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New advances in HKT geometry

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CONTENTS

INTRODUCTION

The search and study of special geometric structures has long since been a fundamental research topic in Riemannian geometry. Such a quest is strictly related to holonomy theory. For instance, the Riemannian holonomy group of a given Riemannian manifold (M, g), that is the holonomy group of the Levi-Civita connection with respect to g, is an important object that detects the presence of additional structure on M as it corresponds to a reduction of the structure group of the frame bundle. The classification of all possible holonomy groups is therefore of obvious interest and has been successfully completed in the works of Cartan [71, 72] and Berger [36], with contributions by other mathematicians.

More in detail, Cartan dealt with Riemannian symmetric spaces, while Berger obtained the list of possible Riemannian holonomy groups for irreducible (non locally symmetric) simply-connected Riemannian manifolds. The possible groups are $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)\mathrm{Sp}(1)$, $\mathrm{Sp}(n)$, G_2, $\mathrm{Spin}(7)$. The group $\mathrm{SO}(n)$ corresponds to generic Riemannian geometry; $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ represent the geometries of Kähler and Calabi-Yau manifolds; $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and $\mathrm{Sp}(n)$ correspond to quaternionic Kähler and hyperkähler geometries; and, finally, G_2 and $\mathrm{Spin}(7)$ are exceptional groups which can only occur in dimensions 7 and 8 respectively. The list originally included $\mathrm{Spin}(9)$ but later Alekseevsky [3] showed that a manifold with $\mathrm{Spin}(9)$ as Riemannian holonomy group is necessarily a Riemannian symmetric space. The classification still needed to show that all the groups in Berger's list actually occur as holonomy groups, a goal that was eventually completed with Bryant and Salamon's [57] examples of metrics with exceptional holonomy.

Such a classification can be looked from a more general perspective in two ways, both relevant to our future discussion. The first perspective comes from looking at non-Riemannian holonomy groups, i.e. the possible holonomy groups of a torsion-free linear connection that does not preserve the Riemannian metric. A result of Hano and Ozeki [162] shows that any closed Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ can occur as the holonomy group of some linear connection. Hence, it makes sense to impose torsion-freeness and restrict the possibilities. Berger presented a list of possible irreducible holonomy groups of a torsion-free connection claiming that at most a finite number of groups was missing from it. Bryant [55] found the first missing group and Chi, Merkulov and Schwachhöfer [85, 86, 87] found more, even an infinite family of them, thus proving wrong Berger's claim. Finally, in 1999 Merkulov and Schwachhöfer [224] reached a complete classification of possible groups and it was shown that every group on such list actually occurs as an holonomy group. See the survey [56] for further details.

The second perspective is driven by theoretical physics, especially in the presence of supersymmetry [173], and calls for investigation of structures with torsion. Thus Kähler, Calabi-Yau, quaternionic Kähler and hyperkähler geometries generalize as follows: instead of looking at the Levi-Civita connection, the focus is moved to a metric linear connection preserving the (hyper)complex structure with holonomy contained in $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)\mathrm{Sp}(1)$, $\mathrm{Sp}(n)$ respectively, however such a connection has (non-vanishing) totally skew-symmetric torsion. The corresponding geometries are called Kähler with torsion (KT), Calabi-Yau with torsion (CYT), quaternionic Kähler with torsion (QKT) and hyperkähler with torsion (HKT).

The last one, namely *HKT geometry*, is the object of the present work and it was introduced by Howe and Papadopoulos in [178] as it arose on some internal spaces of certain supersymmetric sigma models with Wess-Zumino term. HKT manifolds also play a role as moduli spaces for black holes [142], and later they were detected as solutions of five-dimensional de Sitter supergravity [151, 161]. The mathematical interest of HKT geometry is also supported by the necessity of weakening the compatibility conditions of hyperkähler geometry, which tends to be very rigid and restrictive, resulting in a relatively limited number of examples. HKT structures represent a natural class which is larger than hyperkähler ones.

HKT manifolds belong to the family of hypercomplex manifolds, so let us briefly discuss these first. A $4n$-dimensional smooth manifold M is hypercomplex if it allows a $\mathrm{GL}(n,\mathbb{H})$-structure, where $\mathbb{H}$ is the skew field of quaternions. In other words, M is equipped with a triple (I,J,K) of complex structures behaving like the purely imaginary unit quaternions. Obata [235] showed that the integrability of I, J and K is equivalent to the existence of a torsion-free connection that preserves (I,J,K), which is now called the *Obata connection*. Studying the properties of the Obata connection (and its holonomy) is part of the first perspective mentioned above. For instance, it is known [305] that when M has holomorphically trivial canonical bundle $K(M,I)$, the holonomy of the Obata connection lies in $\mathrm{SL}(n,\mathbb{H})$. Such manifolds are called $\mathrm{SL}(n,\mathbb{H})$*-manifolds*. The converse is still an open problem in general, but it is indeed true by a result of Verbitsky [305] when M is compact and admits a HKT structure. The study of complex manifolds with holomorphically trivial canonical bundle has attracted much attention over the years (see e.g. [20, 29, 107, 124, 125, 127, 129, 131, 185, 215, 251]).

Let us now introduce HKT manifolds. As mentioned, a HKT structure on M is the data of a Riemannian metric g which is hyperhermitian, i.e. it is Hermitian with respect to I, J and K together with a linear connection with skew-symmetric torsion that preserves both g and the hypercomplex structure (I,J,K). Such a connection is necessarily the common *Bismut connection* with respect to I, J and K. Moreover, a result of Grantcharov and Poon [148] shows that a HKT structure on a hypercomplex manifold is equivalently defined by a hyperhermitian metric g such that $\Omega := g(J\cdot,\cdot)+ig(K\cdot,\cdot)$ satisfies the condition

$$\partial\Omega = 0\,,$$

where the operator ∂ is taken with respect to I. Already from the outset we observe similarities with the Kähler setting. It is believed that HKT geometry represents the hypercomplex analogue of Kähler geometry and abundance of evidence has been found in this direction. The role of the two operators $d, d^c_I := I^{-1}dI$ is played on hypercomplex manifolds by the two operators ∂ and $\partial_J := J^{-1}\bar\partial J$. Indeed, they anticommute and square to zero, so that cohomology can be performed, as firstly done by Verbitsky [301]. Other interesting cohomologically related results that deepen the similarities with Kähler manifolds have been obtained in [146, 208].

On HKT manifolds a local $\partial\partial_J$-lemma holds, i.e. locally there always exists a smooth real-valued function u such that $\Omega = \partial\partial_J u$, this is due to Banos and Swann [27] whom proved it under a different formalism. The result was spelled out in terms of the operators ∂, ∂_J by Alesker and Verbitsky [17].

As in the Kähler case it becomes natural to wonder if, on a given HKT manifold (M,I,J,K,g,Ω) one could find special metrics belonging to the same "HKT class" of Ω:

$$\mathcal{H}_\Omega = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \Omega + \partial\partial_J\varphi > 0\}\ ,$$

where the inequality means that $\Omega_\varphi := \Omega + \partial\partial_J\varphi$ induces a new HKT metric g_φ on (M,I,J,K). It turns out that in the HKT world the "nicest possible" HKT metric is one that is balanced with respect to all I, J and K, equivalently $\partial\bar\Omega^n_\varphi = 0$ (see [306]). The existence of such a special metric is related to the holonomy of the Obata connection ∇ as it clearly implies holomorphic triviality of the canonical bundle and thus $\mathrm{Hol}(\nabla)\subseteq \mathrm{SL}(n,\mathbb{H})$. A very natural and captivating conjecture emerges as whether or not the converse is true, at least in the compact setting, namely if a compact HKT $\mathrm{SL}(n,\mathbb{H})$-manifold always admits a balanced HKT metric. Such a conjecture can be viewed as the HKT version of the conjecture of Calabi [67] on compact Kähler manifolds, proved by Yau [327] with the method of continuity. As a matter of fact, Alesker and Verbitsky [18] formulated the *quaternionic Calabi conjecture* by wondering if on a HKT $\mathrm{SL}(n,\mathbb{H})$-manifold any complex volume form is the wedge n^{th} power of a HKT metric Ω_φ for some $\varphi \in \mathcal{H}_\Omega$. This problem leads and is equivalent to the so-called *quaternionic Monge-Ampère equation*:

$$(\Omega + \partial\partial_J\varphi)^n = \mathrm{e}^F\Omega^n\,, \qquad F\in C^\infty(M,\mathbb{R})\,, \tag{1}$$

where the datum F satisfies the necessary condition

$$\int_M (e^F - 1)\Omega^n \wedge \bar{\Theta} = 0\,, \tag{2}$$

being Θ a fixed positive (in a quaternionic sense) holomorphic trivialization of the canonical bundle $K(M, I)$. Equation (1) is fully non-linear elliptic of the second order and discussing its solvability lies at the core of this work. Later, the assumption of being a $\mathrm{SL}(n, \mathbb{H})$-manifold was dropped (see [16]) and equation (1) was studied on a general compact HKT manifold, but now it must have the form

$$(\Omega + \partial\partial_J\varphi)^n = b\, e^F \Omega^n\,, \qquad F \in C^\infty(M, \mathbb{R})\,,$$

for some positive constant b, because the necessary condition (2) does no longer make sense. Going a little step further, we observe that the equation can also be studied merely on compact hyperhermitian manifolds.

The most natural approach to tackle the conjecture is by adapting the known results for the complex Monge-Ampère equation. From this point of view, establishing a priori estimates becomes crucial. The C^0 estimates has been proved on general compact HKT manifolds by Alesker and Shelukhin [16] and, with an alternative, much simpler proof, by Sroka [269]. Before that, Alesker was able to show that the conjecture is true on compact flat hyperkähler manifolds [14] and a recent paper by Dinew and Sroka [106] significantly improved this result by removing the assumption of flatness. So far, this is the most general framework under which the quaternionic Monge-Ampère equation has been solved.

The fact that equation (1) appears to be more difficult than the complex Monge Ampère equation is essentially motivated by two facts. First of all, in general there are no "quaternionic coordinates", in the sense that it is not true, for a hypercomplex manifold, that each point allows a neighbourhood isomorphic to an open subset of the flat space $\mathbb{H}^n$. Such a condition, known as *local flatness*, entails the full integrability of the $\mathrm{GL}(n, \mathbb{H})$-structure and is equivalent to flatness of the Obata connection. Furthermore, even if we assume such a condition, working with "quaternionic derivatives" is not particularly nice, as these do not satisfy neither the Leibniz rule, nor the chain rule, in general. Second, a non-hyperkähler HKT manifold cannot be Kähler (see [303]), hence even working from the complex point of view one cannot consider normal coordinates, furthermore the coordinate expression of the equation involves not only the coefficients of the metric but also of the complex structure J, whose presence causes some troubles.

We now outline the content of this thesis.
The first chapter is meant as an introduction to quaternionic linear algebra. The non-commutativity of quaternions imposes to be delicate in the development of linear algebra, however, this is not a major issue and most of the essential theory can be reproved for the skew field of quaternions. On the other hand, a very interesting "breaking point" emerges when dealing with determinants, which cannot be defined coherently via the usual definition. A few possible definitions of quaternionic determinants have been proposed but we shall only be interested in the *Moore determinant* [229]. The Moore determinant compels to restrict to hyperhermitian matrices, i.e. matrices that are Hermitian in a quaternionic sense. Such matrices have well-defined real (right) eigenvalues and the Moore determinant can be defined as the product of them, hence it captures the positivity or negativity of the eigenvalues, which is not the case for other determinants.

In the second chapter, after a brief discussion of G-structures, which we use to quickly review complex structures as a way to fix some notations, we discuss hypercomplex structures and their integrability. We then introduce HKT manifolds and start to study their geometry, giving more details on the argument touched upon in this introduction, i.e. HKT potentials, cohomology and the relation between the holonomy of the Obata connection and the canonical bundle. This leads to consider balanced HKT metrics and the quaternionic Calabi conjecture which is carefully stated along with a description of the current "state of the art" for what regards its solution.

The cohomology of HKT manifolds, especially under the assumption of balancedness resembles quite closely that of Kähler manifolds. The third chapter explores the relation between the cohomologies of ∂ and ∂_J as well as the quaternionic Bott-Chern and Aeppli cohomologies introduced by Grantcharov, Lejmi and Verbitsky [146]. We prove that on a compact balanced HKT manifold all these cohomology groups are isomorphic (Theorem 3.21). Afterwards, we prove formality of the differential graded algebra $(\Lambda^{\bullet,0}(M,I),\partial)$, provided the (global) $\partial\partial_J$-lemma holds (Theorem 3.22), which is true for HKT $\mathrm{SL}(n,\mathbb{H})$-manifolds by a result in [146], this allows to find an obstruction to the existence of a HKT $\mathrm{SL}(n,\mathbb{H})$-structure on complex manifolds (Corollary 3.30). In Section 3.2 we put nilmanifolds and solvmanifolds under our lens and we study the relation between HKT geometry and the presence of an *abelian hypercomplex structure*. Recall that a hypercomplex structure (I,J,K) on a Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ is called abelian if it satisfies $[I\cdot,I\cdot]=[J\cdot,J\cdot]=[K\cdot,K\cdot]=[\cdot,\cdot]$. We recall the results of Barberis, Dotti and Verbitsky [29] for nilmanifolds and we generalize some of them to solvmanifolds, providing evidence for some of the conjectures stated before. For instance we prove that a solvmanifold is $\mathrm{SL}(n,\mathbb{H})$ if and only if its canonical bundle has holomorphic sections (Theorem 3.39). We also show that if a solvmanifold M with a left-invariant hypercomplex structure (I,J,K) such that that either (I,J,K) is abelian or M is $\mathrm{SL}(n,\mathbb{H})$, admits a compatible HKT metric then it also admits a left-invariant balanced HKT metric (Corollaries 3.37 and 3.41). This provides evidence for the quaternionic Calabi conjecture. In a final subsection we also prove that on a Lie group with abelian hypercomplex structure, the restricted holonomy group of the Obata connection is abelian (Theorem 3.50), an interesting fact related to the Merkulov-Schwachhöfer classification.

The remaining chapters are all devoted to solvability results of the quaternionic Monge-Ampère equation. Chapter 4 studies the problem in two cases. The first section focuses on the equation on some 2-step nilmanifolds of (real) dimension 8 which can be naturally viewed as toric fibrations over tori. Under the assumptions that all the data are invariant by the action of the fiber, we prove that the quaternionic Monge-Ampère equation can always be solved (Theorem 4.1). On a related note, in the second section we treat compact HKT manifolds having a foliation of corank 4 that is preserved by the hypercomplex structure. Assuming that the datum is basic with respect to the foliation, the equation rewrites as a semilinear elliptic equation, which is solved by a unique basic function (Theorem 4.12).

In the fifth chapter, inspired by a long-tradition of designing geometric flows as a way to attack partial differential equations, we consider the natural parabolic version of the quaternionic Monge-Ampère equation. More generally, whenever we have an elliptic partial differential equation (PDE) $P(\varphi,F)=0$ with datum $F\in C^\infty(M,\mathbb{R})$ and unknown $\varphi\in C^\infty(M,\mathbb{R})$, the associated parabolic flow is $\frac{\partial}{\partial t}\varphi=P(\varphi,F)$, which is set up by adding time dependence to the unknown, i.e. now $\varphi\in C^\infty(M\times[0,T),\mathbb{R})$. By standard parabolic theory, there always exists a unique maximal solution and one might hope to establish long-time existence and convergence of the flow to a solution of the related elliptic PDE. We proceed in this way, introducing the parabolic quaternionic Monge-Ampère equation and proving that on a compact flat hyperkähler manifold there exists a long-time solution whose normalization converges to a solution of (1) (Theorem 5.1).

One can observe that the quaternionic Monge-Ampère equation belongs to a whole family of fully non-linear elliptic equations which can be treated simultaneously. This approach has a long lasting tradition and goes back to the work of Caffarelli, Nirenberg and Spruck [66]. Inspired by the work of Székelyhidi [280] who studies a class of equations on compact Hermitian manifolds, the sixth chapter deals with a family of equations on compact locally flat hyperhermitian manifolds. Under the assumptions of local flatness, the metric g and the form $\Omega+\partial\partial_J\varphi$ induce hyperhermitian matrices $g_{\bar{r}s}$, $\Omega^{\varphi}_{\bar{r}s}$. The equations we take into account are of the form $f(\lambda(A))=F$, where $F\in C^\infty(M,\mathbb{R})$ is the datum, $A^r_s=g^{\bar{j}r}\Omega^{\varphi}_{\bar{j}s}$ is a hyperhermitian matrix with respect to g with n-tuple of eigenvalues $\lambda(A)$ and f is a real-valued function satisfying some structural assumptions which ensure non-degeneracy and ellipticity of the equation. For instance, for the quaternionic Monge-Ampère equation $f(\lambda_1,\dots,\lambda_n)=\log(\lambda_1\cdots\lambda_n)$. We establish C^0, Laplacian and $C^{2,\alpha}$ a priori estimates for this class of equations under the assumption of having a certain type of subsolution (Propositions 6.4, 6.7 and 6.16). Unfortunately during the Laplacian estimate we need the severe assumption of having a compatible hyperkähler metric.

Similarly to what we discussed above for the parabolic quaternionic Monge-Ampère equation, one can consider the parabolic counterpart of the class of elliptic equations studied in chapter 6. This is done in the seventh chapter following the approach of Phong and Tô [246] who generalized the work of Székelyhidi to the parabolic framework. Our assumptions are essentially the same as for the elliptic case, but here there is a dichotomy and the structural function f can present either a "bounded" behaviour or an "unbounded" one. The bounded case is slightly less nicer, as it requires some additional assumption in order to show the C^0 estimate. Nonetheless, in both cases we prove long-time existence and we show that (the normalization of) its solution converges to a solution of the corresponding elliptic equation (Theorems 7.2 and 7.3).

The last chapter tries to attack the quaternionic Calabi conjecture from a different angle. It is inspired by the variational approach of Berman, Boucksom, Guedj and Zeriahi [39] to the complex Monge-Ampère equation. The underlying idea is that the critical points of the Ding functional are solutions of the Monge-Ampère equation and the problem of solvability is then translated into the variational problem of showing that such a functional admits a maximizer. All this machinery works on plurisubharmonic functions with very weak regularity assumptions, indeed by reducing the smoothness of the family of functions considered, we gain in compactness, making it easier to find maximizers. In order to take advantage of the needed pluripotential environment we need some preliminary work, analogue to the theory developed by Guedj and Zeriahi [156] on compact Kähler manifolds. Eventually we are able to show that the quaternionic Monge-Ampère equation on compact locally flat HKT manifolds always admits a unique weak solution (Theorem 8.31).

CHAPTER 1

QUATERNIONIC LINEAR ALGEBRA

The first chapter serves the purpose of introducing the basic framework of quaternionic linear algebra. Even though quaternions are not a field, once we fix a side for scalar multiplication, the theory goes through almost unaffected, and very many results which are true for vector spaces over a field still hold in the non-commutative setting. Indeed, most of the theory developed in this initial chapter is not a special feature of quaternions and the majority of the results can be extended to more general types of rings. Several results hold in the category of skew fields, sometimes with the requirement that they carry an anti-involution (which plays the role of conjugation). Beside some minor exceptions, we decided to phrase the results in terms of quaternions, leaving aside full generality and keeping the focus on the division ring of quaternions, which is the one of our interest.

As mentioned, the known theory of vector spaces is barely altered by the lack of commutativity, and with little more effort one can still introduce the notions of invertibility of a matrix, linear (in)dependence of vectors, general linear group or special linear group and all these concepts behave very much like the familiar linear algebra. However, sometimes subtle differences emerge. A striking example is the spectral theory, studied at the end of the first section. Within this framework, (right) eigenvalues are not uniquely defined as any other element in the same conjugacy class is again an eigenvalue. On the other hand, the Spectral Theorem for normal matrices can still be obtained.

The determinant is another notion that is heavily affected by the absence of commutativity, indeed, we will see that it is non-trivial to find a satisfactory definition of a determinant, unless we are prepared to give up some properties. Some interesting approaches to these issues have been proposed by Dieudonné [102], Study [274] and other mathematicians, but the determinant that provides the most useful tool in view of Riemannian geometry is the one introduced by Moore [229]. Moore's determinant is the only one that is capable to encompass the positivity (or negativity) of an inner product, but its application is limited to hyperhermitian matrices. For this kind of matrices, which are the quaternionic analogue of Hermitian matrices each conjugacy class of eigenvalues, contains a single real element. This behaviour of the eigenvalues is exactly the reason why the Moore determinant is only valuable on the space of hyperhermitian matrices.

Even though it has attracted the attention of many researchers in the field of Quantum Mechanics (renewed by the book [1]), there is not so much literature dealing carefully with the foundations of quaternionic linear algebra. Our main references for this chapter include the book of Rodman [250], which collects a thorough exposition of quaternionic linear algebra, and the survey of Zhang [330]. We cannot fail to mention the expository article of Farenick and Pidkowich [122] for what regards spectral theory and the excellent survey of Aslaksen [23] which is also our prime source for the topic of quaternionic determinants.

1.1 Basic theory

The present section is divided into two parts. The first one deals with quaternionic modules, linear maps and inner products; the second develops the theory of quaternionic matrices, shedding some light on the structures of the first part and presenting a solid linear algebraic groundwork. In the second part we will deal with the notion of invertibility and the general linear group. The definition of the special linear group is interesting, as, at this level, we do not have at our disposal a determinant function, hence, an alternative definition than the usual one must be adopted. Special importance is then given to hyperhermitian matrices and the notion of (semi)definiteness. We also discuss possible representations of quaternionic matrices into the real algebras of complex and real matrices, which will be incredibly helpful, for instance, in the spectral theory.

1.1.1 Quaternionic modules.

The discovery of quaternions dates back to 1843 and is due to Sir William Rowan Hamilton. The history of how Hamilton came up with the idea of quaternions is well-known and told in innumerable sources and it is not the place here to recall it (among others, we refer the interested reader to [300]). In this subsection we begin with a brief treatment of the basic features of the algebra of quaternions, describe their relation with three-dimensional rotations and build up our way into non-commutative linear algebra.

The algebra of quaternions.

As usual, we will denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers respectively. The space $\mathbb{H}$ of **quaternions** can be defined as the non-commutative associative algebra over $\mathbb{R}$ which is 4-dimensional as a vector space, with basis $(1, i, j, k)$:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},,$$

where i, j, k are the **quaternion units** satisfying the fundamental relations

$$i^2 = j^2 = k^2 = -1\,, \qquad ij = -ji = k\,.$$

The product operation on $\mathbb{H}$ is completely determined by (the distribution laws together with) the relations above, the usual product of real numbers and the condition that these commute with the quaternion units. The center of $\mathbb{H}$ coincides with $\mathbb{R}$, embedded in $\mathbb{H}$ as $\{a + 0i + 0j + 0k \mid a \in \mathbb{R}\}$.

With this product $\mathbb{H}$ has also the structure of a **division ring** (or **skew field**), i.e. a non-commutative ring where every non-zero element has an inverse (see [96, 114]). To see this consider the *quaternionic conjugate* of $q = a + bi + cj + dk$, defined as

$$\bar{q} := a - bi - cj - dk$$

and its *norm*

$$|q| := \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{a^2 + b^2 + c^2 + d^2}\,,$$

then if $q \neq 0$, its inverse is

$$q^{-1} = \frac{\bar{q}}{|q|^2}\,.$$

Checking that $|\cdot|\colon \mathbb{H} \to \mathbb{R}$ is indeed a norm is straightforward, furthermore $|q| = |\bar{q}|$ for any $q \in \mathbb{H}$ and $|pq| = |p||q|$ for any $p, q \in \mathbb{H}$. Observe that conjugation satisfies $\overline{pq} = \bar{q}\bar{p}$ and $q = \bar{q}$ if and only if $q \in \mathbb{R}$.

The field of complex numbers naturally embeds into $\mathbb{H}$ in many ways, we shall fix the identification $\mathbb{C} \cong \{a + bi + 0j + 0k \mid a, b \in \mathbb{R}\}$. We can also write a quaternion $q = a + bi + cj + dk$ in one of the following two forms

$$q = (a + bi) + (c + di)j\,, \qquad q = (a + bi) + j(c - di)\,,$$

emphasizing that $\mathbb{H}$ can also be seen as a vector space over $\mathbb{C}$, however, be aware that $\mathbb{H}$ is not an algebra over $\mathbb{C}$ (because the product of two quaternions is not a $\mathbb{C}$-bilinear operation). Fixing this embedding of $\mathbb{C}$ into $\mathbb{H}$ we have that every $z \in \mathbb{C}$ satisfies $zj = j\bar{z}$.

Rotations and conjugation.

For any quaternion $q = a + bi + cj + dk \in \mathbb{H}$ we call $\mathrm{Re}(q) := a$ its *real* (or *scalar*) *part* and $\mathrm{Im}(q) := bi + cj + dk$ its *imaginary* (or *vector*) part. A purely imaginary quaternion can be seen as a vector in $\mathbb{R}^3$ and under this identification one can define a dot product and a cross product as follows: let $\mathbf{q}_r = b_r i + c_r j + d_r k$ for $r = 1, 2$ be purely imaginary quaternions, then we define

$$\mathbf{q}_1 \cdot \mathbf{q}_2 := b_1 b_2 + c_1 c_2 + d_1 d_2\,, \qquad \mathbf{q}_1 \times \mathbf{q}_2 := (c_1 d_2 - d_1 c_2)i + (d_1 b_2 - b_1 d_2)j + (b_1 c_2 - c_1 b_2)k\,.$$

An interesting feature is that the product of $\mathbf{q}_1$ and $\mathbf{q}_2$ can be expressed in terms of the dot product and the cross product:

$$\mathbf{q}_1 \mathbf{q}_2 = -\mathbf{q}_1 \cdot \mathbf{q}_2 + \mathbf{q}_1 \times \mathbf{q}_2\,. \tag{1.1}$$

Similarly to complex numbers, quaternions allow a polar representation. For every $q \in \mathbb{H}$ with non-zero imaginary part there exists $\alpha \in [0, 2\pi)$ such that

$$q = |q| \left(\cos\frac{\alpha}{2} + \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} \sin\frac{\alpha}{2} \right)\,. \tag{1.2}$$

To see this it is enough to observe that $\mathrm{Re}(q)^2 + |\mathrm{Im}(q)|^2 = |q|^2$, therefore, choosing α such that $\cos(\alpha/2) = \mathrm{Re}(q)/|q|$ we also have $\sin(\alpha/2) = |\mathrm{Im}(q)|/|q|$. Such an observation makes quaternions into an extremely useful tool to model spatial rotations due to the following result:

Theorem 1.1. *The action $\rho_q\colon \mathbb{H} \to \mathbb{H}$ of a unit norm quaternion $q \in \mathbb{H}$ by conjugation*

$$\rho_q(p) := qp\bar{q}\,,$$

fixes the real axis and on $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k \cong \mathbb{R}^3$ acts as a rotation of angle $\alpha = 2\arctan(|\mathrm{Im}(q)|/\mathrm{Re}(q))$ around the axis $\mathrm{Im}(q)$. Such a rotation is clockwise if our line of sight points in the same direction as $\mathrm{Im}(q)$.

Proof. If $\mathrm{Im}(q) = 0$ the statement is trivial, therefore we may assume that q is a non-real quaternion. Furthermore, the fact that the real axis remains fixed is obvious, thus it is enough to prove the theorem assuming $p = \mathbf{p}$ is a purely imaginary quaternion.

We have

$$\rho_q(\mathbf{p}) = (\mathrm{Re}(q) + \mathrm{Im}(q))\mathbf{p}(\mathrm{Re}(q) - \mathrm{Im}(q)) = \mathrm{Re}(q)^2\mathbf{p} + \mathrm{Re}(q)\left(\mathrm{Im}(q)\mathbf{p} - \mathbf{p}\mathrm{Im}(q)\right) - \mathrm{Im}(q)\mathbf{p}\mathrm{Im}(q)\,,$$

but from (1.1) we deduce $\mathrm{Im}(q)\mathbf{p} - \mathbf{p}\mathrm{Im}(q) = 2\mathrm{Im}(q) \times \mathbf{p}$ as well as $\mathrm{Im}(q)\mathbf{p}\mathrm{Im}(q) = -\mathrm{Im}(q)(\mathbf{p} \cdot \mathrm{Im}(q)) + \mathrm{Im}(q) \times \mathbf{p} \times \mathrm{Im}(q)$, therefore

$$\rho_q(\mathbf{p}) = \mathrm{Re}(q)^2\mathbf{p} + 2\mathrm{Re}(q)\mathrm{Im}(q) \times \mathbf{p} + \mathrm{Im}(q)(\mathbf{p} \cdot \mathrm{Im}(q)) - \mathrm{Im}(q) \times \mathbf{p} \times \mathrm{Im}(q)\,.$$

From this formula it follows that $\rho_q(\mathbf{p}) = \mathbf{p}$ whenever $\mathbf{p}$ is along $\mathrm{Im}(q)$, i.e. $\mathbf{p} = a\mathrm{Im}(q)$ for some $a \in \mathbb{R}$. If $\mathbf{p}$ is normal to $\mathrm{Im}(q)$, letting $\mathbf{u} = \mathrm{Im}(q)/|\mathrm{Im}(q)|$ we obtain instead

$$\rho_q(\mathbf{p}) = \mathrm{Re}(q)^2\mathbf{p} + 2\mathrm{Re}(q)|\mathrm{Im}(q)|\mathbf{u} \times \mathbf{p} - |\mathrm{Im}(q)|^2\mathbf{u} \times \mathbf{p} \times \mathbf{u} = \left(\mathrm{Re}(q)^2 - |\mathrm{Im}(q)|^2\right)\mathbf{p} + 2\mathrm{Re}(q)|\mathrm{Im}(q)|\mathbf{u} \times \mathbf{p}\,.$$

Now, from (1.2) we deduce

$$\rho_q(\mathbf{p}) = \left(\cos(\alpha/2)^2 - \sin(\alpha/2)^2\right)\mathbf{p} + 2\cos(\alpha/2)\sin(\alpha/2)\mathbf{u} \times \mathbf{p} = \cos(\alpha)\mathbf{p} + \sin(\alpha)\mathbf{u} \times \mathbf{p}\,.$$

Therefore ρ_q rotates $\mathbf{p}$ on the plane defined by $\mathbf{p}$ and $\mathbf{u} \times \mathbf{p}$ through an angle α.

For a general $\mathbf{p}$ the theorem follows by using $\mathbb{R}$-linearity of ρ_q and decomposing $\mathbf{p}$ into its component along $\mathrm{Im}(q)$ and its component normal to it. □

An automorphism given by conjugation action is called *inner*; on $\mathbb{H}$ it turns out that all automorphisms are of this type. This is a particular case of the Skolem-Noether Theorem [261, 233] which establishes this property for finite-dimensional central simple algebras.

Proposition 1.2. *All automorphisms of the algebra of quaternions are inner.*

Proof. Any automorphism $\eta\colon \mathbb{H} \to \mathbb{H}$ must satisfy $\eta(1) = 1$ and thus fixes the real line. By Theorem 1.1 it is therefore enough to show that the restriction of η to $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k \cong \mathbb{R}^3$ acts as a rotation.

As η preserves real parts we have from (1.1) that η preserves the dot product:

$$\eta(\mathbf{p}) \cdot \eta(\mathbf{q}) = -\mathrm{Re}(\eta(\mathbf{p})\eta(\mathbf{q})) = -\mathrm{Re}(\eta(\mathbf{pq})) = -\eta(\mathrm{Re}(\mathbf{pq})) = \eta(\mathbf{p} \cdot \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}\,.$$

This implies that η restricted to imaginary parts is an element of $\mathrm{O}(3)$. But, again from (1.1) we see that

$$\eta(\mathbf{p}) \times \eta(\mathbf{q}) = \eta(\mathbf{p})\eta(\mathbf{q}) + \eta(\mathbf{p}) \cdot \eta(\mathbf{q}) = \eta(\mathbf{pq}) + \eta(\mathbf{p} \cdot \mathbf{q}) = \eta(\mathbf{p} \times \mathbf{q})\,,$$

whence $\eta|_{\mathbb{R}^3} \in \mathrm{SO}(3)$ as desired. ☐

Observe that when $q \in \mathbb{H}$ has unit norm its action by conjugation coincides with its action by similarity: $\rho_q(p) = qp\bar{q} = qpq^{-1}$. Set

$$\theta(p) := \{qpq^{-1} \mid q \in \mathbb{H}^*\} = \{qp\bar{q} \mid q \in \mathbb{H},\, |q| = 1\}$$

for the *conjugacy class* of $p \in \mathbb{H}$. It will be useful in the future to have a criterion to decide if two quaternions are conjugate. Clearly, if p is real it is the only element in its conjugacy class and conversely. If p is non-real its conjugacy class is infinite and contains exactly two mutually conjugate purely complex elements and all other elements are non-complex quaternions. This last observation, proved below, goes back to Cayley [76]. Our proof follows [23, p. 63], which is geometric in flavour. For other proofs see [53, 330].

Lemma 1.3. *For any non-real $p \in \mathbb{H}$ we have $\theta(p) \cap \mathbb{C} = \{\mu, \bar{\mu}\}$ for some $\mu \in \mathbb{C}$.*

Proof. For any $q \in \mathbb{H}$ with unit norm we know that the conjugation map $\rho_q\colon \mathbb{H} \to \mathbb{H}$, $\rho_q(p) = qp\bar{q}$ fixes the real axis and can be thought of as a rotation of $i\mathbb{R} + j\mathbb{R} + k\mathbb{R} \cong \mathbb{R}^3$ around the axis defined by $\mathrm{Im}(q)$. Therefore, for any non-real $p \in \mathbb{H}$ with unit norm, $\theta(p)$ describes a 2-dimensional sphere orthogonal to the real axis. Clearly $\rho_q(p) = \mathrm{Re}(p) + q\mathrm{Im}(p)q^{-1}$ and by our discussion the conjugacy class of $\mathrm{Im}(p)$ intersects the i-axis only at two points which must have the same norm of $|\mathrm{Im}(p)|$, hence they are $\pm|\mathrm{Im}(p)|i$. We conclude that $\theta(p) \cap \mathbb{C} = \{\mathrm{Re}(p) \pm |\mathrm{Im}(p)|i\}$. ☐

As an application of the lemma above we prove:

Proposition 1.4. *The commutator subgroup of $\mathbb{H}^* := \mathbb{H} \setminus \{0\}$ coincides with the set of quaternions of unit norm.*

Proof. The fact that a commutator has unit norm is obvious. Conversely, let $q \in \mathbb{H}^*$ have unit norm. If $q = 1$ then it is clearly a commutator, if $q = -1$ we can write $q = iji^{-1}j^{-1}$, otherwise by the previous lemma there exists a $\mu \in \mathbb{C}$ such that $q = a\mu a^{-1}$ for some $a \in \mathbb{H}$. Observe that from formula (1.1) it can be deduced that every quaternion can be written as the product of two purely imaginary quaternions. Let $z \in \mathbb{C}$ be such that $z^2 = \mu$ and choose p_1, p_2 purely imaginary and such that $z = p_1p_2$. The fact that q has unit norm implies that all μ, z, p_1 and p_2 have unit norm, in particular $p_i^{-1} = \bar{p}_i = -p_i$ for $i = 1, 2$. From this we conclude

$$q = a\mu a^{-1} = ap_1p_2p_1p_2a^{-1} = ap_1p_2p_1^{-1}p_2^{-1}a^{-1} = (ap_1a^{-1})(ap_2a^{-1})(ap_1a^{-1})^{-1}(ap_2a^{-1})^{-1}$$

which is a commutator. ☐

See also [101, 102] for different proofs of Proposition 1.4.

$\mathbb{H}$-modules.

We now enter into the realm of non-commutative linear algebra. A **right module over** $\mathbb{H}$ (shortly, a **right** $\mathbb{H}$**-module**) is an additive abelian group V together with a function $V \times \mathbb{H} \to V$, $(v, q) \mapsto vq$ such that for all $p, q \in \mathbb{H}$ and $u, v \in V$

1. $(u + v)q = uq + vq$;
2. $v(p + q) = vp + vq$;
3. $(vp)q = v(pq)$;
4. $v1 = v$.

Sometimes, modules over a skew field are still called *vector spaces*. We prefer to keep calling them *modules* to emphasize the lack of commutativity of scalars.

A **left** $\mathbb{H}$**-module** is defined similarly via a function $\mathbb{H} \times V \to V$, $(q, v) \mapsto qv$ satisfying the obvious analogues of the properties listed above. Here we will talk about right modules, everything we say has an obvious analogue for left modules.

Example 1.5. The n-fold direct product $\mathbb{H}^n = \mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}$ is both a left and a right $\mathbb{H}$-module with the natural actions

$$q(v_1, v_2, \ldots, v_n) := (qv_1, qv_2, \ldots, qv_n)\,, \qquad (v_1, v_2, \ldots, v_n)q := (v_1q, v_2q, \ldots, v_nq)\,.$$

A $\mathbb{H}$-module which is both left and right is called a $\mathbb{H}$**-bimodule**.

A map $\alpha \colon V \to W$ between two (right) $\mathbb{H}$-modules is called a **(right)** $\mathbb{H}$**-module homomorphism** if is additive and preserves rescalings, more precisely for all $q \in \mathbb{H}$ and $u, v \in V$

$$\alpha(u + v) = \alpha(u) + \alpha(v)\,, \qquad \alpha(vq) = \alpha(v)q\,.$$

We will also say that α is $\mathbb{H}$**-linear**. A bijective $\mathbb{H}$-linear map is called an **isomorphism** (of $\mathbb{H}$-modules).

Let V be a right $\mathbb{H}$-module and $X \subseteq V$ a subset. Denote $\mathrm{Span}_{\mathbb{H}}(X)$ the smallest submodule of V containing X, if $V = \mathrm{Span}_{\mathbb{H}}(X)$ we say that V is *spanned* by X. The module V is spanned by a subset $X \subseteq V$ if and only if every element of V may be written as a *linear combination* $v_1q_1 + v_2q_2 + \cdots + v_kq_k$ for $q_i \in \mathbb{H}$, $v_i \in X$ and $k \in \mathbb{N}$. If there exists a finite subset spanning V, then V is said to be **finitely generated**.

A subset $X \subseteq V$ is said to be **right linearly dependent** if there exists a collection of distinct elements $v_1, \ldots, v_m \in X$ and a non-zero $q = (q_1, \ldots, q_m) \in \mathbb{H}^m$ such that $\sum_{i=1}^m v_iq_i = 0$, while is called **right linearly independent** if it is not right linearly dependent. A linearly independent subset of V that spans V is called a **basis** of V.

Theorem 1.6. *Every* $\mathbb{H}$*-module* V *satisfies the following properties:*

1. *V has a basis and any two bases of V have the same cardinality. The cardinal number of any basis of V is called the* ***dimension*** *of V, denoted* $\dim(V)$.
2. *Every subset of V that spans it contains a basis of V.*
3. *Every linearly independent subset of V is contained in a basis of V.*
4. *V is free, i.e. it is isomorphic to $\mathbb{H}^{\dim(V)}$.*

Proof. The proof is practically identical to the one for vector spaces over a field, we refer, for instance, to [184, Chaper IV, Section 2]. ☐

The isomorphism of V with $\mathbb{H}^{\dim(V)}$ is not unique. Any choice of a basis of V determines one such isomorphism as follows. Suppose for simplicity that V has finite dimension n and let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis. Any vector $v \in V$ can be written as $v = \sum_{k=1}^n e_kv_k$ for unique $v_1, \ldots, v_n \in \mathbb{H}$ called the *components* of v with respect to $\mathcal{B}$. The correspondence $v \mapsto (v_1, \ldots, v_n)$ determines the isomorphism between V and $\mathbb{H}^n$.

Quaternionic inner products.

Definition 1.7. A **hyperhermitian (sesquilinear) form** over V is a map $g\colon V\times V\to\mathbb{H}$ that

- is additive: $g(u_1+u_2,v)=g(u_1,v)+g(u_2,v)$;
- is $\mathbb{H}$-linear in the second entry $g(u,vq)=g(u,v)q$;
- is hyperhermitian: $g(u,v)=\overline{g(v,u)}$.

A hyperhermitian form is **non-degenerate** if $g(u,v)=0$ for all $v\in V$ implies $u=0$ and it is **positive definite** (resp. **semidefinite**) if $g(u,u)>0$ (resp. $g(u,u)\geq 0$) for all $u\neq 0$. A positive definite hyperhermitian form g will be called a **(quaternionic) inner product** and the pair (V,g) is called a **(quaternionic) inner product space**.

Example 1.8. The standard inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^n$ is defined as

$$\langle u,v\rangle := \sum_{k=1}^{n}\bar{u}_i v_i$$

for every $u=(u_1,\dots,u_n), v=(v_1,\dots,v_n)\in\mathbb{H}^n$.

Remark 1.9. Unlike the complex setting, non-zero symmetric and skew-symmetric bilinear forms do not exist on $\mathbb{H}$-modules. Indeed, if V is a right $\mathbb{H}$-module and g a bilinear form, either symmetric or skew-symmetric, for every $u,v\in V$ and $p,q\in\mathbb{H}$

$$g(up,vq)=g(up,v)q=\pm g(v,up)q=\pm g(v,u)pq=g(u,v)pq$$

but also

$$g(up,vq)=\pm g(vq,up)=\pm g(vq,u)p=g(u,vq)p=g(u,v)qp$$

hence $g\equiv 0$.

Every inner product $g\colon V\times V\to\mathbb{H}$ on a right $\mathbb{H}$-module V induces a norm $\|\cdot\|_g\colon V\to[0,+\infty)$ naturally defined as $\|v\|_g := \sqrt{g(v,v)}$ for every $v\in V$. The usual Cauchy-Schwartz inequality is true:

Lemma 1.10 (Cauchy-Schwartz inequality)**.** *Let $g\colon V\times V\to\mathbb{H}$ be an inner product on the right $\mathbb{H}$-module V, then*

$$|g(u,v)|\leq\|u\|_g\|v\|_g.$$

Proof. Take $u,v\in V$ and $p,q\in\mathbb{H}$, then

$$0\leq\|up-vq\|_g^2=\bar{p}\|u\|_g^2\,p-\bar{p}g(u,v)q-\bar{q}g(v,u)p+\bar{q}\|v\|_g^2\,q\,.$$

Choosing $p=\|v\|_g^2$ and $q=g(v,u)$ one obtains

$$0\leq\|v\|_g^2\left(\|u\|_g^2\|v\|_g^2-|g(u,v)|^2\right)$$

and thus the desired inequality. □

On a quaternionic inner product space (V,g) we say that a basis $(v_1,\dots,v_n)$ is **hyperunitary** (with respect to g) if $g(v_i,v_j)=\delta_{ij}$, where δ_{ij} is the *Kronecker delta*. As usual, one can always adjust a basis to obtain a hyperunitary one:

Proposition 1.11 (Gram-Schmidt process)**.** *Let (V,g) be a quaternionic inner product space and $v_1,\dots,v_n$ a basis. Then there exists a hyperunitary basis $u_1,\dots,u_n$ such that for every $k=1,\dots,n$ we have $\mathrm{Span}_{\mathbb{H}}(v_1,\dots,v_k)=\mathrm{Span}_{\mathbb{H}}(u_1,\dots,u_k)$.*

Proof. Define $u_1=v_1/\|v_1\|_g$ and then inductively

$$u_k := \frac{v_k-\sum_{r=1}^{k-1}u_r\langle u_r,v_k\rangle}{\left\|v_k-\sum_{r=1}^{k-1}u_r\langle u_r,v_k\rangle\right\|_g}\,,\qquad k=2,\dots,n\,.$$

It is clear that the vectors $u_1, \dots, u_n$ form a hyperunitary basis and that for each k we have $\mathrm{Span}_{\mathbb{H}}(v_1, \dots, v_k) = \mathrm{Span}_{\mathbb{H}}(u_1, \dots, u_k)$. □

1.1.2 Quaternionic matrices.

Let V, W be right $\mathbb{H}$-modules of dimension n and m respectively. The set $\mathrm{Hom}_{\mathbb{H}}(V, W)$ of $\mathbb{H}$-linear homomorphisms between V and W is a real vector space (a subspace of $\mathrm{Hom}_{\mathbb{R}}(V, W)$). The choice of bases of V and W induces a isomorphism of real vector spaces between $\mathrm{Hom}_{\mathbb{H}}(V, W)$ and the space $\mathbb{H}^{m,n}$ of $m \times n$ matrices with coefficients in $\mathbb{H}$. Be aware that there is no $\mathbb{H}$-module structure on $\mathrm{Hom}_{\mathbb{H}}(V, W)$ that makes this into an isomorphism of $\mathbb{H}$-modules.

More precisely, to any linear map $\alpha \colon V \to W$ we associate the matrix $M(\alpha) \in \mathbb{H}^{m,n}$ with respect to bases $\mathcal{B} = (e_1, \dots, e_n)$, $\mathcal{C} = (f_1, \dots, f_m)$ of V and W respectively, with entries defined as

$$M(\alpha)_{ij} := f_i^*(\alpha(e_j)) \,,$$

where $(f_i^*) \subseteq \mathrm{Hom}_{\mathbb{H}}(W, \mathbb{H})$ is the *dual basis* to (f_i), i.e. $f_i^*(f_j) = \delta_{ij}$. As usual, we may compute the components of the image of a vector $v \in V$ with respect to $\mathcal{C}$ as the product of the matrix $M(\alpha)$ with the column vector of components of v with respect to $\mathcal{B}$:

$$\alpha(v)_{\mathcal{C}} = M(\alpha) v_{\mathcal{B}} \,, \qquad \text{for every } v \in V \,. \tag{1.3}$$

Also, for any pair of $\mathbb{H}$-linear maps $\alpha \colon V \to W$ and $\beta \colon W \to Z$ we have

$$M(\beta \circ \alpha) = M(\beta) M(\alpha) \,. \tag{1.4}$$

A word of caution is needed here. If we take into account left $\mathbb{H}$-modules instead of right ones we need to adjust things a little. Indeed (1.3), (1.4) and all formulas to come are only true if we define the product of two matrices $M \in \mathbb{H}^{m,n}$, $N \in \mathbb{H}^{n,k}$ as $MN := M \cdot_{\mathrm{op}} N$, where $\cdot_{\mathrm{op}}$ is induced by the opposite product $p \cdot_{\mathrm{op}} q := qp$ for every $p, q \in \mathbb{H}$. Explicitly, if $M = (m_{ij})$ and $N = (n_{rs})$ then

$$(MN)_{ab} = \sum_{l=1}^{n} m_{al} \cdot_{\mathrm{op}} n_{lb} = \sum_{l=1}^{n} n_{lb} m_{al} \,. \tag{1.5}$$

With this adjustment for the matrix product, we can make use of the same formulas of the case of right modules, keeping in mind that if we choose to look at left module structures, products have to be intended as in (1.5). In view of this consideration, it is convenient and customary to work with right modules, which has considerable practical advantage.

Quaternionic linear groups.

We now address the important matter of invertibility of matrices. We say that a matrix $A \in \mathbb{H}^{n,m}$ is **left** (resp. **right**) **invertible** if there exists a matrix $B \in \mathbb{H}^{m,n}$ called the **left** (resp. **right**) **inverse** of A, such that $BA = \mathbb{1}_m$ (resp. $AB = \mathbb{1}_n$), where $\mathbb{1}_m$ denotes the identity matrix of order m. When no confusion can occur we will drop the subscript and simply denote $\mathbb{1}_m$ by $\mathbb{1}$. A priori we do not know if a left inverse is automatically a right inverse and viceversa. Clearly, if a matrix $A \in \mathbb{H}^{n,m}$ has both a left inverse $B \in \mathbb{H}^{m,n}$ and a right inverse $C \in \mathbb{H}^{m,n}$ then they are unique and they coincide:

$$C = C\mathbb{1}_n = CAB = \mathbb{1}_m B = B \,;$$

when this occurs, we call A **invertible** or **non-singular** and $A^{-1} := B = C$ the **inverse** of A.

Example 1.12. If we choose two bases $\mathcal{B} = (e_1, \dots, e_n)$ and $\mathcal{C} = (f_1, \dots, f_n)$ of a right $\mathbb{H}$-module V we obtain two isomorphisms between V and $\mathbb{H}^n$ and it is easy to see that $f_j = \sum_{i=1}^n e_i a_{ij}$ for some $a_{ij} \in \mathbb{H}$. With little effort, we obtain the formula of change of coordinates

$$v_{\mathcal{B}} = P v_{\mathcal{C}} \,, \qquad \text{for every } v \in V \,,$$

where the matrix $P = (a_{ij}) \in \mathbb{H}^{n,n}$ must be invertible and is called the **matrix of change of basis** from $\mathcal{B}$ to $\mathcal{C}$.

In particular, the example above shows that for square matrices there is no distinction between left and right invertibility.

Definition 1.13. The **quaternionic general linear group** of order n, denoted $\mathrm{GL}(n,\mathbb{H})$, is the group of invertible matrices in $\mathbb{H}^{n,n}$.

We do not have yet at our disposal the notion of a determinant, for this reason we cannot define the special linear subgroup $\mathrm{SL}(n,\mathbb{H}) \subseteq \mathrm{GL}(n,\mathbb{H})$ of matrices with determinant equal to 1. A possible alternative definition is offered by looking at elementary transformations.

Define $B_{ij}(q) \in \mathbb{H}^{n,n}$ to be the square matrix obtained from the identity matrix by replacing the (i,j)-entry with $q \in \mathbb{H}$. We distinguish two cases:

- For $i \neq j$ multiplying a matrix on the left by $B_{ij}(q)$ amounts to adding to the i^{th} row the j^{th} row multiplied on the left by q. Similarly multiplying a matrix on the right by $B_{ij}(q)$ amounts to adding to the j^{th} column the i^{th} column multiplied on the right by q. We call $B_{ij}(q)$ an **elementary matrix**. These matrices are invertible because $B_{ij}(q)^{-1} = B_{ij}(-q)$.
- For $i = j$ multiplication on the left by $B_{ii}(q)$ amounts to multiplying the i^{th} row on the left by q while multiplication on the right amounts to multiplying the i^{th} column on the right by q. If $q \neq 0$ these matrices are invertible because $B_{ii}(q)^{-1} = B_{ii}(q^{-1})$.

We first prove a very useful factorization formula.

Lemma 1.14. *Every $A \in \mathrm{GL}(n,\mathbb{H})$ can be factorized as*

$$A = B_{nn}(q)B$$

for some $q \in \mathbb{H}^*$, where B is a product of elementary matrices.

Proof. Since $A \in \mathrm{GL}(n,\mathbb{H})$ we can find a non-zero element a_{1j} in the first row. We may assume $a_{12} \neq 0$ because if this is not the case by summing the j^{th} column to the second we achieve $a_{12} \neq 0$. Now, summing the second column multiplied on the right by $a_{12}^{-1}(1 - a_{11})$ to the first column we obtain a new matrix with $(1,1)$-entry equal to 1 and with suitable operations we can now make all other entries in the first row equal to zero.

Proceeding iteratively for the $2^{\text{nd}}, \ldots, n-1^{\text{th}}$ rows we obtain $a_{jj} = 1$ and $a_{jk} = 0$ for $k \neq j$. For the last row we cannot accomplish this task, however we must have $a_{nn} \neq 0$, for otherwise the last column would be zero which is not possible because A is non-singular. We can then make all the other entries in the last row equal to zero. $\square$

Examples 1.15.

(1) For $i \neq j$ and $q \in \mathbb{H}^*$ denote $M_{ij}(q)$ the matrix obtained from the identity matrix by replacing the (i,i)-entry with q and the (j,j)-entry with q^{-1}. Then $M_{ij}(q)$ is a product of elementary matrices. It is enough to observe that for $i < j$ we have

$$M_{ij}(q) = B_{ji}(-q^{-1})B_{ij}(q-1)B_{ji}(1)B_{ij}(q^{-1}-1)\,,$$

indeed, if $i > j$ we conclude from the identity $M_{ij}(q) = M_{ji}(q^{-1})$.

(2) Whenever $q \in \mathbb{H}^*$ is a commutator, i.e. $q = aba^{-1}b^{-1}$ then $B_{kk}(q)$ can be decomposed as

$$B_{kk}(q) = M_{rk}(a^{-1})M_{rk}(b^{-1})M_{rk}(ba)\,,$$

for any $r \neq k$, which shows that $B_{kk}(q)$ is a product of elementary matrices.

(3) Let S^{ij} be the square matrices obtained by exchanging the i^{th} and the j^{th} rows (equivalently columns) of the identity matrix. In other words the entries of S^{ij} are defined as

$$(S^{ij})_{rs} := \begin{cases} \delta_{rs} & \text{if } r \neq i \text{ and } s \neq j \ , \\ \delta_{js} & \text{if } r = i \ , \\ \delta_{ri} & \text{if } s = j \ . \end{cases} \tag{1.6}$$

Now, multiplication on the left by S^{ij} amounts to exchanging its i^{th} and j^{th} rows, while multiplication on the right amounts to exchanging the i^{th} and j^{th} columns. Since $(S^{ij})^2 = \mathbb{1}$ clearly these matrices are invertible. Furthermore, they are generated by elementary matrices as we have the following decomposition

$$S^{ij} = B_{jj}(-1)B_{ij}(1)B_{ji}(-1)B_{ij}(1)\,.$$

Everything we proved so far only uses the division ring structure of $\mathbb{H}$, in particular, the general linear group makes sense for matrices with coefficients in any division ring R and the analogue of Lemma 1.14 still holds. The following proposition clarifies the relation between the commutator subgroup of $\mathrm{GL}(n, R)$ and the group generated by elementary matrices, hinting to a general definition of the special linear group.

