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# A PDE WITH DRIFT OF NEGATIVE BESOV INDEX AND LINEAR GROWTH SOLUTIONS

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Abstract. This paper investigates a class of PDEs with coefficients in negative Besov spaces and whose solutions have linear growth. We show existence and uniqueness of mild and weak solutions, which are equivalent in this setting, and several continuity results. To this aim, we introduce ad-hoc Besov-Hölder type spaces that allow for linear growth, and investigate the action of the heat semigroup on them. We conclude the paper by introducing a special subclass of these spaces which has the useful property to be separable.

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## 1. Introduction

The objective of this paper is to study existence, uniqueness and continuity results for solutions to a class of parabolic PDEs with negative Besov drifts and unbounded solutions. In particular, the class of parabolic linear PDEs studied in this work is of the form

$$\begin{cases} \partial_t v + \frac{1}{2}\Delta v + \nabla v\, b = \lambda v + g, & \text{on } [0,T] \times \mathbb{R}^d \\ v(T) = v_T, \end{cases} \tag{1}$$

where $\lambda$ is a real parameter, and $b$ and $g$ are continuous functions of time taking values in a negative Besov space $\mathcal{C}^{(-\beta)+}$ with $0 < \beta < \frac{1}{2}$, see definitions and details below. Here the product $\nabla v\, b := \nabla v \cdot b$ needs to be defined using pointwise products, because the term $b$ is a distribution.

Our main motivation to study (1) comes from stochastic analysis. Indeed, PDEs of the form (1) naturally arise in the context of stochastic differential equations, particularly when setting and solving them as *martingale problems*. In the companion paper [11] we will extensively use all results on PDE (1) found in the present paper.

PDEs with distributional coefficients have been studied in the literature before, see for example [3, 5, 8] to name a few. Here we do not require the use of Gubinelli's paracontrolled distributions or Hairer's regularity structures so that the Besov index of the space where the distributional coefficient $b$ lives cannot be lower than $-\frac{1}{2}$. The main novelty is that we allow terminal

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conditions, and hence the solutions, to have linear growth, which is not the case in the existing literature.

For this reason in Section 3 we introduce a suitable class of functions spaces, denoted by $D\mathcal{C}^\alpha$, which contains all functions such that their derivative is an element of $\mathcal{C}^\alpha$. We also investigate the action of the semigroup on these spaces, in particular Schauder's estimates and Bernstein's inequalities in the $D\mathcal{C}^\alpha$ spaces, see Lemmata 3.3 and 3.4. In Section 4 We introduce the notion of weak and mild solutions for this PDE and show that they are equivalent in Proposition 4.5. We then show existence and uniqueness of mild solutions by fixed point arguments in Theorem 4.7, using properties of the heat semigroup. Furthermore we show in Proposition 4.13 some (uniform) bounds on the solution of a special case of the PDE, given by (47). We also exhibiting several continuity results for the solutions with respect to the functions $g, b, v_T$, both in the case when the solutions have linear growth and in the case when they are bounded. This is done in Lemmata 4.17 and 4.19. In the last section we introduce and study a further class of spaces, which are used in our companion paper [11] for applications in stochastic analysis, together with all the results on the PDE. One of the important feature of these spaces is the fact that they are separable, which is not the case for the standard separable Besov-Hölder spaces.

The paper is organised as follows. In Section 2 we introduce the framework in which we work, define some tools like the pointwise product and state some Assumptions. In Section 3 we define some new functions spaces that allow linear growth and derive useful properties of how the heat semigroup acts on them. The PDE (1) is studied in Section 4. In Section 5 we introduce and study a class of Besov type spaces which is separable.

## 2. Setting and preliminary results

2.1. **Function spaces.** We use the notation $C^{0,1} := C^{0,1}([0,T] \times \mathbb{R}^d)$ to indicate the space of functions with gradient in $x$ continuous in $(t,x)$. By a slight abuse of notation we use the same notation $C^{0,1}$ for functions which are $\mathbb{R}^d$-valued. When $f : \mathbb{R}^d \to \mathbb{R}^d$ is differentiable, we denote by $\nabla f$ the matrix given by $(\nabla f)_{i,j} = \partial_i f_j$. When $f : \mathbb{R}^d \to \mathbb{R}$ we denote the Hessian matrix of $f$ by $\text{Hess}(f)$. Given any function $f$ defined on $[0,T] \times \mathbb{R}^d$ we often denote $f(t) := f(t, \cdot)$.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ the space of Schwartz distributions. We denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform on $\mathcal{S}$ and inverse Fourier transform respectively, which are extended to $\mathcal{S}'$ in the standard way. For $\gamma \in \mathbb{R}$ we denote by $\mathcal{C}^\gamma = \mathcal{C}^\gamma(\mathbb{R}^d)$ the Besov space (or Hölder-Zygmund space) defined as

$$\mathcal{C}^\gamma := \left\{ f \in \mathcal{S}' : \sup_{j \in \mathbb{N}} 2^{j\gamma} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_\infty < \infty \right\}, \tag{2}$$

where $(\varphi_j)$ is some partition of unity. $\|\cdot\|_\infty$ denotes the usual sup-norm. For more details see for example [1, Section 2.7]. Note that for $\gamma' < \gamma$ one has $\mathcal{C}^\gamma \subset \mathcal{C}^{\gamma'}$. If $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$ then the space coincides with the classical Hölder space, namely the space of bounded functions with bounded derivatives up to order $\lfloor \gamma \rfloor$ and such that the $\lfloor \gamma \rfloor$th derivative is $(\gamma - \lfloor \gamma \rfloor)$-Hölder continuous. For example if $\gamma \in (0,1)$ the space $\mathcal{C}^\gamma$ can be equipped with the classical $\gamma$-Hölder norm

$$\|f\|_\gamma := \|f\|_\infty + \sup_{x \neq y, |x-y|<1} \frac{|f(x) - f(y)|}{|x-y|^\gamma}, \tag{3}$$

and if $\gamma \in (1,2)$ then norm is given by

$$\|f\|_\infty + \|\nabla f\|_\infty + \sup_{x \neq y, |x-y|<1} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\gamma}. \tag{4}$$

We remark that it is equivalent in the previous formulations of the norms to take the supremum over the whole space rather than on $|x-y| < 1$. Note that we use the same notation $\mathcal{C}^\gamma$ to indicate $\mathbb{R}$-valued functions but also $\mathbb{R}^d$- or $\mathbb{R}^{d\times d}$-valued functions. It will be clear from the context which space is needed.

We denote by $C_T\mathcal{C}^\gamma$ the space of continuous functions on $[0,T]$ taking values in $\mathcal{C}^\gamma$, that is $C_T\mathcal{C}^\gamma := C([0,T]; \mathcal{C}^\gamma)$. For any given $\gamma \in \mathbb{R}$ we denote by $\mathcal{C}^{\gamma+}$ and $\mathcal{C}^{\gamma-}$ the spaces given by

$$\mathcal{C}^{\gamma+} := \cup_{\alpha>\gamma}\mathcal{C}^\alpha, \qquad \mathcal{C}^{\gamma-} := \cap_{\alpha<\gamma}\mathcal{C}^\alpha.$$

Note that $\mathcal{C}^{\gamma+}$ is an inductive space. We will also use the spaces $C_TC^{\gamma+} := C([0,T]; \mathcal{C}^{\gamma+})$. We remark that $f \in C_TC^{\gamma+}$ if and only if there exists $\alpha > \gamma$ such that $f \in C_T\mathcal{C}^\alpha$, see [10, Lemma B.2]. Similarly, we use the space $C_T\mathcal{C}^{\gamma-} := C([0,T]; \mathcal{C}^{\gamma-})$; in particular we observe that if $f \in C_T\mathcal{C}^{\gamma-}$ then for any $\alpha < \gamma$ we have $f \in C_T\mathcal{C}^\alpha$. Note that if $f$ is continuous and such that $\nabla f \in C_T\mathcal{C}^{0+}$ then $f \in C^{0,1}$.

Finally for a general Banach space $(B, \|\cdot\|_B)$ we introduce the family of $\rho$-equivalent norms on $C_TB$, denoted by $\|\cdot\|^{(\rho)}_{C_TB}$ and defined for all $\rho \geq 0$ by $\|f\|^{(\rho)}_{C_TB} = \sup_{t\in[0,T]} e^{-\rho(T-t)}\|f(t)\|_B$. If $\rho = 0$ this is the standard norm in $C_TB$.

2.2. **The heat semigroup in $\mathcal{S}'$.** Let $(P_t)_t$ denote the semigroup generated by $\frac{1}{2}\Delta$ on $\mathcal{S}$, in particular for all $\phi \in \mathcal{S}$ we define $(P_t\phi)(x) := \int_{\mathbb{R}^d} p_t(x-y)\phi(y)\mathrm{d}y$, where the kernel $p$ is the usual heat kernel $p_t(x-y) = \frac{1}{(2\pi t)^{d/2}} \exp\{-\frac{|x-y|^2}{2t}\}$. It is easy to see that $P_t : \mathcal{S} \to \mathcal{S}$. Moreover we can extend it to $\mathcal{S}'$ by dual pairing (and we denote it with the same notation by simplicity). One has $\langle P_t\psi, \phi\rangle = \langle \psi, P_t\phi\rangle$ for each $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$, using the fact that the kernel is symmetric.

Next we state and prove a joint continuity result for the heat semigroup acting on $\mathcal{S}'$. To this aim, we first recall some facts about the Schwartz space $\mathcal{S}'$, which is an inductive space. We recall that [13, Section 7.3] says that for any $\varphi \in \mathcal{S}, f \in \mathcal{S}'$ there exists a constant $C(f)$ and an integer $N \in \mathbb{N}$ such that

$$|\langle \varphi, f\rangle| \leq C(f) \sup_{y\in\mathbb{R}^d, |\alpha|\leq N} |D^\alpha \varphi(y)|\,(|y|^2+1)^N. \tag{5}$$

From this it follows that the space $\mathcal{S}'(\mathbb{R}^d)$ can be expressed as the space $\mathcal{S}'(\mathbb{R}^d) = \cup_{N\in\mathbb{N}} E_N^*$ equipped with the inductive topology, where $E_N$ is the space of smooth functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|\varphi\|_N := \sup_{y\in\mathbb{R}^d, |\alpha|\leq N} |D^\alpha \varphi(y)|\,(|y|^2+1)^N < \infty.$$

**Lemma 2.1.** *Let $f \in C_T\mathcal{S}'(\mathbb{R}^d)$. Then there exists $N \in \mathbb{N}$ and a constant $C(f)$ independent of time such that*

$$\sup_{t\in[0,T]} |\langle f(t), \varphi\rangle| \leq C(f) \sup_{y\in\mathbb{R}^d, |\alpha|\leq N} |D^\alpha \varphi(y)|\,(|y|^2+1)^N.$$

*for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. In particular there exists $N$ such that $f \in C_T E_N^*$.*

*Proof.* Since $t \mapsto f(t)$ is continuous in $\mathcal{S}'(\mathbb{R}^d)$ then $(f(t))_{t\in[0,T]}$ is a compact in $\mathcal{S}'(\mathbb{R}^d)$, so there exists $N$ such that $f : [0,T] \to E_N^*$ and such that $(f(t))_{t\in[0,T]}$ is compact in $E_N^*$ by [10, Remark B.1]. In particular, $(f(t))_{t\in[0,T]}$ is bounded in $E_N^*$, which implies that

$$\sup_{t\in[0,T]} \|f(t)\|_{E_N\to\mathbb{R}} < C(f) < \infty,$$

and thus

$$\sup_{t\in[0,T]} |\langle f(t), \varphi\rangle| \leq C(f)\|\varphi\|_N = C(f) \sup_{y\in\mathbb{R}^d, |\alpha|\leq N} |D^\alpha \varphi(y)|\,(|y|^2+1)^N$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. □

**Lemma 2.2.** *Let $h \in C_T\mathcal{S}'$. Then the function $P_t h(r)$ is jointly continuous in $(t,r) \in [0,T]^2$ with values in $\mathcal{S}'$.*

*Proof.* By means of Fourier transform it is enough to prove that $(r,t) \mapsto \mathcal{F}(P_t h(r))$ is continuous with values in $\mathcal{S}'$. We can write

$$\mathcal{F}(P_t h(r))(\xi) = [\mathcal{F}(\exp(it\cdot))\mathcal{F}h(r)](\xi) = \exp(-\frac{t}{2}\xi^2)\mathcal{F}h(r)(\xi). \tag{6}$$

Expression (6) has to be understood as an element of $\mathcal{S}'$. When $t > 0$ the product of $\xi \mapsto \exp(-\frac{t}{2}\xi^2) \in \mathcal{S}$ and $\mathcal{F}h(r) \in \mathcal{S}'$ belongs to $\mathcal{S}'$. In that case

$$\mathcal{F}(P_t h(r))(\xi) = \langle (\mathcal{F}h(r))(\xi), \exp(\frac{-t\xi^2}{2})\rangle \in \mathbb{R},$$

so that (6) is a function.

