Certain and Uncertain Inference with Indicative Conditionals

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Abstract

This paper develops a trivalent semantics for the truth conditions and the probability of the natural language indicative conditional. Our framework rests on trivalent truth conditions first proposed by W. Cooper and yields two logics of conditional reasoning: (i) a logic C of inference from *certain* premises; and (ii) a logic U of inference from *uncertain* premises. But whereas C is monotonic for the conditional, U is not, and whereas C obeys Modus Ponens, U does not without restrictions. We show systematic correspondences between trivalent and probabilistic representations of inferences in either framework, and we use the distinction between the two systems to cast light, in particular, on McGee's puzzle about Modus Ponens. The result is a unified account of the semantics and epistemology of indicative conditionals that can be fruitfully applied to analyzing the validity of conditional inferences.

1 Introduction and Overview

Research on indicative conditionals (henceforth simply "conditionals") pursues two major projects: the semantic project of determining their *truth conditions*, and the epistemological and pragmatic project of explaining how we should *reason* with them, and when we can *assert* them. The two projects are related: Jackson (1979, p. 589) states that "we should hope for a theory which explains the assertion conditions in terms of the truth conditions" while according to David Lewis (1976, p. 297), "assertability goes by subjective probability", where the value of the latter depends on when a proposition is true or false (see also Adams 1965, pp. 173-174; Jackson 1979, p. 565; Leitgeb 2017, p. 278).

Ideally, we would have a unified treatment of truth conditions and probability of conditionals and, on that basis, a theory of reasoning with conditionals. Here is the standard approach. Suppose A and C are sentential variables of a propositional language $\mathcal{L}$ without conditionals, and $\rightarrow$ denotes the "if... then..." connective. Then, the probability of the sentence $A \to C$ should go by the conditional probability $p(C|A)$ (e.g., Adams 1965, 1975; Stalnaker 1970):

$$p(A \to C) = p(C|A) \qquad \text{(Adams's Thesis)}$$

The idea is that the conditional "if the sun is shining, Mary will go for a walk" seems plausible if and only if it is likely that, given sunshine, Mary is going for a walk.[1] Normative theories of conditionals often recognize Adams's Thesis as a desideratum (e.g., Stalnaker 1970; Adams 1975). The empirical data are more complicated, but Adams's Thesis is well-supported when the antecedent is *relevant* for the consequent (e.g., part of the same discourse: Over, Hadjichristidis, et al. 2007; Skovgaard-Olsen, Singmann, and Klauer 2016).

Unfortunately, David Lewis's well-known triviality result complicates the picture. Lewis (1976) showed that, if (i) the probability of a sentence depends in the standard way on its truth conditions (i.e., as expectation of semantic value),[2] and (ii) the probability function is closed under conditionalization, Adams's Thesis implies $p(A \to C) = p(C)$, whenever A is compatible with both C and its negation. Similar triviality results have been shown by Hájek (1989), Bradley (2000), and Milne (2003). This *reductio ad absurdum* seems to preclude a unified semantic and epistemological treatment of conditionals, at least as far as probability and probabilistic reasoning is concerned.

We show in this paper that this conclusion is premature: we introduce a third truth value ("neither true nor false") and propose trivalent truth conditions for natural language indicative conditionals whose probability validates Adams's Thesis. The probabilistic semantics allows us to define a logic for reasoning with *certain* premises as well as a structurally similar logic for reasoning with *uncertain* premises.

In other words, we argue that *different logics of conditionals suit different epistemic situations*. When no conditionals are involved, the epistemic status of the premises does not matter: deductive logic validates all and only those inferences that preserve maximal certainty, i.e., probability one, and also all and only those inferences that do not increase uncertainty (e.g., Adams 1996a). There is just one notion of valid inference. But conditionals complicate the picture. When premises are supposed as being certain, the inference

[1] The extension of Adams's Thesis to *arbitrary* sentences A and C, possibly involving conditionals, is known as "Stalnaker's Thesis".

[2] In other words, the probability of A corresponds to the total weight of the possible worlds where A is true.

from "if Alice goes to the party, Bob will" to "if Alice and Carol go to the party, Bob will" appears valid. Alice's presence *ensures* Bob's presence no matter his feelings for Carol. This picture changes when the premises are taken to be just *likely* instead of certain: Carol going to the party can make a difference if we think that Alice going to the party does not guarantee that Bob will go in all circumstances. Conditional reasoning for uncertain premises has non-monotonic aspects and may require more than one notion of valid inference (compare Santorio 2022b). Our account explains the difference between certain and uncertain reference by keeping the truth conditions of conditionals constant and by building a definition of probability based on those, while relaxing the definition of logical consequence when going from certain to uncertain reasoning.

We briefly expound the structure of our paper. The first part lays the semantic foundations: Section 2 motivates the trivalent treatment of conditionals and Section 3 introduces specific trivalent truth tables for the indicative conditional and the Boolean connectives, giving reasons to select the trivalent conditional operator first introduced by Cooper 1968. Section 4 defines the (non-classical) probability of trivalent propositions in analogy with defining probability in a conditional-free language.

The second part of the paper focuses on conditional reasoning. From the definition of probability in trivalent semantics, Section 5 and 6 derive two logical consequence relations for certainty-preserving inference (=the logic C) and for inferences that do not increase probabilistic uncertainty (=the logic U). We show that C and U can be characterized as preserving semantic values within trivalent logic, and in Section 7 we examine which principles of conditional logic they validate. In particular, we show that some principles such as Or-to-If or Modus Ponens with nested conditionals are controversisal because they hold in the context of reasoning with certain premises, but fail for uncertain premises.

The third part contains applications, comparisons and evaluations: Section 8 discusses nested conditionals and McGee's objection to Modus Ponens from the vantage point of our semantics and the two separate logics for certain and uncertain inference. Section 9 draws comparisons with other theories. Section 10 highlights the strengths and limits of our account. Appendix A provides proof details.

2 Truth Conditions: The Basic Idea

It is controversial whether indicative conditionals have factual truth conditions and can be treated as expressing propositions (e.g., see the dialogue in Jeffrey and Edgington 1991). According to the non-truth-conditional, probabilistic analysis of conditionals (Adams 1965, 1975; Edgington 1986, 1995, 2009; Over and Baratgin 2017), indicative conditionals do not express propositions; at most they have partial truth conditions.

> [...] the term 'true' has no clear ordinary sense as applied to conditionals, particularly to those whose antecedents prove to be false [...]. In view of the foregoing remarks, it seems to us to be a mistake to analyze the logical properties of conditional statements in terms of their truth conditions. (Adams 1965, pp. 169-170)

Non-truth-conditional accounts *stipulate* that $p(A \to C) = p(C|A)$ and develop a probabilistic theory of reasoning with conditionals on the basis of high probability preservation (called "logic of reasonable inference" by Adams). This move yields a powerful logic for capturing core phenomena of reasoning with simple conditionals, such as their non-monotonic behavior in certain contexts. This success is recognized by truth-conditional accounts (e.g., McGee 1989, p. 485; Ciardelli 2020, p. 544), but the probabilistic approach severs the link between semantics and epistemology. In particular, it does not cover nested conditionals and compounds of conditionals. Moreover, due to the lack of truth conditions, it does not clarify how one can argue and disagree about conditional sentences in a similar way as we do for normal, non-conditional propositions (Bradley 2012, p. 547).

However, even a defender of a non-truth-conditional view such as Adams (1965, p. 187) admits that we feel compelled to say that a conditional "if A, then C" has been *verified* if we observe both A and C, and *falsified* if we observe A and $\neg C$. For example, take the sentence "if it rains, the match will be cancelled"; it seems to be true if it rains and the match is in fact cancelled, and false if the match takes place in spite of rain. Indeed, what else could be required for determining the truth or falsity of the sentence?

This "hindsight problem" (the terminology is from Khoo 2015) is a prima facie reason for treating conditionals as propositions, and assigning them factual truth conditions. Defenders of non-propositional accounts need to explain why observations in our actual world are *sufficient* for the truth or falsity of "if A, then C", and why $A \to C$ behaves so differently when A is false.

Truth-conditional accounts of conditionals address this point. They come in various guises: variably strict conditionals (e.g., Stalnaker 1968), restrictor semantics (e.g., Kratzer 2012), dynamic semantics (e.g., Gillies 2009), information state semantics (e.g., Ciardelli 2020; Santorio 2022a), and many more.[3] Many of these accounts *emulate* Adams's probabilistic logic of reasonable inference, or central parts thereof. For example, truth preservation in Stalnaker's modal framework famously validates the same inference schemes as Adams's logic in their common domain. All of them, however, face a non-trivial task of modelling the probability of conditionals. Truth preservation works in these logics like a *qualitative* plausibility order, but their analysis of the *quantitative* probability of conditionals must, in the light of Lewis's triviality result, deviate systematically from Adams's thesis. Thus, both the truth-conditional and the non-truth-conditional approaches seem to lose out on some important aspects of conditionals.