Proposition 1.16. *Let R be a division ring. Let $S \subseteq \mathrm{GL}(n, R)$ be the subgroup generated by all elementary matrices $B_{ij}(q)$ for $i \neq j$, then $S = [\mathrm{GL}(n, R), \mathrm{GL}(n, R)]$, except when $n = 2$ and $R \cong \mathbb{Z}_2$. Moreover, if $R = \mathbb{K}$ is a field $S = \mathrm{SL}(n, \mathbb{K})$.*

Proof. We want to show that elementary matrices are commutators for $R \ncong \mathbb{Z}_2$. For distinct i, j, k (we need $n \geq 3$ here) it is straightforward to check that $B_{ij}(p) = [B_{ik}(p), B_{kj}(1)]$. When $n = 2$ and $R \ncong \mathbb{Z}_2$ there is an element $q \in R$ such that $q \neq 0, 1$. Set $c = p(q^{-1} - 1)^{-1}$, then $B_{12}(p) = [B_{22}(q), B_{12}(c)]$ and a similar identity holds for $B_{21}(p)$, then we see that $S \subseteq [\mathrm{GL}(n, R), \mathrm{GL}(n, R)]$.

Conversely, to prove that $[\mathrm{GL}(n, R), \mathrm{GL}(n, R)] \subseteq S$, by Lemma 1.14 we only need to show that for every non-zero $p, q \in R$ we have $[B_{nn}(p), B_{nn}(q)] = B_{nn}([p, q]) \in S$ and $[B_{nn}(p), B_{ij}(q)] \in S$ for $i \neq j$. The first fact was already observed in example 1.15(2). To prove the second, simply observe that $[B_{nn}(p), B_{ij}(q)] = \mathbb{1}$ if $i \neq n$ and $j \neq n$ while we have $[B_{nn}(p), B_{in}(q)] = B_{in}(pq^{-1} - p)$ and $[B_{nn}(p), B_{nj}(q)] = B_{jn}(qp - p)$.

Finally, if $R = \mathbb{K}$ is a field, by the usual properties of the determinant we know that elementary operations do not affect the determinant, therefore $S \subseteq \mathrm{SL}(n, \mathbb{K})$. Conversely, from the previous lemma we see that any $A \in \mathrm{SL}(n, \mathbb{K})$ decomposes as $A = B_{nn}(p)B$ where B is a product of elementary matrices, computing the determinant we necessarily have $p = 1$, hence $A = B$ and thus $\mathrm{SL}(n, \mathbb{K}) = S$. □

In view of this result the definition of the special linear group can be extended to any division ring. Coherently to our interest, we state it in terms of quaternions.

Definition 1.17. The **quaternionic special linear group** of order n, denoted $\mathrm{SL}(n, \mathbb{H})$, is the subgroup of $\mathrm{GL}(n, \mathbb{H})$ generated by elementary matrices, or, equivalently, the commutator subgroup of $\mathrm{GL}(n, \mathbb{H})$.

For future reference we state the following technical lemma according to which a matrix of the form $B_{nn}(q)$ lies in the commutator subgroup of $\mathrm{GL}(n, R)$ if and only if q lies in the commutator subgroup of R^*. We omit the proof, which can be found in [244, §16.5].

Lemma 1.18. *Let R be a division ring and $q \in R^*$, where $R^* = R \setminus \{0\}$ is the multiplicative group of R. Then $B_{nn}(q) \in [\mathrm{GL}(n, R), \mathrm{GL}(n, R)]$ if and only if $q \in [R^*, R^*]$.*

Hyperhermitian and hyperunitary matrices.

Transposition and conjugation are not automorphisms nor antiautomorphisms of $\mathbb{H}^{n,n}$, i.e., in general, for $A, B \in \mathbb{H}^{n,n}$

$${}^tA\,{}^tB \neq {}^t(AB) \neq {}^tB\,{}^tA\,, \qquad \bar{A}\bar{B} \neq \overline{AB} \neq \bar{B}\bar{A}\,.$$

Furthermore, they are not well-behaved operations with respect to inverses. However, the two operations combined give rise to an antiautomorphism of $\mathbb{H}^{n,n}$ which commutes with the inversion. Indeed, it is straightforward to check that

$$(AB)^* = B^* A^* \,, \qquad (A^{-1})^* = (A^*)^{-1} \,,$$

where $A^* = {}^t\bar{A}$ denotes the conjugate transpose of the matrix A.

Definition 1.19. A matrix $A \in \mathbb{H}^{n,n}$ is called

- **hyperhermitian** if $A = A^*$;
- **hyperunitary** if $AA^* = \mathbb{1}$.

We denote with $\mathrm{Hyp}(n)$ and $\mathrm{Sp}(n)$ the real vector space of $n \times n$ hyperhermitian matrices and the group of $n \times n$ hyperunitary matrices respectively.

Lemma 1.20. *Let $\langle \cdot, \cdot \rangle$ be the standard quaternionic inner product on $\mathbb{H}^n$. Then, the following conditions are equivalent for $H \in \mathbb{H}^{n,n}$:*

(i) *H is hyperhermitian;*

(ii) *$\langle Hu, v \rangle = \langle u, Hv \rangle$ for every $u, v \in \mathbb{H}^n$;*

(iii) *$\langle v, Hv \rangle$ is real, for every $v \in \mathbb{H}^n$.*

Proof. Since $\langle Hu, v \rangle = \langle u, H^* v \rangle$ the equivalence of the first two assertion is immediate and since $\langle v, Hv \rangle = \overline{\langle Hv, v \rangle}$ it is also clear that they imply the third one.

Conversely, if $\langle v, Hv \rangle$ is real for every $v \in \mathbb{H}^n$, since for every $u, v \in \mathbb{H}^n$

$$\langle u + v, H(u + v) \rangle = \langle u, Hu \rangle + \langle u, Hv \rangle + \langle v, Hu \rangle + \langle v, Hv \rangle$$

we have $\langle u, Hv \rangle + \overline{\langle u, H^* v \rangle} \in \mathbb{R}$. Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{H}^n$. Choosing $(u, v) = (e_r, e_s)$ and $(u, v) = (e_r i, e_s j)$ we get

$$\begin{cases} h_{rs} + h_{sr} \in \mathbb{R} \,, \\ i h_{rs} j + j h_{sr} i \in \mathbb{R} \,, \end{cases}$$

or, equivalently

$$\begin{cases} \mathrm{Im}(h_{rs}) + \mathrm{Im}(h_{sr}) = 0 \,, \\ \mathrm{Im}(i h_{rs} j) + \mathrm{Im}(j h_{sr} i) = 0 \,. \end{cases}$$

Using the first identity we obtain $\mathrm{Im}(i h_{rs} j) + \mathrm{Im}(j h_{sr} i) = (\mathrm{Re}(h_{rs}) - \mathrm{Re}(h_{sr}))\, k$ so that $h_{rs} = \overline{h_{sr}}$. ☐

Arguably, hyperhermitian matrices represent the most interesting class of quaternionic matrices from the geometric point of view. Their geometric interest is evident from the fact that they are in one-to-one correspondence with hyperhermitian forms on a right $\mathbb{H}$-module V. The correspondence is as follows: let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis for V, then a hyperhermitian form $g \colon V \times V \to \mathbb{H}$ induces the hyperhermitian matrix $M(g)$ with entries $M(g)_{ij} = g(e_i, e_j)$. Therefore we have

$$g(u, v) = u_{\mathcal{B}}^* M(g) v_{\mathcal{B}} \,.$$

Furthermore, g is positive (semi)definite if and only if the matrix $M(g)$ is **positive (semi)definite**, i.e. $w^* M(g) w = \langle w, M(g) w \rangle > 0$ (resp. ≥ 0) for every non-zero $w \in \mathbb{H}^n$.

If we take a different basis $\mathcal{C}$ and $P \in \mathrm{GL}(n, \mathbb{H})$ is the matrix of change of basis from $\mathcal{B}$ to $\mathcal{C}$, then $u_{\mathcal{B}} = P u_{\mathcal{C}}$, hence $g(u, v) = u_{\mathcal{C}}^* P^* M(g) P v_{\mathcal{C}}$ so that the matrix $M'(g)$ associated to g with respect to $\mathcal{C}$ is $M'(g) = P^* M(g) P$. Two matrices $A, B \in \mathbb{H}^{n,n}$ such that there exist $Q \in \mathrm{GL}(n, \mathbb{H})$ satisfying $A = Q^* B Q$ are called **congruent**. With a suitable choice of basis we can obtain a simple representative in the congruency class, yielding a convenient choice for the associated matrix.

Proposition 1.21 (Law of Inertia)**.** *For every $H \in \mathrm{Hyp}(n)$ there exists a matrix $P \in \mathrm{SL}(n, \mathbb{H})$ such that $P^* H P$ is diagonal with p positive entries and q negative entries along the diagonal, where the* *pair of numbers (p,q), called the* ***signature*** *is uniquely determined by H. Furthermore, there exists $Q \in \mathrm{GL}(n,\mathbb{H})$ such that*

$$Q^*HQ = \begin{pmatrix} \mathbb{1}_p & & \\ & -\mathbb{1}_q & \\ & & 0 \end{pmatrix} . \tag{1.7}$$

In particular, two hyperhermitian matrices are congruent if and only if they have the same signature.

Proof. We first prove by induction that there exists $P \in \mathrm{SL}(n,\mathbb{H})$ such that P^*HP is diagonal. The case $n = 1$ is trivial. Assume $H \neq 0$, so that there is at least a non-zero row and thus a non-zero corresponding column. Possibly exchanging simultaneously such non-zero row and column with the firsts by means of the matrices $S^{1j} \in \mathrm{SL}(n,\mathbb{H})$ of example 1.15(3), we may assume that the first row and column are non-zero. We may further assume that the first diagonal entry is non-zero, indeed if $h_{11} = 0$ and $h_{1k} \neq 0$ choosing any $q \in \mathbb{H}$ such that $\mathrm{Re}(h_{1k}q) \neq 0$ the matrix $B_{1k}(\bar{q})HB_{k1}(q)$ is again hyperhermitian and has first diagonal entry $2\mathrm{Re}(h_{1k}q) \neq 0$.

Knowing that $h_{11} \neq 0$ we can use it to kill all other entries in the first row and column by left multiplication with $B_{i1}(-h_{i1}h_{11}^{-1})$ and right multiplication with $B_{i1}(-h_{i1}h_{11}^{-1})^* = B_{1i}(-h_{11}^{-1}h_{1i})$ for all $i > 1$. We thus have found a matrix $S \in \mathrm{SL}(n,\mathbb{H})$ such that

$$S^*HS = \begin{pmatrix} h_{11} & 0 \\ 0 & H' \end{pmatrix}$$

where $H' \in \mathrm{Hyp}(n-1)$. By the inductive hypothesis and possibly further switching of rows and columns we conclude that there exists $P \in \mathrm{SL}(n,\mathbb{H})$ such that

$$P^*HP = \begin{pmatrix} d_1 & & & & & & \\ & \ddots & & & & & \\ & & d_p & & & & \\ & & & -d_{p+1} & & & \\ & & & & \ddots & & \\ & & & & & -d_{p+q} & \\ & & & & & & 0 \end{pmatrix}$$

with $d_1, \dots, d_{p+q} > 0$.

To obtain the form (1.7) we need to rescale the diagonal entries, which can be done by left and right multiplication with the matrices $B_{jj}(|d_j|^{-1/2}) \in \mathrm{GL}(n,\mathbb{H})$ for all $j = 1, \dots, p+q$. Observe that in general this step can only be achieved in $\mathrm{GL}(n,\mathbb{H})$, as, by Lemma 1.18 the matrices $B_{jj}(q)$ lie inside $\mathrm{SL}(n,\mathbb{H})$ if and only if q is a commutator (i.e. has unit norm by Proposition 1.4).

Now we prove that the numbers p, q are invariant and do not depend on the matrix P. First of all, $p + q = \mathrm{Rank}(H)$, which is uniquely determined by H. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{H}^n$ and define the submodules $U = \mathrm{Span}_{\mathbb{H}}(Pe_1, \dots, Pe_p)$ and $V = \mathrm{Span}_{\mathbb{H}}(Pe_{p+1}, \dots, Pe_n)$. we have $u^*Hu > 0$ and $v^*Hv \leq 0$ for every $u \in U$ and $v \in V$. Finally, p is the maximal dimension of a submodule of $\mathbb{H}^n$ where H is positive definite, because if there were a larger space, it would have non-zero intersection with V, leading to a contradiction. □

Now we move on to give a brief description of the hyperunitary group $\mathrm{Sp}(n) = \{A \in \mathbb{H}^{n,n} \mid AA^* = \mathbb{1}\}$ which can be equivalently defined as the group of matrices preserving the standard quaternionic inner product in $\mathbb{H}^n$:

$$\mathrm{Sp}(n) = \{A \in \mathbb{H}^{n,n} \mid \langle A\cdot, A\cdot\rangle = \langle\cdot,\cdot\rangle\} .$$

The hyperunitary group is sometimes denoted $\mathrm{U}(n,\mathbb{H})$ or $\mathrm{HU}(n)$ in the literature, but the notation $\mathrm{Sp}(n)$, to which we adhere, is much more common. This is because such a group is also called the *(compact) symplectic group*, however, even though it can be regarded as a subgroup of the complex symplectic group (cf. Subsection 2.1.3), it is not itself "quaternionic symplectic", in the sense of preserving a non-degenerate skew-symmetric $\mathbb{H}$-bilinear form on $\mathbb{H}^n$ (such a form does not exists by Remark 1.9). For this reason we prefer to call it the hyperunitary group, name that is justified at least by two additional reasons, the first one is the next lemma, the second lies in the fact that $\mathrm{Sp}(n)$ can be regarded as the intersection of the three unitary groups determined by i, j and k (see Subsection 2.1.3 for more details).

Lemma 1.22. *For a matrix $A \in \mathbb{H}^{n,n}$ the following are equivalent:*

(i) $A \in \mathrm{Sp}(n)$;

(ii) *A sends hyperunitary bases of $(\mathbb{H}^n, \langle\cdot,\cdot\rangle)$ to hyperunitary bases;*

(iii) *the columns (and/or the rows) of A form a hyperunitary basis of $(\mathbb{H}^n, \langle\cdot,\cdot\rangle)$.*

Proof. If $A \in \mathrm{Sp}(n)$ and $\mathcal{B} = (e_1, \ldots, e_n)$ is a hyperunitary basis of $\mathbb{H}^n$ with respect to the standard inner product $\langle\cdot,\cdot\rangle$ then $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ so that $\mathcal{C} = (Ae_1, \ldots, Ae_n)$ is again hyperunitary, thus (i) implies (ii).

When $\mathcal{B}$ is the canonical basis of $\mathbb{H}^n$, then vectors of $\mathcal{C}$ are columns of A, hence (ii) implies (iii).

Finally if (iii) holds and $\mathcal{B}$ is the canonical basis of $\mathbb{H}^n$ for any $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{H}^n$ we have

$$\langle Au, Av\rangle = \sum_{i,j=1}^{n} \langle Ae_i u_i, Ae_j v_j\rangle = \sum_{i,j=1}^{n} \bar{u}_i \langle Ae_i, Ae_j\rangle v_j = \sum_{i,j=1}^{n} \bar{u}_i \delta_{ij} v_j = \sum_{i=1}^{n} \bar{u}_i v_i = \langle u, v\rangle$$

i.e. $A \in \mathrm{Sp}(n)$. The equivalence of the fact that the rows of A form a hyperunitary basis follows immediately as $A \in \mathrm{Sp}(n)$ if and only if $A^* \in \mathrm{Sp}(n)$. □

One interesting fact that distinguishes $\mathrm{Sp}(n)$ from its real and complex analogues $\mathrm{O}(n)$ and $\mathrm{U}(n)$ is that the notion of "special hyperunitary group" is meaningless:

Lemma 1.23. $\mathrm{Sp}(n)$ *is a subgroup of* $\mathrm{SL}(n, \mathbb{H})$.

Proof. Let $A \in \mathrm{Sp}(n)$, then by Lemma 1.14 we can write $A = B_{nn}(q)B$ for some $q \in \mathbb{H}^*$ and $B \in \mathrm{SL}(n, \mathbb{H})$, therefore

$$\mathbb{1} = A^*A = B^* B_{nn}(\bar{q}) B_{nn}(q) B = B^* B_{nn}(|q|^2) B$$

which implies $B_{nn}(|q|^2) = (B^*)^{-1} B^{-1} \in \mathrm{SL}(n, \mathbb{H})$ and by applying Lemma 1.18 we deduce $|q| = 1$ and thus, from Proposition 1.4 and again Lemma 1.18 we have $B_{nn}(q) \in \mathrm{SL}(n, \mathbb{H})$ from which we conclude $A \in \mathrm{SL}(n, \mathbb{H})$. □

Remark 1.24. More in general one could define hyperhermitian, skew-hyperhermitian and hyperunitary matrices with respect to any non-degenerate hyperhermitian form $g(\cdot,\cdot)$. For example a matrix H is hyperhermitian with respect to g if $g(Hp, q) = g(p, Hq)$. If G is the hyperhermitian matrix associated to g, this is equivalent to the condition $H^* = GHG^{-1}$.

Representations.

In order to overcome some difficulties of working with quaternionic matrices, it will be extremely useful in the future to have a description of them in terms of complex or real matrices. We start by representing complex matrices with real matrices as a preliminary argument, and then move on to present the description of quaternionic matrices.

As usual, there is a vector space isomorphism $\alpha\colon \mathbb{C}^n \to \mathbb{R}^{2n}$ given by

$$\alpha\colon (z^1, \ldots, z^n) \mapsto (x^1, \ldots, x^n, y^1, \ldots, y^n)\,, \qquad \text{where } z^r = x^r + y^r i\,,\ r = 1, \ldots, n\,.$$

Any matrix $Z \in \mathbb{C}^{n,n}$ can be seen as a complex endomorphism of $\mathbb{C}^n$, which in particular is $\mathbb{R}$-linear. Under the isomorphism α, this defines a real endomorphism of $\mathbb{R}^{2n}$ represented by a matrix $\alpha \circ Z \circ \alpha^{-1}$. By a slight abuse of notation we still denote the map that sends Z to $\alpha \circ Z \circ \alpha^{-1}$ with α:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\;Z\;} & \mathbb{C}^n \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathbb{R}^{2n} & \xrightarrow{\;\alpha(Z)\;} & \mathbb{R}^{2n} \end{array}$$

Proposition 1.25. *$\alpha\colon \mathbb{C}^{n,n} \to \mathbb{R}^{2n,2n}$ is an injective morphism of real algebras. Furthermore, if $Z = X + Yi \in \mathbb{C}^{n,n}$ with X, $Y \in \mathbb{R}^{n,n}$, then in the standard bases*

$$\alpha(Z) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

Proof. It is straightforward to check that α is a morphism of real algebras and injectivity is obvious. The explicit form of $\alpha(Z)$ follows directly from the definition looking at the image of the standard basis. For example if $e_1 = (1, 0 \ldots, 0) \in \mathbb{R}^{2n}$ then $\alpha^{-1}(e_1) = (1, 0 \ldots, 0) \in \mathbb{C}^n$ and thus $Z\alpha^{-1}(e_1)$ is the first column of $Z = X + Yi$ so that applying α we obtain the first column of $\alpha(Z)$. ☐

The same approach can be pursued on $\mathbb{H}^n$ seen as a complex vector space with right scalar multiplication, i.e. a scalar $a \in \mathbb{C}$ acts on $z + wj \in \mathbb{H}^n$ with the right product: $(z + wj)a = za + w\bar{a}j$. In this case we consider the isomorphism

$$\beta\colon (q^1, \ldots, q^n) \mapsto (z^1, \ldots, z^n, \bar{w}^1, \ldots, \bar{w}^n)\,, \qquad \text{where } q^r = z^r + w^r j\,,\ r = 1, \ldots, n\,.$$

One could also consider the composition $\alpha \circ \beta$ as an isomorphism of real vector spaces between $\mathbb{H}^n$ and $\mathbb{R}^{4n}$, but we believe it is more natural to choose the following one

$$\gamma\colon (q^1, \ldots, q^n) \mapsto (x_0^1, \ldots, x_0^n, x_1^1, \ldots, x_1^n, x_2^1, \ldots, x_2^n, x_3^1, \ldots, x_3^n)\,,$$

$$\text{where } q^r = x_0^r + x_1^r i + x_2^r j + x_3^r k\,,\ r = 1, \ldots, n\,.$$

In the same spirit of Proposition 1.25 we obtain:

Proposition 1.26. *The induced maps $\beta\colon \mathbb{H}^{n,n} \to \mathbb{C}^{2n,2n}$ and $\gamma\colon \mathbb{H}^{n,n} \to \mathbb{R}^{4n,4n}$ are injective morphisms of real algebras. Furthermore, if $M = Z + Wj = A + Bi + Cj + Dk \in \mathbb{H}^{n,n}$ with Z, $W \in \mathbb{C}^{n,n}$ and A, B, C, $D \in \mathbb{R}^{n,n}$, then in the standard basis*

$$\beta(M) = \begin{pmatrix} Z & -W \\ \bar{W} & \bar{Z} \end{pmatrix}, \qquad \gamma(M) = \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}.$$

Remark 1.27. Here, we have fixed some choices. For instance we could have taken into account the isomorphism $\mathbb{H}^n \to \mathbb{C}^{2n}$, $(q^1, \ldots, q^n) \mapsto (z^1, \ldots, z^n, -\bar{w}^1, \ldots, -\bar{w}^n)$ instead of the one we chose. One can easily check that there are 2 possible choices for α, 2 choices for β (if fixing right scalar multiplication of $\mathbb{C}$ on $\mathbb{H}^n$) and 48 choices for γ if we want the image of these morphisms of real algebras to be block matrices corresponding to the writings $Z = X + Yi \in \mathbb{C}^{n,n}$, $M = Z + Wj = A + Bi + Cj + Dk \in \mathbb{H}^{n,n}$ (cf. [121]).

The isomorphism $\alpha\colon \mathbb{C}^n \to \mathbb{R}^{2n}$ induces on $\mathbb{R}^{2n}$ a structure of a complex vector space, where multiplication by i is given by multiplication with the matrix $\mathcal{I} := \alpha(i\mathbb{1}_n) = \left(\begin{smallmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{smallmatrix}\right)$. In the same way $\beta\colon \mathbb{H}^n \to \mathbb{C}^{2n}$ endows $\mathbb{C}^{2n}$ with a structure of a right $\mathbb{H}$-module, where the action of j is given by the map $\beta \circ j \circ \beta^{-1}$, which is easily checked to be equal to the map $\mathcal{J}(z) := \mathcal{I}\bar{z}$, where $z \in \mathbb{C}^{2n}$. Finally, $\gamma\colon \mathbb{H}^{n,n} \to \mathbb{R}^{4n,4n}$ induces on $\mathbb{R}^{4n}$ the $\mathbb{H}$-bimodule structure with action of i, j, k given by left or right multiplication with $I_0 := \gamma(i\mathbb{1}_n)$, $J_0 := \gamma(j\mathbb{1}_n)$, $K_0 := \gamma(k\mathbb{1}_n)$, i.e.

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & \mathbb{1}_n & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & 0 & 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & -\mathbb{1}_n & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \end{pmatrix}.$$

With this additional structures, we can characterize the images of α, β and γ. Indeed, for example, a matrix $A \in \mathbb{R}^{2n,2n}$ corresponds to a $\mathbb{C}$-linear endomorphism of $\mathbb{R}^{2n}$ if and only if it commutes with the action of i on $\mathbb{R}^{2n}$. Reasoning similarly for $\mathbb{C}^{2n}$ and $\mathbb{R}^{4n}$ as $\mathbb{H}$-modules, we reach the following description:

$$\alpha(\mathbb{C}^{n,n}) = \left\{A \in \mathbb{R}^{2n,2n} \mid \mathcal{I}A = A\mathcal{I}\right\} ,$$
$$\beta(\mathbb{H}^{n,n}) = \left\{A \in \mathbb{C}^{2n,2n} \mid \mathcal{J}A = A\mathcal{J}\right\} = \left\{A \in \mathbb{C}^{2n,2n} \mid \mathcal{I}\bar{A} = A\mathcal{I}\right\} ,$$
$$\gamma(\mathbb{H}^{n,n}) = \left\{A \in \mathbb{R}^{4n,4n} \mid I_0 A = A I_0 \, , \, J_0 A = A J_0 \, , \, K_0 A = A K_0\right\} .$$

Lemma 1.28. *For a matrix $M \in \mathbb{H}^{n,n}$ the following are equivalent:*

(i) *M is invertible (resp. hyperhermitian, hyperunitary, positive (semi)definite);*

(ii) *$\beta(M)$ is invertible (resp. Hermitian, unitary, positive (semi)definite);*

(iii) *$\gamma(M)$ is invertible (resp. symmetric, orthogonal, positive (semi)definite).*

Proof. The lemma follows immediately from the relations $\beta(M^{-1}) = \beta(M)^{-1}$, $\gamma(M^{-1}) = \gamma(M)^{-1}$, $\beta(M^*) = (\beta(M))^*$, $\gamma(M^*) = {}^t(\gamma(M))$. □

Consequently, we can identify $\mathrm{GL}(n, \mathbb{H})$, $\mathrm{SL}(n, \mathbb{H})$ and $\mathrm{Sp}(n)$ with their images under β or γ, thus uncovering their Lie group structures.

Spectral Theory.

When discussing spectral theory, already from the outset one is compelled to look at either left or right eigenvalues and no clear relation between the two sides is available. Looking at $\mathbb{H}^n$ as a right $\mathbb{H}$-module and assuming linear operators act on the left, the natural notion to take into account is that of a *right eigenvalue*, on which we shall focus. With this choice, the theory is almost entirely understood and it has a fairly nice description which goes back to Jacobson [187], Lee [206] and Brenner [53] (an earlier appearance of the Spectral Theorem for normal endomorphisms on quaternionic Hilbert spaces is in Teichmüller's paper [283]). On the other hand, as of today, *left eigenvalues* are still quite obscure and their theory is not fully explored. Our primary source for this treatment is the excellent survey of Farenick-Pidkowich [122].

Definition 1.29. Let $A \in \mathbb{H}^{n,n}$ be a quaternionic matrix. A non-zero vector $q \in \mathbb{H}^n$ is called a **right eigenvector** with **right eigenvalue** $\lambda \in \mathbb{H}$ if it satisfies

$$Aq = q\lambda \, .$$

Remark 1.30. Left eigenvalues, i.e. quaternions $\lambda \in \mathbb{H}$ for which the equation $Aq = \lambda q$ admits a non-zero solution $q \in \mathbb{H}^n$, are less easy to investigate. The fact that left eigenvalues can even be taken into account is solely a consequence of the fact that $\mathbb{H}^n$ is a bimodule. Besides the algebraic inconsistency of considering left eigenvalues on a right module, some issues arising in quaternionic quantum mechanics have been pointed out in [98]. Moreover, finding and studying left eigenvalues is more subtle then right ones. After a wrong affirmation by Lee in a footnote of [206] that left eigenvalues do not exist, Cohn [94, p. 217] was the first to raise the question whether or not they always occur. A positive answer came from Wood [326] with a topological proof (see also [330, Theorem 5.3]).

For further information on the left eigenvalue problem we suggest the reader to consult [123, 181, 183, 214, 219, 262, 330, 331].

The lack of commutativity implies that whenever a matrix has a non-real right eigenvalue it actually allows an infinite family of them. Suppose $q \in \mathbb{H}^n$ satisfies $Aq = q\lambda$ then for every $p \in \mathbb{H}^*$ we have

$$Aqp = q\lambda p = qp(p^{-1}\lambda p)$$

showing that $p^{-1}\lambda p$ is another right eigenvalue. From this point of view it makes sense to consider the conjugacy class $\theta(\lambda)$ of an eigenvalue λ rather than the eigenvalue itself.

From Lemma 1.3 it follows that there is a one-to-one correspondence between conjugacy classes of quaternions and complex numbers with non-negative imaginary part, which we denote by $\mathbb{C}^+$.

Proposition 1.31. *Let $A \in \mathbb{H}^{n,n}$. Then $\lambda \in \mathbb{H}$ is such that $\theta(\lambda)$ is a conjugacy class of right eigenvalues for A if and only if μ is an eigenvalue for $\beta(A)$, where $\mu \in \theta(\lambda) \cap \mathbb{C}^+$. Moreover, if λ is non-real then also $\bar{\mu}$ is an eigenvalue of $\beta(A)$, while if λ is real it has even geometric multiplicity as an eigenvalue of $\beta(A)$.*

Proof. Suppose λ is a right eigenvalue for A, then it follows from the discussion above that $Aq = q\lambda$ if and only if $Aq' = q'\mu$ for some q, $q' \in \mathbb{H}$. Via the isomorphism $\beta\colon \mathbb{H}^n \to \mathbb{C}^{2n}$ (and the induced monomorphism $\beta\colon \mathbb{H}^{n,n} \to \mathbb{C}^{2n,2n}$) this is equivalent to $\beta(A)\beta(q') = \beta(q')\mu$. If λ is non-real we might replace μ with $\bar{\mu}$ to show that this is also an eigenvalue of $\beta(A)$. Finally if $\lambda = \mu = \bar{\mu}$ is real call V the eigenspace of $\beta(A)$ corresponding to λ. Define a map $\sigma\colon V \to \mathbb{C}^{2n}$ as follows. Any eigenvector in V is of the form $\beta(q) = (z, \bar{w})$ for $q = z + wj \in \mathbb{H}^n$, where z, $w \in \mathbb{C}^n$. Set $\sigma(\beta(q)) := (w, -\bar{z})$. Then it is easily checked that $\sigma(V) \subseteq V$ and $\sigma(\beta(q))$ is $\mathbb{C}$-linearly independent from $\beta(q)$. The fact that σ is $\mathbb{C}$-antilinear and $\sigma^2 = -\mathrm{Id}_V$ automatically implies that V is even-dimensional. □

Theorem 1.32. *Every $A \in \mathbb{H}^{n,n}$ has at most n distinct conjugacy classes of right eigenvalues and precisely n of them if counted with multiplicities.*

Proof. We know that $\beta(A)$ has at most $2n$ distinct eigenvalues, furthermore, each conjugate pair of complex eigenvalues of $\beta(A)$ and each real eigenvalue identifies a conjugacy class of right eigenvalues of A by Proposition 1.31. But again, non-real eigenvalues of $\beta(A)$ always appear in conjugate pairs, while real eigenvalues always have even geometric multiplicity. Therefore the distinct conjugacy classes of right eigenvalues of A are at most half of the distinct eigenvalues of $\beta(A)$. □

The existence of a right eigenvalue can also be achieved via topological methods as done by Baker [26] who was inspired by the analogue proof of Wood for left eigenvalues (see Remark 1.30).

Now, we shall proceed to establish the Spectral Theorem. The first step in this direction is represented by Schur's triangularization result.

Proposition 1.33 (Schur's triangularization)**.** *Let $A \in \mathbb{H}^{n,n}$, then there exist T, $U \in \mathbb{H}^{n,n}$ with T upper triangular and $U \in \mathrm{Sp}(n)$, such that*

$$A = U^* T U\,.$$

Furthermore, each entry on the diagonal of T can be chosen to be the unique element in $\mathbb{C}^+$ of a conjugacy class of eigenvalues for A.

Proof. The result is obvious for $n = 1$. We assume that the proposition is true for $n - 1$ and we prove it for n.

Let $\lambda_1 \in \mathbb{C}^+$ be a right eigenvalue of A and choose a corresponding eigenvector $q_1 \in \mathbb{H}^n$ such that $|q_1| = 1$. With the Gram-Schimdt process, extend this to a hyperunitary basis $q_1, \dots, q_n$ of $\mathbb{H}^n$. The matrix V with q_i as its i^{th} column is then unitary. By construction we have that the first column of V^*AV is $(\lambda_1, 0, \dots, 0)$. Let $B \in \mathbb{H}^{n-1,n-1}$ be the matrix obtained from V^*AV removing the first row and the first column. Then by the inductive hypothesis there exist $W \in \mathrm{Sp}(n-1)$ such that $S = W^*BW$ is upper triangular with diagonal entries in $\mathbb{C}^+$. Set $U = V\left(\begin{smallmatrix}1 & 0\\ 0 & W\end{smallmatrix}\right) \in \mathrm{Sp}(n)$, then clearly

$$U^*AU = \begin{pmatrix}1 & 0\\ 0 & W\end{pmatrix}^* V^*AV \begin{pmatrix}1 & 0\\ 0 & W\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & W^*\end{pmatrix}\begin{pmatrix}\lambda_1 & *\\ 0 & B\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & W\end{pmatrix} = \begin{pmatrix}\lambda_1 & *\\ 0 & S\end{pmatrix}$$

which has the desired form. □

Recall that a matrix A is **normal** if it commutes with its conjugate transpose. Hyperhermitian and hyperunitary matrices are instances of normal matrices.

Theorem 1.34 (Spectral Theorem)**.** *If $A \in \mathbb{H}^{n,n}$ is normal, then there exist $U \in \mathrm{Sp}(n)$ such that $D := U^*AU$ is diagonal with entries in $\mathbb{C}^+$ and $\lambda \in \mathbb{H}$ is a right eigenvalue of A if and only if it belongs to the conjugacy class of some diagonal element of D. Furthermore*

1. *if A is hyperhermitian then D has real entries;*
2. *if A is hyperunitary then D has complex entries with unit norm.*

Proof. By Proposition 1.33 there exists $U \in \mathrm{Sp}(n)$ such that $T = U^*AU$ is upper triangular with diagonal entries in $\mathbb{C}^+$. Since we assumed A to be normal, so is $T = (t_{ij})$, implying the identity

$$\sum_{i=1}^{k} |t_{ik}|^2 = \sum_{j=k}^{n} |t_{kj}|^2 .$$

for every $k = 1, \dots, n$. Starting from $k = 1$ this implies $|t_{1j}|^2 = 0$ for all $j > 1$, then for $k = 2$ the equation implies $|t_{2j}|^2 = 0$ for all $j > 2$. Repeating this argument for all k's shows that T is actually diagonal. This concludes the first part of the theorem.

If $\lambda \in \mathbb{H}$ is a right eigenvalue for A and $q \in \mathbb{H}^n$ is one of the corresponding eigenvectors, then $DU^*q = U^*Aq = U^*q\lambda$, i.e. λ is a right eigenvalue for D with eigenvector $p = U^*q$. Let $\mu_1, \dots, \mu_n$ be the diagonal elements of D and write $p = (p_1, \dots, p_n)$, then there exists at least one index i for which $p_i \neq 0$, implying $\mu_i p_i = p_i\lambda$ and thus $\mu_i = p_i\lambda p_i^{-1} \in \theta(\lambda)$.

Conversely, if $\mu \in \mathbb{C}^+$ is a diagonal element of D, which we may assume to be in the $(1,1)$-entry, then, setting $q = (1, 0, \dots, 0)$, we have $Dq = q\mu$. In terms of A such equation reads $AUq = Uq\mu$ so that μ is a right eigenvalue for A.

Finally, the remaining part of the theorem follows from the observation that whenever A is hyperhermitian or hyperunitary, so is D. □

As an interesting application we prove that two hyperhermitian matrices, one of which is positive definite, can always be simultaneously diagonalized via an invertible matrix.

Proposition 1.35. *Let $H_1, H_2 \in \mathrm{Hyp}(n)$ with H_1 positive definite. Then there exists $P \in \mathrm{GL}(n, \mathbb{H})$ such that P^*H_1P is the identity matrix and P^*H_2P is diagonal.*

Proof. From the Law of Inertia (Proposition 1.21) there exists a matrix $Q_1 \in \mathrm{GL}(n, \mathbb{H})$ such that $Q_1^*H_1Q_1$ is the identity. Since $Q_1^*H_2Q_1$ is hyperhermitian it can be diagonalized via a matrix $Q_2 \in \mathrm{Sp}(n)$. Setting $P = Q_1Q_2 \in \mathrm{GL}(n, \mathbb{H})$ we have $P^*H_1P = \mathbb{1}$ and $P^*H_2P = D$, where D is diagonal. □

For further canonical forms and decompositions, such as the Jordan normal form or the polar decomposition see [79, 122, 163, 180, 182, 190, 216, 250, 324, 325, 332].

1.2 Moore determinant.

The naive approach to define the determinant of a quaternionic matrix simply adopting the usual formula for matrices with entries in a field is patently flawed. The formula we are referring to is usually called **Laplace formula** and, when commutativity does not hold, the application of the formula on different rows or columns leads to different results.

Nonetheless, this was the road taken up by Cayley [75] who decided to define the determinant by expanding the Laplace formula along the first column. Let us explain this on a sample 2×2 matrix. The definition of Cayley is simply

$$\mathrm{Cdet}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - cb \,.$$

Cayley observes that if the rows of A are equal the determinant is zero, but if the columns are, in general the determinant need not vanish:

$$\mathrm{Cdet}\begin{pmatrix} a & b \\ a & b \end{pmatrix} := ab - ab = 0 \,, \qquad \mathrm{Cdet}\begin{pmatrix} a & a \\ b & b \end{pmatrix} := ab - ba \,.$$

If we try to ignore this warning we will be forced to face more compelling reasons to reject such an approach. For instance, one of the most important properties about usual determinants is that they vanish exactly on the subset of singular matrices. The Cayley determinant spectacularly fails to do so; for example the matrix $\left(\begin{smallmatrix} k & j \\ i & 1 \end{smallmatrix}\right)$ is easily checked to be invertible but with zero Cayley determinant, whereas its transpose is singular but has non-zero Cayley determinant.

Two quaternionic determinants.

From this discussion is apparent that one needs a cleverer definition of the determinant. Pursuing an axiomatic approach, by singling out some desirable properties of the usual determinant one can indeed have some partially satisfactory notions of determinant. Since we are not particularly interested in this path we refer the reader to Aslaksen [23] for a discussion of this axiomatic perspective. Here, we limit ourselves to briefly present two possible outcomes of this method.

The first determinant function we present was introduced by Dieudonné in 1943. Dieudonné [102] proved that for any division ring R there is an isomorphism

$$\mathrm{GL}(n,R)\big/[\mathrm{GL}(n,R),\mathrm{GL}(n,R)] \cong R^*\big/[R^*,R^*] \,.$$

This isomorphism induces a map $\det_D\colon \mathrm{GL}(n,R)\to R^*/[R^*,R^*]$ defined by setting

$$\det_D(A) := \det_D(B_{nn}(q_A)B_A) = q_A[R^*,R^*]\,,$$

where we used Lemma 1.14 to express A as a product $B_{nn}(q_A)B_A$ for some $B_A\in[\mathrm{GL}(n,R),\mathrm{GL}(n,R)]$. Dieudonné shows that this is a well defined homomorphism with the commutator subgroup as its kernel. The map is easily extended to non-invertible matrices by adjoining the zero to $R^*/[R^*,R^*]$ and declaring that $\det_D$ vanishes on every singular matrix.

Even though $\det_D$ becomes exactly the usual determinant when $R=\mathbb{K}$ is a field, the reader might find this extension of the determinant unsatisfactory, as it does not take values in R itself. However, in the special case of quaternions, due to Proposition 1.4 we see that $\mathbb{H}^*/[\mathbb{H}^*,\mathbb{H}^*]$ is isomorphic to $\mathbb{R}^+$, the set of positive real numbers, via the map $\eta\colon q[\mathbb{H}^*,\mathbb{H}^*]\to|q|$ (actually any power of the norm here will do, yielding a different determinant function). The (quaternionic) **Dieudonné determinant** is the map $\mathrm{Ddet}\colon \mathbb{H}^{n,n}\to[0,+\infty)$ defined as

$$\mathrm{Ddet}(A) := \eta(\det_D(A))\,.$$

The Dieudonné determinant has been studied quite extensively in the literature. We suggest the interested reader to consult [9, 22, 23, 54, 95, 101, 213, 244] and the references cited therein (be aware that [9, 54] contain some wrong statements about transpose matrices as pointed out in [213]).

Earlier than Dieudonné, Study had introduced in 1920 [274] another determinant that takes advantage of the representation of quaternionic matrices as complex ones described in section 1.1.2. The **Study determinant** is the map $\mathrm{Sdet}\colon \mathbb{H}^{n,n}\to[0,+\infty)$ defined as

$$\mathrm{Sdet}(M) := \det(\beta(M))\,.$$

The fact that the Study determinant only takes non-negative real values is a consequence of the fact that $\beta(M)$ is similar to its conjugate, indeed we have seen that $\beta(\mathbb{H}^{n,n})=\{A\in\mathbb{C}^{2n,2n}\mid \bar{A}=\mathcal{I}^{-1}A\mathcal{I}\}$, where $\mathcal{I}=\left(\begin{smallmatrix}0 & -\mathbb{1}_n\\ \mathbb{1}_n & 0\end{smallmatrix}\right)$. This means that if λ is an eigenvalue of $\beta(M)$ then also $\bar\lambda$ is such.

What happens if we use the representation $\gamma\colon \mathbb{H}^{n,n}\to\mathbb{R}^{4n,4n}$ instead of β? By switching the middle block of rows and columns and then exchanging signs to the last block of rows and columns we observe that

$$\det(\gamma(M)) = \det\begin{pmatrix} A & -B & -C & -D\\ B & A & -D & C\\ C & D & A & -B\\ D & -C & B & A\end{pmatrix} = \det\begin{pmatrix} A & -C & -B & D\\ C & A & D & B\\ B & -D & A & -C\\ -D & -B & C & A\end{pmatrix} = \det(\alpha(\beta(M)))$$

but since, for every $Z=X+Yi\in\mathbb{C}^{n,n}$ we have

$$\det(\alpha(Z)) = \det\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix} = \det\begin{pmatrix} X+Yi & Xi-Y\\ Y & X\end{pmatrix} = \det\begin{pmatrix} Z & 0\\ Y & \bar{Z}\end{pmatrix} = \det(Z)\det(\bar{Z}) = |\det(Z)|^2\,,$$

for $Z=\beta(M)$ we get

$$\det(\gamma(M)) = |\det(\beta(M))|^2 = \mathrm{Sdet}(M)^2\,.$$

This observation is due to Bagazgoitia and Lewis [24, 209].

Both determinants are multiplicative and they vanish if and only if evaluated on singular matrices. They actually are one the square of the other, which follows from the following interesting description in terms of the eigenvalues:

Lemma 1.36. *For every $M \in \mathbb{H}^{n,n}$ we have*

$$\mathrm{Ddet}(M) = |\lambda_1||\lambda_2|\cdots|\lambda_n|\,, \tag{1.8}$$
$$\mathrm{Sdet}(M) = |\lambda_1|^2|\lambda_2|^2\cdots|\lambda_n|^2\,, \tag{1.9}$$

where $\theta(\lambda_i)$ are all the conjugacy classes of eigenvalues of M, with multiplicity.

Proof. Observe that all elements within a given conjugacy class have the same norm, therefore the expressions (1.8), (1.9) do not depend on the choice of the λ_i's but only on their conjugacy classes $\theta(\lambda_i)$.

Formula (1.9) follows immediately from Proposition 1.31. Let us prove (1.8). Multiplying any row on the left or any column on the right by $q \in \mathbb{H}$ has the effect of multiplying the Dieudonné determinant by $|q|$ this is because $\mathrm{Ddet}(B_{ii}(q)) = \mathrm{Ddet}(B_{nn}(q)) = |q|$ as it is easily checked. As a consequence, the Dieudonné determinant of any diagonal matrix is the product of the norms of the elements on the diagonal and this fact immediately extends to upper (or lower) triangular matrices because $\mathrm{Ddet}(B_{ij}(q)) = 1$ for $i \neq j$. Using Schur's triangularization (Proposition 1.33), multiplicativity of Ddet and the fact that $\mathrm{Ddet}(\mathrm{Sp}(n)) = 1$ we achieve (1.8). □

Definition of Moore determinant.

If we are willing to restrict the domain of definition of the determinant function there is a really nice candidate, especially for geometric applications. Moore [229] showed that on the space of hyperhermitian matrices $\mathrm{Hyp}(n)$ the definition given by Cayley makes sense, if we specify a certain ordering of the factors in the formula. This kind of determinant is the one we are interested in for future applications, therefore, we give here a detailed presentation.

Fix $H = (h_{ij}) \in \mathbb{H}^{n,n}$. For any cycle $c = (c_1 \;\; \cdots \;\; c_k) \in S_n$ written so that $c_1 > c_r$ for all $r > 1$, define

$$H_c := H_{(c_1 \;\; \cdots \;\; c_k)} := h_{c_1c_2}h_{c_2c_3}\cdots h_{c_{k-1}c_k}h_{c_kc_1}\,.$$

Now, any permutation $\sigma \in S_n$ can be written uniquely as a product of $s(\sigma)$ disjoint cycles

$$\sigma = \sigma^1\sigma^2\cdots\sigma^{s(\sigma)}\,, \qquad \sigma^r = \left(\sigma_1^r \;\; \sigma_2^r \;\; \cdots \;\; \sigma_{l_r}^r\right)\,,\; (r = 1, \ldots, s(\sigma))$$

where the factors are arranged so that $\sigma_1^r > \sigma_t^r$ for all $t > 1$ and $\sigma_1^1 > \sigma_1^2 > \cdots > \sigma_1^{s(\sigma)}$.

Definition 1.37. The **Moore determinant** of a quaternionic matrix $H \in \mathbb{H}^{n,n}$ is defined as the sum

$$\det(H) := \sum_{\sigma \in S_n} |\sigma|\, H_{\sigma^1}H_{\sigma^2}\cdots H_{\sigma^{s(\sigma)}}\,,$$

where $|\sigma| = |\sigma^1||\sigma^2|\cdots|\sigma^{s(\sigma)}| = (-1)^{l_1-1}(-1)^{l_2-1}\cdots(-1)^{l_{s(\sigma)}-1} = (-1)^{n-s(\sigma)}$ denotes the sign of the permutation σ.

Observe that, whenever $H \in \mathbb{C}^{n,n}$ its entries commute and the Moore determinant coincides with the usual determinant, this justifies our use of the notation det for the Moore determinant.

It will be useful to rewrite the Moore determinant in a slightly different form. Consider the following set of ordered partitions of $\{1, \ldots, n\}$

$$\mathcal{P}(n) := \left\{(X_1, \ldots, X_s) \mid X_k \neq \emptyset\,, \coprod_{k=1}^{s} X_k = \{1, \ldots, n\}\,, \max X_1 > \max X_2 > \cdots > \max X_s\right\}\,,$$

and, given any subset $X \subseteq \{1,\ldots,n\}$, let us denote $C(X)$ the set of all cycles of elements in X with the condition to be written so that the first element in the cycle is the biggest. In other words

$$c = (c_1 \ \cdots \ c_l) \in C(X) \iff X = \{c_1,\ldots,c_l\}\,,\, c_1 = \max X\,. \tag{1.10}$$

With this piece of notation we can write

$$\begin{aligned}\det(H) &= \sum_{(X_1,\ldots,X_s)\in\mathcal{P}(n)} \sum_{\substack{\sigma=\sigma^1\sigma^2\cdots\sigma^s\in S_n\\ \sigma^k\in C(X_k)}} |\sigma|\, H_{\sigma^1}H_{\sigma^2}\cdots H_{\sigma^s}\\ &= \sum_{(X_1,\ldots,X_s)\in\mathcal{P}(n)} \left(\sum_{\sigma^1\in C(X_1)} |\sigma^1|\, H_{\sigma^1}\right)\left(\sum_{\sigma^2\in C(X_2)} |\sigma^2|\, H_{\sigma^2}\right)\cdots\left(\sum_{\sigma^s\in C(X_s)} |\sigma^s|\, H_{\sigma^s}\right),\end{aligned}$$

and thus, setting $H_{X_k} := \sum_{\sigma^k\in C(X_k)} |\sigma^k|\, H_{\sigma^k} = (-1)^{|X_k|-1}\sum_{\sigma^k\in C(X_k)} H_{\sigma^k}$, we have

$$\det(H) = \sum_{(X_1,\ldots,X_s)\in\mathcal{P}(n)} H_{X_1}H_{X_2}\cdots H_{X_s}\,. \tag{1.11}$$

Lemma 1.38. *If* $H \in \mathrm{Hyp}(n)$ *then* $\det(H) \in \mathbb{R}$.

Proof. From (1.11) it is enough to prove that for any fixed $\emptyset \neq X \subseteq \{1,\ldots,n\}$ the sum H_X is real. This is a consequence of the observation that for any cycle $c = (c_1 \ \cdots \ c_l)$ we have

$$\overline{H_c} = \overline{h_{c_1c_2}h_{c_2c_3}\cdots h_{c_lc_1}} = \bar{h}_{c_lc_1}\bar{h}_{c_{l-1}c_l}\cdots\bar{h}_{c_1c_2} = h_{c_1c_k}h_{c_lc_{l-1}}\cdots h_{c_2c_1} = H_{(c_1 \ c_l \ c_{l-1} \ \cdots \ c_2)}\,.$$

Indeed, if $l = 1, 2$ then H_c is real, while if $l > 2$, $\overline{H_c}$ is another element in the sum H_X, therefore

$$H_X = \frac{1}{2}\sum_{c\in C(X)} |c|\left(H_c + \overline{H_c}\right) = \sum_{c\in C(X)} |c|\,\mathrm{Re}(H_c)$$

is real. ☐

Remark 1.39. In view of this lemma, whenever we consider the Moore determinant of a hyperhermitian matrix, which we always will from now on, some of the restrictions imposed on the definition can be modified without affecting the determinant. For instance, when computing H_X for $\emptyset \neq X \subseteq \{1,\ldots,n\}$ we imposed in (1.10) that the first element of the cycle be the maximum of X, however, this condition can now be dropped and all we need is to pair the cycles as in the lemma above. Indeed for every $c = (c_1 \ \cdots \ c_l)$

$$\mathrm{Re}(H_c) = \mathrm{Re}\left(H_{(c_1 \ c_2 \ \cdots \ c_l)}\right) = \mathrm{Re}\left(H_{(c_2 \ c_3 \ \cdots \ c_l \ c_1)}\right) = \cdots = \mathrm{Re}\left(H_{(c_l \ c_1 \ c_2 \ \cdots \ c_{l-1})}\right)\,.$$

In general, it will be useful in the future to choose suitably the first element of the cycles, say c_1, then we can clearly also compute H_X as follows

$$H_X = (-1)^{l-1}\sum_{\tau\in S(X\setminus\{c_1\})} H_{(c_1 \ \tau(c_2) \ \tau(c_3) \ \cdots \ \tau(c_l))}\,,$$

where $S(Y)$ denotes the set of permutations of elements in the set Y.

Properties and axiomatic description.

The Moore determinant also allows an axiomatic definition:

Definition 1.40. A function $d\colon \mathrm{Hyp}(n) \to \mathbb{R}$ is called a **hyperhermitian determinant** if it satisfies the following axioms:

- **Axiom 1**. If $H' \in \mathrm{Hyp}(n)$ is obtained from $H \in \mathrm{Hyp}(n)$ by multiplying the i^{th} row on the left by $q \in \mathbb{H}$ and the i^{th} column on the right by $\bar{q}$ (i.e. $H' = B_{ii}(q)HB_{ii}(\bar{q})$), then $d(H') = |q|^2 d(H)$.
- **Axiom 2**. If $H' \in \mathrm{Hyp}(n)$ is obtained from $H \in \mathrm{Hyp}(n)$ by adding the j^{th} row to the i^{th} and then the j^{th} column to the i^{th} for $i \neq j$ (i.e. $H' = B_{ij}(1)HB_{ji}(1)$), then $d(H') = d(H)$.
- **Axiom 3**. $d\left(\begin{smallmatrix} \mathbb{1}_k & 0 \\ 0 & -\mathbb{1}_{n-k} \end{smallmatrix}\right) = (-1)^{n-k}$.

We shall prove in the following that the Moore determinant is the unique hyperhermitian determinant. Along the way we establish some other interesting properties.

