INHALTSVERZEICHNIS

INDEXES

General Introduction

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Introduction

The texts published in this volume can be subdivided into two categories: the first consists of the substantial untitled manuscript which contains, in a sequence numbered by the author, 120 propositions on various subjects, to which Johann Bernoulli applied himself over a very long period of time, namely from 1685 to the first decades of the 18th century; the second is composed of a series of articles and manuscripts devoted to problems on the rectification and transformation of curves, on geodesics and on spherical epicycloids.

I. **The *120 Propositiones***

The manuscript we have entitled *120 propositiones* is a valuable document which allows us to grasp Johann Bernoulli's vast range of scientific interests, and the research he carried out starting from the 1680s, at the time when he was serving his apprenticeship under his brother Jacob, until he reached his full maturity. Moreover, the chronological ordering of the succession of subjects, the internal references to sources consulted and to problems raised, the quotations in letters to his contemporaries and the publication of specific contributions allow us to see the stages of his cultural development and to focus the objectives he pursued and the results he obtained.

The structure of Johann's manuscript is analogous to that of Jacob's scientific notebook entitled *Meditationes, Annotationes, Animadversiones Theologicae et Philosophicae a me J. B. concinnatae et collectae ab anno 1677*. In both cases what we have is a non-organic collection of – often unconnected – problems, regarding Euclidean, Apollonian and Cartesian Geometry, Arithmetic and Theory of numbers, Probability theory, Differential and Integral Calculus, with reflections on Series, on the Rectification of curves, on Differential Equations and on the Calculus of Variations.

It is probable that the initial stimulus to record the results of his studies and mathematical successes came from his brother Jacob, his teacher until the 1690s; Jacob's influence is very clear in the whole of the first part of the manuscript, and can also be seen from the presentation of subjects common to the research of both and from the Synoptic Table on pages pp. 571–573.

The second part deals rather with problems proposed in journals of the period or received through Johann's personal contacts with mathematicians in various countries during his stay in Groningen and subsequently in Basel.

More than half the propositions are from the period between 1685 and January 1692, the date of the publication in the *Acta Eruditorum* of Proposition 68, the second article Johann published under his own name alone (Joh. B. Op. VI).

The previous propositions deal with problems of elementary Euclidean geometry (Prop. 1–5, 11, 18–21), the proof of Ptolemy's theorem on the quadrilateral within a circumference (Prop. 57), exercises in plane trigonometry (Prop. 6–10), properties of conics, in the style of Apollonius (Prop. 30–38), and topics of elementary arithmetic and theory of numbers (Prop. 13, 22, 23, 28, 29, 49, 51, 52, 56).

Some of his research, expounded in Prop. 14, 15, 16 and 17, bring out the influence of J. Wallis' *Arithmetica infinitorum* in their use of the particular terminology adopted (*series primanorum, secundanorum*) and of the symbol for the infinite. Others, such as Prop. 24–27, 39–42, 50 and 53, show that he was familiar with René Descartes' *Géométrie*, in the Latin edition by F. van Schooten, and the use of J. Hudde's methods which appeared in this work.

It is, however, the marked presence of problems which Jacob Bernoulli had already tackled in his scientific notebook that lays stress on the close link with the process by which Johann matured as a scholar. This is clear from Prop. 43–48 on regular polygons inscribed to a circumference, in particular on the triangle, the heptangle, and the enneagon, on which his brother had been engaged in reflection from 1684 to 1688 in the Med. LIX and CXX[1], with procedures identical to those used here.

Propositions 39–42, 55 and 58, in both the statements and the solutions, also return to some problems tackled by Jacob between 1686 and 1689, in Med. CV, CIV, CXIII, CXIV and CXXXVII[2] respectively, with the methods of Cartesian geometry. These propositions were published in 1742 in the fourth volume of Johann's *Opera omnia*.

Prop. 59–67 on series are also the fruit of studies carried out together with his brother, as indeed is clear from an explicit observation by Jacob in the article *Positiones Arithmeticae de Seriebus Infinitis* (7 June 1689), where he quotes Johann's proof on the divergence of the harmonic series[3]. The merits of the method Johann employs compared with Jacob's were also stressed by Johann Bernoulli himself in his letter to Leibniz of 1 December 1696[4]. Not infrequently in this period did Jacob publicly recognise his brother's merits: for

1 Jac. B. *Werke* 2, pp. 313–315, 447–457.
2 Jac. B. *Werke* 2, pp. 407–409, 403–404, 419–420, 421, 490–492.
3 Jac. B. Op. XXXV, *Positio* XVI, *Werke* 4, pp. 56–58.
4 Leibniz, *Math. Schriften* 3, p. 339.

example, in November 1689 in the *Acta Eruditorum*, at the end of the article *Novum theorema pro doctrina sectionum conicarum* (Jac. B. Op. XXXVIII), he inserted an alternative proof given by Johann, and in other writings he quotes his acute observations.

Further confirmation of the investigations carried out together with Jacob and on the same subjects derives from comparison of their manuscripts and of the publications on series which appeared in Basel between 1689 and 1704, subsequently republished as an appendix to the *Ars Conjectandi* in 1713. Specifically, Johann's proposition 59 echoes Jacob's Med. CXXXV, published in Op. XXXV, *Positio* X[5]; proposition 60 echoes Med. CXXXIX, published in Op. XXXV, *Positio* XIV[6]. Propositions 61–62 tackle the series studied in Med. CXLI and CXLIII, which appeared in Op. XXXV, *Positio* XVI and Op. LIV, *Positio* XXII[7].

In proposition 63 of Johann's manuscript we find the series which Jacob drew from a problem on the calculus of games of chance, the subject of Med. CL,[8] and presented in *Positio* XXI of the dissertation *Positionum de Seriebus infinitis Pars altera*, discussed in Basel on 18 November 1692[9]. This also contains the sums of the series examined by Johann in Propositions 64, 65 and 66, with procedures that are entirely analogous to those expounded by Jacob in *Positiones* XXIII, XXIV and XXVI of the same dissertation and in Med. CXLVII[10].

Proposition 68 presents the well-known result of the catacaustic of parallel light rays falling on a circle from a source to infinity, published in the *Acta Eruditorum* in January 1692,[11] whose solution very probably goes back to the end of 1690, before Johann's journey to Geneva and his stay there, for he hints at it in his letter to Jacob on 22 May 1691:

> Vous me ferez aussy savoir … si mon probleme funiculaire n'est pas encore mis dans les actes, de méme que ma solution de la caustique par la Geometrie Cartesienne que je vous ay laissé entre les mains[12].

With proposition 72 we find Leibniz's differential and integral calculus used for the first time to search geometric *loci* of point that satisfy specific proper-

5 Jac. B. *Werke* 4, pp. 186 and 50.
6 Jac. B. *Werke* 4, pp. 188–190 and 52–54.
7 Jac. B. *Werke* 4, pp. 192, 195 and 56–58, 70–71.
8 Jac. B. *Werke* 3, pp. 94–97 and *Werke* 4, pp. 202–205.
9 Jac. B. *Werke* 4, Op. LIV, p. 70.
10 Jac. B. *Werke* 4, Op. LIV, pp. 71–72, 72–75, 76, 197–198.
11 Joh. B. Op. VI, *Opera* I, pp. 52–59 – *Streitschriften*, pp. 10–16, 127–131.
12 Joh. B. *Briefe* 1, p. 105.

ties. In proposition 73, for example, the problem of the funicular or catenary curve is tackled with the same analytic procedure expounded in XII of the *Lectiones* for the Marquis de l'Hôpital[13]. The fact that this topic was the object of the article which appeared in the *Acta Eruditorum* in January 1691, in which it was specified that it had emerged from considerations made together with his brother Jacob, who had then presented it to the public on May 1690, and that it is quoted in the letter mentioned above leads us to date this study at the end of 1690, although in the printed version only the solution and the geometric construction were given, without the analytic expressions.

Proposition 74 seeks the curve envelope of the parabolas described by the trajectories of cannonballs fired from a specific point with constant force (a problem already solved by Torricelli in his *Opera geometrica*[14]), proving that it is a parabola. The same problem appears in Med. CLXVI of Jacob's scientific diary and in the seventh of his *Notae et animadversiones tumultuariae*[15] to the Latin edition of Descartes' *Géométrie* (1695). It is, however, very likely that it was Johann who suggested the study to Jacob, in a lost letter of 10 June 1691 from Geneva, where he briefly mentioned a problem that had been suggested to him by Jean Christophe Fatio de Duillier[16]. In the formulation sent to Leibniz on 2 September 1694, the problem was in the following terms:

> de invenienda curva, quae singulas parabolas a globis ex singulis elevationibus mortarii ejectis descriptas tangit[17].

Johann's result was communicated to his brother in his letter of 27 June 1691:

> J'ay aussi trouvé la nature des courbes Ballistiques par la Geometrie vulgaire, avant que i'eusse reçu votre lettre; mais avec une maniere semblable à celle avec quelle i'ay resolu la nature de la caustique; Mr. Fatio est Témoin[18].

The same problem was to be inserted by de l'Hôpital in article 147 of the *Analyse des infiniment petits* (1696, p. 133), having presented it to Leibniz in his letter of 24 February 1693[19].

13 Joh. B. Op. CXLIX, *Opera* III, p. 426.
14 Florentiae 1644 – *Opere di Evangelista Torricelli*, G. Loria, G. Vassura (eds.), Faenza, Montanari, 1919, vol. II, Prop. XXX, pp. 178–180.
15 Jac. B. *Werke* 2, pp. 505–510 and Jac. B. Op. LXVI, *Opera*, pp. 680–684 – *Werke* 2, pp. 561–566.
16 Joh. B. *Briefe* 1, pp. 106–107.
17 Leibniz, *Math. Schriften* 3, p. 151.
18 Joh. B. *Briefe* 1, p. 111.
19 Leibniz, *Math. Schriften* 2, pp. 225–226.

As emerges from the correspondence with Jacob, the first studies of the *lintearia* curve, the form of a cloth filled with water, go back to June 1691 in Geneva. This was the subject of proposition 76, and Johann took up his studies on this curve once again in Paris in his lessons to de l'Hôpital[20], and subsequently in the context of the controversy with his brother over the isoperimetrical problems. Proposition 77 too, on the nature of the curve described by a ray of light which passes through a dense medium, was presented some months later to de l'Hôpital[21], and later used in the solution to the problem of the brachystochrone[22].

Propositions 79 and 80 have to do with the calculus of games of chance; they are probably linked to the problems of throwing dice, tackled in this period by Jacob Bernoulli in the course of writing the third chapter of the first part of the *Ars conjectandi*. The terminology and the methods used to determine the mathematical expectations of the players are in fact analogous to those his brother employed.

The two subsequent propositions, 81 and 82, are on series and, like the previous two, were published in Volume IV of Johann's *Opera Omnia*[23]. It is possible that they derived from studies of combinatorial analysis connected to the calculus of probability, as A. Weil shows with regard to Jacob's cultural path[24]. Proposition 81, which seeks the maximum term of the binomial $(a + b)^c$, is the object of study of Jacob's Med. CCI, linked by him with a reference to a problem of the doctrine of chances tackled between 1688 and 1690[25].

Proposition 83 tackles the same problem of determining geometric *loci* with Cartesian geometry presented in June 1696 in the *Acta Eruditorum*[26]. Jacob intervened on this very problem in May 1697 with the note *Solutio Problematum Fraternorum, una cum Propositione reciproca aliorum*[27].

Proposition 84 also refers to the search for the equation of a curve subjected to a property analogous to the preceding one, and for this reason Johann proposed it to those who wished to experience the validity of their methods, in

20 Joh. B. Op. CXLIX, *Lectiones* XLIV, XLV, *Opera* III, pp. 512–516.

21 Joh. B. Op. CXLIX, *Lectio* XLVI, *Opera* III, pp. 516–518.

22 Prop. 87, pp. 211–213 h. v., Joh. B. Op. XXXVII, AE Maji 1697, *Opera* I, pp. 187–193 – *Streitschriften*, pp. 263–270 and Joh. B. Op. CIII, *Opera* II, pp. 235–269 – *Streitschriften*, pp. 527–568.

23 Joh. B. Op. CLIII, *Opera* IV, pp. 25–27.

24 Cf. Jac. B. *Werke* 4, Introduction, p. 11.

25 Jac. B. *Werke* 3, p. 89 and Med. CLIa, *Werke* 3, pp. 76–83, 378–383.

26 Joh. B. Op. XXX, *Supplementum defectus Geometriae Cartesianae circa Inventionem Locorum*, AE Junii 1696, pp. 264–265 – *Opera* I, pp. 156–157.

27 Jac. B. Op. LXXV, *Opera*, pp. 775–778 – *Streitschriften* pp. 276–278.

the same article of June 1696: «si qui alii in hisce suarum methodorum vires experiri velint»[28]. The solution, with its proof, was published in 1742 in the fourth volume of the *Opera*[29].

Propositions 85 and 91–96, on the problem of the transformation of curves, are dealt with in § 2.2, which Johann proposed in the *Journal des Savants* in 1703. The first brief reference to this problem is in a letter to Leibniz on 10 June 1702, in which Johann says that it was suggested to him, together with six others, by a Belgian mathematician in 1698.[30]

The problem of the brachystochrone is tackled in proposition 86 with a geometric procedure and in proposition 87 with Leibnizian analysis. The texts of these propositions, slightly recast, were published in the context of the controversy with his brother Jacob on the isoperimetrical problems. To be precise, in February 1718 they appeared at the end of the essay in *Acta Eruditorum*, *De solutionibus quae extant Problematum isoperimetricorum*[31]. A French translation, due to P. Varignon, appeared in the Paris *Mémoires de l'Academie des Sciences*[32]. These propositions were written at latest by 21 July 1696, when Johann sent them to Leibniz for his opinion,[33] and Leibniz advised him not to publish the direct, but only the indirect proof, writing to him on 31 July 1696:

> Caeterum ubi solutionis comprobationem edere vel communicare placebit, suaserim viam illam directam, quam vocas, seu posteriorem, non edi, cum prior sufficiat ad demonstrationem, et posterior praeter necessitatem aliis ante tempus viam aperiat. Praestat enim (ut puto) nonnihil adhuc suspensos alios teneri, ut vel ipsi inveniant aliquid fortasse a nostris diversum, quod augebit scientiam; vel agnoscant, non esse haec tam facilia, ut quidam putant, eoque diligentius has methodos aliquando meditentur[34].

Some months later, on 23 February 1697, Leibniz repeated the same advice:

28 Joh. B. Op. XXX, AE 1696, p. 267 – *Opera* I, p. 158.

29 Joh. B. Op. CLV, *Geometrica, Propositio VII Problema*, *Opera* IV, pp. 40–41.

30 Leibniz, *Math. Schriften* 3, p. 702: «Les courbes paraboliques et hyperboliques de quelques degrez qu'elles soient transformer en d'autres courbes algebraiques, en sorte que les arcs des unes soient egaux aux arcs des autres.» The Belgian mathematician may be Jacques-François Le Poivre (1652–1710).

31 Joh. B. Op. CIII, AE Januarii 1718, pp. 15–31, AE Februarii 1718, pp. 74–88.

32 *Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes sur les Isoperimetres*, Mém. Paris 1718 (1719), pp. 100–138 – *Opera* II, pp. 267–269 – *Streitschriften*, pp. 566–567.

33 Leibniz, *Math. Schriften* 3, pp. 302–309.

34 Leibniz, *Math. Schriften* 3, p. 310.

> Repeto autem, quod initio dixi, recte facturum Te si eam tantum partim solutionis tuae edas, quae Analysin adhuc nonnihil involvit methodo, quam vocas indirectam. Mihi enim (nescio an et Tibi) consultum videtur nondum in interiora admittere homines ingratos, et beneficium postea strenue dissimulaturos[35].

Proposition 90 is also linked to the isoperimetrical problems. It is set out, in fact, in the article *Solutio Problematum quae Jacobus Bernoullius ... fratri proposuit*, which appeared in January 1698 in the *Acta Eruditorum*[36], and in French in Johann's letter to Varignon on 15 October 1697[37]. The remainder of the treatment on the isoperimetrical problems, published in 1718 in Latin and in 1719 in French, appears only in proposition 119 of the manuscript and was written certainly after January 1712, since in the earlier proposition 115 Johann cites the pages of an article of his which had appeared in the Parisian *Mémoires* printed in 1712.

The solution of the linear non-homogeneous differential equation

$$r(y)x\,dy + t(y)\,dy = dx$$

is the subject of study in proposition 88, in the corollary of which Johann tackles the differential equation

$$a\,dy = yp(x)\,dx + by^n q(x)\,dx,$$

suggested by his brother in December 1695 in the *Acta Eruditorum*[38].

The general method presented here was sent to Leibniz on 25 August 1696,[39] and was to be published in the *Acta* in March 1697 in the article *De conoidibus et sphaeroidibus quaedam, Solutio analytica Aequationis in Actis 1695, pag. 553 propositae ...*[40].

Proposition 89 considers the square of segments of a cycloid, and the results obtained were published in the *Acta Eruditorum* in July 1699[41]. This article sparked off a further heated argument with Jacob, who saw two of his own

35 Leibniz, *Math. Schriften* 3, p. 370.

36 Joh. B. Op. XL, *Opera* I, p. 212.

37 Joh. B. *Briefe* 2, p. 142.

38 Jac. B. Op. LXVI, *Explicationes, Annotationes et Additiones ad ea, quae in Actis sup. anni de Curva Elastica, Isochrona Paracentrica, & Velaria ...* AE Decembris 1695, p. 553 – *Opera*, p. 663.

39 Leibniz, *Math. Schriften* 3, pp. 323–324.

40 Joh. B. Op. XXXV, AE Martii 1697, pp. 113–118 – *Opera* I, pp. 174–179.

41 Joh. B. Op. LVIII, *Cycloidis primariae segmenta innumera quadraturam recipientia, aliorumque ejusdem spatiorum quadrabilium determinatio: post varias illius fortunas nunc*

articles alternating with two of his brother's in the pages of the German journal, in September 1699[42], June 1700[43], December 1700[44] and April 1701[45]. Transformations and integrations of differential quantities are the subject of propositions 96, 97, 98, 99, 100 and 101, which Johann sent to Varignon on 5 August 1702, in a letter which sadly has been lost, so that they might be published in the Paris *Mémoires de l'Académie des Sciences*[46].