In this paper, we would like to resolve the impasse by treating "if A, then C" as a *conditional assertion*—i.e., as an assertion about C upon the supposition that A is true. Whereas, when the antecedent is false, the speaker is committed to neither truth nor falsity of the consequent. This view takes into account Adams's observation that "true" has no clear ordinary sense when applied to indicative conditionals; it has been voiced perhaps most prominently by Quine (1950, p. 12, our emphasis):

> An affirmation of the form "if p then q" is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. *If on the other hand the antecedent turns out to have been false, our conditional affirmation is as if it had never been made.*

In other words, asserting a conditional makes an epistemic commitment only in case the antecedent turns out to be true. If it turns out to be false, the assertion is retracted: there is no factual basis for evaluating it (see also Belnap 1970, 1973). Therefore it is classified as neither true nor false. The "gappy" or "defective" truth table of Table 1 interprets this view as a partial assignment of truth values to conditionals (e.g., Reichenbach 1935; de Finetti 1936a; Adams 1975; Baratgin, Over, and Politzer 2013; Over and Baratgin 2017).[4]

[3]The material conditional analysis, endorsed by Jackson and Lewis, claims that the truth conditions of the indicative and the material conditional agree, and that perceived differences are due to pragmatic, not to semantic factors (Jackson 1979; Grice 1989). This approach, however, gives up on a unified picture of truth conditions and probability in the first place. On that account, if sun were unlikely, the probability of "if the sun is shining, Mary is going for a walk" would be close to one regardless of Mary's intentions, which looks unacceptable.

[4]Some accounts also take the conditional probability $p(A|C)$ as a possible semantic value for the conditional $A \to C$ (e.g., McGee 1989; Stalnaker and Jeffrey 1994; Sanfilippo et al. 2020),

Truth value of $A \to C$	$v(C) = 1$	$v(C) = 0$
$v(A) = 1$	1	0
$v(A) = 0$	(neither)	(neither)

Table 1: 'Gappy" or "defective" truth table for a conditional $A \to C$ for a (partial) valuation function in a language with conditional.

However, without a full truth-conditional treatment, such an account is limited: it neither evaluates nested conditionals, nor Boolean compounds of conditionals. If we could complete Table 1 and provide full truth conditions in a satisfactory way, this would greatly increase the scope and descriptive power of conditional reasoning, and facilitate the identification of theorems and valid inferences.

The obvious candidate for such truth conditions is a trivalent truth table, where the absence of commitment to the consequent C is represented by a third truth value. Instead of using partial valuations, we assign a third semantic value, $1/2$ or "indeterminate", when the antecedent is false (See Table 2). This is a recurring idea in the literature, defended, among others, by de Finetti (1936a), Reichenbach (1944), Jeffrey (1963), Cooper (1968), Belnap (1970, 1973), Manor (1975), Farrell (1986), McDermott (1996), Olkhovikov (2002/2016), Cantwell (2008), Rothschild (2014), and Égré, Rossi, and Sprenger (2021a,b).

Truth value of $A \to C$	$v(C) = 1$	$v(C) = 1$
$v(A) = 1$	1	0
$v(A) = 0$	$1/2$	$1/2$

Table 2: Partial trivalent truth table for a conditional $A \to C$ for a partial valuation function in a language with conditional.

This basic idea has to be developed in various directions. Firstly, we need to decide how to extend the truth table of Table 1 to a fully trivalent truth table for $A \to C$ where A and C can also take the value $1/2$ (=neither true nor false, indeterminate). Secondly, we need to decide how to interpret the standard Boolean connectives $\wedge, \vee, \neg$ in the context of propositions which can take three different truth values. Doing so will allow us to deal with nested conditionals, and more generally, with arbitrary compounds of atomic sentences connected by the standard connectives and $\to$. Thirdly, we have to define a probability measure for trivalent propositions and a consequence relation for reasoning

but this analysis reverses the traditional direction of the dependency between the probability and the truth conditions of a sentence: probability should depend on how often we find a sentence to be true, not vice versa.

with certain and uncertain premises. We approach these tasks in turn in the next sections.

3 Trivalent Truth Tables

We start by extending the basic idea of Table 2 to a full trivalent truth table for $A \to C$. The two main options are shown in Table 3 and have been proposed by Bruno de Finetti (1936a) and William Cooper (1968), respectively. We abbreviate the two connectives with "DF" and "CC" (the latter after Cooper-Cantwell).[5] In both of them the value $1/2$ can be interpreted as "neither true nor false", "void", or "indeterminate". There is moreover a systematic duality between those tables: whereas de Finetti treats indeterminate antecedents like false antecedents, Cooper treats them like true ones. Thus, in de Finetti's table the second row copies the third, whereas in Cooper's table it copies the first.

$f_{\to_{DF}}$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
0	$1/2$	$1/2$	$1/2$

$f_{\to_{CC}}$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	1	$1/2$	0
0	$1/2$	$1/2$	$1/2$

Table 3: Truth tables for the de Finetti conditional (left) and the Cooper conditional (right).

Both options can be pursued fruitfully, and the choice between them primarily depends on the results which they yield. Our choice is the Cooper table since it interacts more naturally with our probabilistic treatment of conditionals and the various notions of logical consequence (a detailed analysis is given in Égré, Rossi, and Sprenger 2021a). However, for the arguments made in this section, which concern only simple, non-nested conditionals, there is no difference between the two.[6]

The second choice concerns the definition of the standard logical connectives. A natural option is given by the familiar Łukasiewicz/de Finetti/Strong Kleene truth tables, displayed in Table 4. Conjunction corresponds to the "minimum" of the two values, disjunction to the "maximum", and negation to inversion of the semantic value. In particular, the trivalent analysis ad-

[5] Belnap (1973), Olkhovikov (2002/2016) and Cantwell (2008) rediscovered Cooper's truth table independently.

[6] Intermediate options vary the middle row, e.g., with the triple $\langle 1/2, 1/2, 0\rangle$ (Farrell 1986) or the triple $\langle 1, 1/2, 1/2\rangle$, suggested by a referee. The former option is reviewed in Égré, Rossi, and Sprenger 2021a, while the latter option forsakes the equivalence of $\neg(A \to C)$ and $A \to \neg C$, typically seen as a desirable property.

mits, next to the indicative conditional $A \to C$, a Strong Kleene "material" conditional $A \supset C$, definable as $\neg(A \wedge \neg C)$, or equivalently, $\neg A \vee C$.

	$f_\neg$
1	0
1/2	1/2
0	1

$f_\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$f_\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

Table 4: Strong Kleene truth tables for negation, conjunction, and disjunction.

The Strong Kleene truth table for negation is uncontroversial and also yields the consequence that the conditional commutes with negation (for either the DF- or the CC-conditional): $\neg(A \to C)$ has the same truth table as $A \to \neg C$. This is a very natural choice for interpreting conditional assertions: when we argue about $A \to C$, both sides presuppose the antecedent A and argue about whether we should be committed to C or rather to $\neg C$, given A (see also Ramsey 1929/1990, p. 247).

Unfortunately, the Strong Kleene truth tables for conjunction and disjunction have a very annoying consequence: "partitioning sentences" such as $(A \to B) \wedge (\neg A \to C)$ will always be indeterminate or false (Belnap 1973; Bradley 2002, pp. 368-370). However, a sentence such as:

> If the sun shines tomorrow, John goes to the beach; and if it rains, he goes to the museum.

seems to be true (with hindsight) if the sun shines tomorrow and John goes to the beach. This intuition is completely lost in Strong Kleene semantics, regardless of whether we use the de Finetti or the Cooper table for the conditional. Even worse, "obvious truths" such as $(A \to A) \wedge (\neg A \to \neg A)$ are always classified as indeterminate.

For this reason, we endorse alternative truth tables for conjunction and disjunction, advocated by Cooper (1968) and Belnap (1973). See Table 5. In these truth tables, indeterminate sentences are "truth-value neutral" in Boolean operations: true and false sentences do not change truth value when conjoined or disjoined with an indeterminate sentence. This can be motivated by observing that such sentences do not add determinate content as empirical statements do. We call these connectives *quasi-conjunction* and *quasi-disjunction*. They retain the usual properties of Boolean connectives (associativity, commutativity, the de Morgan laws, etc.), solve the problem of partitioning sentences, and have no substantial disadvantages with respect to Strong Kleene truth tables in conditional logic. Moreover, they have two non-trivial benefits.

First, quasi-disjunction avoids the Linearity principle that $(A \to B) \vee (B \to A)$ cannot be false. This schema was famously criticized by MacColl (1908),

	$f_\neg$
1	0
1/2	1/2
0	1

$f'_\wedge$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	0	0	0

$f'_\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	0
0	1	0	0

Table 5: Truth tables for Strong Kleene negation, paired with quasi-conjunction and quasi-disjunction as defined by Cooper (1968).

who pointed out that neither "if John is red-haired, then John is a doctor", nor "if John is a doctor, then he is red-haired", nor their disjunction seems acceptable in ordinary reasoning. A semantics that qualifies such expressions as either true or indeterminate might thus be considered inadequate. Using quasi-conjunction and quasi-disjunction instead, $(A \to B) \vee (B \to A)$ is false when A is true and B is false (or vice versa).

There is also a principled reason for adopting quasi-conjunction and quasi-disjunction, based on the connection between conditional bets and conditional assertions. How should we evaluate the conjunction of conditional assertions like $(A \to B) \wedge (C \to D)$? The interesting case occurs when A is false, but C and D are true. McGee (1989, 496-501, in particular Theorem 1) shows by a Dutch Book argument that in this case, a bet on $(A \to B) \wedge (C \to D)$ should yield a strictly positive partial return. Also Sanfilippo et al. (2020, p. 156) argue that we should classify the compound bet as winning. Indeed, to the extent that the sentence $(A \to B) \wedge (C \to D)$ is testable, it has been verified when A is false, but C and D are true. All this suggests to treat the assertion $(A \to B) \wedge (C \to D)$ as true rather than indeterminate. Unlike Strong Kleene conjunction, quasi-conjunction allows us to model this line of reasoning.