We now prove that $(t,r) \mapsto \exp(\frac{-t\xi^2}{2})\mathcal{F}h(r)(\xi)$ is continuous with values in $\mathcal{S}'$. By Lemma 2.1 let $N$ be such that $\mathcal{F}h \in C_T E_N^*$ and let $(t_n, r_n) \to$

$(t_0, r_0)$. Let $m \geq N$ to be chosen later. We have (omitting the variable $\xi$ in $\mathcal{F}h(r)$ for brevity)

$$\begin{aligned}
&\|\exp(-\frac{t_n}{2}\xi^2)\mathcal{F}h(r_n) - \exp(-\frac{t_0}{2}\xi^2)\mathcal{F}h(r_0)\|_{E_m^*}\\
\leq&\|\exp(-\frac{t_n}{2}\xi^2)[\mathcal{F}h(r_n) - \mathcal{F}h(r_0)]\|_{E_m^*}\\
+&\|[\exp(-\frac{t_n}{2}\xi^2) - \exp(-\frac{t_0}{2}\xi^2)]\mathcal{F}h(r_0)\|_{E_m^*}\\
=:&I_1(n) + I_2(n).
\end{aligned}$$

We know that

$$I_1(n) = \sup_{\phi\in\mathcal{S}, \|\phi\|_{E_m}\leq 1} |\langle \mathcal{F}h(r_n) - \mathcal{F}h(r_0), \phi \exp(-\frac{t_n}{2}\xi^2)\rangle|.$$

For $\phi \in \mathcal{S}$ we have

$$|\langle \mathcal{F}h(r_n) - \mathcal{F}h(r_0), \phi \exp(-\frac{t_n}{2}\xi^2)\rangle| \leq \|\mathcal{F}h(r_n) - \mathcal{F}h(r_0)\|_{E_N^*} \|\phi \exp(-\frac{t_n}{2}\xi^2)\|_{E_N}$$

and the first term goes to zero as $n \to \infty$ since $\mathcal{F}h \in C_T E_N^*$. We prove that

$$\|\phi \exp(-\frac{t_n}{2}\xi^2)\|_{E_N} \leq C_1 \|\phi\|_{E_{m_1}}, \tag{7}$$

for some $m_1 \geq N$, where $C_1$ is a constant independent of $n$. Let $\alpha$ be a multi index such that $|\alpha| \leq N$. We have

$$(1 + |\xi|^2)^N D^\alpha(\phi \exp(-\frac{t_n}{2}\xi^2))$$

is a linear combination of

$$P(\xi; t_n) D^\gamma \phi(\xi) \exp(-\frac{t_n}{2}\xi^2)$$

where $P(\xi; t_n)$ is a polynomial in $\xi$ with coefficient depending on $t_n$ that can be bounded from above by a polynomial in $\xi$ independent of $t_n$ (possibly depending on $T$) and $|\gamma| \leq N$. It is clear that there exists an integer $m_1$ and a constant $C_1 > 0$ such that $P(\xi; t_n) \leq C_1(1 + |\xi|^2)^{m_1}$. Thus (7) holds.

Concerning $I_2(n)$ we have

$$I_2(n) = \sup_{\phi\in\mathcal{S}, \|\phi\|_{E_m}\leq 1} |\langle \mathcal{F}h(r_0), [\exp(-\frac{t_n}{2}\xi^2) - \exp(-\frac{t_0}{2}\xi^2)]\phi\rangle|,$$

so for $\phi \in \mathcal{S}$ we have

$$\begin{aligned}
&|\langle \mathcal{F}h(r_0), [\exp(-\frac{t_n}{2}\xi^2) - \exp(-\frac{t_0}{2}\xi^2)]\phi\rangle|\\
&\leq \|\mathcal{F}h(r_0)\|_{E_N^*} \frac{t_n - t_0}{2} \|\xi^2 \phi \int_0^1 \exp(-\frac{t_n a + (1-a)t_0}{2}\xi^2) \mathrm{d}a\|_{E_N}.
\end{aligned}$$

Since $t_n - t_0 \to 0$ and $\|\mathcal{F}h(r_0)\|_{E_N^*}$ is finite, it is enough to prove that

$$\|\xi^2 \phi \int_0^1 \exp(-\frac{t_n a + (1-a)t_0}{2}\xi^2) \mathrm{d}a\|_{E_N} \leq C_2 \|\phi\|_{E_{m_2}} \tag{8}$$

for some $m_2$, where $C_2$ is independent of $n$. Let $\alpha$ be a multi index such that $|\alpha| \leq N$. Then

$$(1+|\xi|^2)^N D^\alpha \left( \xi^2 \phi(\xi) \int_0^1 \exp(-\frac{t_n a + (1-a)t_0}{2} \xi^2) \mathrm{d}a \right)$$

is a linear combination of terms of the type

$$P(\xi; t_n) D^\gamma \phi(\xi) \int_0^1 \exp(-\frac{t_n a + (1-a)t_0}{2} \xi^2) \mathrm{d}a$$

where $P(\xi; t_n)$ is a polynomial in $\xi$ with coefficient depending on $t_n$ that can be bounded from above by a polynomial in $\xi$ independent of $t_n$ (possibly depending on $T$) and $|\gamma| \leq N$. As above, there exists an integer $m_2$ and a constant $C_2 > 0$ such that $P(\xi; t_n) \leq C_2(1+|\xi|^2)^{m_2}$. Thus(8) holds.

Finally we conclude that $I_1(n) + I_2(n) \to 0$ as $n \to \infty$ by setting $m = m_1 \vee m_2$ and using the fact that the sequence of seminorms is monotone. □

**Remark 2.3.** *The semigroup $P_t$ and $\nabla$ commute in $\mathcal{S}'$.*

*Indeed let $h \in \mathcal{S}'$. We compute the (generalised) gradient of $P_t h$, that is, for all $\phi \in \mathcal{S}$ we have*

$$\begin{aligned} \langle \nabla P_t h, \phi \rangle &:= -\langle P_t h, div\phi \rangle \\ &= -\langle h, P_t div\phi \rangle \\ &= -\langle h, div P_t \phi \rangle \\ &= \langle \nabla h, P_t \phi \rangle \\ &= \langle P_t \nabla h, \phi \rangle. \end{aligned}$$

2.3. **Estimates in $C^\gamma$ for the heat semigroup.** In this section, we are interested in the action of the semigroup on elements of Besov spaces $\mathcal{C}^\gamma$. These estimates are known as *Schauder's estimates* (for a proof we refer to [4, Lemma 2.5], see also [7] for similar results).

**Lemma 2.4** (Schauder's estimates)**.** *Let $f \in \mathcal{C}^\gamma \subset \mathcal{S}'$ for some $\gamma \in \mathbb{R}$. Then for any $\theta \geq 0$ there exists a constant $c$ such that*

$$\|P_t f\|_{\gamma+2\theta} \leq c t^{-\theta} \|f\|_\gamma \tag{9}$$

*for all $t > 0$.*

*Moreover for $f \in \mathcal{C}^{\gamma+2\theta}$ and for any $\theta \in (0,1)$ we have*

$$\|P_t f - f\|_\gamma \leq c t^\theta \|f\|_{\gamma+2\theta}. \tag{10}$$

Note that from (10), (9) and the semigroup property, it readily follows that if $f \in \mathcal{C}^{\gamma+2\theta}$ for some $0 < \theta < 1$, then for $t > s > 0$ we have

$$\|P_t f - P_s f\|_\gamma \leq c(t-s)^\theta \|f\|_{\gamma+2\theta}. \tag{11}$$

In other words, this means that if $f \in \mathcal{C}^{\gamma+2\theta}$ then $P_\cdot f \in C_T \mathcal{C}^\gamma$ (and in fact it is $\theta$-Hölder continuous in time). We also recall that Bernstein's inequalities hold (see [1, Lemma 2.1] and [7, Appendix A.1]), that is for $\gamma \in \mathbb{R}$ there exists a constant $c > 0$ such that

$$\|\nabla g\|_\gamma \leq c \|g\|_{\gamma+1}, \tag{12}$$

for all $g \in \mathcal{C}^{1+\gamma}$. Using Schauder's and Bernstein's inequalities we can easily obtain a useful estimate on the gradient of the semigroup, as we see below.

**Lemma 2.5.** *Let $\gamma \in \mathbb{R}$ and $\theta \in (0,1)$. If $g \in \mathcal{C}^\gamma$ then for all $t > 0$ we have $\nabla(P_t g) \in \mathcal{C}^{\gamma+2\theta-1}$ and*

$$\|\nabla(P_t g)\|_{\gamma+2\theta-1} \leq c t^{-\theta} \|g\|_\gamma. \tag{13}$$

2.4. **Further properties/tools.** The following is an important estimate which allows to define the pointwise product between certain distributions and functions, which is based on Bony's estimates. For details see [2] or [7, Section 2.1]. Let $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^{-\beta}$ with $\alpha - \beta > 0$ and $\alpha, \beta > 0$. Then the 'pointwise product' $f\, g$ is well-defined as an element of $\mathcal{C}^{-\beta}$ and there exists a constant $c > 0$ such that

$$\|f\, g\|_{-\beta} \leq c \|f\|_\alpha \|g\|_{-\beta}. \tag{14}$$

**Remark 2.6.** *Using* (14) *it is not difficult to see that if $f \in C_T \mathcal{C}^\alpha$ and $g \in C_T \mathcal{C}^{-\beta}$ then the product is also continuous with values in $\mathcal{C}^{-\beta}$, and*

$$\|f\, g\|_{C_T \mathcal{C}^{-\beta}} \leq c \|f\|_{C_T \mathcal{C}^\alpha} \|g\|_{C_T \mathcal{C}^{-\beta}}. \tag{15}$$

## 3. The spaces $D\mathcal{C}^\gamma$ and the action of the semigroup

In this section we introduce some other function spaces that will be central in the analysis of the PDEs in this paper if we are to have solutions with linear growth. The idea is to have functions with the same regularity as the $\mathcal{C}^\gamma$-spaces locally, that allow linear growth at infinity. On these spaces we will show how the heat semigroup acts in terms of regularity, both in the time- and in the space-variable.

For $\gamma \in (0,1)$ we define space $D\mathcal{C}^\gamma$ as

$$D\mathcal{C}^\gamma := \{h : \mathbb{R}^d \to \mathbb{R} \text{ differentiable function s.t. } \nabla h \in \mathcal{C}^\gamma\}.$$

Note that the following inclusion holds:

$$\mathcal{C}^{1+\alpha} \subset D\mathcal{C}^\alpha.$$

On $D\mathcal{C}^\alpha$ we can introduce a topology, induced by the norm

$$\|h\|_{D\mathcal{C}^\gamma} := (|h(0)| + \|\nabla h\|_\gamma). \tag{16}$$

If $h \in D\mathcal{C}^\alpha$ then there exists a constant (which is $h(0)$) and a function $\tilde{h} \in \mathcal{C}^\alpha$ (multidimensional) such that $h(x) = h(0) + x \cdot \tilde{h}$. Indeed, that function $\tilde{h}$ is given by $\int_0^1 \nabla h(ax) \mathrm{d}a$.

**Lemma 3.1.** *$(D\mathcal{C}^\alpha, \|\cdot\|_{D\mathcal{C}^\alpha})$ is a Banach space.*

*Proof.* Let $(h_n)_n$ be a Cauchy sequence in $D\mathcal{C}^\alpha$. Then $h_n \in C^1$ and since $\mathbb{R}$ and $\mathcal{C}^\alpha$ are complete, we know that $h_n(0) \to c \in \mathbb{R}$ and $\nabla h_n \to g$ in $\mathcal{C}^\alpha$ hence uniformly. Now we write $h_n(x) = h_n(0) + x \int_0^1 \nabla h_n(ax)\mathrm{d}a$. We define $h(x) = c + x \int_0^1 g(ax)\mathrm{d}a$, so that $\lim_{n\to\infty} h_n(x) = h(x)$. It is obvious that $c = h(0)$. Now we notice that $\nabla h \in \mathcal{S}'$ so it is left to prove that $\nabla h = g$ in $\mathcal{S}'$ to conclude. For any test function $\phi \in \mathcal{S}$ we have $\langle \nabla h_n, \phi\rangle = \langle h_n, -\mathrm{div}(\phi)\rangle \to \langle h, -\mathrm{div}(\phi)\rangle$ as $n \to \infty$. On the other hand $\langle \nabla h_n, \phi\rangle \to \langle g, \phi\rangle$ hence we conclude $g = \nabla h$. □

Next we study the mapping properties of the semigroup $P_t$ on $D\mathcal{C}^\alpha$ (and on the classical spaces $\mathcal{C}^{\alpha+1}$) for some fixed $\alpha \in (0,1)$. First we prove an inequality that is the analogous of Schauder's estimate (9) with $\theta = 0$ on $D\mathcal{C}^\alpha$.

**Lemma 3.2.** *If $h \in D\mathcal{C}^\alpha$, then*

$$\sup_{s\in[0,T]} \|P_s h\|_{D\mathcal{C}^\alpha} \le c\|h\|_{D\mathcal{C}^\alpha}. \tag{17}$$

*Proof.* Using the definition of the norm in $D\mathcal{C}^\alpha$ we have

$$\|P_s h\|_{D\mathcal{C}^\alpha} = |(P_s h)(0)| + \|\nabla P_s h\|_\alpha =: B_1(s) + B_2(s).$$

Using the kernel of the semigroup and writing $h(x) = h(0) + x \cdot \int_0^1 \nabla h(ax)\mathrm{d}a$ we get

$$\begin{aligned} B_1(s) &= |\int_{\mathbb{R}^d} p_s(y)h(y)\mathrm{d}y| \\ &\le |\int_{\mathbb{R}^d} p_s(y)h(0)\mathrm{d}y| + |\int_{\mathbb{R}^d} p_s(y)y \cdot \int_0^1 \nabla h(ay)\mathrm{d}a\mathrm{d}y| \\ &\le |h(0)|1 + \int_{\mathbb{R}^d} p_s(y)|y| \sup_x |\nabla h(x)|\mathrm{d}y \\ &\le |h(0)| + c\|\nabla h\|_\alpha \le c\|h\|_{D\mathcal{C}^\alpha}. \end{aligned}$$

On the other hand, since $\nabla$ and $P_t$ commute by Remark 2.3, we have

$$B_2(s) = \|\nabla P_s h\|_\alpha = \|P_s \nabla h\|_\alpha \le c\|\nabla h\|_\alpha \le c\|h\|_{D\mathcal{C}^\alpha},$$

having used Schauder's estimate (9). This proves (17). □

**Lemma 3.3.** *Let $\alpha \in (0,1)$.*

(i) *The semigroup $P_t$ maps $\mathcal{C}^{\alpha+1}$ into itself. Moreover if $h \in \mathcal{C}^{1+\alpha+\nu}$ for some $\nu > 0$ such that $\alpha + \nu \in (0,1)$, then $P.h \in C_T\mathcal{C}^{1+\alpha}$.*

(ii) *The semigroup $P_t$ maps $D\mathcal{C}^\alpha$ into itself. Moreover if $h \in D\mathcal{C}^{\alpha+\nu}$ for some $\nu > 0$ such that $\alpha + \nu \in (0,1)$, then $P.h \in C_T D\mathcal{C}^\alpha$.*

*Proof. Item (i)* This is an obvious consequence of Schauder's estimate (Lemma 2.4) and equation (11).