Varignon's time in Normandy held back the translation into French of Johann's article, which was presented to the Académie on 13 December and published in the 1702 volume, printed in 1704[47]. An extract in Latin which includes the slightly recast text of proposition 101, followed in the concluding part by the text of propositions 96–100, was to appear in the *Acta Eruditorum* in January 1703[48].

The subject expounded was linked to the problem of orthogonal trajectories to a family of curves, discussed by his brother Jacob in the same journal in May 1698, as Johann himself states in proposition 101, so these considerations must be dated between May 1698 and August 1702[49].

Proposition 103 is linked to the problem of the transformation of algebraic curves into others of the same length, proposed by Johann in February 1703 and illustrated in § 2.2. Here we see the emergence of the genesis of the reptorial motion, which he had conceived for precisely this problem, and published in 1705.

Propositions 104 and 105 deal with properties relating to the conjugate diameters of conic sections, while 106 uses the analytic method to solve a problem of the ellipsis which the French mathematician Philippe de la Hire had suggested to him, through Varignon, as being particularly difficult[50]. The statement of the problem can be found at the end of Varignon's letter to Jo-

primum detecta, AE Julii 1699, pp. 316–320 – *Opera* I, pp. 322–327 – *Streitschriften*, pp. 393–399.

42 Jac. B. Op. XCII.

43 Joh. B. Op. LX.

44 Jac. B. Op. XCV.

45 Joh. B. Op. LXIX. On this topic cf. *Streitschriften*, Introduction § 18, pp. 111–112, Jac. B. *Werke* 4, Introduction, pp. 30–31.

46 Joh. B. *Briefe* 3, pp. 33, 64–65.

47 Joh. B. Op. LXX, *Solution d'un Problème concernant le calcul intégral, avec quelques abregés par rapport à ce calcul*, Mém. Paris 1702 (1704), pp. 289–297 – *Opera* I, pp. 393–400.

48 Joh. B. Op. LXX, *Problema exhibitum a Jo. Bernoullo*, AE 1703, pp. 26–31 – *Opera* I, pp. 393–400..

49 Cf. H. Goldstine, *Introduction*, to *Streitschriften*, pp. 109–111.

50 Joh. B. *Briefe* 2, pp. 31–32: «voicy un probleme de coniques qu'il vous propose, et qu'il croit fort difficile».

hann Bernoulli on 28 March 1693, and requires the construction of the ellipse tangent to the sides of a given angle and having an axis equal to a given segment. The solution, with the synthetic method, is given by Johann in the manuscript *Problème proposé par M. de la Hire*[51].

Proposition 107 turns on a problem of central forces which, at Johann's suggestion, was proposed by his nephew Nicolaus I to the Italians in 1715, at the conclusion of an article in the *Giornale de' Letterati d'Italia*, published in Venice. This challenge sparked off a heated discussion between the Bernoullis and the mathematicians Sebastiano Checozzi and Bernardino Zendrini, who gave its solution, first in the form of an anonymous Admonition (*Avvertimento*) and then in articles which were criticised at length[52]. The discussions ended by also involving Hermann, accused by the Bernoullis of having suggested the solution to the Italians who maintained excellent relationships with him from the new remises at Frankfurt on the Oder where he was teaching at the time. Problems of central forces are also dealt with in propositions 108 and 109, published together with 107 in volume IV of the *Opera*[53].

Studies on the conditions of integrability of binomial differentials are found in Propositions 110, [110a], 111, 112, 114, 115, 116 and are linked to the considerations Johann Bernoulli sent to George Cheyne concerning his book *Fluxionum Methodus inversa, sive quantitatum fluentium Leges Generaliores* (Londini, 1703), as emerges from the observation at the end of proposition 111. The context in which these propositions were written is described in Johann's letters to Robert Falconer,[54] to Abraham de Moivre[55] and to Cheyne himself.[56] From Bernoulli's notes to the Englishman's work derive the texts (published in the fourth volume of Johann's *Opera*) *Animadversiones in Cl. Georgii Cheynaei Fluxionum Methodum inversam, editam Londini 1703*[57] and *Observationes in Clar. Moivraei Animadversiones in D. Cheynaei Tractatum de Fluxionum Methodo inversa, editas Londini 1704*[58], which are also linked to the Errata cor-

51 Joh. B. Ms. 27a, pp. 333–334 h. v., cf. § II of this Introduction.

52 Cf. S. Mazzone, C. S. Roero, *Jacob Hermann and the diffusion of the Leibnizian Calculus in Italy*, Florence, Olschki, 1997, pp. 123, 167–175, 371–372, 377–381, 397, 415, 419–422.

53 Joh. B. Op. CLXXV, *De lege virium qua fit ut mobile ad centrum descendat temporibus quae sint ut potestates datae distantiarum, a quibus descensum inchoat*, pp. 243–248.

54 Joh. Bernoulli to R. Falconer, 14 August 1703, UB Basel L I a 674: Wollenschläger 1933, pp. 315–317.

55 Joh. Bernoulli to A. de Moivre, 15 November 1704, UB Basel L I a 664, Nr. 1: Wollenschläger 1933, pp. 179–187.

56 Joh. Bernoulli to G. Cheyne, XV Kal. Decembr. 1703 [17 November 1703], UB Basel L I a 673, Nr. 16, pp. 165–168.

57 Joh. B. Op. CLXVII, *Opera* IV, pp. 129–146.

58 Joh. B. Op. CLXVIII, *Opera* IV, pp. 146–160.

rige to his text which Cheyne had printed, under the title of *Addenda et Adnotanda in Libro Fluxionum Georgii Cheynaei*, which Cheyne sent to Bernoulli in April 1704 and which is currently held in Basel at the end of the volume in Johann's possession[59]. These texts were to be described and published in the volume of the *Werke* devoted to the controversy with the English.

Proposition 113 deals with the study of series whose coefficients are figurate numbers and can be dated to a period after 1711, in view of the reference to Isaac Newton's *Analysis per quantitatum series, fluxiones ac differentias* (Londini, 1711).

Finally, Proposition 120, which shows the analytic solution to the problem of the curve described by a projectile in resisting media, was published with a very few variants in May 1721 in the *Acta Eruditorum*[60]. As the text cites Hermann's *Phoronomia*, which Johann had received in September 1715, and this very solution had been communicated to Pierre Remond de Montmort in the letter of 13 July 1719, it must have been written between 1715 and 1719[61].

II. **The problem of Philippe de La Hire**

Among the Bernoulli's manuscripts preserved in Basel there is also a geometrical problem regarding the construction of an ellipse subject to specific conditions (Ms. 27a). It had been suggested to Johann Bernoulli by Philippe de La Hire, through Varignon, in 1693[62]. The problem concerns the construction of the ellipse tangent to two given rays AB, AD and with its centre at point C within these rays and its axis equal to a given segment PQ. Bernoulli solves

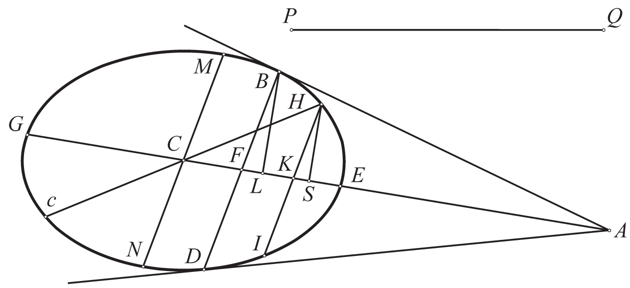
59 UB Basel, Kf. IV. 9.

60 Joh. B. Op. CXXI, *Opera* II, pp. 513–516.

61 UB Basel L I a 665, Nr. 15.

62 Cf. P. Varignon to Joh. Bernoulli, 28 March 1693, Joh. B. *Briefe* 2, pp. 31–32, 34. In 1706 Joh. B. suggested this problem to Jacob Hermann, who sent his geometric-synthetic solution to Jean Christophe Fatio de Duillier (Hermann to Jean Christophe Fatio, Basel 14 August 1706, BPU Geneva MS Fr. 601, ff. 193–194). Hermann sent this solution to Guido Grandi in his letter of 21 December 1708, cf. S. Mazzone, C. S. Roero, *Guido Grandi – Jacob Hermann, Carteggio 1708–1714*, Firenze, Olschki, 1992, pp. 28–29. A construction and the relevant proof of it is kept in an Italian manuscript autograph by Hermann in Venice. Cf. J. Hermann, *Problem on an ellipse*, Venice, Marciana Library (Ms. It. IV 642 – 5503, ff. 7r–8v), published in S. Mazzone, C. S. Roero, *Jacob Hermann and the diffusion of the Leibnizian calculus in Italy*, Firenze, Olschki, 1997, pp. 349–351. Hermann presented the same problem at the Academy of St Petersburg in August 1729 and published it in the *Commentarii*. J. Hermann, *De ellipsi conica cujus axis alteruter datus est, angulo positione et magnitudine dato ita inscribenda, ut centrum ejus intra datum angulum sit etiam positione datum*, CP 4, 1729 (1735), pp. 46–49.

the problem analytically in Prop. 106, by a rather laborious procedure leading to a fourth degree equation. This is how he proceeds:



If $PQ = 2a$ is the given segment, and if $CA = b$, $CE = x$, from the known property of ellipse it follows that $AC : CE = CE : CF$, from which

$$CF = \frac{x^2}{b}$$

and

$$AF = AC - CF = \frac{b^2 - x^2}{b}.$$

Since the angles AFB and FAB are given, the ratio $\dfrac{AF}{BF} = \dfrac{1}{n}$ is also given. From these relations Bernoulli obtains

$$BF = \frac{nb^2 - nx^2}{b}$$

and therefore

$$\frac{GF \times FE}{BF^2} = \frac{x^2}{n^2(b^2 - x^2)} = \frac{p}{q}.$$

Now let H be a generic point on the ellipse, and from the points B and H draw BL and HS perpendicular to AG. Let $CK = z$ and $\dfrac{BF}{FL} = \dfrac{1}{m}$. From the relation

$$\frac{GK \times KE}{HK^2} = \frac{GF \times FE}{BF^2} = \frac{p}{q},$$

since $GK \times KE = x^2 - z^2$, Bernoulli gets

$$HK^2 = \frac{q}{p}(x^2 - z^2)$$

and therefore

$$CH^2 = HS^2 + CS^2 = (HK^2 - KS^2) + (CK + KS)^2 =$$

$$HK^2 + CK^2 + 2CK \times KS = \frac{qx^2 - qz^2 + pz^2}{p} + 2mz\sqrt{\frac{qx^2 - qz^2}{p}}.$$

When H is on the axis of the ellipse, the quantity $CH^2 = a^2$ is maximum with the respect to the variable z. Its differential is therefore equal to zero, so that he obtains

$$z\sqrt{qx^2 - qz^2}(p - q) + m\sqrt{p}qx^2 - 2m\sqrt{p}qz^2 = 0.$$

By squaring the previous equation, Bernoulli finds the biquadratic equation

$$[4m^2pq + (p - q)^2]z^4 - [4m^2pqx^2 + (p - q)^2x^2]z^2 + m^2pqx^4 = 0$$

which gives

$$z^2 = \tfrac{1}{2}x^2 \pm \frac{\frac{1}{2}(p - q)x^2}{\sqrt{p^2 - 2pq + q^2 + 4m^2pq}}.$$

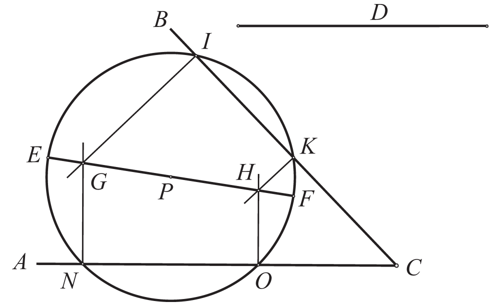
He now substitutes the value of z, found in this way, in the previous expression of $CH^2 = a^2$, and obtains the equation:

$$a^4 - (p + q)a^2 + pq - m^2pq = 0.$$

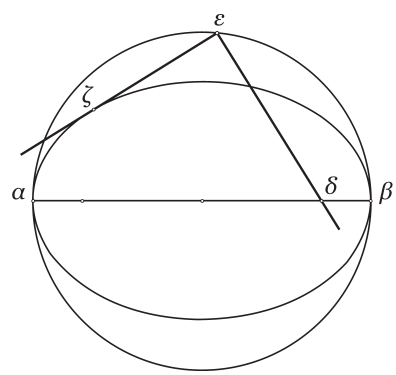
Finally, replacing p, q with their expressions in the variable x, Bernoulli obtains a biquadratic equation in x that gives the solution of the problem.

In the fragment in question, Bernoulli proposes a different solution, simpler and more elegant by far.

In order to determine the ellipse with centre P, axis equal to D and tangent to the rays CA, CB, Bernoulli first considers the circle *IKON* with centre P and diameter D, which meets the sides of the given angle at the points I, K, O, N. The normals to the sides at these points meet in G and H. Then Bernoulli states that G and H are the foci, and that the diameter *EF* is the axis of the ellipse.



The proof, he says, depends on the following beautiful theorem, that he had proved previously: $\alpha\zeta\beta$ be an ellipse and let $\alpha\epsilon\beta$ be the circumference with diameter equal to the axis. From a point ϵ on the circumference, draw the lines $\epsilon\delta$ connecting the point ϵ to one of the foci of the ellipse, and $\epsilon\zeta$ tangent to the ellipse. Then the angle $\zeta\epsilon\delta$ is always right.



Once this theorem is proved, the result follows immediately. Actually, consider the ellipse with foci in G and H and axis *EF*, and the circumference *IKON* with diameter equal to its axis. If from a point I on the circumference we draw the segment *GI* connecting I to one of the foci and the line *IC* perpendicular to it, *IC* will be tangent to the ellipse.

III. **The rectification of curves**

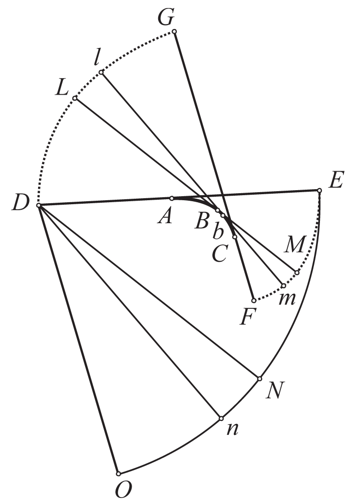
The subject of the rectification of curves often recurs in short texts which Johann Bernoulli published at various times. This is a matter of a series of different problems, ranging from the invention of absolutely rectifiable (algebraic) curves, i.e. such that the length of the arc can be expressed algebraically in function of the point, to the transformation of given algebraic curves into others, also algebraic, of the same length, or rather the same arc element, to the search for algebraic curves whose arc element, added to that of a given curve, will give a rectifiable curve.

1. *The first research*

The first article Johann Bernoulli published on this subject, *Meditatio de dimensione linearum curvarum per circulares*, goes back to 1695, and is to be found in the August issue of the *Acta Eruditorum*[63]. The starting point is a result of Christian Huygens, who in his *Horologium oscillatorium* (1673) had proved that, given a spheroid, it was possible to construct a hyperbolic conoid such that the two surfaces, taken together, should be equal to a circle. Bernoulli takes up this problem again in the unidimensional case, generalising to any curve, and solves the following problem: *Given any curve, find a second curve such that the sum or the difference of their lengths be equal to an arc of circumference.* The construction is rather simple and depends on the following lemma.

Given any curve *ABC*, Bernoulli considers a rigid pole *DE* which, starting from one extreme *A* rests on the curve remaining always tangent until it reaches the tangent position at the other extreme *C*. The two extremes of the pole will then describe two curves *DLG* and *EMF*, whose lengths added together are equal to the length of the arc of circumference of the radius *DE* and whose amplitude is equal to the angle formed by the extreme tangents *DE* and *GF*. This is proved considering the two triangles *LBl* and *MBm* generated by two infinitely close points *L* and *l* on the curve *DG*, and together with these the triangle *NDn*, whose sides *DN* and *Dn* are equal and parallel to *LM* and *lm*. The three triangles thus constructed are similar, given that the two sides are parallel and the straight lines *LM* and *lm* are perpendicular to the curves *DG* and *EF*, whereas *DN* and *Dn* are perpendicular to the arc of

63 Joh. B. Op. XXVI, pp. 335–338 h. v.



the circumference Nn. Hence we have

$$BL : BM = Ll : Mm$$

and therefore, by composition and inversion:

$$LM : (Ll + Mm) = BL : Ll = DN : Nn$$

because of the similarity of the triangles. Since $LM = DN$, then $Ll + Mm = Nn$.

The arc element of the circumference ENO is thus equal to the sum of the arc elements of the two curves, and hence the length of the arc of circumference is equal to the sum of the lengths of the two curves.

The proof of the theorem is now immediate: given the curve DG, to its every point apply the segment DE perpendicular to the curve. The second extreme E will generate a curve EF, whose length, added to that of the given curve, is equal to the arc of circumference EO.

In much the same period Jacob Bernoulli had obtained an analogous result, which he recorded in his scientific notebook[64].

64 Jac. B. Med. CCXXI, *Werke* 5, p. 283.

Three years later, in October 1698, Bernoulli published a second article in the *Acta Eruditorum*, entitled *Theorema universale rectificationi linearum curvarum inserviens*[65], in which he posed the same problem, postulating that the sum of the lengths of the two curves be equal to that of a segment.

The solution is given without proof: if x and y are the coordinates of the given curve, and if we define a new curve by means of parametric coordinates

$$u = x\frac{dy^3}{dx^3},$$
$$v = \frac{3x}{2}\frac{dy^2}{dx^2} - \tfrac{1}{2}\int\frac{dy^2}{dx},$$

the sum of the lengths of the new and of the given curve is equal to

$$x(1 + y'^2)^{3/2}.$$

To justify this result, Bernoulli said that it was possible to calculate the sum of the arc elements of the two curves and check that it is equal to the differential of the quantity assigned. In fact we have

$$dt = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$$
$$ds = \sqrt{du^2 + dv^2} = y'\sqrt{1 + y'^2}(y' + 3xy'')\,dx$$

and the sum of the two arc elements is equal to the differential of the quantity $x(1 + y'^2)^{3/2}$. Specifically, if the starting curve is the generalised parabola of equation $y = \dfrac{x^n}{n}$, the new curve has parametric equations

$$u = x^{3n-2},$$
$$v = \frac{x^{2n-1}}{2}\left(3 - \frac{1}{2n-1}\right),$$

and hence it too is a generalised parabola, of equation

$$v = Au^{\frac{2n-1}{3n-2}}.$$

65 Joh. B. Op. L, pp. 347–351 h. v.

If then $n = 1 + \frac{1}{2p}$, where p is an integer number, the curve is absolutely rectifiable.