Proposition 1.41. *The Moore determinant is a hyperhermitian determinant.*

Proof. We start by proving that it satisfies axiom 1. Fix a matrix $H \in \mathrm{Hyp}(n)$ and an index $i \in \{1, \dots, n\}$. For every partition $(X_1, \dots, X_s) \in \mathcal{P}(n)$ there is only one set X_k containing i. Then, $H'_{X_r} = H_{X_r}$ for every $r \neq k$, where $H' = B_{ii}(q)HB_{ii}(\bar{q})$. It is therefore enough to prove that $H'_{X_k} = |q|^2 H_{X_k}$. In view of Remark 1.39, for all cycles $c = (c_1 \ \cdots \ c_l) \in C(X_k)$ we may assume $c_1 = i$, so that we have

$$H'_X = \sum_{c \in C(X_k)} |c|\, h'_{ic_2} h'_{c_2c_3} \cdots h'_{c_l i} = \sum_{c \in C(X_k)} |c|\, \bar{q} h_{ic_2} h_{c_2c_3} \cdots h_{c_l i} q = \bar{q} H_{X_k} q = |q|^2 H_{X_k} \, .$$

Now we prove that axiom 2 is satisfies. Let $H' = B_{ij}(1)HB_{ji}(1)$ for $i \neq j$. For every partition $(X_1, \dots, X_s) \in \mathcal{P}(n)$, since the terms H'_{X_k} are all real, up to renaming the sets forming the partition we may always assume without loss of generality that $i \in X_1$. Observe that $H'_{X_k} = H_{X_k}$ for every $k > 1$. Similarly, whenever $j \notin X_1$ we may assume $j \in X_2$. Then we split the sum as follows

$$\sum_{(X_1, \dots, X_s) \in \mathcal{P}(n)} H'_{X_1} H'_{X_2} \cdots H'_{X_s} = \sum_{\substack{(X_1, \dots, X_s) \in \mathcal{P}(n) \\ X_1 \ni i,j}} H'_{X_1} H_{X_2} \cdots H_{X_s} + \sum_{\substack{(Y_1, \dots, Y_r) \in \mathcal{P}(n) \\ Y_1 \ni i,\, Y_2 \ni j}} H'_{Y_1} H_{Y_2} \cdots H_{Y_r}$$

and, changing names of the sets in the partitions involved in the second sum on the right-hand side, we regroup it as

$$\sum_{(X_1, \dots, X_s) \in \mathcal{P}(n)} H'_{X_1} H'_{X_2} \cdots H'_{X_s} = \sum_{\substack{(X_1, \dots, X_s) \in \mathcal{P}(n) \\ X_1 \ni i,j}} \left(H'_{X_1} + \sum_{\substack{Y \ni i,\, Z \ni j \\ Y \cup Z = X_1}} H'_Y H_Z \right) H_{X_2} \cdots H_{X_s} \, . \tag{1.12}$$

Now, fixing the partition, suppose $X = X_1 = \{i, j, a_1, \dots, a_r\}$ possibly with $r = 0$. Fix a permutation $\tau \in S(X \setminus \{i, j\})$ and consider the following quantities

$$\begin{aligned}
A_{p,\tau} &:= H'_{(i \ \tau(a_1) \ \tau(a_2) \ \cdots \ \tau(a_p) \ j \ \tau(a_{p+1}) \ \cdots \ \tau(a_r))} = h'_{i\tau(a_1)} h'_{\tau(a_1)\tau(a_2)} \cdots h'_{\tau(a_p)j} h'_{j\tau(a_{p+1})} \cdots h'_{\tau(a_r)i} \\
&= \left(h_{i\tau(a_1)} + h_{j\tau(a_1)}\right) h_{\tau(a_1)\tau(a_2)} \cdots h_{\tau(a_p)j} h_{j\tau(a_{p+1})} \cdots h_{\tau(a_{r-1})\tau(a_r)} \left(h_{\tau(a_r)i} + h_{\tau(a_r)j}\right) \, , \\
B_{p,\tau} &:= H'_{(i \ \tau(a_{p+1}) \ \tau(a_{p+2}) \ \cdots \ \tau(a_r) \ j \ \tau(a_1) \ \cdots \ \tau(a_p))} \\
&= \left(h_{i\tau(a_{p+1})} + h_{j\tau(a_{p+1})}\right) h_{\tau(a_{p+1})\tau(a_{p+2})} \cdots h_{\tau(a_r)j} h_{j\tau(a_1)} \cdots h_{\tau(a_{p-1})\tau(a_p)} \left(h_{\tau(a_p)i} + h_{\tau(a_p)j}\right) \, , \\
C_{p,\tau} &:= H'_{(i \ \tau(a_1) \ \tau(a_2) \ \cdots \ \tau(a_p))} H_{(j \ \tau(a_{p+1}) \ \tau(a_{p+2}) \ \cdots \ \tau(a_r))} \\
&= \left(h_{i\tau(a_1)} + h_{j\tau(a_1)}\right) h_{\tau(a_1)\tau(a_2)} \cdots h_{\tau(a_{p-1})\tau(a_p)} \left(h_{\tau(a_p)i} + h_{\tau(a_p)j}\right) h_{j\tau(a_{p+1})} \cdots h_{\tau(a_r)j} \, , \\
D_{p,\tau} &:= H'_{(i \ \tau(a_{p+1}) \ \tau(a_{p+2}) \ \cdots \ \tau(a_r))} H_{(j \ \tau(a_1) \ \tau(a_2) \ \cdots \ \tau(a_p))} \\
&= \left(h_{i\tau(a_{p+1})} + h_{j\tau(a_{p+1})}\right) h_{\tau(a_{p+1})\tau(a_{p+2})} \cdots h_{\tau(a_{r-1})\tau(a_r)} \left(h_{\tau(a_r)i} + h_{\tau(a_r)j}\right) h_{j\tau(a_1)} \cdots h_{\tau(a_p)j} \, .
\end{aligned}$$

Observe that

$$A_{p,\tau} + B_{p,\tau} - C_{p,\tau} - D_{p,\tau} = h_{i\tau(a_1)} \cdots h_{\tau(a_p)j} h_{j\tau(a_{p+1})} \cdots h_{\tau(a_r)i} + h_{i\tau(a_{p+1})} \cdots h_{\tau(a_r)j} h_{j\tau(a_1)} \cdots h_{\tau(a_p)i} \, ,$$

therefore, multiplying by $(-1)^{r-1}$, summing over $p \in \{0, 1, \dots, r\}$ and $\tau \in S(X \setminus \{i, j\})$ and using again Remark 1.39 we obtain

$$2\left(H'_X + \sum_{\substack{Y\ni i,\, Z\ni j\\ Y\cup Z=X}} H'_Y H_Z\right) = (-1)^{r-1} \sum_{\tau\in S(X\setminus\{i,j\})} \sum_{p=0}^{r} (A_{p,\tau} + B_{p,\tau} - C_{p,\tau} - D_{p,\tau}) = 2H_X$$

which, substituted in (1.12) allows us to conclude.

The fact that axiom 3 is satisfied is obvious. □

Proposition 1.42. *Let $d\colon \mathrm{Hyp}(n) \to \mathbb{R}$ be a hyperhermitian determinant then for every $B \in \mathrm{SL}(n,\mathbb{H})$ and $H \in \mathrm{Hyp}(n)$ we have $d(B^*HB) = d(H)$.*

Proof. Since $\mathrm{SL}(n,\mathbb{H})$ is generated by elementary matrices it is enough to prove the proposition for them. In other words we need to check that whenever $H' \in \mathrm{Hyp}(n)$ is obtained from $H \in \mathrm{Hyp}(n)$ by adding to the i^{th} row the j^{th} multiplied on the left by $q \in \mathbb{H}$ and adding to the i^{th} column the j^{th} multiplied on the right by $\bar{q}$, then $d(H') = d(H)$. This follows from axioms 1 and 2, indeed, H' is obtained from H by performing the following operations:

- Multiply the i^{th} row on the left by q^{-1} and the i^{th} column on the right by $\bar{q}^{-1}$.
- Add the j^{th} row to the i^{th} and then the j^{th} column to the i^{th}.
- Multiply the i^{th} row on the left by q and the i^{th} column on the right by $\bar{q}$.

In terms of the determinant the first operation rescales it by a factor $|q^{-1}|^2 = |q|^{-2}$, the second does not affect it and the third rescales it by a factor $|q|^2$. □

As a consequence, simultaneous switching of two rows and the corresponding columns does not affect the value of a hyperhermitian determinant. This is because the matrices S^{ij}, encoding the operation of switching rows or columns according to the side of the multiplication, lie in $\mathrm{SL}(n,\mathbb{H})$ (and actually in $\mathrm{Sp}(n)$).

Corollary 1.43. *The Moore determinant is the unique hyperhermitian determinant and for every $H \in \mathrm{Hyp}(n)$*

$$\det(H) = \lambda_1\lambda_2\cdots\lambda_n\,,$$

where the λ_i's are the eigenvalues of H.

Proof. Let d be a hyperhermitian determinant and $H \in \mathrm{Hyp}(n)$. It follows from the Spectral Theorem (Theorem 1.34) that there exists $P \in \mathrm{Sp}(n)$ such that P^*HP is diagonal. Let $\lambda_1,\dots,\lambda_n$ be the diagonal entries, then by Proposition 1.42 and axioms 1 and 3 we have $d(H) = d(S^*HS) = \lambda_1\cdots\lambda_n$. Therefore the hyperhermitian determinant is unique and necessarily equal to the Moore determinant. □

Corollary 1.44. *For every $M \in \mathbb{H}^{n,n}$ and $H \in \mathrm{Hyp}(n)$ we have*

$$|\det(H)| = \mathrm{Ddet}(H)\,, \qquad \mathrm{Sdet}(M) = \mathrm{Ddet}(M)^2\,, \qquad \det(M^*M) = \mathrm{Sdet}(M)\,.$$

Proof. The first two formulas are an immediate consequence of Proposition 1.36. To prove the third observe that $\mathrm{Ddet}(M^*) = \mathrm{Ddet}(M)$ and that M^*M is positive semi-definite, hence $\det(M^*M) \geq 0$ (see Proposition 1.45), therefore

$$\det(M^*M) = |\det(M^*M)| = \mathrm{Ddet}(M^*M) = \mathrm{Ddet}(M^*)\mathrm{Ddet}(M) = \mathrm{Ddet}(M)^2 = \mathrm{Sdet}(M)\,.$$ □

The identity between Moore and Study determinants can be used to extend the Study determinant to non-square matrices $M \in \mathbb{H}^{n,m}$ by setting $\mathrm{Sdet}(M) := \det(M^*M)$. This is sometimes called the *double determinant* (see [330] and references therein).

Further references discussing properties of the Moore determinant and its relations with other determinants are [9, 78, 188, 230, 243, 297, 298, 299]. Finally, we mention that the Moore determinant also admits a definition in terms of the Pfaffian (see [115, 116, 271]).

Eigenvalues of hyperhermitian matrices.

Proposition 1.45. *A matrix $H \in \mathrm{Hyp}(n)$ is positive semidefinite (resp. definite) f and only if all its eigenvalues are non-negative (resp. positive).*

Proof. The proposition follows from Lemma 1.28, Proposition 1.31 and the analogue result for complex matrices, however, we shall give here a direct proof.

If H is positive semidefinite (resp. definite) and $\lambda \in \mathbb{R}$ is one of its eigenvalues with eigenvector $q \in \mathbb{H}^n$, then $\lambda \|q\|^2 = \langle q, q\lambda\rangle = \langle q, Hq\rangle \geq 0$ (resp. > 0) implying $\lambda \geq 0$ (resp. $\lambda > 0$).

Conversely, suppose all eigenvalues $\lambda_1, \dots, \lambda_n$ of H are non-negative (resp. positive). Let $U \in \mathrm{Sp}(n)$ be such that $D = U^*HU$ is diagonal with $\lambda_1, \dots, \lambda_n$ as its entries. For a non-zero $q \in \mathbb{H}^n$ let $p = U^*q = U^{-1}q$, then we have $\langle q, Hq\rangle = \langle Up, HUp\rangle = \langle p, U^*HUp\rangle = \langle p, Dp\rangle = \sum_{k=1}^n \lambda_k |p_k|^2 \geq 0$ (resp. > 0), where $p = (p_1, \dots, p_n)$. □

Sometimes, we shall write $H > 0$ and $H \geq 0$ to mean that the matrix $H \in \mathrm{Hyp}(n)$ is positive definite and positive semidefinite respectively. On the set of hyperhermitian matrices we can thus define a partial order relation by setting

$$H_1 \geq H_2 \quad \Longleftrightarrow \quad H_1 - H_2 \geq 0\,,$$

from Lemma 1.28 it is clear that $H_1 \geq H_2$ if and only if $\beta(H_1) \geq \beta(H_2)$ if and only if $\gamma(H_1) \geq \gamma(H_2)$. The inequality $H_1 \geq H_2$ imposes the following constraint in terms of the eigenvalues: denote $\lambda_1^i \geq \lambda_2^i \geq \cdots \geq \lambda_n^i$ the eigenvalues of H_i for $i = 1, 2$ then

$$\lambda_r^1 \geq \lambda_r^2\,, \qquad \text{for all } r = 1, \dots, n\,. \tag{1.13}$$

Be aware that the converse does not hold, for instance $H_1 = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and $H_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ satisfy the inequality (1.13) but it is not true that $H_1 - H_2 \geq 0$.

The sum of the eigenvalues of a hyperhermitian matrices is equal to its trace. This is evidently true for real diagonal matrices. Given any hyperhermitian matrix $H \in \mathrm{Hyp}(n)$, from the Spectral Theorem there exists a matrix $P \in \mathrm{Sp}(n)$ such that $D = P^*HP = P^{-1}HP$ is real diagonal with the eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal. However, for matrices over the quaternions the usual identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ is no longer true in general. On the other side, if we take real parts it does become true, as a consequence of the fact that $\mathrm{Re}(pq) = \mathrm{Re}(qp)$ for every $p, q \in \mathbb{H}$. Therefore

$$\mathrm{tr}(H) = \mathrm{Re}\,\mathrm{tr}(H) = \mathrm{Re}\,\mathrm{tr}(PDP^{-1}) = \mathrm{Re}\,\mathrm{tr}(DP^{-1}P) = \mathrm{tr}(D) = \sum_{i=1}^n \lambda_i\,.$$

In particular the eigenvalues and the diagonal entries of a hyperhermitian matrix are strictly related. Wondering whether or not it is possible to construct a hyperhermitian matrix with arbitrary eigenvalues and diagonal, one is led to the following:

Proposition 1.46 (Schur-Horn Theorem)**.** *Let $\mu = (\mu_1, \dots, \mu_n), \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ be such that $\mu_1 \geq \cdots \geq \mu_n$ and $\lambda_1 \geq \cdots \geq \lambda_n$. There exists a hyperhermitian matrix B with diagonal μ and eigenvalues λ if and only if*

$$\sum_{i=1}^j \mu_i \leq \sum_{i=1}^j \lambda_i\,, \qquad \textit{for all } j = 1, \dots, n \qquad \textit{and} \qquad \sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i\,. \tag{1.14}$$

Proof. A hyperhermitian matrix B satisfies the assumptions of the lemma if and only if there exists $C \in \mathrm{Sp}(n)$ such that $B = C^*DC$ where D is the diagonal matrix with diagonal λ. In particular μ is the diagonal of B if and only if $\mu = T\lambda$ where $T = (|c_{rs}|^2)$. Since $C \in \mathrm{Sp}(n)$, the matrix T is doubly stochastic. By the Birkhoff theorem [40] $\mu = T\lambda$, where T is doubly stochastic, if and only if T lies in the convex hull of the set of all permutation matrices. In other words B exists if and only if μ lies in the convex hull of the vectors obtained by permuting the entries of λ, which is known to be equivalent to (1.14) (see e.g. [164, Theorem 46]). □

Differentiating determinants.

We shall prove a Jacobi's formula for the Study determinant which follows from the usual Jacobi's formula for the complex determinant. Let $A \in \mathrm{GL}(n, \mathbb{H})$ and $B \in \mathbb{H}^{n,n}$, then

$$\begin{aligned}\mathrm{Sdet}_*|_A(B) &= \frac{d}{dt}\mathrm{Sdet}(A+tB)\big|_{t=0} = \frac{d}{dt}\det(\beta(A+tB))\big|_{t=0} = \det(\beta(A))\mathrm{tr}\left(\beta(A)^{-1}\beta(B)\right) \\ &= \mathrm{Sdet}(A)\mathrm{tr}\left(\beta\left(A^{-1}B\right)\right) = 2\,\mathrm{Sdet}(A)\mathrm{Re}\,\mathrm{tr}\left(A^{-1}B\right)\end{aligned}$$

We stress here that the parameter t can only be real. Since the Dieudonné determinant is the positive square root of the Study determinant we also deduce

$$\mathrm{Ddet}_*|_A(B) = \frac{d}{dt}\mathrm{Ddet}(A+tB)\big|_{t=0} = \frac{d}{dt}\mathrm{Sdet}(A+tB)^{1/2}\big|_{t=0} = \mathrm{Ddet}(A)\mathrm{Re}\,\mathrm{tr}\left(A^{-1}B\right) .$$

If we also assume that A and B are positive definite, then so is $A + tB$, for $t > 0$, therefore $\det(A + tB) = \mathrm{Ddet}(A + tB)$ and we conclude

$$\det_*|_A(B) = \det(A)\mathrm{Re}\,\mathrm{tr}\left(A^{-1}B\right) .$$

Choosing $A = A(t)$ and $B = \frac{d}{dt}A(t)$ we also obtain:

Proposition 1.47. *Let $A(t)$ be a curve in the cone of positive definite hyperhermitian matrices. Then the following* ***Jacobi's formula*** *holds:*

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\mathrm{Re}\,\mathrm{tr}\left(A(t)^{-1}\frac{d}{dt}A(t)\right) . \tag{1.15}$$

CHAPTER 2

HYPERCOMPLEX AND HKT MANIFOLDS

In this second chapter we introduce the kind of geometry with torsion which is the main object of our study. HKT manifolds, where HKT is a shorthand for *hyperkähler with torsion*, arose from supersymmetric theoretical physics [178] and soon became object of study from the mathematical point of view [148]. They belong to the realm of hypercomplex geometry and present strong and deep similarities with Kähler manifolds. HKT geometry is rich and worthwhile to investigate.

Here, we approach HKT manifolds from the point of view of geometrical G-structures and then move on to describe their basic properties, describing the notion of HKT forms and HKT potentials. Afterwards we briefly explore Hodge theory and cohomology. An important role is played by the canonical bundle which always admits smooth global sections but these in general are not holomorphic. Some relations between the canonical bundle and the holonomy of the Obata connection (the unique torsion free connection preserving the hypercomplex structure [235]) are established, but their link is not fully understood yet, hence some conjectures emerge naturally.

One of these conjectures, introduced by Alesker and Verbitsky [18], is the perfect analogue of the Calabi conjecture and is equivalent to an equation of Monge-Ampère–type. Its solution would give a way to find balanced HKT metrics, which play the role of "quaternionic Calabi-Yau" metrics. Attacking the Calabi problem in HKT geometry seems to be more complicated then its complex counterpart and is the main focus of this work.

2.1 Preliminary notions.

We briefly introduce the framework of G-structures, which essentially goes back to Cartan [70, 73] and was carefully implemented by Chern [81, 82]. Subsequently, we describe complex and hypercomplex structures in order to fix notations and conventions.

For more information on G-structures consult [196, 228, 272]. Essential references for quaternionic G-structures are [4, 5, 6, 7, 8, 252, 253].

2.1.1 G-structures and integrability.

Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A **$G$-structure** on a smooth manifold M is a reduction of the structure group of the frame bundle $\mathrm{GL}(M)$ from $\mathrm{GL}(n, \mathbb{R})$ to G.

If G can be described as the stabilizer of a tensor T on $\mathbb{R}^n$ then there is a one-to-one correspondence between G-structures on M and global tensor fields S on M that allow a pointwise identification $(T_xM, S_x) \cong (\mathbb{R}^n, T)$ (see [196, Proposition 1.1]). The argument generalizes to groups G that are common stabilizers of a finite number of tensor fields. For instance, this means that $\mathrm{O}(n)$-structures on M are in one-to-one correspondence with Riemannian metrics.

Tensor bundles can be realized as vector bundles associated to $\mathrm{GL}(M)$ (via a representation induced by the canonical representation $\rho\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^n)$ which, for simplicity, we still call ρ). In particular there is a one-to-one correspondence between connections on the frame bundle and linear connections on TM. Furthermore, any linear connection ∇ on TM induces a linear connection on all tensor bundles (and by abuse of notation we will still denote such connections with ∇).

Let $\pi\colon Q \to M$ be a G-structure on M. A connection on Q can always be extended to a connection on $\mathrm{GL}(M)$ that in turn induces a linear connection on TM (cf. [197, Chapter II, Proposition 6.1]). However the converse breaks down: given a linear connection ∇ on TM the corresponding connection on $\mathrm{GL}(M)$ does not always reduce to Q, when this occurs we say that ∇ and Q are **compatible** and we call ∇ a **G-connection**.

Given a linear connection ∇ on M and $u \in \mathrm{GL}(M)$ we can find a G-structure Q compatible with ∇ such that $u \in Q$ if and only if the holonomy group $\mathrm{Hol}(\nabla)$ is contained in G. If it exists Q is unique (see [194, Proposition 2.6.3]).

This fact, together with the Holonomy Principle allows to deduce the following: let ∇ be a linear connection on TM and assume M admits a G-structure, where G is isomorphic to the common stabilizer of a finite number of tensor fields $S_1, \dots, S_k$. Then ∇ is a G-connection if and only if $S_1, \dots, S_k$ are parallel with respect to ∇, moreover whenever this happens $\mathrm{Hol}(\nabla) \subseteq G$.

Now we fix a G-structure $\pi\colon Q \to M$ and inspect the space of G-connections on Q. It is easy to see that the difference of two linear connections is tensorial and hence it is a section of $\mathrm{End}(M) \otimes T^*M$. We expect the difference of two G-connections to take values in a smaller space. Define the *adjoint bundle* $\mathrm{Ad}(Q)$ as the vector bundle associated to Q via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. It can be seen that the set of all G-connections is an affine space over $\Gamma(T^*M \otimes \mathrm{Ad}(Q))$ (see [253, p. 16]). Thus, in general, many different G-connections can be found on Q. What about torsion-free G-connections then?

Let ∇ and $\tilde{\nabla}$ be two G-connections, then $\alpha := \nabla - \tilde{\nabla} \in \Gamma(T^*M \otimes \mathrm{Ad}(Q))$, since $\mathrm{Ad}(Q)$ has standard fiber $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R}) \cong \mathrm{End}(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ we can regard $\mathrm{Ad}(Q)$ as a vector subbundle of $T^*M \otimes TM$, in particular α is a tensor field of type $(1,2)$. It follows that $T^{\nabla} - T^{\tilde{\nabla}} = \delta(\alpha)$, where $\delta\colon T^*M \otimes \mathrm{Ad}(Q) \to \Lambda^2 T^*M \otimes TM$ is the antisymmetrization on the covariant components. Explicitly: $\delta(\alpha)(X,Y) = \alpha_X Y - \alpha_Y X$ for every $X, Y \in TM$, where we use the notation $\alpha_X := \alpha(X) \in \mathrm{Ad}(Q)$.

Definition 2.1. Let $\pi\colon Q \to M$ be a G-structure and ∇ a G-connection on TM. The **intrinsic torsion** of Q is the equivalence class

$$\tau(Q) := \left[T^{\nabla}\right] \in \Gamma\left(\Lambda^2 T^*M \otimes TM \big/ \delta(T^*M \otimes \mathrm{Ad}(Q)) \right).$$

The G-structure Q is said to be **torsion-free** if $\tau(Q) = 0$.

It follows from our discussion that the definition depends only on the G-structure Q. The set of torsion-free G-connections on Q is in one-to-one correspondence with $\mathrm{Ad}(Q)^{(1)} := \mathrm{Ker}(\delta)$ called the **first prolongation** of $\mathrm{Ad}(Q)$. The intrinsic torsion is thus a measure of how obstructed the G-structure is to allow a compatible torsion-free connection. A manifold M admits a torsion-free G-structure if and only if there is a torsion-free linear connection ∇ on TM such that $\mathrm{Hol}(\nabla) \subseteq G$ (see [194, Proposition 2.6.3]).

Whenever $G \subseteq \mathrm{O}(n)$ is a closed Lie subgroup defined as the stabilizer of some tensors $S_1, \dots, S_k$ we can describe the intrinsic torsion of any G-structure Q on M in terms of the covariant derivative of those tensors with respect to the Levi-Civita connection of the Riemannian metric induced by the G-structure (see [253, Corollary 2.2]).

It turns out that the intrinsic torsion is just the first of a whole family of intrinsic objects that one can take into account on G-structures as obstructions to integrability. For instance, the difference of the curvatures of two torsion-free G-connections ∇ and $\tilde{\nabla} = \nabla + \alpha$ lies in $\mathrm{Im}(\delta^{(1)})$, where here

$$\delta^{(1)}\colon T^*M \otimes \mathrm{Ad}(Q)^{(1)} \to \Lambda^2 T^*M \otimes \mathrm{Ad}(Q)$$

denotes the antisymmetrization on the first prolongation $\mathrm{Ad}(Q)^{(1)} \subseteq T^*M \otimes \mathrm{Ad}(Q)$ of the adjoint bundle, i.e. $\delta^{(1)}(\beta)(X,Y) = \beta_X Y - \beta_Y X$. Moreover, the curvature of a torsion-free connection satisfies the Bianchi identity: $\delta'(R^\nabla) = 0$, where

$$\delta' \colon \Lambda^2 T^*M \otimes \mathrm{Ad}(Q) \to \Lambda^3 T^*M \otimes TM$$

is the map $\delta'(\gamma)(X,Y,Z) := \gamma(X,Y)Z + \gamma(Y,Z)X + \gamma(Z,X)Y$. Hence the equivalence class $[R^\nabla]$ modulo $\mathrm{Im}(\delta^{(1)})$ in $\mathrm{Ker}(\delta')$ does not depend on ∇ but only on the G-structure.

Definition 2.2. Let $\pi \colon Q \to M$ be a torsion-free G-structure. The **intrinsic curvature** of Q is the equivalence class

$$R(Q) := \left[R^\nabla\right] \in \Gamma\left({\mathrm{Ker}(\delta')}\big/{\mathrm{Im}(\delta^{(1)})} \right),$$

where ∇ is any torsion-free G-connection on TM.

An *isomorphism* of two G-structures $\pi_1 \colon Q_1 \to M_1$ and $\pi_2 \colon Q_2 \to M_2$ is a diffeomorphism $f \colon M_1 \to M_2$ such that $f_*(Q_1) = Q_2$, where f_* is the induced diffeomorphism between $\mathrm{GL}(M_1)$ and $\mathrm{GL}(M_2)$. So, for example, an isomorphism of two $\mathrm{O}(n)$-structures is an isometry. An isomorphism of a G-structure with itself is called an *automorphism*. Two G-structures as above are called *locally equivalent* if at any point $x \in M_1$ there exists a neighborhood U and an isomorphism f such that the restriction of Q_1 over U is isomorphic to the restriction of Q_2 over $f(U)$.

A very relevant topic in the theory of G-structures is the *local equivalence problem*, which focuses on characterizing the conditions under which two G-structures are locally equivalent. Of course, a case of particular interest is that of local equivalence with the standard flat structure. The *standard flat G-structure* on $\mathbb{R}^n$ is the set of all frames obtained from the standard one by the action of $G \subseteq \mathrm{GL}(n,\mathbb{R})$.

Definition 2.3. A G-structure is called **integrable** or **locally flat** if it is locally equivalent to the standard flat G-structure.

Integrability of a G-structure Q can also be equivalently defined by saying that around any point $x \in M$ one can find *admissible* local coordinates $(x^1, \dots, x^n)$, where the word admissible signifies that the corresponding frame $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ seen as a cross section of $\mathrm{GL}(M)$ over U actually is a cross section of Q over U.

Whenever G is the stabilizer of some tensor $T \in T^r_s(\mathbb{R}^n)$, we have the following neat description: let $\pi \colon Q \to M$ be a G-structure over M, then Q is integrable if and only if each point of M allows local coordinates with respect to which the tensor $S \in T^r_s(M)$ induced by T has constant components (see [196, Proposition 1.1]).

The integrability problem is strictly related to the intrinsic torsion and curvature; as a matter of fact, they belong to a whole tower of tensors $\tau^k(Q)$, called the $k^{\mathbf{th}}$ **(intrinsic) structure tensors**, the non-vanishing of which represents an obstruction for our G-structure to take locally the model form as on $\mathbb{R}^n$.

Let Q be a G-structure on M. We say that Q is m**-integrable** if $\tau^k(Q) = 0$ for all $k < m$ and **formally integrable** if $\tau^k(Q) = 0$ for all k. The standard flat G-structure has vanishing intrinsic structure tensors, therefore we have that formal integrability is necessary in order to have local flatness. The converse is in general false (see [160]), however, it is true in many reasonable cases, for instance it is true when the Lie algebra of G is of finite type. Let us explain what it means. Consider again the map $\delta \colon (\mathbb{R}^n)^* \otimes \mathfrak{g} \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ and define inductively the $k^{\mathbf{th}}$ **prolongation** of $\mathfrak{g}$ as $\mathfrak{g}^{(k)} := \left(\mathfrak{g}^{(k-1)}\right)^{(1)} \subseteq (\mathbb{R}^n)^* \otimes \mathfrak{g}^{(k-1)}$, then $\mathfrak{g}$ is of finite type if $\mathfrak{g}^{(j)} = 0$ for some j.

Theorem 2.4 (Guillemin [159])**.** *If the Lie algebra $\mathfrak{g}$ of G is of finite type, then a G-structure is integrable if and only if it is formally integrable.*

2.1.2 Complex structures.

Before discussing G-structures related to quaternionic groups, we briefly review some facts of complex and Kähler geometry which will be helpful to fix notations and conventions.

$\mathrm{GL}(n, \mathbb{C})$**-structures.**

We recall that the group $\mathrm{GL}(n, \mathbb{C})$ can be seen as the stabilizer of a $(1,1)$-tensor I_0 on $\mathbb{R}^{2n}$ acting on $\mathrm{GL}(2n, \mathbb{R})$ as $A \mapsto I_0 A I_0^{-1}$. Therefore a $\mathrm{GL}(n, \mathbb{C})$-structure on a $2n$-dimensional manifold M is determined by an endomorphism of the tangent bundle I that squares to $-\mathrm{Id}_{TM}$. The $(1,1)$-tensor I is called an *almost complex structure*.

The *Nijenhuis tensor* associated to the almost complex structure I is defined as

$$N_I(X,Y) := \frac{1}{4}\left([X,Y] + I[IX,Y] + I[X,IY] - [IX,IY]\right) .$$

It is well-known that a $\mathrm{GL}(n, \mathbb{C})$-structure $\pi\colon Q \to M$ is torsion-free if and only if $N_I = 0$, where I is the almost complex structure induced by Q. The intrinsic torsion is the only obstruction to the integrability of the $\mathrm{GL}(n, \mathbb{C})$-structure. This is the celebrated Theorem of Newlander and Nirenberg [232]. Whenever a $\mathrm{GL}(n, \mathbb{C})$-structure is torsion-free we say that the induced almost complex structure I is *integrable*, or we simply drop the term “almost” and call it a *complex structure* on M. The pair (M, I) is accordingly referred to as an **(almost) complex manifold**.

The complexified tangent space $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$ of an almost complex manifold (M, I) decomposes in a direct sum of eigenbundles of I: $TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$, where $I|_{T^{1,0}M} = i\,\mathrm{Id}$ and $I|_{T^{0,1}M} = -i\,\mathrm{Id}$. Moreover

$$T^{1,0}M = \{X - iIX \mid X \in TM\} , \qquad T^{0,1}M = \{X + iIX \mid X \in TM\} = \overline{T^{1,0}M} .$$

Consequently the vector bundles of complex differential forms decompose naturally as

$$\Lambda^k T^*M_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q}T^*M ,$$

where $\Lambda^{p,q}T^*M := \Lambda^p(T^*M)^{1,0} \otimes_{\mathbb{C}} \Lambda^q(T^*M)^{0,1}$ is the space of (complex) forms of type (or bidegree) (p,q) (with respect to I) or simply (p,q)-forms. Note that $\overline{\Lambda^{p,q}T^*M} = \Lambda^{q,p}T^*M$. We will write

$$\Lambda^k(M) := \Gamma(\Lambda^k T^*M) , \qquad \Lambda^{p,q}(M) := \Gamma(\Lambda^{p,q}T^*M)$$

for the corresponding spaces of sections, which we still call k-forms and (p,q)-forms respectively. When we have multiple (almost) complex structures, we may denote with a subscript the one with respect to which we are considering forms and fields, for instance $\Lambda^{p,q}_I(M)$ and $T^{1,0}_I M$ are the space of (p,q)-forms and the $(1,0)$-tangent bundle with respect to I.

The wedge product sends a (p,q)-form and a (r,s)-form to a $(p+r, q+s)$-form and the almost complex structure extends to a map on differential k-forms as

$$(I\beta)(X_1, \dots, X_k) = \beta(IX_1, \dots, IX_k) .$$

Be aware that some authors prefer to define the action of an almost complex structure on k-forms by setting $(I\beta)(X_1, \dots, X_k) = \beta(I^{-1}X_1, \dots, I^{-1}X_k)$.

The $\mathbb{C}$-linear extension $d\colon \Lambda^k(M)_{\mathbb{C}} \to \Lambda^{k+1}(M)_{\mathbb{C}}$ of the exterior differential satisfies

$$d\colon \Lambda^{p,q}(M) \to \Lambda^{p+2,q-1}(M) \oplus \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \oplus \Lambda^{p-1,q+2}(M).$$

This means that we can decompose it accordingly as $d = \mu + \partial + \bar\partial + \bar\mu$. On an almost complex manifold (M, I) the following are equivalent:

(i) I is integrable;

(ii) $d = \partial + \bar\partial$;

(iii) $\mu = 0$.

When any of the previous conditions holds, then $d^2 = 0$ implies $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$. We call ∂ and $\bar\partial$ the *Dolbeault operators* (with respect to I).

Sometimes it will also be useful to take into account the twisted exterior differential operator $d^c_I := d^c_I = I^{-1}dI$ which satisfies $d^c_I = i(\bar{\partial} - \partial)$ as well as $dd^c_I + d^c_I d = 0$ (if and only if I is integrable).

U(n)-structures.

A complex structure I_0 on the vector space $\mathbb{R}^{2n}$ is compatible with an inner product g_0 if it is orthogonal, i.e. $g_0(I_0\cdot, I_0\cdot) = g_0$. In this case, the tensor

$$\omega_{I_0} := g_0(I_0\cdot, \cdot)$$

becomes a 2-form. The group U(n) can be seen as the stabilizer of any two of I_0, g_0, ω_{I_0}:

$$\mathrm{U}(n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{C})\,, \tag{2.1}$$

therefore a U(n)-structure on a manifold M is given by a Riemannian metric g, a g-orthogonal almost complex structure I and the non-degenerate 2-form $\omega_I := g(I\cdot, \cdot)$, called the *fundamental form*. Any two of these objects determines the third.

Whenever g satisfies $g(I\cdot, I\cdot) = 0$ we say that it is *Hermitian*. An almost complex manifold always admits a Hermitian metric as, given a Riemannian metric g the metric $\frac{1}{2}(g + g(I\cdot, I, \cdot))$ is Hermitian. We call the triple (M, I, g), where g is Hermitian, an **almost Hermitian manifold** and an **Hermitian manifold** whenever I is integrable.

The intrinsic torsion of (g, I, ω_I) identifies with $\nabla^g I$ or, equivalently, $\nabla^g \omega_I$, where ∇^g is the Levi-Civita connection with respect to g. For $n \geq 3$ representation theory allows to split the space where the intrinsic torsion lives in the direct sum of four spaces W_1, W_2, W_3, W_4 which are irreducible with respect to the action of U(n). This allows to classify U(n)-structures into $2^4 = 16$ classes determined by suitable conditions imposed on I and ω_I. This has been done by Alfred Gray and Luis M. Hervella [149].

The following conditions on an almost Hermitian manifold (M, I, g) are equivalent:

$$\nabla^g I = 0\,, \qquad \nabla^g \omega_I = 0\,, \qquad N_I = 0 \text{ and } d\omega_I = 0$$

and they are satisfied if and only if the U(n)-structure is torsion-free, in which case it is called a *Kähler structure*, (M, I, g) is called a **Kähler manifold** and $\mathrm{Hol}(\nabla^g) \subseteq \mathrm{U}(n)$.

By uniqueness of the Levi-Civita connection, on a Kähler manifold there is only one U(n)-connection. Gauduchon [135] studied U(n)-connections on Hermitian manifolds showing that they form an affine subspace of the space of linear connections. Among these, Gauduchon also singled out and studied the so-called *canonical Hermitian connections*, distinguished by some constraints imposed on the torsion tensor. Nowadays these are also known as *Gauduchon connections*. The space of canonical connections is at most one dimensional and collapses to a single point if and only if the Hermitian manifold is Kähler. When the manifold is not Kähler the family of Gauduchon connections includes many distinguished interesting connections.

As an example we mention the **Bismut connection**. The name stems from the fact that Bismut [41] used it to prove a local index theorem, however it has been pointed out that it was Strominger [273] that first discussed its existence three years before, giving it the name H-connection.

The Bismut connection is the U(n)-connection ∇^I on a Hermitian manifold (M, I, g) whose torsion is totally skew-symmetric, in the sense that the tensor

$$c(X, Y, Z) := g(X, T^\nabla(Y, Z))$$

is a 3-form. Since ∇^I preserves the metric, it is uniquely determined by its torsion which can be expressed in terms of the fundamental form as

$$c(X, Y, Z) = d^c_I \omega_I(X, Y, Z)\,. \tag{2.2}$$

The Bismut connection is sometimes called by physicists the KT connection (which is a shorthand for *Kähler with torsion*). If the torsion c is closed we say that the KT-structure is *strong*, otherwise we call it *weak*. Strong Kähler with torsion (SKT) structures represent one of the most studied classes of $\mathrm{U}(n)$-structures with non-vanishing intrinsic torsion.

$\mathrm{SU}(n)$**-structures.**

Let us take coordinates $(z^1,\dots,z^n)$ on $\mathbb{C}^n$, inducing the coframe $(dz^1,\dots,dz^n)$. Define

$$\langle\cdot,\cdot\rangle_0=\sum_{k=1}^n dz^k\odot d\bar z^k\,,\qquad I_0\frac{\partial}{\partial z^r}=i\frac{\partial}{\partial z^r}\,(r=1,\dots,n)\,,\qquad \omega_{I_0}=\frac{i}{2}\sum_{k=1}^n dz^k\wedge d\bar z^k\,,$$

where $dz^k\odot d\bar z^k:=\frac{1}{2}(dz^k\otimes d\bar z^k+d\bar z^k\otimes dz^k)$. Then one can check that $(\langle\cdot,\cdot\rangle_0,I_0,\omega_{I_0})$ is a $\mathrm{U}(n)$-structure and $h=\langle\cdot,\cdot\rangle_0-i\omega_{I_0}$ is the standard Hermitian product on $\mathbb{C}^n$. The Lie group of special unitary matrices is by definition the intersection of $\mathrm{U}(n)$ and $\mathrm{SL}(n,\mathbb{C})$. Since $\mathrm{SL}(n,\mathbb{C})$ can be seen as the stabilizer in $\mathrm{GL}(n,\mathbb{C})$ of the complex volume form of type $(n,0)$

$$\psi_0:=dz^1\wedge dz^2\wedge\cdots\wedge dz^n\,,$$

a $\mathrm{SU}(n)$-structure (also called *special Hermitian structure*) on a smooth $2n$-dimensional manifold M is defined by the data of a $\mathrm{U}(n)$-structure (g,I,ω_I) together with a nowhere vanishing complex $(n,0)$-form ψ satisfying the normalization condition:

$$\psi\wedge\bar\psi=(-1)^{\frac{n(n+1)}{2}}\frac{(2i)^n}{n!}\omega^n.$$

For $n=3$ we have $\mathrm{SU}(3)=\mathrm{Sp}(6,\mathbb{R})\cap\mathrm{SL}(n,\mathbb{C})$ and an $\mathrm{SU}(3)$-structure can be completely described in terms of the symplectic form ω and the real part of the complex volume form ψ (see [172]).

The $\mathrm{SU}(n)$-structure is torsion-free if and only if

$$\nabla^g\omega=\nabla^g\psi=0\,,$$

for $n\geq 4$ these conditions are also equivalent to $d\omega=d(\mathrm{Re}(\psi))=0$ (see [62]), while for $n=3$ they are equivalent to $d\omega=d(\mathrm{Re}(\psi))=d(\mathrm{Im}(\psi))=0$ (see [88]). Observe that $\mathrm{SU}(1)=1$ and $\mathrm{SU}(2)\cong\mathrm{Sp}(1)$, to be treated below. Whenever the $\mathrm{SU}(n)$-structure is torsion-free we have $\mathrm{Hol}(g)\subseteq\mathrm{SU}(n)$ and actually, for a Kähler manifold (M,g,J) we have $\mathrm{Hol}^0(g)\subseteq\mathrm{SU}(n)$ if and only if $\mathrm{Ric}_g\equiv 0$ (see [194, Proposition 7.1.1]).

When a manifold M is equipped with a torsion-free $\mathrm{SU}(n)$-structure it is called a **Calabi-Yau manifold** this is because the groundbreaking Calabi-Yau theorem was historically crucial in order to find compact examples (cf. Chapter 2.3).

2.1.3 Hypercomplex structures.

Here, we begin the treatment of hypercomplex structures, starting to apply the machinery of quaternionic linear algebra developed in the first chapter.

$\mathrm{GL}(n,\mathbb{H})$**-structures.**

In this subsection we start dealing with $\mathrm{GL}(n,\mathbb{H})$-structures. On the flat space $\mathbb{H}^n$ we can take into account the standard left hypercomplex structure. Choosing quaternionic coordinates $(q^1,\dots,q^n)$, i.e. a basis for the left $\mathbb{H}$-module $\mathbb{H}^n$ we may write them in terms of the corresponding real coordinates $(x_0^1,\cdots,x_0^n,x_1^1,\cdots,x_1^n,x_2^1,\cdots,x_2^n,x_3^1,\cdots,x_3^n)$, where $q^r=x_0^r+ix_1^r+jx_2^r+kx_3^r$. Under this identification $\mathbb{H}^n\cong\mathbb{R}^{4n}$ the hypercomplex structure (I_0,J_0,K_0) induced on $\mathbb{R}^{4n}$ acts as $I_0(x_0^r)=x_1^r$, $I_0(x_2^r)=x_3^r$, $J_0(x_0^r)=x_2^r$, $J_0(x_1^r)=-x_3^r$ and $K_0(x_0^r)=x_3^r$, $K_0(x_1^r)=x_2^r$ for $r=1,\dots,n$. In terms of (block) matrices

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & -\mathbb{1} & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where $\mathbb{1} = \mathbb{1}_n$ is the $n \times n$ identity matrix. Equivalently, I_0, J_0, K_0 correspond to the action of the unit quaternions i, j, k on $\mathbb{R}^{4n}$ under the map $\gamma\colon \mathbb{H}^n \to \mathbb{R}^{4n}$ introduced in Subsection 1.1.2. From this perspective we can write $\mathrm{GL}(n, \mathbb{H})$ as the stabilizer of (I_0, J_0, K_0):

$$\mathrm{GL}(n, \mathbb{H}) = \left\{ A \in \mathrm{GL}(4n, \mathbb{R}) \mid A = I_0^{-1} A I_0 = J_0^{-1} A J_0 = K_0^{-1} A K_0 \right\} . \tag{2.4}$$

This leads to the following definition: an *almost hypercomplex structure* on a smooth manifold M is a $\mathrm{GL}(n, \mathbb{H})$-structure, i.e. a triple (I, J, K) of almost complex structures satisfying the quaternionic relations:

$$IJ = -JI = K . \tag{2.5}$$

A *hypercomplex structure* on M is an almost hypercomplex structure with I, J and K integrable. An **(almost) hypercomplex manifold** is a smooth manifold equipped with an (almost) hypercomplex structure.

Remark 2.5. In recent literature some authors prefer to choose the right action of the almost hypercomplex structure (I, J, K) on the tangent bundle, making TM into a right $\mathbb{H}$-module. As said in precedence, this agrees with the most common convention for $\mathbb{H}$-modules, which has indeed some practical advantages in that framework. However, the traditional choice in hypercomplex geometry is the one that makes TM into a left $\mathbb{H}$-module. In the present work, we adhere to the second convention and let (I, J, K) act on the left. Be aware that the choice of the side of the action results in some sign differences in certain formulas and definitions.

If a manifold M allows an almost hypercomplex structure its dimension must be a multiple of 4. From (2.5) it is evident that almost hypercomplex structures can be defined in terms of two of the three complex structures, being the third defined *a posteriori* as the product of the other two. Furthermore, integrability of these two almost complex structures implies integrability of the third, indeed we have the following result originally proved by Obata [235]:

Proposition 2.6. *Let (M, I, J, K) be an almost hypercomplex manifold. Then, if any two of the Nijenhuis tensors vanish then so does the third.*

Proof. The proposition follows from the identities

$$N_I = 2P_I(N_J + N_K) , \qquad N_J = 2P_J(N_K + N_I) , \qquad N_K = 2P_K(N_I + N_J) ,$$

where P_L is the projection of a tensor of type $(1, 2)$ on its $(0, 2)$-component with respect to the almost complex structure L:

$$P_L(S) := \frac{1}{4} \left(S + LS(\cdot, L\cdot) + LS(L\cdot, \cdot) - S(L\cdot, L\cdot) \right) . \qquad \square$$

We now address the integrability problem for $\mathrm{GL}(n, \mathbb{H})$-structures. First of all, we prove that they are of finite type. More precisely, the intrinsic torsion and curvature are the only structure tensors that do not trivially vanish.

Lemma 2.7. *Let $\mathfrak{gl}(n, \mathbb{H})$ be the Lie algebra of $\mathrm{GL}(n, \mathbb{H})$. Then the first prolongation $\mathfrak{gl}(n, \mathbb{H})^{(1)}$ is zero.*

Proof. Let $\delta\colon (\mathbb{R}^{4n})^* \otimes \mathfrak{gl}(n, \mathbb{H}) \to \Lambda^2 (\mathbb{R}^{4n})^* \otimes \mathbb{R}^{4n}$ be the antisymmetrization map. From the description (2.4) of $\mathrm{GL}(n, \mathbb{H})$ it is evident that

$$\mathfrak{gl}(n, \mathbb{H}) = \{ A \in \mathfrak{gl}(4n, \mathbb{R}) \mid A = I_0^{-1} A I_0 = J_0^{-1} A J_0 = K_0^{-1} A K_0 \} .$$

Let $\alpha \in \mathfrak{gl}(n,\mathbb{H})^{(1)} = \mathrm{Ker}(\delta)$, then we have $I_0 \circ \alpha_X = \alpha_X \circ I_0$ and $J_0 \circ \alpha_X = \alpha_X \circ J_0$ for every $X \in \mathbb{R}^{4n}$, as well as $\alpha_X Y = \alpha_Y X$ for every $X, Y \in \mathbb{R}^{4n}$. Therefore

$$\alpha_X Y = -J_0 \alpha_X (J_0 Y) = -J_0 \alpha_{J_0 Y} X = J_0 I_0 \alpha_{J_0 Y}(I_0 X) = -I_0 J_0 \alpha_{I_0 X}(J_0 Y) = I_0 \alpha_{I_0 X} Y = -\alpha_Y X\,,$$

i.e. $\alpha = 0$. □

It turns out that the intrinsic torsion is encoded precisely in the integrability of the three complex structures I, J, K:

Theorem 2.8 (Obata [235])**.** *An almost hypercomplex structure (I, J, K) on M is hypercomplex if and only if there exists on M a torsion-free* $\mathrm{GL}(n,\mathbb{H})$*-connection, i.e. a linear connection ∇ such that*

$$\nabla I = \nabla J = \nabla K = 0\,.$$

Moreover ∇ is necessarily unique and is called the ***Obata connection****.*

Proof. Set $J_1 = I$, $J_2 = J$, $J_3 = K$ and consider the connection ∇ such that

$$\begin{aligned}\nabla_X Y := \frac{1}{12} \sum_{(\alpha,\beta,\gamma)} J_\alpha \left([J_\beta X, J_\gamma Y] - [J_\gamma X, J_\beta Y]\right) + \frac{1}{6}\sum_{\alpha=1}^{3} J_\alpha \left([J_\alpha X, Y] - [X, J_\alpha Y]\right) \\ + \frac{1}{2}[X,Y] + \frac{1}{2} T(X,Y)\,,\end{aligned} \tag{2.6}$$

for any $X, Y \in TM$, where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$ and $T := -\frac{2}{3}(N_{J_1} + N_{J_2} + N_{J_3})$.

A tedious but straightforward calculation shows that ∇ preserves J_α for $\alpha = 1, 2, 3$ and has torsion $T^\nabla = T$.

We claim that T^∇ can be taken as a representative for the intrinsic torsion of the $\mathrm{GL}(n,\mathbb{H})$-structure. Consider the projection $p\colon T^*M \otimes TM \to \mathrm{Ad}(Q)$ such that $p(S) := \frac{1}{4}\left(S - J_1 S J_1 - J_2 S J_2 - J_3 S J_3\right)$ and the antisymmetrization $\delta\colon T^*M \otimes \mathrm{Ad}(Q) \to \Lambda^2 T^*M \otimes TM$. Since the kernel of the surjective map $\delta \circ (\mathrm{Id}_{T^*M} \otimes p)\colon \Lambda^2 T^*M \otimes TM \to \mathrm{Im}(\delta)$ is a $\mathrm{GL}(n,\mathbb{H})$-invariant complement to the image of δ in $\Lambda^2 T^*M \otimes TM$, it is enough to prove that $(\mathrm{Id} \otimes p)(T) = 0$. For every $X \in TM$ we have $N_{J_\alpha}(X, J_\alpha \cdot) = -J_\alpha N_{J_\alpha}(X, \cdot)$ which implies

$$p(T(X,\cdot)) = -\frac{2}{3}\sum_{\alpha=1}^{3} p(N_{J_\alpha}(X,\cdot)) = 0\,.$$

This proves that 1-integrability of the $\mathrm{GL}(n,\mathbb{H})$-structure, torsion-freeness of ∇ and integrability of the almost complex structures I, J and K are all equivalent to each other.

Uniqueness of the Obata connection immediately follows from Lemma 2.7. □

The identification of the intrinsic torsion of a given hypercomplex structure (I, J, K) with the sum of the Nijenhuis tensors (up to a constant) is essentially due to Obata [236] (see also Bonan [47]). The explicit formula (2.6) for the Obata connection, to our knowledge, was found by Alekseevsky and Marchiafava [4], however, (when it is torsion-free) the Obata connection can also be written as

$$\nabla_X Y = \frac{1}{2}\left([X,Y] + I[IX,Y] - J[X,JY] + K[IX,JY]\right)\,. \tag{2.7}$$

Indeed, one can easily check that ∇I, ∇J, ∇K and T^∇ can all be expressed in terms of the Nijenhuis tensors of I, J and K. This simplified formula is due to Soldatenkov [263].

Differently from the complex case, where torsion-freeness ensures integrability of the $\mathrm{GL}(n,\mathbb{C})$-structure, for $\mathrm{GL}(n,\mathbb{H})$-structures integrability is also obstructed by the intrinsic curvature. Since the Obata connection is the unique torsion-free $\mathrm{GL}(n,\mathbb{H})$-connection, the intrinsic curvature can be identified with its curvature and we obtain:

Theorem 2.9 (Obata [235])**.** *On a hypercomplex manifold (M, I, J, K) the following are equivalent:*

(i) *The torsion-free* $\mathrm{GL}(n,\mathbb{H})$*-structure* (I,J,K) *is locally flat;*

(ii) *The Obata connection* ∇ *is flat;*

(iii) M *has quaternionic affine transition functions.*

Proof. The equivalence of the first two assertions follows from Theorem 2.4 and Lemma 2.7, the equivalence of the third is due to Sommese [266]. □

Locally flat hypercomplex structures were first studied by Sommese [266] and were termed "quaternionic". As mentioned earlier, such a condition is equivalent to require that the hypercomplex manifolds (M,I,J,K) is locally isomorphic to $\mathbb{H}^n$.

Examples.

The most trivial example of hypercomplex manifold is that of (open subsets of) $\mathbb{H}^n$ with the standard left action of i,j,k, or, equivalently, $\mathbb{R}^{4n}$ with the flat hypercomplex structure given by (2.3). Since the hypercomplex structure is invariant with respect to the sum of two elements of $\mathbb{H}^n$ it descends to a hypercomplex structure on the quotient by a lattice isomorphic to $\mathbb{Z}^{4n}$. We thus obtain compact hypercomplex tori.

$\mathbb{H}^n\setminus\{0\}$ inherits from $\mathbb{H}^n$ the standard hypercomplex structure acting from the left. Fix a quaternion $q\in\mathbb{H}^*$ such that $|q|\neq 1$ and consider the integer group $\langle q\rangle$ generated via right multiplication by q on $\mathbb{H}^n\setminus\{0\}$. Since the hypercomplex structure acts on the left, it commutes with the action of $\langle q\rangle$, therefore it descends to a hypercomplex structure on the quotient $(\mathbb{H}^n\setminus\{0\})/\langle q\rangle$. This quotient is compact, as any coset allows a representative with norm smaller than 1. The quotient $(\mathbb{H}^n\setminus\{0\})/\langle q\rangle$ is called the **quaternionic Hopf manifold** and it is diffeomorphic to $S^1\times S^{4n-1}$ (see [240], where the authors also compute the moduli space for the hypercomplex structures on $S^1\times S^{4n-1}$).

Kato [195] studied complex Hopf surfaces determining which ones admit hypercomplex structures. Kato showed more generally that the only compact locally flat 4-dimensional manifolds are tori and certain Hopf surfaces.

Less trivial examples came from the works of Spindel, Sevrin, Troost, Van Proeyen [267] and Joyce [193]. They discovered that a compact Lie group, when multiplied by a torus of a suitable dimension, carries a hypercomplex structure. Let us briefly overview how this works. Joyce shows that, by a covering group argument, we may further assume that the Lie group is semisimple. Let then G be a compact semisimple Lie group and H a maximal torus in it. Within this framework, structure theory can be performed, which allows to obtain a suitable decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g}=\mathfrak{b}\oplus\bigoplus_{j=1}^{m}\mathfrak{d}_j\oplus\bigoplus_{j=1}^{m}\mathfrak{f}_j,$$

where $\mathfrak{b}$ is abelian of dimension $\mathrm{Rank}(G)-m$, $\mathfrak{d}_j\subseteq\mathfrak{g}$ are subalgebras isomorphic to $\mathfrak{su}(2)$, and $\mathfrak{f}_j\subseteq\mathfrak{g}$ are subspaces satisfying the conditions:

1. $[\mathfrak{d}_j,\mathfrak{b}]=0$ and $\mathfrak{b}\oplus\bigoplus_{j=1}^{n}\mathfrak{d}_j$ contains the Lie algebra of H;
2. $[\mathfrak{d}_j,\mathfrak{f}_i]=0$ for $j<i$;
3. $[\mathfrak{d}_j,\mathfrak{f}_j]\subseteq\mathfrak{f}_j$ and this Lie bracket action of $\mathfrak{d}_j$ on $\mathfrak{f}_j$ is isomorphic to the direct sum of a certain (finite) amount of copies of the action of $\mathfrak{su}(2)$ on $\mathbb{C}^2$ by matrix multiplication from the left.

Such a decomposition will be called a **Joyce decomposition** of the Lie algebra $\mathfrak{g}$.