For all these reasons, we adopt quasi-conjunction, quasi-disjunction, Strong Kleene negation and the Cooper truth table for the conditional in the remainder of this paper. Our object-language is the language of propositional logic $\mathcal{L}$, supplemented with a primitive conditional connective $\to$, and is indicated as $\mathcal{L}^{\to}$. A *Cooper valuation* is a function $v : \mathcal{L}^{\to} \mapsto \{0, 1/2, 1\}$ that assigns a semantic value to all sentences of $\mathcal{L}^{\to}$ in agreement with the Cooper truth-tables, i.e. it interprets $\neg$ as the strong Kleene negation, $\wedge$ and $\vee$ as Cooper's quasi-conjunction and quasi-disjunction respectively, and $\to$ as Cooper's conditional. Note finally that all combinations of conditional and conjunctions surveyed in this section validate Import-Export: $(A \wedge B) \to C$ and $A \to (B \to C)$ are extensionally equivalent formulas.

4 Probability for Trivalent Propositions

Epistemologists capture the standing of a proposition A by the *probability* of A, reflecting the agent's evidence for and against A. When we identify propositions with sets of possible worlds, the probability of a proposition A is the cumulative credence assigned to all possible worlds where A is true.

Trivalent semantics for conditionals implements the same approach using a slight twist. As with bivalent probability, we start with a set of possible worlds W with an associated algebra $\mathcal{A}$, and a weight or credence function $c : \mathcal{A} \to [0, 1]$ defined on the measurable space $(W, \mathcal{A})$. This function represents the subjective plausibility of a particular element of the algebra, i.e., a set of possible worlds. Our use of possible worlds is devoid of metaphysical baggage and instrumental to define credence functions, as is customary in probabilistic semantics: for us, possible worlds are just Cooper valuations. [7] Moreover, we assume that any algebra $\mathcal{A}$ includes the singletons of worlds, i.e., for every $w \in W$, $\{w\} \in \mathcal{A}$. Finally, we assume that the credence function c is finitely additive with $c(\emptyset) = 0$, and $c(W) = 1$.

We now identify propositions with sentences of $\mathcal{L}^{\to}$ and define a (non-classical) probability function $p : \mathcal{L}^{\to} \longmapsto [0, 1]$, taking into account that sentences of $\mathcal{L}^{\to}$ can receive three values: true, false, or indeterminate.[8] For convenience, define

$$A_T = \{w \in W \mid v_w(A) = 1\} \qquad A_I = \{w \in W \mid v_w(A) = 1/2\}$$
$$A_F = \{w \in W \mid v_w(A) = 0\}$$

as the sets of possible worlds where A is valued as true, false or indeterminate, relative to (Cooper) valuation functions $v_w : \mathcal{L}^{\to} \mapsto \{0, 1/2, 1\}$, indexed by the possible worlds they represent.

In analogy to bivalent probability, we derive the probability of a (conditional) proposition A from the (conditional) betting odds on A: how much more likely is a bet on A to be won than to be lost? For this comparison, two quantities are relevant: (1) the cumulative weight of the worlds where A is true (i.e., $c(A_T)$), and (2) the cumulative weight of the worlds where A is false, i.e., $c(A_F)$). The *decimal odds* on A are $O(A) = (c(A_T) + c(A_F))/c(A_T)$,

[7] Notably, this does not make the interpretation of the conditional modal or non-truth-funtional: at each world w, the truth-value of $A \to C$ is given by a Cooper valuation.

[8] If you do not like to use the term "probability" in a non-classical framework, because you prefer to reserve it for standard bivalent probability, just replace it by "degree of assertability" or a similar term. This is the choice of McDermott (1996), whose definition is identical to ours. Also Cantwell (2006) proposes the same definition of trivalent probability on the basis of different truth conditions.

indicating the factor by which the bettor's stake is multiplied in case A occurs and she wins the bet. Then we calculate the probability of A from the decimal odds on A by the familiar formula $p(A) = 1/O(A)$, yielding

$$p(A) := \frac{c(A_T)}{c(A_T) + c(A_F)} \qquad \text{if } \max(c(A_T), c(A_F)) > 0. \qquad \text{(Probability)}$$

Hence, the probability of a sentence corresponds to its expected semantic value, *restricted to the worlds where the sentence takes classical truth value*. Additionally, we let $p(A) = 1$ whenever $c(A_T) + c(A_F) = 0$, i.e., if it is certain that A takes the value $1/2$ (e.g., when A is $\bot \to \top$).

In other words, the trivalent probability of A is the ratio between the credence assigned to the worlds where A is true, and the credence assigned to the worlds where A has classical truth value. Worlds where A takes indeterminate truth value are neglected for calculating the probability of A, except when they take up the whole space. For conditional-free sentences A and their Boolean compounds, this corresponds to the classical picture since $W = A_T \cup A_F$, or equivalently, $A_I = \emptyset$.

The idea behind (Probability) is the same that motivates classical operational definitions of probability: a proposition is assertable, or probable, *to the degree that we can rationally bet* on it, i.e., to the degree that betting on this proposition will, in the long run, provide us with gains rather than losses (e.g., Sprenger and Hartmann 2019). This is a good reason for calling the object defined by equation (Probability) a "probability", or a measure of the plausibility of a proposition.

The structural properties of $p : \mathcal{L}^{\to} \longmapsto [0, 1]$ resemble the standard axioms of probability:

(1) $p(\top) = 1$ and $p(\bot) = 0$.

(2) $p(A) = 1 - p(\neg A)$.

(3) $p(A \vee B) \leq p(A) + p(B)$. The equality $p(A \vee B) = p(A) + p(B)$ holds if and only if $A_T \cap B_T = \emptyset$ and $A_I = B_I$.[9]

Just like standard probability, our trivalent probability is not additive, but *subadditive*. Equality holds here exactly when A and B are incompatible and they take classical truth values in the same set of worlds. The main difference to the standard picture is that the probability of a conjunction can *exceed* the probability of a conjunct. In other words, the "and-drop" inference from $X \wedge Y$ to Y will not always preserve probability.

[9]The "only if" direction presupposes that $p(A) > 0$ and $p(B) > 0$.

However, on the betting interpretation of probability, this makes sense: when A and B are false and C is true, the bet on $(A \to B) \wedge C$ yields a positive return, while the bet on $A \to B$ is called off. So we should not expect that in all circumstances $p((A \to B) \wedge C) \leq p(A \to B)$, in notable difference to bivalent probability, and some non-classical probability functions (for a survey, see Williams 2016). Exactly the same phenomenon—the failure of "and-drop" in the context of conditional reasoning—was demonstrated in recent experiments by Santorio and Wellwood (2023). Of course, $p(A \wedge B) \leq p(A)$ will hold as long as A and B are conditional-free sentences.

On this definition of probability, we obtain for conditional-free sentences $A, C \in \mathcal{L}$ that

$$p(A \to C) = \frac{c(A_T \cap C_T)}{c(A_T)} = \frac{p(A \wedge C)}{p(A)} = p(C|A) \qquad \text{(Adams's Thesis)}$$

as for conditional-free sentences, $p(X) = c(X_T)$, and because for bivalent A and C, $\frac{c(A \to C)_T}{c(A \to C)_T + c(A \to C)_F} = \frac{c(A_T \cap C_T)}{c(A_T)}$. That is, instead of *postulating* Adams's Thesis as a desideratum on the probability of a conditional, as in Stalnaker (1970) and Adams (1975, p. 3), we obtain it immediately from the semantics of trivalent conditionals, and the definition of probability as the inverse of rational betting odds.[10] The well-known triviality results by Lewis (1976) and others are blocked since they depend on an application of the (bivalent) Law of Total Probability, which does not hold for trivalent, non-classical probability functions (Lassiter 2020).[11] Equipped with a definition of probability, we now proceed to characterizing logical consequence relations for certain and uncertain inference.

5 Certain Inference

For a conditional-free propositional language $\mathcal{L}$ with only two truth values, valid inferences preserve the truth of the premises—or equivalently, they preserve *certainties* (i.e., probability one: Leblanc 1979). In a trivalent setting,

[10]For more discussion of Adams's Thesis, including experimental evidence for and against, see Stalnaker 1968; Adams 1975; Dubois and Prade 1994; Douven and Verbrugge 2010, 2013; Evans et al. 2007; Over, Hadjichristidis, et al. 2007; Égré and Cozic 2011; Over 2016; Skovgaard-Olsen, Singmann, and Klauer 2016.