2. *Transformations and rectifications*

The same type of problem is the subject of a text published in the *Opera* under the title *De transformationibus et rectificationibus curvarum*[66], to be found already in Prop. 91–95 of the *120 Propositiones*. This article also tackles the transformation of (algebraic) curves into other (algebraic) curves of the same length.

The method is always the same: write the square $ds^2 = dx^2 + dy^2$ of the element of arc length of the given curve as the sum of the squares which, separately integrated, give a parametric representation of the curve sought. For example, if the given curve has equation $y = f(x)$ we have

$$ds^2 = (1 + f'^2(x))\, dx^2$$

and it is a matter of writing the quantity $1 + f'^2(x)$ as the sum $a^2(x) + b^2(x)$, with $a(x)$ and $b(x)$ integrable.

If $A(x)$ and $B(x)$ are two primitives of a and b respectively, the curve of parametric equations $u = A(x)$, $v = B(x)$ is the one sought.

In the case of a parabola, of equation $y = x^n$ (problem I)), the quantity $1 + n^2x^{2n-2}$ must be written as the sum of two squares. One possibility is to take

$$a(x) = \pm(nx^{n-1} - 1) \quad \text{and} \quad b(x) = \pm\sqrt{2nx^{n-1}},$$

obtaining the parametric equations

$$u = \pm(x^n - x), \quad v = \pm\frac{2}{n+1}\sqrt{2n}x^{\frac{n+1}{2}}.$$

Specifically, if $n = 2$, then $u = \pm(x^2 - x)$, $v = \pm\frac{4}{3}x^{\frac{3}{2}}$ and we will have a curve of equation

$$81v^4 = \pm 432auv^2 + 144a^2v^2 \pm 256au^3,$$

where the unit a has been introduced for dimensional homogeneity.

66 Joh. B. Op. CLXIV, *Opera* IV, pp. 92–98; Ms. 27, *120 Propositiones*, Prop. 91–[95a] 96, pp. 221–229 h. v.

If, instead, the given curve is a hyperbola of equation $y = x^{-n}$ (problem II), we have $1 + f'^2(x) = 1 + n^2x^{-2n-2}$, which breaks down into the sum of two squares $a^2(x)+b^2(x)$, with $a(x) = -nx^{-n-1}+1$ and $b(x) = \sqrt{2nx^{-n-1}}$. Thus the curve sought has parametric equations $u = x^{-n} - x$, $v = \frac{2}{1-n}\sqrt{2n}x^{\frac{1-n}{2}}$.

The transformation does not work in the case of the equilateral hyperbola ($n = 1$), and Bernoulli finds a further decomposition assuming $a = mx^p - x^{-2}$. We then have

$$a^2 = x^{-4} - 2mx^{p-2} + m^2x^{2p} \quad \text{and} \quad b^2 = 2mx^{p-2} - m^2x^{2p} + 1.$$

For this last quantity to be a square it is necessary that $8mx^{p-2} = m^4x^{4p}$, and hence $p - 2 = 4p$ and $8m = m^4$, from which $p = -\frac{2}{3}$ and $m = 2$. We then have

$$a(x) = 2x^{-\frac{2}{3}} - x^{-2} \quad \text{and} \quad b(x) = 1 - 2x^{-\frac{4}{3}}.$$

Integrating, we obtain the parametric equation of the new curve:

$$u = x^{-1} + 6x^{\frac{1}{3}} \quad \text{and} \quad v = x + 6x^{-\frac{1}{3}}.$$

With the same technique we find a curve whose length is equal to the area below a given curve. If the equation of this curve is $y = f(x)$ it will be necessary to split the function $f^2(x)$ into the sum of two integrable squares: $f^2 = a^2 + b^2$. For example (problem III), if $f(x) = \sqrt{1 - x^2}$, we can take

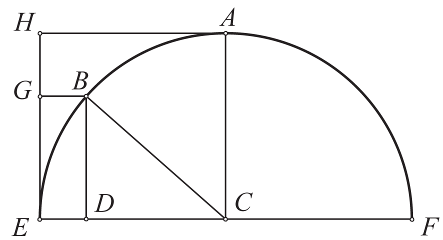
$$a(x) = \pm(1 - mx^2) \quad \text{and} \quad b(x) = x\sqrt{2m - 1 - m^2x^2}.$$

Integrating, the curve sought will have parametric equations:

$$u = \pm\left(x - \frac{m}{3}x^3\right), \quad v = \frac{2m - 1 - m^2x^2}{3m^2}\sqrt{2m - 1 - m^2x^2}.$$

The other two problems tackled are similar to those in the 1698 article; indeed, problem IV had already been solved in this context and in Prop. 94 of the *120 Propositiones*. As for problem V, which was also tackled in the manuscript just cited (Prop. 95), it consists of finding a curve whose arc element, added to that of a circumference, will give a rectifiable curve.

Let a be the radius AC of the circumference and s the arc AB. Thus the area of the sector CAB is $\frac{1}{2}as$, and the problem can be reduced to finding a curve whose length, multiplied by $\frac{1}{2}a$, will give the area of the figure $AHGB$. In fact, in this case the sum of the lengths of the arcs of the circumference of



the curve, multiplied by $\frac{1}{2}a$, gives the area of the rectilinear figure $AHGBCA$. If $x = CD$, the differential of the area $AHGB$ will be equal to $\dfrac{ax - x^2}{\sqrt{a^2 - x^2}}\,dx$, which, divided by $\frac{1}{2}a$, will give the element of the curve sought:

$$ds = \frac{2x - \frac{2x^2}{a^2}}{\sqrt{a^2 - x^2}}\,dx.$$

Finally, ds^2 must be divided into the sum of two squares with integrable roots. Bernoulli breaks down ds^2 into the sum of the squares

$$\text{of} \quad \frac{\frac{mx}{\sqrt{a}} + \frac{nx^2}{\sqrt{a^3}}}{\sqrt{a-x}} \quad \text{and} \quad \text{of} \quad \frac{\frac{px}{\sqrt{a}} + \frac{qnx^2}{\sqrt{a^3}}}{\sqrt{a-x}}.$$

Making the sum of their squares equal to

$$\frac{4x^2 - \frac{8x^3}{a} + \frac{4x^4}{a^2}}{a^2 - x^2},$$

we obtain for the constants m, n, p and q the values

$$n = q = -m = -p = \pm\sqrt{2}.$$

Introducing these values into the preceding formulae and integrating, we find the curve of coordinates

$$\frac{\pm 4a^2 \pm 2ax \mp 6x^2}{15a\sqrt{a}}\sqrt{2a - 2x} \quad \text{and} \quad \frac{\pm 12a^2 \mp 6ax \pm 2x^2}{5a\sqrt{a}}\sqrt{2a + 2x}.$$

An alternative solution is obtained by supposing $ED = x$. In this case the differential of the area $AHGB$ is $\dfrac{x^2 - ax}{\sqrt{2ax - x^2}}$. It is therefore a matter of breaking down the square of this latter quantity divided by $\frac{1}{2}a$ into the sum of two squares, whose roots will be integrable. With similar calculations, Johann Bernoulli finds the curve of coordinates

$$\frac{\mp 8a^2 \mp 2ax \mp 2x^2}{5a\sqrt{a}}\sqrt{4a - 2x} \quad \text{and} \quad \frac{\mp 10ax \pm 6x^2}{15a\sqrt{a}}\sqrt{2x}.$$

3. *A new formula*

Within the framework of the theory of reptorial curves, Bernoulli obtains a new formula for the rectification of curves, published only in the fourth volume of the *Opera*[67]. Bernoulli gives no details of the proof, simply saying that it derives from the nature of the reptorial curve. Given a curve γ and calling its arc element *ds*, we indicate the tangent versor with τ and the normal versor with ν:

$$\tau = \frac{d\mathbf{X}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$$
$$\nu = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right).$$

If r is the radius of curvature, we have $r\,d\tau = \nu\,ds$, and hence

$$d\tau = \frac{ds}{r}\left(\frac{dy}{ds}, -\frac{dx}{ds}\right) = \left(\frac{dy}{r}, -\frac{dx}{r}\right).$$

Then

$$\int_\gamma ds = \int_\gamma \frac{dx^2 + dy^2}{ds} = \int_\gamma \left(\frac{dx}{ds}\,dx + \frac{dy}{ds}\,dy\right) =$$
$$= \int_\gamma (\tau, d\mathbf{X}) = \int_\gamma d(\tau, \mathbf{X}) - \int_\gamma (\mathbf{X}, d\tau) = (\tau, \mathbf{X})|_P^Q + \int_\gamma \frac{y\,dx - x\,dy}{r}$$

where P and Q are the extremes of γ.

67 Joh. B. Op. CLXIII, pp. 433–438 h. v.

Now if the vector τ at the extremes is perpendicular to $\mathbf{X}$ (with an appropriate choice of axes, this happens if the tangents at the extremes of the curve are orthogonal), the length of γ is given by the integral

$$\int_\gamma \frac{y\,dx - x\,dy}{r}.$$

In reality, Bernoulli writes $b - y$ in place of y, and hence

$$L(\gamma) = \int_\gamma \frac{(b-y)\,dx + x\,dy}{r}.$$

The result applies immediately to the elastic curve, whose equation is

$$y = \int_0^x \frac{t^2 dt}{\sqrt{1-t^4}}$$

where x is variable between 0 and 1, and y between 0 and

$$b = \int_0^1 \frac{t^2 dt}{\sqrt{1-t^4}}.$$

For this curve $r = \dfrac{1}{2x}$, and since

$$\int 2x(b-y)\,dx = x^2(b-y) + \int x^2 dy$$

we have, for the length of the portion of the curve from the first extreme till the point (x, y) the expression

$$x^2(b-y) + 3\int_0^x \frac{t^4 dt}{\sqrt{1-t^4}}.$$

For the total length, setting $x = 1$ and $y = b$ in the above formula, we get

$$L(\gamma) = 3\int_0^1 \frac{x^4 dx}{\sqrt{1-x^4}}.$$

On the other hand,

$$ds = \frac{dx}{\sqrt{1-x^4}}$$

and thus we have the relation

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 3\int_0^1 \frac{x^4 dx}{\sqrt{1-x^4}}.$$

More generally, if the equation of the curve is

$$y = \int_0^x \frac{t^n dt}{\sqrt{1-t^{2n}}}$$

its length, given by the new formula, will be

$$L(\gamma) = (n+1)\int_0^1 \frac{x^{2n} dx}{\sqrt{1-x^{2n}}},$$

from which it follows that

$$\int_0^1 \frac{dx}{\sqrt{1-x^{2n}}} = (n+1)\int_0^1 \frac{x^{2n} dx}{\sqrt{1-x^{2n}}}.$$

In the event that the tangents at the extremes of the curve are not orthogonal, a corrective term is necessary. We arrange the axes in such a way that the tangent of the first extreme of the curve will make a right angle with the x-axis, and suppose that the tangent at the other extreme makes an acute angle with the y-axis. In this case the scalar product $(\tau, \mathbf{X})$ does not vanish at point Q, but is equal to the length of the projection of $\mathbf{X}$ on the tangent. Bernoulli calls this length t, and thus

$$L(\gamma) = t + \int_\gamma \frac{(b-y)\,dx + x\,dy}{r}.$$

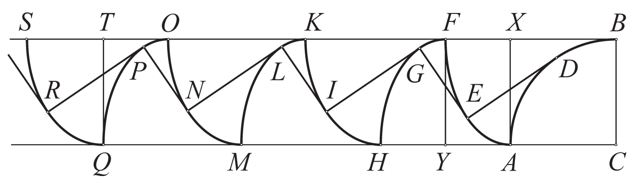
4. *The « Schediasma Cyclometricum»*

Another text which refers to the rectification of curves – but in this case of a particular curve, the circumference – was published in the fourth volume of the *Opera* under the title *De Evolutione successiva et alternante Curvae cujuscunque in infinitum continuata, tandem Cycloidem generante, Schediasma Cyclometricum*[68]. Bernoulli had concerned himself in various circumstances – albeit marginally – with the problem of giving a good approximation of the

68 Joh. B. Op. CLXV, pp. 439–449 h. v.

length of the circumference[69]; here he approaches it from a rather original viewpoint.

The starting point consists in the observation that given any curve, if we trace its involute, and then the involute of the involute starting from the final point, and then again the involute of the second, and so on, always starting from the point of arrival, the succession of curves thus obtained will converge to a cycloid. Moreover, this convergence will be very rapid, so much so that after a small number of repetitions the curve will not be perceptibly distinguishable from the cycloid. Johann Bernoulli offers no justification for this statement, saying merely that «its truth will readily appear to anyone who reflects attentively». On the other hand it is not so much the proof of this result that interests Bernoulli, as its application to the rectification of the circumference.



For this, let the starting curve be a quarter of a circle ADB of radius 1, and let AEF, FGH, HIK, KLM, MNO, OPQ, QRS, etc. be the successive involutes. Starting from point D, trace the tangents DE, EG, GI, IL, LN, LP, etc. which will each prove to be perpendicular to the next. Then if $z = AD$,

$$\frac{1}{dz} = \frac{ED}{d(AE)} = \frac{GE}{d(HG)} = \frac{IG}{d(HI)} = \frac{LI}{d(ML)} = \frac{NL}{d(MN)} = \dots$$

Now, if a marks the length of the circumference ADB and $b, c, e, f, \dots$ the lengths of the odd-numbered curves AEF, HIK, MNO, QRS, ..., because of the properties of the involutes the tangents ED, IG, NL, etc. will be equal to the arcs AD, HG, ML, etc., while the segments GE, LI, PN will be equal to the arcs FE, KI, ON, that is to say to $b - AE$, $c - HI$, $e - MN$. The above relations can then be written

$$\frac{1}{dz} = \frac{z}{d(AE)} = \frac{b - AE}{d(HG)} = \frac{HG}{d(HI)} = \frac{c - HI}{d(ML)} = \frac{ML}{d(MN)} = \dots$$

69 See for instance the letters with Leibniz on the *reptoria* curve, between 1707 and 1709, in Leibniz, *Math. Schriften* 3, pp. 811–844.

or

$$
\begin{aligned}
d(AE) &= z\,dz\\
d(HG) &= (b - AE)\,dz\\
d(HI) &= HG\,dz\\
d(ML) &= (c - HI)\,dz\\
d(MN) &= ML\,dz
\end{aligned}
$$

from which one gets easily

$$
\begin{aligned}
AE &= \frac{z^2}{2}\\
HG &= bz - \frac{z^3}{2\times 3}\\
HI &= \frac{bz^2}{2} - \frac{z^4}{2\times 3\times 4}\\
ML &= cz - \frac{bz^3}{2\times 3} + \frac{z^5}{2\times 3\times 4\times 5}\\
MN &= \frac{cz^2}{2} - \frac{bz^4}{2\times 3\times 4} + \frac{z^6}{2\times 3\times 4\times 5\times 6}\\
QP &= ez - \frac{cz^3}{2\times 3} + \frac{bz^5}{2\times 3\times 4\times 5} - \frac{z^7}{2\times 3\times 4\times 5\times 6\times 7}\\
QR &= \frac{ez^2}{2} - \frac{cz^4}{2\times 3\times 4} + \frac{bz^6}{2\times 3\times 4\times 5\times 6} - \frac{z^8}{2\times 3\times 4\times 5\times 6\times 7\times 8}
\end{aligned}
$$

and so on.

Now if $z = a$, the point D coincides with B and we will have

$$
\begin{aligned}
AEF &= b = \frac{a^2}{2}\\
HGF &= ba - \frac{a^3}{3!} = \frac{2a^3}{3!}\\
HIK &= c = \frac{ba^2}{2} - \frac{a^4}{4!} = \frac{5a^4}{4!}\\
MLK &= ca - \frac{ba^3}{3!} + \frac{a^5}{5!} = \frac{16a^5}{5!}
\end{aligned}
$$

$$MNO = e = \frac{ca^2}{2} - \frac{ba^4}{4!} + \frac{a^6}{6!} = \frac{61a^6}{6!}$$
$$QPO = ea - \frac{ca^3}{3!} + \frac{ba^5}{5!} - \frac{a^7}{7!} = \frac{272a^7}{7!}$$
$$QRS = f = \frac{ea^2}{2} - \frac{ca^4}{4!} + \frac{ba^6}{6!} - \frac{a^8}{8!} = \frac{1385a^8}{8!}$$
etc.

At this point, Bernoulli takes, in place of the curve QRS, a semicycloid, generated by a circumference of diameter ST. From the properties of the cycloid we have $QRS = 2ST$, and $QT = BC = 1$ is the semicircumference of diameter ST. Hence $QT : 2ST = ADB : 2AC = a : 2$ and thus

$$a^9 = \frac{2 \times 8!}{1385} = \frac{16128}{277}.$$

Extracting the ninth root we obtain[70] $a = \frac{\pi}{2} = 1.570805$ from which it follows that $\pi = 3.141610$.

It is even possible to avoid the extraction of the root, since after a certain number of iterations we see that the involutes will be substantially equal. If, for example, $QRS = QPO$, we have

$$\frac{1385a^8}{8!} = \frac{272a^7}{7!}$$

from which it follows that

$$a = \frac{8 \times 272}{1385} = 1.57112$$

which gives, for π, the value 3.14224.

70 Bernoulli gives for the ninth root the value $\frac{135}{86}$, from which he obtains for π the value $\frac{135}{43}$, i.e. about 3.1395, considerably lower than the truth.