Now, let $r=\mathrm{Rank}(G)$, and denote $T^{2m-r}\cong\mathrm{U}(1)^{2m-r}$ the $2m-r$-dimensional torus, so that the Lie algebra of $T^{2m-r}\times G$ decomposes as

$$(2m-r)\mathfrak{u}(1)\oplus\mathfrak{g}\cong\mathbb{R}^m\oplus\bigoplus_{j=1}^{m}\mathfrak{d}_j\oplus\bigoplus_{j=1}^{m}\mathfrak{f}_j\,.$$

Let $(e_1, \ldots, e_n)$ be the standard basis for $\mathbb{R}^n$ and let φ_j be an isomorphism from $\mathfrak{su}(2) = \mathfrak{sp}(1)$ to $\mathfrak{d}_j$. Consider a basis i_1, i_2, i_3 of $\mathfrak{su}(2)$ such that

$$[i_1, i_2] = 2i_3\,, \qquad [i_3, i_1] = 2i_2\,, \qquad [i_2, i_3] = 2i_1\,. \tag{2.8}$$

We can regard $\mathrm{Span}_{\mathbb{H}}(e_j, \varphi_j(i_1), \varphi_j(i_2), \varphi_j(i_3))$ as a copy of $\mathbb{H}$. From here we define a hypercomplex structure $I_1, I_2, I_3 \in \mathrm{End}((2m-r)\mathfrak{u}(1) \oplus \mathfrak{g})$ as follows.

(a) Let I_1, I_2, I_3 act on $\mathbb{R}^m \oplus \bigoplus_{j=1}^m \mathfrak{d}_j$ as

$$I_a(e_j) = \varphi_j(i_a), \qquad I_a(\varphi_j(i_a)) = -e_j, \qquad I_a(\varphi_j(i_b)) = \varphi_j(i_c), \qquad I_a(\varphi_j(i_c)) = -\varphi_j(i_b),$$

whenever (a, b, c) is an even permutation of $(1, 2, 3)$.

(b) Let I_1, I_2, I_3 act on $\mathfrak{f}_j$ as $I_a(v) = [\varphi_j(i_a), v]$, for each $v \in \mathfrak{f}_j$ and $a = 1, 2, 3$.

By definition, it is clear that (I_1, I_2, I_3) is a hypercomplex structure on $\mathbb{R}^m \oplus \bigoplus_{j=1}^m \mathfrak{d}_j$, the fact that it is also a hypercomplex structure on $\bigoplus_{j=1}^m \mathfrak{f}_j$ follows from the third property of Joyce decompositions, as the action of $\mathfrak{d}_j$ on $\mathfrak{f}_j$ by conjugation is isomorphic to a finite amount of copies of the action of $\mathrm{Im}(\mathbb{H})$ on $\mathbb{H}$ and (b) is nothing but a way to write down this isomorphism.

At this point Joyce uses an argument due to Samelson [254] to prove that I_a must be integrable and therefore, by left-translations, induces a homogeneous complex structure on $T^{2m-r} \times G$. Therefore (I_1, I_2, I_3) extends to a homogeneous hypercomplex structure on $T^{2m-r} \times G$.

According to [34, 104] all invariant hypercomplex structures on compact Lie groups are obtained from Joyce's construction.

Example 2.10. The Lie algebra of SU(3) can be written as

$$\mathfrak{su}(3) = \left\{ \begin{pmatrix} D & f \\ -\bar{f}^t & -\mathrm{tr}(D) \end{pmatrix} \;:\; D \in \mathfrak{u}(2) \text{ and } f \in \mathbb{C}^2 \right\},$$

therefore it splits accordingly:

$$\mathfrak{su}(3) = \mathfrak{b} \oplus \mathfrak{d} \oplus \mathfrak{f}\,,$$

where

- $\mathfrak{d} \cong \mathfrak{sp}(1)$ is the space of matrices with zero f and $\mathrm{tr}(D)$;
- $\mathfrak{f}$ consists of matrices with zero D;
- $\mathfrak{b} \cong \mathbb{R}$ is the set of diagonal matrices commuting with $\mathfrak{d}$.

We have

$$[\mathfrak{b}, \mathfrak{d}] = 0\,, \quad [\mathfrak{b}, \mathfrak{f}] = \mathfrak{f}\,, \quad [\mathfrak{d}, \mathfrak{f}] = \mathfrak{f}\,, \quad [\mathfrak{f}, \mathfrak{f}] = \mathfrak{b} \oplus \mathfrak{d}\,, \quad [\mathfrak{d}, \mathfrak{d}] = \mathfrak{d}\,.$$

Let us consider the following basis of $\mathfrak{su}(3)$:

$$X_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \quad X_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}.$$

Following Joyce's construction a hypercomplex structure (I, J, K) on SU(3) is then defined by the following relations:

(a) on $\mathfrak{b} \oplus \mathfrak{d} = \langle X_1, X_2, X_3, X_4 \rangle$ as $IX_1 = X_2$, $IX_3 = X_4$, $JX_1 = X_3$, $JX_2 = -X_4$;

(b) on $\mathfrak{f} = \langle X_5, X_6, X_7, X_8 \rangle$ as $Iv = [X_2, v]$, $Jv = [X_3, v]$, $Kv = [X_4, v]$ for every $v \in \mathfrak{f}$.

With a similar technique, Joyce also constructs hypercomplex structures on homogeneous spaces. For instance quaternionic Hopf manifolds can be seen as a particular case of this construction when regarded as the product $\mathrm{U}(1) \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$.

Joyce presented other geometric constructions in [192, 193], most notably hypercomplex quotients (in analogy to Kähler [222], hyperkähler [173] and quaternionic Kähler [133] reductions) and a way of twisting associated bundles via a quaternionic instanton, further studied by Pedersen, Poon and Swann [241].

Twistor theory was exploited by Grantcharov, Pedersen and Poon [147, 238, 239, 240] to study deformations of hypercomplex structures. In particular in [240] the authors use deformation theory to construct inhomogeneous hypercomplex structures on Joyce's examples.

Finally, we mention a series of examples due to Boyer, Galicki and Mann on Stiefel manifolds [50, 51] and on certain circle bundles over 3-Sasakian manifolds [52].

$\mathrm{Sp}(n)$-structures.

Consider again quaternionic coordinates $(q^1, \dots, q^n)$ on $\mathbb{H}^n$ with the standard left hypercomplex structure and let $(x_0^1, \dots, x_0^n, x_1^1, \dots, x_1^n, x_2^1, \dots, x_2^n, x_3^1, \dots, x_3^n)$ be the corresponding real coordinates on $\mathbb{R}^{4n} \cong \mathbb{H}^n$. The hypercomplex structure (I_0, J_0, K_0) defined in (2.3) is compatible with the standard inner product $\langle \cdot, \cdot \rangle_0$ on $\mathbb{R}^{4n}$, in the sense that

$$\langle I_0 \cdot, I_0 \cdot \rangle_0 = \langle J_0 \cdot, J_0 \cdot \rangle_0 = \langle K_0 \cdot, K_0 \cdot \rangle_0 = \langle \cdot, \cdot \rangle_0 \,.$$

From this compatibility condition we see that there are three fundamental forms $\omega_{I_0}, \omega_{J_0}, \omega_{K_0}$ on $\mathbb{H}^n$, furthermore there is a non-degenerate complex form

$$\Omega_0 := \omega_{J_0} + i\omega_{K_0} \,.$$

In this set of coordinates we have

$$\langle \cdot, \cdot \rangle_0 = \sum_{k=1}^{n} \sum_{p=0}^{3} dx_p^k \otimes dx_p^k \,, \qquad \omega_{I_0} = \sum_{k=1}^{n} (dx_0^k \wedge dx_1^k + dx_2^k \wedge dx_3^k) \,,$$

$$\omega_{J_0} = \sum_{k=1}^{n} (dx_0^k \wedge dx_2^k - dx_1^k \wedge dx_3^k) \,, \qquad \omega_{K_0} = \sum_{k=1}^{n} (dx_0^k \wedge dx_3^k + dx_1^k \wedge dx_2^k) \,,$$

Identifying $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ with the complex coordinates $(z^1, \dots, z^{2n})$, where $z^{2k-1} = x_0^k + ix_1^k$ and $z^{2k} = x_2^k + ix_3^k$, one could easily verify that

$$\Omega = \sum_{k=1}^{n} dz^{2k-1} \wedge dz^{2k} \,. \tag{2.9}$$

Observe that

$$\langle \cdot, \cdot \rangle_0 + i\omega_{I_0} + j\omega_{J_0} + k\omega_{K_0} = \bar{h} + \Omega j = \sum_{k=1}^{n} dq^k \otimes d\bar{q}^k$$

is the standard quaternionic inner product on $\mathbb{H}^n$.

The hyperunitary group $\mathrm{Sp}(n)$ is the subgroup of $\mathrm{GL}(n, \mathbb{H})$ that preserves $\langle \cdot, \cdot \rangle_0$, in particular it preserves all the object we hereby introduced and it can be seen as their common stabilizer. We can therefore regard $\mathrm{Sp}(n)$ in multiple ways, obtaining the hypercomplex analogue of (2.1):

$$\mathrm{Sp}(n) = \mathrm{GL}(n, \mathbb{H}) \cap \mathrm{O}(4n) = \mathrm{O}(4n) \cap \mathrm{Sp}(2n, \mathbb{C}) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{GL}(n, \mathbb{H}) \,.$$

Another possible description of $\mathrm{Sp}(n)$ is as the intersection of the three unitary groups $\mathrm{Stab}(I_0, \omega_{I_0})$, $\mathrm{Stab}(J_0, \omega_{J_0})$, $\mathrm{Stab}(K_0, \omega_{K_0})$, which motivates the name of *hyperunitary group*.

A $\mathrm{Sp}(n)$-structure on a $4n$-dimensional manifold is given by the data (I, J, K, g), where (I, J, K) is an almost hypercomplex structure and g a *hyperhermitian metric*, i.e. a Riemannian metric such that

$$g(I\cdot, I\cdot) = g(J\cdot, J\cdot) = g(K\cdot, K\cdot) = g\,.$$

Of course we also have the induced fundamental forms $\omega_I, \omega_J, \omega_K$ and the non-degenerate complex form

$$\Omega := \omega_J + i\omega_K\,. \tag{2.10}$$

The collection of these data is also called an *almost hyperhermitian structure*. If I, J, K are integrable we drop the word "almost". An **(almost) hyperhermitian manifold** is one equipped with an (almost) hyperhermitian structure.

The intrinsic torsion of a $\mathrm{Sp}(n)$-structure can be identified with the three tensors $\nabla^g I, \nabla^g J, \nabla^g K$, or equivalently, the three tensors $\nabla^g\omega_I, \nabla^g\omega_J, \nabla^g\omega_K$ or else $\nabla^g\Omega$. In other words the following conditions are equivalent to torsion-freeness:

$$\nabla^g I = \nabla^g J = \nabla^g K = 0\,, \qquad \nabla^g\omega_I = \nabla^g\omega_J = \nabla^g\omega_K = 0\,, \qquad \nabla^g\Omega = 0\,.$$

In particular we have $\mathrm{Hol}(\nabla^g) \subseteq \mathrm{Sp}(n)$. Whenever the Obata connection preserves some Riemannian metric g, it automatically induces a torsion-free $\mathrm{Sp}(n)$-structure also known as a *hyperkähler structure*. Accordingly, a manifold with a $\mathrm{Sp}(n)$-structure with vanishing intrinsic torsion is called a **hyperkähler manifold**. The name is due to Calabi [68], who also proposed the term *hypercomplex*.

Since $\mathrm{Hol}(\nabla^g) \subseteq \mathrm{Sp}(n) \subseteq \mathrm{SU}(2n)$ hyperkähler manifolds are always Calabi-Yau, and thus, Ricci-flat (see also Subsection 2.2.4).

The conditions for the vanishing of the torsion can be slightly relaxed. Indeed Hitchin [171] showed that the closure of $\omega_I, \omega_J, \omega_K$ implies the vanishing of the Nijenhuis tensors of I, J, K, hence (M, I, g), (M, J, g) and (M, K, g) are automatically Kähler. In particular, if (M, I, J, K, g) is hyperhermitian the concise condition

$$d\Omega = 0 \tag{2.11}$$

is sufficient to give hyperkähler.

Verbitsky [303] proved that any hypercomplex manifold (M, I, J, K) that carries a Kähler metric on (M, I) admits a hyperkähler structure (not necessarily compatible with the original Kähler structure).

Hypercomplex and hyperkähler manifolds are respectively, subfamilies of the more general concepts of *quaternionic* and *quaternionic Kähler* manifolds. These can be described as manifolds admitting a $\mathrm{GL}(n, \mathbb{H})\mathrm{GL}(1, \mathbb{H})$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$-structure respectively. The terminology is standard, as of today, however, some years passed before it established itself, hence, early literature might refer to a hypercomplex structure as a quaternionic one, e.g..

Our attention is focused on hypercomplex manifolds and we shall not describe here quaternionic structures. We only report here that the quaternionic projective space $\mathbb{HP}^n$ is a quaternionic manifold, however it is not hypercomplex, as it does not even admit an almost complex structure. For $n = 1$ we have $\mathbb{HP}^1 \cong S^4$ and it is known that the only spheres admitting almost complex structures are S^2 and S^6 (cf. [117, 174]). The general case is due to Massey [223], although the case $n \geq 4$ was already established by Hirzebruch [170]

Following the ideas of the Gray-Hervella classification, Cabrera and Swann [61, 63] give a nice and detailed description of the possible classes of quaternionic geometries.

The class of $\mathrm{Sp}(n)$-structures with non-vanishing intrinsic torsion which is most interesting for us is that of *hyperkähler structures with torsion*, to be introduced in the next subsection.

2.1.4 HKT structures.

Definition of HKT manifolds.

We have seen that KT geometry deals with Hermitian manifolds together with a $\mathrm{U}(n)$-connection with skew-symmetric torsion. In analogy to this, one defines *hyperkähler geometry with torsion*, abbreviated to *HKT geometry*, as the geometry of hyperhermitian manifolds (M, I, J, K, g) with a $\mathrm{Sp}(n)$-connection with skew-symmetric torsion, called the *HKT connection*. The existence of such a connection is equivalent to require that the three Bismut connections $\nabla^I, \nabla^J, \nabla^K$ coincide.

Geometries with torsion are of particular interest for some supersymmetric sigma models in theoretical physics, especially in presence of the so-called Wess-Zumino term [134, 177]. Indeed, in such a case the internal space of the sigma model has a linear connection with skew-symmetric torsion and holonomy in either $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$, depending on the number of supersymmetries that leave the sigma model action invariant. Thus, we either have KT or HKT geometry. HKT manifolds also emerged as moduli spaces for a certain class of black holes [142], and recently, as base spaces of "timelike" solutions of five-dimensional de Sitter supergravity [151, 161].

HKT manifolds were first studied by Howe and Papadopoulos [178]. As discovered by Grantcharov and Poon [148], a HKT structure can also be equivalently expressed in terms of a nice differential equation involving the complex volume form Ω defined in (2.10):

Proposition 2.11. *On a hyperhermitian manifold (M, I, J, K, g) the following conditions are equivalent:*

(i) $\partial\Omega = 0$;

(ii) $Id\omega_I = Jd\omega_J = Kd\omega_K$;

(iii) $\nabla^I = \nabla^J = \nabla^K$.

It is nowadays customary to use this alternative description as a definition.

Definition 2.12. Let (M, I, J, K, g) be a hyperhermitian manifold and consider the complex 2-form $\Omega := \omega_J + i\omega_K$. Then the metric g, the form Ω and the manifold M are called **hyperkähler with torsion** or, shortly, **HKT** if

$$\partial\Omega = 0\,. \tag{2.12}$$

The name stems from the fact that a hyperhermitian manifold is hyperkähler if only if (2.11) holds, i.e. Ω is closed. Therefore the intrinsic torsion of the $\mathrm{Sp}(n)$-structure does not fully vanish for HKT (non hyperkähler) manifolds. As of today, the name hyperkähler with torsion is generally considered to be misleading, therefore the acronym HKT is preferred.

Since the three Bismut connections coincide the common torsion is $c = d^c_I\omega_I = d^c_J\omega_J = d^c_K\omega_K$ by (2.2). In particular, if one of the Hermitian structures $(g, I), (g, J)$ or (g, K) on a HKT manifold (M, I, J, K, g) is Kähler, then the other two are also Kähler, and thus the $\mathrm{Sp}(n)$-structure is torsion-free. The HKT-structure is called **strong** or **weak** according as the torsion is closed or not.

Cabrera and Swann [64], studying the intrinsic torsion of almost quaternionic Hermitian manifolds were able to weaken the integrability assumptions:

Proposition 2.13. *On an almost hyperhermitian manifold (M, I, J, K, g) the following conditions are equivalent:*

(i) $\partial\Omega = 0$ *and* $N_J = 0$ *(or* $N_K = 0$*)*;

(ii) $Jd\omega_J = Kd\omega_K$ *and* $N_J = 0$ *(or* $N_K = 0$*)*;

(iii) $Id\omega_I = Jd\omega_J = Kd\omega_K$;

(iv) I, J, K *are integrable and* $\nabla^I = \nabla^J = \nabla^K$.

Examples.

We already observed that HKT manifolds are a generalization of hyperkähler manifolds. From now on, unless declared otherwise we shall use the term HKT manifold to mean HKT non-hyperkähler manifold.

Every 4-dimensional hyperhermitian manifold is automatically HKT because (2.12) is trivially satisfied. Boyer [49] classified compact hyperhermitian manifolds of dimension 4 up to conformal equivalence. The only possibilities are tori with the flat metric, $K3$ surfaces and those locally flat quaternionic Hopf surfaces studied by Kato [195] with their standard locally conformally flat metric. The manifolds of the first two classes are hyperkähler, while those of the third are not, however they are always locally conformally hyperkähler.

We have seen that compact Lie groups, whenever multiplied by a torus of a suitable dimension always admit a homogeneous hypercomplex structure. Whenever the group G is semisimple, the Cartan-Killing form B is a negative-definite inner product on the Lie algebra $\mathfrak{g}$. The (opposite of the) Cartan-Killing form can be extended to a hyperhermitian metric on $T^{2m-r}G$ (see [148, Lemma 2]), where $r = \mathrm{Rank}(G)$. As we now show, such hyperhermitian metric is actually HKT. This observation is originally due to Opfermann and Papadopoulos [237], who also generalized the construction to certain homogeneous spaces, showing that they carry HKT structures.

Let g be the left-translation on $T^{2m-r} \times G$ of the hyperhermitian inner product on $(2m-r)\mathfrak{u}(1) \oplus \mathfrak{g}$.

Proposition 2.14. *The hyperhermitian metric g on $T^{2m-r} \times G$ is strong HKT.*

Proof. Consider the left-invariant connection ∇ on $T^{2m-r} \times G$ such that all left-invariant vector fields are parallel. Since the metric g and the hypercomplex structure (I_1, I_2, I_3) on $T^{2m-r} \times G$ are left-invariant they are preserved by ∇. The torsion tensor is $T^\nabla(X,Y) = -[X,Y]$, therefore

$$c(X,Y,Z) = -B([X,Y],Z) \tag{2.13}$$

which is known to be antisymmetric, i.e. a 3-form. It follows that the Bismut connections of I_1, I_2 and I_3 all coincide with ∇. Therefore the metric g is HKT. To see that it is strong simply compute the exterior differential of c:

$$\begin{aligned} dc(X,Y,Z,W) = {} & c([X,Y],Z,W) - c([X,Z],Y,W) + c([X,W],Y,Z) \\ & + c([Y,Z],X,W) - c([Y,W],X,Z) + c([Z,W],X,Y) \end{aligned}$$

using the Jacobi identity and (2.13) we obtain

$$dc(X,Y,Z,W) = c([X,Y],W,Z) + c([Z,W],X,Y) = c(Z,W,[Y,X]) - c(Y,X,[Z,W]) = 0$$

as desired. □

For instance the list of compact simple Lie groups yields the following list of strong HKT Lie groups:

$$\mathrm{SU}(2k+1)\,,\quad T^1\times\mathrm{SU}(2k)\,,\quad T^k\times\mathrm{SO}(2k+1)\,,\quad T^{2k}\times\mathrm{SO}(4k)\,,\quad T^{2k-1}\times\mathrm{SO}(4k+2)\,,\quad T^k\times\mathrm{Sp}(k)\,,$$
$$T^2\times\mathrm{E}_6\,,\qquad T^7\times\mathrm{E}_7\,,\qquad T^4\times\mathrm{F}_4\,,\qquad T^2\times\mathrm{G}_2\,.$$

Example 2.15. Let us give some details for the case of SU(3). In example 2.10 we described the homogeneous hypercomplex structure in terms of a basis of the Lie algebra $\mathfrak{su}(3) = \langle X_1, \ldots, X_8\rangle$.

Let $(X^1, \ldots, X^8)$ be the dual basis of $(X_1, \ldots X_8)$ and let

$$Z^1 = \frac{1}{2}(X^1 + iX^2)\,,\quad Z^2 = \frac{1}{2}(X^3 + iX^4)\,,\quad Z^3 = -\frac{1}{2}(X^5 + iX^6)\,,\quad Z^4 = \frac{1}{2}(X^7 + iX^8)$$

be the induced unitary coframe with respect to (g, I), where $g = \sum_{k=1}^8 X^k \otimes X^k$. The HKT form can then be expressed as

$$\Omega = Z^{12} + Z^{34}\,.$$

To check that this is ∂-closed we compute the non-zero brackets of vectors in $\{X_1, \ldots, X_8\}$. They are

$$
\begin{aligned}
&[X_5, X_6] = X_1 + X_2\,, \qquad [X_7, X_8] = X_2 - X_1\,, \qquad [X_3, X_4] = 2X_2\,,\\
&\frac{1}{2}[X_2, X_4] = [X_5, X_7] = -[X_6, X_8] = -X_3\,,\\
&\frac{1}{2}[X_2, X_3] = -[X_5, X_8] = -[X_6, X_7] = X_4\,,\\
&\frac{1}{3}[X_1, X_6] = [X_2, X_6] = -[X_3, X_7] = -[X_4, X_8] = -X_5\,,\\
&\frac{1}{3}[X_1, X_5] = [X_2, X_5] = -[X_3, X_8] = [X_4, X_7] = X_6\,,\\
&\frac{1}{3}[X_1, X_8] = -[X_2, X_8] = -[X_3, X_5] = -[X_4, X_6] = X_7\,,\\
&\frac{1}{3}[X_1, X_7] = -[X_2, X_7] = -[X_3, X_6] = [X_4, X_5] = -X_8\,,
\end{aligned}
$$

therefore we have

$$
\partial Z^1 = 0\,, \qquad \partial Z^2 = -2Z^{12} - 2Z^{34}\,, \qquad \partial Z^3 = -(1+3i)Z^{13}\,, \qquad \partial Z^4 = (3i-1)Z^{14}
$$

and $\Omega = -\frac{1}{2}\partial Z^2$ is evidently HKT.

Barberis and Fino [30] gave an interesting procedure that allows to construct new HKT Lie algebras starting from others of half the dimension by using quaternionic representations. The strong (respectively, weak, hyperkähler, balanced) condition is preserved under this construction. Many new examples can be produced applying their technique.

A remarkable family of left-invariant examples was detected by Dotti and Fino [112]. Let us quickly recall that a complex structure I on a Lie algebra $\mathfrak{g}$ is called **abelian** if $\mathfrak{g}^{1,0}$ (and thus $\mathfrak{g}^{0,1}$) is abelian. Equivalently I is "orthogonal" with respect to the Lie bracket: $[I\cdot, I\cdot] = [\cdot,\cdot]$. A hypercomplex structure (I, J, K) on $\mathfrak{g}$ is called **abelian** if each of I, J, K is abelian.

Dotti and Fino observed that every abelian hypercomplex structure on a (non-abelian) Lie group G gives rise to a left-invariant weak HKT structure on G. Under the construction of Barberis and Fino [30] abelianness of the hypercomplex structure is never preserved, unless the quaternionic representation used to produce the new example is trivial. We will investigate the properties of these special kind of hypercomplex structures later in Section 3.2 for the time being we limit ourselves to observe that this kind of structures is in certain sense complementary to those found by Joyce, indeed they can only occur on solvable Lie groups which cannot be compact, unless they are tori. More precisely, we have the following result of Anatoliĭ Petravchuk [242]:

Proposition 2.16. *If $\mathfrak{g}$ admits an abelian complex structure then it is 2-step solvable.*

Proof. Let $A = \mathfrak{g}^{1,0}$ and $B = \mathfrak{g}^{0,1}$. Since A and B are abelian it is enough to prove that

$$
[[A, B], [A, B]] = 0\,.
$$

Take arbitrary $a_1, a_2 \in A$ and $b_1, b_2 \in B$ and write $[a_1, b_2] = a_3 + b_3$, $[a_2, b_1] = a_4 + b_4$ with $a_3, a_4 \in A$ and $b_3, b_4 \in B$. The computation is just a multiple usage of Jacobi's identity and the fact that A and B are abelian:

$$
\begin{aligned}
[[a_1, b_1], [a_2, b_2]] &= [[[a_1, b_1], a_2], b_2] - [[[a_1, b_1], b_2], a_2] = -[[[b_1, a_2], a_1], b_2] + [[[b_2, a_1], b_1], a_2]\\
&= [[b_4, a_1], b_2] - [[a_3, b_1], a_2] = -[[a_1, b_2], b_4] + [[b_1, a_2], a_3] = -[a_3, b_4] - [b_4, a_3] = 0\,,
\end{aligned}
$$

which is the desired identity. □

Example 2.17. Dotti and Fino [110] classified non-abelian 8-dimensional 2-step nilpotent Lie groups admitting an abelian hypercomplex structure. The only such groups are

$$
N_1 = H_1(2) \times \mathbb{R}^3\,, \qquad N_2 = H_2(1) \times \mathbb{R}^2\,, \qquad N_3 = H_3(1) \times \mathbb{R}\,,
$$

where $H_i(n)$ denotes the real ($i = 1$), complex ($i = 2$), and quaternionic ($i = 3$) Heisenberg group:

$$H_1(n) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & \mathbb{1}_n & {}^t c \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{R}^n\,, b \in \mathbb{R} \right\}, \quad H_2(n) = \left\{ \begin{pmatrix} 1 & v & w \\ 0 & \mathbb{1}_n & {}^t z \\ 0 & 0 & 1 \end{pmatrix} \mid v, z \in \mathbb{C}^n\,, w \in \mathbb{C} \right\},$$

$$H_3(n) = \left\{ \begin{pmatrix} 1 & p & \operatorname{Im}(q) - \frac{1}{2}\|p\|^2 \\ 0 & \mathbb{1}_n & -{}^t\bar{p} \\ 0 & 0 & 1 \end{pmatrix} \mid p \in \mathbb{H}^n\,, q \in \mathbb{H} \right\}.$$

Each N_i contains a canonical co-compact lattice Γ_i by taking integer coordinates, hence the nilmanifolds $M_i = \Gamma_i \backslash N_i$ are compact HKT.

We also remark that Dotti and Fino extended their result by classifying hypercomplex 8-dimensional nilpotent Lie algebras [113].

Other examples were obtained by Verbitsky [302] as a by-product of hyperholomorphic bundles.

Finally, we mention some examples of hypercomplex manifolds not admitting any HKT structure. Such manifold were initially conjectured not to exist, but the first example came from Fino and Grantcharov [124] using a symmetrization procedure to obtain left-invariant structures from non-invariant ones. Other examples and new obstructions to the existence of HKT metrics followed, for instance in [29, 146, 208, 264]. We cannot fail to mention the remarkable twist construction due to Swann [278, 279], its argument allowed to construct the first simply-connected compact hypercomplex manifolds not admitting any HKT metric.

2.2 Potentials, cohomology and canonical bundles.

In this section we report on some basic results in HKT geometry. After some analytic preliminaries dealing with the notions of quaternionic Hessian and Laplacian, we overview the theory of HKT potentials whose existence was proved by Banos and Swann [27]. After this, we recall the cohomological results of Verbitsky [301] who exploited an analogy between the Kähler and HKT worlds to investigate Hodge theory. Finally, we report on the interplay between the holonomy of the Obata connection, the existence of holomorphic sections of the canonical bundle and the existence of a balanced HKT metric.

2.2.1 Quaternionic analysis.

The twisted Dolbeault operator.

On a hypercomplex manifold (M, I, J, K) there are two important differential operators acting on forms. The first is the usual Dolbeault operator ∂ which we will always consider with respect to I. If one wishes to study complex cohomology then ∂ can be paired with its conjugate $\bar{\partial}$. However, we are of course interested in the whole hypercomplex structure (I, J, K), therefore there is another operator which is of the utmost importance for us, which is obtained by twisting $\bar{\partial}$ via J. We call

$$\partial_J := J^{-1}\bar{\partial}J$$

the **twisted Dolbeault operator**, being $\bar{\partial}$ taken with respect to I.

Let us remark a few facts regarding the operator ∂_J. We claim that since J anticommutes with I it switches the type of forms, i.e. $J\colon \Lambda_I^{p,q}(M) \to \Lambda_I^{q,p}(M)$. We first prove that $J\colon T^{1,0}M \to T^{0,1}M$ as for any $X \in TM$

$$J(X - iIX) = JX - iJIX = JX + iIJX$$

is clearly of type $(0,1)$. Since J preserves the wedge product it is enough to show the claim for $(1,0)$-forms. Let $\alpha \in \Lambda^{1,0}(M)$ and $Z \in T^{0,1}M$, then

$$I(J\alpha)(Z) = \alpha(JIZ) = -\alpha(IJZ) = -i\alpha(JZ) = -i(J\alpha)(Z)\,,$$

i.e. $J\alpha \in \Lambda^{0,1}(M)$.

As a consequence of the claim ∂_J sends (p,q)-forms to $(p+1,q)$-forms, just like ∂. Furthermore, the pair of operators (∂, ∂_J) resembles very closely the pair (d, d^c) in complex geometry, for instance we have the following:

Proposition 2.18. *An almost hypercomplex manifold (M,I,J,K) is hypercomplex if and only if the operators $\partial, \bar\partial, \partial_J, \bar\partial_J := \overline{\partial_J} = J^{-1}\partial J$ square to zero and anticommute with each other.*

Proof. Since

$$\partial = \frac{1}{2}(d + id^c_I)\,, \qquad \bar\partial = \frac{1}{2}(d - id^c_I)$$

we also have

$$\partial_J = \frac{1}{2}(d^c_J + id^c_K)\,, \qquad \bar\partial_J = \frac{1}{2}(d^c_J - id^c_K)\,.$$

The proposition is thus just a consequence of the fact that integrability of I, J and K is equivalent to $(d^c_I)^2 = (d^c_J)^2 = (d^c_K)^2 = 0$ or $dd^c_I + d^c_I d = dd^c_J + d^c_J d = dd^c_K + d^c_K d = 0$ (which is also equivalent to $d^c_I d^c_J + d^c_J d^c_I = d^c_J d^c_K + d^c_K d^c_J = d^c_K d^c_I + d^c_I d^c_K = 0$). ☐

Real and positive forms.

In this section we discuss the notion of q-realness and q-positivity for differential forms. These concepts were introduced and studied by Alesker and Verbitsky [11, 17, 308] in complete analogy to the notion of positive forms in complex geometry.

Let (M,I,J,K) be an hypercomplex manifold and recall that the anticommutation property of I and J implies that J sends (p,q)-forms to (q,p)-forms (with respect to I). Composing with the complex conjugation we obtain the operator

$$\mathcal{J} := J \circ \bar{\ }\colon \Lambda^{p,q}(M) \to \Lambda^{p,q}(M)\,.$$

Furthermore $\mathcal{J}^2|_{\Lambda^{p,q}(M)} = (-1)^{p+q}\,\mathrm{Id}$, therefore when $p+q$ is even $\mathcal{J}$ is an involution of $\Lambda^{p,q}(M)$. The eigenbundle of $\mathcal{J}$ corresponding to the eigenvalue 1 is of particular interest.

Definition 2.19. Let $p+q$ be even. A form $\alpha \in \Lambda^{p,q}(M)$ is called **quaternionic real** (shortly **q-real**) if $\mathcal{J}\alpha = \alpha$.

We shall be primarily concerned with forms of type $(2k,0)$. In this case q-realness can be characterized as follows: $\alpha \in \Lambda^{2k,0}(M)$ is q-real if and only if $\alpha(Z_1, \mathcal{J}Z_1, \dots, Z_k, \mathcal{J}Z_k)$ is real for any $Z_1, \dots, Z_k \in \Gamma(T^{1,0}M)$. The equivalence follows from the identity

$$\mathcal{J}\alpha(Z_1, \mathcal{J}Z_1, \cdots, Z_k, \mathcal{J}Z_k) = \bar\alpha(JZ_1, J^2\bar Z_1, \cdots, JZ_k, J^2\bar Z_k) = \overline{\alpha(Z_1, \mathcal{J}Z_1, \cdots, Z_k, \mathcal{J}Z_k)}\,.$$

This allows to define a notion of positivity for q-real $(2k,0)$-forms:

Definition 2.20. A q-real form $\alpha \in \Lambda^{2k,0}(M)$ is called **quaternionic (semi)positive** (shortly **q-(semi)positive**) if

$$\alpha(Z_1, \mathcal{J}Z_1, \cdots, Z_k, \mathcal{J}Z_k) > 0 \ (\geq 0)$$

for every non-vanishing $Z_1, \dots, Z_k \in T^{1,0}M$. Equivalently,

$$\alpha(X_1, JX_1, \dots, X_k, JX_k) > 0 \ (\geq 0)$$

for every non-vanishing $X_1, \dots, X_k \in TM$.

The equivalence of the two conditions is easily obtained by writing $Z_j = X_j - iIX_j$ for $X_j \in TM$ and $j = 1, \dots, k$ and using that α is of type $(2k, 0)$ so that

$$\alpha(Z_1, \mathcal{J}Z_1, \cdots, Z_k, \mathcal{J}Z_k) = 4^k \alpha(X_1, JX_1, \cdots, X_k, JX_k)\,.$$

The terminology we adopt may differ from the most common one used in literature. For instance semi-positivity is generally called weak positivity and our notion of positivity is mainly called strict weak positivity.

Example 2.21. Given any $(1, 0)$-form β the form $-\beta \wedge \mathcal{J}\beta$ is always q-positive. More generally any $(2k, 0)$-form of the type $(-\beta_1 \wedge \mathcal{J}\beta_1) \wedge \cdots \wedge (-\beta_k \wedge \mathcal{J}\beta_k)$ for $\beta_j \in \Lambda^{1,0}(M)$ is q-positive.

This definition of positivity for q-real $(2n, 0)$-forms agrees with the natural orientation on the bundle of q-real $(2n, 0)$-forms. To see this, fix a point $x \in M$ and choose a basis $(e_1, \dots, e_n)$ of T_xM as a left $\mathbb{H}$-module, then $z_{2j-1} = e_j + iIe_j$, $z_{2j} = \mathcal{J}z_{2j-1} = Je_j + iKe_j$ form a complex basis and the corresponding coframe $(dz^1, \dots, dz^{2n})$ induces the orientation form

$$\Theta = dz^1 \wedge \cdots \wedge dz^{2n}$$

which is q-positive by Example 2.21. Choose another basis $(f_1, \dots, f_n)$ on T_xM inducing the complex coframe $(dw^1, \dots, dw^{2n})$ in the same fashion as before. Since the bases $(e_1, \dots, e_n)$ and $(f_1, \dots, f_n)$ are related by a map $A \in \mathrm{Aut}_{\mathbb{H}}(T_xM) \cong \mathrm{GL}(n, \mathbb{H})$ the corresponding complex bases $(z_1, \dots, z_{2n})$, $(w_1, \dots, w_{2n})$ are related by $\beta(A)$, where $\beta \colon \mathbb{H}^{n,n} \to \mathbb{C}^{2n,2n}$ is the representation of Subsection 1.1.2. Therefore we get

$$dw^1 \wedge \cdots \wedge dw^{2n} = \det(\beta(A))\Theta = \mathrm{Sdet}(A)\Theta$$

and the Study determinant always takes positive values. This shows that the announced orientation is intrinsic.

It is not hard to check that a q-real $(2k, 0)$-form α is q-positive if and only if

$$\alpha \wedge (-\beta_1 \wedge \mathcal{J}\beta_1) \wedge \cdots \wedge (-\beta_{n-k} \wedge \mathcal{J}\beta_{n-k})$$

is a positive orientation form for every $\beta_1, \dots, \beta_{n-k} \in \Lambda^{1,0}(M)$.

Take a hyperhermitian metric g on (M, I, J, K), then we can take into account the form $\Omega \in \Lambda^{2,0}(M)$ defined in (2.10), since $J\omega_J = \omega_J$ and $J\omega_K = -\omega_K$ the form Ω is q-real. But it is also q-positive:

$$\Omega(Z, \mathcal{J}Z) = g(JZ, J\bar{Z}) + ig(KZ, J\bar{Z}) = g(Z, \bar{Z}) - ig(IZ, \bar{Z}) = 2|Z|_g^2\,.$$

Therefore any hyperhermitian metric induces a q-positive $(2, 0)$-form and the converse is also true, indeed if Ω is such a form, we can define a metric polarizing the quadratic form $Q(X) = \Omega(X, JX)$, i.e.

$$g(X, Y) := \frac{1}{2}\big(\Omega(X + Y, J(X + Y)) - \Omega(X, JX) - \Omega(Y, JY)\big) = \frac{1}{2}\big(\Omega(X, JY) + \Omega(Y, JX)\big)\,.$$

Such expression is clearly symmetric, moreover g is real-valued because Ω is q-real, it is I-Hermitian because Ω is of type $(2, 0)$:

$$g(IX, IY) = \frac{1}{2}\big(\Omega(IX, JIY) + \Omega(IY, JIX)\big) = -\frac{1}{2}\big(\Omega(IX, IJY) + \Omega(IY, IJX)\big) = g(X, Y)$$

and it is clearly J-Hermitian:

$$g(JX, JY) = \frac{1}{2}\big(\Omega(JX, J^2Y) + \Omega(JY, J^2X)\big) = g(X, Y)$$

therefore it is also K-Hermitian:

$$g(KX, KY) = g(IJX, IJY) = g(JX, JY) = g(X, Y)\,,$$

Finally, g is positive (semi)definite if and only if Ω is q-(semi)positive, because $g(X, X) = \Omega(X, JX)$.

We thus have showed:

Theorem 2.22 (Alesker-Verbitsky [17]). *On a hypercomplex manifold (M, I, J, K) there is a one-to-one correspondence between the bundle of (real parts of) hyperhermitian forms on TM and the bundle of q-real $(2,0)$-forms on M. Such correspondence is explicitly given by*

$$\{\textit{hyperhermitian forms on } TM\} \to \{\textit{q-real } (2,0) \textit{ forms on } M\}$$
$$g \mapsto g(J\cdot,\cdot) + ig(K\cdot,\cdot)$$
$$\frac{1}{2}\big(\Omega(\cdot, J\cdot) - \Omega(J\cdot,\cdot)\big) \leftarrow\!\shortmid \Omega$$

Furthermore a hyperhermitian form is positive (semi)definite if and only if the corresponding $(2,0)$-form is q-(semi)positive.

In view of this correspondence the form Ω defined in (2.10) completely determines the metric, therefore we shall often forget the metric and refer to (M, I, J, K, Ω) as a (almost) hyperhermitian manifold.

Quaternionic Hessian.

Take real coordinates (x_p^r), $p = 0,1,2,3$, $r = 1,\dots n$, on $\mathbb{R}^{4n} \cong \mathbb{H}^n$ corresponding to quaternionic coordinates:

$$q^r := \sum_{p=0}^{3} x_p^r e_p\,,$$

where, in order to simplify the notation, we denote the unit quaternions $1, i, j, k$ with e_0, e_1, e_2, e_3.

We can then introduce the *quaternionic derivatives* $\partial_{\bar{q}^r}$ and ∂_{q^s}, sometimes also called *Cauchy-Riemann-Fueter* operators, acting on smooth $\mathbb{H}$-valued functions as follows

$$\partial_{\bar{q}^r} u := \sum_{i=0}^{3} e_i\, \partial_{x_i^r} u\,, \qquad \partial_{q^r} u := \partial_{x_0^r} u\, e_0 - \sum_{i=1}^{3} \partial_{x_i^r} u\, e_i\,;$$

The operators ∂_{q^r} and $\partial_{\bar{q}^s}$ commute, but they do not satisfy the Leibniz rule nor the chain rule, so care must be taken during computations.

For any real-valued $u\colon \mathbb{H}^n \to \mathbb{R}$, the matrix of second derivatives

$$\mathrm{Hess}_{\mathbb{H}} u := \frac{1}{4}(\partial_{\bar{q}^r}\partial_{q^s} u)_{\bar{r}s}$$

is called the *quaternionic Hessian* of u. The reason of the normalization constant $\frac{1}{4}$ will become clear in the future (cf. Lemma 2.23). This is a hyperhermitian matrix, because $\partial_{\bar{q}^r}$ and ∂_{q^s} commute.

Using the vector fields ∂_{q^r} corresponding in the natural way to the operator ∂_{q^r} we can locally regard every q-real $(2,0)$-form Ω on M as a hyperhermitian matrix $(\Omega_{\bar{r}s}) \in \mathrm{Hyp}(n)$. Indeed, Ω corresponds bijectively to a hyperhermitian form g, which we can extend $\mathbb{H}$-sesquilinearly, i.e.

$$g(\lambda X, Y) = \bar{\lambda} g(X,Y)\,, \qquad g(X, \lambda Y) = g(X,Y)\lambda\,, \qquad \text{for every } \lambda \in \mathbb{H}\,,\ X, Y \in TM\,.$$

Then, g induces the hyperhermitian matrix $g_{\bar{r}s} := g(\partial_{q^r}, \partial_{q^s})$.

Observe that on a hypercomplex manifold (M, I, J, K) for any real-valued $u \in C^\infty(M,\mathbb{R})$ the $(2,0)$-form $\partial\partial_J u$ is q-real:

$$J\overline{\partial\partial_J u} = J\bar{\partial}J^{-1}\partial u = -J^{-1}\bar{\partial}J\partial u = -\partial_J\partial u = \partial\partial_J u\,.$$

The hyperhermitian matrix associated to $\partial\partial_J u$ whenever M is locally flat, is the quaternionic Hessian:

Lemma 2.23. *Let (M, I, J, K) be a locally flat hypercomplex manifold, $u \in C^\infty(M,\mathbb{R})$. Then the hyperhermitian matrix associated to $\partial\partial_J u$ is $\mathrm{Hess}_{\mathbb{H}} u$. In particular $\partial\partial_J u$ is q-(semi)positive if and only if $\mathrm{Hess}_{\mathbb{H}} u$ is (semi)positive definite.*

For a proof see [17].

Without the assumption of local flatness there is no possibility to construct the quaternionic Hessian in the sense above, however the hyperhermitian form associated to $\partial\partial_J u$ can still be viewed as an Hessian:

Lemma 2.24. *Let (M, I, J, K) be a hypercomplex manifold, $u \in C^\infty(M, \mathbb{R})$ and h the hyperhermitian form associated to $\partial\partial_J u$ (via Theorem 2.22). Then*

$$h = \frac{1}{4}(1 + I + J + K)\nabla^2 u\,,$$

where $\nabla^2 u$ is the Hessian of u with respect to the Obata connection.

Proof. Since both sides of the identity are hyperhermitian forms, it is enough to show it along the diagonal, i.e. $h(X, X) = \frac{1}{4}(1 + I + J + K)\nabla^2 u(X, X)$ for each $X \in \Gamma(TM)$. Clearly

$$4h(X, X) = 4\partial\partial_J u(X, JX) = (d+id^c_I)(d^c_J+id^c_K)u(X, JX) = (dd^c_J + d^c_K d^c_I + idd^c_K + id^c_I d^c_J)\, u(X, JX)\,.$$

We now treat the first term separately

$$\begin{aligned} dd^c_J u(X, JX) &= dJ^{-1}du(X, JX) = -dJdu(X, JX) \\ &= -X(Jdu(JX)) + JX(Jdu(X)) + Jdu([X, JX]) \\ &= \nabla_X(\nabla_X u) + \nabla_{JX}(\nabla_{JX} u) + \nabla_{J[X,JX]} u \\ &= \nabla_X(\nabla_X u) + \nabla_{JX}(\nabla_{JX} u) + \nabla_{J\nabla_X JX} u - \nabla_{J\nabla_{JX} X} u \\ &= \nabla^2 u(X, X) + \nabla^2 u(JX, JX) \end{aligned}$$

where we used that ∇ is torsion-free and $\nabla J = 0$. With a similar computation we arrive at the identities

$$\begin{aligned} d^c_K d^c_I u(X, JX) &= \nabla^2 u(IX, IX) + \nabla^2 u(KX, KX)\,, \\ dd^c_K u(X, JX) &= \nabla^2 u(X, IX) + \nabla^2 u(JX, KX)\,, \\ d^c_I d^c_J u(X, JX) &= -\nabla^2 u(X, IX) - \nabla^2 u(JX, KX) \end{aligned}$$

which imply the result. □

It will be convenient to have the explicit pointwise expression of $\partial\partial_J u$ in holomorphic coordinates $(z^1, \ldots, z^{2n})$ around some given point. We may assume that $Jdz^{2k-1} = -d\bar{z}^{2k}$ for $k = 1, \ldots, n$, then

$$\begin{aligned} \partial\partial_J u =& \partial J^{-1}\bar{\partial} u = \partial J^{-1}\sum_{r=1}^{2n} u_{\bar{r}} d\bar{z}^r = \partial \sum_{r=1}^{2n}(-1)^r u_{\overline{r-(-1)^r}} dz^r = \sum_{r,s=1}^{2n}(-1)^r u_{s\,\overline{r-(-1)^r}} dz^s \wedge dz^r \\ =& \sum_{r,s=1}^{n}(u_{2s-1\,\overline{2r-1}} + u_{2r\,\overline{2s}})dz^{2s-1} \wedge dz^{2r} + \sum_{s<r}(u_{2r-1\,\overline{2s}} - u_{2s-1\,\overline{2r}})dz^{2s-1} \wedge dz^{2r-1} \\ &+ \sum_{s<r}(u_{2s\,\overline{2r-1}} - u_{2r\,\overline{2s-1}})dz^{2s} \wedge dz^{2r} \end{aligned}$$

in particular, at a point where the complex Hessian $(u_{s\bar{r}})$ is diagonal we have

$$\partial\partial_J u = \sum_{r=1}^{n}(u_{2r-1\,\overline{2r-1}} + u_{2r\,\overline{2r}})dz^{2r-1} \wedge dz^{2r} \tag{2.14}$$

We can also link the n^{th} wedge power of $\partial\partial_J u$ with the Moore determinant of the quaternionic Hessian:

$$(\partial\partial_J u)^n = \det(\mathrm{Hess}_{\mathbb{H}} u)\,\Omega^n\,.$$

For future reference we establish a more general result involving the elementary symmetric functions σ_r. Recall that for each $r = 1, \dots, n$

$$\sigma_r(\lambda) = \sum_{1 \le i_1 < \cdots < i_r \le n} \lambda_{i_1} \cdots \lambda_{i_r}\,, \qquad \text{for all } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n\,.$$

Lemma 2.25. *Let (M, I, J, K, g, Ω) be a locally flat hyperhermitian manifold. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the n-tuple of eigenvalues of* $\mathrm{Hess}_{\mathbb{H}} u$ *with respect to g, where $u \colon M \to \mathbb{R}$. Then*

$$\binom{n}{k} \frac{(\partial \partial_J u)^k \wedge \Omega^{n-k}}{\Omega^n} = \sigma_k(\lambda)\,.$$

Proof. We work in holomorphic coordinates $(z^1, \dots, z^{2n})$ at a point where (the hermitian matrix associated to) g is half of the identity and the complex Hessian of u is diagonal. Let $q^r = z^{2r-1} + j z^{2r}$ be quaternionic coordinates. If the complex Hessian is diagonal, then so is the quaternionic Hessian, because in general for a real-valued u it takes the form

$$\mathrm{Hess}_{\mathbb{H}} u = \frac{1}{4} \partial_{\bar{q}^r} \partial_{q^s} u = (\partial_{\bar{z}^{2r-1}} + j \partial_{\bar{z}^{2r}})(\partial_{z^{2s-1}} - j \partial_{\bar{z}^{2s}}) u = u_{\overline{2r-1}\, 2s-1} + u_{2r\, \overline{2s}} + j(u_{\overline{2r}\, 2s-1} - u_{2r-1\, \overline{2s}})\,.$$

This implies, by using (2.14), that Ω and $\partial \partial_J u$ at the given point take the form

$$\Omega = \sum_{i=1}^n dz^{2i-1} \wedge dz^{2i}\,, \qquad \partial \partial_J u = \sum_{i=1}^n \lambda_i dz^{2i-1} \wedge dz^{2i}\,.$$

With these assumptions we compute

$$\begin{aligned}
(\partial \partial_J u)^k \wedge \Omega^{n-k} &= \sum_{\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}} \lambda_{i_1} \cdots \lambda_{i_k} dz^{2i_1 - 1} \wedge dz^{2i_1} \wedge \cdots \wedge dz^{2j_{n-k}-1} \wedge dz^{2j_{n-k}} \\
&= (n-k)! \sum_{\substack{i_1, \dots, i_k = 1 \\ i_1, \dots, i_k \text{ distinct}}}^n \lambda_{i_1} \cdots \lambda_{i_k} dz^1 \wedge \cdots \wedge dz^{2n} \\
&= \frac{(n-k)! k!}{n!} \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} \Omega^n = \frac{1}{\binom{n}{k}} \sigma_k(\lambda) \Omega^n
\end{aligned}$$

which is the desired formula. ☐

Quaternionic Laplacian.

A straightforward verification shows that the diagonal elements of the quaternionic Hessian of a function $u \colon \mathbb{H}^n \to \mathbb{R}$ are $\partial_{\bar{q}^r} \partial_{q^r} u = \sum_{p=0}^3 \partial^2_{x^r_p} u$, in particular the trace of the quaternionic Hessian is, up to a constant, the usual Laplacian:

$$\mathrm{tr}(\mathrm{Hess}_{\mathbb{H}} u) = \frac{1}{4} \sum_{r=1}^n \partial_{\bar{q}^r} \partial_{q^s} u = \frac{1}{4} \sum_{r=1}^n \sum_{p=0}^3 \partial^2_{x^r_p} u = \frac{1}{4} \Delta u\,.$$

As observed in the first chapter the trace of a hyperhermitian matrix coincides with the sum of the eigenvalues. So, the Laplacian represents the sum of the eigenvalues of the quaternionic Hessian.

If the inner product g is not the standard one, the trace of $\mathrm{Hess}_{\mathbb{H}} u$ with respect to g need not be real, and in order to retain the interpretation of the Laplacian as the sum of the eigenvalues we need to take the real part. Let us explain this point in more detail. If $G, B \in \mathrm{Hyp}(n)$ are hyperhermitian matrices then $A = G^{-1} B$ is hyperhermitian with respect to G, i.e. $A = G^{-1} A^* G$. In this case the eigenvalues of A are still well-defined and real, but the trace of A is in general no longer real, however, it is straightforward to verify that $\mathrm{Re}\,\mathrm{tr}(A) = \mathrm{Re}\,\mathrm{tr}_G(B)$ is still the sum of the eigenvalues of A.

We thus may define the *quaternionic Laplacian* Δ_g on $\mathbb{H}^n$ with respect to the inner product g on $\mathbb{H}^n$ to be the operator acting on smooth functions $u\colon \mathbb{H}^n \to \mathbb{R}$ as

$$\Delta_g u := \operatorname{Re}\operatorname{tr}_g(\mathrm{Hess}_{\mathbb{H}} u) = \operatorname{Re}\operatorname{tr}(G^{-1}\mathrm{Hess}_{\mathbb{H}} u)\,,$$

where G is the hyperhermitian matrix associated to g.

All of the above has of course a local definition on locally flat hyperhermitian manifolds, once we have chosen a coordinate neighborhood corresponding to quaternionic local coordinates. However, there is also an intrinsic description, thanks to Lemma 2.25, which allows to generalize the concept on hyperhermitian manifolds that are not locally flat.

Definition 2.26. On a hyperhermitian manifold (M, I, J, K, g, Ω) the **quaternionic Laplacian** with respect to g is the second order linear operator

$$\Delta_g u := n\frac{\partial\partial_J u \wedge \Omega^{n-1}}{\Omega^n}\,.$$

From Lemma 2.24 and Lemma 2.25 we can also express the quaternionic Laplacian as the Laplacian of the Obata connection:

Corollary 2.27. *On a hyperhermitian manifold* (M, I, J, K, g, Ω)

$$\Delta_g u = \operatorname{tr}_g(\nabla^2 u)\,,$$

where $\operatorname{tr}_g(\nabla^2 u) = \sum_{k=1}^{4n} \nabla^2 u(e_k, e_k)$*, where* e_k *is an adapted orthonormal basis with respect to* g*, i.e.* $e_{4r-2} = Ie_{4r-3}$*,* $e_{4r-1} = Je_{4r-r}$*,* $e_{4r} = Ke_{4r-3}$ *for* $r = 1, \dots, n$*. In particular, the quaternionic Laplacian is a uniformly elliptic operator.*

Proof. The expression for Δ_g is clear. To see that it is an elliptic operator we compute its symbol $\sigma_{\Delta_g}\colon T^*M \to \mathbb{R}$:

$$\sigma_{\Delta_g}(\xi) = g^{ij}\xi_i\xi_j = |\xi|_g^2\,, \qquad \text{for all } \xi \in T^*M\,,$$

therefore Δ_g is uniformly elliptic with ellipticity constant 1. □

Unfortunately, in general, the quaternionic Laplacian is not self-adjoint (with respect to the L^2 product), and its integral does not vanish. However, under the additional assumption that the canonical bundle K_M of (M, I) allows a q-positive holomorphic trivialization the quaternionic Laplacian indeed satisfies these properties. We will discuss in detail the existence of a q-positive holomorphic section of K_M in subsection 2.2.4.