[11]Bradley (2000) proposes a different triviality result: arguably we want indicative conditionals to satisfy the Preservation Condition—if $p(A) > 0$ and $p(C) = 0$, then $p(A \to C) = 0$—, but for this to hold in full generality, we need to posit strong logical dependencies between a conditional and its components, thus trivializing the conditional. This is indeed so for bivalent accounts, but our trivalent account implies the Preservation Condition as a theorem without having a vicious dependency between the truth values of A, C and $A \to C$.

however, there is no canonical notion of "truth preservation": it could amount to preserving strict truth (i.e., semantic value 1), to preserving non-falsities (i.e., semantic value greater than 0), or to a combination of both. It is simply not clear what valid inference amounts to. But there is a canonical extension of *certainty-preserving inference* to $\mathcal{L}^{\rightarrow}$: whenever all premises have probability one, as defined in the previous section, the conclusion should have probability one, too. We call this logic C like "inference with certain premises". Formally:

Definition 1 (Valid Inference in C). *For any set of formulas $\{\Gamma, B\} \subseteq \mathcal{L}^{\rightarrow}$, the inference from Γ to B is C-valid, in symbols $\Gamma \models_{\mathsf{C}} B$, if and only if for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0, 1]$: if $p(A) = 1$ for all $A \in \Gamma$, then also $p(B) = 1$.*

In its spirit, this definition of logical consequence is similar to theories of conditional inference based on preserving acceptability in context (e.g., Gillies 2009; Santorio 2022a,b)—probability 1 is just a specific way of expressing which propositions are *accepted*, and valid inference amounts to preservation of (full) acceptance (e.g., Stalnaker 1975, p. 271). In fact, the properties of C largely agree with Santorio's preferred system (though not with Gillies's)—but without the limitation to a language involving at most simple conditionals.

Based on the probabilistic characterization of the logic of certain inference C, we can derive which trivalent logic corresponds to it: an inference is C-valid if and only if non-falsity is preserved in passing from Γ to B. Equivalently, we cannot assign a designated value (1 or $1/2$) to the premises without assigning it to the conclusion, too. This is the main result of this section.

Proposition 1 (Trivalent Characterization of C). *For any set $\{\Gamma, B\} \subset \mathcal{L}^{\rightarrow}$, $\Gamma \models_{\mathsf{C}} B$ if and only if for all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$:*

for every A in Γ, if $v(A) \geq 1/2$, then $v(B) \geq 1/2$.

In other words, C preserves truth in the (weak) sense that we cannot infer a false conclusion from a set of non-false premises. Equivalently, if the conclusion is false, one of the premises must have been false. We have thus established an analogous result to the equivalence between truth-preserving and certainty-preserving inference in standard propositional logic.

C satisfies principles such as $B \models_{\mathsf{C}} A \rightarrow B$, i.e., if we are certain that Bob comes to the party, then we are also certain that Bob comes to the party if Alice does. While this inference is fallacious when premises are uncertain, it is valid in any context where we have *verified* the premise—either empirically or by mathematical proof.[12] We also have Conditional Proof and other characteristic principles of deductive reasoning in C, such as Modus Ponens, Modus Tollens and the Law of Identity ($\models_{\mathsf{C}} A \to A$). On the other hand, problematic inferences such as the inference from $\neg A$ to $A \to B$ are blocked. Finally, the laws of classical logic in the conditional-free language $\mathcal{L}$ (=the Boolean fragment of $\mathcal{L}^{\to}$) are also theorems of C, if we restrict ourselves to bivalent valuations. [13]

C retains Disjunctive Syllogism ($A \vee B, \neg A \models B$), but gives up Disjunction Introduction ($A \models A \vee B$). However, the counterexample necessarily involves the semantic value $1/2$: when we restrict ourselves to classical *valuations* of atomic sentences, the only invalid instances of $A \models A \vee B$ occur when A is itself a conditional with a false antecedent. This shows that exceptions to the otherwise intuitive rule of Disjunction Introduction addition are quite modest; in fact, Santorio and Wellwood (2023) present theoretical and empirical arguments why Disjunction Introduction *should* fail in these circumstances.

Finally, characterizing C as preserving two designated semantic values ($D = \{1, 1/2\}$) is not only of theoretical interest, but greatly simplifies the study of this logic: for deciding theorems and valid inferences it suffices to look at the truth tables. Section 7 studies the theorems and valid inferences in more detail and compares certain inference with C to uncertain inference where instead of certainty, high probability is preserved. Notably, these properties depend on interpreting the conditional using the Cooper truth table: if we had instead paired the de Finetti truth table with preserving non-falsity, we would have lost Modus Ponens—arguably a substantial drawback for a logic that generalizes deductive logic to certain inference with conditionals.

At this point, the reader may ask what would have happened if we had adopted *strict* truth preservation (i.e., preservation of semantic value 1) as the condition for logical consequence. This logic, let us call it $\models_{\mathsf{P}}$, preserves strictly positive probability in passing from the premise to the conclusions:

Proposition 2 (Characterization of Possibility-Preserving Inference)**.** *Suppose $A, B \in \mathcal{L}^{\to}$ and there exists at least one probability function where $p(B) < 1$. Then the following two characterizations of $A \models_{\mathsf{P}} B$ are equivalent:*

1. *For all Cooper valuations $v : \mathcal{L}^{\to} \longmapsto \{0, 1/2, 1\}$ such that $v(A) = 1$, it is also the case that $v(B) = 1$.*

[12] This behavior of the conditional is similar to the conditional developed in state space semantics, e.g., by Leitgeb (2017).

[13] C is a paraconsistent logic almost equivalent to Cooper's—his propositional logic of Ordinary Discourse—except that we do not restrict C to bivalent valuations. Égré, Rossi, and Sprenger (2021a), who study the entire family of trivalent consequence relations and provide a different argument in favor of C, call it QCC/TT.

2. *For all credence functions* $c : \mathcal{A} \longmapsto \mathbb{R}$ *with* $c(A_I) < 1$ *and associated probability function* $p : \mathcal{L}^{\to} \longmapsto [0, 1]$*: if* $p(A) > 0$*, then* $p(B) > 0$*.*

In other words, $A \models_{\mathsf{P}} B$ if and only if B is a real possibility (i.e., $p(B) > 0$) in all probability functions that make A a real possibility. While C preserves non-falsity and probabilistic certainties, P preserves strict truth and probabilistic possibilities (see also Adams 1996b).[14] Therefore it also satisfies characteristic principles of (conditional) possibility logic, such as the inference from $A \to B$ to $B \to A$. The fact that it satisfies such principles (and fails plausible theorems such as $\models A \to A$) is also a good argument why preservation of (strict) truth is not an adequate consequence relation for reasoning with conditionals. We now move to the main contribution of this paper: developing an account of non-monotonic reasoning with conditionals when premises are uncertain.

6 Uncertain Inference

Certain inference with conditionals is arguably monotonic: when we know B for certain, or when we suppose it as holding no matter what, we also know that B is the case under the condition that A. However, when we move to *uncertain* inference, where only high probability or degree of assertability is preserved, things change. We may accept, assert, or find plausible B, but reject B under the condition that A. For example, the conditional "if Real Madrid faces Juventus in their next match, then Real Madrid will win" sounds highly plausible, whereas "if Real Madrid faces Juventus in their next match but most of their players are sick, then Real Madrid will win" seems much less plausible. A logic of inference with *uncertain* premises U should therefore, unlike the logic C, be non-monotonic, i.e., we cannot infer from $A \to C$ that $A \wedge B \to C$ for any A, B and $C \in \mathcal{L}^{\to}$.[15]

The canonical definition of validity for single-premise inference in a logic of uncertain inference preserves probability, as a proxy for rational acceptance or assertability (e.g. Adams 1975). In other words, the probability of the premise A must never exceed the probability of the conclusion B. Almost all logics of uncertain reasoning agree on this criterion for single-premise inference, which is the natural analogue of truth preservation in certain reasoning. We therefore adopt it as our definition of single-premise logical consequence in uncertain reasoning:

[14] This logic is called QCC/SS in the classification system proposed by Égré, Rossi, and Sprenger (2021a).

[15] The structural rule of Weakening (that is, inferring $A, B \models C$ from $A \models C$) will remain valid in our logic U. However, the rule $\neg A \models A \to B$ fails in it, for the same reasons that make it fail in C.

Definition 2 (Valid Single-Premise Inference in U). *For formulas $A, B \in \mathcal{L}^{\rightarrow}$: $A \models_{\mathsf{U}} B$ if and only if $p(A) \leq p(B)$ for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$ based on credence functions $c : \mathcal{A} \longmapsto \mathbb{R}^{\geq 0}$.*

Corollary 1. *$\models_{\mathsf{U}} B$ if and only $p(B) = 1$ for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$ based on credence functions $c : \mathcal{A} \rightarrow \mathbb{R}^{\geq 0}$.*

Corollary 2. *C and U have the same theorems.*

It is easy to show that this inference criterion has the following characterization in trivalent logic:

Proposition 3 (Equivalent Characterizations of Valid Single-Premise Inference in U). *For $A, B \in \mathcal{L}^{\rightarrow}$, the following are equivalent:*

(1) $A \models_{\mathsf{U}} B$

(2) For all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$, $v(A) \leq v(B)$, or $\models_{\mathsf{C}} B$. In other words, if $v(A) = 1$ then $v(B) = 1$, and if $v(A) \geq 1/2$, then $v(B) \geq 1/2$.