IV. The problem challenge of 1703 and the controversy with Craig

The transformation of curves is also the focus of the controversy in which Johann Bernoulli, between 1703 and 1710, was engaged with the English mathematician John Craig. The starting point is a problem which Bernoulli claimed to have received from «a Belgian mathematician[71]» and which he sent, together with six more, to Pierre Varignon and to Gottfried Wilhelm Leibniz on 6 May and 10 June 1702 respectively[72]. As he believed it to be of a certain importance, in January 1703 he begged Varignon to bring it to the attention of mathematicians in the *Journal des Savants*:

> Pour remplir ce vuide, je vous fais part d'un problème tres general que quelque mathematicien de mon voisinage m'a proposé et que j'ay resolu. ... Il me le proposa donc en ces termes generaux *Datam curvam algebraicam id est vulgo geometricam transformare in infinitas alias curvas etiam geometricas sed diversae speciei, singulas scilicet longitudine aequales propositae* ... Vous voyez de quelle importance ce probleme est, d'autant que la solution donne une grande ouverture pour la rectification et la reduction de courbes; car une courbe etant rectifiable, on aura une infinité d'autres de differentes especes qui le seront aussy ... Si vous jugez ce probleme aussy digne qu'il me paroit, je vous prie de le proposer publiquement aux scavans, on leur donne tout le temps de cette année cy. On souhaiteroit leur solution, car on espere d'en profiter pour le bien de la geometrie interieur.[73]

On 12 February 1703 a *Problème à résoudre* was published in the Paris and Amsterdam editions of the *Journal des Savants*, in the following terms:

71 Bernoulli gives no further information about this Belgian mathematician who could be Jacques-François Le Poivre.

72 In Joh. B. Op. LXXIV, pp. 359–373 h. v., Bernoulli states that he solved it shortly after receiving it «ab egregio quodam Belgii mathematico» and dates the solution back to four years before it was actually published, in summer 1705 in the AE. His letter to Varignon from Groningen (6 May 1702) has been lost, but its contents are known from Varignon's answer (Paris 24 May 1702, Joh. B. *Briefe* 2, p. 314) and from later missives to Leibniz (Groningen, 10 June 1702, Leibniz *Math. Schriften* 3, pp. 699, 701–702) and to Varignon (Groningen, 20 January 1703, Joh. B. *Briefe* 3, pp. 54–55). In the attachment to the letter to Leibniz the problem in question is the third of the seven presented there and is formulated as follows: «Les courbes paraboliques et hyperboliques de quelques degrez qu'elles soient transformer en d'autres courbes algebraiques, en sorte que les arcs des unes soient egaux aux arcs des autres», Leibniz, *Math. Schriften* 3, p. 702. Subsequently, as we shall see, the statement was changed into a more general formulation. Johann Bernoulli's first solution on this subject appears in Prop. 85 of the manuscript *120 Propositiones*, Ms. 27, pp. 206–208 h. v.

73 Joh. Bernoulli to P. Varignon, 20 January 1703, Joh. B. *Briefe* 3, p. 55.

> Une courbe algébraïque (vulgairement appellée géométrique) étant donnée, la transformer en une infinité d'autres aussi géométriques, mais d'especes differentes, lesquelles soient chacune de même longueur que la proposée[74].

The possibility cannot be excluded that among Johann Bernoulli's objectives was a new challenge to his brother, as appears from brief references in his correspondence with Varignon,[75] and with Jacob Hermann[76], but Jacob, despite these urgings, decided not to concern himself with it.[77] Some attempts at solution, which proved mistaken or unsatisfactory, and remarks on the difficulties encountered, emerge from the correspondence between Johann Bernoulli and various mathematicians between 1703 and 1705. The commonest flaw in the solutions attempted by Varignon,[78], by Hermann[79] and later by Abraham de Moivre[80] was that they had not ruled out the possibility that the new curve was nothing but the previous one, in a different coordinate system[81].

1. *Leibniz's solution*

The first person to send Bernoulli a correct and appropriate solution was Leibniz, in a letter (in Latin) of 3 January 1704, to which he attached a text in French, containing the same solution ready for publication in the *Journal des Savants*, although it was never, in fact, published[82]. His method is based

74 Joh. B. Op. LXXII, p. 353 h. v.

75 P. Varignon to Joh. Bernoulli, 17 February 1703, Joh. B. *Briefe* 3, p. 56; Joh. Bernoulli to P. Varignon, 20 March 1703, Joh. B. *Briefe* 3, pp. 68–69.

76 Joh. Bernoulli to J. Hermann, 10 March 1703, UB Basel ms. L I a 659, Nr. 2.

77 Joh. Bernoulli to J. Hermann, 26 January 1704, UB Basel ms. L I a 659, Nr. 4; J. Hermann to Joh. Bernoulli, 13 February 1704, UB Basel ms. L I a 659, Nr. 4*; Joh. Bernoulli to J. Hermann, 28 June 1704, UB Basel ms. L I a 659, Nr. 5.

78 P. Varignon to Joh. Bernoulli, 7 May 1703, Joh. B. *Briefe* 3, p. 87.

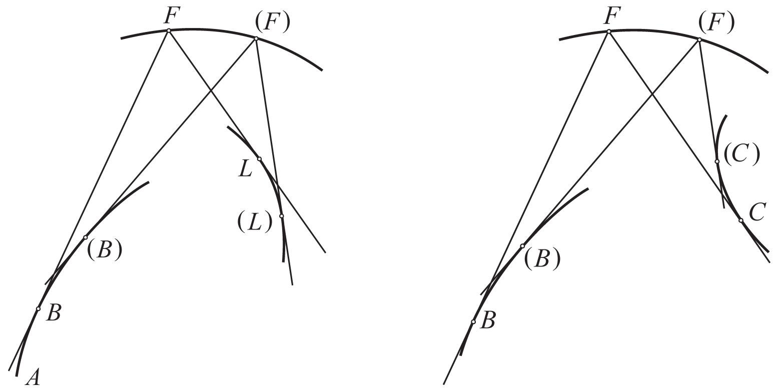
79 J. Hermann to Joh. Bernoulli, 14 November 1703 and 22 December 1703, UB Basel ms. L I a 659, Nr. 2*, 3*.

80 A. de Moivre to Joh. Bernoulli, 27 July 1705, UB Basel L I a 664, Nr. 3*: Wollenschläger pp. 210–212.

81 On the history of this problem in various cultural contexts, cf. H. Krieger, *Konstruction bogengleicher algebraischer Kurven in historischer Sicht*, Centaurus, 16, 1972, pp. 92–162 and C. S. Roero, *Johann Bernoulli's and Leibniz's solutions to the 1703 Problem on the Transformation of Algebraic Curves*, H. Breger, J. Herbst, S. Erdner (eds.), VIII Internationaler Leibniz-Kongress, *Einheit in der Vielheit*, Vorträge 2, Hannover 2006, pp. 848–855.

82 Leibniz, *Math. Schriften* 3, pp. 732–736. The original is preserved in UB Basel ms. L I a 19.2, ff. 215r–216r and an autograph copy in LA Hanover ms. LBr 57, f. 94r–95v.

on the theory of involutes and caustics, which was greatly developed at this time[83].



Leibniz considers three curves: the one given by $B(B)$, a curve $F(F)$ which acts as a mirror, and another, $C(C)$, enveloped by rays of light reflected from $F(F)$. The curves $B(B)$ and $C(C)$ are of the same length and are both involutes of $F(F)$ (this is why Leibniz also calls them «co-evolutes») and if this datum is algebraic, the caustic $C(C)$ will be algebraic too. A specific example of application of the method, in which the mirror curve is an ellipse, is offered by Bernoulli in his letter of 27 December 1705[84].

2. *Craig's first solution*

A year later a solution by John Craig, in a handful of lines, appeared in the *Philosophical Transactions*[85]. Craig reduces the problem to finding a new curve whose arc element is equal to the given curve. If v and s are the coordinates of the given curve, and x, y those of the new curve, then

$$dv^2 + ds^2 = dx^2 + dy^2.$$

83 G. F. de l'Hôpital, *Analyse des infiniment petits*, Sections V–VII, Paris 1696, pp. 71–130.

84 Leibniz, *Math. Schriften* 3, pp. 778–781. The original is preserved in UB Basel ms. L I a 19.2, ff. 239r–240r and an autograph copy in LA Hanover ms. LBr 57, f. 147v. Johann Bernoulli told Hermann of this solution by Leibniz, and Hermann passed on the information to the Italian mathematician Guido Grandi (cf. S. Mazzone, C. S. Roero, *Guido Grandi – Jacob Hermann, Carteggio (1708–1714)*, Firenze, Olschki, 1992, pp. 173–175) who was to make use of it in his book *Risposta apologetica* . . . , Lucca 1712, p. 282.

85 {J. Craig} Op. 73, pp. 355–358 h. v.

Let $dx = dv - m\,dz$ we have

$$dy = \sqrt{ds^2 + 2m\,dv\,dz - m^2 dz^2}.$$

If in this last equation ds is replaced with its expression in function of v and dv, obtained from the equation of the curve, and if dz (expressed in terms of v and dv) is chosen in such a way that dx and dy be algrebraically integrable, the functions $x(v)$ and $y(v)$ thus determined are the coordinates of the curve required.

In support of his solution, Craig gives four very simple examples: when the given curve is respectively a parabola, a circumference, an ellipse and a cubic parabola.

3. *Bernoulli's reaction and his solution with «reptorius» motion*

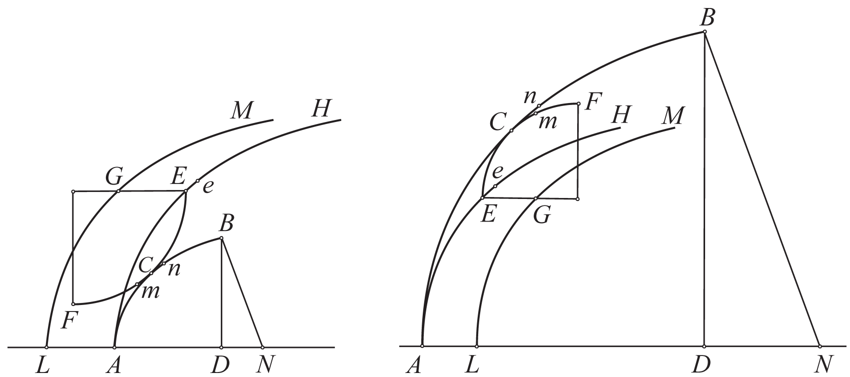
Johann Bernoulli's answer was to arrive more than a year later, and was to be published in the August 1705 issue of the *Acta Eruditorum*[86]. The first pages are devoted to a criticism of Craig's supposed universal method, which, says Bernoulli, reduces the problem to one no less difficult, namely how to choose dz, in such a way that dx and dy be integrable. It is true that the method worked in the examples, but it is as though someone who had found the solution of certain second- and third-degree equations, claimed to have found the general solution of all algebraic equations. On the other hand, the problem Craig tackled is more difficult than the one posed, which actually asked only that the two curves be of the same length, whereas Craig demands that the single arc elements be equal[87].

Once the criticism of Craig's article has been completed and a glancing reference has been made to Leibniz's method with the co-evolutes, Bernoulli turns to expounding his own solution, for which he introduces a new method of generation of curves by *motus reptorius*[88].

86 Joh. B. Op. LXXIV, pp. 359–373 h. v.

87 Bernoulli made harsh criticisms of Craig's solution in his letters to de Moivre (15 November 1704, UB Basel L I a 664, Nr. 1: Wollenschläger pp. 186–187; 11 July 1705, UB Basel L I a 664, Nr. 2: Wollenschläger p. 200), to Varignon (13 June 1705, Joh. B. *Briefe* 3, p. 159) and to Leibniz (25 July 1705, Leibniz *Math. Schriften* 3, p. 769).

88 On this motion, see Eugene Prouhet, *Sur le courbes engendrées par le mouvement de reptation, pour servir d'eclaircissement à plusieurs passages des Oeuvres de Jean Bernoulli* and *Sur les reptoires*, Nouvelles Annales de Mathématiques, 1854, pp. 274–282 and 335–348, to which we refer for a more thorough mathematical analysis.



Given two curves, one fixed and the other mobile, the latter is made to run always parallel to itself, in such a way that it be always tangent to the fixed curve. The definition of the motion is split in two, according to whether the mobile curve touches with its convexity the convex part of the fixed curve, as in the first figure (*motus obreptionis*) or its concave part as in the second figure (*motus subreptionis*). A point on the plane of the mobile curve will describe a new curve, which is called *reptoria*.

We see immediately that two different points describe congruent *reptoriae* curves (Lemma II). Bernoulli proves that if the generating curves are algebraic, the *reptoria* curve too will be algebraic (Lemma I); its length is equal to the sum (in the case of *motus obreptionis*) or the difference (in the case of *motus subreptionis*) of the lengths of the generating curves (Theorem I).

The proof comes from the definition of the *motus reptorius*.

Let ACB be the fixed curve and FCE the mobile curve, and let E be the point which describes the *reptoria* by *motus obreptionis*. Let Cm be an infinitesimal element of the curve FCE, and let n be the point on ACB, infinitely close to C, in which the tangent will be parallel to the tangent in m to FCE. When point m goes into n, point E will be moved to e and will have travelled an infinitesimal arc Ee parallel to mCn. Thus $Ee = mn = mC + Cn$; the arc element of the *reptoria* is equal to the sum of the arc elements of the generating curves. If now the curves are of equal width, i.e. if the angles formed by the normals to the curves at their endpoints are equal, they will simultaneously absorb each other in the *motus reptorius* and hence the total length of the *reptoria* is equal to the sum of the lengths of the curves.

The *reptoria* solves the problem posed. In fact, given an algebraic curve, divide it into two parts of equal width which move one on the other in *motus reptorius*. The resulting curve will be algebraic and its length will be the sum of the two, i.e. equal to the length of the given curve. The same procedure can be applied to this new curve, obtaining a second, then a third and so on; hence infinite curves all of the same length as the given curve. These curves are all different, except in one case, when the initial curve is an arc of circumference. For this case Bernoulli finds a particular solution, from which with the previous procedure infinite curves of the same length are obtained.

Bernoulli's article did not bring the controversy with Craig to an end, for Craig was to take up the question again, as we shall see, some years later.

Meanwhile, *motus reptorius* was the subject, at the beginning of 1707 and of 1709, of an exchange of letters between Johann Bernoulli and Leibniz, partially published in the *Miscellanea Berolinensia*[89]. In the first letter, dated 15 January 1707, under the title *Inventa de Appropinquationibus promtis ad metiendas figuras per Motus repentis considerationem exhibitis*, Johann examines the case in which the curves of the *motus reptorius* are two equal ellipses, but one rotated by 90° with respect to the other, in such a way that the semi-axes be exchanged.

Bernoulli begins his letter by finding the equation of the *reptoria* curve. Calling a and b the semi-axes of the immobile ellipse, the semi-axes of the mobile ellipse will then be b and a, and the two ellipses, as regards their centres, will have respectively the equations[90]

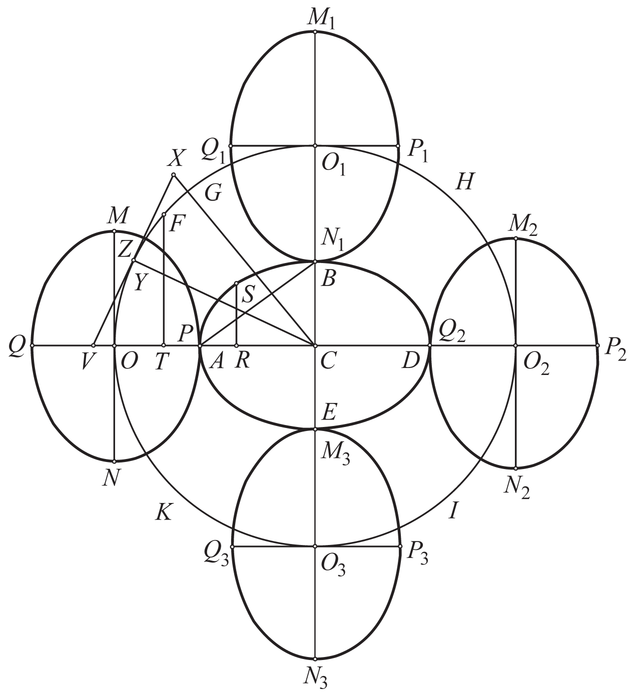
$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{b^2} + \frac{z^2}{a^2} = 1.$$

Now let $S = (x, z)$ be a point of the fixed ellipse E. The angular coefficient of the line tangent in S is

$$t = -\frac{x}{z}\frac{b^2}{a^2}.$$

89 Joh. B. Op. LXXVII, pp. 375–381 h. v., which is an extract from Joh. Bernoulli's letter to Leibniz, 15 January 1707, Leibniz *Math. Schriften* 3, pp. 803–809, the original of which is held in LA Hanover, ms. LBr. 57, ff. 165r–167r; Leibniz Op. 78, p. 383 h. v., which is an extract from Leibniz's letter to Joh. Bernoulli, 1 February 1707, Leibniz *Math. Schriften* 3, pp. 811–812, the original of which is held in UB Basel ms. L I a 19,2 ff. 250–251; Joh. B. Op. LXXIX, pp. 385–387 h. v., extract from Joh. Bernoulli's letter to Leibniz, 23 March 1707, Leibniz *Math. Schriften* 3, pp. 812–814, an autograph copy of which is held in UB Basel ms. L I a 18, ff. 166–167; Joh. B. Op. LXXX, pp. 429–430 h. v., extract from Joh. Bernoulli's letter to Leibniz, 15 April 1709, Leibniz *Math. Schriften* 3, pp. 842–843.

90 We use the variable z because, as we shall see, Bernoulli uses the symbol y for another magnitude.



Similarly, the angular coefficient of the tangent to the mobile ellipse E_1 at the point $S_1 = (x_1, z_1)$ is

$$t_1 = -\frac{x_1}{z_1}\frac{a^2}{b^2}.$$

In *motus reptorius*, the points S and S_1 will touch if the two coefficients t and t_1 are equal, i.e. if

$$z_1 = \frac{a^4 z}{b^4 x} x_1.$$

The values of x_1 and z_1 are found by making the point S_1 be on E_1, i.e. $\frac{x_1^2}{b^2} + \frac{z_1^2}{a^2} = 1$. We find

$$x_1 = \frac{b^4 x}{\sqrt{b^6 x^2 + a^6 z^2}},$$

$$z_1 = \frac{a^4 z}{\sqrt{b^6 x^2 + a^6 z^2}}.$$

At this point it is not difficult to find the equation of the serpentine curve described by the centre of the mobile ellipse. In fact $X = x + x_1$ and $Z = z + z_1$, so the generic point of the *reptoria* has coordinates

$$X = x + \frac{b^4 x}{\sqrt{b^6 x^2 + a^6 z^2}},$$
$$Z = z + \frac{a^4 z}{\sqrt{b^6 x^2 + a^6 z^2}},$$

or, writing with Bernoulli, $y = \dfrac{a}{b} z$,

$$X = x + \frac{b^3 x}{\sqrt{b^4 x^2 + a^4 y^2}},$$
$$Z = \frac{b}{a} y + \frac{a^3 y}{\sqrt{b^4 x^2 + a^4 y^2}}.$$

The new curve, he says, is contained between two circumferences of radius $a + b$ and $\sqrt{2a^2 + 2b^2}$, and is tangent to the first at the points $(0, \pm(a + b))$, $(\pm(a + b), 0)$ and to the second at four intermediate points between these two. As a result, its length, equal to twice the length of the ellipse of semi-axes a and b, is included between those of the circumferences found. Specifically, if the ellipse has semi-axes 4 and 5, the radii of the circumferences are 9 and $\sqrt{82}$, which differ by less than one part in 160, so that the length of the ellipse is included between those of the circumferences of radii $\frac{9}{2}$ and $\frac{1}{2}\sqrt{82}$.