Lemma 2.28. *Let* (M, I, J, K, g, Ω) *be a compact hyperhermitian manifold such that* K_M *admits a q-positive holomorphic global section* Θ*. Then, for any* $u, v \in C^\infty(M, \mathbb{R})$

$$\int_M \Delta_g u\, \Omega^n \wedge \bar{\Theta} = 0$$

and

$$\int_M \Delta_g uv\, \Omega^n \wedge \bar{\Theta} = \int_M u\Delta_g v\, \Omega^n \wedge \bar{\Theta}\,.$$

Proof. First, let us observe that $\Omega^n \wedge \bar{\Theta}$ is a positive real volume because J acts trivially on top forms and Ω and Θ are q-real:

$$\overline{\Omega^n \wedge \bar{\Theta}} = \bar{\Omega}^n \wedge \Theta = J\Omega^n \wedge J\bar{\Theta} = J(\Omega^n \wedge \bar{\Theta}) = \Omega^n \wedge \bar{\Theta}\,.$$

Now, using the definition of Δ_g, integration by parts and Stokes' Theorem we have

$$\int_M \Delta_g u\, \Omega^n \wedge \bar{\Theta} = \int_M \partial\partial_J u \wedge \Omega^{n-1} \wedge \bar{\Theta} = \int_M \partial\left(\partial_J u \wedge \Omega^{n-1} \wedge \bar{\Theta}\right) + \int_M \partial_J u \wedge \partial(\Omega^{n-1} \wedge \bar{\Theta}) = 0\,;$$

where, $\partial(\Omega^{n-1}\wedge\bar{\Theta}) = (n-1)\partial\Omega\wedge\Omega^{n-2}\wedge\bar{\Theta} + \Omega^{n-1}\wedge\partial\bar{\Theta} = 0$ because of the HKT condition on Ω and the holomorphicity of Θ.

Similarly, one can observe (again using that J acts trivially on top forms) that ∂_J essentially behaves as ∂ and satisfies Stokes' Theorem, hence

$$\begin{aligned}\int_M \Delta_g uv\,\Omega^n\wedge\bar{\Theta} &= \int_M v\partial\partial_J u\wedge\Omega^{n-1}\wedge\bar{\Theta} = -\int_M \partial v\wedge\partial_J u\wedge\Omega^{n-1}\wedge\bar{\Theta} = -\int_M u\partial_J\partial v\wedge\Omega^{n-1}\wedge\bar{\Theta}\\ &= \int_M u\partial\partial_J v\wedge\Omega^{n-1}\wedge\bar{\Theta} = \int_M u\Delta_g v\,\Omega^n\wedge\bar{\Theta}\end{aligned}$$

and the lemma is proved. □

2.2.2 HKT potentials.

Definition of HKT potential.

The notion of a potential for HKT forms was proposed by Grantcharov and Poon [148] in analogy to the usual Kähler potential. If (M,I,g) is a Kähler manifold with Kähler form $\omega = g(I\cdot,\cdot)$ a (possibly local) potential function for ω is a function u such that

$$\omega = i\partial\bar{\partial}u = \frac{1}{2}dd^c u\,.$$

If (M,I,J,K,g) is hyperkähler then we have three Kähler forms and a function u is a hyperkähler potential if it is a potential for each of them, i.e.

$$\omega_I = \frac{1}{2}dd^c_I u\,,\qquad \omega_J = \frac{1}{2}dd^c_J u\,,\qquad \omega_K = \frac{1}{2}dd^c_K u\,.$$

Definition 2.29. Let (M,I,J,K,Ω) be a HKT manifold. A (possibly local) **potential** for the HKT form (or the HKT metric) is a smooth real-valued function u such that

$$\Omega = \partial\partial_J u\,.$$

The definition is equivalent to require $g = \frac{1}{4}(1+I+J+K)\nabla^2 u$ by Lemma 2.24 and this holds if and only if any of the following expressions holds:

$$\omega_I = \frac{1}{4}\left(dd^c_I + d^c_J d^c_K\right)u\,,\qquad \omega_J = \frac{1}{4}\left(dd^c_J + d^c_K d^c_I\right)u\,,\qquad \omega_K = \frac{1}{4}\left(dd^c_K + d^c_I d^c_J\right)u\,. \tag{2.15}$$

Seeing that all expressions in (2.15) are equivalent is just a matter of calculations by expanding the terms, the equivalence with Definition 2.29 is immediate as

$$\partial\partial_J u = \frac{1}{4}(d+id^c_I)(d^c_J+id^c_K)u = \frac{1}{4}\left(dd^c_J + d^c_K d^c_I + idd^c_K + id^c_I d^c_J\right)u\,.$$

The identities (2.15) imply (after a short computation) that a hyperkähler potential is in particular an HKT potential.

Example 2.30. On the flat space $\mathbb{H}^n$ the standard hyperhermitian metric g admits as HKT potential the function

$$u\colon q = (q^1,\dots,q^n)\mapsto \frac{1}{2}\|q\|^2 = \frac{1}{2}\sum_{j=1}^n |q^j|^2\,,$$

indeed, this is actually an hyperkähler potential. Take holomorphic coordinates $(z^1,\dots,z^{2n})$ such that

$q^j = z^{2j} + z^{2j+1}j$ then $u(q) = \frac{1}{2}\sum_{j=1}^{2n} |z^j|^2$ and by (2.14) we conclude

$$\partial\partial_J u = \frac{1}{2}\partial\partial_J\left(\sum_{j=1}^{2n} z^j \bar{z}^j\right) = \sum_{j=1}^{n} dz^{2j-1}\wedge dz^{2j} = \Omega\,.$$

Existence of local potentials.

Proposition 2.31. *Every hypercomplex manifold (M, I, J, K) locally admits a HKT metric.*

Proof. Let u be a local Kähler potential for (M, I), then $g = \frac{1}{2}dd^c_I u(\cdot, I\cdot)$ is a I-Hermitian metric with respect to which (2.15) holds and thus $\Omega = \omega_J + i\omega_K = \partial\partial_J u$ locally. □

On the other hand global potentials do not always exist, for instance they never occur on compact manifolds, as follows from a standard argument involving the maximum principle. Indeed, if u were a global HKT potential of a HKT metric g on a compact hypercomplex manifold (M, I, J, K), then setting $\Omega = \partial\partial_J u$ for the corresponding HKT form we would have

$$\Delta_g u = n\frac{\partial\partial_J u\wedge\Omega^{n-1}}{\Omega^n} \equiv n \geq 0\,.$$

Since the quaternionic Laplacian is a second order linear elliptic operator without free term, by the maximum principle u must be constant, which is a contradiction.

In order to give an idea of the proof that on HKT manifolds local potentials always exist, we need to introduce the Salamon complex. Let (M, I, J, K) be an almost hypercomplex manifold and let S^2 be the sphere of almost complex structures on M. While investigating the structure of the more general class of quaternionic manifolds, Salamon [252] introduces the bundles

$$A^k(M) := \sum_{L\in S^2}\Lambda^{k,0}_L T^*M\,,\qquad B^k(M) := \bigcap_{L\in S^2}\left(\Lambda^{k-1,1}_L T^*M\oplus\cdots\oplus\Lambda^{1,k-1}_L T^*M\right).$$

Observe that for each $L \in S^2$ also $-L \in S^2$ and $\Lambda^{k,0}_{-L}T^*M = \Lambda^{0,k}_L T^*M$. Clearly

$$\Lambda^k(M) = A^k(M)\oplus B^k(M)\,,$$

where we are denoting in the same way the bundles $A^k(M)$ and $B^k(M)$ and the relative spaces of sections. Let $\pi\colon \Lambda^k(M) \to A^k(M)$ be the orthogonal projection and consider the composition

$$D := \pi\circ d\colon A^k(M) \to A^{k+1}(M)\,. \tag{2.16}$$

Theorem 2.32. *The almost hypercomplex structure is hypercomplex if and only if*

$$0 \longrightarrow A^0 \xrightarrow{D=d} A^1 \xrightarrow{D} \cdots \longrightarrow A^{2n} \longrightarrow 0 \tag{2.17}$$

is a complex, i.e. $D^2 = 0$. In this case we call (2.17) *the* ***quaternionic Dolbeault complex*** *or the* ***Salamon complex*** *and* (2.16) *the* ***Salamon differential****.*

We remark the analogy with the corresponding result for an almost complex structure and the corresponding Dolbeault operator $\bar\partial$.

Assume from now on that (I, J, K) is hypercomplex. For simplicity we shall work with the complex structure I, but the following would work taking any $L \in S^2$. Setting $A^{p,q}_I(M) = A^{p+q}(M)\cap\Lambda^{p,q}_I(M)$ we deduce the Hodge decomposition

$$A^k(M) = \bigoplus_{p+q=k} A^{p,q}_I(M)\,. \tag{2.18}$$

Observe that $A_I^{p,0}(M) = \Lambda_I^{p,0}(M)$. More generally, Verbitsky proved in [304] using a slightly different formalism that $A_I^{p,q}(M)$ is isomorphic to $\Lambda_I^{p+q,0}(M)$. Actually, he proves more: let us split the Salamon differential $D = D^{1,0} + D^{0,1}$ according to the Hodge decomposition (2.18), i.e.

$$D^{1,0}\colon A_I^{p,q}(M) \to A_I^{p+1,q}(M)\,, \qquad D^{0,1}\colon A_I^{p,q}(M) \to A_I^{p,q+1}(M)\,,$$

then the Salamon complex, after Hodge decomposition, becomes a bicomplex $(A_I^{\bullet,\bullet}(M), D^{1,0}, D^{0,1})$ and this is isomorphic to the complex $(\Lambda_I^{\bullet,0}(M), \partial, \partial_J)$. In view of the result of Verbitsky Theorem 2.32 can be seen as a consequence of Proposition 2.18.

After some partial results [226, 247] Banos and Swann were able to prove the following *local $\partial\partial_J$-lemma* thus establishing existence of local HKT potentials:

Theorem 2.33 (Banos-Swann [27])**.** *Let (M, I, J, K) be a hypercomplex manifold. A q-real form $\Omega \in \Lambda^{2,0}(M)$ can be written locally as*

$$\Omega = \partial\partial_J u$$

for some smooth real-valued local function u, if and only if it is HKT, i.e. $\partial\Omega = 0$.

Sketch of proof. The only if part is obvious. By [301, Theorem 5.7] (see also [27]) Ω is HKT if and only if $D\omega_I = 0$, where D is the Salamon differential. The converse is thus a consequence of the following two facts:

- Any D-closed form is locally D-exact.
- A HKT form admits a local potential if and only if ω_I is locally D-exact.

The first fact is due to Mamone Capria and Salamon [221] and uses twistor theory (see also [27]). The second is proved as follows. If $\Omega = \partial\partial_J u$, then $\omega_I = \frac{1}{2}(d\eta - Jd\eta)$, where $\eta = \frac{1}{2}d_I^c u$. Since $d\eta = \frac{1}{2}dd_I^c u$ is a $(1,1)$-form with respect to I

$$D\eta = \pi(d\eta) = (d\eta)^{2,0} + (d\eta)^{0,2} + \frac{1}{2}\left((d\eta)^{1,1} - J(d\eta)^{1,1}\right) = \frac{1}{2}\left(d\eta - Jd\eta\right) = \omega_I\,.$$

Conversely, if $\omega_I = D\eta$ for some η, we must have $\omega_I = \frac{1}{2}(d\eta - Jd\eta)$, because ω_I is of type $(1,1)$, in particular also $d\eta \in \Lambda^{1,1}(M)$. From the local dd_I^c-lemma, there exists a local function u such that $d\eta = \frac{1}{2}dd_I^c u$, implying the claim. □

2.2.3 The Dolbeault differential graded algebra of HKT manifolds.

We already observed that the triple $(\Lambda^{\bullet,q}(M), \partial, \partial_J)$ forms a cochain complex for every fixed q. This slightly differs from the complex case, as here we obtain a single complex, while in the complex setting ∂ and $\bar\partial$ give rise to a double complex. In this subsection we shall study such cochain complex in detail when $q = 0$.

Kodaira relations.

There is a deep analogy between $(\Lambda^{\bullet,0}(M), \partial)$ and the de Rham differential graded algebra of a Kähler manifold. The role of the de Rham differential is played by the Dolbeault differential on $\Lambda^{\bullet,0}(M)$ and the Kähler form is replaced by the $(2,0)$-form Ω. The existence of the non-degenerate ∂-closed form Ω naturally leads to consider the Lefschetz operator $L = L_\Omega = \frac{\Omega}{2} \wedge -$. Recall the definition of the Hodge star operator $*\colon \Lambda^{p,q}(M) \to \Lambda^{2n-q,2n-p}(M)$ by the relation:

$$\alpha \wedge *\bar\beta = g(\alpha,\beta)\mathrm{Vol}_g\,, \qquad \text{for every } \alpha,\beta \in \Lambda^{p,q}(M)\,,$$

where g here is the Hermitian product induced by the Riemannian metric on $\Lambda^{p,q}(M)$ and

$$\mathrm{Vol}_g = \frac{\Omega^n \wedge \bar\Omega^n}{(n!)^2}$$

is the standard Riemannian volume form. Then we actually have a Lefschetz triple (L, Λ, H), where $\Lambda = \Lambda_\Omega = *^{-1}\bar{L}* = *^{-1}L_{\bar{\Omega}}*$ is the adjoint to L and $H = [L, \Lambda]$.

Remark 2.34. The normalization of the Lefschetz operator as the wedge product with $\Omega/2$ instead of Ω is needed to have an actual Lefschetz triple.

The Lefschetz triple generates an $\mathfrak{sl}(2, \mathbb{C})$ action on $\Lambda^{\bullet,0}(M)$ and the following relations hold:

$$[L, \partial] = [L, \partial_J] = 0\,, \qquad [H, \partial] = \partial\,, \qquad [H, \partial_J] = \partial_J\,.$$

Verbitsky uses these properties to shows that the Lie superalgebra generated by $\{L, \Lambda, H, \partial, \partial_J\}$ is isomorphic to the de Rham superalgebra on a Kähler manifold (M, I, ω), generated by $\{L_\omega, \Lambda_\omega, H_\omega, d, d^c_I\}$.

Let us recall here that a *Lie superalgebra* is a pair $(A, [\cdot,\cdot])$ such that

- A is a $\mathbb{Z}_2$-graded vector space: $A = A^0 \oplus A^1$ an element $a \in A$ is called *pure* if $a \in A^0$ or $A \in A^1$, when this holds we denote $\deg(a) = i$ if $a \in A^i$;
- $[\cdot,\cdot]\colon A \times A \to A$ is a bilinear operator which is *graded anti-commutative*, i.e. for pure elements $a, b \in A$
$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba\,,$$
and satisfies the *graded Jacobi identity*, i.e. for pure elements $a, b, c \in A$
$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]]\,.$$
The operator $[\cdot,\cdot]$ is called *supercommutator*.
- The supercommutator is compatible with the grading, in the sense that if $a, b \in A$ are pure, then so is $[a, b]$ and $\deg([a, b]) = \deg(a)\deg(b) \mod \mathbb{Z}_2$.

Verbitsky also considered the adjoints

$$\partial^* = - * \bar{\partial} *\,, \qquad \partial_J^* = - * \bar{\partial}_J *$$

and obtains

$$[\partial^*, L] = \partial_J - \theta_J \wedge -\,,$$

where $\theta_J = J\bar{\theta}$ being θ the uniquely defined $(1, 0)$-form satisfying the relation

$$\partial\bar{\Omega}^n = \theta \wedge \bar{\Omega}^n\,.$$

By duality and twisting with $\mathcal{J} = J \circ \bar{}$ this yields:

Proposition 2.35. *Let (M, I, J, K, Ω) be a HKT manifold. Then, the following identities hold*

$$[L, \partial^*] = -\delta_J\,, \qquad [\Lambda, \partial] = \delta_J^*\,, \qquad [L, \partial_J^*] = \delta\,, \qquad [\Lambda, \partial_J] = -\delta^*\,,$$

where $\delta = \partial + \theta \wedge -$, $\delta_J = J^{-1}\bar{\delta}J = \partial_J - \theta_J \wedge -$, $\delta^* = - * \bar{\delta} *$, $\delta_J^* = - * \bar{\delta}_J *$.

The $(1, 0)$-form θ.

Before we continue, let us describe more in detail the form θ. Let (M, I, J, K, g, Ω) be a hyperhermitian manifold and $K_M = K(M, I) = \Lambda^{2n,0}(M)$ the canonical bundle of (M, I). Observe that K_M is always topologically trivial because the form Ω is non-degenerate and Ω^n provides a nowhere vanishing global section.

Call α the connection 1-form of the Obata connection ∇ with respect to the trivialization determined by Ω^n, in other words

$$\nabla\Omega^n = \alpha \otimes \Omega^n$$

such form is real, as for any $X \in \Gamma(TM)$

$$\alpha(X)J\Omega^n = J\nabla_X\Omega^n = \nabla_X J\Omega^n = \nabla_X\bar{\Omega}^n = \bar{\alpha}(X)\bar{\Omega}^n = \bar{\alpha}(X)J\Omega^n\,. \tag{2.19}$$

Observe that the form θ we encountered earlier is precisely the $(1,0)$-component of α, indeed, since ∇ is torsion-free the exterior differential of $\bar{\Omega}^n$ coincides with the alternation of $\nabla\bar{\Omega}^n$, i.e. let $\mathrm{Alt}\colon \Lambda^{2n}(M)\otimes\Lambda^1(M)\to\Lambda^{2n+1}(M)$ denote the exterior product, then $d\bar{\Omega}^n=\mathrm{Alt}(\nabla\bar{\Omega}^n)=\alpha\wedge\bar{\Omega}^n$ and thus

$$\theta\wedge\bar{\Omega}^n=\partial\bar{\Omega}^n=d\bar{\Omega}^n=\mathrm{Alt}(\nabla\bar{\Omega}^n)=\alpha\wedge\bar{\Omega}^n=\alpha^{1,0}\wedge\bar{\Omega}^n$$

which means $\theta=\alpha^{1,0}$.

Such a form is ∂-closed and satisfies the identity

$$\partial\theta_J=\partial_J\theta\,, \tag{2.20}$$

where $\theta_J:=J\bar{\theta}$ in particular $\partial_J\theta$ is q-real. Let us start with the closure:

$$0=\partial^2\bar{\Omega}^n=\partial(\theta\wedge\bar{\Omega}^n)=\partial\theta\wedge\bar{\Omega}^n-\cancel{\theta\wedge\theta\wedge\bar{\Omega}^n}\,,$$

the identity (2.20) is equally easy:

$$0=(\partial\partial_J+\partial_J\partial)\bar{\Omega}^n=-\partial(\theta_J\wedge\bar{\Omega}^n)+\partial_J(\theta\wedge\bar{\Omega}^n)=-\partial\theta_J\wedge\bar{\Omega}^n+\cancel{\theta_J\wedge\theta\wedge\bar{\Omega}^n}+\partial_J\theta\wedge\bar{\Omega}^n+\cancel{\theta\wedge\theta_J\wedge\bar{\Omega}^n}$$

thus we deduce $\partial_J\theta-\partial\theta_J=0$, as desired. Finally (2.20) implies q-realness:

$$J\overline{\partial_J\theta}=\partial J\bar{\theta}=\partial\theta_J=\partial_J\theta\,.$$

The normalized HKT superalgebra.

The superalgebras $\langle L,\Lambda,H,\partial,\partial_J,\delta^*,\delta_J^*\rangle$ and $\langle L,\Lambda,H,\partial^*,\partial_J^*,\delta,\delta_J\rangle$ are isomorphic, however they are distinct, hence they are not closed by the operation of duality via the Hodge star. In order to fix this, Verbitsky considers the *"normalized" superalgebra*:

$$\mathfrak{g}^\theta:=\left\langle L,\Lambda,H,\partial^\theta,\partial^\theta_J,(\partial^\theta)^*,(\partial^\theta_J)^*\right\rangle\,,$$

where

$$\partial^\theta=\frac{\partial+\delta}{2}=\partial+\frac{1}{2}\theta\wedge-\,,\qquad \partial^\theta_J=\frac{\partial_J+\delta_J}{2}=\partial_J-\frac{1}{2}\theta_J\wedge-\,,$$
$$(\partial^\theta)^*=-*\bar{\partial}^\theta*=\frac{\partial^*+\delta^*}{2}\,,\qquad (\partial^\theta_J)^*=-*\bar{\partial}^\theta_J*=\frac{\partial^*_J+\delta^*_J}{2}\,.$$

In the trivialization determined by Ω^n we have seen that the Obata connection on K_M can be written as $\nabla=d+\theta+\bar{\theta}$. Let $K_M^{1/2}$ be a square root of K_M determined by the above trivialization. Then the holomorphic structure on $K_M^{1/2}$ is defined by the connection $\nabla_{1/2}=d+\frac{1}{2}\theta+\frac{1}{2}\bar{\theta}$. Furthermore, with the chosen trivialization, one can identify the Dolbeault complex

$$\left(\Lambda^{\bullet,0}(M)\otimes K_M^{1/2},\nabla^{1,0}_{1/2}=\partial+\frac{1}{2}\theta\right)$$

of forms with values in $K_M^{1/2}$ with the complex $(\Lambda^{\bullet,0}(M),\partial^\theta)$. In other words, the Dolbeault differential of $K_M^{1/2}$ is equal to the normalized HKT differential ∂^θ.

Using the properties of θ it is easily checked that

$$[\partial^\theta,\partial^\theta]=[\partial^\theta_J,\partial^\theta_J]=[\partial^\theta,\partial^\theta_J]=[L,\partial^\theta]=[L,\partial^\theta_J]=0\,,\qquad [H,\partial^\theta]=\partial^\theta\,,\qquad [H,\partial^\theta_J]=\partial^\theta_J\,,$$

hence, once again, $\mathfrak{g}^\theta$ is isomorphic to the de Rham superalgebra of a Kähler manifold and we have:

Proposition 2.36 (Misha Verbitsky [301])**.** *Let (M,I,J,K,Ω) be a compact HKT manifold and $K_M^{1/2}$ the square root of the canonical bundle constructed above via the trivialization determined by Ω^n. Then*

1. *The normalized Laplacians*

$$\Delta_{\partial^\theta} := [\partial^\theta, (\partial^\theta)^*] = \partial^\theta(\partial^\theta)^* + (\partial^\theta)^*\partial^\theta\,, \qquad \Delta_{\partial^\theta_J} := [\partial^\theta_J, (\partial^\theta_J)^*] = \partial^\theta_J(\partial^\theta_J)^* + (\partial^\theta_J)^*\partial^\theta_J$$

coincide and their kernel is identified with the cohomology of $(K_M^{1/2}, \nabla^{1,0}_{1/2})$ *(or, equivalently, of* $(\Lambda^{\bullet,0}(M), \partial^\theta)$*).*

2. *The normalized Laplacian commutes with the action of the* $\mathfrak{sl}(2,\mathbb{C})$*-triple* (L, Λ, H)*.*

3. *The Hard Lefschetz isomorphism* $L^{n-i}\colon H^i(K_M^{1/2}) \to H^{2n-1}(K_M^{1/2})$ *holds, as well as the Serre duality* $H^{2n-i}(K_M^{1/2}) \cong H^i(K_M^{1/2})^*$*.*

Other cohomological results that strengthen the parallel between Kähler and HKT geometry have been proved. For instance, Grantcharov, Lejmi and Verbitsky [146] proved that a quaternionic surface with holonomy of the Obata connection inside $\mathrm{SL}(2,\mathbb{H})$ is HKT if and only if $H^{1,0}(M, I)$ is even-dimensional. This is the analogue of a classical result independently proved by Buchdahl [58] and Lamari [204]. Lejmi and Weber also explored further quaternionic cohomologies [208] (see also [207]) and some obstructions they provide to the existence of HKT structures.

2.2.4 $\mathrm{SL}(n,\mathbb{H})$-manifolds and the balanced condition.

Computations with the Hodge star operator.

Let (M, I, J, K, g, Ω) be a hyperhermitian manifold. Recall that the Hodge star operator is defined by the relation:

$$\alpha \wedge *\bar\beta = g(\alpha,\beta)\frac{\Omega^n \wedge \bar\Omega^n}{(n!)^2}\,, \qquad \text{for every } \alpha,\beta \in \Lambda^{p,q}(M)\,.$$

It is easy to compute

$$*\Omega = \frac{4}{(n-1)!\,n!}\Omega^n \wedge \bar\Omega^{n-1}\,.$$

We shall also need to compute the Hodge star of an arbitrary $(1,0)$-form α. In order to perform this calculation we first need to establish the following:

$$\frac{\alpha\wedge\beta\wedge\Omega^{n-1}}{\Omega^n} = \frac{1}{2n}g(\alpha, J\bar\beta)\,, \qquad \text{for every } \alpha,\beta \in \Lambda^{1,0}(M)\,, \tag{2.21}$$

We have

$$\alpha\wedge\beta\wedge\Omega^{n-1}\wedge\bar\Omega^n = g\left(\alpha\wedge\beta, *(\bar\Omega^{n-1}\wedge\Omega^n)\right)\mathrm{Vol}_g = \frac{(n-1)!\,n!}{4(n!)^2}g\left(\alpha\wedge\beta,\Omega\right)\Omega^n\wedge\bar\Omega^n \tag{2.22}$$

now, we compute $g(\alpha\wedge\beta,\Omega)$ in local holomorphic coordinates:

$$g(\alpha\wedge\beta,\Omega) = g^{i\bar j}g^{r\bar s}\alpha_i\beta_r(\overline{J^{\bar a}_j g_{s\bar a} + iK^{\bar b}_j g_{s\bar b}}) = g^{i\bar j}\alpha_i\beta_r(J^r_{\bar j} - iK^r_{\bar j}) = 2g(\alpha, J\bar\beta) \tag{2.23}$$

because β is of type $(1,0)$. So, combining (2.22) and (2.23) we obtain (2.21). It is now easy to compute the Hodge star of a form $\alpha \in \Lambda^{1,0}(M)$:

$$\alpha\wedge *\bar\alpha = g(\alpha,\alpha)\mathrm{Vol}_g = -\frac{2n}{(n!)^2}\alpha\wedge J\bar\alpha\wedge\Omega^{n-1}\wedge\bar\Omega^n$$

so

$$*\alpha = -\frac{2}{(n-1)!\,n!}J\alpha\wedge\Omega^n\wedge\bar\Omega^{n-1}\,.$$

Balanced HKT manifold.

On a Hermitian manifold (M, I, g) the **Lee form** is the 1-form defined by

$$\theta^I_{\text{Lee}} := -Id^*\omega_I\,,$$

where $d^* = -*d*$ is the formal adjoint of the exterior differential and $\omega_I = g(I\cdot,\cdot)$ is the fundamental form of the Hermitian structure. Equivalently, the Lee form is the unique form such that $d\omega_I^{n-1} = \theta\wedge\omega_I^{n-1}$, where n is the complex dimension of M. Such form was introduced by Lee [205] and it bears interesting properties with respect to conformal transformations. For instance, whenever $\theta_{\text{Lee}} = 0$, i.e. ω_I is coclosed we say that the Hermitian manifold is **balanced**. The balanced condition $d\omega_I^{n-1} = 0$ is the sole of the form $d\omega_I^k = 0$ with $k < n$ that does not imply that ω_I is Kähler [149]. Balanced metrics have been studied extensively since the paper of Michelsohn [225]. Alessandrini and Bassanelli [19] proved that unlike the Kähler condition balancedness is invariant under modifications. For further references see the survey [128].

On a hyperhermitian manifold (M, I, J, K, g) we naturally have three Lee forms $\theta^I_{\text{Lee}}, \theta^J_{\text{Lee}}, \theta^K_{\text{Lee}}$. In some circumstances they all coincide and whenever this happens we shall call the common form $\theta_{\text{Lee}} = \theta^I_{\text{Lee}} = \theta^J_{\text{Lee}} = \theta^K_{\text{Lee}}$, the *Lee form of the hyperhermitian manifold*. This is for instance true under the HKT assumption (see [186]). We thus call **balanced HKT** an HKT manifold with vanishing Lee form.

We have already introduced the $(1,0)$-form θ such that

$$\partial\bar\Omega^n = \theta\wedge\bar\Omega^n\,.$$

We called θ such form because it happens to be strictly related to the Lee form:

Lemma 2.37. *On a HKT manifold (M, I, J, K, Ω) we have*

$$\theta_{\text{Lee}} = \theta + \bar\theta\,.$$

Proof. Since $\theta_{\text{Lee}} = -Jd^*\omega_J = -Kd^*\omega_K$ then

$$d^*\Omega = d^*\omega_J + id^*\omega_K = J\theta_{\text{Lee}} + iK\theta_{\text{Lee}} = J(\theta_{\text{Lee}} + iI\theta_{\text{Lee}})\,,$$

on the other hand using the formulas involving the Hodge star that we proved above, and the HKT condition (2.12) we obtain

$$\begin{aligned} d^*\Omega = -*d*\Omega &= -\frac{4}{(n-1)!\,n!}*d\left(\Omega^n\wedge\bar\Omega^{n-1}\right) = -\frac{4}{(n-1)!\,n!}*\left(\bar\partial\Omega^n\wedge\bar\Omega^{n-1}\right)\\ &= -\frac{4}{(n-1)!\,n!}*\left(\bar\theta\wedge\Omega^n\wedge\bar\Omega^{n-1}\right) = 2J\bar\theta \end{aligned}$$

hence $\theta + \bar\theta = \theta_{\text{Lee}}$. □

As a consequence we recover the following result of Verbitsky [306]:

Corollary 2.38. *A HKT manifold (M, I, J, K, Ω) is balanced if and only if Ω^n is holomorphic.*

$\mathrm{SL}(n,\mathbb{H})$**-manifolds and their canonical bundle.**

Any hyperhermitian manifold (M, I, J, K, Ω) naturally admits a trivialization of the canonical bundle K_M provided by Ω^n. Hence, the canonical bundle is always topologically trivial, however, it is not, in general holomorphically trivial. In this paragraph we present the relation between some properties of K_M (for instance flatness and more importantly holomorphic triviality) and some conditions involving the Obata connection such as the presence of a global parallel section.

We begin with a definition. Since we can regard $\mathrm{SL}(n,\mathbb{H})$ as the intersection of $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{SL}(2n,\mathbb{C})$ a hyperhermitian manifold admits a $\mathrm{SL}(n,\mathbb{H})$-structure if and only if there exists a section of K_M which is parallel with respect to the Obata connection, equivalently:

Definition 2.39. A hyperhermitian manifold (M, I, J, K, Ω) is called a $\mathrm{SL}(n,\mathbb{H})$**-manifold** if the holonomy group of the Obata connection is contained in $\mathrm{SL}(n,\mathbb{H})$.

We now prove a general lemma:

Lemma 2.40. *A connection on a trivial line bundle is flat if and only if the connection* 1*-form is closed. Furthermore, the bundle admits a global parallel section if and only if the connection* 1*-form is exact.*

Proof. Since the bundle is trivial any connection can be written as $\nabla = d + \omega$, where d is the trivial connection and ω the connection 1-form. The curvature form can be expressed as $R^\nabla = d\omega - \omega\wedge\omega = d\omega$, therefore ∇ is flat if and only if ω is closed.

Now, since the bundle is trivial any section can be seen as a function $T \in C^\infty(M,\mathbb{R})$. Let T be a nowhere vanishing parallel section, and assume it is positive, then $0 = \nabla T = dT + \omega T$ implying $\omega = -T^{-1}dT = -d(\log T)$, i.e. ω is exact. Conversely, if $\omega = df$ is exact the section e^{-f} satisfies $\nabla \mathrm{e}^{-f} = d\mathrm{e}^{-f} + df\mathrm{e}^{-f} = -df\mathrm{e}^{-f} + df\mathrm{e}^{-f} = 0$. □

Proposition 2.41. *On a hyperhermitian manifold (M, I, J, K, Ω) the following are equivalent:*

(i) *The Obata connection ∇ is Ricci-flat.*

(ii) *The Obata connection ∇^{K_M} induced on the canonical bundle $K_M = K(M, I)$ is flat.*

(iii) $\bar\partial\theta + \partial\bar\theta = 0$.

(iv) *The restricted holonomy group of ∇ is contained in* $\mathrm{SL}(n,\mathbb{H})$.

If Ω is HKT the above conditions are also equivalent to:

(v) *The Lee form is closed.*

Sketch of proof. The equivalence of (ii) and (iii) follows from the previous lemma as the connection 1-form of ∇^{K_M} is $\theta + \bar\theta$. Recall that θ is ∂-closed and thus $d(\theta + \bar\theta) = \bar\partial\theta + \partial\bar\theta$. Moreover, in the HKT case $\theta + \bar\theta = \theta_{\mathrm{Lee}}$ (Lemma 2.37) hence the equivalence with (v) follows.

The equivalence of (i) and (iv) is proved by Alekseevsky and Marchiafava [8, Theorem 5.6].

We show that (ii) is equivalent to (iv). If a matrix $A \in \mathrm{GL}(n,\mathbb{H})$ acts on $\mathbb{H}^n \cong \mathbb{C}^{2n}$ the induced map on $\Lambda^{2n,0}\mathbb{C}^{2n}$ corresponds to multiplication by $\mathrm{Sdet}(A)$ (cf. [305]). Therefore we have $\mathrm{Hol}^0(\nabla^{K_M}) = \mathrm{Sdet}\,\mathrm{Hol}^0(\nabla)$, showing that $\mathrm{Hol}^0(\nabla) \subseteq \mathrm{SL}(n,\mathbb{H}) = \mathrm{Ker}(\mathrm{Sdet})$ if and only if $\mathrm{Hol}^0(\nabla^{K_M})$ is trivial, i.e. ∇^{K_M} is flat. □

As a corollary, we recover the result of Berger [36] that all hyperkähler manifolds are Ricci-flat, indeed in this case, the Obata connection coincides with the Levi-Civita connection.

We now discuss the global counterpart of Proposition 2.41, i.e. when the full holonomy group lies inside $\mathrm{SL}(n,\mathbb{H})$. Before we prove such a result we need to establish a preliminary lemma due to Barberis, Dotti and Verbitsky [29, Theorem 3.2] (see also [306, Lemma 4.3] and [278, Proposition 5.4]):

Lemma 2.42. *Any q-real holomorphic $(2n, 0)$-form on a hypercomplex manifold is parallel with respect to the Obata connection.*

Proof. Let Θ be a q-real holomorphic $(2n, 0)$-form and α the connection 1-form in the trivialization of Θ, i.e. $\nabla\Theta = \alpha \otimes \Theta$, where ∇ is the Obata connection. Using that ∇ is torsion-free we get

$$0 = \bar\partial\Theta = d\Theta = \mathrm{Alt}(\nabla\Theta) = \alpha \wedge \Theta = \alpha^{0,1} \wedge \Theta\,.$$

But since Θ is q-real we see as in (2.19) that α is real, so $\alpha^{0,1} = 0$ implies $\alpha = 0$, meaning that Θ is parallel. □

Proposition 2.43. *On a hyperhermitian manifold (M, I, J, K, Ω) the following are equivalent:*

(i) *There exists a q-positive holomorphic $(2n, 0)$-form on M.*

(ii) $\theta = \partial f$.

(iii) *The holonomy of the Obata connection is contained in* $\mathrm{SL}(n,\mathbb{H})$.

If Ω *is HKT the above conditions are also equivalent to:*

(iv) *The Lee form is exact.*

Proof. The equivalence of (ii) and (iii) (and also (iv), if the case) follows from Lemma 2.40.

If (iii) holds there exists a parallel section Θ of K_M. Observe that Θ can be assumed to be q-real, because if it is not we can replace it with the q-real form $\Psi = \Theta + J\bar{\Theta}$ which is parallel:

$$\nabla\Psi = \cancel{\nabla\Theta} + \nabla J\bar{\Theta} = J\nabla\bar{\Theta} = J\overline{\nabla\Theta} = 0\,.$$

Being parallel it is nowhere vanishing, hence we can assume it is q-positive. Since ∇ is torsion-free we have $(\nabla^K)^{0,1} = \bar{\partial}$, in particular $0 = (\nabla^K)^{0,1}\Psi = \bar{\partial}\Psi$. Hence (iii) implies (i).

Conversely, if $\Theta = f\Omega^n$ is a q-positive holomorphic $(2n,0)$-form we have $0 = \bar{\partial}\Theta = (\bar{\partial}f + f\bar{\theta})\wedge\Omega^n$, i.e. $\theta = -\partial(\log f)$. Therefore (i) implies (ii). ☐

The equivalence of (iii) and (iv) above on HKT manifolds is due to Ivanov and Petkov [186].

Corollary 2.44 (Misha Verbitsky [305])**.** *A* $\mathrm{SL}(n,\mathbb{H})$*-manifold has holomorphically trivial canonical bundle*

Verbitsky raised the question if the converse is true in general and no counterexamples are known so far. We phrase this as a conjecture:

Conjecture 2.45. *Every hyperhermitian manifold with holomorphically trivial canonical bundle* K_M *is* $\mathrm{SL}(n,\mathbb{H})$*, i.e. admits a global section of* K_M *which is parallel with respect to the Obata connection.*

The conjecture was partially confirmed by Verbitsky [305] providing a proof for compact HKT manifolds as a consequence of the Hodge theory he developed.

Another interesting observation arises by looking at balanced HKT manifolds. By Corollary 2.38 a balanced HKT manifold has always holomorphically trivial canonical bundle, furthermore, the trivialization is provided by Ω^n which is q-positive and thus, by Proposition 2.43 the manifold is $\mathrm{SL}(n,\mathbb{H})$. The converse is false in general, as there exist compact HKT $\mathrm{SL}(n,\mathbb{H})$-manifolds that are not balanced [30, Examples 6.1 and 6.2]. It is for instance true for all HKT nilmanifolds and HKT $\mathrm{SL}(n,\mathbb{H})$-solvmanifolds with left-invariant abelian hypercomplex structure (cf. Chapter 3). However, all known counterexamples still admit a different HKT metric that is balanced. One is therefore led to conjecture that this is always the case, at least in the compact setting (cf. [306]):

Conjecture 2.46. *Every compact HKT* $\mathrm{SL}(n,\mathbb{H})$*-manifold admits a balanced HKT metric.*

We shall address this conjecture much more in detail starting from the next Section, where it will be reinterpreted as the quaternionic analogue of the famous Calabi conjecture.

We observe here that, at least, the $\mathrm{SL}(n,\mathbb{H})$ condition implies that the metric is conformal to a balanced hyperhermitian metric, indeed let (M,I,J,K,g,Ω) be a compact HKT $\mathrm{SL}(n,\mathbb{H})$-manifold, then by Proposition 2.43 $\theta_{\mathrm{Lee}} = df$ and the conformally rescaled metric $g' = \mathrm{e}^{-f/(2n-1)}g$ is still hyperhermitian. Furthermore, all the Lee forms vanish, to see this set $\omega'_L := g'(L\cdot,\cdot) = \mathrm{e}^{-f/(2n-1)}\omega_L$ for $L = I, J, K$, then

$$d(\omega'_L)^{2n-1} = d(\mathrm{e}^{-f}\omega_L^{2n-1}) = (\theta_{\mathrm{Lee}} - df)\wedge(\omega'_L)^{2n-1} = 0\,.$$

However, the conformal rescaling destroys the HKT condition, indeed g' is HKT if and only if f is constant, because $\Omega' = \omega'_J + i\omega'_K = \mathrm{e}^{-f/(2n-1)}\Omega$ is HKT if and only if $\partial f = 0$, which, by compactness of M, means that f is constant.

Let us also underline that Verbitsky [306] showed that a balanced HKT manifold (M,I,J,K,Ω) with quaternionic dimension $n \geq 3$ admits no strong HKT metric of the form $\Omega + \partial\partial_J\varphi$ unless $d\Omega = 0$ and M is hyperkähler. Thus, in view of Conjecture 2.46 we expect no strong HKT metric on $\mathrm{SL}(n,\mathbb{H})$-manifolds. We remark that examples of strong HKT manifold are very scarce, we are aware of no other example than the examples of Joyce and the ones that can be produced via the doubling construction of Barberis and Fino [30]. In particular it remains an open problem whether or not there exist solvmanifolds with strong HKT metrics.

We conclude this chapter by reporting the observation of Barberis, Dotti and Verbitsky [29] that the hypercomplex structures on Joyce's examples are never $\mathrm{SL}(n, \mathbb{H})$. More generally, if (M, I, J, K) is a hypercomplex manifold such that $\pi \colon (M, I) \to B$ is a principal torus fibration over a base B which is a Fano manifold (the anticanonical bundle K_B^{-1} is ample) then K_M can never admit holomorphic sections. Indeed, the adjunction formula yields $K_M \cong \pi^* K_B$ because the fiber is a torus and the canonical bundle of the torus is trivial, and since K_B^{-1} is ample $\pi^* K_B^{-k}$ has sections for some k, therefore K_M cannot have any. To see that Joyce's examples belongs to this class we refer to [307] where it is proved that they are tori fibrations over a homogeneous rational manifold.

This discussion includes the quaternionic Hopf manifold seen as the product $\mathrm{SU}(2) \times \mathrm{U}(1)$. Observe that such manifold has flat Obata connection by construction, thus in particular has restricted holonomy in $\mathrm{SL}(n, \mathbb{H})$, but, as just remarked, not the global holonomy. The only example of Joyce's of which we explictly know the holonomy is $\mathrm{SU}(3)$. Soldatenkov [263] proved that the holonomy of the Obata connection on $\mathrm{SU}(3)$ with the hypercomplex structure in Example 2.10 is $\mathrm{GL}(2, \mathbb{H})$.

Conjecture 2.47. *All the examples of Joyce have full holonomy of the Obata connection* $\mathrm{GL}(n, \mathbb{H})$*.*

2.3 The quaternionic Calabi conjecture.

Since Yau proved the Calabi conjecture in [327], other Calabi-Yau-type problems have been introduced in various geometric contexts. Here we overview the so-called *quaternionic Calabi conjecture* in HKT geometry formulated by Alesker and Verbitsky [18]. Within this analogy, the "quaternionic Calabi-Yau metrics" are the balanced HKT metrics.

2.3.1 Statement of the conjecture.

"Hyperhermitian Ricci form".

Let (M, I, J, K, Ω) be a hyperhermitian manifold. In local holomorphic coordinates we have

$$\frac{\Omega^n}{n!} = \mathrm{pf}(\Omega) dz^1 \wedge \cdots \wedge dz^{2n}$$

where $\mathrm{pf}(\Omega)$ denotes the Pfaffian of the skew-symmetric complex matrix (Ω_{rs}) induced by Ω in the given holomorphic coordinates, i.e. $\Omega = \sum_{r<s} \Omega_{rs} dz^r \wedge dz^s$. We define a q-real ∂-closed $(1,0)$-form ρ, which in the given holomorphic coordinates is expressed as

$$\rho = -\partial \partial_J \log \mathrm{pf}(\Omega)\,.$$

Observe that this is globally defined, i.e. it does not depend on the choice of coordinates.

More generally, If Φ is a q-positive $(2n,0)$-form then

$$\Phi = \varphi\, dz^1 \wedge \cdots \wedge dz^{2n}$$

for a locally defined smooth positive function φ and we may set $\rho(\Phi) := -\partial\partial_J \log \varphi$. We have

$$\rho(\Omega^n) - \rho(\Phi) = \partial\partial_J \log \frac{\Phi}{\Omega^n}$$

where $\partial\partial_J \log(\Phi/\Omega^n)$ is a globally defined $\partial\partial_J$-exact form. Therefore, the quaternionic Bott-Chern cohomology class of $\rho = \rho(\Omega^n)$ does not depend on the choice of the complex volume form.

We could also consider another perspective. Let θ be the $(1,0)$-form such that $\partial\bar{\Omega}^n = \theta \wedge \bar{\Omega}^n$, then one can show that $\rho = \partial_J \theta$. Since Φ is q-positive there exists a smooth positive real-valued function f

such that $\Phi = f\Omega^n$ and thus

$$\partial\bar{\Phi} = (\partial f + f\theta)\wedge\bar{\Omega}^n = (f^{-1}\partial f + \theta)\wedge\bar{\Phi} = (\partial \log f + \theta)\wedge\bar{\Phi}$$

which shows that

$$\rho(\Phi) = -\partial\partial_J \log f + \partial_J\theta$$

hence, again we see that $[\rho(\Phi)] = [\rho(\Omega^n)]$ in terms of quaternionic Bott-Chern cohomology classes.

We shall denote with $c_{2,0}^{\mathrm{BC}}(M) := [\rho] \in H^{2,0}_{\mathrm{BC}}(M)$ the quaternionic Bott-Chern cohomology class of which $\partial_J\theta$ is a representative.

The conjecture.

Let (M, I, g) be a Kähler manifold. The *Ricci form* (of the Levi-Civita connection) is the closed $(1,1)$ form $\rho := \mathrm{Ric}(I\cdot,\cdot)$. It turns out that $\frac{1}{2\pi}\rho$ can be taken as a representative for the *first Chern class* $c_1(M)$ of M. Eugenio Calabi, conjectured [67] that the Ricci form of a compact Kähler manifold can be prescribed.

Theorem 2.48 (Calabi-Yau)**.** *Let (M, I) be a compact complex manifold admitting a Kähler metric g with associated Kähler form ω. Let ρ' be a closed real $(1,1)$-form such that its de Rham cohomology class is $[\rho'] = 2\pi c_1(M) \in H^2(M,\mathbb{R})$, where $c_1(M)$ is the first Chern class. Then there exist a unique Kähler metric g' in (M, I) with associated Kähler form ω' such that $[\omega'] = [\omega]$ and ρ' is the Ricci form of g'.*

Calabi himself already proved that the conjecture has at most one solution, but a full proof of existence had to wait more than 20 years. After some work by Calabi, Aubin, Borguignon, Nirenberg and many others, Yau completed the proof of such a groundbreaking conjecture [327].

A full account of the proof, very close to the original one can be found in [194], however, the treatment has become somewhat outdated in certain parts. Over the years the argument of Yau underwent various improvements from many people and the solution has been significantly simplified. We refer, among others, to the lecture notes [44] or the book [158].

The Calabi-Yau Theorem has important consequences, for instance, when (M, I, g) is a compact Kähler manifold with vanishing first chern class. Then we may choose $\rho' = 0$ and the theorem guarantees the existence of a Ricci-flat Kähler metric on M. This allows to find examples of compact Riemannian manifolds with (Riemannian) holonomy group $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$. The first are the so-called **Calabi-Yau manifolds**, the second are hyperkähler manifolds.

One can repeat the steps leading to the Calabi conjecture in the realm of HKT geometry. First, we take into account the space of HKT potentials. The existence of local potentials for HKT forms, established by Banos and Swann [27] opens the possibility to investigate (pluri)potential theory on HKT manifolds. Let (M, I, J, K, Ω) be a HKT manifold. In analogy with the complex case, the space of *quaternionic Ω-plurisubharmonic functions* has been introduced

$$\mathcal{H}_\Omega = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \Omega_\varphi := \Omega + \partial\partial_J\varphi > 0\}\,, \tag{2.24}$$

where the inequality "$\Omega_\varphi > 0$" stands for the q-positivity of Ω_φ, so that it induces a new hyperhermitian metric on (M, I, J, K) which we denote g_φ.

It is natural to wonder if, within the space (2.24) one can find a φ such that the HKT form Ω_φ is in some sense preferable. We have seen that the most desirable HKT metrics one could have are balanced ones. Alesker and Verbitsky [18, 306] proposed to mimic the approach used to prove the Calabi-Yau theorem in order to show that a compact HKT $\mathrm{SL}(n,\mathbb{H})$-manifold (M, I, J, K, Ω) always admits a balanced HKT form in the same Bott-Chern class of Ω.

In view of the formulation above of the Calabi-Yau Theorem, we phrase the quaternionic Calabi conjecture in a more general setting:

Conjecture 2.49. *Let (M, I, J, K, Ω) be a compact HKT manifold. If $\rho' \in \Lambda^{2,0}(M)$ is q-real and such that $[\rho'] = c_{2,0}^{\mathrm{BC}}(M)$. Then there exists a unique HKT metric g' with HKT-Ricci form ρ' and HKT form Ω', where $[\Omega'] = [\Omega] \in H^{2,0}_{\mathrm{BC}}(M)$.*

We know that if M is $\mathrm{SL}(n,\mathbb{H})$ the canonical bundle is holomorphically trivial, hence, necessarily $\rho=0$ and thus $c_{2,0}^{\mathrm{BC}}(M)=0$. In particular, if the conjecture is true, on a compact HKT $\mathrm{SL}(n,\mathbb{H})$ manifold (M,I,J,K,Ω) there always exists a balanced HKT metric g' compatible with the hypercomplex structure (I,J,K). Under this light, balanced HKT metrics are the perfect quaternionic parallel of Calabi-Yau's metrics.

We now rephrase the conjecture in terms of a fully non-linear PDE. First of all, the conditions $[\rho']=c_{2,0}^{\mathrm{BC}}(M)=[\rho]$ and $[\Omega']=[\Omega]$ can be expressed by writing $\rho'=\rho+\partial\partial_J F$ and $\Omega'=\Omega+\partial\partial_J\varphi$ for some $F,\varphi\in C^\infty(M,\mathbb{R})$ unique up to an additive constant. If we impose the condition $\int_M\varphi\,\Omega^n\wedge\bar\Omega^n=0$ (or $\sup_M\varphi=0$) then φ is uniquely determined. Since $(\Omega')^n$ and Ω^n are both q-positive there must be a positive function $f\in C^\infty(M,\mathbb{R})$ such that $(\Omega')^n=f\Omega^n$, but then

$$\partial\partial_J\log f=\partial\partial_J\log\frac{(\Omega')^n}{\Omega^n}=\partial\partial_J\log\mathrm{pf}(\Omega)-\partial\partial_J\log\mathrm{pf}(\Omega')=\partial_J\theta'-\partial_J\theta=\rho'-\rho=-\partial\partial_J F$$

it follows that $\log f-F=\log b$, i.e. $f=b\,\mathrm{e}^F$ where $b>0$ is a constant.

A priori one would need to require Ω_φ to be a q-positive form. Let us quickly observe that the condition $\Omega+\partial\partial_J\varphi>0$ is actually redundant for any solution φ of the conjecture. Indeed, the form $\Omega_\varphi^n=b\,\mathrm{e}^F\Omega^n$ is nowhere vanishing and q-positive, furthermore, at a minimum point of φ we have $\partial\partial_J\varphi\geq0$ and by continuity Ω_φ must be q-positive everywhere on M.

Summing up all the above, we can restate the conjecture as follows:

Conjecture 2.50. *Let (M,I,J,K,Ω) be a compact HKT manifold. For any $F\in C^\infty(M,\mathbb{R})$ there exists a unique pair $(\varphi,b)\in C^\infty(M,\mathbb{R})\times\mathbb{R}_+$ such that*

$$(\Omega+\partial\partial_J\varphi)^n=b\,\mathrm{e}^F\Omega^n\,,\qquad\sup_M\varphi=0\,. \tag{2.25}$$

Observe that when M is $\mathrm{SL}(n,\mathbb{H})$ and Θ is a q-positive holomorphic $(2n,0)$-form, the constant b is uniquely determined by Stokes' Theorem as

$$b\int_M\mathrm{e}^F\Omega^n\wedge\bar\Theta=\int_M(\Omega+\partial\partial_J\varphi)^n\wedge\bar\Theta=\int_M\Omega^n\wedge\bar\Theta\,,$$

therefore, in this case, it can be "absorbed" inside the datum F and (2.25) can be written as

$$(\Omega+\partial\partial_J\varphi)^n=\mathrm{e}^F\Omega^n\,,\qquad\sup_M\varphi=0\,,$$

where $F\in C^\infty(M,\mathbb{R})$ satisfies the necessary condition

$$\int_M(\mathrm{e}^F-1)\Omega^n\wedge\bar\Theta=0\,.$$

These are actually the original assumptions of Alesker and Verbitsky, when they formulated the *quaternionic Calabi conjecture.*

The formulation of the conjecture in terms of an equation is suitable for further generalization. This is also motivated by the success of the complex Monge-Ampère equation on compact (almost) Hermitian manifolds [288, 93, 329, 92].

Conjecture 2.51. *Let (M,I,J,K,Ω) be a compact hyperhermitian manifold. For any $F\in C^\infty(M,\mathbb{R})$ there exists a unique pair $(\varphi,b)\in C^\infty(M,\mathbb{R})\times\mathbb{R}_+$ such that*

$$(\Omega+\partial\partial_J\varphi)^n=b\,\mathrm{e}^F\Omega^n\,,\qquad\sup_M\varphi=0\,. \tag{2.26}$$

The quaternionic Monge-Ampère equation.

Here, we write down the local expression for the equation

$$(\Omega + \partial\partial_J\varphi)^n = b\,\mathrm{e}^F\Omega^n\,, \tag{2.27}$$

this will motivate the terminology of *quaternionic Monge-Ampère equation*. We thus assume that (M, I, J, K, Ω) is a locally flat HKT manifold. Let u be a HKT potential for Ω and denote with $G = \mathrm{Hess}_{\mathbb{H}}u$ the hyperhermitian matrix associated to $\partial\partial_J u = \Omega$. From Lemma 2.25 we know that

$$\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} = \frac{(\partial\partial_J(u+\varphi))^n}{(\partial\partial_J u)^n} = \frac{\det(G + \mathrm{Hess}_{\mathbb{H}}\varphi)}{\det G}$$

therefore we can write (2.27) as

$$\det(G + \mathrm{Hess}_{\mathbb{H}}\varphi) = b\,\mathrm{e}^F\det(G)\,,$$

which is an equation of Monge-Ampère type.

The study of the quaternionic Monge-Ampère equation in the flat case, precedes the conjecture of Alesker and Verbitsky. Indeed, the Dirichlet problem associated to (2.27) was already considered by Alesker in [9, 10] and solved on strictly pseudoconvex domains in quaternionic sense, when the boundary data are continuous and the right-hand side is continuous up to the boundary. Some years later Zhu [335], Kołodziej, Sroka [201, 270] and Wan [314] obtained weak solutions and some regularity results.

For further references the interested reader is referred to [271].

2.3.2 Solving the quaternionic Monge-Ampère equation.

Ellipticity and uniqueness.

First, we observe that the quaternionic Monge-Ampère equation, although being fully non-linear, it is elliptic, indeed the linearization of the operator

$$P\colon C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})\,, \quad P(\varphi) = \log\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} - \log(b) - F\,.$$

is the quaternionic Laplacian $\Delta_\varphi := \Delta_{g_\varphi}$:

$$P_{*|\varphi}(\psi) = n\frac{\partial\partial_J\psi\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n} = \Delta_\varphi\psi\,,$$

which we have seen to be elliptic.