(3) $A \models_{\mathsf{C}} B$ and $A \models_{\mathsf{P}} B$, or $\models_{\mathsf{C}} B$;

(4) $A \models_{\mathsf{C}} B$ and $\neg B \models_{\mathsf{C}} \neg A$, or $\models_{\mathsf{C}} B$;

Condition (2) expresses that the semantic value of the conclusion must not fall below the semantic value of the premise in all possible valuations. By Proposition 1 and Proposition 2, this is equivalent to the conjunction of $A \models_{\mathsf{C}} B$ and $A \models_{\mathsf{P}} B$ (or $\neg B \models \neg A$), i.e., both certainties and possibilities are preserved.[16] Thus, U validates fewer inferences than C. The proposition states that all these conditions are equivalent to demanding that the conclusion be at least as probable as the premise for all probability functions.

Extending this criterion to multi-premise inference $\Gamma \models B$, for $\Gamma \subseteq \mathcal{L}^{\rightarrow}$, is non-trivial. Should the probability of B not fall below the minimum probability of the premises? Should it follow Adams's uncertainty preservation criterion (Adams 1975, 1996b)? Should B be at least as plausible as the conjunction of the premises? Since there is no intuitively best candidate here, we believe that the choice should depend on the logical properties of the proposed criterion. We propose that $\Gamma \models_{\mathsf{U}} B$ if and only if for a subset $X \subseteq \Gamma$ of the premises, *the probability of the (quasi-)conjunction of the elements of X never exceeds the probability of the conclusion*, regardless of the choice of the probability function. Formally:

[16] Égré, Rossi, and Sprenger (2021a) call this logic QCC/SS∩TT since it preserves both strict and tolerant truth value (=both strict truths and non-falsities). This is one of the logics entertained in Belnap (1973).

Definition 3 (Valid Multi-Premise Inference in U)**.** *For a set of formulas $\Gamma \subseteq \mathcal{L}^{\rightarrow}$ and a formula $B \in \mathcal{L}^{\rightarrow}$: $\Gamma \models_{\mathsf{U}} B$ if and only if there is a finite subset of the premises $\Delta \subseteq \Gamma$ such that for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$, $p(\bigwedge_{A_i \in \Delta} A_i) \leq p(B)$.*

We define validity by means of existential quantification over (possibly improper) *subsets* of Γ, in order to preserve the fact that a set of premises entails each of its members, namely $\Gamma \models_{\mathsf{U}} A$ for any $A \in \Gamma$ (compare Dubois and Prade 1994, p. 1729). If we required instead that the quasi-conjunction of *all* members of Γ have lower probability than B, we would no longer have that $A, B \models_{\mathsf{U}} A$ for every B, despite the fact that $A \models_{\mathsf{U}} A$ for every A.[17]

There are also principled reasons for adopting this definition. First of all, Definition 3 allows us to extend the equivalence between probabilistic inference and a trivalent consequence relation from the single-premise to the multi-premise case:

Proposition 4 (Equivalent Characterizations of Valid Multi-Premise Inference in U)**.** *For $\Gamma \subseteq \mathcal{L}^{\rightarrow}$ and $B \in \mathcal{L}^{\rightarrow}$, the following are equivalent:*

(1) $\Gamma \models_{\mathsf{U}} B$.

(2) Either $\models_{\mathsf{C}} B$, or there is a finite subset of premises $\Delta \subseteq \Gamma$ such that the semantic value of B is, for all Cooper valuations v, at least as high as the semantic value of the quasi-conjunction of the premises: $v(\bigwedge_{A_i \in \Delta} A_i) \leq v(B)$.

(3) Either $\models_{\mathsf{C}} B$, or there is a finite subset of premises $\Delta \subseteq \Gamma$ such that $\bigwedge_{A_i \in \Delta} A_i \models_{\mathsf{C}} B$ and $\bigwedge_{A_i \in \Delta} A_i \models_{\mathsf{P}} B$.

(4) Either $\models_{\mathsf{C}} B$, or there is a finite subset of premises $\Delta \subseteq \Gamma$ such that $\bigwedge_{A_i \in \Delta} A_i \models_{\mathsf{C}} B$ and $\neg B \models_{\mathsf{C}} \bigvee_{A_i \in \Delta} \neg A_i$.

As for C, the equivalence of (1) with (2), (3) and (4) is not only attractive from a computational point of view, but it also connects probabilistic reasoning with conditionals to the trivalent semantics that defines their truth conditions in the first place.

Secondly, Proposition 4 also provides sound and complete calculi for the logic U for free. For instance, since Cooper (1968) has a sound and complete Hilbert-style calculus for C, this automatically translates, thanks to Proposition 4, into a sound and complete calculus for U. Validity in U is nothing else but the combination of two valid consequence relations in C. Alternatively,

[17] In other words, although the logic would remain reflexive, it would not be structurally monotonic. We are indebted David Over for discussion on this topic.

still using Proposition 4, tableau- and sequent-style sound and complete axiomatizations of U can be extracted from Égré, Rossi, and Sprenger (2021b).

Thirdly and finally, defining multi-premise inference in this way yields an attractive set of valid inferences with uncertain premises, as we will see in the next two sections.

7 Properties of U

Constitutive and Generally Desirable Principles in Uncertain Inference		C	U
Logical Truth	$\models A \to \top$	✓	✓
Law of Identity	$\models A \to A$	✓	✓
Supraclassicality (Laws)	(for A without $\to$) if $\models_{\mathsf{CL}} A$, then $\models A$	(✓)	(✓)
Left Logical Equivalence	if $A \models_{\mathsf{C}} B$, $B \models_{\mathsf{C}} A$, then $A \to C \models B \to C$	✓	✓
Stronger-Than-Material	$A \to B \models A \supset B$	(✓)	(✓)
Conjunctive Sufficiency	$A, B \models A \to B$	✓	(✓)
AND	$A \to B, A \to C \models A \to (B \land C)$	✓	✓
OR	$A \to C, B \to C \models (A \lor B) \to C$	✓	(✓)
Cautious Transitivity	$A \to B, (A \land B) \to C \models A \to C$	✓	(✓)
Cautious Monotonicity	$A \to B, A \to C \models (A \land C) \to B$	✓	✓
Rational Monotonicity	$A \to B, \neg(A \to \neg C) \models (A \land C) \to B$	✓	✓
Reciprocity	$A \to B, B \to A \models (A \to C) \equiv (B \to C)$	✓	(✓)
Right Weakening	if $B \models_{Q} C$, then $A \to B \models A \to C$	✓	(✓)
Rule of Conditional K	if $A_1, \ldots, A_n \models_{\mathsf{C}} C$, then $(B \to A_1), \ldots, (B \to A_n) \models (B \to C)$	✓	(✓)
Optional and Disputed Principles			
Supraclassicality (Inferences)	if $\Gamma \models_{\mathsf{CL}} B$ then $\Gamma \models B$	✗	✗
Modus Ponens	$A \to B, A \models B$	✓	(✓)
Modus Tollens	$A \to B, \neg B \models \neg A$	(✓)	(✓)
Simplifying Disjunctive Antecedents	$(A \lor B) \to C \models (A \to C) \land (B \to C)$	(✓)	(✓)
Import-Export	$A \to (B \to C)$ if and only if $(A \land B) \to C$	✓	✓
Or-to-If	$\neg A \lor B \models A \to B$	✓	✗
Conditional Excluded Middle	$\models (A \to B) \lor (A \to \neg B)$	✓	✓
Connexive Principles (optional)			
Aristotle's Thesis	$\models \neg(\neg A \to A)$	✓	✓
Boethius's Thesis	$\models (A \to C) \to \neg(A \to \neg C)$	✓	✓
Undesirable Principles			
Contraposition	$A \to C \models \neg C \to \neg A$	(✓)	✗
Monotonicity	$A \to C \models (A \land B) \to C$	✓	✗
Transitivity	$A \to B, B \to C \models A \to C$	✓	✗

Table 6: Overview of Inference Principles involving conditionals in uncertain inference. In the rightmost columns, it is shown whether C and U validate the principle generally (✓), only for bivalent valuations of the sentential variables (✓in parentheses), or not at all (✗).

We now evaluate the logic U in terms of the inference schemes it validates, using the principles in Table 6, taken from the survey article by Egré and Rott

(2021).[18] The principles above the first horizontal line are generally considered to be desirable, or at least not harmful, in uncertain reasoning with conditionals. The principles between the lines—e.g., Modus Ponens, Or-To-If, Import-Export, and Conditional Excluded Middle—are typically a bone of contention between theorists. We also include some tautologies that are distinctive for connexive logics. The principles at the bottom—Contraposition, Monotonicity and Transitivity—are characteristic of most monotonic logics, and logics of deductive inference in particular, but should *not* be satisfied by a non-monotonic logic of uncertain reasoning with conditionals (for compelling counterexamples, see Adams 1965). So we should expect that these principles are satisfied by C, but not by U.

Table 6 evaluates, in the rightmost columns, C and U with respect to all these principles. We cannot discuss each of them in detail, but we make some general observations. Many desirable or non-harmful principles are satisfied by U without restriction, whereas some of them only hold for bivalent ("atom-classical") valuations of at least one sentential variable. This means that when all sentences are conditional-free, the inference is valid; only when one of the sentences contains a conditional connective (so that it can take the third truth value), it is possible that the inference fails. When we compare U to *classical* conditional logics (i.e., logics where all valuations are bivalent, such as Stalnaker-Lewis logics), we can consider the principles valid since making a comparison presupposes bivalent valuations. Specifically, U recovers all valid inferences of System P, which is a classical benchmark for conditional logics (Adams 1975; Kraus, Lehmann, and Magidor 1990).[19] Moreover, both C and U validate connexive principles such as Aristotle's Thesis ($\neg(\neg A \to A)$) and Boethius's Thesis ($(A \to C) \to \neg(A \to \neg C)$).