Bernoulli does not prove these statements, which can be checked by considering the maximum and minimum of the distance of the points of the curve from the origin, or in other words the function

$$f(x) = x^2 \left(1 + \frac{b^3}{\sqrt{b^4 x^2 + a^4 (a^2 - x^2)}}\right)^2 + (a^2 - x^2) \left(\frac{b}{a} + \frac{a^3}{\sqrt{b^4 x^2 + a^4 (a^2 - x^2)}}\right)^2$$

which is the square of the distance. If $v^2 = b^4x^2 + a^4(a^2 - x^2)$, then

$$f(x) = \frac{(a^6 - v^2)\left(1 + \dfrac{b^3}{v}\right)^2 + (v^2 - a^2b^4)\left(\dfrac{b}{a} + \dfrac{a^3}{v}\right)^2}{a^4 - b^4} = \frac{g(v)}{a^4 - b^4},$$

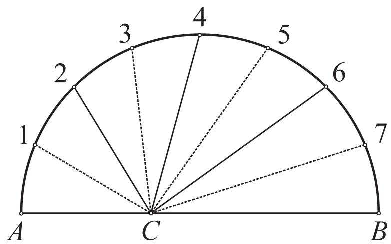
and it is a matter of finding the maximum and minimum of the function $g(v)$ in the interval $ab^2 \le v \le a^3$.

Setting the first derivative equal to zero and simplifying, we arrive at the equation

$$v^4 - a^2bv^3 + a^6b^3v - a^8b^4 = 0,$$

which in the interval being considered has a single solution $v = a^2b$. The value of the function $f(x)$ at this point is $2(a^2 + b^2)$, while at the extremes of the interval the function has the value $(a + b)^2$. Hence we have in conclusion a curve with four «humps», one in each quadrant, and four «troughs» at the intersection with the axes. At this point Bernoulli constructs a new *reptoria* moving on itself the previous one, or rather the homothetic curve of coefficient $\frac{1}{2}$ in order to preserve its length, after aligning a hump with a trough. He thus obtains a curve with eight humps, lying between two circumferences which are even closer than the previous ones. Continuing this process, curves are obtained with 16, 32, etc. humps, of a length equal to a multiple of that of the ellipse, and which approximate ever more closely to the circumference. In the preceding example, the second *reptoria* is between the circumferences of radii $\frac{9}{2} + \frac{1}{2}\sqrt{82}$ and $\sqrt{41 + \frac{1}{2}\sqrt{6562}}$; the third between those of radii $\frac{9}{4} + \frac{1}{4}\sqrt{82} + \frac{1}{2}\sqrt{41 + \frac{1}{2}\sqrt{6562}}$ and $\sqrt{41 + \frac{1}{2}\sqrt{6562}}$, which differ by less than one in 56000.

Leibniz's answer was not slow to arrive, and on 1 February 1707 he suggested to begin, rather than from an ellipse, from a rectifiable curve, for example from an epicycloid. In this way all the subsequent *reptoriae* curves would be rectifiable, and it would be possible to obtain excellent approximations of the circumference. On 23 March, Bernoulli replied that in place of the ellipse any closed curve could be taken, for instance one formed by two arcs of a parabola. Indeed, it is not even necessary to start from a closed curve; any arc of curve can be the starting point, as long as in the initial configuration the two (equal) curves are tangent to each other. In the case, the subsequent *reptoriae* will approximate an arc of a circle of the same amplitude. Finally, coming back to the *reptoria* generated by an ellipse, Bernoulli remarks that if this has an infinitely small semi-axis, i.e. if it is reduced to a segment counted twice, the subsequent *reptoriae* will be regular polygons, of 4, 8, 16, etc. sides respectively.



Two years later, in a letter dated 15 April 1709,[91] Bernoulli explicitly calculated the radii of the circle circumscribed to and inscribed in the *reptoria* with 2^n humps, generated starting from the ellipse. To this end he divided in 2^n equal parts the circumference of diameter AB, equal to the sum $AC + CB$ of the semi-axes of the ellipse, and draw the segments $C_1, C_2, \ldots, C_{2^n-1}$ connecting the point C to the points of division. The arithmetic mean of the segments in odd places $(C_1 + C_3 + \ldots + C_{2^n-1}) : 2^{n-1}$ is the radius of the circle circumscribed to, whereas the arithmetic mean of the segments in even places and of the radius R of the circumference, $(C_2 + C_4 + \ldots + C_{2^n-2} + R) : 2^{n-1}$ is the radius of the circle inscribed in the *reptoria* with 2^{n-1} humps, whose length is equal to that of the ellipse.

4. *Craig's second solution*

After the criticisms made in 1705 in the *Acta Eruditorum* Craig prepared his response, which was published in the March–April 1708 issue of the *Philosophical Transactions* and republished in 1710 in the Leipzig journal[92]. It was a question of rebutting the statement made by Bernoulli, who had stressed the difficulty of writing the square of the arc element $dz^2 + ds^2$ of the given curve as $dx^2 + dy^2$, the sum of the squares of the differentials of the unknown curve, subject to the condition that dx and dy be integrable. Craig remarks that it is a trivial problem: recalling with Diophantus that the integers $m^2 + n^2$, $m^2 - n^2$ and $2mn$ form a Pythagorean triple, setting

$$dx = \frac{(m^2 - n^2)dz + 2mn\,ds}{m^2 + n^2}, \quad dy = \frac{(n^2 - m^2)ds + 2mn\,dz}{m^2 + n^2}$$

91 Joh. B. Op. LXXX, pp. 429–430 h. v.

92 {J. Craig} Op. 81, pp. 389–391 h. v.

then

$$dx^2 + dy^2 = dz^2 + ds^2.$$

Hence if z and s are the coordinates of the given curve,

$$x = \frac{(m^2 - n^2)z + 2mns}{m^2 + n^2} \quad \text{and} \quad y = \frac{(n^2 - m^2)s + 2mnz}{m^2 + n^2}$$

will be the coordinates of a new curve, which obviously will be of the same length as the given curve.

Having solved the problem in such a surprisingly simple fashion, Craig went on to criticise Bernoulli's solution, which he felt must be regarded as a mechanical solution, since it used motion.

5. *Bernoulli's reply and Craig's retraction*

Bernoulli's reply is contained in a letter to William Burnet dated 9 January 1709, an extract from which was to be published the following year in the first volume of the *Miscellanea Berolinensia*[93]. Believing that he had found a new curve of the same length as the given curve, Craig did not realise, Bernoulli pointed out, that it was a matter of the same curve, written in a different system of coordinates. He admitted that he himself had at first fallen into the same misconception, but added that he had realised it immediately, as indeed is clear from the footnote to Prop. 85 of the manuscript of the *120 Propositiones*. The same thing had happened to Hermann and to De Moivre, who – fortunately for them – had told Johann Bernoulli before publishing the solution, and thus had been advised of the error. Two years later, in an article in the *Philosophical Transactions*[94], Craig was to admit his mistake and recognise the correctness of Bernoulli's solution. The controversy ended without bloodshed. However, at the height of the dispute between Leibniz and Newton over the priority of the invention of infinitesimal calculus, in a letter to Leibniz Bernoulli came back to this problem, stressing that the English had not been able to solve it:

> ... sicuti Cheynaeus quondam inepte iactavit, nihil nempe intra hos 20 vel 30 annos prodiisse in lucem, quae non sint iteratae repetitiones vel ad summum levia tantum corollaria eorum, quae Newtonus iam pridem invenerit, quasi nobis amplius relictum fuisse, vel nullius esse pretii, quod

93 Joh. B. Op. LXXXII, pp. 427–428 h. v.

94 {J. Craig} Op. 83, p. 431 h. v.

subinde a nobis nihil amplius relictum fuisse, vel nullius esse pretii, quod subinde a nobis publicatum extat, et cuius in Newtonianis ne vestigium quidem videre est: qualia sunt quae de Catenariis, Velariis, Isochronis paracentricis, Brachystochronis, de novis proprietatibus Cycloidis, de eius segmentis innumeris quadrabilibus, de Calculo exponentialium seu percurrentium eosque differentiandi modo, de Coevolutarum dimensione, de Motu traptorio, de reptorio, de Curvarum reductione ad circulares, de earum transformatione, et de innumeris aliis, quae Angli pro parte tentarunt, sed omni suo calculo fluxionum adiuti irresoluta reliquerunt, quod vel ex solo problemate Catenariae et Curvarum transformandarum patet, cui pertinaciter et longo tempore insudantes, aliud nihil quam turpes paralogismos produxerunt[95].

V. **Geodesics**

The problem of the equation of lines of minimum length on a given surface appears for the first time in Johann Bernoulli's writings in the form of a problem he proposed in the *Journal des Savants*[96] on 26 August 1697. It was the first of a group of six problems brought to the attention of scholars, limited to the surfaces of conoids or spheroids. The following year, his brother Jacob gave a solution in the *Acta Eruditorum*[97], and Johann commented on this solution a few months later[98]. Further references to the problem can be found in the correspondence with Leibniz, where Johann notes the fundamental property of geodesics, namely that the plane osculatory to the curve is orthogonal to the surface[99]. Although in his letter to Leibniz he claims to have found the general equation of geodesics, the first surviving writing on this subject is from the last months of 1728, when Johann sent the solution to the Swedish mathematician Samuel Klingenstjerna (1698–1765). The latter wrote a review which was to be inserted in the fourth volume of Johann's *Opere*, the manuscript of which is preserved in UB Basel, I a 12.4, ff. 256r–265r. Almost simultaneously, the

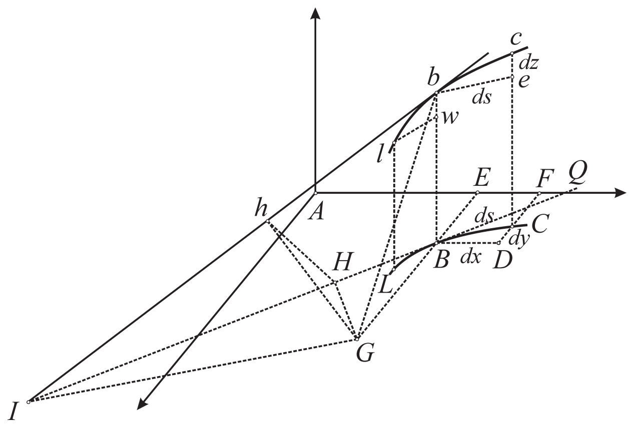
95 Joh. Bernoulli to Leibniz, 29 July 1713, Leibniz, *Math. Schriften* 3, p. 916.

96 Joh. B. Op. XXXIX, *Opera* I, pp. 204–205 – *Streitschriften*, pp. 292–293. Cf. Joh. B. *Briefe* 2, pp. 120–121.

97 Jac. B., *Solutio sex problematum fraternorum in Ephem. Gall. 26 Aug. 1697 propositorum*, AE Maij 1698, pp. 226–230, *Opera*, pp. 796–806 – *Streitschriften*, pp. 332–336. Cf. also *Streitschriften*, Introduction, pp. 87–94.

98 Joh. B. Op. LII, *Annotata in solutiones fraternas problematum quorundam suorum ...*, AE Octobris 1698, pp. 466–474 – *Streitschriften*, pp. 383–392.

99 Joh. Bernoulli to Leibniz, 16/26 August 1698, Leibniz *Math. Schriften* 3, p. 532.



problem was put to Leonhard Euler by way of Daniel Bernoulli, who was living in St Petersburg[100] at the time. We shall see Euler's solution in due course; for the moment, let us begin with Bernoulli's.

1. *Johann Bernoulli's solution*

As we have said, the starting point is the orthogonality between the plane tangent to the surface and the plane osculating the curve, the latter being determined by three infinitely close points l, b and c on the curve.

If α and β are the angles that the osculating plane π and the tangent plane bGI make with the vertical plane, we have $\alpha + \beta = \dfrac{\pi}{2}$ and hence

$$\tan\alpha \tan\beta = 1.$$

It is then a matter of expressing this last equation in terms of the coordinates and their differentials. Let L, B and C be the projections of the points l, b and c on the plane xy. We put $BD = EF = dx$, $DC = dy$, $ce = dz$, and $LB = BC = be = ds = \sqrt{dx^2 + dy^2} = \text{constant}$[101].

The calculation of $\tan\beta$ is relatively easier, and is tackled first. To identify the plane tangent to the surface, Bernoulli takes the straight lines bI tangent

100 Joh. Bernoulli's letter to his son Daniel of 10/21 May 1728 on this subject has been lost, but reference is made to it by L. Euler writing to Joh. Bernoulli, 18 February 1729, Euler *Opera* IV/2, p. 92.

101 This is equivalent to taking as independent variable the arc length of the projection of the curve on the horizontal plane xy.

to the geodesic and bG tangent to the curve obtained by intersecting the given surface with the plane[102] $x = x_B$.

The straight line IB, projection of bI on the horizontal plane, is tangent to the projection curve LBC. Drawing GH perpendicular to IB, from the similarity between the triangles[103] CBD and BGH, it follows that $BC : BD = BG : HG$ and $BC : CD = BG : HB$, so that $HG = T\dfrac{dx}{ds}$ and $HB = T\dfrac{dy}{ds}$.

The triangles ceb and bBI are also similar, hence $ce : eb = bB : BI$, from which $BI = z\dfrac{ds}{dz}$. Consequently, $HI = BI - HB = z\dfrac{ds}{dz} - T\dfrac{dy}{ds}$.

The triangles bce and IHh are also similar, since $\widehat{HhI}$ and $\widehat{bec}$ are right angles and $\widehat{hIH} = \widehat{cbe}$. It follows that $bc : ce = IH : Hh$, and since $bc = \sqrt{ds^2 + dz^2}$, we have

$$Hh = \left(z\frac{ds}{dz} - T\frac{dy}{ds}\right)\frac{dz}{\sqrt{ds^2 + dz^2}} = \frac{z\,ds^2 - T\,dy\,dz}{ds\sqrt{ds^2 + dz^2}}.$$

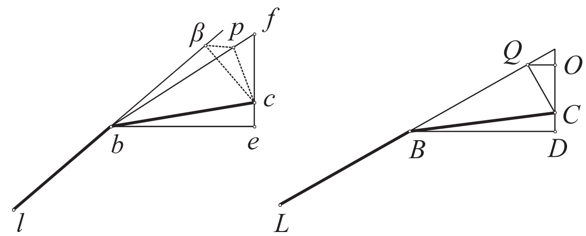
Finally, since GH is perpendicular to the plane IBb and Hh is perpendicular to bI, intersection of the planes IBb and IGb, the triangle HhG is in a plane orthogonal to both IBb and to IGb, with the result that $\widehat{GhH} = \beta$. As $\widehat{GHh}$ is a right angle, we have

$$\tan\beta = \frac{HG}{hH} = T\frac{dx\sqrt{ds^2 + dz^2}}{z\,ds^2 - T\,dy\,dz}. \tag{1}$$

We now come to the calculation of $\tan\alpha$. The first equation is obtained considering the vertical planes which pass through the two arc elements lb and bc. Let $lw = be = ds$ be the horizontal projections of lb and cb, and let $bw = dz$, $ce = dz'$. Let $ef = bw = dz$ and on the straight line bf mark $bp = bc$. Then $fc = ef - ce = dz - dz' = -d^2z$.

102 If the equation of the surface is $z = f(x, y)$, the subtangent will be given by $T = BG = \dfrac{f(x_B, y_B)}{f_y(x_B, y_B)}$.

103 As was usual at the time, Bernoulli identifies the infinitesimal arc of the curve with the corresponding infinitesimal part of the tangent.



The triangles bce and cfp are similar; in fact $\widehat{bcp}$ and $\widehat{bpc}$ are right angles (the angle $\widehat{pbc}$ is infinitesimal), hence $\widehat{fcp} = \widehat{cbe}$. Therefore $bc : be = fc : cp$, from which

$$cp = -\frac{ds\, d^2 z}{\sqrt{ds^2 + dz^2}}.$$

Now let us compare the configuration on the curve with the configuration on the projection on the plane xy.

Let $BQ = BC$ and let QO be perpendicular to DC. Correspondingly, let[104] $b\beta = bc = bp$. The straight line $b\beta$ has the same inclination as bp, and thus the segment $p\beta$ is horizontal, and is projected on to BQ. The triangles BCD and OQC are similar; in fact, $\widehat{BCQ}$ and $\widehat{QOC}$ are right angles, and therefore $\widehat{QCO} = \widehat{OBD}$. Thus we have $BD : BC = OC : QC$, and as $OC = OD - CD = BW - CD = d^2 y$, then

$$QC = p\beta = \frac{d^2 y\, ds}{dx}.$$

Now note that both cp and $c\beta$ are orthogonal to bc, and thus the triangle $cp\beta$ rests on a plane orthogonal to both the osculating plane and the vertical plane IBb. Since $\widehat{cp\beta}$ is a right angle, we have

$$\tan\alpha = \frac{p\beta}{cp} = -\frac{d^2 y\,\sqrt{ds^2 + dz^2}}{dx\, d^2 z}. \tag{2}$$

From the equation $\tan\alpha\ \tan\beta = 1$, it follows that

$$-\frac{T\, dx\,\sqrt{ds^2 + dz^2}}{z\, ds^2 - T\, dy\, dz}\, d^2 y\, \frac{\sqrt{ds^2 + dz^2}}{dx\, d^2 z} = 1$$

104 *Nota bene*: $b\beta$ is on the plane *lbw*, while bp is on the plane *bce*.

and in conclusion

$$T(ds^2 + dz^2)\, d^2y = (T\, dy\, dz - z\, ds^2)\, d^2z. \tag{3}$$

Thus far, the manuscript written by Klingenstjerna. Later, having learned of Euler's results on the same topic, Bernoulli added a number of comments and specifications[105].