*Item (ii)* Let $h \in D\mathcal{C}^\alpha \subset \mathcal{S}'$. Let $t \ge 0$ be fixed. By Remark 2.3 $\nabla P_t h = P_t \nabla h$, so that $\nabla P_t h \in \mathcal{C}^\alpha$ (and this automatically implies that $P_t h$ is a differentiable function of $x$).

Next we show that $t \mapsto P_t h$ is continuous with values in $D\mathcal{C}^\alpha$ if $h \in D\mathcal{C}^{\alpha+\nu}$. We need to show that for each $t \geq 0$ we have

$$\begin{aligned} &\|P_{t+\varepsilon}h - P_t h\|_{D\mathcal{C}^\alpha} \\ &= |(P_{t+\varepsilon}h)(0) - (P_t h)(0)| + \|\nabla P_{t+\varepsilon}h - \nabla P_t h\|_\alpha \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{18}$$

Concerning first term in (18) we note that since $h \in D\mathcal{C}^{\alpha+\nu}$ then $\nabla h$ belongs to $C^{\alpha+\nu}$, and $\|\nabla h\|_\infty \leq \|\nabla h\|_{\mathcal{C}^{\alpha+\nu}}$. We observe that for any $t \geq 0$ and $x \in \mathbb{R}^d$ we have $(P_t h)(x) = \mathbb{E}[h(W_t^x)]$ where $(W_t^x)$ is a Brownian motion starting at $W_0 = x$. Hence

$$\begin{aligned} |(P_{t+\varepsilon}h)(0) - (P_t h)(0)| &= |\mathbb{E}[h(W_{t+\varepsilon}^0) - h(W_t^0)]| \\ &\leq \mathbb{E}[|h(W_{t+\varepsilon}^0) - h(W_t^0)|] \\ &\leq \|\nabla h\|_\infty \mathbb{E}[|W_{t+\varepsilon}^0 - W_t^0|] \\ &= \|\nabla h\|_{\alpha+\nu} \mathbb{E}[|W_\varepsilon^0|] \\ &= \sqrt{\tfrac{2}{\pi}}\varepsilon^{\frac{1}{2}} \|\nabla h\|_{\alpha+\nu}. \end{aligned} \tag{19}$$

The second term in (18) can be bounded by

$$\|\nabla P_{t+\varepsilon}h - \nabla P_t h\|_\alpha \leq c\varepsilon^{\nu/2} \|\nabla h\|_{\alpha+\nu} \tag{20}$$

by using the fact that $\nabla$ and $P_t$ commute by Remark 2.3 together with (11) $\theta = \nu/2$.

Putting (19) and (20) together we get

$$\|P_{t+\varepsilon}h - P_t h\|_{D\mathcal{C}^\alpha} \leq c\varepsilon^{\frac{\nu\wedge 1}{2}} \|\nabla h\|_{\alpha+\nu} \leq c\varepsilon^{\frac{\nu\wedge 1}{2}} \|h\|_{D\mathcal{C}^{\alpha+\nu}},$$

which shows $P_\cdot h \in C_T D\mathcal{C}^\alpha$ as wanted. □

**Lemma 3.4.** *Let $\alpha \in (0,1)$.*

(i) *Let $h \in C_T\mathcal{C}^{\alpha+1}$. Then $\int_\cdot^T P_{s-\cdot}h(s)\mathrm{d}s \in C_T\mathcal{C}^{\alpha+1}$ and $\|\int_\cdot^T P_{s-\cdot}h(s)\mathrm{d}s\|_{C_T\mathcal{C}^{\alpha+1}} \leq c\|h\|_{C_T\mathcal{C}^{\alpha+1}}$.*

(ii) *Let $h \in C_T D\mathcal{C}^\alpha$. Then $\int_\cdot^T P_{s-\cdot}h(s)\mathrm{d}s \in C_T D\mathcal{C}^\alpha$ and $\|\int_\cdot^T P_{s-\cdot}h(s)\mathrm{d}s\|_{C_T D\mathcal{C}^\alpha} \leq c\|h\|_{C_T D\mathcal{C}^\alpha}$.*

*Proof.* We first show that given $h \in C_T D\mathcal{C}^\alpha$ (resp. $h \in C_T\mathcal{C}^{\alpha+1}$), then $\int_0^{T-\cdot} P_s h(s+\cdot)\mathrm{d}s \in C_T D\mathcal{C}^\alpha$ (resp. $\int_0^{T-\cdot} P_s h(s+\cdot)\mathrm{d}s \in C_T\mathcal{C}^{\alpha+1}$), which is equivalent to the first part of the claim in (ii) (resp. in (i)). To this aim, let $t_n \to t_0$. We have

$$\begin{aligned} &\int_0^{T-t_n} P_s h(s+t_n)\mathrm{d}s - \int_0^{T-t_0} P_s h(s+t_0)\mathrm{d}s \\ &= \int_0^{T-t_0} P_s[h(s+t_n) - h(s+t_0)]\mathrm{d}s + \int_{T-t_0}^{T-t_n} P_s h(s+t_n)\mathrm{d}s. \end{aligned} \tag{21}$$

We denote by $\delta(h;s)$ the modulus of continuity of $h$ in $D\mathcal{C}^\alpha$ (resp. in $\mathcal{C}^{\alpha+1}$). Then the first integral in (21) is bounded in the $D\mathcal{C}^\alpha$-norm using (17) (resp.

in the $\mathcal{C}^{\alpha+1}$-norm using (9) with $\theta = 0$ and $\gamma = \alpha + 1$) to get

$$\begin{aligned}
\| \int_0^{T-t_0} & P_s[h(s+t_n) - h(s+t_0)]\mathrm{d}s\|_{D\mathcal{C}^\alpha} \\
&\leq \int_0^{T-t_0} \|P_s[h(s+t_n) - h(s+t_0)]\|_{D\mathcal{C}^\alpha}\mathrm{d}s \\
&\leq c\int_0^{T-t_0} \|h(s+t_n) - h(s+t_0)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \\
&= c\int_0^{T-t_0} \delta(h; t_n - t_0)\mathrm{d}s \\
&= c(T-t_0)\delta(h; t_n - t_0),
\end{aligned}$$

respectively

$$\| \int_0^{T-t_0} P_s[h(s+t_n) - h(s+t_0)]\mathrm{d}s\|_{\mathcal{C}^{\alpha+1}} \leq c(T-t_0)\delta(h; t_n - t_0),$$

which tends to 0 as $n \to \infty$. The second integral in (21) is bounded again using (17) (resp. using (9) with $\theta = 0$ and $\gamma = \alpha + 1$) to get

$$\begin{aligned}
\| \int_{T-t_0}^{T-t_n} P_s h(s+t_n)\mathrm{d}s\|_{D\mathcal{C}^\alpha} &\leq \left| \int_{T-t_0}^{T-t_n} \|P_s h(s+t_n)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \right| \\
&\leq c\left| \int_{T-t_0}^{T-t_n} \|h(s+t_n)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \right| \\
&= c\left| \int_{T-t_0}^{T-t_n} \|h\|_{C_T D\mathcal{C}^\alpha}\mathrm{d}s \right| \\
&= c|t_0 - t_n| \|h\|_{C_T D\mathcal{C}^\alpha},
\end{aligned}$$

respectively

$$\| \int_{T-t_0}^{T-t_n} P_s h(s+t_n)\mathrm{d}s\|_{\mathcal{C}^{\alpha+1}} \leq c|t_0 - t_n| \|h\|_{C_T\mathcal{C}^{\alpha+1}},$$

which tends to 0 as $n \to \infty$.

It is left to prove that $\| \int_\cdot^T P_{s-\cdot} h(s)\mathrm{d}s\|_{C_T D\mathcal{C}^\alpha} \leq c\|h\|_{C_T D\mathcal{C}^\alpha}$ for point (ii) (resp. $\| \int_\cdot^T P_{s-\cdot} h(s)\mathrm{d}s\|_{C_T\mathcal{C}^{\alpha+1}} \leq c\|h\|_{C_T\mathcal{C}^{\alpha+1}}$ for point (i)). Using again (17) (resp. (9) with $\theta = 0$) we have

$$\begin{aligned}
\| \int_\cdot^T P_{s-\cdot} h(s)\mathrm{d}s\|_{C_T D\mathcal{C}^\alpha} &= \sup_{t\in[0,T]} \| \int_t^T P_{s-t} h(s)\mathrm{d}s\|_{D\mathcal{C}^\alpha} \\
&\leq \sup_{t\in[0,T]} \int_t^T \|P_{s-t} h(s)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \\
&\leq c \sup_{t\in[0,T]} \int_t^T \|h(s)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \\
&\leq cT\|h\|_{C_T D\mathcal{C}^\alpha},
\end{aligned}$$

respectively

$$\| \int_{\cdot}^{T} P_{s-\cdot} h(s) \mathrm{d}s \|_{C_T \mathcal{C}^{\alpha+1}} \leq cT \|h\|_{C_T \mathcal{C}^{\alpha+1}},$$

which is the claim. □

In fact, it turns out that in the $\mathcal{C}^\alpha$ spaces a stronger continuity result will be needed, which is the following.

**Lemma 3.5.** *If $h \in C_T \mathcal{C}^{(-\beta)+}$ then $\int_0^{\cdot} P_{\cdot - s} h(s) \mathrm{d}s \in C_T \mathcal{C}^{1+\alpha}$ with any $\alpha \in [\beta, 1-\beta)$.*

*Proof.* This is the adaptation of [9, Lemma 3.2] in the special case $h \in C_T \mathcal{C}^{-\beta} \subset L^\infty([0,T]; \mathcal{C}^{-\beta})$. □

Analogously as for the $\mathcal{C}^{\gamma+}$-spaces, for $\gamma > 0$ we also introduce the spaces

$$D\mathcal{C}^{\gamma+} := \cup_{\alpha > \gamma} D\mathcal{C}^{\alpha}, \qquad D\mathcal{C}^{\gamma-} := \cap_{\alpha < \gamma} D\mathcal{C}^{\alpha}.$$

We will also use the spaces $C_T D\mathcal{C}^{\gamma+} := C([0,T]; D\mathcal{C}^{\gamma+})$. We remark that $f \in C_T D\mathcal{C}^{\gamma+}$ if and only if there exists $\alpha > \gamma$ such that $f \in C_T D\mathcal{C}^{\alpha}$, see [10, Remark B.1]. Similarly, we use the space $C_T D\mathcal{C}^{\gamma-} := C([0,T]; D\mathcal{C}^{\gamma-})$; we observe in particular that if $f \in C_T D\mathcal{C}^{\gamma-}$ then for any $\alpha < \gamma$ we have $f \in C_T D\mathcal{C}^{\alpha}$.

## 4. Main results

In this section we prove existence, uniqueness, continuity properties and various bounds for solutions to a class of parabolic PDEs with unbounded terminal condition. This means that said solutions too are unbounded, indeed they live in the space $C_T D\mathcal{C}^{\beta+}$. We also consider a special case of this class where terminal conditions are bounded, hence also the solutions are bounded, i.e. they live in $C_T \mathcal{C}^{(1+\beta)+}$.

### 4.1. Assumptions.

We introduce here various assumptions concerning distribution-valued functions ($b$ respectively $g$) needed below in the paper.

**Assumption A1.** *Let $0 < \beta < 1/2$ and $b \in C_T \mathcal{C}^{(-\beta)+}(\mathbb{R}^d)$. In particular $b \in C_T \mathcal{C}^{-\beta}(\mathbb{R}^d)$. Notice that $b$ is a column vector.*

Next we introduce two assumptions concerning $g$ and $v_T$.

**Assumption A2.** *We suppose that $g \in C_T \mathcal{C}^{(-\beta)+}$ and $v_T \in D\mathcal{C}^{(1-\beta)-}$.*

**Assumption A3.** *We suppose that $g \in C_T \mathcal{C}^{(-\beta)+}$ and $v_T \in \mathcal{C}^{(2-\beta)-}$.*

The main difference between Assumption A3 and Assumption A2 is that in the latter we allow the terminal condition to be unbounded, in particular we can choose $v_T = \mathrm{id}$, while in the former the identity function is excluded.

4.2. **A class of PDEs with drifts in Besov spaces.** Let $b$ fulfill Assumption A1 for the rest of Section 4. Let $v_T \in \mathcal{S}'$ and $t \mapsto g(t,\cdot)$ be continuous in $\mathcal{S}'$. We consider here PDEs of the form

$$\left\{ \begin{array}{l} \partial_t v + \frac{1}{2}\Delta v + \nabla v\, b = \lambda v + g \\ v(T) = v_T. \end{array} \right. \tag{22}$$

We consider weak and mild solutions, both defined in the space $C_T D\mathcal{C}^\beta$, as detailed below. To shorten notation, we define

$$G(v) := \lambda v + g.$$

**Definition 4.1.** *Let $v \in C_T D\mathcal{C}^\beta$. We say that $v$ is a weak solution of* (22) *if for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have that $v$ satisfies*

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) v_T(x)\mathrm{d}x - \int_{\mathbb{R}^d} \varphi(x) v(t,x)\mathrm{d}x + \int_t^T \int_{\mathbb{R}^d} \frac{1}{2}\Delta\varphi(x) v(s,x)\mathrm{d}x\,\mathrm{d}s \\ + \int_t^T \int_{\mathbb{R}^d} \varphi(x)\left(\nabla v(s,x) b(s,x)\right)\mathrm{d}x\mathrm{d}s = \int_t^T \int_{\mathbb{R}^d} \varphi(x) G(v)(s,x)\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{23}$$

*for all $t \in [0,T]$.*

Notice that the notation $\int_{\mathbb{R}^d} \varphi(x)\left(\nabla v(s,x) b(s,x)\right)\mathrm{d}x$ is only formal because $\nabla v(s,\cdot) b(s,\cdot)$ is a distribution. In practice when we write the integral we mean the dual pairing with $\varphi$, namely $\langle \varphi, \nabla v(s) b(s)\rangle$, where the pairing in $\mathcal{S}, \mathcal{S}'$ is well-defined as an element in $\mathcal{C}^{(-\beta)+}$ via the pointwise product (14).

