a sufficient (but not necessary) condition for the equivalence of two geometric figures. On the other hand, Cavalieri's principle becomes also necessary when the two solids have equal heights, therefore, if we derive the equivalence of the bowl and cone from a different proof, we can deduct the equivalence of the indivisibles and verify it with the law of the lever. Thus, they are initially guided in applying sufficient Cavalieri's principle in the classical proof of the equivalence (i.e. the equality in volumes) between the sphere and its circumscribed hollowed-out cylinder (Fig.2).

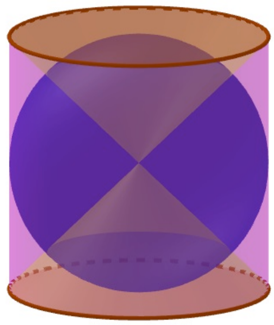

Figure 2: the setting for the classical proof

Students are then divided into eight groups and assigned the following task:

- *prove* the equivalence of the cone and bowl deriving it from the classical proof and connect it to that of the corresponding flat sections;
- assuming that the common height of the solids is 7.5 cm, *calculate* the exact measurements of the assigned flat sections (see Fig. 3) and cut them out using the provided foam sheets (Fig. 4);
- *verify* the equivalence of the flat sections using Archimedes' procedure: equality of mechanical moments (Fig. 4) and obtain the equivalence of the indivisibles from this result.

In this frame of work, the proof part comes before the verification in order to draw students' attention to the conceptual difference between the two steps.

To assign a specific level to each group, a dynamic figure created using GeoGebra was shared with the class: students could drag the horizontal line to their assigned level and refer to the ruler to read the measure indicating the position of their indivisibles (Fig.3).

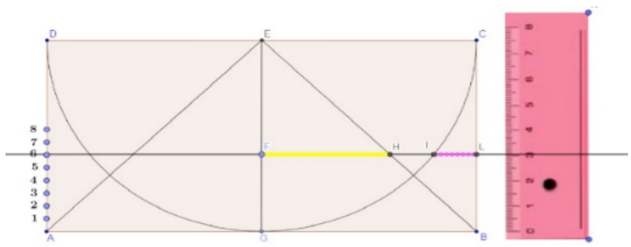

Figure 3: the assignment of the sections

Working in groups, students had then to figure out the math part needed to appraise the measurements of their indivisibles and cut them out from the provided foam sheets. Finally, they constructed the lever arms using pierced wooden sticks and checked for the expected equilibrium (Fig.4).

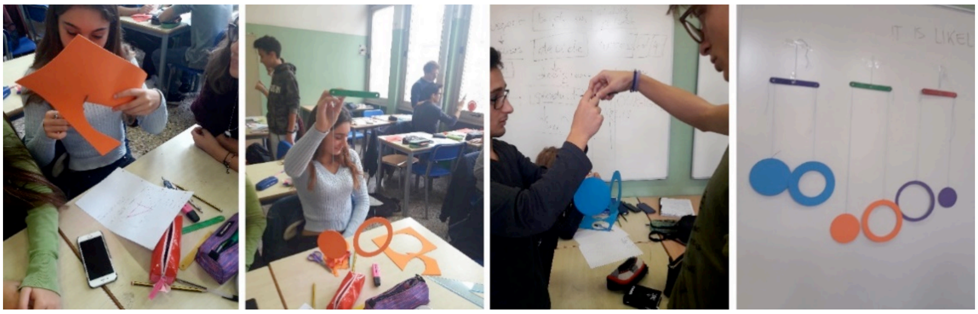

Figure 4: the indivisibles construction and the equilibrium checking

In conclusion, class discussion allowed the sharing of results and findings and then each group summarized his conclusions in a written essay at home, reports were submitted and graded by the teacher.

Comparing aims to students productions

Excerpts from field notes, students' lab reports (indented quotations) and reflections emerged during class discussion, made it possible to assess whether Furinghetti and Radford's cited goal of "enhancing the students' development of mathematical thinking" had been met or at least approached.

1 - Cavalieri's principle as a necessary condition:

> "Since the bowl and the cone have the same volume and the same height, if we cut off both solids with a plane parallel to the common base, for the Cavalieri's principle [the sections] will have equivalent surfaces."

This excerpt supports the impression that students are aware of the additional hypothesis needed to apply the Cavalieri principle as a necessary condition.

2 - The Maths behind it: the calculations of the measures of the sections

During this evaluation part, weaker students had a hard time calculating the exact measures of their indivisibles, but they were strongly motivated to succeed in order to finalise their construction, while a group of stronger ones underestimated the problem and cut wrong sections, but were confronted with their mistake once the lever arm was not balanced. These episodes sustain the effectiveness of the laboratory approach.

3 - The Physics behind it: verification vs proof

> "With Archimedes' procedure, we verify in a physical way that the section of the cone (circle) is equivalent to that of the bowl (annulus) […] To carry out Archimedes' procedure we need material sections."

The accurate use of the term "verify" enforces the idea that students correctly grasped the difference in question and that, although Cavalieri's indivisibles are one dimension less than the continuum they generate, we need to give them some thickness in order to make Cavalieri and Archimedes meet.

4 - Mathematical equivalence vs Numerical coincidence: comparing surfaces

> "The two results have a minimal difference due to the approximations made during the carrying out the calculations"

This shows that students are aware of the necessity of allowing for errors and of the difference between irrational numbers and their rational approximations.

5 - Indivisibles or infinitesimals? The birth of modern calculus

During the discussion, the utterance of a student: "an indivisible is a plane figure of infinitesimal thickness" eased the shift from historical to epistemic awareness. In fact, it triggered a metamathematical (in Pech's sense) class talk about the difference between indivisibles and infinitesimals and about the path that, from Cavalieri to Newton and Leibniz, allows the morphing from indivisibles to infinitesimals, that leads to the birth of modern Calculus. In the discussion students' attention was drawn to the fact that Cavalieri knew that this method of summing lines into areas and areas into volumes could hide some pitfalls, but that, embodying the experimental thrust of that historical period, "he was less interested in questions as to the precise nature or existence of indivisibles, than in their pragmatic use as a device for obtaining computational results. Rigour, he wrote in the Exercitationes, is the affair of philosophy rather than mathematics" (Edwards, 1979, p.104).

Results and conclusions

Observing the processes, the outcomes of this activity seem to support the efficacy of a history-inspired laboratory strategy to engage students and promote their mathematical activity. The final verification through the physical perception of the equilibrium deeply involved them, they rejoiced visibly when the lever arms stood still in the equilibrium position and they wondered whether Archimedes had felt the same way. Archimedes, Galileo and Cavalieri emerged from the past and from the stillness of the textbook to act as workmates, engaging students, humanizing the learning process, and fostering the students' awareness of the relevance of the history of mathematics and its role within our culture.

Several different levels of mathematical bearings were synergically integrated with the historical aspect, that students eventually had to convey in their final papers: a geometric one to understand the cross-section image of the bowl and grasp the 3D shape of the sections, a symbolic one to derive a formula to obtain the sizes of the required sections and a numerical one to finally compute the exact measurements. Artefacts, tools and sign systems acted as effective means for the construction of knowledge: these rich stimuli, together with the act of constructing the mathematical objects themselves, activated a range of intertwined referents which effectively supported the students' learning, prompting them to review the fundamental difference between verification and proof and in the same time hopefully smoothing the path for the future introduction of Calculus.

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