2. *Euler's solution*

Euler's complete solution was published in the 1728 volume of the *Commentarii Academiae Scientiarum Imperialis Petropolitanae*[106] which came out in 1732; but he told Johann Bernoulli of it – though omitting the proofs – in his letter of 18 February 1729. Here Euler starts from the equation of the surface written in differential form:

$$P\, dy = Q\, dz + R\, dx$$

where he assumes x as independent variable (i.e. dx = constant) in order to obtain the equation of the geodesics[107]

$$\frac{Q\, d^2y + P\, d^2z}{Q\, dy + P\, dz} = \frac{dy\, d^2y + dz\, d^2z}{dx^2 + dy^2 + dz^2}.$$

Starting from this equation, Euler discusses certain specific cases in this letter. First of all, if the given surface is a cylinder over a curve whose equation is

$$P\, dy = Q\, dz,$$

the equation of the geodesics becomes

$$\frac{dy\, d^2y + dz\, d^2z}{dy^2 + dz^2} = \frac{dy\, d^2y + dz\, d^2z}{dx^2 + dy^2 + dz^2}$$

105 These repeated interventions made themselves felt: Bernoulli several times changed notations, creating no small difficulty for the reader.

106 L. Euler, E. 9, *De linea brevissima in superficie quacumque duo quaelibet puncta jungente*, CP III 1728 (1732), pp. 110–120 – Euler *Opera* I/25, pp. 1–12.

107 L. Euler to Joh. Bernoulli, 18 February 1729, Euler *Opera* IV/2, pp. 92–94. Note that Euler writes t, x and y instead of x, y and z. We have changed the notations here to render them uniform with those Bernoulli had used previously.

from which we derive $dy\, d^2y + dz\, d^2z = 0$, or in other terms

$$dy^2 + dz^2 = \text{cost.} = n^2 dx^2.$$

If instead the surface concerned is a surface of revolution around the x-axis, its equation is

$$y\, dy = -z\, dz + R\, dx$$

and therefore

$$\frac{y\, d^2y - z\, d^2z}{y\, dy - z\, dz} = \frac{dy\, d^2y + dz\, d^2z}{dx^2 + dy^2 + dz^2}.$$

Remembering that $d^2x = 0$, this equation can be integrated, obtaining

$$\log(y\, dz - z\, dy) = \log\sqrt{dx^2 + dy^2 + dz^2} + c$$

and hence[108]

$$y\, dz - z\, dy = a\sqrt{dx^2 + dy^2 + dz^2}.$$

If we set $w^2 = y^2 + z^2$ and $du^2 = dx^2 + dy^2 + dz^2$, the previous equation can be written in the more concise form

$$du = \frac{w\sqrt{dx^2 + dw^2}}{\sqrt{w^2 - a^2}}.$$

Specifically, for the sphere we have $w^2 + x^2 = b^2$, and hence

$$du = \frac{b\, dx}{\sqrt{b^2 - a^2 - x^2}}$$

from which Euler claims that he can deduce that the geodesics are great circles.

Finally, if the surface is a cone, its equation is of the kind

$$\frac{x}{y} = F(y, z)$$

with F a homogeneous function of degree 0. Thus we have

$$y\, dx - x\, dy = y^2 dF = y^2(M\, dy + N\, dz)$$

108 Euler omits the integration constant c, but still obtains the factor a in the subsequent equation.

which, compared with the general equation $P\,dy = Q\,dz + R\,dx$, gives $P = My^2 + x$ and $Q = -Ny^2$.

On the other hand, as F is homogeneous, $yM + zN = 0$, which together with the preceding equation supplies the values of M and N:

$$M = \frac{z(y\,dx - x\,dy)}{y^2(z\,dy - y\,dz)}, \quad N = -\frac{y(y\,dx - x\,dy)}{y^2(z\,dy - y\,dz)}$$

and hence those of P and Q:

$$P = \frac{zy\,dx - xy\,dz}{z\,dy - y\,dz}, \quad Q = \frac{y^2 dx - xy\,dy}{z\,dy - y\,dz}.$$

Finally, we obtain the equation

$$\frac{z\,dx\,d^2z - x\,dz\,d^2z + y\,dx\,d^2y - x\,dy\,d^2y}{z\,dx\,dz - x\,dz^2 + y\,dx\,dy - x\,dy^2} = \frac{dy\,d^2y + dz\,d^2z}{dx^2 + dy^2 + dz^2}.$$

This last equation can be integrated, taking $dx^2 + dy^2 + dz^2 = ds^2$ and $x^2 + y^2 + z^2 = w^2$. Remembering that $d^2x = 0$, the right-hand side is equal to $\dfrac{d^2s}{ds}$, while as

$$y\,dy + z\,dz = w\,dw - x\,dx \quad \text{and} \quad dy^2 + dz^2 = ds^2 - dx^2,$$

the denominator of the left-hand side can be written as

$$w\,dw\,dx - x\,ds^2.$$

As for the numerator, we have

$$\begin{aligned} w\,d^2w + dw^2 - ds^2 &= d(w\,dw) - ds^2 = \\ &= d(x\,dx + y\,dy + z\,dz) - dx^2 - dy^2 - dz^2 = \\ &= y\,d^2y + z\,d^2z \end{aligned}$$

and hence in conclusion the equation of the geodesics on a conic surface becomes

$$\frac{w\,dx\,d^2w + dx\,dw^2 - dx\,ds^2 - x\,ds\,d^2s}{s\,dw\,dx - x\,ds^2} = \frac{d^2s}{ds}$$

or in other terms

$$ds = \frac{(w\,d^2w + dw^2)\,ds - w\,dw\,d^2s}{ds^2} = d\,\frac{w\,dw}{ds}.$$

This last equation admits an immediate integral:

$$s = \frac{w\,dw}{ds},$$

from which

$$s^2 = w^2 + c = x^2 + y^2 + z^2 + c.$$

This last equation can be used in single cases to obtain the properties of the geodesics.

3. *Bernoulli's remarks*

At the time of publication of the work in the fourth volume of the *Opera*, Johann Bernoulli added a series of observations which specified and simplified what was to be found in Klingenstjerna's manuscript.

In the first place, Bernoulli comments that it would have been possible to cut the surface with the plane $y = y_B$. In this case the resulting equation is obtained from (3) putting x in place of y and writing θ in place of T:

$$\theta(ds^2 + dz^2)\,d^2x = (\theta\,dx\,dz - z\,ds^2)\,d^2z. \tag{4}$$

If we now take $ds =$ constant (i.e. if we parameterise according to the length of the arc of the projection curve on the plane xy), then $dx\,d^2x + dy\,d^2y = 0$, and hence multiplying (4) by $-T\,dx : dy$ we obtain:

$$T\theta(ds^2 + dz^2)\,d^2y = \left(-\theta T\frac{dz\,dx^2}{dy} + Tz\frac{ds^2dx}{dy}\right)d^2z.$$

On the other hand, for any surface the equation

$$\frac{\theta\,dy + T\,dx}{T\theta} = \frac{dz}{z}$$

is valid, hence $T\theta\, dz = z(\theta\, dy + T\, dx)$, which, when introduced into the preceding equation, gives

$$T\theta(ds^2 + dz^2)\, d^2y = \left(-z\theta\, dx^2 - Tz\frac{dx^3}{dy} + Tz\frac{ds^2 dx}{dy}\right) d^2z =$$
$$= \left(-z\theta\, dx^2 + Tz\frac{dx}{dy}(ds^2 - dx^2)\right) d^2z =$$
$$= (-\theta\, dx^2 + T\, dx\, dy)z\, d^2z \tag{5}$$

since $ds^2 - dx^2 = dy^2$. Remembering that $T\theta\, dz = z(\theta\, dy + T\, dx)$, the final equation becomes

$$(\theta\, dy + T\, dx)(ds^2 + dz^2)\, d^2y = (-\theta\, dx + T\, dy)\, dx\, dz\, d^2z$$

or, since $dx\, d^2x + dy\, d^2y = 0$,

$$\frac{dz\, d^2z}{ds^2 + dz^2} = \frac{\theta\, dy + T\, dx}{-\theta\, dx + T\, dy}\frac{d^2y}{dx} = \frac{\theta\, d^2x - T\, d^2y}{\theta\, dx - T\, dy}. \tag{6}$$

Johann notes that he could have arrived at the same equation directly from (3). In fact, multiplying by θ, and remembering that $T\theta\, dz = z(\theta\, dy + T\, dx)$, we obtain

$$T\theta(ds^2 + dz^2)\, d^2y = (\theta z\, dy^2 + Tz\, dy\, dx - \theta z\, ds^2)\, d^2z =$$
$$= (-\theta\, dx^2 + T\, dy\, dx)z\, d^2z,$$

namely equation (5).

These equations may take on a simpler form with the use of Euler's notations[109], specifically writing the equation of the surface in the form $P\, dy = Q\, dz + R\, dx$. Then $T = Qy/P$ and $\theta = -Qy/R$, and equations (6), (3) and (4)

109 Note however that the formulae do not coincide, since Bernoulli takes ds = constant, whereas Euler assumes dx = constant.

become respectively

$$\frac{dz\,d^2z}{dx^2+dy^2+dz^2}=\frac{P\,d^2x+R\,d^2y}{P\,dx+R\,dy}$$
$$\frac{d^2z}{dx^2+dy^2+dz^2}=\frac{Q\,d^2y}{Q\,dy\,dz-P\,dx^2-P\,dy^2}$$
$$\frac{d^2z}{dx^2+dy^2+dz^2}=\frac{Q\,d^2x}{Q\,dx\,dz+R\,dx^2+R\,dy^2}.$$

In a second note, Bernoulli comments that the problem admits an immediate generalisation: instead of making the osculating plane of the unknown curve orthogonal to the surface, one may assume that it makes a given angle ϕ. In fact, in place of $\alpha+\beta=\frac{\pi}{2}$ we may simply write $\alpha+\beta=\phi$, or

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,\tan\beta}=\tan\phi=n.$$

Remembering the equations (1) and (2), we may easily derive the differential equation

$$(T\,dx^2d^2z-z\,ds^2d^2y+T\,dy\,dz\,d^2y)\sqrt{ds^2+dz^2}=$$
$$=nz\,ds^2dx\,d^2z-nT\,dx\,dy\,dz\,d^2z+nT\,dx\,d^2y(ds^2+dz^2)$$

or alternatively, proceeding as above,

$$(\theta\,dx\,dy\,d^2z+T\,dx^2d^2z-\theta\,dx\,dz\,d^2y+T\,dz\,dy\,d^2y)\sqrt{ds^2+dz^2}=$$
$$=n\theta\,dx^2dz\,d^2z-nT\,dx\,dy\,dz\,d^2z+(n\theta\,dy\,d^2y+nT\,dx\,d^2y)(ds^2+dz^2).$$

Equation (3) of the geodesics is of the second order. In some cases, however, it is reduced to the first order; for example, when the given surface is a cylinder with generatrices parallel to the x-axis. In this case we have $T=+\infty$ and equation (3) becomes $(ds^2+dz^2)d^2y=dy\,dz\,dz$, or

$$\frac{d^2y}{dy}=\frac{dz\,d^2z}{ds^2+dz^2}.$$

Recalling that $ds =$ constant, we therefore have $\log dy^2 = \log(ds^2 + dz^2) + c$, and hence

$$n\,dy^2 = ds^2 + dz^2 = dx^2 + dy^2 + dz^2.$$

In our case, z is function only of x, and so $dz = p\,dx$. Thus $(n-1)dy^2 = (1 + p^2)dx^2$, and hence

$$\sqrt{n-1}\,dy = \sqrt{1+p^2}\,dx.$$

The quantity on the right-hand side is the element of the arc of the curve $z = z(x)$. If $A(x)$ is its length, we have

$$y\sqrt{n-1} = A(x).$$

4. *Surfaces of revolution*

Having commented that it is possible to arrive at equation (6) without calculating the angle β (but the calculations are no less lengthy than those which preceded), Bernoulli tackles the specific case of surfaces of revolution around the z-axis. In this case the cylindrical coordinates[110] ϕ, ρ and z are more convenient, linked to x, y, z by the equations

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z. \end{aligned}$$

Reasoning in a manner similar to the general case, Bernoulli calculates

$$\tan\beta = \tan\widehat{GhH} = -\frac{d\rho\sqrt{ds^2 + dz^2}}{\rho\,d\phi\,dz}$$

and

$$\tan\alpha = \frac{\rho\,ds\,d^2\phi + 2d\rho\,ds\,d\phi}{d\rho}.$$

110 In place of ϕ, ρ and z, Bernoulli continues to use x, y and z. In the interests of clarity we have uniformly used the modern notation.

Since $\alpha + \beta = \dfrac{\pi}{2}$, we have $\tan\alpha\ \tan\beta = 1$ and hence we arrive at the equation

$$(\rho\, d^2\phi + 2d\rho\, d\phi)(ds^2 + dz^2) = \rho\, d\phi\, dz\, d^2 z$$

which can also be written in the form

$$\frac{\rho^2 d^2\phi + 2\rho\, d\rho\, d\phi}{\rho^2 d\phi} = \frac{1}{2}\frac{2dz\, d^2 z}{ds^2 + dz^2}. \tag{7}$$

This last equation admits an immediate integral:

$$\log \rho^2 d\phi = \log b + \log \sqrt{ds^2 + dz^2}$$

or

$$\rho^2 d\phi = b\sqrt{ds^2 + dz^2}.$$

On the other hand, in a surface of revolution z is a function only of ρ, hence $dz = p(\rho)\, d\rho$. Recollecting that $ds^2 = d\rho^2 + \rho^2 d\phi^2$, we will thus have the equation

$$d\phi = \frac{b\, d\rho}{\rho}\sqrt{\frac{1 + p^2}{\rho^2 - b^2}}. \tag{8}$$

This is an equation with separable variables, and hence the determination of the projection curve is reduced to the quadratures. Once this curve has been found, the geodesic on the surface is immediate, at least in principle.

Three examples illustrate the method, when the surface is 1. a horizontal plane; 2. a cone, and 3. a sphere. In the first case, we have $z =$ constant and hence $p = 0$. Equation (8) thus becomes

$$d\phi = \frac{b\, d\rho}{\rho\sqrt{\rho^2 - b^2}}$$

and integrating

$$c + \phi = \arctan \frac{\sqrt{\rho^2 - b^2}}{b}.$$

It readily follows from this that

$$\rho \cos(c + \phi) = b$$

and hence the geodesic is a straight line, whose projection on the plane xy has the equation

$$x \cos c - y \sin c = b.$$

If the surface is a cone, we have $z = n\rho$ and hence $p = n =$ constant. Equation (8) thus becomes

$$d\phi = \sqrt{1+n^2}\frac{b\,d\rho}{\rho\sqrt{\rho^2 - b^2}}$$

whose solution is

$$\frac{c+\phi}{\sqrt{1+n^2}} = \arctan\frac{\sqrt{\rho^2-b^2}}{b}.$$

Reasoning as above, and selecting the origin of the angles in such a way as to have $c = 0$, we thus obtain the equation

$$\rho \cos\frac{\phi}{\sqrt{1+n^2}} = b.$$

In this case the projection curve is not explicitly expressed in terms of the variables x and y; Bernoulli, however, gives it a construction by points. If $\alpha = \dfrac{\phi}{\sqrt{1+n^2}}$, the quantity ρ proves equal to $b \sec\alpha$. Then, taking the circumference with centre A and radius $AL = b$, take an arbitrary angle α and draw the straight line $AD = \sec\alpha$. This done, take an angle $\widehat{LAF} = \alpha\sqrt{1+n^2}$, and on the straight line AF obtain the segment $AB = AD$. Point B will be on the projection curve.

The example of the sphere is the most complex of the three. If a is the radius, we have $z = \sqrt{a^2-\rho^2}$ and $p = -\dfrac{\rho}{\sqrt{a^2-\rho^2}}$. Equation (8) thus becomes

$$d\phi = \frac{ab\,d\rho}{\rho\sqrt{\rho^2-b^2}\sqrt{a^2-\rho^2}}.$$

In order to integrate the second member, Bernoulli carries out a series of changes of variable in succession[111]:

$$\rho = \frac{ab}{u},$$
$$u^2 = cv,$$
$$v = \frac{a^2 + b^2}{2c} - t,$$

and arrives at the equation

$$d\phi = \frac{dt}{2\sqrt{\alpha^2 - t^2}},$$

with $\alpha = \dfrac{a^2 - b^2}{2c}$.

Integrating, we get

$$2\phi = \arcsin \frac{t}{\alpha}$$

or

$$t = \frac{a^2 + b^2}{2c} - \frac{a^2 b^2}{c\rho^2} = \sin 2\phi.$$

Multiplying by ρ^2, we obtain

$$\frac{\rho^2(a^2 + b^2) - 2a^2b^2}{a^2 - b^2} = 2\rho^2 \sin\phi \, \cos\phi$$

and returning to the variables x and y,

$$2xy(a^2 - b^2) = (x^2 + y^2)(a^2 + b^2) - 2a^2b^2,$$

or else

$$a^2(x^2 + y^2 - 2xy) + b^2(x^2 + y^2 + 2xy) = 2a^2b^2.$$

111 The constant c in the last two formulae is not essential, serving merely as a guarantee of homogeneity.

At this point Johann carries out the change of variables

$$x + y = \frac{u}{\sqrt{2}},$$
$$x - y = \frac{v}{\sqrt{2}},$$

and arrives at the equation

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

Bernoulli stops at this equation, without saying that the projection of the geodesic is an ellipse with major axis equal to the radius a of the sphere nor – which is more important – that consequently the geodesics on the sphere are arcs of great circles, a fact which Cramer, in a footnote, proves by recourse to a property of spherical trigonometry.