**Definition 4.2.** *Let $v \in C_T D\mathcal{C}^\beta$. We say that $v$ is a mild solution of* (22) *if $v$ satisfies*

$$v(t) = P_{T-t} v_T + \int_t^T P_{s-t}\left(\nabla v(s) b(s)\right)\mathrm{d}s - \int_t^T P_{s-t}(G(v)(s))\mathrm{d}s, \tag{24}$$

*for all $t \in [0,T]$.*

Note that for each $s \in [0,T]$ the product $\nabla v(s)\, b(s)$ appearing in (23) and (24) is well-defined as an element of $\mathcal{C}^{(-\beta)+}$ using the pointwise product (14), thanks to Assumption A1. Indeed since $v \in C_T D\mathcal{C}^\beta$ and $b \in C_T \mathcal{C}^{-(\beta)+}$ we can always choose $\varepsilon > 0$ such that $b \in C_T \mathcal{C}^{-\beta+\varepsilon}$ so that $-\beta + \varepsilon + \beta = \varepsilon > 0$ and (15) holds. Moreover both integrals are well-defined as Bochner integrals with values in $\mathcal{S}'$ because $(s,r) \mapsto P_s h(r)$ is jointly continuous with values in $\mathcal{S}'$ (where $h$ is either $\nabla v\, b$ or $G(v)$, and the continuity follows from Lemma 2.2).

For future use, it is convenient to properly define the singular operator $\mathcal{L}$, formally given by $\mathcal{L}f = \partial_t f + \frac{1}{2}\Delta f + \nabla f\, b$.

**Definition 4.3.** *Let $b$ satisfy Assumption A1. The operator $\mathcal{L}$ is defined as*

$$\begin{array}{rccl} \mathcal{L}: & \mathcal{D}^0_{\mathcal{L}} & \to & \{\mathcal{S}'\text{-valued continuous functions}\} \\ & f & \mapsto & \mathcal{L}f := \dot f + \frac{1}{2}\Delta f + \nabla f\, b, \end{array}$$

*where*

$$\mathcal{D}_{\mathcal{L}}^0 := C_T D\mathcal{C}^\beta \cap C^1([0,T];\mathcal{S}').$$

*Here* $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *and the function* $\dot{f} : [0,T] \to \mathcal{S}'$ *is the time-derivative of* $f$*. Note also that* $\nabla f\, b$ *is well-defined and continuous using* (15) *and Assumption A1. The Laplacian* $\Delta$ *is intended in the weak sense.*

**Remark 4.4.** *We observe that if* $v \in C_T D\mathcal{C}^\beta$ *is a weak solution, then it is automatically differentiable in time with continuous derivative in* $\mathcal{S}'$*, hence* $v \in \mathcal{D}_{\mathcal{L}}^0$*. The same is true for* $v \in C_T\mathcal{C}^{1+\beta}$ *by the inclusion of the spaces.*

Using the operator $\mathcal{L}$ defined in Definition 4.3 and Remark 4.4, we see that PDE (22) rewrites as

$$\begin{cases} \mathcal{L}v = \lambda v + g \\ v(T) = v_T. \end{cases}$$

**Proposition 4.5.** *Weak and mild solutions of* (22) *are equivalent in* $C_T D\mathcal{C}^\beta$*.*

*Proof. (i) mild implies weak.* Let $v \in C_T D\mathcal{C}^\beta$ be a mild solution. For any $\varphi \in \mathcal{S}$ we have

$$\begin{aligned} \int_t^T \langle v(s), \frac{1}{2}\Delta\varphi\rangle \mathrm{d}s &= \int_t^T \langle P_{T-s}v_T, \frac{1}{2}\Delta\varphi\rangle \mathrm{d}s \\ &\quad + \int_t^T \int_s^T \langle P_{r-s}\nabla v(r) b(r), \frac{1}{2}\Delta\varphi\rangle \mathrm{d}r\mathrm{d}s \\ &\quad - \int_t^T \int_s^T \langle P_{r-s} G(v)(r), \frac{1}{2}\Delta\varphi\rangle \mathrm{d}r\mathrm{d}s. \end{aligned} \tag{25}$$

The first term on the RHS of (25) gives

$$\begin{aligned} \int_t^T \langle P_{T-s}v_T, \frac{1}{2}\Delta\varphi\rangle \mathrm{d}s &= \int_t^T \langle \frac{1}{2}\Delta P_{T-s}v_T, \varphi\rangle \mathrm{d}s \\ &= \int_0^{T-t} \langle \frac{\mathrm{d}}{\mathrm{d}s} P_s v_T, \varphi\rangle \mathrm{d}s \\ &= \langle P_{T-t}v_T, \varphi\rangle - \langle v_T, \varphi\rangle. \end{aligned}$$

The second and third terms on the RHS of (25) give

$$\begin{aligned}\int_t^T \int_s^T &\langle P_{r-s}[\nabla v(r) b(r) - G(v)(r)], \frac{1}{2}\Delta\varphi\rangle \mathrm{d}r\mathrm{d}s \\ &= \int_t^T \int_0^{r-t} \langle P_s[\nabla v(r) b(r) - G(v)(r)], \frac{1}{2}\Delta\varphi\rangle \mathrm{d}s\mathrm{d}r \\ &= \int_t^T \int_0^{r-t} \langle \frac{\mathrm{d}}{\mathrm{d}s} P_s[\nabla v(r) b(r) - G(v)(r)], \varphi\rangle \mathrm{d}s\mathrm{d}r \\ &= \int_t^T \langle P_{r-t}[\nabla v(r) b(r) - G(v)(r)], \varphi\rangle \mathrm{d}r \\ &\quad - \int_t^T \langle [\nabla v(r) b(r) - G(v)(r)], \varphi\rangle \mathrm{d}r.\end{aligned}$$

Putting these into (25) we get

$$\begin{aligned}\int_t^T \langle v(s), \frac{1}{2}\Delta\varphi\rangle \mathrm{d}s =& \langle P_{T-t} v_T, \varphi\rangle - \langle v_T, \varphi\rangle + \int_t^T \langle P_{r-t}[\nabla v(r) b(r) - G(v)(r)], \varphi\rangle \mathrm{d}r \\ & - \int_t^T \langle [\nabla v(r) b(r) - G(v)(r)], \varphi\rangle \mathrm{d}r \\ =& \langle v(t), \varphi\rangle - \langle v_T, \varphi\rangle + \int_t^T \langle [\nabla v(r) b(r) - G(v)(r)], \varphi\rangle \mathrm{d}r\end{aligned}$$

which shows that $v$ is also a weak solution.

*(ii) weak implies mild.* We proceed as follows. Given a weak solution $v \in C_T D\mathcal{C}^\beta$ that satisfies (23) we define

$$u(t) := P_{T-t} v_T + \int_t^T P_{s-t} \left(\nabla v(s) b(s)\right) \mathrm{d}s - \int_t^T P_{s-t} G(v)(s) \mathrm{d}s. \tag{26}$$

We see that $u$ is a mild solution of the heat equation with extra source terms involving $v$, more specifically of

$$\partial_t u + \frac{1}{2}\Delta u = G(v) - \nabla v\, b; \qquad u(T) = v_T.$$

By using (i) with $\lambda = 0$ and $g = G(v) - \nabla v\, b$ we have that $u$ is also a weak solution of the above PDE. Now we take the difference $\bar{v} = v - u$ and see that $\bar{v}$ fulfills

$$\bar{v}(t, \cdot) = -\int_t^T \frac{1}{2}\Delta \bar{v}(s, \cdot) \mathrm{d}s; \qquad \bar{v}(T) = 0,$$

hence $\bar{v}$ is a weak solution of the heat equation with zero terminal condition so we have $\bar{v} = 0$, which implies that $u = v$ and so $u$ is a mild solution by (26). □

4.3. **Linear growth solutions.** In this subsection we consider equation (22) and pick a terminal condition $v_T$ fulfilling Assumption A2. We will show below that solutions of (22) exist in the space $C_T D\mathcal{C}^{(1-\beta)-}$ and are unique in the space $C_T D\mathcal{C}^{\beta+}$. If furthermore the terminal condition is bounded (Assumption A3) then the solution will also be bounded.

Given $\rho \geq 0$, we introduce an equivalent norm in $C_T D\mathcal{C}^\alpha$, respectively $C_T\mathcal{C}^{\alpha+1}$, defined as

$$\|f\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} := \sup_{t\in[0,T]} e^{-\rho(T-t)} \left(|f(t,0)| + \|\nabla f(t)\|_\alpha\right), \tag{27}$$

respectively

$$\|f\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} := \sup_{t\in[0,T]} e^{-\rho(T-t)} \left(\sup_x |f(t,x)| + \|\nabla f(t)\|_\alpha\right). \tag{28}$$

Notice that those norms are equivalent to those defined in (16) (resp. (4)). With these norms the pointwise products estimates corresponding to those from Remark 2.6 will become, for $\alpha > \beta$,

$$\|fg\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq c\|f\|^{(\rho)}_{C_T\mathcal{C}^\alpha}\|g\|_{C_T\mathcal{C}^{-\beta}}. \tag{29}$$

We start with a preliminary result.

**Lemma 4.6.** *Let $\ell \in C_T\mathcal{C}^{-\beta}$ and $\rho \geq 1$. Then for every $t \in [0,T]$ and for every $\alpha \in [\beta, 1-\beta)$ we have*

$$\|\int_t^T P_{s-t}\ell(s)\mathrm{d}s\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq c\|\ell\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}}, \tag{30}$$

*and in particular,*

$$\|\int_t^T P_{s-t}\ell(s)\mathrm{d}s\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} \leq c\|\ell\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}},$$

*where $c$ depends on $\alpha$ and $\beta$.*

*Proof.* We recall that for $f \in C_T\mathcal{C}^{\alpha+1}$ then $f \in C_T D\mathcal{C}^\alpha$ and $\|f\|_{C_T D\mathcal{C}^\alpha} \leq \|f\|_{C_T\mathcal{C}^{\alpha+1}}$ by (27) and (28). For this reason, we will only prove (30). We bound each term in the $\rho$-equivalent norm (28) in $\mathcal{C}^{\alpha+1}$ separately. Let us denote by $f(t,x) := \int_t^T (P_{s-t}\ell(s))(x)\mathrm{d}s$. The sup term in (28) gives

$$
\begin{aligned}
\sup_x |f(t,x)| &= \sup_x \left| \int_t^T (P_{s-t}\ell(s))(x)\mathrm{d}s \right| \\
&= \left\| \int_t^T P_{s-t}\ell(s)\mathrm{d}s \right\|_\infty \\
&\leq \left\| \int_t^T P_{s-t}\ell(s)\mathrm{d}s \right\|_\alpha \\
&\leq \int_t^T \|P_{s-t}\ell(s)\|_\alpha \mathrm{d}s \\
&\leq c \int_t^T (s-t)^{-\frac{\alpha+\beta}{2}} \|\ell(s)\|_{-\beta}\mathrm{d}s,
\end{aligned}
$$

having used (9) from Lemma 2.4. Now multiplying by $e^{-\rho(T-t)}$ and taking the supremum over $t$, using (28) we get

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\rho(T-t)} \sup_x |f(t,x)| \\
&\qquad \leq c \sup_{t\in[0,T]} \int_t^T e^{-\rho(s-t)}(s-t)^{-\frac{\alpha+\beta}{2}} e^{-\rho(T-s)} \|\ell(s)\|_{-\beta}\mathrm{d}s \\
&\qquad \leq c\|\ell\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \sup_{t\in[0,T]} \int_t^T e^{-\rho(s-t)}(s-t)^{-\frac{\alpha+\beta}{2}}\mathrm{d}s. \qquad (31)
\end{aligned}
$$

The latter integral can be bounded noting that $\theta := \frac{\alpha+\beta}{2} < 1$ by choice of $\alpha$, thus

$$
\begin{aligned}
\int_t^T e^{-\rho(s-t)}(s-t)^{-\theta}\mathrm{d}s &\leq \int_0^\infty e^{-s\rho}s^{-\theta}\mathrm{d}s \\
&\leq \int_0^\infty e^{-x}x^{-\theta}\rho^{-1+\theta}\mathrm{d}x \\
&= \Gamma(-\theta+1)\rho^{-1+\theta},
\end{aligned}
$$

where

$$
\Gamma(\eta) := \int_0^\infty e^{-x}x^{\eta-1}\mathrm{d}x \qquad (32)
$$

denotes the Gamma function. Thus (31) gives

$$
\sup_{t\in[0,T]} e^{-\rho(T-t)} \sup_x |f(t,x)| \leq c\|\ell\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-2}{2}}, \qquad (33)
$$

where $c$ depends on $\alpha$ and $\beta$.

The term with the $\alpha$-norm of $\nabla f$ in (28) is bounded with similar computations as above but using (13) in place of (9) to get

$$\sup_{t\in[0,T]} e^{-\rho(T-t)} \|\nabla f(t)\|_\alpha$$
$$\leq c \sup_{t\in[0,T]} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta+1}{2}} \|\ell(s)\|_{-\beta} \mathrm{d}s.$$

Proceeding as between (31) and (33) and using the fact that $\frac{\alpha+\beta+1}{2} < 1$, we get

$$\sup_{t\in[0,T]} e^{-\rho(T-t)} \|\nabla f(t)\|_\alpha \leq c\|\ell\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}}. \tag{34}$$

Combining (33) and (34), and using the fact that $\rho^{\frac{\alpha+\beta-2}{2}} \leq \rho^{\frac{\alpha+\beta-1}{2}}$ since $\rho \geq 1$, we conclude. □

**Theorem 4.7.** *Let $b$ satisfy Assumption A1.*

(i) *Let $v_T$ and $g$ satisfy Assumption A2. Then there exists a mild solution $v$ to* (22) *in $C_T D\mathcal{C}^{(1-\beta)-}$ which is unique in $C_T D\mathcal{C}^{\beta}$.*
(ii) *Let $v_T$ and $g$ satisfy Assumption A3 (in particular $v_T$ is bounded). Then the unique mild solution $v$ of PDE* (22) *is also bounded, more precisely $v \in C_T\mathcal{C}^{(2-\beta)-}$.*

**Remark 4.8.** *One could relax Assumption A2 (resp. Assumption A3) for $v_T$ and only ask that $v_T \in D\mathcal{C}^{\beta+}$ (resp. $v_T \in \mathcal{C}^{(1+\beta)+}$). In this case the unique solution would no longer belong to $C_T D\mathcal{C}^{(1-\beta)-}$ (resp. $C_T\mathcal{C}^{(2-\beta)-}$) but only to $C_T D\mathcal{C}^{\beta+}$ (resp. $C_T\mathcal{C}^{(1+\beta)+}$).*

*Proof of Theorem 4.7.* We start with an arbitrary $\alpha \in (\beta, 1-\beta)$. The case $\alpha = \beta$ will be explained at the end of the proof. Let $\mathcal{T}$ denote the solution operator, namely for $v \in C([0,T]; D\mathcal{C}^\alpha)$ we define $\mathcal{T}v$ as

$$\mathcal{T}v(t) := P_{T-t}v_T + \int_t^T P_{s-t}\left(\nabla v(s) b(s)\right) \mathrm{d}s - \int_t^T P_{s-t}(\lambda v(s) + g(s))\mathrm{d}s. \tag{35}$$

We prove both items of the theorem in two steps, first showing stability and then the contraction property. Notice that Assumption A3 implies Assumption A2.