Principles that are typically considered problematic—Monotonicity, Contraposition, Transitivity, (Egré and Rott 2021)—are indeed *not* valid in U. These principles do not even hold when we restrict U to bivalent valuations of sentential variables. However, they *do* (mainly) hold in our logic of certain inference C, in line with our view of C as a generalization of classical deductive logic to a language with a conditional.

Most interesting are the six principles in the middle. Supraclassicality fails because C does not support Explosion, e.g., while $A \wedge \neg A \models_{\mathsf{CL}} B$ holds for any two sentences A and B, it is not the case that $A \wedge \neg A \models_{\mathsf{C}} B$. However,

[18] We use C as an appropriate generalization of classical deductive logic in formulating principles like Left Logical Equivalence or Right Weakening.

[19] Adams (1975) characterized his logic of uncertain inference by seven syntactic principles whose combination is known as System P: the Law of Identity, AND, OR, Cautious Monotonicity, Left Logical Equivalence, and Right Weakening.

all classical laws are theorems of both C and U when restricted to bivalent valuations. Modus Ponens and Modus Tollens hold for conditional-free sentences, but break down for nested conditionals—in line with McGee's famous objections (see the next section for a detailed analysis). Also Simplification of Disjunctive Antecedent is preserved for bivalent valuations only.

Import-Export holds unrestrictedly, since $A \to (B \to C)$ and $(A \wedge B) \to C$ have exactly the same truth conditions. The principle is intuitively plausible: "it appears to be a fact of English usage, confirmed by numerous examples, that we assert, deny, or profess ignorance of a compound conditional $A \to (B \to C)$ under precisely the circumstances under which we assert, deny, or profess ignorance of $(A \wedge B) \to C$" (McGee 1989, p. 489). Experimental evidence seems to confirm this attitude (van Wijnbergen-Huitink, Elqayam, and Over 2015). Indeed, the main motivation for giving up Import-Export—e.g., in Stalnaker-Lewis semantics, but also in the probabilistic semantics of Sanfilippo et al. (2020)—is not its implausibility, but the pressure from Gibbard's and Lewis's triviality results, where Import-Export is an important premise. Some accounts therefore restrict the validity of Import-Export to simple conditionals and set up an error theory of why we infer from there to the general validity of the principle (e.g., Mandelkern 2020). By contrast, both C and U can incorporate Import-Export since the triviality results do not apply to these logics (Égré, Rossi, and Sprenger 2022).

Conditional Excluded Middle (CEM) is a validity of C, and is therefore valid in U as well. Numerous analyses of indicatives endorse CEM (e.g., Stalnaker 1980; Williams 2010; Ciardelli 2020; Santorio 2022a), but there are also notable opponents (e.g., Gillies 2009; Kratzer 2012). A natural way to argue for CEM is to note that it is an immediate consequence of commutation with negation, i.e., the semantic equivalence between $\neg(A \to B)$ and $A \to \neg B$, which also holds in our system. To see this, note that $(A \to B) \vee \neg(A \to B)$—an instance of the Law of Excluded Middle—immediately entails $(A \to B) \vee (A \to \neg B)$, that is CEM.

Finally, a crucial difference between C and U concerns the relation of the indicative to the material conditional $A \supset B := \neg A \vee B$ (read as the quasi-disjunction of A and B). On the one hand, $A \supset B \models_{\mathsf{C}} A \to B$, i.e., if we know that Alice or Bob ordered a beer, then, if we learn that Alice did not order a beer, we can infer that Bob did so. This apparently valid Or-to-If inference is a classical argument for analyzing the indicative conditional in line with the material conditional, and C captures this intuition. However, this inference is *invalid* when we infer the conditional from an uncertain disjunction. A good illustration of this failure is given by Edgington (1986, p. 191): if I am 90% confident that it is 8 o'clock, then I am at least as confident that it is 8 or 11 o'clock, but that does not give me the same confidence that if it is not 8 then it is 11 o'clock. Indeed, Or-to-If fails in U, as we want to have it. Actually, neither does the material conditional imply the indicative conditional in U, nor vice versa.

However, the simple, non-nested indicative conditional often appears to be more demanding to assert than the material conditional (e.g., Gibbard 1980; Gillies 2009). Can our account then explain this "Stronger-Than-Material" intuition? Yes—because for bivalent valuations that use only classical truth values, $A \to B$ entails $A \supset B$ in both C and in U. In the context of uncertain reasoning with conditional-free statements, $p(A \to B) = p(B|A) \leq p(A \supset B)$ is a theorem. In summary, we have Or-to-If as a valid principle for reasoning from certain premises, but not from uncertain premises; nonetheless, we show that why $A \to B$ is less acceptable than $A \supset B$ whenever antecedent and consequent are conditional-free sentences.

8 Modus Ponens, Tollens, and Import-Export

Modus Ponens appears invariably valid in inference from *certain* premises, but a famous counterexample by McGee (1985) challenges its validity in inference from *uncertain* premises. It concerns the 1980 U.S. presidential elections.

> If a Republican wins the election, then, if Reagan does not win, Anderson will win.
>
> A Republican will win the election.
>
> ---
>
> Therefore, if Reagan does not win the election, Anderson will.

At some point before the elections, the two premises were commonly accepted: Ronald Reagan was predicted to win the election, and Anderson was the runner-up behind Reagan in the Republicans' primary race. By Modus Ponens we infer that if Reagan does not win, Anderson will. The logical form of that inference is: from $A \to (B \to C)$ and A, infer, by Modus Ponens, $B \to C$. However, in the polls Anderson was actually trailing both Reagan and Carter, the democrat incumbent. Therefore, if Reagan was not elected president, the best prediction would be that Carter would be elected, contradicting the conclusion.

McGee's counterexample has generated a large amount of literature concerning the validity of Modus Ponens.[20] As stressed by McGee, the intuitive appeal of the counterexample depends crucially on the use of nested conditionals. In particular, Stern and Hartmann (2018) show that when the major premise of Modus Ponens is a nested conditional, the probability loss in inferring to the conclusion can be much higher than when we apply Modus Ponens to non-nested premises. For bivalent propositions A and B, the term

$$p(B) = p(B|A)p(A) + p(B|\neg A)(1 - p(A)) \tag{1}$$

is, by the Law of Total Probability, well controlled by the values of $p(A)$ and $p(B|A)$—the values that represent the probability of the two premises of Modus Ponens. For example, if both values exceed .9, then $p(B) \geq .81$, so the product of the two probabilities is still a reasonably high value.

However, in the case of right-nested conditionals, the probability of the conclusion of Modus Ponens is poorly controlled:

$$p(C|B) = p(C|A \wedge B)p(A|B) + p(C|\neg A \wedge B)(1 - p(A|B)) \tag{2}$$

Suppose that premises are highly plausible, e.g. $p(A) \geq .9$ and $p(C|A \wedge B) \geq .9$, where the latter probability has been calculated by applying Import-Export and Adams's Thesis to $A \to (B \to C)$. Then you can still assign extremely low values to three of the four probabilities on the right hand side of equation (2), and derive a very low value of $p(C|B)$. Therefore the probability loss is more pronounced in McGee's example than when we apply Modus Ponens to simple conditionals.

Our logics mirror this diagnosis: Modus Ponens is valid in C, i.e., in certain inference, and valid in U for bivalent valuations, i.e., when all involved propositional constants are classical. However, U does *not* validate the unrestricted form of Modus Ponens, and in fact, the only countermodel to the schema $A \to B, A \models B$ is $v(A) = 1$ and $v(B) = 1/2$ (i.e., B is a conditional with false antecedent).[21] The same kind of analysis can be applied to showing

[20] Sinnott-Armstrong, Moor, and Fogelin (1986) respond that the conclusion should be evaluated as a material conditional—which would be a plausible proposition—, and argue that the burden is on McGee to show that this interpretation of the conditional is inadequate. But this defensive strategy is threatened by the strong theoretical and empirical arguments against the material conditional view, in particular the paradoxes of material implication, and the fact that judgments on the probability or assertability of $A \to C$ align with $p(C|A)$, not with $p(\neg A \vee C)$ (e.g., Over, Hadjichristidis, et al. 2007).

[21] Suppose that "A Republican will win" is true if and only if Reagan or Anderson wins. The main conditional then has probability 1 (since Or-to-If is valid in C), the disjunction has high probability, and the consequent has a low probability. Thus, nested Modus Ponens in McGee-type examples fails if and only if the associated Or-to-If inference fails. The fact that

that Modus Tollens, i.e., the schema $A \to B, \neg B \models \neg A$, is valid for simple conditionals, but not for arbitrary nested conditionals.

Since Import-Export features crucially in McGee's counterexample (e.g., in Stern and Hartmann's probabilistic reconstruction), philosophers and logicians have often faced a choice between both principles. For example, Stalnaker (1968) and Lewis (1973b) give up Import-Export, but retain Modus Ponens. So does Mandelkern (2020), who restricts the validity of Import-Export.[22] Our trivalent framework makes the opposite and arguably more natural choice: like McGee (1989), we let Import-Export be unrestrictedly valid and restrict the validity of Modus Ponens. This account does not only give a convincing analysis of McGee-style examples, which are typically recognized as a problem for Modus Ponens in uncertain reasoning, but also agrees with psychological evidence in favor of Import-Export and simple Modus Ponens.