5. *Jacob Bernoulli's solution and Johann's criticisms*

As we said at the beginning, the problem of the geodesics on a surface of revolution had been posed by Johann in the summer of 1697. A few months later, Jacob Bernoulli had given the following solution. Let $z(\rho)$ be the equation of the surface in cylindrical coordinates[112], and let t be the segment of the tangent to a meridian (intersection of the surface with the halfplane $\phi =$ constant) from the tangency point to the z-axis.

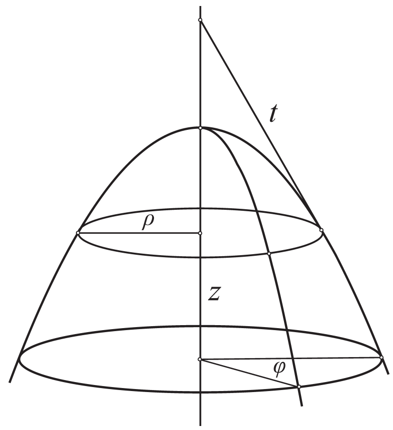
Then, taking

$$\phi = \int \frac{at\,d\rho}{\rho^2\sqrt{\rho^2 - a^2}}, \tag{9}$$

the point of the cylindrical coordinates $(\rho, \phi, z(\rho))$ lies on the geodesic.

Jacob offers no justification for this result, which he applies only to the case of the cone. In his response, Johann states that at the basis of his brother's solution is the principle that the difference between two angles which the solution curve makes with two infinitely close meridians is equal to the angle between the two tangents to the meridian at the points of intersection. This property is obviously valid only for surfaces of revolution, whereas Johann maintains that he has a general solution valid for any surface.

112 Jacob writes x instead of ρ, indicates the arc ϕ with CF and does not introduce the coordinate z, which appears only implicitly by means of the tangent t. Johann uses Jacob's notations, calling z the arc ϕ.



In the text published in his *Opera*, Johann returns to Jacob's solution both to compare it with his own and show that they coincide, and to offer a proof that depends on the principle set forth. As regards the comparison between the two solutions, we need only remark that

$$t = \rho \frac{\sqrt{d\rho^2 + dz^2}}{d\rho} = \rho\sqrt{1 + p^2}.$$

Introducing this value into the previous formula, we at once obtain (8), only changing b into a.

Once the agreement between the two formulae has been established, Johann goes on to the proof according to the above principle. To this end he calculates, on the one hand, the tangent of the angle γ between the geodesic and the meridian:

$$\tan\gamma = -\frac{\rho\, d\phi}{\sqrt{d\rho^2 + dz^2}}$$

and on the other, the angle δ between the two tangents[113]:

$$\delta = \rho\frac{d\phi}{t} = \frac{d\rho\, d\phi}{\sqrt{d\rho^2 + dz^2}}.$$

113 As it is infinitesimal, the angle δ coincides with its tangent.

In view of what has been said, this angle will be equal to the differential of γ. Writing $d\tau = \sqrt{d\rho^2 + dz^2}$, we have

$$\frac{d\rho\, d\phi}{d\tau} = -\frac{d\dfrac{\rho\, d\phi}{d\tau}}{1 + \left(\dfrac{\rho\, d\phi}{d\tau}\right)^2}.$$

Setting $v = \dfrac{\rho\, d\phi}{d\tau}$, the equation becomes

$$\frac{v\, d\rho}{\rho} = -\frac{dv}{1 + v^2}$$

from which it follows that

$$\log\frac{a}{\rho} = \log v - \log\sqrt{1 + v^2}$$

and hence

$$\frac{a}{\rho} = \frac{v}{\sqrt{1 + v^2}}.$$

In conclusion we have

$$\frac{a}{\sqrt{\rho^2 - a^2}} = v = \frac{\rho\, d\phi}{d\tau} = \frac{\rho^2\, d\phi}{t\, d\rho}$$

which immediately gives Jacob's formula (9).

6. *Leibniz's method*

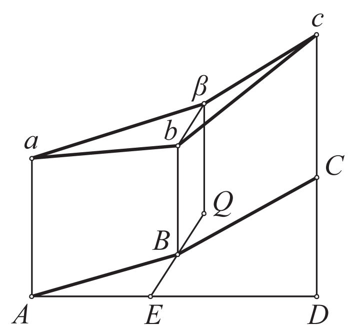
In a letter, of 29 July 1698, Leibniz had suggested another approach to the general problem[114]. The given surface was considered as a polyhedron with infinitesimal plane sides; taking two points R and S on the curve, belonging to two contiguous faces, it was a matter of finding a point T on the segment common to the two faces, in such a way that the sum $RT + TS$ would be minimal. From this, Leibniz said, an equation would be derived which would define the required curve. As is evident, it was a matter of a programme rather than of a solution, and in his reply Bernoulli calls it «your method, or rather

114 Leibniz to Joh. Bernoulli, 29 July 1698, Leibniz, *Math. Schriften* 3, pp. 526–527.

the basis of some method»[115], stating that it was the first idea that had struck him, but it did not lead to the construction of the curve.

Subsequently he certainly had to change his mind, because at the end of the text in the *Opera* we find a new solution to the problem of the geodesics, precisely according to the lines proposed by Leibniz.

Bernoulli considers three infinitely close points a, b and c on the curve, and the corresponding projections A, B and C on the plane xy.



Let

$$AE = f, \quad EB = m, \quad Bb - Aa = c,$$
$$ED = g, \quad DC = n, \quad Cc - Bb = e,$$

we have $ab = \sqrt{f^2 + m^2 + c^2}$, $bc = \sqrt{g^2 + n^2 + e^2}$. The sum

$$ab + bc = \sqrt{f^2 + m^2 + c^2} + \sqrt{g^2 + n^2 + e^2}$$

must be minimal, when point b is made to vary, while a and c remain fixed. Bernoulli makes b vary on the plane $x = x_B$; calling the corresponding point β we will then have

$$a\beta = \sqrt{f^2 + (m + dm)^2 + (c + dc)^2}.$$

Analogously,

$$\beta c = \sqrt{g^2 + (n - dm)^2 + (e - dc)^2}.$$

115 Joh. Bernoulli to Leibniz, 16/26 August 1698, Leibniz *Math. Schriften* 3, p. 532. «Methodus tua vel potius basis alicujus methodi ... »

Since $ab + bc$ is minimal, the first variation must vanish, and hence

$$\frac{m\,dm + c\,dc}{\sqrt{f^2 + m^2 + c^2}} - \frac{n\,dm + e\,dc}{\sqrt{g^2 + n^2 + e^2}} = 0.$$

Now let T be the subtangent to the intersection curve of the surface with the plane $x = x_B$. We have $T = z\dfrac{dm}{dc}$, which, when introduced into the preceding equation leads to

$$\frac{mT + cz}{\sqrt{f^2 + m^2 + c^2}} = \frac{nT + ez}{\sqrt{g^2 + n^2 + e^2}}.$$

This last can also be written in the form

$$\begin{aligned} &T\left(\frac{m}{\sqrt{f^2 + m^2 + c^2}} - \frac{n}{\sqrt{g^2 + n^2 + e^2}}\right) \\ &= z\left(\frac{e}{\sqrt{g^2 + n^2 + e^2}} - \frac{c}{\sqrt{f^2 + m^2 + c^2}}\right). \end{aligned}$$

Apart from the sign, the terms within brackets are the differences of the quantities $\dfrac{m}{\sqrt{f^2 + m^2 + c^2}}$ and $\dfrac{c}{\sqrt{f^2 + m^2 + c^2}}$. Bearing in mind that the points a, b and c are infinitely close, these two quantities are simply

$$\frac{dy}{\sqrt{dx^2 + dy^2 + dz^2}} \quad \text{and} \quad \frac{dz}{\sqrt{dx^2 + dy^2 + dz^2}},$$

so that in conclusion we have

$$T\,d\left(\frac{dy}{\sqrt{dx^2 + dy^2 + dz^2}}\right) + z\,d\left(\frac{dz}{\sqrt{dx^2 + dy^2 + dz^2}}\right) = 0.$$

Specific forms of the equation of the geodesics are obtained by making an appropriate choice of the independent variable. If $ds = \sqrt{dx^2 + dy^2} =$ constant, we come back to Bernoulli and Klingenstjerna's equation (3); if instead we take as constant the arc element $\sqrt{dx^2 + dy^2 + dz^2}$, we find the equation

$$T\,d^2y + z\,d^2z = 0.$$

In any case it is generally impossible to reduce the equation to one of first order.

VI. **Spherical epicycloids**

1. *Hermann's article*

The problem of the spherical epicycloids originated long before in the famous problem posed several decades earlier by Vincenzo Viviani[116], who had challenged the geometers to cut out a portion from a sphere in such a way that the remaining surface be quadrable. As we know, the problem, which according to Viviani's intentions was to mark a defeat for the new infinitesimal calculus, was solved without difficulty by the exponents of the new analysis, first of all by Leibniz[117].

A quarter of a century later, a similar problem was posed by Jacob Hermann under the pseudonym of Carl Ernest Offenburg: in the April 1718 issue of the *Acta Eruditorum* he proposed to «perforate a hemisphere with an arbitrary number of oval windows, each of which to have an absolutely (i.e. algebraically) rectifiable perimeter»[118].

This problem does not appear to have had the same impact as Viviani's; it was only in 1726, six years after he had posed it, that Hermann decided to publish the solution[119] in the *Commentarii* of the St Petersburg Academy. The curves Hermann found are *spherical epicycloids*, generated by a point on a circumference which lies on a sphere and rolls on a parallel of the same sphere. More precisely:

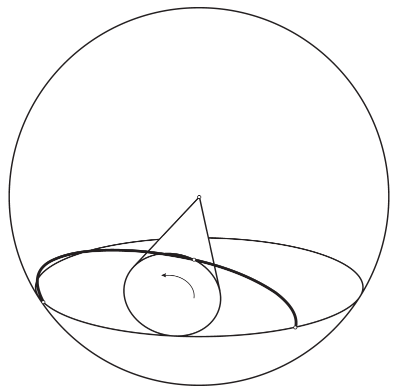
116 *Die 4 April. 1692 Aenigma Geometricum de Miro Opificio Testudinis Quadrabilis Hemisphaericae a D. Pio Lisci Pusillo Geometra propositum*, Florence – AE Junii 1692, pp. 274–275, PT 17, 1692/93, pp. 585–586.

117 G. W. Leibniz, *Constructio testudinis quadrabilis hemisphaericae*, AE 1692, pp. 275–279 – Leibniz *Math. Schriften* 5, pp. 274–278. Cf. C. S. Roero, *Sull'Aenigma Geometricum Florentinum*, Jac. B. *Werke* 2, pp. 623–635; *Leibniz and the temple of Viviani. His prompt reply and the repercussions in the field of mathematics*, Annals of Science 47, 1990, pp. 423–443; *La matematica tra gli affari di stato, nel Granducato di Toscana, alla fine del XVII secolo*, Bollettino di Storia delle Scienze Matematiche XI, 1991, pp. 85–142.

118 C. E. Offenburg, *Annotationes in Epistolam mensi Julio Act. Erud. superioris anni insertam, una cum solutione Problematis in ea propositi. Accedit geminum problema Clarissimo Epistolae Autori vicissim propositum a Carolo Ernesto Offenburgio*, AE Aprilis 1718, p. 175: «Testitudinem Hemisphaericam tot fenestris ovalibus perforare quot libuerit, sed iis tamen quarum unaquaeque peripheriam independenter a rectificatione arcuum circularium absolute rectificabilem habeat».

119 {J. Hermann} Op. 141, pp. 451–458 h. v.

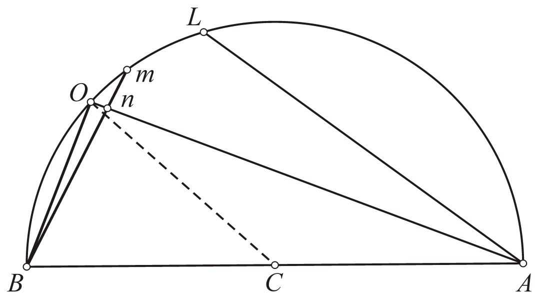
A spherical epicycloid is the curve described on the surface of a sphere by a given point on the circumference of the base of a right cone while the base of the cone rolls on the circumference of a fixed circle, and the vertex of the cone remains fixed in the center of the sphere (whose radius is equal to the side of the cone)[120].



The base of the cone is called the *generating circle* or *mobile circle*, and the parallel on the sphere is called the *fixed circle*. If this is not a great circle, we have two epicycloids, according to whether the generating circle is above or below the plane of the fixed circle. Hermann believes he can prove that all these epicycloids are rectifiable. His – very simple – proof is based on two lemmas. The first is none other than Carnot's theorem: if p and q are two sides of a triangle and α is the angle they form, the length of the third side is

$$\sqrt{p^2 + q^2 - 2pq\cos\alpha}.$$

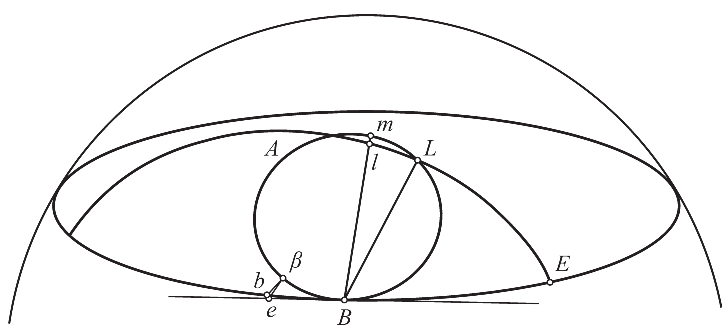
120 {J. Hermann} Op. 141, pp. 451–458 h. v.



The second lemma is a sort of geometrical integration: in the semicircle BLA the sum of the products of the chords BO up to BL by the arc element Om, in formulae the integral $\int BO\,ds$, is equal to the product $AB \times (AB - AL)$.

This can immediately be verified. In fact, calling R the radius of the semicircle and setting $2\vartheta = \widehat{BCO}$ and $2\vartheta_0 = \widehat{BCL}$, we have $BO = 2R\sin\vartheta$, $ds = 2R\,d\vartheta$, and hence

$$\int BO\,ds = 4R^2 \int_0^{\vartheta_0} \sin\vartheta\,d\vartheta = 4R^2(1 - \cos\vartheta_0) = AB \times (AB - AL).$$



Hermann's proof proceeds considering the figure, in which ALB is the generating circle of radius b, EBb the fixed circle of radius a, Be the tangent to both at the point B, and L the point which describes the epicycloid.

When the circle ABL turns (anticlockwise), the arc element $B\beta$ revolves around B covering the parallel arc Bb. Meanwhile the segment BL describes a sector LBl similar to the sector $B\beta b$. Thus we have $BL : Ll = B\beta : b\beta$, and hence $Ll \times B\beta = BL \times b\beta$.

If we now set $Bb = Be = B\beta = Lm = ds$, we have $eb = \dfrac{ds^2}{a}$, $e\beta = \dfrac{ds^2}{b}$, and by the first lemma, calling ϑ the angle of inclination $\widehat{be\beta}$, we get

$$b\beta = \frac{ds^2}{ab}\sqrt{a^2 + b^2 - 2ab\cos\vartheta}.$$

Thus we have

$$ab \times Ll = ds \times BL \times \sqrt{a^2 + b^2 - 2ab\cos\vartheta} = BL \times Lm \times \sqrt{a^2 + b^2 - 2ab\cos\vartheta}$$

and by the second lemma the length of the arc EL will be

$$\begin{aligned} &\frac{1}{ab} AB \times (AB - AL)\sqrt{a^2 + b^2 - 2ab\cos\vartheta} \\ &= 2\frac{\sqrt{a^2 + b^2 - 2ab\cos\vartheta}}{a} \times (AB - AL). \end{aligned}$$

2. *Johann Bernoulli's first memoir*

Hermann's proof contained a basic error, as a result of his neglecting the angle of contact bBe, a second-order infinitesimal and thus comparable to eb and $e\beta$. Consequently, the statement that «the sector LBl is similar to the sector $B\beta b$» is not generally valid and the reasoning collapses; specifically, the epicycloids are algebraically rectifiable in only one case.

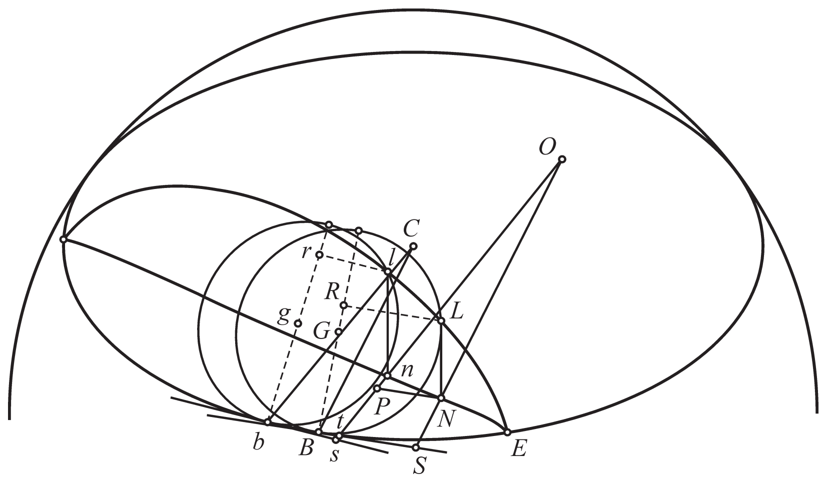
In fact things are far more complex than might appear from Hermann's article. Six years were to pass before Johann Bernoulli's two memoirs, published by the Paris Académie des Sciences, settled the question[121]. We begin with the first of these memoirs.

121 Joh. B. Op. CXLII and Op. CXLIII, pp. 507–519 h. v. and pp. 529–536 h. v. A version of these two articles in Latin was sent by Joh. Bernoulli to P. L. M. de Maupertuis as an attachment to a letter of 8 May 1732, UB Basel, L I a 662, Nr. 20, in which Bernoulli writes: «Je vous y ai promis ma solution directe du Probleme des courbes algebriques et rectifiables à tracer sur la surface spherique. Pour m'acquitter de cette promesse, voici le cahier cy joint qui contient ma methode: peutetre y trouverés vous de quoi repaitre votre curiosité par la singularité de l'analyse dont je me suis servi». In UB Basel L I a 12.3, ff. 150–153, there is an autograph copy, in Latin, of the text sent to Maupertuis and translated into French. Cf. also P. M. de Maupertuis, *Solution du mesme probleme et de quelques autres de cette espece,* Mém. Paris 1732 (1735), pp. 255–256. With this, ff. 154–156, is a first draft of the same work, again in Latin, in slightly reduced form. f. 157 contains the figures, which are redrawn on the following page.