Next, it is straightforward to show that solutions to the quaternionic Monge-Ampère equation (2.26) on a compact hyperhermitian manifold are in general unique. This can, for instance, be observed as follows: let $(\varphi_1, b_1), (\varphi_2, b_2)$ be two solutions to (2.26) with $b_1 \geq b_2$. Setting $\Omega_i = \Omega + \partial\partial_J\varphi_i$ we have that

$$\partial\partial_J(\varphi_1 - \varphi_2)\wedge\sum_{k=0}^{n-1}\Omega_1^k\wedge\Omega_2^{n-1-k} = \Omega_1^n - \Omega_2^n = (b_1 - b_2)\mathrm{e}^F\Omega^n \geq 0\,.$$

On the left hand-side we have a second order linear elliptic operator without free term applied to $\varphi_1 - \varphi_2$ and from the maximum principle and the fact that $\sup_M\varphi_1 = \sup_M\varphi_2 = 0$ it follows $\varphi_1 = \varphi_2$. Hence we have also $b_1 = b_2$ and the uniqueness follows.

Method of continuity.

The most natural approach to attack the problem is the method of continuity, much in the same spirit of Yau's proof of the Calabi conjecture [327]. The idea of such a technique is to interpolate between the equation

$$\frac{(\Omega + \partial\partial_J \varphi)^n}{\Omega^n} = b\,\mathrm{e}^F$$

we wish to solve and another one which is easier to solve, for instance

$$\frac{(\Omega + \partial\partial_J \varphi)^n}{\Omega^n} = 1\,,$$

which has the obvious solution $\varphi \equiv 0$. We then consider a one-parameter family of equations

$$\frac{(\Omega + \partial\partial_J \varphi_t)^n}{\Omega^n} = b_t\,\mathrm{e}^{tF} \tag{$*_t$}$$

with continuous dependence on $t \in [0, 1]$.

At this point, the solvability comes down to prove a connectedness argument: consider the set

$$S = \{t \in [0, 1] \mid (*_t) \text{ has a solution } (\varphi_t, b_t) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+\}$$

then we only need to prove that $1 \in S$, which will immediately follow if we show that S is connected. As we observed $0 \in S$, therefore S is non-empty.

Now, two standard arguments are usually employed to show that S is both open and closed. Openness is the easy part: take $t' \in S$ and let $(\varphi_{t'}, b_{t'})$ be the corresponding solution of $(*)_{t'}$, then, in order to show that a small neighborhood of t' is all contained in S one usually considers the linearization of the equation at the solution $(\varphi_{t'}, b_{t'})$ between some Banach spaces and tries to apply the inverse function theorem.

The proof of closedness is in general the hard part which requires *a priori estimates*. The idea is to take an arbitrary sequence $(t_j) \subseteq S$ and show that its limit $t' = \lim_{j\to\infty} t_j$ still lies in S. If we denote (φ_{t_j}, b_{t_j}) the solutions of $(*)_{t_j}$ we may wish to extract from these a subsequence which is convergent to a solution of $(*)_{t'}$. This is done by proving that some Banach norms of all solutions φ_t are bounded by a constant under control, which allows to show that they lie in a compact subset of the Banach space, thus implying the existence of a convergent subsequence.

The a priori estimates are the core of the method of continuity and they represent the most involved part of the proof. In general the norms adopted to start the machinery of the method of continuity are the $C^{k,\alpha}$ norms for some α. To achieve these estimates one starts from the C^0 bound, continues with the gradient bound, then the Laplacian bound, equivalent to the C^2 bound and with standard Evans-Krylov theory one achieves the $C^{2,\alpha}$-estimate, which is improved to a $C^{k+2,\alpha}$-estimate for all k via bootstrapping and Schauder estimates (see Subsection 2.3.3 below for further details).

Current progress towards the proof.

So far there are only partial results about the solvability of the quaternionic Calabi conjecture.

In [18], where the problem is proposed, the authors use a Moser iteration technique such as the one originally used by Yau to obtain an a priori C^0 estimate for solutions to the quaternionic Monge-Ampère equation. This approach requires to have a holomorphic section of the canonical bundle. Later, Alesker and Shelukhin [15] were able to prove the same estimate but under the different assumption that the hypercomplex structure is locally flat. They generalized their work in [16] showing that the C^0-estimate holds on any compact HKT manifold. Recently, Sroka [269] provided a much shorter proof, using a Cherrier-type inequality, following the work of Tosatti and Weinkove [287, 288] on the complex Monge-Ampère equation, which in turn is based on a previous work of Cherrier [84]. Therefore we have:

Theorem 2.52 (Alesker-Shelukhin-Sroka)**.** *Let (M, I, J, K, Ω) be a compact HKT manifold. There exists a constant $C > 0$ such that for any solution $\varphi \in C^\infty(M, \mathbb{R})$ of* (2.25)

$$\|\varphi\|_{C^0} \leq C$$

where C depends only on the HKT structure and $\sup_M F$.

The higher order estimates seem to be more tricky. The first full result of solvability is due to Alesker [14], who assumed not only that the hypercomplex structure is locally flat but that there exists a compatible hyperkähler metric. These two assumptions entail flatness of the hyperkähler metric, in the sense that the full Riemann curvature tensor vanishes. By Bieberbach's theorem on compact, flat Riemannian manifolds this reduces to proving the conjecture on finite covers of a torus. These severe assumptions are fully exploited in the proof of the Laplacian estimate, where normal coordinates are used.

Theorem 2.53 (Alesker). *Let (M, I, J, K, Ω) be a compact flat hyperkähler manifold. For any $F \in C^\infty(M, \mathbb{R})$ there exists a unique pair $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ such that*

$$(\Omega + \partial\partial_J \varphi)^n = b\, \mathrm{e}^F \Omega^n\,, \qquad \sup_M \varphi = 0\,.$$

However, Alesker, adapting an argument of Błocki [42] (see also [44]) was able to obtain the $C^{2,\alpha}$-estimate dropping the hyperkähler assumption, we state this as a seperate result for future reference:

Theorem 2.54 (Alesker). *Let (M, I, J, K, g) be a $4n$-dimensional compact HKT manifold whose underlying hypercomplex structure is locally flat. Suppose $\varphi \in C^2(M, \mathbb{R})$ is a solution to the quaternionic Monge-Ampère equation* (2.25). *Then*

$$\|\varphi\|_{C^{2,\alpha}} \leq C$$

for some $\alpha \in (0,1)$ and a positive constant C, both depending only on the HKT structure, $\|F\|_{C^2}$, $\|\varphi\|_{C_0}$ and $\|\Delta_g \varphi\|_{C^0}$, where Δ_g is the quaternionic Laplacian.

In a recent paper, Dinew and Sroka [106] were able to improve the result of Alesker by giving a complete proof of the conjecture on compact hyperkähler manifolds, so far, this is the most general result available regarding the solvability of the quaternionic Monge-Ampère equation.

Theorem 2.55 (Dinew-Sroka). *Let (M, I, J, K, Ω) be a compact hyperkähler manifold. For any $F \in C^\infty(M, \mathbb{R})$ there exists a unique pair $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ such that*

$$(\Omega + \partial\partial_J \varphi)^n = b\, \mathrm{e}^F \Omega^n\,, \qquad \sup_M \varphi = 0\,.$$

2.3.3 An analytic toolbox.

In this subsection we collect some useful results used in the proofs of the a priori estimates.

Alexandrov-Bakelman-Pucci estimate.

Historically, the C^0 bound for the complex Monge-Ampère equation was the last one to be proved. This was done by Yau via Moser iteration. In the following years different techniques and various alternative proofs have been developed.

Here we present an argument based on the Alexandrov-Bakelman-Pucci estimate. The idea goes back to Cheng and Yau and a simpler approach has been found by Błocki [43]. The argument we present below is the reinterpretation of Székelyhidi [280] of the Alexandrov-Bakelman-Pucci estimate, which, as of today is the simplest and most immediate one.

Let M be a compact oriented m-dimensional manifold. Suppose $\varphi \in C^\infty(M, \mathbb{R})$ is a solution of some partial differential equation satisfying $\sup_M \varphi = 0$ for simplicity. Let $x_0 \in M$ be a point at which u attains its minimum and take a coordinate chart centered at x_0 which, without loss of generality, we identify with a ball $B_1(0)$ of radius 1 with coordinates $(x^1, \ldots, x^m)$. We now fix $\varepsilon > 0$ and define

$$\psi = \varphi + \varepsilon \sum_{i=1}^m (x^i)^2\,.$$

The auxiliary function ψ satisfies the assumptions of the following Alexandrov-Bakelman-Pucci–type estimate (see [280, Proposition 10]):

Proposition 2.56. *Let $B_1(0) \subseteq \mathbb{R}^m$ denote the unit ball centered at the origin. Assume that $\psi \in C^2(\mathbb{R}^m)$ satisfies $\psi(0) + \varepsilon \leq \min_{\partial B_1(0)} \psi(x)$. Then there exists a constant c_m depending only on m such that*

$$\varepsilon^m \leq c_m \int_{\Gamma_\varepsilon} \det(D^2\psi)\,,$$

where $D^2\psi$ is the (real) Hessian of ψ and

$$\Gamma_\varepsilon = \left\{ x \in B_1(0) \mid \psi(y) \geq \psi(x) + \nabla\psi(x)\cdot(y-x),\ \forall y \in B_1(0),\ |\nabla\psi(x)| < \frac{\varepsilon}{2} \right\}.$$

At this point one needs to obtain an estimate for $\det(D^2\psi)$ over Γ_ε, which has to be done using the equation satisfied by φ together with the definition of Γ_ε. If one manages to prove such a bound, then

$$\varepsilon^m \leq C|\Gamma_\varepsilon|\,,$$

where $|\Gamma_\varepsilon| = \int_{\Gamma_\varepsilon} dx$. Since on Γ_ε we have $\psi \leq \psi(0) + \varepsilon/2$ then $\varphi \leq \inf_M \varphi + \varepsilon/2 = \|\varphi\|_{C^0} + \varepsilon/2$ and thus for any $p > 0$

$$\|\varphi\|_{L^p} \geq \left(\|\varphi\|_{C^0} + \frac{\varepsilon}{2}\right)|\Gamma_\varepsilon| \geq C\left(\|\varphi\|_{C^0} + 1\right)$$

which shows that the C^0-estimate follows from an L^p-estimate.

A L^p-estimate is generally easy to find with the aid of the following (see [143, Theorem 8.18]):

Theorem 2.57 (Weak Harnack Inequality)**.** *Let $R > 0$ and fix an integer $m > 2$. Assume $u \in C^2(\mathbb{R}^m)$ is non-negative on $B_R(0)$ and such that $\Delta u(x) \leq f(x)$ for some $f \in C^0(\mathbb{R}^m)$ and all $x \in B_R(0)$. Consider $1 \leq p < m/(m-2)$, and $q > m$. Then there exists a positive constant $C = C(m, R, p, q)$ such that*

$$r^{-m/p}\|u\|_{L^p(B_{2r}(0))} \leq C\left(\inf_{x \in B_r(0)} u(x) + r^{2-2m/q}\|f\|_{L^{q/2}(B_R(0))}\right),$$

for any $0 < r < R/4$.

Evans-Krylov theory.

One of the important simplifications of the proof of the Calabi conjecture came from an estimate obtained independently by Evans [118] and Krylov [202]. The original proof required to obtain a third order estimate, while this can now be skipped thanks to the Evans-Krylov theory.

They proved that if ψ is a solution to a uniformly elliptic, fully non-linear, convex (or concave), equation

$$P(D^2\psi) = 0$$

in the ball $B_1(0)$ then $\psi \in C^{2,\alpha}(B_{1/2}(0))$ and

$$\|\psi\|_{C^{2,\alpha}} \leq C$$

where $C > 0$ and $\alpha > 0$ depend only on $\|\psi\|_{C^0}$, $\|D^2\psi\|_{C^0}$ and the ellipticity of P.

Such a result is extremely powerful and simplifies many arguments. However, when working on (hyper)complex manifolds one typically goes from a (quaternionic) Laplacian bound to a bound for the (quaternionic) complex Hessian and Evans-Krylov theory cannot be directly applied.

As a workaround, two possible approaches have been pursued in the literature. Either the proof of the Evans-Krylov Theorem is adapted to the specific setting at hand (which, for instance, has been done by Alesker for the proof of Theorem 2.54) or an estimate for the real Hessian is obtained (as done, e.g. by Dinew and Sroka in [106]).

We also stress that the issue with applying Evans-Krylov Theorem in this type of problems always reduces to verifying *uniform* ellipticity of the operator at the function assumed to be the solution. This is the main reason why in the literature one can find various Evans-Krylov–type results (e.g. [90, 286]).

Schauder theory and bootstrapping.

As mentioned, when the $C^{2,\alpha}$-estimate is at hand, a standard argument of bootstrapping allows to obtain estimates of any higher order. The key result to do so is the following *Schauder estimate* (see [194, Theorem 1.4.2]):

Theorem 2.58. *Let M be a compact Riemannian manifold and E_1, E_2 two vector bundles on M of the same dimension. Let $P\colon E_1 \to E_2$ be a linear elliptic differential operator of order k. If the coefficients of P are of class $C^{r,\alpha}$ for some $\alpha \in (0,1)$ and $\psi \in C^{k,\alpha}(E_1)$ is a solution of the equation*

$$P(\psi) = F$$

with datum $F \in C^{r,\alpha}(E_2)$, then $\psi \in C^{k+r,\alpha}(E_1)$ and

$$\|\psi\|_{C^{k+r,\alpha}} \leq C\left(\|F\|_{C^{r,\alpha}} + \|\psi\|_{C^0}\right)$$

for some constant $C > 0$ that does not depend on ψ and F.

Suppose now we have a solution of a second order linear elliptic equation $P(\psi) = F$ with smooth datum $F \in C^\infty(M, \mathbb{R})$ and assume there are estimates $\|\psi\|_{C^0} \leq C$, $\|\psi\|_{C^{2,\alpha}} \leq C$ for a constant C not depending on ψ. We can now differentiate the equation and regard it as another linear elliptic equation in the first derivatives of ψ with coefficients in $C^{0,\alpha}(M)$ depending on the second derivatives of ψ. Schauder estimate now implies that the derivatives of ψ are bounded in $C^{2,\alpha}$ norm and we can differentiate the equation again obtaining a $C^{2,\alpha}$-estimate for the second derivatives of ψ. Reiterating this argument, since F is smooth we achieve estimates for the derivatives of ψ of any given order.

CHAPTER 3

HODGE THEORY, FORMALITY AND BALANCED METRICS

The purpose of this chapter is twofold: first, we aim to explore deeper cohomological properties of HKT manifolds, especially for the subclass of balanced HKT manifolds; second, we analyse in detail HKT structures on hypercomplex solvmanifolds and nilmanifolds which provide a useful benchmark to use to test problems and conjectures. These two topics occupy the two sections this chapter is divided in. We shall show that all HKT nilmanifolds are balanced, and while this is not true in general for solvmanifolds, at least we can show that balancedness is implied by the assumption of being $\mathrm{SL}(n, \mathbb{H})$ (when the hypercomplex structure is left-invariant). We thus have plenty of examples of balanced HKT manifolds, to which results of the first section apply.

This chapter is essentially an account of the preprint [136].

3.1 Cohomology of balanced HKT manifolds.

The main objective of this section is to explore the analogies of HKT geometry with Kähler geometry from a cohomological point of view. More precisely, let (M, I, J, K, Ω) be a compact HKT manifold and denote with $\Lambda^{p,q}(M) = \Lambda_I^{p,q}(M)$ the space of (p, q)-forms with respect to I. We have two important cochain complexes: $(\Lambda^{\bullet,q}(M), \partial, \partial_J)$ and $(\Lambda^{\bullet,q}(M), \partial, \partial^\Lambda)$ for every fixed q, where $\partial^\Lambda := [\partial, \Lambda]$, being Λ the adjoint of $L := \frac{\Omega}{2} \wedge -$.

In both cases, we will restrict to study the case $q = 0$ under suitable assumptions, showing a behavior similar to Kähler manifolds. One can study cohomology groups and Hodge theory from a "complex point of view" on $(\Lambda^{\bullet,0}(M), \partial, \partial_J)$ or from a "symplectic point of view" on $(\Lambda^{\bullet,0}(M), \partial, \partial^\Lambda)$. Some of the analysis is essentially algebraic and relies only on few properties of the structures under investigation. Indeed, part of the results can be contextualized in the more general setting of Lefschetz spaces and this approach encompasses at once some of the theory of Kähler and HKT cohomology.

Building on the work of Lefschetz spaces by Tomassini and Wang [284] we define a generalization of the Hodge star operator, which allows us to take into account formal adjoints and Laplacians. Several relations among the Laplacians, the spaces of harmonic forms and the associated cohomology groups, together with Hard Lefschetz properties, are proved. Moreover, we show that for a compact HKT $\mathrm{SL}(n, \mathbb{H})$-manifold the differential graded algebra $(\Lambda^{\bullet,0}(M), \partial)$ is formal and this will lead to an obstruction for the existence of an HKT $\mathrm{SL}(n, \mathbb{H})$-structure (I, J, K, Ω) on a compact complex manifold (M, I).

The results presented in this section have the same spirit of (and are inspired by) the work done in [146, 208, 284, 301].

3.1.1 Lefschetz spaces.

The framework of Tomassini and Wang.

We start by recalling the main definitions and results from Tomassini and Wang [284] (see also [319]).

Definition 3.1. Let $A = \bigoplus_{p=0}^{2n} A^p$ be a direct sum of complex vector spaces. Let L be a $\mathbb{C}$-linear endomorphism of A such that $L(A^p) \subseteq A^{p+2}$ for $p = 0, \ldots, 2n-2$ and $L(A^{2n-1}) = L(A^{2n}) = 0$. We say that (A, L) is a *Lefschetz space* if L satisfies the Hard Lefschetz Condition (HLC), i.e.

$$L^{n-p} \colon A^p \to A^{2n-p}$$

is an isomorphism for all $p = 0, \ldots, n$.

If a Lefschetz space (A, L) is equipped with a $\mathbb{C}$-linear endomorphism d such that $d(A^p) \subseteq A^{p+1}$ for $p = 0, \ldots, 2n-1$, while $d(A^{2n}) = 0$ we call the triple (A, L, d) a *differential Lefschetz space*. If moreover $d^2 = 0$ then the triple (A, L, d) is called a *Lefschetz complex*.

On a Lefschetz space we say that $\alpha \in A^p$ is a *primitive form* if $p \leq n$ and $L^{n-p+1}\alpha = 0$. By the HLC immediately follows the decomposition into primitive forms (see [319]), more precisely, for every $\alpha \in A^p$ there exist unique primitive $\alpha^k \in A^{p-2k}$ such that

$$\alpha = \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{1}{k!} L^k \alpha^k \,. \tag{3.1}$$

As a generalization of the symplectic star operator Tomassini and Wang introduced the *Lefschetz star operator* $*_L \colon A \to A$, acting on a primitive form $\beta \in A^p$ as follows:

$$*_L \frac{1}{k!} L^k \beta := (-1)^{1+2+\cdots+p} \frac{1}{(n-p-k)!} L^{n-p-k} \beta \,.$$

Clearly the definition is then extended by linearity to any $\alpha \in A^p$ via the Lefschetz decomposition (3.1). Notice that $*_L^2 = 1$.

The starting point of the discussion by Tomassini and Wang is the following general Demailly-Griffiths-Kähler identity [284, Theorem A].

Theorem 3.2. *Let (A, L, d) be a differential Lefschetz space and $\Lambda = *_L^{-1} L *_L$ the dual Lefschetz operator. Define $d^\Lambda \in \operatorname{End}(A)$ by*

$$d^\Lambda|_{A^p} := (-1)^{p+1} *_L d *_L \,,$$

and assume that $[L, [d, L]] = 0$, then

$$[d^\Lambda, L] = d + [\Lambda, [d, L]] \,, \qquad [d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L] \,.$$

Notice that if (A, L, d) is a Lefschetz complex then $d^2 = 0$ implies that $(d^\Lambda)^2 = 0$. In case $[d, L] = 0$, one also obtains that

$$[d, d^\Lambda] = 0.$$

Therefore, on a Lefschetz space with $[d, L] = 0$ one has that the triple (A, d, d^Λ) is a double complex.

We summarize here the main consequences which we are interested in (cf. [284, Theorems 3.3, 3.5]).

Theorem 3.3. *Let (A, L, d) be a Lefschetz complex. Suppose $[d, L] = 0$ and denote with $\mathcal{H}_L^p$ the space of* Lefschetz harmonic *p-forms, i.e. elements $\alpha \in A^p$ such that*

$$d\alpha = 0 = d^\Lambda \alpha.$$

Then $(\mathcal{H}_L^\bullet, L)$ and $(\mathcal{H}_L^\bullet, \Lambda)$ satisfy the HLC. Furthermore the following are equivalent:

- *$(A^\bullet, L)$ satisfies the dd^Λ-lemma, i.e.,*

$$\operatorname{Ker} d \cap \operatorname{Ker} d^\Lambda \cap (\operatorname{Im} d + \operatorname{Im} d^\Lambda) = \operatorname{Im} dd^\Lambda$$

- *There is a Lefschetz harmonic representative in each cohomology class of* $H^{\bullet}_{d}$*;*
- $(H^{\bullet}_{d}, L)$ *satisfies the HLC;*
- $(H^{\bullet}_{d^{\Lambda}}, \Lambda)$ *satisfies the HLC.*

A Hodge star-type operator.

In this paragraph, we wish to push a little further the work of Tomassini and Wang. We introduce in the picture (a generalization of) the Hodge star operator and use it to define formal adjoints and some Laplacians in a fairly general context. Some identities between these Laplacians are then obtained.

Definition 3.4. A *Lefschetz (differential) graded algebra* is a (differential) Lefschetz space $A = \bigoplus_{p=0}^{2n} A^p$ which is also a graded algebra that is generated by A^1 over $\mathbb{C}$.

In order to provide the promised algebraic treatment, we need a complex structure acting on our Lefschetz space. Let A be a Lefschetz graded algebra and assume that A^1 is equipped with an endomorphism $\mathcal{J}$ such that $\mathcal{J}^2 = -\operatorname{Id}$. We extend the action of $\mathcal{J}$ on A by setting on homogeneous elements

$$\mathcal{J}(\alpha_1 \cdots \alpha_k) = \mathcal{J}\alpha_1 \cdots \mathcal{J}\alpha_k\,, \qquad \text{for every } \alpha_1, \ldots, \alpha_k \in A^1\,,$$

and then extending by $\mathbb{C}$-linearity. Let us denote with

$$A^{1,0} := \{\alpha \in A^1 \mid \mathcal{J}\alpha = i\alpha\}\,, \qquad A^{0,1} := \{\alpha \in A^1 \mid \mathcal{J}\alpha = -i\alpha\}\,,$$

the $\pm i$-eigenspaces of $\mathcal{J}$ on A^1. Putting

$$A^{p,q} := \operatorname{Span}_{\mathbb{C}}\left((A^{1,0})^p \otimes (A^{0,1})^q\right)$$

we see that

$$A^k = \bigoplus_{p+q=k} A^{p,q}$$

and the complex structure $\mathcal{J}$ acts on $A^{p,q}$ as $\mathcal{J}\alpha = i^{p-q}\alpha$. In other words, considering the natural projection $\Pi^{p,q}\colon A \to A^{p,q}$, we have

$$\mathcal{J} = \sum_{p,q} i^{p-q}\Pi^{p,q}.$$

We make the assumption that $\mathcal{J}L = L\mathcal{J}$ and consequently introduce a generalization of the Hodge star operator by setting

$$* := \mathcal{J}*_L = *_L\mathcal{J} \tag{3.2}$$

or equivalently

$$*\frac{1}{k!}L^k\beta := (-1)^{1+2+\cdots+p}\frac{1}{(n-p-k)!}L^{n-p-k}\mathcal{J}\beta\,, \tag{3.3}$$

for a primitive $\beta \in A^p$ and then extend the definition on all A by bilinearity via the Lefschetz decomposition (3.1). It follows that

$$*^2|_{A^p} = \mathcal{J}^2|_{A^p} = (-1)^p\,.$$

Remark 3.5. Let (M, J, ω) be an almost Kähler manifold, namely ω is a symplectic structure on a smooth manifold M and J is a compatible almost complex structure. Clearly when J is integrable and so (M, J) is a complex manifold then (M, J, ω) is a Kähler manifold. Set $L = \omega \wedge -$ for the usual Lefschetz operator and let $A = \bigoplus_{p=0}^{2n} A^p$ be the Lefschetz graded algebra of differential forms on M. The almost complex structure $J\colon TM \to TM$ naturally induces a complex structure $\mathcal{J}$ on A^1. Since ω is a $(1,1)$-form we have $\mathcal{J}L = L\mathcal{J}$ and the description above is coherent with the well known almost Kähler case. Indeed, formula (3.3), where $*$ is the usual Hodge operator, is sometimes referred to as the Weil relation [321].

Now, suppose A is equipped with a differential d. Consider the dual Lefschetz operator

$$\Lambda = *_L^{-1} L *_L = *^{-1} \mathcal{J} L \mathcal{J}^{-1} * = *^{-1} L *$$

and define as before the "Lefschetz adjoint" of d, i.e. $d^\Lambda \in \mathrm{End}(A)$ given by $d^\Lambda|_{A^p} := (-1)^{p+1} *_L d *_L$. Then, by Theorem 3.2, if $[d, L] = 0$ we have $d^\Lambda = [d, \Lambda]$ and $d = [d^\Lambda, L]$.

We may take the "Hodge adjoints"

$$d^* = - * d * , \qquad d^{\Lambda *} = - * d^\Lambda * ,$$

and obtain also $d^* = [\Lambda, d^{\Lambda *}]$.

Now, we consider the following operators and we aim to study the relations between them:

$$\Delta_d = dd^* + d^* d , \qquad \Delta_{d^\Lambda} = d^\Lambda d^{\Lambda *} + d^{\Lambda *} d^\Lambda ,$$

$$\Delta^{\mathrm{BC}}_{d^\Lambda} = d^* d + d^{\Lambda *} d^\Lambda + d d^\Lambda d^{\Lambda *} d^* + d^{\Lambda *} d d^* d^\Lambda + d^* d^\Lambda d^{\Lambda *} d + d^{\Lambda *} d^* d d^\Lambda ,$$

$$\Delta^{\mathrm{BC}}_{d^{\Lambda *}} = d^* d + d^\Lambda d^{\Lambda *} + d d^{\Lambda *} d^\Lambda d^* + d^\Lambda d d^* d^{\Lambda *} + d^* d^{\Lambda *} d^\Lambda d + d^\Lambda d^* d d^{\Lambda *} .$$

We will denote with $\mathcal{H}^\bullet_d$ and $\mathcal{H}^\bullet_{d^\Lambda}$ the kernels of Δ_d and Δ_{d^Λ} respectively. All these operators where originally introduced for symplectic manifolds in [294].

Proposition 3.6. *In the previous assumptions it holds*

$$\Delta_d = \Delta_{d^\Lambda} - [\Lambda, [d, d^{\Lambda *}]] .$$

In particular, if $[d, d^{\Lambda *}] = 0$ the kernels of Δ_d and Δ_{d^Λ} coincide, namely for every p we have

$$\mathcal{H}^p_d = \mathcal{H}^p_{d^\Lambda} .$$

Proof. Using $[\Lambda, d^{\Lambda *}] = d^*$ and $[d, \Lambda] = d^\Lambda$ we obtain

$$\Delta_d = [d, d^*] = [d, [\Lambda, d^{\Lambda *}]] = [[d, \Lambda], d^{\Lambda *}] - [\Lambda, [d, d^{\Lambda *}]] = [d^\Lambda, d^{\Lambda *}] - [\Lambda, [d, d^{\Lambda *}]] = \Delta_{d^\Lambda} - [\Lambda, [d, d^{\Lambda *}]] ,$$

as desired □

Proposition 3.7. *If $[d, d^{\Lambda *}] = 0 = [d, L]$, then*

$$\begin{aligned} \Delta^{\mathrm{BC}}_{d^\Lambda} &= \Delta_{d^\Lambda} \Delta_{d^\Lambda} + d^* d + d^{\Lambda *} d^\Lambda \\ &= \Delta^{\mathrm{BC}}_{d^{\Lambda *}} + d^{\Lambda *} d^\Lambda - d^\Lambda d^{\Lambda *} . \end{aligned}$$

Proof. Notice that under our assumptions we also have $[d^*, d^\Lambda] = 0$ and $[d^*, d^{\Lambda *}] = 0$. We start by considering $\Delta_{d^\Lambda} \Delta_{d^\Lambda}$. By Proposition 3.6

$$\Delta_{d^\Lambda} \Delta_{d^\Lambda} = \Delta_d \Delta_{d^\Lambda} = d d^* d^\Lambda d^{\Lambda *} + d d^* d^{\Lambda *} d^\Lambda + d^* d d^\Lambda d^{\Lambda *} + d^* d d^{\Lambda *} d^\Lambda = (I) + (II) + (III) + (IV) .$$

We will treat the four terms separately. Using that $[d, d^{\Lambda *}] = 0$ and $[d, d^\Lambda] = 0$

$$\begin{aligned} (I) &= d d^* d^\Lambda d^{\Lambda *} = - d d^\Lambda d^* d^{\Lambda *} = d d^\Lambda d^{\Lambda *} d^* , & (II) &= d d^* d^{\Lambda *} d^\Lambda = - d d^{\Lambda *} d^* d^\Lambda = d^{\Lambda *} d d^* d^\Lambda , \\ (III) &= d^* d d^\Lambda d^{\Lambda *} = - d^* d^\Lambda d d^{\Lambda *} = d^* d^\Lambda d^{\Lambda *} d , & (IV) &= d^* d d^{\Lambda *} d^\Lambda = - d^* d^{\Lambda *} d d^\Lambda = d^{\Lambda *} d^* d d^\Lambda . \end{aligned}$$

Now, putting the four terms together we have

$$\Delta_{d^\Lambda} \Delta_{d^\Lambda} = \Delta_d \Delta_{d^\Lambda} = d d^\Lambda d^{\Lambda *} d^* + d^{\Lambda *} d d^* d^\Lambda + d^* d^\Lambda d^{\Lambda *} d + d^{\Lambda *} d^* d d^\Lambda = \Delta^{\mathrm{BC}}_{d^\Lambda} - d^* d - d^{\Lambda *} d^\Lambda .$$

Furthermore using again that $[d, d^{\Lambda*}] = 0$ and $[d, d^{\Lambda}] = 0$ we obtain

$$\begin{aligned}\Delta^{\mathrm{BC}}_{d^\Lambda} =& d^*d + d^{\Lambda*}d^{\Lambda} + dd^{\Lambda}d^{\Lambda*}d^* + d^{\Lambda*}dd^*d^{\Lambda} + d^*d^{\Lambda}d^{\Lambda*}d + d^{\Lambda*}d^*dd^{\Lambda}\\ =& d^*d + d^{\Lambda*}d^{\Lambda} + d^{\Lambda}dd^*d^{\Lambda*} + dd^{\Lambda*}d^{\Lambda}d^* + d^{\Lambda}d^*dd^{\Lambda*} + d^*d^{\Lambda*}d^{\Lambda}d\\ =& d^{\Lambda*}d^{\Lambda} - d^{\Lambda}d^{\Lambda*} + \Delta^{\mathrm{BC}}_{d^{\Lambda*}}\end{aligned}$$

as desired. □

Remark 3.8. Let (M, J, ω) be an almost Kähler manifold, then $[d, d^{\Lambda*}] = 0$ if and only if $[d, d^c] = 0$ if and only if J is integrable. In such a case (M, J, ω) is a Kähler manifold and we recover the usual equalities for the Laplacians in Propositions 3.6 and 3.7.

3.1.2 Application to HKT manifolds.

Naturally, the quaternionic Dolbeault, Bott-Chern and Aeppli cohomology groups can be defined:

$$H^{p,0}_{\partial}(M) := \frac{\mathrm{Ker}(\partial|_{\Lambda^{p,0}(M)})}{\partial\Lambda^{p-1,0}(M)}, \qquad H^{p,0}_{\partial_J}(M) := \frac{\mathrm{Ker}(\partial_J|_{\Lambda^{p,0}(M)})}{\partial_J\Lambda^{p-1,0}(M)},$$

$$H^{p,0}_{\mathrm{BC}}(M) := \frac{\mathrm{Ker}(\partial|_{\Lambda^{p,0}(M)}) \cap \mathrm{Ker}(\partial_J|_{\Lambda^{p,0}(M)})}{\partial\partial_J\Lambda^{p-2,0}(M)},$$

$$H^{p,0}_{\mathrm{A}}(M) := \frac{\mathrm{Ker}(\partial\partial_J|_{\Lambda^{p,0}(M)})}{\partial\Lambda^{p-1,0}(M) + \partial_J\Lambda^{p-1,0}(M)},$$

when M is compact all these groups are finite-dimensional [146], indeed, as usual, once fixed an hyperhermitian metric, one can show that each of these cohomology groups is isomorphic to the kernel of the following Laplacians acting on $(p, 0)$-forms

$$\Delta_{\partial} := \partial\partial^* + \partial^*\partial, \qquad \Delta_{\partial_J} := \partial_J\partial_J^* + \partial_J^*\partial_J,$$

$$\Delta_{\mathrm{BC}} := \partial^*\partial + \partial_J^*\partial_J + \partial\partial_J\partial_J^*\partial^* + \partial_J^*\partial^*\partial\partial_J + \partial_J^*\partial\partial^*\partial_J + \partial^*\partial_J\partial_J^*\partial,$$

$$\Delta_{\mathrm{A}} := \partial\partial^* + \partial_J\partial_J^* + \partial\partial_J\partial_J^*\partial^* + \partial_J^*\partial^*\partial\partial_J + \partial\partial_J^*\partial_J\partial^* + \partial_J\partial^*\partial\partial_J^*.$$

For each of these we denote with a calligraphic letter the corresponding space of harmonic forms, thus, for instance, $\mathcal{H}^{p,0}_{\partial}(M) := \mathrm{Ker}(\Delta_{\partial}|_{\Lambda^{p,0}(M)})$.

It is well known that on a compact Kähler manifold the spaces of Dolbeault, Bott-Chern and Aeppli-harmonic forms all coincide. We exploit the general theory of Lefschetz spaces to prove that the analogue result is also true for balanced HKT manifolds (Theorem 3.21). We remark that the equality of Δ_{∂_J} and Δ_{∂} on balanced HKT manifolds already follows from Proposition 2.35 by Verbitsky. Along the way we shall also study the Hard Lefschetz condition on these spaces (see Theorems 3.9 and 3.17).

"Symplectic" Hodge theory on HKT manifolds.

Let (M, I, J, K) be a $4n$-dimensional compact hypercomplex manifold and $\Omega \in \Lambda^{2,0}(M)$ a non-degenerate $(2, 0)$-form on (M, I). As usual we set

$$L \colon \Lambda^{r,0}(M) \to \Lambda^{r+2,0}(M), \qquad L := \frac{\Omega}{2} \wedge -$$

for the Lefschetz operator. Then $(\Lambda^{\bullet,0}(M), L)$ is a Lefschetz space. Moreover, if we consider as differential operator ∂ (always taken with respect to I), since I is integrable, $\partial^2 = 0$ and so $(\Lambda^{\bullet,0}(M), L, \partial)$ defines a Lefschetz complex. If Ω satisfies $\partial\Omega = 0$ then

$$[\partial, L] = 0.$$

Denote with $\mathcal{H}_L^{p,0}(M)$ the space of *Lefschetz harmonic* $(p,0)$-forms, i.e. forms $\alpha \in \Lambda^{p,0}(M)$ such that $\partial\alpha = 0 = \partial^\Lambda\alpha$, where $\partial^\Lambda = [\partial, \Lambda]$.

We can therefore apply the results of the previous section to infer:

Theorem 3.9. *Let (M, I, J, K, Ω) be a compact hypercomplex manifold and $\Omega \in \Lambda^{2,0}(M)$ a non-degenerate $(2,0)$-form on (M, I) such that $\partial\Omega = 0$. Then $(\mathcal{H}_L^{\bullet,0}(M), L)$ and $(\mathcal{H}_L^{\bullet,0}(M), \Lambda)$ satisfy the HLC. Furthermore the following are equivalent:*

- *$(\Lambda^{\bullet,0}(M), L)$ satisfies the $\partial\partial^\Lambda$-lemma, i.e.,*

$$\operatorname{Ker}\partial \cap \operatorname{Ker}\partial^\Lambda \cap (\operatorname{Im}\partial + \operatorname{Im}\partial^\Lambda) = \operatorname{Im}\partial\partial^\Lambda \,;$$

- *There is a Lefschetz harmonic representative in each Dolbeault cohomology class of $H_\partial^{\bullet,0}(M)$;*
- *$(H_\partial^{\bullet,0}(M), L)$ satisfies the HLC;*
- *$(H_{\partial^\Lambda}^{\bullet,0}(M), \Lambda)$ satisfies the HLC.*

Moreover, by the general results in the previous section we obtain:

Proposition 3.10. *Let (M, I, J, K, Ω) be a compact hypercomplex manifold and $\Omega \in \Lambda^{2,0}(M)$ a non-degenerate $(2,0)$-form on (M, I) such that $\partial\Omega = 0$. Then,*

$$\Delta_\partial = \Delta_{\partial^\Lambda} - [\Lambda, [\partial, \partial^{\Lambda *}]] \,.$$

In particular, if $[\partial, \partial^{\Lambda *}] = 0$

$$\Delta_\partial = \Delta_{\partial^\Lambda} \,,$$

and for every p we have

$$\mathcal{H}^p_\partial(M) = \mathcal{H}^p_{\partial^\Lambda}(M) \,.$$

Moreover, if $[\partial, \partial^{\Lambda *}] = 0$

$$\Delta^{\mathrm{BC}}_{\partial^\Lambda} = \Delta^{\mathrm{BC}}_{\partial^{\Lambda *}} + \partial^{\Lambda *}\partial^\Lambda - \partial^\Lambda\partial^{\Lambda *} = \Delta_{\partial^\Lambda}\Delta_{\partial^\Lambda} + \partial^*\partial + \partial^{\Lambda *}\partial^\Lambda \,.$$

"Complex" Hodge theory on HKT manifolds.

If we further assume that Ω is q-positive, in the sense that $J\Omega = \bar{\Omega}$ and $\Omega(Z, J\bar{Z}) > 0$ for every $Z \in T_I^{1,0}M$, $Z \neq 0$, then it must be the HKT form corresponding to a HKT metric g on (M, I, J, K). If (M, I, J, K, g, Ω) is HKT by Proposition 2.35 we have

$$[\partial^*, L] = \partial_J - \theta_J \wedge -$$

where $\theta_J = J\bar{\theta}$, being θ the 1-form such that $\partial\bar{\Omega}^n = \theta \wedge \bar{\Omega}^n$. Notice that (M, I, J, K, g, Ω) is balanced if and only if $\theta_J = 0$ and so for balanced HKT manifolds we have

$$[\partial^*, L] = \partial_J$$

and actually we can specialize Proposition 2.35 to the following:

Proposition 3.11. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold. Then, the following identities hold:*

$$[\partial^*, L] = \partial_J \,, \qquad [\partial, \Lambda] = -\partial_J^* \,, \qquad [L, \partial_J^*] = \partial \,, \qquad [\Lambda, \partial_J] = -\partial^* \,.$$

Now, we shall show that the framework of the previous subsection can be used to study hypercomplex cohomologies. First of all, we set $\mathcal{J}\alpha = J\bar{\alpha}$ for every $\alpha \in \Lambda^{1,0}(M)$, thus $\mathcal{J}$ is a complex structure on $\Lambda^{1,0}(M)$ and naturally extends to $\Lambda^{p,0}(M)$ by imposing compatibility with the wedge product. Since Ω is q-real we have $\mathcal{J}L = L\mathcal{J}$ and we can use (3.2) to define a Hodge-type operator.

We warn the reader that in this framework the operator defined by (3.2) slightly differs from the usual Hodge operator. To distinguish them, let us denote here $* : \Lambda^{p,0}(M) \to \Lambda^{2n-p,0}(M)$ the operator defined in (3.2) and $\hat{*} : \Lambda^{p,q}(M) \to \Lambda^{2n-q,2n-p}(M)$ the usual Hodge star operator, then one can easily show that

$$\alpha \wedge *\beta = g(\alpha,\beta)\frac{\Omega^n}{n!}\,, \qquad \text{for every } \alpha,\beta \in \Lambda^{p,0}(M)\,,$$

where g is the Hermitian product induced by the Riemannian metric on $\Lambda^{p,0}(M)$, while, by definition,

$$\alpha \wedge \hat{*}\bar{\beta} = g(\alpha,\beta)\frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}\,, \qquad \text{for every } \alpha,\beta \in \Lambda^{p,0}(M)\,.$$

However, we can identify the formal adjoints of ∂ and ∂_J with respect to $*$ and $\hat{*}$ in the following way.

Suppose M is a $\mathrm{SL}(n,\mathbb{H})$-manifold and fix a q-positive holomorphic $(2n,0)$-form Θ. Define the following L^2-products:

$$(\alpha,\beta)_1 := \int_M g(\alpha,\beta)\frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2} = \int_M \alpha \wedge \hat{*}\bar{\beta}\,, \qquad (\alpha,\beta)_2 := \int_M g(\alpha,\beta)\frac{\Omega^n}{n!} \wedge \bar{\Theta} = \int_M \alpha \wedge *\beta \wedge \bar{\Theta}\,,$$

Then the adjoint of ∂ and ∂_J with respect to $(\cdot,\cdot)_1$ are $\partial^{\hat{*}} = -\hat{*}\bar{\partial}\hat{*}$ and $\partial_J^{\hat{*}} = -\hat{*}\bar{\partial}_J\hat{*}$, while those with respect to $(\cdot,\cdot)_2$ are $\partial^* = - * \partial *$ and $\partial_J^* = - * \partial_J *$ (cf. [208]). Since Θ is q-positive, there exists a real-valued function $f > 0$ such that $\Theta = f\frac{\Omega^n}{n!}$, moreover, the holomorphicity of Θ translates into the condition $\partial f + f\theta = 0$. Now, observe that $(\cdot,\cdot)_2 = (f\cdot,\cdot)_1$ thus

$$(\alpha,\partial^*\beta)_2 = (\partial\alpha,\beta)_2 = (f\partial\alpha,\beta)_1 = (\partial(f\alpha) - \partial f \wedge \alpha,\beta)_1 = (\alpha,\partial^{\hat{*}}\beta)_2 + (\theta\wedge\alpha,\beta)_2$$

and similarly, working with ∂_J^* and $\partial_J^{\hat{*}}$ one obtains

$$(\alpha,\partial_J^*\beta)_2 = (\alpha,\partial_J^{\hat{*}}\beta)_2 - (\theta_J \wedge \alpha,\beta)_2\,.$$

In particular if M is balanced then $\theta = \theta_J = 0$ and f is constant, so that the two L^2-products coincide up to a constant and $\partial^* = \partial^{\hat{*}}$ and $\partial_J^* = \partial_J^{\hat{*}}$. In particular the usual Laplacians obtained by means of the Riemannian Hodge star operator coincide with those Laplacians considered above and the related results can be applied.

In particular, if M is compact and if $\alpha \in \Lambda^{p,0}(M)$ one immediately obtains

$$\begin{cases} \alpha \in \mathcal{H}^{p,0}_{\partial}(M) & \iff \quad \partial\alpha = 0\,, \quad \partial^*\alpha = 0\,; \\ \alpha \in \mathcal{H}^{p,0}_{\partial_J}(M) & \iff \quad \partial_J\alpha = 0\,, \quad \partial_J^*\alpha = 0\,; \\ \alpha \in \mathcal{H}^{p,0}_{BC}(M) & \iff \quad \partial\alpha = 0\,, \quad \partial_J\alpha = 0\,, \quad \partial_J^*\partial^*\alpha = 0\,; \\ \alpha \in \mathcal{H}^{p,0}_{A}(M) & \iff \quad \partial^*\alpha = 0\,, \quad \partial_J^*\alpha = 0\,, \quad \partial\partial_J\alpha = 0\,. \end{cases}$$

Proposition 3.11 shows that $\partial^\Lambda = -\partial_J^*$ and we readily obtain from Proposition 3.6.

Proposition 3.12. *Let (M,I,J,K,Ω) be a compact balanced HKT manifold, then*

$$\Delta_{\partial_J} = \Delta_\partial\,.$$

In particular, the spaces of harmonic forms coincide, namely for every p we have

$$\mathcal{H}^{p,0}_{\partial_J}(M) = \mathcal{H}^{p,0}_{\partial}(M)\,.$$

Remark 3.13. If we do not assume the compact HKT manifold (M,I,J,K,Ω) to be balanced we would have, in general

$$[\partial^*, L] = \partial_J - \theta_J \wedge -\,.$$

Setting $\tau(\alpha) := \theta_J \wedge \alpha$ and $\psi(\alpha) := \theta \wedge \alpha$, then, we would get

$$\partial^\Lambda = [\partial,\Lambda] = -\partial_J^* + \tau^*\,.$$

In particular $[\partial, \partial^{\Lambda *}] = -\partial\theta_J \wedge -$ and in such a case the Laplacians Δ_∂, $\Delta_{\partial^\Lambda}$ and Δ_{∂_J} do not coincide, and in fact by a direct computation one gets

$$\Delta_{\partial_J} = \Delta_\partial - [\psi^*, \partial] + [\partial_J, \tau^*].$$

Notice that $\psi^* = \iota_{\theta^\sharp}$ and $\tau^* = \iota_{\theta_J^\sharp}$.

We also observe that the condition $[\partial, \partial^{\Lambda *}] = 0$, i.e. $\partial\theta_J = 0$ is only satisfied when the manifold is balanced, indeed $\partial\theta_J = 0$ is equivalent to $\partial_J\theta = 0$, from which we obtain

$$\begin{aligned}\partial_J\partial\bar{\Omega}^n &= \partial_J(\theta \wedge \bar{\Omega}^n) = \partial_J\theta \wedge \bar{\Omega}^n - \theta \wedge \partial_J\bar{\Omega}^n = -\theta \wedge J^{-1}\bar{\partial}\Omega^n \\ &= -\theta \wedge J^{-1}(\bar{\theta} \wedge \Omega^n) = \theta \wedge J\bar{\theta} \wedge \bar{\Omega}^n\,.\end{aligned}$$

Therefore by integrating, using (2.21) and the HKT condition we infer

$$0 = \int_M \partial_J\bar{\Omega}^n \wedge \partial\Omega^{n-1} = \int_M \partial_J\partial\bar{\Omega}^n \wedge \Omega^{n-1} = \int_M \theta \wedge J\bar{\theta} \wedge \bar{\Omega}^n \wedge \Omega^{n-1} = -\frac{1}{2n}\int_M \|\theta\|_g^2\, \Omega^n \wedge \bar{\Omega}^n$$

and the claim follows.

As a consequence of Proposition 3.12 we obtain isomorphisms for the associated cohomology groups (cf. [208, Proposition 2.3] where it is noticed that an isomorphism, induced by J and conjugation with respect to I, holds in general for hypercomplex manifolds).

Corollary 3.14. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold, then*

$$H^{p,0}_{\partial_J}(M) \simeq H^{p,0}_{\partial}(M)\,.$$

In particular, we have the equalities $h^{p,0}_{\partial_J}(M) = h^{p,0}_{\partial}(M)$.

Invoking Proposition 3.7 we obtain that, similarly to the Kähler case, the Laplacians Δ_{BC} and $\Delta_{\partial_J} = \Delta_\partial$ are related.

Proposition 3.15. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold, then*

$$\begin{aligned}\Delta_{\mathrm{BC}} &= \Delta_{\partial_J}\Delta_{\partial_J} + \partial^*\partial + \partial_J^*\partial_J \\ &= \Delta_\partial\Delta_\partial + \partial^*\partial + \partial_J^*\partial_J\,.\end{aligned}$$

In particular, the spaces of harmonic forms coincide, namely for every p we have

$$\mathcal{H}^{p,0}_{\mathrm{BC}}(M) = \mathcal{H}^{p,0}_{\partial_J}(M)\,.$$

Consequently we obtain isomorphisms for the associated cohomology groups.

Corollary 3.16. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold, then for every p,*

$$H^{p,0}_{\mathrm{BC}}(M) \simeq H^{p,0}_{\partial_J}(M) \simeq H^{p,0}_{\partial}(M)\,.$$

In particular, we have the equalities $h^{p,0}_{\mathrm{BC}}(M) = h^{p,0}_{\partial_J}(M) = h^{p,0}_{\partial}(M)$.

Notice that these results are the analogue of the ones proved in [256] for compact Kähler manifolds.

As a consequence of the previous results we prove that, under the same hypothesis, the Hard Lefschetz condition holds for the cohomologies $H^{\bullet,0}_{\partial}(M)$, $H^{\bullet,0}_{\partial_J}(M)$, $H^{\bullet,0}_{\mathrm{BC}}(M)$, thus generalizing [29, Proposition 4.7].

Theorem 3.17. *Let (M, I, J, K, Ω) be a compact $4n$-dimensional balanced HKT manifold, then for every i,*

$$L^{n-i} : \mathcal{H}^{i,0}_{\partial}(M) \to \mathcal{H}^{2n-i,0}_{\partial}(M),$$

$$L^{n-i} : \mathcal{H}^{i,0}_{\partial_J}(M) \to \mathcal{H}^{2n-i,0}_{\partial_J}(M)\,,$$

$$L^{n-i} : \mathcal{H}^{i,0}_{\mathrm{BC}}(M) \to \mathcal{H}^{2n-i,0}_{\mathrm{BC}}(M)$$

are isomorphisms. In particular $h^{i,0}_{\mathrm{BC}} = h^{i,0}_{\partial} = h^{i,0}_{\partial_J} = h^{2n-i,0}_{\mathrm{BC}} = h^{2n-i,0}_{\partial} = h^{2n-i,0}_{\partial_J}$.

Proof. In view of Propositions 3.12, 3.15 it is sufficient to prove that

$$L^{n-i} : \mathcal{H}^{i,0}_{\partial}(M) \to \mathcal{H}^{2n-i,0}_{\partial}(M)$$

are isomorphisms. Notice that by hypothesis $\partial\Omega = \partial_J\Omega = 0$, hence

$$[\partial, L] = 0\,, \qquad [\partial_J, L] = 0\,.$$

Let $\alpha \in \mathcal{H}^{i,0}_{\partial}(M) = \mathcal{H}^{i,0}_{\partial_J}(M)$. Then,

$$\partial\alpha = 0\,, \quad \partial_J\alpha = 0\,, \quad \partial^*\alpha = 0\,, \quad \partial_J^*\alpha = 0.$$

As a consequence

$$\partial(L^{n-i}\alpha) = L^{n-i}\partial\alpha = 0,$$

and, using $[\partial^*, L] = \partial_J$,

$$\partial^*(L^{n-i}\alpha) = L^{n-i}\partial^*\alpha + (n-i)L^{n-i-1}\partial_J\alpha = 0.$$

Hence, $L^{n-i}\alpha \in \mathcal{H}^{2n-i,0}_{\partial}(M)$. The result follows from Ω being non-degenerate. □

Notice that combining this result with Theorem 3.9 we have

Corollary 3.18. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold, then the $\partial\partial^\Lambda$-lemma holds and there exists a Lefschetz harmonic representative in each Dolbeault cohomology class of $H^{\bullet,0}_{\partial}(M)$.*

Proposition 3.19. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold, then for every p we have*

$$\mathcal{H}^{p,0}_{\mathrm{BC}}(M) = \mathcal{H}^{p,0}_{\mathrm{A}}(M).$$

Proof. We first show the inclusion $\mathcal{H}^{p,0}_{\mathrm{BC}}(M) \subseteq \mathcal{H}^{p,0}_{\mathrm{A}}(M)$. Let $\alpha \in \mathcal{H}^{p,0}_{\mathrm{BC}}(M)$. By Propositions 3.12, 3.15 $\alpha \in \mathcal{H}^{p,0}_{\partial}(M) = \mathcal{H}^{p,0}_{\partial_J}(M)$, namely

$$\partial\alpha = 0\,, \quad \partial_J\alpha = 0\,, \quad \partial^*\alpha = 0\,, \quad \partial_J^*\alpha = 0.$$

Hence, $\alpha \in \mathcal{H}^{p,0}_{\mathrm{A}}(M)$. The opposite inclusion $\mathcal{H}^{p,0}_{\mathrm{A}}(M) \subseteq \mathcal{H}^{p,0}_{\mathrm{BC}}(M)$ follows from Theorem 3.17 and [146, Remark 21], indeed for every p,

$$h^{p,0}_{\mathrm{BC}}(M) = h^{2n-p,0}_{\mathrm{BC}}(M) = h^{p,0}_{\mathrm{A}}(M).$$

□

As a corollary we have

Corollary 3.20. *Let (M, I, J, K, Ω) be a compact balanced HKT manifold, then for every p,*

$$H^{p,0}_{\mathrm{BC}}(M) \simeq H^{p,0}_{\mathrm{A}}(M).$$

We summarize the results of Propositions 3.12, 3.15, 3.19 collecting them into a single theorem:

Theorem 3.21. *On a compact balanced HKT manifold M the spaces of harmonic forms all coincide:*

$$\mathcal{H}^{p,0}_{\partial}(M) = \mathcal{H}^{p,0}_{\partial_J}(M) = \mathcal{H}^{p,0}_{\mathrm{BC}}(M) = \mathcal{H}^{p,0}_{\mathrm{A}}(M)\,.$$

In particular there are isomorphisms

$$H^{p,0}_{\partial}(M) \cong H^{p,0}_{\partial_J}(M) \cong H^{p,0}_{\mathrm{BC}}(M) \cong H^{p,0}_{\mathrm{A}}(M)$$

and equalities $h^{p,0}_{\partial} = h^{p,0}_{\partial_J} = h^{p,0}_{\mathrm{BC}} = h^{p,0}_{\mathrm{A}} = h^{2n-p,0}_{\partial} = h^{2n-p,0}_{\partial_J} = h^{2n-p,0}_{\mathrm{BC}} = h^{2n-p,0}_{\mathrm{A}}$ *for every p.*

3.1.3 Formality of HKT manifolds.

In this subsection we study formality for compact hypercomplex manifolds. It is well known that formality in the sense of Sullivan is an obstruction to Kählerness, more precisely compact complex manifolds satisfying the $\partial\bar{\partial}$-lemma are formal (see [99]). However, notice that the HKT condition does not imply formality, indeed there are examples of non tori nilmanifolds that are HKT but it is well know that non tori nilmanifolds are not formal in the sense of Sullivan [167].