**Step 1 - stability.** We suppose Assumption A2 (resp. Assumption A3). We show that $\mathcal{T} : C_T D\mathcal{C}^\alpha \to C_T D\mathcal{C}^\alpha$ (resp. $\mathcal{T} : C_T\mathcal{C}^{\alpha+1} \to C_T\mathcal{C}^{\alpha+1}$).
The term $P_{T-t}v_T \in D\mathcal{C}^\alpha$ (resp. $P_{T-t}v_T \in \mathcal{C}^{\alpha+1}$) is continuous in $t$ by Lemma 3.3, item (ii) (resp. item (i)) since $v_T \in D\mathcal{C}^{\alpha+\nu}$ (resp. $v_T \in \mathcal{C}^{1+\alpha+\nu}$) for all $\nu > 0$ such that $\alpha + \nu < 1 - \beta$ by Assumption A2 (resp. Assumption A3). Since $v \in C_T D\mathcal{C}^\alpha$ (resp. $v \in C_T\mathcal{C}^{\alpha+1}$) and $b \in C_T\mathcal{C}^{(-\beta)+}$, then by Remark 2.6 $\nabla v b \in C_T\mathcal{C}^{(-\beta)+}$. Moreover $g \in C_T\mathcal{C}^{(-\beta)+}$ by assumption. Thus we can apply Lemma 3.5 to deduce that $\int_\cdot^T P_{s-\cdot}\left(\nabla v(s) b(s)\right) \mathrm{d}s + \int_\cdot^T P_{s-\cdot} g(s)\mathrm{d}s \in C_T\mathcal{C}^{\alpha+1} \subset C_T D\mathcal{C}^\alpha$.

Finally by Lemma 3.4 item (ii) (resp. item (i)), $t \mapsto \int_t^T P_{s-t}\lambda v(s)\mathrm{d}s$ is continuous with values in $D\mathcal{C}^\alpha$ (resp. $\mathcal{C}^{\alpha+1}$).

**Step 2 - contraction.** Next we show that $\mathcal{T}$ is a contraction in $C_T D\mathcal{C}^\alpha$ (resp. $C_T\mathcal{C}^{\alpha+1}$).

To this aim it is convenient to use the equivalent norm in $C_T D\mathcal{C}^\alpha$ (resp. $C_T\mathcal{C}^{\alpha+1}$) introduced in (27) (resp. (28)). Let $v_1, v_2 \in C_T D\mathcal{C}^\alpha$ (resp. $v_1, v_2 \in C_T\mathcal{C}^{\alpha+1}$). Then

$$\begin{aligned}\mathcal{T}v_1(t) - \mathcal{T}v_2(t) &= \int_t^T P_{s-t}\left((\nabla v_1(s) - \nabla v_2(s))b(s)\right)\mathrm{d}s \\ &\quad + \lambda\int_t^T P_{s-t}(v_1(s) - v_2(s))\mathrm{d}s \\ &=: B_1(t) + B_2(t). \end{aligned} \tag{36}$$

We consider $B_1$ first. By Lemma 4.6 with $\ell = \nabla(v_1 - v_2)b$ and using (29) we get

$$\begin{aligned}\|B_1\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} &= \|\int_t^T P_{s-t}\left(\nabla(v_1 - v_2)(s)b(s)\right)\mathrm{d}s\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} \\ &\le c\|\nabla(v_1 - v_2)b\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}} \\ &\le c\|b\|_{C_T\mathcal{C}^{-\beta}}\|\nabla(v_1 - v_2)\|^{(\rho)}_{C_T\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}}, \\ &\le c\|b\|_{C_T\mathcal{C}^{-\beta}}\|v_1 - v_2\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}}, \end{aligned} \tag{37}$$

respectively

$$\|B_1\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \le c\|b\|_{C_T\mathcal{C}^{-\beta}}\|v_1 - v_2\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}\rho^{\frac{\alpha+\beta-1}{2}}. \tag{38}$$

We now bound $B_2$ in (36). We use Lemma 3.2 (resp. Schauder's estimate (9) with $\theta = 0$) to get

$$\begin{aligned}\|B_2\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} &= \sup_{t\in[0,T]} e^{-\rho(T-t)}\|\lambda\int_t^T P_{s-t}(v_1(s) - v_2(s))\mathrm{d}s\|_{D\mathcal{C}^\alpha} \\ &\le \lambda\sup_{t\in[0,T]}\int_t^T e^{-\rho(T-t)}\|P_{s-t}(v_1(s) - v_2(s))\|_{D\mathcal{C}^\alpha}\mathrm{d}s \\ &\le \lambda\sup_{t\in[0,T]}\int_t^T e^{-\rho(s-t)}ce^{-\rho(T-s)}\|v_1(s) - v_2(s)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \\ &\le c\lambda\sup_{t\in[0,T]}\int_t^T e^{-\rho(s-t)}\|v_1 - v_2\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\mathrm{d}s \\ &\le c\lambda\|v_1 - v_2\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\rho^{-1} \\ &\le c\lambda\|v_1 - v_2\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}}, \end{aligned} \tag{39}$$

respectively

$$\|B_2\|^{(\rho)}_{\mathcal{C}^{\alpha+1}} \leq c\lambda \|v_1 - v_2\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \rho^{\frac{\alpha+\beta-1}{2}}. \tag{40}$$

Combining (37) and (39) (resp. (38) and (40)) and plugging them in (36) we get

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|^{(\rho)}_{C_TD\mathcal{C}^\alpha} \leq c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})\rho^{\frac{\alpha+\beta-1}{2}} \|v_1 - v_2\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}, \tag{41}$$

respectively

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})\rho^{\frac{\alpha+\beta-1}{2}} \|v_1 - v_2\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}. \tag{42}$$

Now choosing $\rho$ large enough so that (recalling that $\frac{\alpha+\beta-1}{2} < 0$)

$$c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})\rho^{\frac{\alpha+\beta-1}{2}} \leq \frac{1}{2}, \tag{43}$$

we get

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|^{(\rho)}_{C_TD\mathcal{C}^\alpha} \leq \frac{1}{2} \|v_1 - v_2\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}, \tag{44}$$

respectively

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq \frac{1}{2} \|v_1 - v_2\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}, \tag{45}$$

for all $v_1, v_2 \in C_TD\mathcal{C}^\alpha$ (resp. $v_1, v_2 \in C_T\mathcal{C}^{\alpha+1}$). By Banach fixed point theorem we conclude that there exists a unique fixed point $v \in C_TD\mathcal{C}^\alpha$ (resp. in $C_T\mathcal{C}^{\alpha+1}$) of $\mathcal{T}$, which is the unique mild solution $v \in C_TD\mathcal{C}^\alpha$ to (22).

Since this is true for all $\alpha \in (\beta, 1-\beta)$, then under Assumption A3 existence holds in the smaller space $C_TD\mathcal{C}^{(1-\beta)-}$. At this point we observe that we can choose $\alpha = \beta$ in all computations above, but one must replace $\|b\|_{C_T\mathcal{C}^{-\beta}}$ with $\|b\|_{C_T\mathcal{C}^{-\beta+\varepsilon}}$ for some small $\varepsilon$ such that $2\beta - \varepsilon + 1 > 0$, and the powers $\rho^{\frac{\alpha+\beta-1}{2}}$ must be replaced by $\rho^{\frac{2\beta-\varepsilon-1}{2}}$. In conclusion (44) and (45) still hold for $\alpha = \beta$, hence and uniqueness holds in the larger space $C_TD\mathcal{C}^\beta$, which proves item (i). Moreover when Assumption A2 holds then the unique solution belongs to $C_T\mathcal{C}^{(2-\beta)-}$. □

**Lemma 4.9.** *Let $b$ satisfy Assumption A1, $v_T$ and $g$ satisfy Assumption A3 and let $\lambda > 0$. Let $\alpha \in (\beta, 1-\beta)$ such that $v_T \in \mathcal{C}^{\alpha+1}$. Let $v$ be the unique solution of* (22) *given in Theorem 4.7 item (ii) and Remark 4.8. Then there exists an increasing function $R_\lambda$ such that*

$$\|v\|_{C_T\mathcal{C}^{\alpha+1}} \leq R_\lambda(\|b\|_{C_T\mathcal{C}^{-\beta}})(\|v_T\|_{\mathcal{C}^{\alpha+1}} + \|g\|_{C_T\mathcal{C}^{-\beta}}).$$

*Proof.* For the map $\mathcal{T}$ defined in (35) we have

$$\|\mathcal{T}v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq \|\mathcal{T}v - \mathcal{T}0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} + \|\mathcal{T}0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}.$$

Using (42) with $v_1 = v$ and $v_2 = 0$ we get

$$\|\mathcal{T}v - \mathcal{T}0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})\rho^{-\theta} \|v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}},$$

where $\theta = \frac{1-\alpha-\beta}{2} > 0$. On the other hand

$$\mathcal{T}0 = P_{T-t}v_T + \int_t^T P_{s-t}g(s)\mathrm{d}s,$$

so using Lemma 4.6

$$\begin{aligned}\|\mathcal{T}0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &\le \|P_{T-t}v_T\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} + \|\int_\cdot^T P_{s-\cdot}g(s)\mathrm{d}s\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \\ &\le \|P_{T-t}v_T\|_{C_T\mathcal{C}^{\alpha+1}} + c\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{-\theta} \\ &\le \|v_T\|_{\mathcal{C}^{\alpha+1}} + c\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{-\theta}.\end{aligned}$$

Combining the estimates above we have

$$\|\mathcal{T}v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \le c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})\rho^{-\theta}\|v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} + \|v_T\|_{\mathcal{C}^{\alpha+1}} + c\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{-\theta}.$$

Choosing $\rho = [2c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})]^{1/\theta}$ so that $c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})\rho^{-\theta} = \frac{1}{2}$ we get

$$\|\mathcal{T}v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \le \frac{1}{2}\|v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} + \|v_T\|_{\mathcal{C}^{\alpha+1}} + \frac{c}{2c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})}\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}.$$

Since $v$ is a solution then $\mathcal{T}v = v$ and we get

$$\begin{aligned}\|v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &\le 2\|v_T\|_{\mathcal{C}^{\alpha+1}} + 2\frac{1}{2(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})}\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \\ &\le 2\|v_T\|_{\mathcal{C}^{\alpha+1}} + \frac{1}{\lambda}\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}. \qquad (46)\end{aligned}$$

Using $\|v\|_{C_T\mathcal{C}^{\alpha+1}} = \sup_{t\in[0,T]} e^{\rho(T-t)}e^{-\rho(T-t)}\|v(t)\|_{\mathcal{C}^{\alpha+1}} \le e^{\rho T}\|v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}$, the bound (46) and $\|g\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \le \|g\|_{C_T\mathcal{C}^{\alpha+1}}$ we get

$$\begin{aligned}\|v\|_{C_T\mathcal{C}^{\alpha+1}} &\le e^{\rho T}\|v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \\ &\le 2e^{\rho T}\|v_T\|_{\mathcal{C}^{\alpha+1}} + e^{\rho T}\frac{1}{\lambda}\|g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \\ &\le 2e^{\rho T}\max\{1, \frac{1}{\lambda}\}(\|v_T\|_{\mathcal{C}^{\alpha+1}} + \|g\|_{C_T\mathcal{C}^{-\beta}}).\end{aligned}$$

Recall that we chose $\rho = [2c(\lambda + \|b\|_{C_T\mathcal{C}^{-\beta}})]^{1/\theta}$ and since $\theta > 0$ the result follows with $R_\lambda(x) := 2\exp\{[2c(\lambda + x)]^{1/\theta}T\}\max\{1, \frac{1}{\lambda}\}$. □

A special case of interest of PDE (22) is the following. Let $\mathrm{id}_i(x) = x_i$, which clearly belongs to $\mathcal{D}^0_{\mathcal{L}}$, see Definition 4.3. Thus $\mathcal{L}\,\mathrm{id}_i$ is well-defined and gives $\mathcal{L}\,\mathrm{id}_i = b_i$. An immediate consequence of Theorem 4.7 point (i) with $\lambda = 0, v_T = x_i, g = b_i$ is the following corollary, taking into account that $\mathrm{id}_i \in C_TD\mathcal{C}^\beta$.