9 Comparisons

The trivalent treatment of indicative conditionals is first sketched in Reichenbach (1935) and de Finetti (1936a,b). A more detailed motivation of this approach, including an overview of the main consequence relations of interest, is given by Belnap (1970, 1973), but none of these authors provides a fully worked out account of the logic and epistemology of conditionals. The first complete trivalent account of a logic of conditionals is due to Cooper (1968), who originally created system C. However, Cooper restricts it to bivalent valuations of the sentential variables, without applying it to the entire language $\mathcal{L}^{\to}$, and does not connect it to the probability of conditionals. Cantwell (2008) investigates the logical consequence relation of C (=preservation of non-falsity), but uses Strong Kleene connectives for conjunction and disjunction. Moreover, his treatment of "non-bivalent probability" ends up with an altogether different probabilistic logic (Cantwell 2006).

Most similar to our approach, both in spirit and content, are the trivalent accounts developed by Dubois and Prade (1994) and McDermott (1996).

McGee's argument is analyzed as valid in C and as invalid in U is also in accordance with the ambivalence generally felt regarding whether the argument is valid or not; specifically, also Neth (2019) and Santorio (2022b) distinguish between the validity of Modus Ponens in certain and uncertain inference.

[22] More precisely, Mandelkern shows that a conditional satisfying Conditional Introduction (i.e., the meta-inference from $\Gamma, A \models B$ to $\Gamma \models A \to B$) and both Modus Ponens and Import-Export is equivalent to the material conditional. He suggests to restrict the scope of Import-Export to cases where the "middle proposition" B in $A \to (B \to C)$ does not contain a conditional.

However, these authors stick to de Finetti's original truth table and (in the case of McDermott) use Strong Kleene truth tables for conjunction and disjunction. The semantic features are thus quite different. On the level of inferences, many features are similar, but McDermott's logic validates Transitivity ($A \to B$, $B \to C$, therefore $A \to C$). While this is acceptable and even desirable in the framework of *certain* inference, it is arguably problematic when reasoning from *uncertain* premises since the probability of $p(C|A)$ is in no way controlled by $p(C|B)$ and $p(B|A)$; in fact, it can be arbitrarily low. Suppose that you live in a very sunny, dry place. Consider the sentences A = "it will rain tomorrow", B = "I will work from home", C = "I will work on the balcony". Clearly, both $A \to B$ and $B \to C$ are highly plausible, but $A \to C$ isn't. This structural feature offers, in our view, a decisive reason to prefer our model to McDermott's. Dubois and Prade avoid that feature, but like Adams and Cooper, they restrict their account to the flat fragment of $\mathcal{L}^{\to}$, i.e., allowing only simple, non-nested conditionals.

	Trivalent Logics		Bivalent Logics		
Inference Principle	U	MD	P	VC	C2
Stronger-Than-Material	(✓)	✓	✓	✓	✓
Conjunctive Sufficiency	(✓)	✓	✓	✓	✓
OR	(✓)	✓	✓	✓	✓
Cautious Transitivity	(✓)	✓	✓	✓	✓
Transitivity	✗	✓	✗	✗	✗
Modus Ponens	(✓)	(✓)	✓	✓	✓
Modus Tollens	(✓)	(✓)	✓	✓	✓
Import-Export	✓	✓	N/A	✗	✗
SDA	(✓)	✓	N/A	✗	✗
Rational Monotonicity	✓	(✓)	N/A	✓	✓
Conditional Excluded Middle	✓	✓	N/A	✗	✓

Table 7: Comparison of the logic U with alternative conditional logics, restricted to inference principles where not all of the logics agree. The surveyed alternatives are System P, Lewis' VC, Stalnaker's C2, and McDermott's MD.

On the side of reasoning, our logic U generalizes the benchmark account of uncertain reasoning developed in Adams's (1975) monograph *The Logic of Conditionals*. In this book, Adams equates the probability of a conditional $A \to C$ with the conditional probability $p(C|A)$, and develops a probabilistic logic of uncertain reasoning with conditionals on that basis. The descriptive accuracy of the predictions of Adams's logic is acknowledged both by philosophers and by psychologists of reasoning (e.g., McGee 1989, pp. 487-488; Ciardelli 2020, p. 544; Over, Hadjichristidis, et al. 2007; Over and Baratgin 2017), but due the lack of general truth conditions for compounds and Boolean combinations of conditionals, it has limited scope. The incompleteness of the theory has encouraged more ambitious theorists to pursue different roads (e.g., modal semantics or dynamic semantics). Our account recovers all the inferences in Adams's logic of reasonable inference without suffering from these restrictions. Specifically, some principles that Adams needs to postulate as axioms, such as the equation $p(A \to B) = p(B|A)$ (for $A, B \in \mathcal{L}$) or the Import-Export Principle, emerge as *corollaries* of our semantics. This makes our account more unified and coherent than Adams's.

We conclude our comparisons with a note on other truth-conditional approaches. The classical modal semantics for a conditional $A \to C$ defines it as true if C is true at the closest possible A-world (e.g., as defined by Stalnaker's selection function or Lewisian spheres: Stalnaker 1968, 1975; Lewis 1973b,a; McGee 1989). If A is true in the actual world, the truth value of the conditional corresponds to the truth value of the consequent, as in our analysis. The fundamental difference emerges when A is false: while we assign a third truth value to the conditional, modal theorists assign a classical truth value, essentially based on epistemic considerations ("is C the case in a plausible world, or set of worlds, where A is the case?"). In other words, Stalnaker-Lewis semantics creates a disparity between the case where A is true, where truth conditions are factual, and the case where A is false, where truth conditions depend on considerations of plausibility and normality. On our approach, epistemological considerations are relevant for assertion and reasoning, but truth conditions are entirely factual.

Modern developments of modal semantics go beyond possible-world selection functions. Their common denominator is to evaluate a conditional $A \to C$ as true if C is true in all relevant contexts selected by the antecedent A (e.g., Kratzer 1986; Mandelkern 2019). Specifically, dynamic and information state semantics implement this idea by *updating* on A (e.g., Gillies 2009; Santorio 2022a). These accounts integrate the semantics of "if... then..." with the semantics of other modal operators, but they struggle to give a quantitative analysis of the probability of conditional which squares with the truth conditions and yields Adams's thesis (though see Goldstein and Santorio 2021). The connection to probabilistic reasoning, and the distinction between certain and uncertain inference, is therefore easier to make for us than for them. Moreover, in order to obtain full truth conditions that are stronger than the material conditional, Gibbard's (1980) triviality result forces modal accounts to give up Import-Export (or another very plausible principle such as Supraclassicality), limiting them in their ability to analyze complex conditionals. As explained in Section 7, the trivalent account does not need to make similar concessions (see also Lassiter 2020; Égré, Rossi, and Sprenger 2022).

10 Conclusions

The trivalent analysis in this paper closes the gap between the truth conditions of conditionals, their probabilistic semantics, and our (certain and uncertain) reasoning with them. Specifically, we propose two logics that generalize the concept of valid inference to reasoning with conditionals: C explicates conditional reasoning with certain premises, U explicates conditional reasoning with uncertain premises. Although C is a paraconsistent logic, all theorems of classical logic are also theorems of C when restricted to bivalent valuations. The combination of C and U avoids Gibbard's and Lewis's triviality results, and provides a unified framework for conditional reasoning, in light with the observation that some inference schemes (e.g., Or-To-If, nested Modus Ponens) appear valid in certain and invalid in uncertain reasoning.

Summarizing the main features and results of our approach according to topics:

Truth Conditions The indicative conditional expresses a conditional commitment to the consequent, retracted if the antecedent turns out false. This interpretation motivates a fully truth-functional trivalent analysis of the conditional. Following Cooper, we group indeterminate antecedents with true ones, and interpret conjunction and disjunction according to his truth tables for quasi-conjunction and -disjunction.

Probability The probability of a sentence of $\mathcal{L}^{\rightarrow}$ is the ratio of the weight of possible worlds where it is true, divided by the weight of possible worlds where it is either true or false. Adams's Thesis $p(A \rightarrow C) = p(C|A)$ for conditional-free sentences follows as a corollary and need not be postulated as an axiom.

Certain Inference Conditional reasoning from *certain* premises is captured by the logic C, which can be characterized as preservation of maximal probability, and equivalently as preservation of non-falsity in trivalent semantics (Proposition 1).

Uncertain Inference Conditional reasoning with *uncertain* premises is captured by the logic U, which preserves probability between the quasi-conjunction of the premises and the conclusion. Equivalently, U preserves truth *and* non-falsity for all trivalent valuations of the premises and the conclusion (Proposition 3 and 4).

Combining these semantic and epistemological elements delivers a coherent and fruitful framework. Specifically, we can use it to analyze and to explain the controversy about the validity of Modus Ponens, Or-to-If, Import-Export and other important inference principles.

More work needs to be done. The most urgent projects are to explore whether this analysis can in any way be connected to the semantics and epistemology of counterfactuals, and to integrate our analysis with an account of modal operators in natural language, such as "must" and "might". Possible ways of achieving this are to find an equivalent modal semantics, or to embed the present trivalent approach into a modal framework. We leave these issues for further research.