We first recall some definitions.
Let $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ be two differential graded algebras (DGA for short) over a field $\mathbb{K}$. A *DGA-homomorphism* between $\mathcal{A}$ and $\mathcal{B}$ is a $\mathbb{K}$-linear map $f : \mathcal{A} \to \mathcal{B}$ such that

- $f(\mathcal{A}^i) \subset \mathcal{B}^i$;
- $f(\alpha \cdot \beta) = f(\alpha) \cdot f(\beta)$;
- $d_{\mathcal{B}} \circ f = f \circ d_{\mathcal{A}}$.

Any DGA-homomorphism $f : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ induces a DGA-homomorphism in cohomology

$$H(f) : (H^\bullet(\mathcal{A}, d_{\mathcal{A}}), 0) \to (H^\bullet(\mathcal{B}, d_{\mathcal{B}}), 0)\,.$$

A DGA-homomorphism $f : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ is called *quasi-isomorphism* if $H(f)$ is an isomorphism. Two DGA $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ are said to be *equivalent* if there exists a sequence of quasi-isomorphisms of the following form:

$$\begin{array}{ccccccccc} & & (\mathcal{C}_1, d_{\mathcal{C}_1}) & & & & \cdots & & (\mathcal{C}_n, d_{\mathcal{C}_n}) & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\ (\mathcal{A}, d_{\mathcal{A}}) & & & & (\mathcal{C}_2, d_{\mathcal{C}_2}) & & & & \cdots & & & & (\mathcal{B}, d_{\mathcal{B}})\,. \end{array}$$

Finally, a DGA $(\mathcal{A}, d_{\mathcal{A}})$ is called *formal* if $(\mathcal{A}, d_{\mathcal{A}})$ is equivalent to a DGA $(\mathcal{B}, d_{\mathcal{B}} = 0)$.

We are about to show that for a compact hypercomplex manifold M instead of $(\Lambda^\bullet(M), d)$, the appropriate DGA to consider in this context is $(\Lambda^{\bullet,0}(M), \partial)$ by proving the following

Theorem 3.22. *Let (M, I, J, K) be a compact hypercomplex manifold satisfying the $\partial\partial_J$-lemma, then the DGA $(\Lambda^{\bullet,0}(M), \partial)$ is formal.*

Preliminary lemmas.

In order to prove Theorem 3.22 we will need three lemmas.

Lemma 3.23. *Let (M, I, J, K) be a compact hypercomplex manifold satisfying the $\partial\partial_J$-lemma, then the natural inclusion*

$$i : \left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right) \to \left(\Lambda^{\bullet,0}(M),\, \partial\right)$$

is a DGA quasi-isomorphism.

Proof. Notice that $\left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right)$ is a DGA and the inclusion

$$i : \left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right) \to \left(\Lambda^{\bullet,0}(M),\, \partial\right)$$

is a morphism of DGAs. We are left to prove that the map induced in cohomology

$$H_\partial(i) : H_\partial\left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right) \to H_\partial^{\bullet,0}(M)$$

is an isomorphism.
We first prove that $H_\partial(i)$ is injective. Fix k, and let $[\alpha] \in H_\partial\left(\Lambda^{k,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right)$ such that $H_\partial(i)([\alpha]) = [\alpha]_\partial = 0$, hence

$$\alpha \in \operatorname{Ker}\partial_J \cap \operatorname{Im}\partial = \operatorname{Im}\partial\partial_J$$

i.e., $\alpha = \partial\left(\partial_J \beta\right)$ for some form $\beta \in \Lambda^{k-2,0}(M)$ and clearly $\partial_J\beta \in \Lambda^{k-1,0}(M) \cap \operatorname{Ker}\partial_J$, hence

$$[\alpha] = 0 \in H^{k,0}_{\partial}\left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right)$$

and so $H_\partial(i)$ is injective.

We now prove that $H_\partial(i)$ is surjective. Let $a \in H^{k,0}_{\partial}(M)$, $a = [\alpha]$ with $\partial\alpha = 0$. Consider,

$$\partial_J \alpha \in \operatorname{Im}\partial_J \cap \operatorname{Ker}\partial = \operatorname{Im}\partial_J\partial$$

hence $\partial_J\alpha = \partial_J\partial\beta$ for some β. Therefore, $\partial_J(\alpha - \partial\beta) = 0$ and $\partial(\alpha - \partial\beta) = \partial\alpha = 0$. This means that $\alpha - \partial\beta$ defines a class in $H^{k,0}_{\partial}\left(\Lambda^{\bullet,0}M \cap \operatorname{Ker}\partial_J,\, \partial\right)$ and

$$H_\partial(i)([\alpha - \partial\beta]) = [\alpha - \partial\beta]_\partial = [\alpha] = a\,,$$

concluding the proof. □

Lemma 3.24. *Let (M, I, J, K) be a compact hypercomplex manifold satisfying the $\partial\partial_J$-lemma, then the natural projection*

$$p : \left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right) \to \left(H^{\bullet,0}_{\partial_J}(M),\, \partial\right)$$

is a DGA quasi-isomorphism.

Proof. Notice that the projection

$$p : \left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right) \to \left(H^{\bullet,0}_{\partial_J}(M),\, \partial\right)$$

is a morphism of DGAs. We are left to prove that the map induced in cohomology

$$H_\partial(p) : H^{\bullet,0}_{\partial}\left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right) \to H^{\bullet,0}_{\partial}\left(H^{\bullet,0}_{\partial_J}(M),\, \partial\right)$$

is an isomorphism.

We first prove that $H_\partial(p)$ is injective. Fix k, and let $[\alpha] \in H^{k,0}_{\partial}\left(\Lambda^{\bullet,0}(M) \cap \operatorname{Ker}\partial_J,\, \partial\right)$ such that $H_\partial(p)([\alpha]) = 0$. Hence,

$$\alpha \in \operatorname{Im}\partial \cap \operatorname{Ker}\partial_J = \operatorname{Im}\partial\partial_J$$

i.e., $\alpha = \partial\left(\partial_J \beta\right)$ for some form $\beta \in \Lambda^{k-2,0}(M)$ and clearly $\partial_J\beta \in \Lambda^{k-1,0}(M) \cap \operatorname{Ker}\partial_J$, hence

$$[\alpha] = 0 \in H^{\bullet,0}_{\partial}\left(H^{\bullet,0}_{\partial_J}(M),\, \partial\right)$$

and so $H_\partial(p)$ is injective.

The surjectivity of $H_\partial(p)$ is immediate. □

Lemma 3.25. *Let (M, I, J, K) be a compact hypercomplex manifold satisfying the $\partial\partial_J$-lemma, then ∂ is the trivial operator on $H^{\bullet,0}_{\partial_J}(M)$.*

Proof. Fix k and let $a = [\alpha]_{\partial_J} \in H^{k,0}_{\partial_J}(M)$, namely $\partial_J\alpha = 0$. Now

$$\partial a = [\partial\alpha]_{\partial_J}$$

and

$$\partial\alpha \in \operatorname{Im}\partial \cap \operatorname{Ker}\partial_J = \operatorname{Im}\partial_J\partial$$

so $\partial\alpha = \partial_J\partial\beta$ for some β, giving $\partial a = [\partial_J\partial\beta]_{\partial_J} = 0 \in H^{k+1,0}_{\partial_J}(M)$, concluding the proof. □

Proof of formality and consequences.

Now we are able to prove Theorem 3.22.

Proof. Under the assumptions and as a consequence of the lemmas 3.23, 3.24, 3.25 we have the following diagram of quasi-isomorphisms of DGAs,

$$\begin{array}{ccc} & \left(\Lambda^{\bullet,0}(M)\cap \operatorname{Ker}\partial_J,\, \partial\right) & \\ \overset{i}{\underset{\text{q-is}}{\swarrow}} & & \overset{p}{\underset{\text{q-is}}{\searrow}} \\ \left(\Lambda^{\bullet,0}(M),\, \partial\right) & & \left(H^{\bullet,0}_{\partial_J}(M),\, 0\right) \end{array}.$$

hence, by definition, $\left(\Lambda^{\bullet,0}(M),\, \partial\right)$ is a formal DGA. □

In [146, Theorem 6] Grantcharov, Lejmi and Verbitsky proved that the $\partial\partial_J$-lemma always holds on compact HKT $\mathrm{SL}(n,\mathbb{H})$ manifolds. As a consequence of this and Theorem 3.22 we obtain:

Corollary 3.26. *Let (M, I, J, K, Ω) be a compact HKT $\mathrm{SL}(n,\mathbb{H})$-manifold, then the DGA $(\Lambda^{\bullet,0}(M), \partial)$ is formal.*

We recall now the definition of triple Massey products of a DGA in our setting.

Definition 3.27. Let $\mathfrak{a} = [\alpha] \in H^{p,0}_{\partial}(M)$, $\mathfrak{b} = [\beta] \in H^{q,0}_{\partial}(M)$ and $\mathfrak{c} = [\gamma] \in H^{r,0}_{\partial}(M)$ such that $\mathfrak{a} \cup \mathfrak{b} = 0 \in H^{p+q,0}_{\partial}(M)$ and $\mathfrak{b} \cup \mathfrak{c} = 0 \in H^{q+r,0}_{\partial}(M)$; more precisely suppose that $\alpha \wedge \beta = \partial\lambda$ and $\beta \wedge \gamma = \partial\mu$ for some $\lambda \in \Lambda^{p+q-1,0}$, $\mu \in \Lambda^{q+r-1,0}$. The *triple ∂-Massey triple product* of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ is defined as

$$\langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\rangle := [\lambda \wedge \gamma - (-1)^p \alpha \wedge \mu] \in \frac{H^{p+q+r-1,0}_{\partial}(M)}{H^{p+q-1,0}_{\partial}(M) \cup H^{r,0}_{\partial}(M) + H^{p,0}_{\partial}(M) \cup H^{q+r-1,0}_{\partial}(M)}.$$

Then, since for a formal DGA the associated Massey products vanish we have the following

Corollary 3.28. *Let (M, I, J, K) be a compact hypercomplex manifold satisfying the $\partial\partial_J$-lemma, then the triple ∂-Massey products vanish.*

Hence, we have

Theorem 3.29. *Let (M, I, J, K, Ω) be a compact HKT $\mathrm{SL}(n,\mathbb{H})$-manifold, then the triple ∂-Massey products vanish.*

In particular, triple ∂-Massey products are an obstruction to the existence of a HKT $\mathrm{SL}(n,\mathbb{H})$-structure on a compact hypercomplex manifold. More precisely,

Corollary 3.30. *Let (M, I) be a $4n$-dimensional compact complex manifold such that there exists a non trivial ∂-Massey product, then (M, I) does not admit any complex structures J, K such that (M, I, J, K) is hypercomplex and admits a HKT $\mathrm{SL}(n,\mathbb{H})$-structure.*

We shall delay examples on which we could apply our results to the next section, where we deal with nilmanifolds and solvmanifolds. Here, we only present one example which has the purpose to show that the converse of Corollary 3.28 (and hence Theorem 3.29) does not hold in general.

Example 3.31. Consider SU(3) equipped with the homogeneous hypercomplex structure (I, J, K) of Example 2.10 and the compatible HKT metric of Example 2.15. By [263] the holonomy of the Obata connection on SU(3) is $\mathrm{GL}(2,\mathbb{H})$ and, in fact, we claim that the $\partial\partial_J$-lemma cannot hold on SU(3).

To see this, we observe that from Example 2.15, there exists a unitary coframe $\{Z^1, \dots, Z^4\}$ of $(1,0)$-forms (with respect to I) on the Lie algebra of SU(3) such that the HKT form is

$$\Omega = Z^{12} + Z^{34} = -\frac{1}{2}\partial Z^2.$$

Now, if the $\partial\partial_J$-lemma hold we would have that $\Omega = \partial\partial_J f$ for some function f, but since the HKT form is q-positive, by E. Hopf's maximum principle f would be constant and thus $\Omega = 0$ which is a contradiction.

On the other hand the triple ∂-Massey products are all zero because the same coframe satisfies

$$\partial Z^1 = 0, \qquad \partial Z^2 = -2Z^{12} - 2Z^{34}, \qquad \partial Z^3 = -(1+3i)Z^{13}, \qquad \partial Z^4 = (3i-1)Z^{14},$$

which shows that $H^{1,0}_{\partial}(M) \simeq \langle Z^1 \rangle$ and $H^{i,0}_{\partial}(M) = 0$ for $i > 1$.

3.2 HKT nilmanifolds and solvmanifolds.

In this section we briefly overview the main results of Barberis, Dotti and Verbitsky [29], dealing with (hyper)complex nilmanifold. Some of these results is hereby extended to hypercomplex solvmanifolds. In particular, we will see that Conjecture 2.45 is true for hypercomplex solvmanifolds (Theorem 3.39). We also show that Conjecture 2.46 is verified for solvmanifolds with left-invariant hypercomplex structure (Corollary 3.41). Another sufficient condition on solvmanifolds with a left-invariant hypercomplex to have a balanced metric is given in Corollary 3.37.

The second subsection is dedicated to the study of the curvature and the holonomy group of the Obata connection ∇ on solvable Lie groups with an abelian hypercomplex structure. We shall prove that the holonomy algebra of ∇ is always abelian (Theorem 3.50).

3.2.1 Balanced HKT and $\mathrm{SL}(n, \mathbb{H})$-solvmanifolds.

Nilmanifolds and solvmanifolds.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and consider the *lower central series* and the *derived series* defined inductively by $= \mathfrak{g}_0 := \mathfrak{g} =: \mathcal{D}^0(\mathfrak{g})$ and

$$\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}] , \qquad \mathcal{D}^k(\mathfrak{g}) = [\mathcal{D}^{k-1}(\mathfrak{g}), \mathcal{D}^{k-1}(\mathfrak{g})] .$$

Recall that G (and $\mathfrak{g}$) is (**k-step**) **nilpotent** (resp. (**k-step**) **solvable**) if there exists a k such that $\mathfrak{g}_k = 0$ and $\mathfrak{g}_{k-1} \neq 0$ (resp. $\mathcal{D}^k(\mathfrak{g}) = 0$ and $\mathcal{D}^{k-1}(\mathfrak{g}) \neq 0$). Observe that any nilpotent Lie group is solvable.

A **nilmanifold** (resp. **solvmanifold**) $\Gamma \backslash G$ is the quotient of a simply connected nilpotent (resp. solvable) Lie group G with a left-invariant Riemannian metric by a lattice Γ, i.e. a discrete co-compact subgroup. There is a bijective correspondence between left-invariant tensor fields on a Lie group G and tensors of the same type on the Lie algebra $\mathfrak{g}$. Furthermore, every such tensor descends to the quotient nilmanifold (resp. solvmanifold).

The existence of lattices is not always guaranteed. For the case of nilpotent Lie groups a classical result of Mal'cev [220] shows that a simply connected nilpotent Lie group has a lattice if and only if there exists a basis of the Lie algebra with rational structure constants. This characterization fails for solvable Lie groups, and no general result ensuring the existence of a lattice is known. Such a problem is investigated by Bock [46] for low dimensions. All we know is that a necessary condition for a Lie group G to have a lattice is *unimodularity*, i.e. all the adjoint operators ad_X must have vanishing trace for all $X \in \mathfrak{g}$ (see [227]).

Nilmanifolds and solvmanifolds often provide fruitful examples and counterexamples because almost everything can be regarded by looking at invariant objects, and the analysis need only to be carried out at the Lie algebra level. For example, by a result of Nomizu [234], the de Rham cohomology of nilmanifolds is isomorphic to the cohomology of its Lie algebra and thus can be computed using left-invariant forms. Unfortunately there are counterexamples for what regards solvmanifolds, but it is still true in some special cases [169, 231]. Under certain assumptions also other cohomologies can be computed via invariant forms, we refer to the introduction of [21] for further details and references. On the other hand both nilmanifolds and solvmanifolds present some rigidities, for instance, the well-known result of Benson and Gordon [35] states that a nilmanifold with a Kähler structure is necessarily a torus (see also the generalization of Hasegawa [168] for solvmanifolds).

Hypercomplex nilmanifolds.

Here we summarize the main results of the paper by Barberis, Dotti and Verbitsky [29], which deals with hypercomplex nilmanifolds.

Anytime a Lie group G is equipped with an almost complex structure I the complexification of the Lie algebra splits into its $(1,0)$ and $(0,1)$-parts, call them $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$. By Newlander-Nirenberg theorem I is integrable if and only if $\mathfrak{g}^{1,0}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$. It is remarkable that complex nilmanifold always have holomorphically trivial canonical bundle [29, 74].

The complex structure I is called abelian if the subalgebra $\mathfrak{g}^{1,0}$ is abelian; equivalently

$$[I\cdot, I\cdot] = [\cdot, \cdot]\,.$$

It is clear that an abelian almost complex structure is necessarily integrable. In [29] is reported that condition of abelianity was introduced in Barberis' Ph.D. thesis. Abelian complex structures became particularly interesting very soon as they are easier to inspect. By the result of Petravchuk (Proposition 2.16), abelian complex structure can only occur on 2-step solvable Lie algebras.

When we equip a solvable Lie algebra with a hypercomplex structure (I, J, K) such that one of the three complex structures, say I, is abelian, then automatically all the other ones are [111]. Since the canonical bundle of a $4n$-dimensional nilmanifold is holomorphically trivial there exists a holomorphic global section Θ, by the following result of Fino, Otal and Ugarte [125] it must be left-invariant:

Proposition 3.32. *A nowhere vanishing holomorphic $(n,0)$-form on a solvmanifold with left-invariant complex structure is left-invariant.*

Now, going back to the holomorphic $(2n,0)$-form Θ, we see that the form $J\bar{\Theta}$ must again be holomorphic because it is a left-invariant section of the canonical bundle and such section is uniquely determined up to a multiplicative constant. It follows that $\Theta + J\bar{\Theta}$ is q-real and holomorphic, hence by Lemma 2.42 we deduce:

Theorem 3.33. *Every hypercomplex nilmanifold is $\mathrm{SL}(n,\mathbb{H})$ and has holomorphically trivial canonical bundle.*

Dotti and Fino showed in [112] that any hyperhermitian metric on a solvable Lie algebra with left-invariant hypercomplex structure (i.e. every complex structure is left-invariant) is weak HKT and then, it gives rise to a left-invariant HKT structure on the corresponding solvable Lie group by left-translation. Under this light, a stronger condition than being $\mathrm{SL}(n,\mathbb{H})$ actually holds for hypercomplex nilmanifolds, indeed:

Theorem 3.34. *Every HKT nilmanifold is balanced.*

Dotti and Fino also proved that the hypercomplex structure of any 2-step nilmanifold admitting a left-invariant HKT metric is abelian. This was generalized to any nilpotency step in [29]:

Theorem 3.35. *A hypercomplex nilmanifold with an HKT structure has abelian hypercomplex structure.*

Sketch of proof. Let $N = \Gamma\backslash G$ a hypercomplex nilmanifold with a HKT structure and let $\mathfrak{g}$ be the Lie algebra of G. Suppose by contradiction that $\mathfrak{g}^{1,0}$ is not abelian, along the lines of [35] one can show that this implies that the Lefschetz map

$$L^{n-1}\colon H^{1,0}_{\partial}(N) \to H^{2n-1,0}_{\partial}(N)$$

is not surjective, since N is balanced this contradicts Proposition 2.36. □

Hypercomplex solvmanifolds.

The following result can be seen as a generalization of Theorem 3.34 on solvmanifolds:

Theorem 3.36. *Let $(\Gamma\backslash G, I, J, K, \Omega, g)$ be a $4n$-dimensional solvmanifold with a left-invariant abelian HKT structure. Then g is balanced.*

Proof. We will denote with (I, J, K, Ω, g) the induced structure on G. By hypothesis, Ω is HKT, hence the Bismut connections associated to I, J, K coincide and we will denote them uniquely with ∇^B. Since $\Gamma\backslash G$ is a solvmanifold then G is unimodular. Hence, by [29, Lemma 2.4] the common Lee form θ_{Lee} of G is given by

$$\theta_{\text{Lee}}(X) = \frac{1}{2}\text{tr}\left(J\nabla^B_{JX}\right), \qquad \text{for every } X \in \mathfrak{g}\,.$$

Now we argue as in the proof of [29, Proposition 4.11] to show that $\theta_{\text{Lee}} = 0$ and so g is balanced.

Let $X_1, IX_1, JX_1, KX_1, \ldots, X_n, IX_n, JX_n, KX_n$ be an orthonormal basis of $\mathfrak{g}$. Now using that ∇^B preserves (I, J, K) and that g is hyperhermitian we have

$$\begin{aligned}
\text{tr}\left(J\nabla^B_{JX}\right) &= \sum_{j=1}^{n}\left(g(J\nabla^B_{JX}X_j, X_j) + g(J\nabla^B_{JX}IX_j, IX_j) + g(J\nabla^B_{JX}JX_j, JX_j) + g(J\nabla^B_{JX}KX_j, KX_j)\right)\\
&= \sum_{j=1}^{n}\left(g(J\nabla^B_{JX}X_j, X_j) + g(JI\nabla^B_{JX}X_j, IX_j) + g(\nabla^B_{JX}JX_j, X_j) + g(JK\nabla^B_{JX}X_j, KX_j)\right)\\
&= \sum_{j=1}^{n}\left(g(J\nabla^B_{JX}X_j, X_j) - g(IJ\nabla^B_{JX}X_j, IX_j) + g(\nabla^B_{JX}JX_j, X_j) - g(KJ\nabla^B_{JX}X_j, KX_j)\right)\\
&= \sum_{j=1}^{n}\left(g(\nabla^B_{JX}JX_j, X_j) - g(\nabla^B_{JX}JX_j, X_j) + g(\nabla^B_{JX}JX_j, X_j) - g(\nabla^B_{JX}JX_j, X_j)\right) = 0\,,
\end{aligned}$$

therefore $\theta_{\text{Lee}} = 0$ and g is balanced. □

Corollary 3.37. *Let $(\Gamma\backslash G, I, J, K)$ be a solvmanifold with a left-invariant abelian hypercomplex structure. Suppose that there exists an HKT structure Ω on $(\Gamma\backslash G, I, J, K)$. Then there exists a balanced abelian HKT structure on $\Gamma\backslash G$.*

Proof. By [124] there exists a invariant HKT structure $\tilde{\Omega}$ on $(\Gamma\backslash G, I, J, K)$. Now, by Theorem 3.36 we have that $\tilde{\Omega}$ is balanced. □

Remark 3.38. Notice that, differently from the nilpotent case, the converse of Theorem 3.36 is not true. Indeed, in [30] it is provided an example of a balanced HKT solvmanifold with an hypercomplex structure that is not abelian.

We have said that Barberis, Dotti and Verbitsky proved that hypercomplex nilmanifolds are all $\mathrm{SL}(n, \mathbb{H})$ (Theorem 3.33) and their proof actually works to show that this occurs on solvmanifolds if and only if their canonical bundle is holomorphically trivial.

Theorem 3.39. *Let $(M := \Gamma\backslash G, I, J, K, g)$ be a solvmanifold with holomorphically trivial canonical bundle, then the holonomy of the Obata connection ∇ is contained in* $\mathrm{SL}(n, \mathbb{H})$.

Proof. Let η be a nowhere vanishing holomorphic section of the canonical bundle, which is necessarily invariant by Proposition 3.32. The fact that $\mathrm{Hol}(\nabla) \subseteq \mathrm{SL}(n, \mathbb{H})$, follows from the fact that $\eta + J\bar{\eta}$ is parallel with respect to ∇, which is a consequence of Lemma 2.42. □

Similarly to the case of nilmanifolds, when a $\mathrm{SL}(n, \mathbb{H})$-solvmanifold admits an invariant HKT structure it is automatically balanced:

Theorem 3.40. *Let $(M := \Gamma\backslash G, I, J, K, g)$ be a solvmanifold with holomorphically trivial canonical bundle and invariant HKT structure. Then g is balanced.*

Proof. Let $\bar{\eta}$ be a non-vanishing ∂-closed section of $\Lambda^{0,2n}(M)$, then by Proposition 3.32 $\bar{\eta}$ is invariant, hence

$$\bar{\Omega}^n = c\,\bar{\eta}$$

with c constant. Since $d\bar{\eta} = 0$, then $d\bar{\Omega}^n = 0$ and so $\partial\bar{\Omega}^n = 0$ proving that g is balanced. □

As a consequence we confirm the conjecture by Verbitsky on solvmanifolds with invariant hypercomplex structure.

Corollary 3.41. *Let $(M := \Gamma\backslash G, I, J, K)$ be a $\mathrm{SL}(n,\mathbb{H})$-solvmanifold with invariant hypercomplex structure. Suppose that there exists an HKT metric on M. Then there exists a balanced HKT structure on M.*

Proof. Since $(M := \Gamma\backslash G, I, J, K, g)$ is an HKT solvmanifold with a $\mathrm{SL}(n,\mathbb{H})$ structure, then the canonical bundle of M is holomorphically trivial and, by [124], there exists an invariant HKT structure on M. Hence, by the previous result the associated Hermitian metric is balanced. □

Examples and triple ∂-Massey products.

We begin with an example of a nilmanifold admitting hypercomplex structures but not admitting HKT structures:

Example 3.42. Consider the nilmanifold $M = \Gamma\backslash G$ whose structure equations of the Lie algebra $\mathfrak{g}$ of G are given by (see also [208, Example 1])

$$de^1 = de^2 = de^3 = de^4 = de^5 = 0\,, \quad de^6 = e^{12} + e^{34}\,, \quad de^7 = e^{13} - e^{24}\,, \quad de^8 = e^{14} + e^{23}\,,$$

where we use the standard notation $e^{ij} = e^i \wedge e^j$. Define the following complex structure

$$Ie^1 = e^2\,, \quad Ie^3 = e^4\,, \quad Ie^5 = e^6\,, \quad Ie^7 = e^8\,.$$

Then a co-frame for invariant $(1,0)$-forms on M is given by

$$\varphi^1 = e^1 - ie^2\,, \quad \varphi^2 = e^3 - ie^4\,, \quad \varphi^3 = e^5 - ie^6\,, \quad \varphi^4 = e^7 - ie^8$$

and the complex structure equations become

$$d\varphi^1 = d\varphi^2 = 0\,, \quad d\varphi^3 = -\frac{1}{2}(\varphi^{1\bar{1}} + \varphi^{2\bar{2}})\,, \quad d\varphi^4 = \varphi^{12}\,.$$

Then, the conjugate Dolbeault cohomology in bidegree $(p,0)$ is given by (cf. [208, Example 1])

$$H^{1,0}_{\partial}(M) \simeq \left\langle \varphi^1, \varphi^2, \varphi^3 \right\rangle\,, \qquad H^{2,0}_{\partial}(M) \simeq \left\langle \varphi^{13}, \varphi^{23}, \varphi^{14}, \varphi^{24} \right\rangle\,, \qquad H^{3,0}_{\partial}(M) \simeq \left\langle \varphi^{134}, \varphi^{234}, \varphi^{124} \right\rangle\,.$$

We now construct a non trivial triple ∂-Massey product. Take $[\varphi^1] \in H^{1,0}_{\partial}(M)$, $[\varphi^2] \in H^{1,0}_{\partial}(M)$ and $[\varphi^2] \in H^{1,0}_{\partial}(M)$. Notice that $\varphi^1 \wedge \varphi^2 = \partial\varphi^4$ and $\varphi^2 \wedge \varphi^2 = 0$. Hence, the ∂-Massey product is given by

$$[\varphi^4 \wedge \varphi^2] \in \frac{H^{2,0}_{\partial}(M)}{[\varphi^1] \cup H^{1,0}_{\partial}(M) + H^{1,0}_{\partial}(M) \cup [\varphi^2]}\,.$$

and this class is clearly non trivial. Therefore, by Corollary 3.30 the complex manifold (M, I) does not admit any complex structures J, K such that the nilmanifold (M, I, J, K) is hypercomplex and admits a HKT structure (such structure should be $\mathrm{SL}(n,\mathbb{H})$ since M is a nilmanifold). One can confront this result with [208, Example 1] where a specific hypercomplex structure (I, J, K) is constructed and it is showed that it does not admit any HKT metric.

Notice that, in fact, if a nilmanifold N admits an invariant HKT structure (I, J, K, Ω) then the complex structures I, J, K are abelian and in such a case the triple ∂-Massey products are trivial. Indeed, we prove in general the following:

Theorem 3.43. *Let $N = \Gamma\backslash G$ be a $2n$-dimensional nilmanifold and let I be an invariant abelian complex structure on N. Then, the triple ∂-Massey products are all zero.*

Proof. Since I is an invariant abelian complex structure on N, there exists a co-frame of invariant $(1,0)$-forms $\{\varphi^i\}_{i=1,\dots,n}$ on (N,I) such that

$$\partial\varphi^i = 0\,, \quad \text{for } i = 1,\dots,n\,.$$

Since I is abelian, by [97] the Dolbeault cohomology of N can be computed using only invariant forms, hence

$$H^{r,0}_{\partial}(N) \simeq H^{r,0}_{\partial}(\mathfrak{g}^{\mathbb{C}}) = \langle \varphi^{i_1} \wedge \dots \wedge \varphi^{i_r} \rangle_{1\leq i_1<\dots<i_r\leq n}$$

for $r = 1,\dots,n$, where, denoting with $\mathfrak{g} = \mathrm{Lie}(G)$, $H^{\bullet,\bullet}_{\partial}(\mathfrak{g}^{\mathbb{C}})$ denotes the cohomology of the differential bigraded algebra $\Lambda^{\bullet,\bullet}(\mathfrak{g}^{\mathbb{C}})^*$ with respect to the operator ∂.
In order to construct a triple ∂-Massey product let $\mathfrak{a} = [\alpha] \in H^{p,0}_{\partial}(N)$, $\mathfrak{b} = [\beta] \in H^{q,0}_{\partial}(N)$ such that $\mathfrak{a}\cup\mathfrak{b} = 0 \in H^{p+q,0}_{\partial}(N)$, hence $\mathfrak{a}\cup\mathfrak{b} = 0 \in H^{p+q,0}_{\partial}(\mathfrak{g}^{\mathbb{C}})$, namely there exists an invariant $(p+q-1,0)$-form λ such that

$$\alpha \wedge \beta = \partial\lambda\,.$$

But, on invariant $(r,0)$-forms the operator ∂ vanishes and so we can take the primitive $\lambda = 0$ itself. A similar conclusion is obtained taking the third class in the definition of ∂-Massey products. This means that we cannot construct non trivial ∂-Massey products since both λ and μ in the definition of ∂-Massey products would be zero. ☐

An immediate consequence of this result combined with Theorem 3.35 is the following:

Theorem 3.44. *Let $N = \Gamma\backslash G$ be a $4n$-dimensional nilmanifold and let (I,J,K,Ω) be an invariant HKT structure on N. Then, the triple ∂-Massey products are all zero.*

Therefore, a relevant application of Corollary 3.30 should be given on solvmanifolds.

Example 3.45. Consider the 8-dimensional almost abelian Lie algebra $\mathfrak{g}$ with structure equations

$$[e_8, e_2] = e_4, \quad [e_8, e_3] = e_5.$$

Let G be the associated solvable simply connected Lie group. Then, by [46] G admits a lattice Γ such that $S := \Gamma\backslash G$ is a solvmanifold. Define the complex structure setting as global co-frame of $(1,0)$-forms

$$\varphi^1 = e^1 + ie^8\,, \quad \varphi^2 = e^2 + ie^3\,, \quad \varphi^3 = e^4 + ie^5\,, \quad \varphi^4 = e^6 + ie^7\,.$$

The complex structure equations become

$$d\varphi^1 = d\varphi^2 = d\varphi^4 = 0\,, \quad d\varphi^3 = \frac{i}{2}\varphi^{12} + \frac{i}{2}\varphi^{2\bar{1}}\,.$$

We now construct a non trivial triple ∂-Massey product. Take $[\varphi^1] \in H^{1,0}_{\partial}(S)$, $[\varphi^2] \in H^{1,0}_{\partial}(S)$ and $[\varphi^2] \in H^{1,0}_{\partial}(S)$. Notice that $\varphi^1 \wedge \varphi^2 = \partial(-2i\,\varphi^3)$ and $\varphi^2 \wedge \varphi^2 = 0$. Hence, the ∂-Massey product is given by

$$[-2i\,\varphi^3 \wedge \varphi^2] \in \frac{H^{2,0}_{\partial}(S)}{[\varphi^1] \cup H^{1,0}_{\partial}(S) + H^{1,0}_{\partial}(S) \cup [\varphi^2]}\,.$$

and this class is clearly non trivial. Therefore, by Corollary 3.30 the complex manifold (S,I) does not admit any complex structures J,K such that the solvmanifold (S,I,J,K) is hypercomplex and admits a $\mathrm{SL}(n,\mathbb{H})$ HKT structure.

3.2.2 Curvature and holonomy.

The abelianness of a hypercomplex structure makes really simple computations involving the Obata connection. Here we shall compute explicitly the curvature of the Obata connection on a Lie algebra $\mathfrak{g}$

with abelian hypercomplex structure (I, J, K). We will then exploit this to observe that the holonomy of the Obata connection must be abelian. The results of this subsection are part of unpublished work together with Misha Verbitsky.

Curvature.

Recall the expression (2.7) of Soldatenkov for the Obata connection. Under the assumption of (I, J, K) being abelian, this can be rewritten as

$$\nabla_X Y = \frac{1}{2}\left([X,Y] + I[IX,Y] + J[JX,Y] + K[KX,Y]\right) , \qquad \text{for every } X, Y \in \mathfrak{g}\,.$$

We begin by computing the curvature:

Proposition 3.46. *The Obata connection on a Lie algebra $\mathfrak{g}$ with abelian hypercomplex structure satisfies*

$$[\nabla_X, \nabla_Y] = -\nabla_{[X,Y]}$$

for any $X, Y \in \mathfrak{g}$. In particular the curvature is given by

$$R(X,Y) = 2[\nabla_X, \nabla_Y] = -2\nabla_{[X,Y]}\,.$$

Proof. Let $(J_1, J_2, J_3) = (I, J, K)$ be the hypercomplex structure on $\mathfrak{g}$. First we compute

$$\begin{aligned} 4\nabla_X\nabla_Y Z &= 2\nabla_X\left([Y,Z] + \sum_{\alpha=1}^{3} J_\alpha[J_\alpha Y, Z]\right) \\ &= [X,[Y,Z]] + \sum_{\alpha=1}^{3}[X, J_\alpha[J_\alpha Y,Z]] + \sum_{\alpha=1}^{3} J_\alpha[J_\alpha X,[Y,Z]] \ + \sum_{\alpha,\beta=1}^{3} J_\beta[J_\beta X, J_\alpha[J_\alpha Y, Z]] \end{aligned}$$

splitting the last sum according as $\alpha = \beta$ or $\alpha \neq \beta$ yields

$$\begin{aligned} \sum_{\alpha,\beta=1}^{3} J_\beta[J_\beta X, J_\alpha[J_\alpha Y,Z]] &= \sum_{\alpha=1}^{3} J_\alpha[J_\alpha X, J_\alpha[J_\alpha Y,Z]] + \sum_{\alpha\neq\beta} J_\beta[J_\beta X, J_\alpha[J_\alpha Y, Z]] \\ &= -\sum_{\alpha=1}^{3} J_\alpha[X,[Y,J_\alpha Z]] + \sum_{\alpha\neq\beta} J_\beta[J_\alpha J_\beta X,[J_\alpha J_\beta Y, J_\beta Z]] \end{aligned}$$

therefore

$$\begin{aligned} 4\nabla_X\nabla_Y Z = {}& [X,[Y,Z]] - \sum_{\alpha=1}^{3}[J_\alpha X,[J_\alpha Y,Z]] + \sum_{\alpha=1}^{3} J_\alpha\left([J_\alpha X,[J_\alpha Y, J_\alpha Z]] - [X,[Y,J_\alpha Z]]\right) \\ &+ \sum_{\alpha\neq\beta} J_\beta[J_\alpha J_\beta X,[J_\alpha J_\beta Y, J_\beta Z]]\,. \end{aligned}$$

Using Jacobi's identity we then finally obtain

$$\begin{aligned} \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z &= \frac{1}{4}\left([[X,Y],Z] - 3[[X,Y],Z] + 2\sum_{\beta=1}^{3} J_\beta[[X,Y], J_\beta Z]\right) \\ &= -\frac{1}{2}\left([[X,Y],Z] + \sum_{\beta=1}^{3} J_\beta[J_\beta[X,Y],Z]\right) = -\nabla_{[X,Y]}Z \end{aligned}$$

which concludes the proof of the proposition. □

Endow the space $\mathfrak{g}^\nabla := \{\nabla_X \mid X \in \mathfrak{g}\} \subseteq \mathrm{End}(\mathfrak{g})$ with the opposite of the commutator as a Lie bracket. By Proposition 3.46 it is immediate to see that the Jacobi identity of this Lie bracket follows directly from the Jacobi identity on $(\mathfrak{g}, [\cdot,\cdot])$. Therefore $\mathfrak{g}^\nabla$ is a Lie algebra. Furthermore, this yields a representation $\rho\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, naturally defined by $\rho_X = \nabla_X$, which is onto $\mathfrak{g}^\nabla$.

Proposition 3.47. *If $\mathfrak{g}$ is nilpotent then so is $\mathfrak{g}^\nabla$. Furthermore, if the nilpotency step of $\mathfrak{g}$ is k, then the nilpotency step of $\mathfrak{g}^\nabla$ is at most $k-1$.*

Proof. The first assertion is obvious. Observe that the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ lies in $\mathrm{Ker}(\rho)$, hence $\mathfrak{g}^\nabla \cong \mathfrak{g}/\mathrm{Ker}(\rho) \subseteq \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ which shows that if $\mathfrak{g}$ is k-step nilpotent $\mathfrak{g}^\nabla$ is at most $(k-1)$-step nilpotent. □

It is evident that whenever $\mathfrak{g}$ is 2-step nilpotent the Obata connection is necessarily flat, equivalently, $\mathfrak{g}^\nabla$ is abelian. The converse is not true in general, we shall give a counterexample inspired by a 3-step nilpotent non-integrable example of Dotti and Fino in [111]:

Example 3.48. Consider the 3-step nilpotent 12-dimensional Lie algebra $\mathfrak{g} = \langle e_1, \dots, e_{12}\rangle$ with abelian hypercomplex structure

$$Ie_{4i-3} = e_{4i-2}\,, \qquad Je_{4i-3} = e_{4i-1}\,, \qquad Ke_{4i-3} = e_{4i}\,, \qquad i = 1,2,3\,,$$

and non-zero Lie brackets

$$\begin{aligned}
&[e_1,e_2] = -[e_3,e_4] = -e_{10}\,,\\
&[e_1,e_4] = -[e_2,e_3] = e_{12}\,,\\
&[e_1,e_9] = [e_2,e_{10}] = [e_3,e_{11}] = [e_4,e_{12}] = -e_6\,,\\
&[e_1,e_{11}] = [e_2,e_{12}] = -[e_3,e_9] = -[e_4,e_{10}] = -e_8\,.
\end{aligned}$$

Then one can check that $[\mathfrak{g},\mathfrak{g}] \subseteq \mathrm{Ker}(\rho)$, therefore $\mathfrak{g}^\nabla \cong \mathfrak{g}/\mathrm{Ker}(\rho) \subseteq \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian and thus the Obata connection is flat on $\mathfrak{g}$.

This is in contrast to what happens when the Obata connection ∇ preserves an indefinite Riemannian metric. Observe that the Riemannian metric has to be indefinite, otherwise the Lie algebra is abelian [35]. In this case ∇ coincides with the Levi-Civita connection and $\mathfrak{g}$ is 2-step nilpotent if and only if ∇ is flat. This was proved by Bajo and Sanmartín [25] who also show that such structures can be at most 3-step nilpotent. The only restriction on the nilpotency step that abelianness of the hypercomplex structure imposes is that it can be at most equal to the quaternionic dimension [110]. Dotti and Barberis provided examples with arbitrary nilpotency step in [28]; it is easy to check that these have flat Obata connection.

Holonomy.

We now prove a lemma that will allow us to prove abelianness of the holonomy group.

Lemma 3.49. *The curvature of the Obata connection satisfies:*

$$\nabla_X R(Y,Z) = 6\nabla_{[X,[Y,Z]]} = -3R(X,[Y,Z])\,.$$

Proof. We use Proposition 3.46 and the Jacobi identity

$$\begin{aligned}
\nabla_X R(Y,Z)W &= \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W\\
&= -2\nabla_X\nabla_{[Y,Z]}W + 2\nabla_{[\nabla_X Y,Z]}W + 2\nabla_{[Y,\nabla_X Z]}W + 2\nabla_{[Y,Z]}\nabla_X W\\
&= -2[\nabla_X,\nabla_{[Y,Z]}]W + 2\nabla_{[\nabla_X Y,Z]+[Y,\nabla_X Z]}W\\
&= 6\nabla_{[X,[Y,Z]]}W\,.
\end{aligned}$$

The last term is computed by using the explicit formula for the Obata connection:

$$\begin{aligned}
2[\nabla_X Y, Z] + 2[Y, \nabla_X Z] &= [[X,Y],Z] + \sum_{\alpha=1}^{3}[J_\alpha[J_\alpha X, Y], Z] + [Y,[X,Z]] + \sum_{\beta=1}^{3}[Y, J_\beta[J_\beta X, Z]] \\
&= [X,[Y,Z]] + \sum_{\alpha=1}^{3}\left([[X, J_\alpha Y], J_\alpha Z] + [J_\alpha Y, [X, J_\alpha Z]]\right) \\
&= 4[X,[Y,Z]]\,,
\end{aligned}$$

where we used again the Jacobi identity. □

Theorem 3.50. *The restricted holonomy group of the Obata connection on a Lie group with abelian hypercomplex structure is abelian.*

Proof. From the Ambrose-Singer holonomy theorem the holonomy algebra is generated by the curvature endomorphisms and all its covariant derivatives. But the previous lemma shows that all covariant derivatives of the curvature are again curvature endomorphisms. Therefore the holonomy algebra is $\mathfrak{hol}(\nabla) = [\mathfrak{g}^\nabla, \mathfrak{g}^\nabla]$. Since we know by Proposition 2.16 that $\mathfrak{g}$ is necessarily 2-step solvable then also $\mathfrak{g}^\nabla$ is such. In other words $[\mathfrak{hol}(\nabla), \mathfrak{hol}(\nabla)] = 0$. □

CHAPTER 4

THE QUATERNIONIC CALABI CONJECTURE: TWO RESULTS OF SOLVABILITY

Here, we present two cases on which the quaternionic Calabi conjecture can be solved. The first, deals with specific examples of 2-step nilmanifolds viewed as tori fibrations. The solution on these examples is new because they cannot be Kähler due to Benson and Gordon [35], hence, they do not belong to the spaces on which the conjecture was solved by Alesker [14] and Dinew and Sroka [106].

The second setting is, in some sense, inspired by the first, and treats compact HKT manifolds having a foliation that is preserved by the hypercomplex structure. In order to manage the terms we make the assumption that such a foliation is of corank 4.

The two sections collect the results of [137, 138] respectively.

4.1 Abelian hypercomplex 8-dimensional nilmanifolds viewed as tori fibrations.

The work of the present section takes off where [59, 60, 126, 289, 291, 309] stopped. Those articles studied the symplectic Calabi-Yau conjecture [109, 322] on torus fibrations in the case the problem's data admits certain symmetries. In the same spirit, we study the quaternionic Monge-Ampère equation on compact quotients of on 8-dimensional 2-step nilmanifolds M endowed with an *abelian* hypercomplex structure. We show that on these manifolds, regarded as tori fibrations, the quaternionic Calabi-Yau problem can always be solved for any data that is invariant under the action of a 3-torus.

4.1.1 Preliminaries.

Overview.

By a result of Dotti and Fino [110] the only non-abelian 8-dimensional 2-step nilpotent Lie groups admitting an abelian hypercomplex structure are

$$N_1 = H_1(2) \times \mathbb{R}^3\,, \qquad N_2 = H_2(1) \times \mathbb{R}^2\,, \qquad N_3 = H_3(1) \times \mathbb{R}\,,$$

where $H_i(n)$ denotes the real ($i = 1$), complex ($i = 2$), and quaternionic ($i = 3$) Heisenberg group (cf. example 2.17). As we shall see each N_i contains a canonical co-compact lattice Γ_i, and the nilmanifold $M_i = \Gamma_i \backslash N_i$, i.e. the quotient of N_i by Γ_i, inherits the structure of a principal T^3-bundle over a 5-dimensional torus T^5 and also an HKT structure (I, J, K, g). In view of [35] the nilmanifolds M_i are not Kähler, since a compact nilmanifold admits a Kähler metric if and only if it is a torus.

Moreover, the canonical bundle of (M_i, I) is holomorphically trivial (Theorem 3.33) and M_i carries a left-invariant holomorphic volume form Θ. Hence it is quite natural to wonder whether the Alesker-Verbitsky conjecture might hold on these spaces.

Our main result is the following:

Theorem 4.1. *The quaternionic Monge-Ampère equation*

$$(\Omega + \partial\partial_J\varphi)^n = \mathrm{e}^F\Omega^n\,. \tag{4.1}$$

on (M_i, I, J, K, g) can be solved for every T^3-invariant map $F \in C^\infty(M_i, \mathbb{R})$ satisfying

$$\int_M (\mathrm{e}^F - 1)\,\Omega^n \wedge \bar\Theta = 0\,. \tag{4.2}$$

Since we are assuming F is invariant under the action of the fibre T^3, it can be regarded as a smooth function on the base T^5. Furthermore, we shall see that condition (4.2) can be written as

$$\int_{T^5} (\mathrm{e}^F - 1) dx^1 \cdots dx^5 = 0\,. \tag{4.3}$$

By imposing the same invariance property on the HKT potential φ, we reduce the quaternionic Monge-Ampère equation on (M_i, I, J, K, g) to

$$(\varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1)(\varphi_{55} + 1) - \varphi_{15}^2 - \varphi_{25}^2 - \varphi_{35}^2 - \varphi_{45}^2 = \mathrm{e}^F\,, \tag{4.4}$$

where φ_{rs} denotes the second derivative of φ in the real coordinates $x^r, x^s \in \{x^1, \ldots, x^5\}$ on T^5. Then we prove that equation (4.4) has a solution $\varphi \in C^\infty(T^5)$ whenever F satisfies (4.3).

We mention that Fusi studied in [130] a class of fully non-linear elliptic equations on tori which includes and generalizes the solvability of (4.4).

Writing the equation.

Let G be an 8-dimensional Lie group with a left-invariant hypercomplex structure (I, J, K). Assume that I is *abelian*, meaning

$$[IX, IY] = [X, Y], \qquad \text{for every} \quad X, Y \in \mathfrak{g}\,,$$

where $\mathfrak{g}$ is the Lie algebra of G. Recall that this is equivalent to requiring that the Lie algebra $\mathfrak{g}^{1,0}$ of left-invariant vector fields of type $(1,0)$ on (G, I) is abelian. It also implies that any left-invariant $(p,0)$-form on (G, I) is ∂-closed. If g is a left-invariant Riemannian metric on G compatible with (I, J, K), the hyperhermitian structure (I, J, K, g) is HKT because the corresponding form Ω is ∂-closed.

As mentioned, by [110] the only 8-dimensional nilpotent, non-abelian, Lie groups carrying a left-invariant HKT structure (I, J, K, g) such that (I, J, K) is abelian are

$$N_1 = H_1(2) \times \mathbb{R}^3\,, \qquad N_2 = H_2(1) \times \mathbb{R}^2\,, \qquad N_3 = H_3(1) \times \mathbb{R}\,,$$

where

$$H_1(2) = \left\{ \begin{pmatrix} 1 & x^1 & x^4 & y^1 \\ 0 & 1 & 0 & x^3 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad H_2(1) = \left\{ \begin{pmatrix} 1 & x^1 + ix^2 & y^3 + iy^2 \\ 0 & 1 & x^4 + ix^3 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$H_3(1) = \left\{ \begin{pmatrix} 1 & q & h - \frac{1}{2}q\bar q \\ 0 & 1 & -\bar q \\ 0 & 0 & 1 \end{pmatrix} \mid q = x^1 + ix^4 + jx^3 + kx^2, h = iy^3 + jy^2 + ky^1 \right\}.$$

Above, $x^1, \ldots, x^4, y^1, y^2, y^3 \in \mathbb{R}$.

Note that each group N_i is diffeomorphic to $\mathbb{R}^8$, and there are global coordinates

$$N_1 = H_1(2)_{x^1,\dots,x^4,y^1} \times \mathbb{R}^3_{y^2,y^3,x^5}\,, \qquad N_2 = H_2(1)_{x^1,\dots,x^4,y^2,y^3} \times \mathbb{R}^2_{y^1,x^5}\,,$$
$$N_3 = H_3(1)_{x^1,\dots,x^4,y^1,y^2,y^3} \times \mathbb{R}_{x^5}\,.$$

The Lie algebras of the N_i's can be characterized in terms of left-invariant frames $(e_1,\dots,e_8)$ (corresponding to the ordered coordinates $x_1,x_2,x_3,x_4,y_1,y_2,y_3,x_5$) satisfying the following structure equations:

N_1: $[e_1,e_2] = -[e_3,e_4] = e_5$, and all other brackets vanish;

N_2: $[e_1,e_3] = [e_2,e_4] = e_6,\ [e_1,e_4] = -[e_2,e_3] = e_7$, and all other brackets vanish;

N_3: $[e_1,e_2] = -[e_3,e_4] = e_5,\ [e_1,e_3] = [e_2,e_4] = e_6,\ [e_1,e_4] = -[e_2,e_3] = e_7$, and all other brackets vanish.

In each case, using the frame $(e_1,\dots,e_8)$ we can define the left-invariant HKT structure as consisting of the standard metric

$$g = \sum_{r=1}^{8} e^r \otimes e^r$$

and the hypercomplex structure (I,J,K) defined by

$$Ie_1 = e_2\,, \qquad Je_1 = e_3\,, \qquad Ke_1 = e_4\,, \qquad Ie_5 = e_6\,, \qquad Je_5 = e_7\,, \qquad Ke_5 = e_8\,.$$

Let us fix co-compact lattices

$$\Gamma_1 = \mathbb{Z}^3 \times \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b^t \\ 0 & 0 & 1 \end{pmatrix} \mid a,b \in \mathbb{Z}^2,\, c \in \mathbb{Z} \right\} \subset N_1\,;$$

$$\Gamma_2 = \mathbb{Z}^2 \times \left\{ \begin{pmatrix} 1 & z & u \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \mid u,z,w \in \mathbb{Z} + i\mathbb{Z} \right\} \subset N_2\,;$$

$$\Gamma_3 = \mathbb{Z} \times \left\{ \begin{pmatrix} 1 & q & h - \frac{1}{2}q\bar{q} \\ 0 & 1 & -\bar{q} \\ 0 & 0 & 1 \end{pmatrix} \mid q \in \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z}\,, \quad h \in i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z} \right\} \subset N_3\,.$$

For $r = 1,2,3$ we denote by $M_r = \Gamma_r \backslash N_r$ the compact nilmanifold obtained by quotienting N_r by Γ_r. The left-invariant quadruple (I,J,K,g) on N_r induces an HKT structure on M_r. Let $\{Z_1,\dots,Z_4\}$ indicate the left-invariant $(1,0)$-frame $Z_r = \frac{1}{2}(e_{2r-1} - iIe_{2r-1})$, $r = 1,\dots,4$, and denote by $\{\zeta^1,\dots,\zeta^4\}$ the dual $(1,0)$-coframe. We deduce the following identity, holding for every smooth real map φ on M_r:

$$\begin{aligned}
\partial\partial_J\varphi =& \partial J^{-1}\bar\partial\varphi = -\partial J\left(\bar Z_1(\varphi)\bar\zeta^1 + \bar Z_2(\varphi)\bar\zeta^2 + \bar Z_3(\varphi)\bar\zeta^3 + \bar Z_4(\varphi)\bar\zeta^4\right)\\
=& \partial\left(\bar Z_1(\varphi)\zeta^2 - \bar Z_2(\varphi)\zeta^1 + \bar Z_3(\varphi)\zeta^4 - \bar Z_4(\varphi)\zeta^3\right)\\
=& \left(Z_1\bar Z_1(\varphi) + Z_2\bar Z_2(\varphi)\right)\zeta^{12} + \left(Z_3\bar Z_2(\varphi) - Z_1\bar Z_4(\varphi)\right)\zeta^{13} + \left(Z_4\bar Z_2(\varphi) + Z_1\bar Z_3(\varphi)\right)\zeta^{14}\\
&- \left(Z_3\bar Z_1(\varphi) + Z_2\bar Z_4(\varphi)\right)\zeta^{23} + \left(Z_2\bar Z_3(\varphi) - Z_4\bar Z_1(\varphi)\right)\zeta^{24} + \left(Z_3\bar Z_3(\varphi) + Z_4\bar Z_4(\varphi)\right)\zeta^{34}\,.
\end{aligned}$$

Since

$$\Omega = \zeta^{12} + \zeta^{34}\,,$$

it follows that

$$\begin{aligned}
(\Omega + \partial\partial_J\varphi)^2 = 2\Big(&\left(Z_1\bar Z_1(\varphi) + Z_2\bar Z_2(\varphi) + 1\right)\left(Z_3\bar Z_3(\varphi) + Z_4\bar Z_4(\varphi) + 1\right)\\
&- \left(Z_3\bar Z_2(\varphi) - Z_1\bar Z_4(\varphi)\right)\left(Z_2\bar Z_3(\varphi) - Z_4\bar Z_1(\varphi)\right)\\
&- \left(Z_4\bar Z_2(\varphi) + Z_1\bar Z_3(\varphi)\right)\left(Z_3\bar Z_1(\varphi) + Z_2\bar Z_4(\varphi)\right)\Big)\zeta^{1234}\,.
\end{aligned} \tag{4.5}$$

Furthermore, every manifold M_i is naturally a principal T^3-bundle over T^5 with projection

$$\pi\colon M_i \to T^5_{x^1,\dots,x^5}\,.$$

More in detail, if $\mathfrak{n}_i$ is the Lie algebra of N_i we have a short exact sequence:

$$0 \longrightarrow [\mathfrak{n}_i,\mathfrak{n}_i] \longrightarrow \mathfrak{n}_i \longrightarrow \mathfrak{n}_i/[\mathfrak{n}_i,\mathfrak{n}_i] \longrightarrow 0 \tag{4.6}$$

where $[\mathfrak{n}_i,\mathfrak{n}_i] = \mathrm{Span}(e_5,e_6,e_7)$ and $\mathfrak{n}_i/[\mathfrak{n}_i,\mathfrak{n}_i] \cong \mathrm{Span}(e_1,e_2,e_3,e_4,e_8)$. Since N_i is 2-step nilpotent the commutator lies in the center, therefore both $[\mathfrak{n}_i,\mathfrak{n}_i]$ and $\mathfrak{n}_i/[\mathfrak{n}_i,\mathfrak{n}_i]$ are abelian subalgebras. Exponentiating and quotienting by the lattice Γ_i the short exact sequence 4.6 induces the principal fibration

$$0 \longrightarrow T^3_{y^1,y^2,y^3} \longrightarrow M_i \longrightarrow T^5_{x^1,\dots,x^5} \longrightarrow 0.$$

A smooth function on M_i is invariant under the action of the principal fibre T^3 if and only if it depends only on the five coordinates $\{x^1,\dots,x^5\}$. What is more, T^3-invariant functions on M_i are naturally identified with functions on T^5. As mentioned above, for a T^3-invariant real map F condition (4.2) becomes (4.3). Further assuming that the HKT potential φ is T^3-invariant, writing (4.5) in terms of real derivative and renormalizing in a suitable way, the quaternionic Monge-Ampère equation (4.1) can be written as (4.4) on T^5.