**Corollary 4.10.** *The function $id_i$ is the solution of $\mathcal{L}v = b_i$; $v(T) = id_i$ (unique in $C_TD\mathcal{C}^\beta$).*

4.4. **Properties of the solution: bounds and continuity.** Another particular case of interest of PDE (22) is given when $g$ is chosen to be the $i$th component of the drift $b$ and the terminal condition is zero. We denote by $u_i$ the solution in this case, that is

$$\begin{cases} \partial_t u_i + \frac{1}{2}\Delta u_i + \nabla u_i b = \lambda u_i - b_i \\ u_i(T) = 0. \end{cases} \tag{47}$$

**Remark 4.11.** *Since PDE (47) is a special case of (22) where $g = -b_i$ and $v_T = 0$, by Theorem 4.7 the solution $u_i$ exists in $C_T\mathcal{C}^{(2-\beta)-}$ and is unique in $C_T D\mathcal{C}^{\beta}$ (indeed Assumption A3 is automatically satisfied for $v_T$ and $g$ if $b$ satisfies Assumption A1).*

**Remark 4.12.** *Let $b \in C_T\mathcal{C}^{0+}$. Then the unique solution $u$ to (47) coincides with the classical solution in $C^{1,2+\nu}$ (see* [12, Theorem 5.1.9]*, see also* [11, Theorem A.3]*).*

*Indeed, if $b \in C_T\mathcal{C}^{0+}$ then $b \in C^{0,\nu}([0,T]\times\mathbb{R}^d)$ for some $\nu > 0$ by* [11, Remark A.2]*, so by* [12, Theorem 5.1.9] *there exists a (unique) solution $\bar u$ in $C^{1,2+\nu}$ to PDE (47). Moreover $b \in C_T\mathcal{C}^{0+} \subset C_T\mathcal{C}^{(-\beta)+}$ hence $u$ is the unique solution of (47) in $C_T\mathcal{C}^{(1+\beta)+}$. We moreover have the inclusion $C^{1,2+\nu} \subset C_T\mathcal{C}^{(1+\beta)+}$, thus $\bar u = u \in C^{1,2+\nu}$.*

**Proposition 4.13.** *Let $b$ satisfy Assumption A1, in particular $b \in C_T\mathcal{C}^{-\beta+\varepsilon}$ for some $\varepsilon > 0$ such that $\theta := \frac{1+2\beta-\varepsilon}{2} < 1$. Let $u_i$, $i = 1, \dots, d$ be the unique solution of (47) as given in Remark 4.11. Then the following holds.*

(i) *The solution $u_i$ is bounded in $(t,x)$, that is, there exists a constant $c$ such that*
$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u_i(t,x)| \le c.$$
(ii) *There is a constant $C(\beta,\varepsilon)$ such that choosing $\lambda$ with*
$$\lambda^{1-\theta} \ge C(\beta,\varepsilon)\|b\|_{C_T\mathcal{C}^{-\beta+\varepsilon}}, \tag{48}$$
*then we have*
$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla u_i(t,x)| \le \frac{1}{2}.$$

*Proof.* For simplicity of notation we drop the subscript $i$ in the rest of the proof. We know that $u \in C_T\mathcal{C}^{1+\beta}$ by Remark 4.11.

*Item (i)* By (4) we have
$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t,x)| \le \sup_{t\in[0,T]} \|u(t)\|_{1+\beta} = \|u\|_{C_T\mathcal{C}^{1+\beta}} < \infty.$$

*Item (ii)* By (3) we have
$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla u(t,x)| \le \sup_{t\in[0,T]} \|\nabla u(t)\|_{\beta}.$$

Assume now (we will show it below) that the unique solution $u$ of (47) is also a solution of the integral equation

$$u(t) = \int_t^T e^{-\lambda(s-t)} P_{s-t} \left( \nabla u(s) b(s) \right) \mathrm{d}s - \int_t^T e^{-\lambda(s-t)} P_{s-t} b(s) \mathrm{d}s. \tag{49}$$

From (49) we take the gradient on both sides and calculate its norm in $\mathcal{C}^\beta$. We use Schauder's estimates (9), Bernstein's inequality (12), and the fact that $\nabla u(s) b(s), b(s) \in \mathcal{C}^{-\beta+\varepsilon}$ by pointwise product (14) to get

$$\begin{aligned}
\|\nabla u(t)\|_\beta \leq & \int_t^T \|\nabla (e^{-\lambda(s-t)} P_{s-t} \left( \nabla u(s) b(s) \right))\|_\beta \mathrm{d}s \\
& + \int_t^T \|\nabla (e^{-\lambda(s-t)} P_{s-t} b(s))\|_\beta \mathrm{d}s \\
\leq & c \int_t^T (\|e^{-\lambda(s-t)} P_{s-t} \left( \nabla u(s) b(s) \right) \|_{\beta+1} + \|e^{-\lambda(s-t)} P_{s-t} b(s)\|_{\beta+1}) \mathrm{d}s \\
\leq & c \int_t^T e^{-\lambda(s-t)} (s-t)^{-\frac{1+2\beta-\varepsilon}{2}} \left( \|\nabla u(s)\|_\beta + 1 \right) \|b(s)\|_{-\beta+\varepsilon} \mathrm{d}s \\
\leq & c \int_t^T e^{-\lambda(s-t)} (s-t)^{-\frac{1+2\beta-\varepsilon}{2}} \mathrm{d}s (1 + \sup_{s\in[0,T]} \|\nabla u(s)\|_\beta) \|b\|_{C_T \mathcal{C}^{-\beta+\varepsilon}},
\end{aligned}$$

where $c$ varies from line to line but it depends only on $\beta$ and $\varepsilon$. Since $\theta := \frac{1+2\beta-\varepsilon}{2} < 1$ by assumption, the integral is bounded from above by $\Gamma(1-\theta)\lambda^{\theta-1}$ by a change of variable $\tilde{s} = \lambda(s-t)$ and using the definition of the Gamma function (32). We get

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_\beta \leq c\Gamma(1-\theta)\lambda^{\theta-1} (1 + \sup_{t\in[0,T]} \|\nabla u(t)\|_\beta) \|b\|_{C_T \mathcal{C}^{-\beta+\varepsilon}},$$

that is

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_\beta (1 - c\lambda^{\theta-1}\Gamma(1-\theta) \|b\|_{C_T \mathcal{C}^{-\beta+\varepsilon}}) \leq c\Gamma(1-\theta)\lambda^{\theta-1} \|b\|_{C_T \mathcal{C}^{-\beta+\varepsilon}}$$

and choosing $\lambda$ according to (48) with $C(\beta,\varepsilon) = 3c\Gamma(1-\theta)$ we have

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_\beta \leq \frac{c\Gamma(1-\theta)\lambda^{\theta-1} \|b\|_{C_T \mathcal{C}^{-\beta+\varepsilon}}}{1 - c\Gamma(1-\theta)\lambda^{\theta-1} \|b\|_{C_T \mathcal{C}^{-\beta+\varepsilon}}} \leq \frac{1}{2},$$

as wanted.

It is left to prove that (49) holds. We can multiply both sides of (49) by $e^{-\lambda t}$ to obtain

$$e^{-\lambda t} u(t) = \int_t^T e^{-\lambda s} P_{s-t} \left( \nabla u(s) b(s) \right) \mathrm{d}s - \int_t^T e^{-\lambda s} P_{s-t} b(s) \mathrm{d}s.$$

Setting $\hat{b}(s) := e^{-\lambda s} b(s)$ we observe that the equation above writes

$$e^{-\lambda t} u(t) = \int_t^T P_{s-t} \left( \nabla e^{-\lambda s} u(s) b(s) \right) \mathrm{d}s - \int_t^T P_{s-t} \hat{b}(s) \mathrm{d}s,$$

which is the mild form of the PDE (recall that mild and weak solutions are equivalent in $C_T D\mathcal{C}^\beta$ by Proposition 4.5)

$$\left\{ \begin{array}{l} \partial_t v + \frac{1}{2}\Delta v + \nabla v \, b = -\hat{b} \\ v(T) = 0, \end{array} \right. \tag{50}$$

where $v(t) := e^{-\lambda t} u(t)$. Therefore to show that (49) holds it is enough to show that if $u$ is a weak solution of (47), then $v(t) = e^{-\lambda t} u(t)$ is a weak solution of (50). For $u$ weak solution of (47) then $u \in C^1([0,T]; \mathcal{S}')$ and (50) readily holds by time-differentiation. Moreover $v \in C_T D\mathcal{C}^\beta$ since $u \in C_T D\mathcal{C}^\beta$. □

Next we consider another special case of PDE (22). Let us define the vector-valued function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ as

$$\phi(t,x) := u(t,x) + x, \tag{51}$$

where $u = (u_1, \dots, u_d)^\top$ and $u_i$ is the solution of (47), unique in the sense of Remark 4.11, for $i = 1, \dots, d$. We define $\phi$ as a column vector.

**Theorem 4.14.** *Each component $\phi_i$, for $i = 1, \dots, d$, of the function $\phi$ defined in* (51) *is the unique solution of*

$$\left\{ \begin{array}{l} \mathcal{L}\phi_i = \lambda(\phi_i - id_i) \\ \phi_i(T) = id_i \end{array} \right. \tag{52}$$

*in $C_T D\mathcal{C}^\beta$.*

*Proof.* Using the linearity of the PDEs for $u_i$ and $\mathrm{id}_i$ (see Corollary 4.10 and Remark 4.11) it is easy to check that each component $\phi_i$, for $i = 1, \dots d$ solves (52). By Theorem 4.7 item (i) we also have that $\phi_i$ is the unique solution of (52). □

**Proposition 4.15.** *Let $\phi$ be given by* (51)*. Then $\phi \in \mathcal{D}^0_{\mathcal{L}}$ and the time-derivative $\dot{\phi}_i$ is in $C_T \mathcal{C}^{(-\beta)-}$ for all $i = 1, \dots d$.*

*Proof.* In this proof we drop the subscript $i$ for ease of writing.

By Theorem 4.14 and Remark 4.4 we have $\phi \in \mathcal{D}^0_{\mathcal{L}}$. Using (52) we get $\mathcal{L}\phi = \lambda(\phi - \mathrm{id})$ with $\phi(T) = \mathrm{id}$, therefore concerning the time-derivative $\dot{\phi}$ we have

$$\int_0^t \dot{\phi}(s,\cdot)\mathrm{d}s = -\int_0^t \frac{1}{2}\Delta\phi(s,\cdot)\mathrm{d}s - \int_0^t \nabla\phi(s,\cdot)\, b(s,\cdot)\mathrm{d}s + \int_0^t \lambda u(s,\cdot)\mathrm{d}s.$$

Since $\phi \in C_T\mathcal{C}^{(2-\beta)-}$ by Remark 4.8, we have $\Delta\phi \in C_T\mathcal{C}^{(-\beta)-}$ and $\nabla\phi \in C_T\mathcal{C}^{(1-\beta)-}$. Moreover $b \in C_T\mathcal{C}^{(-\beta)+}$, so $\nabla\phi\, b \in C_T\mathcal{C}^{(-\beta)+}$ by (14), and $u \in C_T\mathcal{C}^{(2-\beta)-}$. Thus $\dot{\phi} \in C_T\mathcal{C}^{(-\beta)-}$. □

In the following proposition we show that $\phi$ enjoys other useful properties when $\lambda$ is large enough.

**Proposition 4.16.** *Let $\phi$ be given by* (51)*.*

(i) *We have* $\phi \in C^{0,1}$ *and* $\nabla\phi \in C_T\mathcal{C}^{(1-\beta)-}$. *In particular* $\nabla\phi$ *is uniformly bounded.*

(ii) *For* $\lambda$ *as in Proposition 4.13 we have that* $\phi(t,\cdot)$ *is invertible for all* $t \in [0,T]$, *with the (space-)inverse denoted by*

$$\psi := \phi^{-1}(t,\cdot). \tag{53}$$

*Moreover* $\psi \in C^{0,1}$, $\nabla\psi$ *is uniformly bounded and* $\nabla\psi(t,\cdot) \in \mathcal{C}^{(1-\beta)-}$ *for all* $t \in [0,T]$ *and* $\sup_{t\in[0,T]} \|\nabla\psi(t,\cdot)\|_{1-\alpha} < \infty$ *for all* $\alpha < \beta$.

*Proof. Item (i).* The fact that $\phi \in C^{0,1}$ follows from the fact that both id and $u$ are in $C^{0,1}$, since $u \in C_T\mathcal{C}^{(2-\beta)-}$ by Theorem 4.7. By the same regularity property of $u$ we also have $\nabla\phi \in C_T\mathcal{C}^{(1-\beta)-}$.

*Item (ii).* To show that $\phi(t,\cdot)$ is invertible one can proceed like in the proof of [6, Lemma 22]. This proof uses the fact that $|\nabla u(t,x)| \leq \frac{1}{2}$ for $\lambda$ satisfying (48) from Proposition 4.13. We can also easily see that $\psi \in C^{0,1}$. Indeed $\nabla\phi$ is non-degenerate, $\nabla\psi = \nabla\phi(\psi)^{-1}$ so that $(t,x) \mapsto \nabla\psi(t,\cdot)$ is continuous since $\phi \in C^{0,1}$ and $\psi \in C^{0,1}$. Here the superscript $-1$ denotes the matrix inverse. Finally we prove that $\nabla\psi(t,\cdot) \in \mathcal{C}^{(1-\beta)-}$ for all $t \in [0,T]$. We drop the time variable by ease of notation. We notice that $|\nabla\phi|$ is lower bounded by $\frac{1}{2}$ because $\nabla\phi = \nabla u + \mathrm{id}$, hence $|(\nabla\phi)^{-1}|$ is bounded by some constant $C$ independent of time and so $|\nabla\psi|$ is bounded, where $|\cdot|$ denotes the Frobenious norm. Therefore $\psi$ is Lipschitz. Using the fact that $\nabla\phi \in C_T\mathcal{C}^{(1-\beta)-}$ , $|\nabla\phi^{-1}|$ is bounded and that $\psi$ is Lipschitz, we have for $y,z \in \mathbb{R}^d$

$$\begin{aligned}
|\nabla\psi(y) - \nabla\psi(z)| &= |\nabla\phi(\psi(y))^{-1} - \nabla\phi(\psi(z))^{-1}| \\
&= |\nabla\phi(\psi(z)))^{-1} \left(\nabla\phi(\psi(z)) - \nabla\phi(\psi(y))\right) \nabla\phi(\psi(y))^{-1}| \\
&\leq |\nabla\phi(\psi(z)))^{-1}|\, |\nabla\phi(\psi(z)) - \nabla\phi(\psi(y))|\, |\nabla\phi(\psi(y))^{-1}| \\
&\leq C|\nabla\phi(\psi(z)) - \nabla\phi(\psi(y))| \\
&\leq C|\psi(z) - \psi(y)|^{1-\beta-\nu} \\
&\leq C|z-y|^{1-\beta-\nu},
\end{aligned}$$

for any $\nu > 0$, where we recall that $C$ does not depend on time. □

We now state and prove a continuity result for PDEs with bounded or unbounded solutions.