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A Proofs of the Propositions

Given a model, consisting of a nonempty set of worlds W and a valuation function v, recall that $A_T, A_I, A_F \subseteq W$ denote the set of possible worlds where A is true, indeterminate, and false, respectively. Here and in the remainder, we identify possible worlds with complete valuation functions to all sentences in the language $\mathcal{L}_{\rightarrow}$.

Proposition 1 (Trivalent Characterization of C)**.** *For any set $\{\Gamma, B\} \subset \mathcal{L}^{\rightarrow}$, $\Gamma \models_{\mathsf{C}} B$ if and only if for all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$:*

> *for every A in Γ, if $v(A) \geq 1/2$, then $v(B) \geq 1/2$.*

Proof. "$\Rightarrow$". Suppose $A \models_{\mathsf{C}} B$. This means that for every model, $B_F \subseteq A_F$. Suppose now that $p(A) = 1$ for some probability function p: by (Probability), this requires $c(A_F) = 0$. But since $B_F \subseteq A_F$, and the measure properties of c, also $c(B_F) \leq c(A_F) = 0$ and hence $p(B) = 1$.

"$\Leftarrow$". Suppose that for any p with $p(A) = 1$, also $p(B) = 1$. Suppose further that $A \not\models_{\mathsf{C}} B$, i.e., there is a model and a world $w \in B_F$ with $w \notin A_F$. Choose c such that $c(w) = 1$, i.e., w has maximal credence, and in particular, $c(w') = 0\,, \forall w' \neq w$. Then $c(A_F) = c(B_T) = 0$, and

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \frac{c(A_T)}{c(A_T) + 0} = 1, \text{ but}$$
$$p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1} = 0,$$

contradicting what we have assumed. Hence it must be the case that $A \models_{\mathsf{C}} B$.

The generalization to more than one premise is straightforward since $A_1, \ldots A_n \models_{\mathsf{C}} B$ if and only if $\bigwedge A_i \models_{\mathsf{C}} B$. □

Proposition 2 (Characterization of Possibility-Preserving Inference)**.** *Suppose $A, B \in \mathcal{L}^{\rightarrow}$ and there exists at least one probability function where $p(B) < 1$. Then the following two characterizations of $A \models_{\mathsf{P}} B$ are equivalent:*

1. *For all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$ such that $v(A) = 1$, it is also the case that $v(B) = 1$.*

2. *For all credence functions $c : \mathcal{A} \longmapsto \mathbb{R}$ with $c(A_I) < 1$ and associated probability function $p : \mathcal{L}^{\rightarrow} \longmapsto [0, 1]$: if $p(A) > 0$, then $p(B) > 0$.*

Proof. "$\Rightarrow$". Suppose $A \models_{\mathsf{QCC/SS}} B$. This means that for every model, $A_T \subseteq B_T$. Suppose now that $p(A) > 0$; since we have excluded the case $c(A_I) = 1$

we have strictly positive credence that A is true. In other words, $c(A_T) > 0$. Since $A_T \subseteq B_T$, it follows that $c(B_T) \geq c(A_T) > 0$, and hence $p(B) > 0$.
"$\Leftarrow$". Suppose $A_T \neq \emptyset$ (otherwise the proof is trivial). We suppose further that $A \not\models_{\mathsf{QCC/SS}} B$, i.e., there is a world $w \in A_T$ with $w \notin B_T$. Moreover, by assumption ($=B$ is no theorem of C) there must be a world $w' \in B_F$. Then we choose $c(w) = c(w') = 1/2$ (for the case $w = w'$, choose $c(w) = 1$) and infer

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \left(1 + \frac{c(A_F)}{c(A_T)}\right)^{-1} > 0,$$

and moreover, since $w, w' \notin B_T$,

$$p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1/2} = 0,$$

contradicting what we assumed. Hence $A \models_{\mathsf{QCC/SS}} B$. $\square$

Proposition 3 (Equivalent Characterizations of Valid Single-Premise Inference in U)**.** *For $A, B \in \mathcal{L}^{\rightarrow}$, the following are equivalent:*

(1) $A \models_{\mathsf{U}} B$

(2) For all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$, $v(A) \leq v(B)$, or $\models_{\mathsf{C}} B$. In other words, if $v(A) = 1$ then $v(B) = 1$, and if $v(A) \geq 1/2$, then $v(B) \geq 1/2$.

(3) $A \models_{\mathsf{C}} B$ and $A \models_{\mathsf{P}} B$, or $\models_{\mathsf{C}} B$;

(4) $A \models_{\mathsf{C}} B$ and $\neg B \models_{\mathsf{C}} \neg A$, or $\models_{\mathsf{C}} B$;

Proof. We reason by cases and begin with the case $\models_{\mathsf{C}} B$. In this case, $p(B) = 1$ and hence, (1), (2) and (3) are all true. In the remainder, we can therefore neglect this case and assume that there is at least a world $w \in B_F$. We simplify and unify notation and write "$\models_{\mathsf{SS}}$" instead of "$\models_{\mathsf{QCC/SS}}$", and "$\models_{\mathsf{TT}}$" instead of "$\models_{\mathsf{QCC/TT}}$" or "$\models_{\mathsf{C}}$". First, we show the equivalence of (2) and (3).

(2)$\Rightarrow$(3): By assumption, we already have $A \models_{\mathsf{TT}} B$. Suppose $\neg B \models_{\mathsf{C}} \neg A$; this means that $(\neg A)_F \subseteq (\neg B)_F$, or equivalently, $A_T \subseteq B_T$. But the latter is the same as $A \models_{\mathsf{SS}} B$. So both the SS- and the TT-entailment holds between A and B.

(3)$\Rightarrow$(2): Suppose $A \models_{\mathsf{QCC/SS \cap TT}} B$. This implies $A \models_{\mathsf{TT}} B$ trivially; we still have to show $\neg B \models_{\mathsf{TT}} \neg A$. But since $A \models_{\mathsf{QCC/SS}} B$, we have $A_T \subseteq B_T$ and hence $(\neg A)_F \subseteq (\neg B)_F$. The latter is equivalent to $\neg B \models_{\mathsf{TT}} \neg A$.

(3)⇒(1): By assumption, $A_T \subseteq B_T$ and $B_F \subseteq A_F$. Hence, $c(A_T) \leq c(B_T)$ and $c(A_F) \geq c(B_F)$. Thus, for all probability functions $p : L_{\rightarrow} \rightarrow [0,1]$,

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \left(1 + \frac{c(A_F)}{c(A_T)}\right)^{-1} \leq \left(1 + \frac{c(B_F)}{c(B_T)}\right)^{-1} = p(B).$$

(1)⇒(3): Let us first deal with the case $A_T = \emptyset$. In that case, $A \models_{\mathsf{SS}} B$ is trivially satisfied. The only way for (3) to be false is if there is a $w \in A_I \cap B_F$, such that $A \models_{\mathsf{TT}} B$ fails. However, in that case, we can assign $c(w) = 1$, obtaining $p(A) = 1$ and $p(B) = 0$. So (1) would fail, too. For this reason, we can presuppose in the remainder that $A_T \neq \emptyset$.

We now prove the converse, i.e., ¬(3) ⇒ ¬(1). Assume first that $A \not\models_{\mathsf{QCC/SS}} B$, i.e., $A_T \cap (B_F \cup B_I) \neq \emptyset$.

Case 1: $A_T \cap B_F \neq \emptyset$. Choose a $w \in A_T \cap B_F$ and a probability distribution with $c(w) = 1$, yielding $p(A) = 1$ and $p(B) = 0$. So ¬(1) holds.

Case 2: $A_T \cap B_F = \emptyset$. Choose a $w \in A_T \cap B_I$. However, since B is by assumption no theorem of QCC/TT, we know that there is a $w' \in B_F$. Assign the credences $c(w) = c(w') = 1/2$. Then we obtain the following counterexample to (1):

$$\begin{aligned} p(A) &= \frac{c(A_T)}{c(A_T) + c(A_F)} \geq \frac{1/2}{1/2 + c(A_F)} \geq 1/2 \\ p(B) &= \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1/2} = 0. \end{aligned}$$

Now assume that $A \not\models_{\mathsf{TT}} B$, i.e., $B_F \cap (A_T \cup A_I) \neq \emptyset$. If there is a $w \in B_F \cap A_T$, we are done: simply assign maximal credence to this world, and we obtain that $p(A) > p(B)$. If there is only a $w \in B_F \cap A_I$, by contrast, we assign $c(w) = 1/2$, and moreover, we choose an arbitrary $w' \in A_T \cap (B_T \cup B_I)$ with $c(w') = 1/2$. Such a w' must exist since we have assumed $A_T \neq \emptyset$. Then, we construct a counterexample to (1) as follows:

$$\begin{aligned} p(A) &= \frac{c(A_T)}{c(A_T) + c(A_F)} = \frac{1/2}{1/2 + 0} = 1 \\ p(B) &= \frac{c(B_T)}{c(B_T) + c(B_F)} \leq \frac{1/2}{1/2 + 1/2} = 1/2 \end{aligned}$$

□

The proof of Proposition 4 proceeds exactly as the proof of Proposition 3, with the (quasi-)conjunction $A_1 \wedge \ldots \wedge A_n$ taking the role of A. Since there are no structural differences, we omit it.