As has been remarked, the problem is rather complicated, and demands a subtle analysis of the second-order infinitesimals. In order to facilitate reading, we will divide the argument into several parts.

2.1. *Position of the problem*

Two infinitely close positions of the generating circle are to be considered; the first has its centre in G and is tangent at B to the fixed circle with centre C; the second has its centre in g and touches the fixed circle at b.



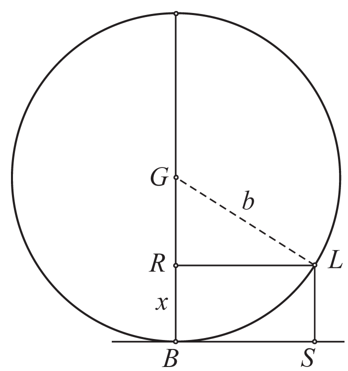
The intersection of the planes of the two circles (mobile and fixed) is the straight line tangent to both: *BS* in the first case and *bs* in the second. It is clear that *BS* is perpendicular to both *CB* and *GB* (radii of the two circles) and *bs* is perpendicular to *Cb* and to *gb*. The angle $\vartheta = \widehat{CBG} = \widehat{Cbg}$ is the constant angle of inclination of the two planes.

Now let L be a point on the epicycloid, l the infinitely close point, and let *LR* and *lr* be the perpendiculars to the radii *BG* and *bg* of the mobile circle. In addition, let N and n be the projections of L and l on the fixed plane, and let the curve *ENn* be the projection on this plane of the epicycloid *ELl*. From L and l we draw the perpendiculars *LS* and *ls* to the tangents *BS* and *bs*, and from the points S and s the perpendiculars on the fixed plane to the tangents *BS* and *bs*, which meet at the point O.

It is clear that *BS* is perpendicular to the plane of the straight lines *CB* and *BG*, and also to the plane of the straight lines *SN* and *SL*; analogously, *bs* is perpendicular to the plane of the straight lines *Cb* and *bg*, as well as to the plane of the straight lines *sn* and *sl*. Consequently the angles $\widehat{OSL}$ and $\widehat{Osl}$ will both be equal to the inclination ϑ.

Finally, indicating with t the intersection point of the straight lines sO and BS, the triangle SOt is similar to tbs, which in its turn is similar to BCb. In addition, let NP be drawn on the fixed plane parallel to St.

2.2. *On the plane of the mobile circle*



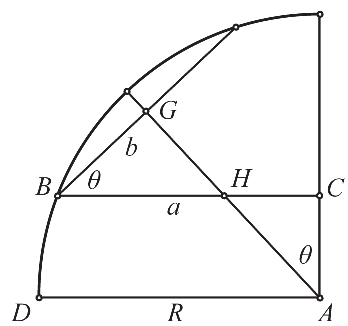
Assuming $LS = BR = x$ and $GB = b$, we will have $RL = BS = \sqrt{2bx - x^2}$ and hence

$$d\,LS = dx, \quad d\,BS = d\,RL = \frac{b - x}{\sqrt{2bx - x^2}}\,dx.$$

Calling φ the angle $\widehat{BGL} = \arccos\big(1 - \dfrac{x}{b}\big)$, then $d\varphi = \dfrac{dx}{\sqrt{2bx - x^2}}$ and hence

$$d\,\widehat{BL} = b\,d\varphi = \frac{b\,dx}{\sqrt{2bx - x^2}}.$$

2.3. *On the meridian plane*



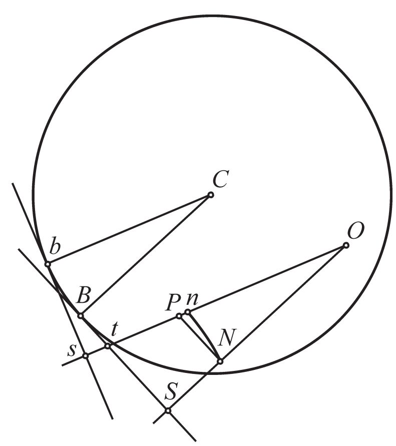
The radius R of the sphere can be found in function of the radii a of the fixed circle and b of the generating circle, and of the angle of inclination ϑ. In fact we have

$$a = BH + HC = \frac{b}{\cos\vartheta} + AC\tan\vartheta = \frac{b}{\cos\vartheta} + \sqrt{R^2 - a^2}\tan\vartheta$$

from which

$$R = \frac{\sqrt{a^2 + b^2 - 2ab\cos\vartheta}}{\sin\vartheta}.$$

2.4. *On the fixed plane*



We have $d\widehat{BL} = \widehat{bl} - \widehat{BL} = \widehat{bE} - \widehat{BE}$ (because of the nature of the epicycloid) $= \widehat{Bb} = \dfrac{b\,dx}{\sqrt{2bx - x^2}}$ (by 2.2) and hence, since up to higher-order infinitesimals $d\,BS = bs - BS = bt - BS = bB - St$, we have, again by 2.2,

$$St = d\widehat{BL} - d\,BS = \frac{x\,dx}{\sqrt{2bx - x^2}}.$$

2.5. *Similar triangles*

a) The triangles SOt and BCb are similar. In fact, neglecting infinitesimals, we have $Ot = OS$, $CB = Cb$, and the angle in O is equal to the angle in C, since their sides are parallel.
b) The triangles sbt and SOt are similar. In fact, neglecting second-order infinitesimals, we have $bs = bt$, and the angles in s and S are equal.

From a) it follows that $Bb : St = CB : OS$, or

$$\frac{b\,dx}{\sqrt{2bx - x^2}} : \frac{x\,dx}{\sqrt{2bx - x^2}} = a : OS$$

and hence

$$OS = \frac{ax}{b}.$$

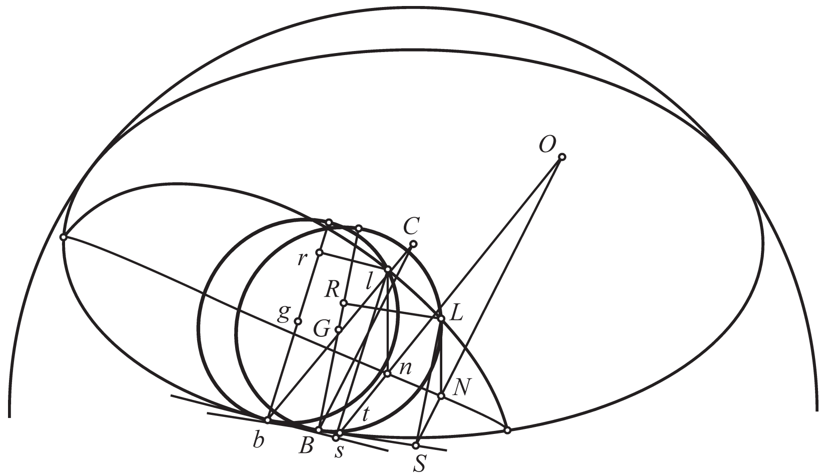
Moreover, it follows from a) and b) that the triangles sbt and BCb are similar, and hence $BC : bs = Bb : st$, or, bearing in mind that $bs = BS$ up to higher-order infinitesimals,

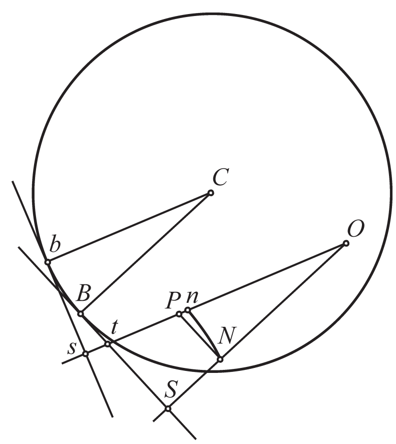
$$a : \sqrt{2bx - x^2} = \frac{b\,dx}{\sqrt{2bx - x^2}} : st$$

and hence

$$st = \frac{b\,dx}{a}.$$

2.6. *Projections*





Since $SN = SL\cos\vartheta = x\cos\vartheta$, we have $d\,SN = \cos\vartheta\,dx$. But $d\,SN = sn - SN =$ (up to second-order infinitesimals) $sn - tP = st + Pn$. It follows that

$$Pn = \cos\vartheta\,dx - st = \frac{a\cos\vartheta - b}{a}\,dx.$$

On the other hand $OS : ON = St : NP$, and since (by 2.5 and 2.4)

$$\begin{aligned}
OS &= \frac{ax}{b}\\
ON &= OS - SN = \frac{ax}{b} - x\cos\vartheta = \frac{a - b\cos\vartheta}{b}x\\
St &= \frac{x\,dx}{\sqrt{2bx - x^2}}
\end{aligned}$$

then

$$NP = \frac{ON \times St}{OS} = \frac{a - b\cos\vartheta}{a}\frac{x\,dx}{\sqrt{2bx - x^2}}.$$

Moreover, since $LN = SL\sin\vartheta = x\sin\vartheta$, we have $d\,LN = \sin\vartheta\,dx$.

2.7. *Arc elements*

It is now possible to express the arc elements of the epicycloid ELl and of its projection ENn on the fixed plane. For the latter we have

$$Nn = \sqrt{NP^2 + Pn^2} = \sqrt{\left(\frac{a - b\cos\vartheta}{a}\right)^2 \frac{x}{2b - x} + \left(\frac{a\cos\vartheta - b}{a}\right)^2}\,dx$$

whereas for the first we have

$$\begin{aligned}
Ll &= \sqrt{Nn^2 + d\,LN^2} =\\
&= \sqrt{\left(\frac{a - b\cos\vartheta}{a}\right)^2 \frac{x}{2b - x} + \left(\frac{a\cos\vartheta - b}{a}\right)^2 + \sin^2\vartheta}\,dx =\\
&= \sqrt{\frac{2b(a^2 + b^2 - 2ab\cos\vartheta) - b^2\sin^2\vartheta\,x}{2b - x}}\,\frac{dx}{a}.
\end{aligned}$$

2.8. *Reduction to the quadrature of the hyperbola*

In conclusion, the length of the epicycloid depends on the calculation of integrals of the form

$$\int\sqrt{\frac{\alpha - \beta x}{\gamma - x}}\,dx$$

which, by a change of the variable $x = \gamma - u^2$, becomes

$$-2\int \sqrt{\alpha - \beta\gamma + \beta u^2}\, du.$$

Since $\beta = b^2 \sin^2 \vartheta > 0$, this last integral can be expressed by means of the quadrature of the hyperbola, and hence by means of logarithms.

2.9. *Rectifiable epicycloids*

Thus spherical epicycloids are rectifiable algebraically only in a small number of cases, contrary to what Hermann had affirmed. One of these is when $\vartheta = 0$, i.e. when the mobile circle lies on the same plane as the fixed circle. The resultant curve is the epicycloid of the circle, for which

$$Ll = \sqrt{\frac{2b(a-b)^2}{2b-x}}\,\frac{dx}{a}$$

and hence

$$EL = \frac{2(a-b)}{a}\left(2b - \sqrt{4b^2 - 2bx}\right).$$

A second case of rectifiability occurs when $a = \infty$, i.e. when the fixed circumference is a straight line and the rotating cone becomes a cylinder. The curve obtained is the ordinary cycloid, and thus

$$Ll = \sqrt{\frac{x}{2b-x} + 1}\, dx = \sqrt{\frac{2b}{2b-x}}\, dx.$$

Apart from these trivial cases, the epicycloid will be algebraically rectifiable when $a - b\cos\vartheta = 0$, in other words when the mobile circle is a great circle of the sphere. If this occurs, then

$$Ll = \sqrt{\left(\frac{a\cos\vartheta - b}{a}\right)^2 + \sin^2\vartheta}\, dx = \tan\vartheta\, dx$$

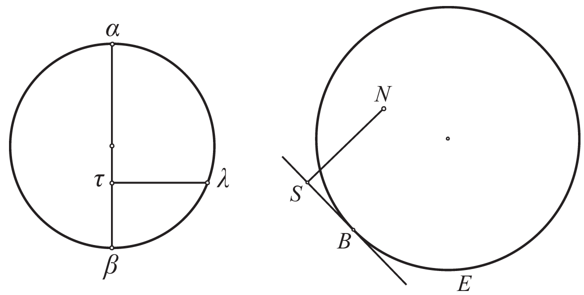
and hence

$$EL = x\tan\vartheta.$$

All that has been said works in the same way if the inclination is greater than $\frac{\pi}{2}$; in this case the epicycloid is never algebraically rectifiable, because the condition $a - b\cos\vartheta = 0$ cannot be satisfied.

2.10. *Construction by points*

Points can be constructed on the projection *ENn* on the fixed plane by taking a circle equal to the mobile circle. Having fixed a point λ on this, let β be a generic point, $\beta\alpha$ the diameter passing through β and τ the projection of λ on $\beta\alpha$.



Now on the fixed circle take an arc $\widehat{EB} = \widehat{\beta\lambda}$, on the tangent in B a segment $BS = \tau\lambda$, and from S draw perpendicularly a segment of length $SN = \beta\tau \sin\vartheta$.

Point N is in the projection. In fact by construction we have $\widehat{\beta\lambda} = \widehat{EB} = \widehat{BL}$, and hence $\tau\beta = x$ and $SN = x\cos\vartheta$, as in 2.6.

Once points have been constructed on the projection we need only raise the perpendicular until it meets the sphere in order to have the same number of points on the epicycloid.

2.11. *Lengthened or shortened epicycloids*

If point L is within or outside the generating circle we have lengthened or shortened epicycloids. This also happens when, while point L rotates at a constant angular speed, the mobile circle slides along the fixed circle at constant speed. If the ratio between these speeds is $1 : n$, the epicycloid is lengthened if $n > 1$ and shortened if $n < 1$. In any case we have[122]:

$$Nn = \sqrt{\frac{[(n-1)ab + (a - nb\cos\vartheta)x]^2}{2bx - x^2} + (a\cos\vartheta - nb)^2}\,\frac{dx}{a}$$

122 In the autograph ms. UB Basel L I a 12.3, f. 162r Bernoulli gives the details of the related calculations.

and

$$Ll = \sqrt{\frac{[(n-1)ab + (a - nb\cos\vartheta)x]^2}{2bx - x^2} + a^2 - 2nab\cos\vartheta + n^2b^2}\,\frac{dx}{a}.$$

Neither of these curves is algebraically rectifiable.

3. *Another point of view*

In an immediately following article, *Sur les Courbes algébriques & rectifiables tracées sur une surface sphérique*[123], which appeared in the same volume of the Paris Académie, Johann Bernoulli took up a different viewpoint, which was more direct and immediate than the earlier one, and less closely linked to Hermann's work on spherical epicycloids.

The problem, we may recall, consists of drawing on the unitary sphere an algebraic curve, whose arc element is algebraically integrable. To this end, Bernoulli expresses the equation of the curve in the form $z = \varphi(s)$, where z is the height of the point of the curve on the equatorial plane and s is the length of the arc of the projection on this plane. In fact, he chooses a priori the form of the function φ in the simplest manner, taking $\varphi(s) = a + bs$. If polar coordinates are introduced on the equatorial plane, with vector radius y and angle x, we have

$$\begin{aligned} ds^2 &= y^2\,dx^2 + dy^2 \\ z &= \sqrt{1-y^2} = \varphi(s) = a + bs \\ dz &= \frac{-y\,dy}{\sqrt{1-y^2}} = \varphi'(s)\,ds = b\,ds \end{aligned}$$

and hence

$$b^2(y^2\,dx^2 + dy^2) = \frac{y^2}{1-y^2}\,dy^2$$

123 Joh. B. Op. CXLIII. The Latin version of this article was sent to Maupertuis together with the letter dated of May the 8^{th} 1732. In UB Basel ms. L I a 12.3 two versions are conserved, with slight differences, of the manuscript sent to France, both autograph, precisely in ff. 159r–160r and 161r–v, 162v.

from which we may easily derive

$$dx = \frac{m^2 y\, dy}{\sqrt{(1-y^2)(m^2y^2-1}} - \frac{dy}{y\sqrt{(1-y^2)(m^2y^2-1}}$$

with $m^2 = \dfrac{1+b^2}{b^2}$.

The right-hand-side terms are integrated without difficulty, albeit with a certain prolixity in the calculations, obtaining the equation of the projection curve in the from

$$x = f(y) = f_1(y) + f_2(y)$$

with

$$f_1(y) = \frac{m}{2}\arcsin\frac{2m^2y^2 - m^2 - 1}{m^2-1} = \frac{m}{2}A$$

$$f_2(y) = \frac{1}{2}\arcsin\frac{2-y^2(m^2+1)}{y^2(m^2-1)} = \frac{1}{2}B.$$

Hence the equation of the projection curve is

$$2x = mA + B.$$

If we want this to be algebraic, the preceding equation should be written as an algebraic equation in between[124] y and $\sin x$. This is possible if m is a rational number. In fact the sine of the sum of two angles A and B, as well as that of a multiple and a submultiple of an angle A, is a rational function of $\sin A$ and $\sin B$. Consequently the sine of $mA + B$ is an algebraic function of $\sin A$ and $\sin B$, which in their turn are algebraic functions of y. On the other hand $\sin 2x$ is an algebraic function of $\sin x$, and thus if m is rational the projection curve is algebraic.

The same obviously happens for the curve of the sphere, which furthermore also proves to be algebraically integrable. In fact if $d\sigma$ is the arc element of the curve, then

$$d\sigma = \sqrt{ds^2 + dz^2} = \sqrt{1+b^2}\, ds = mb\, ds$$

and the integrability of $d\sigma$ is a consequence of the integrability of ds.

124 We recall that if ξ and η are the Cartesian coordinates on the equatorial plane, we have $y^2 = \xi^2 + \eta^2$ and $\sin x = \dfrac{\eta}{y}$.

In reality, Bernoulli comments, the curves found in this way are precisely the spherical epicycloids in the integrable case, i.e. those generated by a great circle which rotates on a lesser one. In fact in these epicycloids we have $d\sigma = \dfrac{ds}{\sin\vartheta}$, and hence we need only take $mb \sin\vartheta = 1$.