**Lemma 4.17.** *Let Assumption A1 hold. Let* $\lambda > 0$ *be fixed. Let* $b^n$ *be a sequence converging to* $b$ *in* $C_T\mathcal{C}^{-\beta}$, $g^n \to g$ *in* $C_T\mathcal{C}^{-\beta}$. *Then*

(i) *if* $v_T^n \to v_T$ *in* $D\mathcal{C}^{(1-\beta)-}$ *then* $v^n \to v$ *in* $C_TD\mathcal{C}^{(1-\beta)-}$;

(ii) *if* $v_T^n \to v_T$ *in* $\mathcal{C}^{(2-\beta)-}$ *then* $v^n \to v$ *in* $C_T\mathcal{C}^{(2-\beta)-}$,

*where* $v^n$ *is the unique solution of* (22) *with* $b$ *replaced by* $b^n$, $g$ *replaced by* $g^n$ *and* $v_T$ *replaced by* $v_T^n$.
*In particular* $\nabla v^n \to \nabla v$ *in* $C_T\mathcal{C}^{(1-\beta)-}$.

*Proof.* We show both items at the same time.

To show that $v^n \to v$ in $C_T D\mathcal{C}^{(1-\beta)-}$ (resp. in $C_T\mathcal{C}^{(2-\beta)-}$) we have to show that for all $\alpha < 1-\beta$ such that $v^n \to v$ in $C_T D\mathcal{C}^\alpha$ (resp. in $C_T\mathcal{C}^{1+\alpha}$). Since $v^n_T \to v_T$ in $D\mathcal{C}^{(1-\beta)-}$ (resp. in $\mathcal{C}^{(2-\beta)-}$) for all $\alpha < 1-\beta$ such that $v^n_T \to v_T$ in $D\mathcal{C}^\alpha$ (resp. in $\mathcal{C}^{1+\alpha}$ ), we fix any $\alpha < 1-\beta$. We show that $\|v^n - v\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} \to 0$ (resp. $\|v^n - v\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \to 0$) as $n \to \infty$, where the superscript $(\rho)$ denotes the $\rho$-equivalent norm introduced in Section 2. Using the definition of mild solution we have

$$
\begin{aligned}
&v^n(t) - v(t) = P_{T-t}(v^n_T - v_T)\\
&+ \int_t^T P_{s-t}\big(\nabla v^n(s) b^n(s) + \nabla v(s) b^n(s) - \nabla v(s) b^n(s) - \nabla v(s) b(s)\big)\mathrm{d}s\\
&+ \int_t^T P_{s-t}(g^n(s) - g(s))\mathrm{d}s + \lambda \int_t^T P_{s-t}(v^n(s) - v(s))\mathrm{d}s.
\end{aligned}
$$

Let us calculate the $\|\cdot\|_{D\mathcal{C}^\alpha}$-norm (resp. $\|\cdot\|_{\mathcal{C}^{1+\alpha}}$-norm) of the quantity above:

$$
\begin{aligned}
\|v^n - v\|^{(\rho)}_{C_T D\mathcal{C}^\alpha} &= \sup_{0\le t\le T} e^{-\rho(T-t)} \|v^n(t) - v(t)\|_{D\mathcal{C}^\alpha}\\
&\le \sup_{0\le t\le T} e^{-\rho(T-t)} \|P_{T-t}(v^n_T - v_T)\|_{D\mathcal{C}^\alpha}\\
&\quad + \Big\|\int_t^T P_{s-t}\big((\nabla v^n(s) - \nabla v(s)) b^n(s)\big)\mathrm{d}s\Big\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\\
&\quad + \Big\|\int_t^T P_{s-t}\big(\nabla v(s)(b^n(s) - b(s))\big)\mathrm{d}s\Big\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\\
&\quad + \Big\|\int_\cdot^T P_{s-\cdot}(g^n(s) - g(s))\mathrm{d}s\Big\|^{(\rho)}_{C_T D\mathcal{C}^\alpha}\\
&\quad + \lambda \sup_{0\le t\le T} e^{-\rho(T-t)} \Big\|\int_t^T P_{s-t}(v^n(s) - v(s))\mathrm{d}s\Big\|_{D\mathcal{C}^\alpha}\\
&=: B_1 + B_2 + B_3 + B_4 + B_5,
\end{aligned}
$$

(respectively $\|v^n - v\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} =: B_1 + B_2 + B_3 + B_4 + B_5$, where the norm in $D\mathcal{C}^\alpha$ is substituted by the one in $\mathcal{C}^{1+\alpha}$).
The terms $B_1$ and $B_5$ are bounded using Lemma 3.2 (resp. (9) with $\theta = 0$) to get

$$
\begin{aligned}
B_1 &\le \sup_{0\le t\le T} \|P_t(v^n_T - v_T)\|_{D\mathcal{C}^\alpha} \le c\|v^n_T - v_T\|_{D\mathcal{C}^\alpha},\\
B_5 &\le \lambda \int_t^T e^{-\rho(s-t)} e^{-\rho(T-s)} \|v^n(s) - v(s)\|_{D\mathcal{C}^\alpha}\mathrm{d}s \le c\rho^{-1}\|v^n - v\|^{(\rho)}_{C_T D\mathcal{C}^\alpha},
\end{aligned}
$$

(respectively similar estimates where the norm in $D\mathcal{C}^\alpha$ is substituted by the one in $\mathcal{C}^{1+\alpha}$).

For $B_2$ and $B_3$ we apply Lemma 4.6 and (29) twice and for the term $B_4$ we only apply Lemma 4.6 to get

$$B_2 \leq c\|b^n\|_{C_T\mathcal{C}^{-\beta}}\|\nabla(v^n - v)\|^{(\rho)}_{C_T\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}} \leq c\|b^n\|_{C_T\mathcal{C}^{-\beta}}\|v^n - v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}},$$
$$B_3 \leq c\|b^n - b\|_{C_T\mathcal{C}^{-\beta}}\|\nabla v\|^{(\rho)}_{C_T\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}} \leq c\|b^n - b\|_{C_T\mathcal{C}^{-\beta}}\|v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}},$$
$$B_4 \leq c\|g^n - g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}},$$

(respectively similar estimates where the norm in $D\mathcal{C}^\alpha$ is substituted by the one in $\mathcal{C}^{1+\alpha}$). Thus we have

$$\begin{aligned}\|v^n - v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha} \leq{}& c\|v^n_T - v_T\|_{D\mathcal{C}^\alpha}\\ &+ c\|b^n\|_{C_T\mathcal{C}^{-\beta}}\|v^n - v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}}\\ &+ c\|b^n - b\|_{C_T\mathcal{C}^{-\beta}}\|v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}}\\ &+ c\|g^n - g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}} + c\rho^{-1}\|v^n - v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha},\end{aligned}$$

(respectively similar estimates where the norm in $D\mathcal{C}^\alpha$ is substituted by the one in $\mathcal{C}^{1+\alpha}$).

Similarly to (43) but replacing $\|b\|_{C_T\mathcal{C}^{-\beta}}$ with $\sup_n \|b^n\|_{C_T\mathcal{C}^{-\beta}}$, we choose $\rho \geq 1$ such that

$$c(1 + \sup_n \|b^n\|_{C_T\mathcal{C}^{-\beta}})\rho^{\frac{\alpha+\beta-1}{2}} \leq \frac{1}{2},$$

so that combining the estimates above and moving to the left-hand side the terms involving $v^n - v$ we get

$$\begin{aligned}\frac{1}{2}\|v^n - v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha} \leq{}& c\|v^n_T - v_T\|_{D\mathcal{C}^\alpha} + c\|g^n - g\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}}\\ &+ c\|b^n - b\|_{C_T\mathcal{C}^{-\beta}}\|v\|^{(\rho)}_{C_TD\mathcal{C}^\alpha}\rho^{\frac{\alpha+\beta-1}{2}},\end{aligned}$$

(respectively similar estimates where the norm in $D\mathcal{C}^\alpha$ is substituted by the one in $\mathcal{C}^{1+\alpha}$). The proof is concluded. □

**Remark 4.18.** *Following the proof of Lemma 4.17, it is easy to see that a slightly weaker convergence remains valid under slightly weaker assumptions, namely*

(i) *if $v^n_T \to v_T$ in $D\mathcal{C}^{\beta+}$ then $v^n \to v$ in $C_TD\mathcal{C}^{\beta+}$;*
(ii) *if $v^n_T \to v_T$ in $\mathcal{C}^{(1+\beta)+}$ then $v^n \to v$ in $C_T\mathcal{C}^{(1+\beta)+}$.*

*In particular $\nabla v^n \to \nabla v$ in $C_T\mathcal{C}^{\beta+}$.*

**Lemma 4.19.** *Let $b^n \to b$ in $C_T\mathcal{C}^{-\beta}$. Let $\lambda$ be such that*

$$\lambda^{1-\theta} = C(\beta,\varepsilon)\max\{\sup_n \|b^n\|_{C_T\mathcal{C}^{-\beta+\varepsilon}}, \|b\|_{C_T\mathcal{C}^{-\beta+\varepsilon}}\} \tag{54}$$

*with $\theta := \frac{1+2\beta-\varepsilon}{2} < 1$ and $C(\beta,\varepsilon)$ chosen according to Proposition 4.13 item (ii). Let $\phi^n$ be defined as in (51) but with $b$ replaced by $b^n$ and let $\psi^n$ be the (space-)inverse of $\phi^n$ as in (53). Then we have*

(i) *$u^n \to u, \nabla u^n \to \nabla u, \phi^n \to \phi$ and $\psi^n \to \psi$ uniformly on $[0,T] \times \mathbb{R}^d$;*
(ii) *$\|\nabla \phi^n\|_\infty$ and $|\phi^n(0,0)|$ are uniformly bounded in $n$.*

*Proof.* We choose $\lambda$ according to (54) as done in (48). This implies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla u^n(t,x)| \le \frac{1}{2} \tag{55}$$

by Proposition 4.13 part (ii).

*Item (i)* By Lemma 4.17 part (ii) we have $u_n \to u$ in $C_T\mathcal{C}^{\alpha+1}$ thus $u_n \to u$ and $\nabla u_n \to \nabla u$, uniformly on $[0,T] \times \mathbb{R}^d$. Since $\phi_n - \phi = u_n - u$, then also $\phi_n \to \phi$ uniformly on $[0,T] \times \mathbb{R}^d$.

The rest of the proof follows the same ideas of [6, Lemma 24, part (iii)]. We recall the basic elements of the proof for ease of reading. Let us prove the uniform convergence of $\psi_n$ to $\psi$. Given $y \in \mathbb{R}^d$, we know that for every $t \in [0,T]$ and $n \in \mathbb{N}$ there exist $x(t), x_n(t) \in \mathbb{R}^d$ such that

$$x(t) + u(t, x(t)) = y$$
$$x_n(t) + u_n(t, x_n(t)) = y$$

and we have called $x(t)$ and $x_n(t)$ by $\psi(t,y)$ and $\psi_n(t,y)$ respectively. Then from (55) we get

$$\begin{aligned}
|x_n(t) - x(t)| \le & \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla u_n(t,x)|\, |x_n(t) - x(t)| \\
& + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u_n(t,x) - u(t,x)| \\
\Rightarrow |x_n(t) - x(t)| \le & 2 \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u_n(t,x) - u(t,x)|,
\end{aligned}$$

namely

$$|\psi_n(t,y) - \psi(t,y)| \le 2 \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u_n(t,x) - u(t,x)|,$$

which implies that $\psi_n \to \psi$ uniformly on $[0,T] \times \mathbb{R}^d$.

*Item (ii)* To show that $\|\nabla \phi^n\|_\infty$ is bounded uniformly in $n$ we simply observe that $\nabla \phi^n(t,x) = \mathrm{id} + \nabla u^n(t,x)$ and use (55).

To prove that $|\phi^n(0,0)| = |u^n(0,0)|$ is uniformly bounded we observe that $u^n \to u$ in $C_T D\mathcal{C}^{(1-\beta)-}$ by Lemma 4.17 part (i), hence there exists $\alpha < 1-\beta$ such that $u^n \to u$ in $C_T D\mathcal{C}^\alpha$ and so

$$\sup_{n\ge1} |u^n(0,0)| \le c \sup_{n\ge1} \|u^n\|_{C_T D\mathcal{C}^\alpha},$$

which concludes the proof. □

## 5. On some separable Besov-Hölder type spaces

In the companion paper [11] we use a special class of PDEs like (1) for some applications in stochastic analysis. In particular, the PDE plays a role in the formulation of the martingale problem for stochastic differential equations with distributional drifts $b$. For more details on the latter, see

[11, Section 4]. The class of PDEs that we use in [11] are PDEs of the form $\mathcal{L}f = g$, where the element $g$ is a function (instead of a distribution) that, most importantly, lives in a space which is separable. The spaces $C_T\mathcal{C}^{0+}$ would be the natural choice since it contains only functions, but it is not separable. It would be separable if one restricted them to functions with compact support, however the class $C_T\mathcal{C}_c^{0+}$ of functions in $C_T\mathcal{C}^{0+}$ with compact support is not closed under the topology of $C_T\mathcal{C}^{0+}$ and not rich enough for our purpose. Thus here we introduce and investigate a further class of function spaces, namely the closure of $C_T\mathcal{C}_c^{0+}$ with respect to the topology of $C_T\mathcal{C}^{0+}$. These spaces turn out to be separable and rich enough to be used in our application to stochastic analysis. In this section, we prove some useful results about these space, most importantly separability.

**Lemma 5.1.** *Let $f$ be a Schwartz distribution with compact support. We have $f * p_t \in \mathcal{S}$ for all $t > 0$.*

*Proof.* We will show that the Fourier transform $\mathcal{F}(p_t * f)$ of $p_t * f$ is in $\mathcal{S}$. Since $f$ is a compactly supported Schwartz distribution we apply [14, Theorem 26, page 91] to write $f$ as the finite sum $\sum_\nu \partial^\nu h$ with $h$ some continuous function with compact support. By linearity it is enough to show that $\mathcal{F}(\partial^\nu h * p_t) \in \mathcal{S}$, where $h$ some continuous function with compact support. In this case we have

$$\mathcal{F}(\partial^\nu h * p_t) = \mathcal{F}(h * \partial^\nu p_t) = \mathcal{F}(h)\mathcal{F}(\partial^\nu p_t),$$

and this belongs to $\mathcal{S}$ since $\mathcal{F}\partial^\nu p_t \in \mathcal{S}$ and $\mathcal{F}h \in C_b^\infty$ by an easy calculation. □

We denote by $C_c = C_c(\mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued continuous functions with compact support. For $\gamma \geq 0$ we denote by $\mathcal{C}_c^\gamma = \mathcal{C}_c^\gamma(\mathbb{R}^d)$ the space of elements in $\mathcal{C}^\gamma$ with compact support. Similarly when $\gamma$ is replaced by $\gamma+$ or $\gamma-$, for $\gamma \geq 0$. When defining the domain of the martingale problem we will work with spaces of functions which are the limit of functions with compact support, so that they are Banach space. More precisely, let us denote by $\bar{\mathcal{C}}_c^\gamma = \bar{\mathcal{C}}_c^\gamma(\mathbb{R}^d)$ the space

$$\bar{\mathcal{C}}_c^\gamma := \{f \in \mathcal{C}^\gamma \text{ such that } \exists (f_n)_n \subset \mathcal{C}_c^\gamma \text{ and } f_n \to f \text{ in } \mathcal{C}^\gamma\}.$$

As above we denote the inductive space and intersection space as

$$\bar{\mathcal{C}}_c^{\gamma+} := \cup_{\alpha>\gamma}\bar{\mathcal{C}}_c^\alpha, \qquad \bar{\mathcal{C}}_c^{\gamma-} := \cap_{\alpha<\gamma}\bar{\mathcal{C}}_c^\alpha.$$

We also introduce the space $C_T\bar{\mathcal{C}}_c^{\gamma+}$ and observe that $f \in C_T\bar{\mathcal{C}}_c^{\gamma+}$ if and only if there exists $\alpha > \gamma$ such that $f \in C_T\bar{\mathcal{C}}_c^\alpha$, by [10, Remark B.1 part (ii)].

We will state and prove several useful properties of such spaces. Let us start by showing that $C_T\bar{\mathcal{C}}_c^\gamma$ is an algebra.

**Proposition 5.2.** *The space $C_T\bar{\mathcal{C}}_c^\gamma$ is an algebra for $\gamma \in (0,1)$.*

*Proof.* Let $f, g \in C_T\bar{\mathcal{C}}_c^\gamma$. By [10, Remark B.1], we know that there exists a sequence $(f_n)_n \subset C_T\mathcal{C}_c^\gamma$ (resp. $(g_n)_n$) such that $f_n \to f$ (resp. $g_n \to g$) in

$C_T\mathcal{C}^\gamma$. Clearly $f_n g_n \in C_T\mathcal{C}_c^\gamma$ so it remains to show that $f_n g_n \to fg$ in $C_T\mathcal{C}^\gamma$. We have $f_n g_n - fg = (f_n - f)g_n + f(g_n - g)$ so it is enough to show that $(f_n - f)g_n \to 0$ and $f(g_n - g) \to 0$ in $C_T\mathcal{C}^\gamma$. We show the first term only, as the second can be handled the same (but easier). Using the norm (3) we need to bound two terms. The first one is $\sup_{t\in[0,T]} \|(f_n(t,\cdot) - f(t,\cdot))g_n(t,\cdot)\|_\infty$ and it clearly converges to 0 by assumptions on $f_n, g_n$. As for the Hölder seminorm for all $t \in [0,T]$ we have

$$\begin{aligned}
&|(f_n - f)(t,x)g_n(t,x) - (f_n - f)(t,y)g_n(t,y)|\\
&\le |[(f_n - f)(t,x) - (f_n - f)(t,y)]g_n(t,x)|\\
&+ |(f_n - f)(t,y)[g_n(t,x) - g_n(t,y)]|\\
&\le \|f_n - f\|_{C_T\mathcal{C}^\gamma}|x-y|^\gamma \sup_{t,x}|g_n(t,x)|\\
&+ \sup_{t,x}|(f_n - f)(t,x)|\|g_n\|_{C_T\mathcal{C}^\gamma}|x-y|^\gamma.
\end{aligned}$$

Using this we conclude that

$$\sup_{t\in[0,T]}\sup_{x\neq y}\frac{|(f_n - f)(t,x)g_n(t,x) - (f_n - f)(t,y)g_n(t,y)|}{|x-y|^\gamma} \to 0,$$

by the fact that $f_n \to f$ uniformly and $\|f_n - f\|_{C_T\mathcal{C}^\gamma}$ and $\|g_n\|_{C_T\mathcal{C}^\gamma}$ are bounded. □

**Lemma 5.3.** *We have*

$$\mathcal{S} \subset \bar{\mathcal{C}}_c^{\gamma+} \tag{56}$$

*for $\gamma \in \mathbb{R}$. In particular, $\mathcal{S}$ is included in the closure $\bar{C}_c$ of the space of continuous functions with compact support $C_c$ with respect to the topology of uniform convergence.*

*Proof.* It is enough to show the claim for every $\gamma \ge 0$. We only prove (56) since the closure of the space of continuous functions with compact support $C_c(\mathbb{R}^d)$ with respect to the topology of uniform convergence contains $\bar{\mathcal{C}}_c^{\gamma+}$.

Let $\chi : \mathbb{R} \to \mathbb{R}_+$ be a smooth function such that

$$\chi(x) = \begin{cases} 0 & x \ge 0\\ 1 & x \le -1\\ \in (0,1) & x \in (-1,0). \end{cases}$$

We set $\chi_n : \mathbb{R}^d \to \mathbb{R}$ as $\chi_n(x) := \chi(|x| - (n+1))$. In particular

$$\chi_n(x) = \begin{cases} 0 & |x| \ge n+1\\ 1 & |x| \le n\\ \in (0,1) & \text{otherwise}. \end{cases}$$

Let $f \in \mathcal{S}$. We set $f_n(x) := f(x)\chi_n(x)$. Clearly $f_n \in \mathcal{C}_c^{\gamma+}$.

*Step 1.* For any multi-index $m$ we first show that $D^m f_n \to D^m f$ uniformly.

Notice that $D^m(f_n - f) = D^m(f(1 - \chi_n))$ is a finite sum of terms of the form $D^l f D^k(1 - \chi_n)$ for some finite $|l|, |k| \le |m|$. One can show that

$\sup_x |D^k(1-\chi_n)(x)| \leq \|D^k\chi\|_\infty$ by the definition of $\chi_n$. Let $\varepsilon > 0$. Since $f \in \mathcal{S}$ there exists $n(\varepsilon)$ such that for all $|x| > n(\varepsilon)$ then $|D^l f(x)| < \varepsilon$ for all $l$ such that $|l| \leq |m|$. For $|x| > n(\varepsilon)$ we have

$$|D^l f(x) D^k(1-\chi_n)(x)| \leq \|D^k\chi\|_\infty \varepsilon.$$

This shows uniform convergence of $D^l f D^k(1-\chi_n)$ to 0, hence uniform convergence of $D^m(f_n - f)$ to zero.

*Step 2.* Let $\alpha \in (0,1)$. For any multi-index $m$ it remains to show that

$$\sup_{|x-y|<1} \frac{|D^m(f(1-\chi_n))(x) - D^m(f(1-\chi_n))(y)|}{|x-y|^\alpha}$$

converges to 0 as $n \to \infty$. We clearly have that

$$\frac{|D^m(f(1-\chi_n))(x) - D^m(f(1-\chi_n))(y)|}{|x-y|^\alpha} \leq \|\nabla D^m(f(1-\chi_n))\|_\infty |x-y|^{1-\alpha}$$

by finite increments theorem, hence we reduce to Step 1. □

**Lemma 5.4.** (i) *For any $\gamma \in \mathbb{R}$ the space $\mathcal{S}$ is dense in $\bar{\mathcal{C}}_c^{\gamma+}$.*
(ii) *$\mathcal{S}$ is dense in $\bar{C}_c$.*

*Proof. Item (i)* We observe that $\mathcal{S} \subset \bar{\mathcal{C}}_c^{\gamma+}$, see Lemma 5.3. Let $\gamma \in \mathbb{R}$ and $f \in \bar{\mathcal{C}}_c^{\gamma+}$. By the definition of the space we can reduce to the case $f \in \mathcal{C}_c^{\gamma+}$. We mollify $f$ using the heat semigroup $P_\varepsilon$, that is we consider $P_\varepsilon f = p_\varepsilon * f$ where $p_\varepsilon$ is the heat kernel. By Lemma 5.1 we have $P_\varepsilon f \in \mathcal{S}$. By (10) we also have that $P_\varepsilon f \to f$ in $\mathcal{C}^{\gamma+}$.

*Item (ii)* The result follows from the fact that $P_\varepsilon f \to f$ uniformly, for $f \in C_c$ and that $\mathcal{S} \subset \bar{C}_c$ by Lemma 5.3. □

The next three lemmata will be used below to prove that the spaces are separable.

**Lemma 5.5.** *Let $f : [0,1] \to B$ where $(B, \|\cdot\|)$ is a Banach space. Then the sequence $(f_n)_n$ defined by $f_n(t) := \sum_{j=0}^n f(\frac{j}{n}) t^j (1-t)^{n-j} \binom{n}{j}$ converges uniformly to $f$.*

*Proof.* The polynomials $(f_n)_n$ are also know as Bernstein polynomials, often denoted by $B_n(f,t)$ that is

$$B_n(f,t) := f_n(t) := \sum_{j=0}^n f(\frac{j}{n}) t^j (1-t)^{n-j} \binom{n}{j}. \tag{57}$$

Bernstein polynomials have the property that they can be expressed as expectations of suitable random variables, which is useful in the computations below. In particular, let $U_1, \dots, U_n \sim U(0,1)$ be independent uniform r.v.s and let

$$S_n(t) := \frac{1}{n} \sum_{j=1}^n \mathbb{1}_{[0,t)}(U_j).$$

Since $S_n(t)$ is a binomial r.v. with parameter $n$ and $t$, then clearly

$$B_n(f,t) = \mathbb{E}[f(S_n(t))]. \tag{58}$$

Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there exists $\delta > 0$ such that if $|t-s| \leq \delta$ then $\|f(t) - f(s)\| \leq \varepsilon$. Let $t \in [0,1]$, by (58) we have

$$\begin{aligned}\|f_n(t) - f(t)\| =& \|\mathbb{E}[f(S_n(t)) - f(t)]\| \\ \leq& \mathbb{E}[\|f(S_n(t)) - f(t)\| \mathbb{1}_{\{|S_n(t)-t| \leq \delta\}}] \\ &+ \mathbb{E}[\|f(S_n(t)) - f(t)\| \mathbb{1}_{\{|S_n(t)-t| > \delta\}}] \\ =:& I_1(t) + I_2(t).\end{aligned}$$

Now

$$I_1(t) \leq \varepsilon \mathbb{P}(|S_n(t) - t| \leq \delta) \leq \varepsilon.$$

Concerning $I_2(t)$, being $S_n(t)$ a binomial random variable with parameter $n$ and $t$,

$$\operatorname{Var}(S_n(t)) = \frac{1}{n}(t - t^2).$$

Using this and by Chebyshev inequality we get

$$\begin{aligned}I_2(t) \leq& 2\|f\|_\infty \mathbb{P}(|S_n(t) - t| > \delta) \\ \leq& 2\|f\|_\infty \frac{\operatorname{Var}(S_n(t) - t)}{\delta^2} \\ \leq& 2\|f\|_\infty \frac{(t-t^2)}{n\delta^2}.\end{aligned}$$

Now taking the supremum over $t \in [0,1]$ we get $\sup_{t\in[0,1]} I_2(t) \leq \frac{1}{2}\|f\|_\infty \frac{1}{n\delta^2}$ and putting this together with the bound for $I_1(t)$ we obtain

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} \|f_n(t) - f(t)\| \leq \limsup_{n\to\infty} \sup_{t\in[0,1]} (I_1(t) + I_2(t)) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is concluded. □

**Lemma 5.6.** *Let $E$ be an inductive space of the form $E = \cup_{\alpha\in\mathbb{N}} E_\alpha$, with $E_\alpha$ Banach space. If $E$ is separable then $C_T E$ is separable.*

*Proof.* Without loss of generality we choose $T = 1$. Let $f \in C_T E$ and we consider the functions

$$f_n(t) := \sum_{j=0}^{n} f(\frac{j}{n}) t^j (1-t)^{n-j} \binom{n}{j}.$$

We now use the fact that $C_T E = \cup_{\alpha\in\mathbb{N}} C_T E_\alpha$ by [10, Remark B.1], where the space $C_T E_\alpha$ can be equipped with the norm $\sup_t \|f(t)\|_{E_\alpha}$. By this fact, there exists $\alpha$ such that $f \in C_T E_\alpha$, in particular $f_n \in C_T E_\alpha$ for all $n$. By Lemma 5.5 $f_n$ converges to $f$ in $C_T E_\alpha$, which by the fact stated above implies it converges also in $C_T E$. We have thus reduced our problem to polynomials of the form $\sum_{j=1}^n a_j t^j$ with $a_j \in E_\alpha$. We conclude the proof by using the fact that $E$ is separable, thus there exists a countable dense subset of $E$, say $\mathcal{P}$, so that every polynomial $\sum_{j=1}^n a_j t^j$ can be approached by a sequence of polynomials of the type $\sum_{j=1}^n q_j t^j$ with $q_j \in \mathcal{P}$. □

**Lemma 5.7.** (i) *For any $\gamma \in \mathbb{R}$ the space $\bar{\mathcal{C}}_c^{\gamma+}$ is separable.*
(ii) *$\bar{C}_c$ is separable.*

*Proof.* This follows from Lemma 5.4. □

**Corollary 5.8.** *The space $C_T\bar{\mathcal{C}}_c^{\gamma+}$ is separable for any $\gamma \in \mathbb{R}$.*

*Proof.* Notice that by definition $\bar{\mathcal{C}}_c^{\gamma}$ is a Banach space and the inductive space $\bar{\mathcal{C}}_c^{\gamma+}$ is separable by Lemma 5.7, so we conclude using Lemma 5.6. □

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