Remark 4.2. The Lie algebras of the 2-step nilpotent Lie groups N_i all have 4-dimensional center $\mathfrak{z} = \langle e_5,e_6,e_7,e_8\rangle$. Therefore the nilmanifolds M_i can be regarded in a natural way as principal T^4-bundles over a torus T^4 if we project onto the first four coordinates $(x^1,\dots,x^4)$. From this point of view, requiring all data to be invariant under the action of the fibre T^4 implies that the resulting equation can be written as the following Poisson equation on the base T^4

$$\Delta\varphi = \varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} = \mathrm{e}^F - 1\,.$$

And this can be solved using standard techniques.

From this point on we shall focus on equation (4.4). In order to simplify the notation let us set

$$A = \varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1\,, \quad B = \varphi_{55} + 1\,.$$

Lemma 4.3. *If $\varphi \in C^2(T^5)$ is a solution to* (4.4)*, then $A > 0, B > 0$ and*

$$0 < 2\mathrm{e}^{F/2} \le \Delta\varphi + 2\,. \tag{4.7}$$

Proof. From equation (4.4) we infer $AB \ge \mathrm{e}^F > 0$. Hence A and B have the same sign. At a point p_0 where φ attains its minimum we must have $\varphi_{55}(p_0) \ge 0$. This implies $B > 0$ and then $A > 0$. Finally, by using $A^2 + B^2 \ge 2AB$ we obtain

$$(\Delta\varphi + 2)^2 = (A + B)^2 \ge 4AB \ge 4\mathrm{e}^F > 0\,.$$

Taking the square root produces (4.7). □

Proposition 4.4. *Equation* (4.4) *is elliptic. More precisely, if $\varphi \in C^2(T^5)$ denotes a solution to* (4.4) *then*

$$A\xi_5^2 + B(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) - 2\sum_{i=1}^{4} \varphi_{i5}\xi_i\xi_5 \ge \lambda(\varphi)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2) \tag{4.8}$$

for every $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) \in \mathbb{R}^5$, where

$$\lambda(\varphi) = \frac{1}{2}\left(A + B - \sqrt{(A+B)^2 - 4\mathrm{e}^F}\right).$$

Proof. The principal symbol of the linearized equation at a solution φ equals

$$A\xi_5^2 + B(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) - 2\varphi_{15}\xi_1\xi_5 - 2\varphi_{25}\xi_2\xi_5 - 2\varphi_{35}\xi_3\xi_5 - 2\varphi_{45}\xi_4\xi_5$$

and the corresponding matrix is

$$P(\varphi) = \begin{pmatrix} B & 0 & 0 & 0 & -\varphi_{15} \\ 0 & B & 0 & 0 & -\varphi_{25} \\ 0 & 0 & B & 0 & -\varphi_{35} \\ 0 & 0 & 0 & B & -\varphi_{45} \\ -\varphi_{15} & -\varphi_{25} & -\varphi_{35} & -\varphi_{45} & A \end{pmatrix}.$$

Since, by (4.4),

$$\begin{aligned}\det(P(\varphi) - \lambda I) &= (B-\lambda)^3 \left((A-\lambda)(B-\lambda) - (\varphi_{15}^2 + \varphi_{25}^2 + \varphi_{35}^2 + \varphi_{45}^2)\right) \\ &= (B-\lambda)^3 \left(\lambda^2 - (A+B)\lambda + \mathrm{e}^F\right),\end{aligned}$$

the eigenvalues are $\lambda = B$ and

$$\lambda_\pm = \frac{1}{2}\left(A + B \pm \sqrt{(A+B)^2 - 4\mathrm{e}^F}\right).$$

Now $(A+B)^2 - 4\mathrm{e}^F \geq (A+B)^2 - 4AB = (A-B)^2$, so that

$$0 < \lambda_- \leq B \leq \lambda_+ .$$

This proves the claim. □

4.1.2 Proof of Theorem 4.1.

C^0-estimate.

Although the a priori C^0-estimate for equation (4.4) can be deduced from the C^0-estimate of the quaternionic Monge-Ampère equation, as shown in [16, 18, 269], we shall prove this fact using an argument that is specific to our setup.

Now, let us identify functions on T^5 with functions $\varphi\colon \mathbb{R}^5 \to \mathbb{R}$ that are periodic in each variable. Denote by $C^n(T^5)$ the Banach space of functions $\varphi\colon T^5 \to \mathbb{R}$ with C^n-norm

$$\|\varphi\|_{C^n} = \max_{|I|\leq n} \sup_{x\in\mathbb{R}^5} |\partial^I \varphi(x)|$$

where $I = \{i_1, \ldots, i_5\}$. We are adopting the multi-index notation $\partial^I = \partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3}\partial_4^{i_4}\partial_5^{i_5}$ with $|I| = i_1 + i_2 + i_3 + i_4 + i_5$. For $\alpha \in (0,1)$ we also consider the Banach space $C^{n,\alpha}(T^5)$ of functions $\varphi \in C^n(T^5)$ with Hölder-continuous derivatives of order n:

$$\|\varphi\|_{C^{n,\alpha}} = \max\{\|\varphi\|_{C^n}, |\varphi|_{C^{n,\alpha}}\} < \infty,$$

where

$$|\varphi|_{C^{n,\alpha}} = \max_{|I|=n} \sup_{x\in\mathbb{R}^3} \sup_{0<|h|\leq 1} \frac{|\partial^I\varphi(x+h) - \partial^I\varphi(x)|}{|h|^\alpha}.$$

Set

$$C_*^k(T^5) = \left\{\varphi \in C^k(T^5) \mid \int_K \varphi = 0\right\}$$

where

$$K = \left[-\frac{1}{2}, \frac{1}{2}\right]^5.$$

Proposition 4.5. *Assume that $F \in C^0(T^5)$ satisfies* (4.3). *Let $\varphi \in C_*^2(T^5)$ be a solution to* (4.4).

Then there is a positive constant C, depending on $\|F\|_{C^0}$ only, such that

$$\|\varphi\|_{C^0} \leq C\,. \tag{4.9}$$

Proof. Let $x_0 \in \mathbb{R}^5$ be a point where φ attains its minimum on K. Fix $\varepsilon > 0$ and define

$$u(x) = \varphi(x) - \max_K \varphi + 4\varepsilon|x - x_0|^2\,. \tag{4.10}$$

Then

$$u(x_0) + \varepsilon = \varphi(x_0) - \max_K \varphi + \varepsilon \leq \min_{|x-x_0|=1/2} \varphi(x) - \max_K \varphi + \varepsilon = \min_{|x-x_0|=1/2} u(x)$$

and by Proposition 2.56 we have

$$\varepsilon^5 \leq c_5 \int_{\Gamma_\varepsilon} \det(D^2 u)\,. \tag{4.11}$$

Differentiating (4.10) twice gives $D^2u = D^2\varphi + 8\varepsilon I$. Hence we may rewrite equation (4.4) as

$$(u_{11} + u_{22} + u_{33} + u_{44} - 32\varepsilon + 1)(u_{55} - 8\varepsilon + 1) - u_{15}^2 - u_{25}^2 - u_{35}^2 - u_{45}^2 = \mathrm{e}^F\,. \tag{4.12}$$

Now, on Γ_ε the function u is convex, therefore the Hessian matrix $D^2u(x)$ is non-negative for all $x \in \Gamma_\varepsilon$. In particular $u_{ii}(x) \geq 0$ for all $i = 1, \dots, 5$ and every $x \in \Gamma_\varepsilon$. In addition,

$$u_{ii}(x)u_{55}(x) - u_{i5}^2(x) \geq 0, \qquad \text{for all } i = 1, \dots, 5, \text{ and every } x \in \Gamma_\varepsilon\,. \tag{4.13}$$

Set $\varepsilon = \varepsilon_0 = 1/48$, so that from (4.13) and (4.12) we obtain, for every $x \in \Gamma_{\varepsilon_0}$,

$$\begin{aligned}
\frac{\Delta u(x)}{5} &\leq \frac{5}{6}(u_{11}(x) + u_{22}(x) + u_{33}(x) + u_{44}(x)) + \frac{1}{3}u_{55}(x) \\
&\leq \left(u_{11}(x) + u_{22}(x) + u_{33}(x) + u_{44}(x) + \frac{1}{3}\right)\left(u_{55}(x) + \frac{5}{6}\right) - \sum_{i=1}^4 u_{i5}^2(x) - \frac{5}{18} \\
&= \mathrm{e}^{F(x)} - \frac{5}{18} \leq \mathrm{e}^{\max_K F}.
\end{aligned}$$

Using again the fact that D^2u is non-negative on Γ_ε, the arithmetic-geometric mean inequality forces

$$\det(D^2u(x)) \leq \left(\frac{\Delta u(x)}{5}\right)^5 \leq \mathrm{e}^{5\max_K F}, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,. \tag{4.14}$$

At last, (4.11) and (4.14) imply

$$\left(\frac{1}{48}\right)^5 c_5 \leq \int_{\Gamma_{\varepsilon_0}} \det(D^2u) \leq \mathrm{e}^{5\max_K F}|\Gamma_{\varepsilon_0}|\,,$$

i.e.

$$|\Gamma_{\varepsilon_0}| \geq C\,, \tag{4.15}$$

where $C > 0$ depends on $\max_K F$.

Now observe that

$$u(x) \leq u(x_0) - \nabla u(x) \cdot (x_0 - x) \leq u(x_0) + \frac{\varepsilon_0}{4}\,, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,,$$

that is

$$\varphi(x) - \max_K \varphi + 4\varepsilon_0|x - x_0|^2 \leq \varphi(x_0) - \max_K \varphi + \frac{\varepsilon_0}{4} = \min_K \varphi - \max_K \varphi + \frac{\varepsilon_0}{4}\,, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,.$$

This implies

$$\max_K \varphi - \min_K \varphi \leq \max_K \varphi - \varphi(x) + 1\,, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,.$$

It follows that for every $p \geq 1$

$$\left(\max_K \varphi - \min_K \varphi\right) |\Gamma_{\varepsilon_0}|^{1/p} \leq \left(\int_{\Gamma_{\varepsilon_0}} \left(\max_K \varphi - \varphi + 1\right)^p\right)^{1/p} = \left\|\max_K \varphi - \varphi + 1\right\|_{L^p(\Gamma_{\varepsilon_0})},$$

and since $\Gamma_{\varepsilon_0} \subseteq B_{1/2}(x_0) \subseteq K + x_0$, we have

$$\left\|\max_K \varphi - \varphi + 1\right\|_{L^p(\Gamma_{\varepsilon_0})} \leq \left\|\max_K \varphi - \varphi + 1\right\|_{L^p(K)}.$$

Therefore, since $\int_K \varphi = 0$, we have $\|\varphi\|_{C^0} \leq \max_K \varphi - \min_K \varphi$. Then (4.15) implies

$$\|\varphi\|_{C^0} \leq \max_K \varphi - \min_K \varphi \leq C^{-1/p}\left(\left\|\max_K \varphi - \varphi\right\|_{L^p(K)} + 1\right), \qquad \forall p \geq 1. \tag{4.16}$$

By (4.7) we see that $\Delta(\max_K \varphi - \varphi) \leq 2$, and since $\max_K \varphi - \varphi \geq 0$ we can apply Theorem 2.57 on a ball centered at x_0 such that $\varphi(x_0) = \max_K \varphi$ with $\max_K \varphi - \varphi$ in place of u, $m = 5$, $p = 4/3$, $q = 6$, $r = 1/2$ and $R = 3$. This eventually shows there exists a positive constant C satisfying

$$\left\|\max_K \varphi - \varphi\right\|_{L^{4/3}(K)} \leq C\left(\inf_K \left(\max_K \varphi - \varphi\right) + \|2\|_{L^3(K)}\right) = 2C. \tag{4.17}$$

Estimate (4.9) now follows from (4.16) with $p = 4/3$ and (4.17). □

C^0-estimate for the Laplacian.

In this section we shall prove a C^0-estimate for the Laplacian of φ. The technique we employ is an adaptation of that found in [60].

Lemma 4.6. *Let φ be a C^2 function on the n-torus T^n, fix $\mu \in \mathbb{R}$ and pick a point p_0 where $\Phi = (\Delta\varphi + 2)\mathrm{e}^{-\mu\varphi}$ attains its maximum value. Define*

$$\eta_{ij} = \mu(\Delta\varphi + 2)(\varphi_{ij} + \mu\varphi_i\varphi_j) - \Delta\varphi_{ij}, \quad i, j = 1, \ldots, n.$$

Then

$$\eta_{ii}(p_0) \geq 0, \quad \text{and } \sqrt{\eta_{ii}\eta_{jj}} \geq |\eta_{ij}| \text{ at } p_0,$$

for every $i, j = 1, \ldots, n$.

Proof. We begin by differentiating Φ:

$$\Phi_j = \mathrm{e}^{-\mu\varphi}\left(\Delta\varphi_j - \mu(\Delta\varphi + 2)\varphi_j\right)$$

and

$$\Phi_{ij} = -\mu\mathrm{e}^{-\mu\varphi}\left(\varphi_i\Delta\varphi_j + \Delta\varphi_i\varphi_j\right) + \mu^2\mathrm{e}^{-\mu\varphi}(\Delta\varphi + 2)\varphi_i\varphi_j + \mathrm{e}^{-\mu\varphi}\left(\Delta\varphi_{ij} - \mu(\Delta\varphi + 2)\varphi_{ij}\right).$$

Since

$$\nabla\Phi = 0, \qquad \mathrm{Hess}(\Phi) = (\Phi_{ij}) \leq 0 \qquad \text{at } p_0,$$

we infer

$$\Delta\varphi_j = \mu(\Delta\varphi + 2)\varphi_j \qquad \text{at } p_0 \tag{4.18}$$

and thus $\Phi_{ij}(p_0) = -\eta_{ij}(p_0)$. The Sylvester criterion then yields

$$\eta_{ii}(p_0) \geq 0, \qquad \text{and} \qquad \eta_{ij}^2(p_0) \leq \eta_{ii}\eta_{jj}(p_0)$$

for every $1 \leq i, j \leq n$ and the claim follows. □

Proposition 4.7. *Let $F \in C^2(T^5)$ satisfy (4.3). There exists a positive constant C, depending on $\|F\|_{C^2}$ only, such that*

$$\|\Delta\varphi\|_{C^0} \leq C(1 + \|\varphi\|_{C^1}) \tag{4.19}$$

for any solution $\varphi \in C^4_*(T^5)$ to (4.4).

Proof. For starters,

$$\Delta \mathrm{e}^F = \Delta A B + A \Delta B + 2\nabla A \cdot \nabla B - 2\sum_{i=1}^{4}\left(|\nabla\varphi_{i5}|^2 + \varphi_{i5}\Delta\varphi_{i5}\right). \tag{4.20}$$

Let p_0 and η_{ij} be as in Lemma 4.6 with

$$\mu = \frac{\varepsilon}{\max(\Delta\varphi + 2)}$$

and $\varepsilon \in (0,1)$ to be determined later. Then by using (4.8) with

$$\xi_i = \mathrm{sgn}(\varphi_{i5})\sqrt{\eta_{ii}},\ i = 1, \dots, 4, \quad \xi_5 = \sqrt{\eta_{55}},$$

we find

$$\mu(\Delta\varphi + 2)\left(A(\varphi_{55} + \mu\varphi_5^2) + B\sum_{i=1}^{4}(\varphi_{ii} + \mu\varphi_i^2)\right) - A\underbrace{\Delta\varphi_{55}}_{\Delta B} - B\underbrace{\sum_{i=1}^{4}\Delta\varphi_{ii}}_{\Delta A} - 2\sum_{i=1}^{4}\varphi_{i5}\xi_i\xi_5 \geq 0.$$

at p_0. Lemma 4.6 now implies

$$\varphi_{i5}\xi_i\xi_5 = |\varphi_{i5}|\sqrt{\eta_{ii}}\sqrt{\eta_{55}} \geq \varphi_{i5}\eta_{i5}, \quad \text{at } p_0,$$

i.e.

$$\varphi_{i5}\xi_i\xi_5 \geq \varphi_{i5}\left(\mu(\Delta\varphi + 2)(\varphi_{i5} + \mu\varphi_i\varphi_5) - \Delta\varphi_{i5}\right) \text{ at } p_0.$$

Therefore we obtain

$$\begin{aligned}\mu(\Delta\varphi + 2)\left(A(\varphi_{55} + \mu\varphi_5^2) + B\sum_{i=1}^{4}(\varphi_{ii} + \mu\varphi_i^2)\right) - 2\sum_{i=1}^{4}\varphi_{i5}\left(\mu(\Delta\varphi + 2)(\varphi_{i5} + \mu\varphi_i\varphi_5)\right)\\ \geq A\Delta B + B\Delta A - 2\sum_{i=1}^{4}\varphi_{i5}\Delta\varphi_{i5}, \quad \text{at } p_0.\end{aligned}$$

By (4.20), and the definition of A, B, at the point p_0 we have

$$\begin{aligned}\Delta \mathrm{e}^F &\leq \mu(\Delta\varphi + 2)\left(A(B-1) + B(A-1)\right) + 2\nabla A \cdot \nabla B\\ &\quad + \mu^2(\Delta\varphi + 2)\left(A\varphi_5^2 + B\sum_{i=1}^{4}\varphi_i^2\right) - 2\mu(\Delta\varphi + 2)\sum_{i=1}^{4}\left(\varphi_{i5}^2 + \mu\varphi_{i5}\varphi_i\varphi_5\right)\\ &= 2\mu(\Delta\varphi + 2)\left(AB - \sum_{i=1}^{4}\varphi_{i5}^2\right) - \mu(\Delta\varphi + 2)(A + B) + 2\nabla A \cdot \nabla B\\ &\quad + \mu^2(\Delta\varphi + 2)\left(A\varphi_5^2 + B(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2) - 2\sum_{i=1}^{4}\varphi_{i5}\varphi_i\varphi_5\right)\\ &\leq 2\mu(\Delta\varphi + 2)\mathrm{e}^F - \mu(\Delta\varphi + 2)^2 + 2\nabla A \cdot \nabla B + 2\mu^2(\Delta\varphi + 2)\left(A\varphi_5^2 + B(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2)\right).\end{aligned}$$

Observe that in the last inequality we used (4.8) with $\xi_i = \varphi_i(p_0)$ for $i = 1, \dots, 4$ and $\xi_5 = -\varphi_5(p_0)$.

By (4.18) we then have

$$\mu^2(\Delta\varphi+2)^2|\nabla\varphi|^2 = |\nabla\Delta\varphi|^2 = |\nabla(A+B)|^2 = |\nabla A|^2+|\nabla B|^2+2\nabla A\cdot\nabla B \geq 2\nabla A\cdot\nabla B\,, \quad \text{at } p_0\,,$$

and with the help of

$$A\varphi_5^2+B(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2) \leq A|\nabla\varphi|^2+B|\nabla\varphi|^2 = (\Delta\varphi+2)|\nabla\varphi|^2$$

we deduce

$$\mu(\Delta\varphi(p_0)+2)^2 \leq -\Delta \mathrm{e}^F(p_0)+2\mu(\Delta\varphi(p_0)+2)\mathrm{e}^{F(p_0)}+3\mu^2(\Delta\varphi(p_0)+2)^2|\nabla\varphi(p_0)|^2. \tag{4.21}$$

Let us set

$$m=\Delta\varphi(p_0)+2\,,\quad \varphi_0=\varphi(p_0)\,.$$

Since p_0 is a maximum point for Φ, clearly

$$\max\Phi = m\mathrm{e}^{-\mu\varphi_0}\,.$$

From (4.21) we obtain

$$\mu m^2 \leq \left\|\Delta \mathrm{e}^F\right\|_{C^0}+2\mu m\|\mathrm{e}^F\|_{C^0}+3\mu^2m^2\|\nabla\varphi\|^2_{C^0}. \tag{4.22}$$

Now fix a point p_1 where $\Delta\varphi+2$ reaches its maximum, and call $\varphi_1=\varphi(p_1)$. Then

$$m\leq \max(\Delta\varphi+2) = \mathrm{e}^{\mu\varphi_1}\Phi \leq m\mathrm{e}^{\mu(\varphi_1-\varphi_0)} \leq m\mathrm{e}^{2\mu\|\varphi\|_{C^0}}. \tag{4.23}$$

By the definition of μ and inequality (4.7) we have

$$2\mu=\frac{2}{\max(\Delta\varphi+2)}\,\varepsilon \leq \frac{1}{\mathrm{e}^{\min(F/2)}}\,\varepsilon \leq \mathrm{e}^{-\min(F/2)}\,,$$

hence by (4.23)

$$\varepsilon\exp\left(-\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right) \leq \varepsilon\mathrm{e}^{-2\mu\|\varphi\|_{C^0}} = \mu\max(\Delta\varphi+2)\mathrm{e}^{-2\mu\|\varphi\|_{C^0}} \leq \mu m$$

and also

$$\exp\left(-\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right)\max(\Delta\varphi+2) \leq \mathrm{e}^{-2\mu\|\varphi\|_{C^0}}\max(\Delta\varphi+2)\leq m\,.$$

Next we multiply the last two inequalities and use (4.22), recalling that $\mu m\leq\varepsilon$, to the effect that

$$\varepsilon\exp\left(-2\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right)\max(\Delta\varphi+2) \leq \left\|\Delta \mathrm{e}^F\right\|_{C^0}+2\varepsilon\|\mathrm{e}^F\|_{C^0}+3\varepsilon^2\|\nabla\varphi\|^2_{C^0}\,.$$

Put otherwise,

$$\|\Delta\varphi\|_{C^0} \leq \exp\left(2\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right)\left(\frac{1}{\varepsilon}\left\|\Delta \mathrm{e}^F\right\|_{C^0}+2\|\mathrm{e}^F\|_{C^0}+3\varepsilon\|\nabla\varphi\|^2_{C^0}\right),$$

and by choosing

$$\varepsilon=\frac{1}{1+\|\nabla\varphi\|_{C^0}}$$

the claim is straighforward. □

The next theorem will provide us with an a priori C^1-estimate for φ. Together with Proposition 4.7 it will give an a priori C^0-bound for $\Delta\varphi$.

Theorem 4.8. *For all solutions $\varphi\in C^4_*(T^5)$ of equation (4.4) with $F\in C^2(T^5)$ satisfying (4.3) there* *exists a positive constant C, depending on $\|F\|_{C^2}$ only, such that*

$$\|\varphi\|_{C^1} \leq C. \tag{4.24}$$

Proof. Fix $0 < \alpha < 1$ and $p = \frac{5}{1-\alpha} > 5$. Morrey's inequality says

$$\|\varphi\|_{C^{1,\alpha}} \leq C_1 \|\varphi\|_{W^{2,p}}$$

for some positive constant C_1 depending only on α. Elliptic L^p-estimates for the Laplacian also generate another constant C_2, still depending on α only, such that

$$\|\varphi\|_{W^{2,p}} \leq C_2 \left(\|\varphi\|_{L^p} + \|\Delta\varphi\|_{L^p}\right) .$$

If $\varphi \in C^2(T^5)$, the C^0-estimate (4.9) for φ and bound (4.19) for $\Delta\varphi$ imply

$$\|\varphi\|_{L^p} + \|\Delta\varphi\|_{L^p} \leq \|\varphi\|_{C_0} + \|\Delta\varphi\|_{C^0} \leq C_3 + C_4(1 + \|\varphi\|_{C^1}) .$$

Using standard interpolation theory (see [143, section 6.8]), for any $\varepsilon > 0$ there is a constant $P_\varepsilon > 0$ such that

$$\|\varphi\|_{C^1} \leq P_\varepsilon \|\varphi\|_{C^0} + \varepsilon \|\varphi\|_{C^{1,\alpha}}, \qquad \text{for every } \varphi \in C^{1,\alpha}(T^5) .$$

Putting all this together, we obtain

$$\|\varphi\|_{C^1} \leq P_\varepsilon C_0 + \varepsilon K_0 \left(C_3 + C_4(1 + \|\varphi\|_{C^1})\right) = P_\varepsilon C_3 + \varepsilon C_5(C_3 + C_4) + \varepsilon C_5 C_4 \|\varphi\|_{C^1} ,$$

for some positive constant C_5, again depending on α only. This produces (4.24) once we choose

$$\varepsilon < \frac{1}{C_5 C_1} .$$

□

Corollary 4.9. *Assume that $F \in C^2(T^5)$ satisfies* (4.3) *and let $\varphi \in C^4_*(T^5)$ be a solution to* (4.4)*. Then there exists a positive constant C, depending on $\|F\|_{C^2}$ only, such that*

$$\|\Delta\varphi\|_{C^0} \leq C .$$

$C^{2,\alpha}$-estimate.

The $C^{2,\alpha}$-estimate for our equation (4.4) can now be deduced directly from the general result of Alesker, Theorem 2.54. The HKT structures we are considering on M_r are flat for the Obata connection because N_r is 2-step nilpotent (cf. [110, Proposition 6.1]). Hence the underlying hypercomplex structure is locally flat. Moreover, for T^3-invariant functions the quaternionic Laplacian acts as a multiple of the usual Laplace operator, hence Theorem 2.54 and Corollary 4.9 imply:

Proposition 4.10. *Assume $F \in C^2(T^5)$ satisfies* (4.3)*. For every solution $\varphi \in C^4_*(T^5)$ to equation* (4.4) *there exist $\alpha \in (0,1)$ and a positive constant C, depending on $\|F\|_{C^2}, \|\varphi\|_{C^0}$ only, such that*

$$\|\varphi\|_{C^{2,\alpha}} \leq C .$$

Proof of Theorem 4.1.

In this section we shall use the previously established a priori estimates in order to prove the following result. This will then imply Theorem 4.1.

Theorem 4.11. *Let $F \in C^\infty(T^5)$ satisfy* (4.3)*. Then equation* (4.4) *admits a solution $\varphi \in C^\infty_*(T^5)$.*

Proof. For $t \in [0,1]$ we define

$$F_t = \log(1 - t + te^F)$$

and set

$$S_t = \{\varphi \in C^\infty_*(T^5) \mid (\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1)(\varphi_{55}+1)-\varphi_{15}^2-\varphi_{25}^2-\varphi_{35}^2-\varphi_{45}^2 = \mathrm{e}^{F_t}\}\,,$$

and $S=\bigcup_{t\in[0,1]} S_t$. Clearly $0\in S_0$, and S_1 is the set of smooth solutions of (4.4). We thus need to show that $S_1\neq\emptyset$. For any $t\in[0,1]$ the map F_t satisfies (4.3) and

$$\max_{t\in[0,1]} \|F_t\|_{C^2} < \infty\,.$$

Proposition 4.10 therefore implies there exists $\alpha\in(0,1)$ such that

$$\sup_{\varphi\in S} \|\varphi\|_{C^{2,\alpha}} < \infty\,. \tag{4.25}$$

Let

$$\tau = \sup\{t\in[0,1] \mid S_t\neq\emptyset\}\,.$$

We claim that $S_\tau\neq\emptyset$ and $\tau=1$.

- $S_\tau\neq\emptyset$. Let $\{t_k\}\subseteq[0,1]$ be an increasing sequence converging to τ, and for any $k\in\mathbb{N}$ we fix $\varphi_k\in S_{t_k}$. Condition (4.25) implies that $\{\varphi_k\}$ is a sequence in $C^{2,\alpha}_*(T^5)$, so by the Ascoli-Arzelà Theorem there exists a subsequence $\{\varphi_{k_j}\}$ converging to some ψ in $C^{2,\alpha/2}_*(T^5)$. The function ψ satisfies

  $$(\psi_{11}+\psi_{22}+\psi_{33}+\psi_{44}+1)(\psi_{55}+1)-\psi_{15}^2-\psi_{25}^2-\psi_{35}^2-\psi_{45}^2 = \mathrm{e}^{F_\tau}\,.$$

  In view of Proposition 4.4, equation (4.4) is elliptic, and elliptic regularity (see e.g. [282, Theorem 4.8, Chapter 14]) implies that ψ is in fact C^∞. Therefore $S_\tau\neq\emptyset$, as required.

- $\tau=1$. Assume, by contradiction, that $\tau<1$, and consider the non-linear operator

  $$T\colon C^{2,\alpha}_*(T^5)\times[0,1]\to C^{0,\alpha}_*(T^5)$$

  defined by

  $$T(\varphi,t) = (\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1)(\varphi_{55}+1)-\varphi_{15}^2-\varphi_{25}^2-\varphi_{35}^2-\varphi_{45}^2 - \mathrm{e}^{F_t}\,.$$

  Since $S_\tau\neq\emptyset$, there exists $\psi\in C^\infty_*(T^5)$ such that $T(\psi,\tau)=0$. Let $L\colon C^{2,\alpha}_*(T^5)\to C^{0,\alpha}_*(T^5)$ be the first variation of T with respect to the first variable. Then

  $$Lu = Au_{55} + B(u_{11}+u_{22}+u_{33}+u_{44}) - 2C_1u_{15} - 2C_2u_{25} - 2C_3u_{35} - 2C_4u_{45}$$

  where

  $$A = (\psi_{11}+\psi_{22}+\psi_{33}+\psi_{44}+1)\,, \qquad B=(\psi_{55}+1)\,, \qquad C_i=\psi_{i5}\,,$$

  which implies that L is elliptic since $\psi\in S_\tau$. The strong maximum principle guarantees L is injective because $L\varphi=0$ forces φ to be constant. Furthermore, ellipticity implies that L has closed range, and Schauder Theory together with the method of continuity (see [143, Theorem 5.2]) ensures L is surjective. Hence by the Implicit Function Theorem there exists $\varepsilon>0$ such that for every fixed $t\in(\tau-\varepsilon,\tau+\varepsilon)$, equation

  $$T(\varphi,t)=0$$

  has a solution φ, which is additionally smooth by elliptic regularity. Therefore $S_t\neq\emptyset$ for every $t\in(\tau,\tau+\varepsilon)$, which contradicts the maximality of τ. □

4.1.3 Further Developments.

The manifold M_2, for instance, can be regarded as a T^2-bundle over T^6, so it is quite natural to wonder whether Theorem 4.1 might extend to T^2-invariant functions (instead of T^3-invariant).

T^2-invariance.

We shall next describe this setup for M_2 and point out the differences from the T^3-invariant setting considered in Theorem 4.1.

From (4.5) the quaternionic Monge-Ampère equation (4.1) on (M_2, I, J, K, g) reduces to the following PDE on the 6-dimensional base T^6 when the map F is T^2-invariant

$$\begin{aligned}(\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1)(\varphi_{55}+\varphi_{66}+1)\\ -(\varphi_{35}-\varphi_{26})^2-(\varphi_{45}-\varphi_{16})^2-(\varphi_{46}+\varphi_{15})^2-(\varphi_{36}+\varphi_{25})^2=\mathrm{e}^F\,,\end{aligned} \tag{4.26}$$

where φ is an unknown function in $C^\infty(T^6)$. By calling

$$A=\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1\,,\qquad\qquad B=\varphi_{55}+\varphi_{66}+1$$

and

$$a_1=\varphi_{35}-\varphi_{26}\,,\qquad a_2=\varphi_{45}-\varphi_{16}\,,\qquad a_3=\varphi_{46}+\varphi_{15}\,,\qquad a_4=\varphi_{36}+\varphi_{25}\,,$$

we may rewrite (4.26) as

$$AB-\sum_{i=1}^4 a_i^2=\mathrm{e}^F\,. \tag{4.27}$$

The above is elliptic and

$$\begin{aligned}B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2)-2a_1(\xi_3\xi_5-\xi_2\xi_6)-2a_2(\xi_4\xi_5-\xi_1\xi_6)\\ -2a_3(\xi_4\xi_6+\xi_1\xi_5)-2a_4(\xi_3\xi_6+\xi_2\xi_5)>0\,,\end{aligned} \tag{4.28}$$

for every $\xi\in\mathbb{R}^6$, $\xi\neq 0$.

In order to show that (4.26) can be solved, we need only prove an a priori C^0-estimate for the Laplacian of the solutions to (4.26). The natural approach consists in adapting the proof of Proposition 4.7 by mixing Lemma 4.6 with the ellipticity of the equation. In this case, however, it seems that condition (4.28) should be replaced with a stronger assumption, one implied by the estimate

$$2(|a_2a_3|+|a_1a_4|)<\mathrm{e}^F\,. \tag{4.29}$$

Applying the Laplacian operator to both sides of (4.27) we get

$$B\Delta A+A\Delta B+2\nabla A\cdot\nabla B-2\sum_{k=1}^4\left(|\nabla a_k|^2+a_k\Delta a_k\right)=\Delta\mathrm{e}^F,$$

which readily implies

$$\Delta\mathrm{e}^F\leq B\Delta A+A\Delta B+2\nabla A\cdot\nabla B-2\sum_{k=1}^4 a_k\Delta a_k\,. \tag{4.30}$$

Let p_0 be a maximum point for $(\Delta\varphi+2)\mathrm{e}^{-\mu\varphi}$, as in Lemma 4.6, and

$$\mu=\frac{1}{\max(\Delta\varphi+2)}\,\frac{1}{1+\|\nabla\varphi\|_{C^0}}\,.$$

Using (4.18), we see that the following relation holds at p_0

$$\mu^2(\Delta\varphi+2)^2|\nabla\varphi|^2 = |\nabla\Delta\varphi|^2 = |\nabla(A+B)|^2 = |\nabla A|^2 + |\nabla B|^2 + 2\nabla A\cdot\nabla B \geq 2\nabla A\cdot\nabla B\,,$$

i.e.,

$$2\nabla A\cdot\nabla B \leq \mu^2(\Delta\varphi+2)^2|\nabla\varphi|^2\,. \tag{4.31}$$

To produce an upper bound for $B\Delta A + A\Delta B - 2\sum_{k=1}^4 a_k\Delta a_k$ we consider η_{ij} as in Lemma 4.6 and

$$\xi_i = \sqrt{\eta_{ii}}\,.$$

Then at p_0 we have

$$\xi_i\xi_j \geq |\eta_{ij}|\,.$$

Moreover,

$$\begin{aligned}|a_1|(\xi_3\xi_5+\xi_2\xi_6) &\geq |a_1|\Big\{|\mu(\Delta\varphi+2)(\varphi_{35}+\mu\varphi_3\varphi_5)-\Delta\varphi_{35}| + |\mu(\Delta\varphi+2)(\varphi_{26}+\mu\varphi_2\varphi_6)-\Delta\varphi_{26}|\Big\}\\ &\geq a_1\Big\{\mu(\Delta\varphi+2)(\varphi_{35}+\mu\varphi_3\varphi_5)-\Delta\varphi_{35}-\mu(\Delta\varphi+2)(\varphi_{26}+\mu\varphi_2\varphi_6)+\Delta\varphi_{26}\Big\}\\ &= \mu(\Delta\varphi+2)(a_1^2+a_1\mu(\varphi_3\varphi_5-\varphi_2\varphi_6)) - a_1\Delta a_1\end{aligned}$$

at p_0, i.e.,

$$|a_1|(\xi_3\xi_5+\xi_2\xi_6) \geq \mu(\Delta\varphi+2)(a_1^2+\mu a_1(\varphi_3\varphi_5-\varphi_2\varphi_6)) - a_1\Delta a_1$$

at p_0. Similarly,

$$\begin{aligned}|a_2|(\xi_4\xi_5+\xi_1\xi_6) &\geq \mu(\Delta\varphi+2)(a_2^2+\mu a_2(\varphi_4\varphi_5-\varphi_1\varphi_6)) - a_2\Delta a_2\,,\\ |a_3|(\xi_4\xi_6+\xi_1\xi_5) &\geq \mu(\Delta\varphi+2)(a_3^2+\mu a_3(\varphi_4\varphi_6+\varphi_1\varphi_5)) - a_3\Delta a_3\,,\\ |a_4|(\xi_3\xi_6+\xi_2\xi_5) &\geq \mu(\Delta\varphi+2)(a_4^2+\mu a_4(\varphi_3\varphi_6+\varphi_2\varphi_5)) - a_4\Delta a_4\,,\end{aligned}$$

at p_0. If we add up the last four inequalities and use (4.28) with $\xi_k = \varphi_k$ for $k = 1,\dots,4$ and $\xi_5 = -\varphi_5$, $\xi_6 = -\varphi_6$, we end up with

$$\begin{aligned}&2|a_1|(\xi_3\xi_5+\xi_2\xi_6) + 2|a_2|(\xi_4\xi_5+\xi_1\xi_6) + 2|a_3|(\xi_4\xi_6+\xi_1\xi_5) + 2|a_4|(\xi_3\xi_6+\xi_2\xi_5) \geq\\ &\mu(\Delta\varphi+2)\left(\sum_{k=1}^4(2a_k^2-\mu B\varphi_k^2) - \mu A(\varphi_5^2+\varphi_6^2)\right) - 2\sum_{k=1}^4 a_k\Delta a_k\end{aligned}$$

at p_0.

To handle the last inequality we need the following estimate

$$\begin{aligned}&B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2) + A(\xi_5^2+\xi_6^2) \geq\\ &\qquad 2|a_1|(\xi_3\xi_5+\xi_2\xi_6) + 2|a_2|(\xi_4\xi_5+\xi_1\xi_6) + 2|a_3|(\xi_4\xi_6+\xi_1\xi_5) + 2|a_4|(\xi_3\xi_6+\xi_2\xi_5)\,.\end{aligned} \tag{4.32}$$

Notice this is stronger than (4.28).

In fact, if we assume (4.32), then

$$B\sum_{k=1}^4\xi_k^2 + A(\xi_5^2+\xi_6^2) \geq \mu(\Delta\varphi+2)\left(\sum_{k=1}^4(2a_k^2-\mu B\varphi_k^2) - \mu A(\varphi_5^2+\varphi_6^2)\right) - 2\sum_{k=1}^4 a_k\Delta a_k$$

at p_0 and, keeping in mind the definition of ξ_k,

$$B\sum_{k=1}^4\xi_k^2 + A(\xi_5^2+\xi_6^2) = \mu(\Delta\varphi+2)\left(A\sum_{k=5}^6(\varphi_{kk}+\mu\varphi_k^2) + B\sum_{k=1}^4(\varphi_{kk}+\mu\varphi_k^2)\right) - A\Delta B - B\Delta A\,,$$

at p_0.

Therefore

$$\mu(\Delta\varphi+2)\left(A\sum_{k=5}^{6}(\varphi_{kk}+\mu\varphi_k^2)+B\sum_{k=1}^{4}(\varphi_{kk}+\mu\varphi_k^2)\right)-A\Delta B-B\Delta A\geq$$
$$\mu(\Delta\varphi+2)\left(\sum_{k=1}^{4}(2a_k^2-\mu B\varphi_k^2)-\mu A(\varphi_5^2+\varphi_6^2)\right)-2\sum_{k=1}^{4}a_k\Delta a_k\,,$$

at p_0, which implies

$$\begin{aligned}A\Delta B+B\Delta A-2\sum_{k=1}^{4}a_k\Delta a_k &\leq \mu(\Delta\varphi+2)\left(A\sum_{k=5}^{6}(\varphi_{kk}+2\mu\varphi_k^2)+B\sum_{k=1}^{4}(\varphi_{kk}+2\mu\varphi_k^2)-2\sum_{k=1}^{4}a_k^2\right)\\ &\leq \mu(\Delta\varphi+2)\left(2AB-(A+B)+2\mu(A+B)|\nabla\varphi|^2-2\sum_{k=1}^{4}a_k^2\right)\\ &= \mu(\Delta\varphi+2)\left(2\mathrm{e}^F-(\Delta\varphi+2)+2\mu(\Delta\varphi+2)|\nabla\varphi|^2\right)\,,\end{aligned}$$

at p_0. In other terms,

$$A\Delta B+B\Delta A-2\sum_{k=1}^{4}a_k\Delta a_k\leq\mu(\Delta\varphi+2)\left(2\mathrm{e}^F-(\Delta\varphi+2)+2\mu(\Delta\varphi+2)|\nabla\varphi|^2\right)\tag{4.33}$$

at p_0. From (4.30), (4.31) and (4.33) we finally deduce

$$\mu(\Delta\varphi+2)^2\leq-\Delta\mathrm{e}^F+2\mu(\Delta\varphi+2)\mathrm{e}^F+3\mu^2(\Delta\varphi+2)^2|\nabla\varphi|^2,$$

at p_0. At this juncture the a priori C^0-estimate for $\Delta\varphi$ can be obtained as we did in Subsection 4.1.2.

Let us point out that requiring (4.32) for every $\xi\in\mathbb{R}^6$ is equivalent to (4.29). Indeed the quadratic form

$$\begin{aligned}Q(\xi)&=B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2)\\ &\quad-2|a_1|(\xi_3\xi_5+\xi_2\xi_6)-2|a_2|(\xi_4\xi_5+\xi_1\xi_6)-2|a_3|(\xi_4\xi_6+\xi_1\xi_5)-2|a_4|(\xi_3\xi_6+\xi_2\xi_5)\end{aligned}$$

has matrix

$$\begin{pmatrix}B&0&0&0&-|a_3|&-|a_2|\\0&B&0&0&-|a_4|&-|a_1|\\0&0&B&0&-|a_1|&-|a_4|\\0&0&0&B&-|a_2|&-|a_3|\\-|a_3|&-|a_4|&-|a_1|&-|a_2|&A&0\\-|a_2|&-|a_1|&-|a_4|&-|a_3|&0&A\end{pmatrix},$$

which is positive definite if and only if

$$B^4\left(\left(A-B^{-1}\sum_{k=1}^{4}a_k^2\right)^2-4B^{-2}\left(|a_2a_3|+|a_1a_4|\right)^2\right)>0$$

since $B>0$. A direct computation tells that the last condition is equivalent to

$$2(|a_2a_3|+|a_1a_4|)<\mathrm{e}^F\,.$$

S^1-invariance.

In analogy to the above discussion, the manifold M_1 arises as an S^1-bundle over a T^7-torus, and the function F may be chosen to be S^1-invariant. If so, the quaternionic Monge-Ampère equation (4.1) reads

$$\begin{aligned}(\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1)(\varphi_{55}+\varphi_{66}+\varphi_{77}+1)\\ -(\varphi_{45}-\varphi_{16}-\varphi_{27})^2-(\varphi_{35}+\varphi_{17}-\varphi_{26})^2\\ -(\varphi_{36}+\varphi_{47}+\varphi_{25})^2-(\varphi_{46}-\varphi_{37}+\varphi_{15})^2=\mathrm{e}^F\,,\end{aligned}$$

where φ is an unknown function in $C^\infty(T^7)$.

Setting

$$A=\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1\,,\qquad B=\varphi_{55}+\varphi_{66}+\varphi_{77}+1$$

and

$$\begin{aligned}a_1&=\varphi_{45}-\varphi_{16}-\varphi_{27}\,, & a_2&=\varphi_{35}+\varphi_{17}-\varphi_{26}\,,\\ a_3&=\varphi_{36}+\varphi_{47}+\varphi_{25}\,, & a_4&=\varphi_{46}-\varphi_{37}+\varphi_{15}\,,\end{aligned}$$

the equation turns into

$$AB-\sum_{i=1}^4 a_i^2=\mathrm{e}^F\,. \tag{4.34}$$

The above is elliptic and

$$\begin{aligned}B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2+\xi_7^2)-2a_1(\xi_4\xi_5-\xi_1\xi_6-\xi_2\xi_7)\\ -2a_2(\xi_3\xi_5+\xi_1\xi_7-\xi_2\xi_6)-2a_3(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)-2a_4(\xi_4\xi_6-\xi_3\xi_7+\xi_1\xi_5)>0\,,\end{aligned} \tag{4.35}$$

for every $\xi\in\mathbb{R}^7$, $\xi\neq 0$.

We proceed as in the previous case, and choose p_0 and η_{ij} as in Lemma 4.6 and

$$\mu=\frac{1}{\max(\Delta\varphi+2)}\,\frac{1}{1+\|\nabla\varphi\|_{C^0}}\,,$$

resulting in

$$\Delta\mathrm{e}^F\leq B\Delta A+A\Delta B+\mu^2(\Delta\varphi+2)^2|\nabla\varphi|^2-2\sum_{k=1}^4 a_k\Delta a_k\,,\qquad\text{at } p_0\,.$$

Set $\xi_i=\sqrt{\eta_{ii}}$ and apply Lemma 4.6 to obtain

$$\begin{aligned}&|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)\\ &\geq|a_1|\Big\{|\mu(\Delta\varphi+2)(\varphi_{45}+\mu\varphi_4\varphi_5)-\Delta\varphi_{45}|+|\mu(\Delta\varphi+2)(\varphi_{16}+\mu\varphi_1\varphi_6)-\Delta\varphi_{16}|\\ &\quad+|\mu(\Delta\varphi+2)(\varphi_{27}+\mu\varphi_2\varphi_7)-\Delta\varphi_{27}|\Big\}\\ &\geq a_1\Big\{\mu(\Delta\varphi+2)(\varphi_{45}+\mu\varphi_4\varphi_5)-\Delta\varphi_{45}-\mu(\Delta\varphi+2)(\varphi_{16}+\mu\varphi_1\varphi_6)+\Delta\varphi_{16}\\ &\quad-\mu(\Delta\varphi+2)(\varphi_{27}+\mu\varphi_2\varphi_7)+\Delta\varphi_{27}\Big\}\\ &=\mu(\Delta\varphi+2)(a_1^2+a_1\mu(\varphi_4\varphi_5-\varphi_1\varphi_6-\varphi_2\varphi_7))-a_1\Delta a_1\end{aligned}$$

at p_0, i.e.

$$|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)\geq\mu(\Delta\varphi+2)(a_1^2+\mu a_1(\varphi_4\varphi_5-\varphi_1\varphi_6-\varphi_2\varphi_7))-a_1\Delta a_1$$

at p_0. From that we deduce

$$
\begin{aligned}
|a_2|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6) &\geq \mu(\Delta\varphi+2)(a_2^2+\mu a_2(\varphi_3\varphi_5+\varphi_1\varphi_7-\varphi_2\varphi_6))-a_2\Delta a_2\,,\\
|a_3|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5) &\geq \mu(\Delta\varphi+2)(a_3^2+\mu a_3(\varphi_3\varphi_6+\varphi_4\varphi_7+\varphi_2\varphi_5))-a_3\Delta a_3\,,\\
|a_4|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5) &\geq \mu(\Delta\varphi+2)(a_4^2+\mu a_4(\varphi_4\varphi_6-\varphi_3\varphi_7+\varphi_1\varphi_5))-a_4\Delta a_4\,,
\end{aligned}
$$

at p_0. The sum of the previous four inequalities, together with (4.35), yields

$$
\begin{aligned}
&2|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)+2|a_2|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6)\\
&\qquad +2|a_3|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)+2|a_4|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5)\\
&\qquad\qquad \geq \mu(\Delta\varphi+2)\left(\sum_{k=1}^{4}(2a_k^2-\mu B\varphi_k^2)-\mu A\sum_{k=5}^{7}\varphi_k^2\right)-2\sum_{k=1}^{4}a_k\Delta a_k
\end{aligned}
$$

at p_0.

We need the following estimate

$$
\begin{aligned}
B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2+\xi_7^2)-2|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)&\\
-2|a_2|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6)-2|a_3|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)&\\
-2|a_4|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5)>0\,,&
\end{aligned}
\tag{4.36}
$$

at p_0, which is stronger than (4.35). Once this has been established, the result follows.

To prove (4.36) one has to show that the quadratic form

$$
\begin{aligned}
Q(\xi)=B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2+\xi_7^2)-2|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)&\\
-2|a_2|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6)-2|a_3|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)-2|a_4|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5)&
\end{aligned}
$$

on $\mathbb{R}^7$ is positive definite. This is equivalent to demanding two things:

$$
\mathrm{e}^{2F}-4(|a_2a_3|+|a_1a_4|)^2>0\,,
$$

$$
\begin{aligned}
&\mathrm{e}^{3F}-4\mathrm{e}^{F}\Big((|a_2a_3|+|a_1a_4|)^2+(|a_1a_3|+|a_2a_4|)^2+(|a_1a_2|+|a_3a_4|)^2\Big)\\
&-16\,(|a_2a_3|+|a_1a_4|)\,(|a_1a_3|+|a_2a_4|)\,(|a_1a_2|+|a_3a_4|)>0\,.
\end{aligned}
$$

We wrap up this overview of our future plans by observing that there exist torus fibrations whose hypercomplex structure is not locally flat. On these spaces Alesker's Theorem cannot be applied, so once the C^0-estimate of the Laplacian is at hand one needs to prove the $C^{2,\alpha}$-estimate by alternative arguments.

We expect that the study of the equation on these explicit examples will give new insight for the handling of the general case.

4.2 Foliated HKT manifolds.

Pursuing the approach of the previous section we study here the quaternionic Monge-Ampère equation on HKT manifolds admitting an HKT foliation having corank 4. We show that in this setting the quaternionic Monge-Ampère equation has always a unique solution for every basic datum. This approach includes the study of the equation on SU(3).

4.2.1 Rewriting the equation.

Setting and statement of the main result.

When the bundles considered in the previous section are regarded as T^4-bundles over a T^4, then the equation reduces to the classical Poisson equation on the base (see Remark 4.2). Here we generalize the construction to foliated HKT manifolds, where the foliation replaces the role of the fiber. More precisely we consider the following setting:

We say that a foliation $\mathcal{F}$ on an HKT manifold (M, I, J, K, g) is an *HKT foliation* if

$$T_x\mathcal{F} \text{ is } (I_x, J_x, K_x)\text{-invariant for every } x \text{ in } M,$$

where $T\mathcal{F}$ denotes the vector bundle induced by $\mathcal{F}$. A function f is called *basic* with respect to a foliation $\mathcal{F}$ if $X(f) = 0$ for every $X \in \Gamma(\mathcal{F})$, where $\Gamma(\mathcal{F})$ is the space of smooth sections of $T\mathcal{F}$. We denote by $C^k_B(M)$ the space of real C^k basic functions on $(M, \mathcal{F})$. Our main result is the following:

Theorem 4.12. *Let (M, I, J, K, g) be a compact HKT manifold and let $\mathcal{F}$ be an HKT foliation of real corank 4 on M. Then the quaternionic Monge-Ampère equation*

$$(\Omega + \partial\partial_J\varphi)^n = b\,\mathrm{e}^F\Omega^n\,, \qquad \int_M \varphi \mathrm{Vol}_g = 0\,, \quad F \in C^\infty_B(M). \tag{4.37}$$

has a unique solution for every basic datum $F \in C^\infty_B(M)$. Moreover the solution is necessarily basic.

Rewriting the equation.

Now we consider the framework of Theorem 4.12: let (M, I, J, K, g) be a compact HKT manifold equipped with an HKT foliation $\mathcal{F}$ of real corank 4. We have the following

Lemma 4.13. *Let $\varphi \in C^2_B(M)$. Then*

$$\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} = \Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1\,,$$

where Δ is the Riemannian Laplacian of g and $Q \in \Gamma(T^*M \otimes T^*M)$ is negative semi-definite.

Proof. Since $\mathcal{F}_x$ is I_x-invariant for every $x \in M$, then $T\mathcal{F} \otimes \mathbb{C}$ splits as $T\mathcal{F} \otimes \mathbb{C} = T^{1,0}\mathcal{F} \otimes T^{0,1}\mathcal{F}$. Let $\{Z_1, \dots, Z_{2n}\}$ be a local g-unitary frame with respect to I such that

$$\langle Z_3, \dots, Z_{2n}\rangle = \Gamma(T^{1,0}\mathcal{F})\,.$$

Let us denote the conjugate $\bar{Z}_r$ by $Z_{\bar{r}}$ for every $r = 1, \dots, 2n$ and suppose

$$J(Z_{2k-1}) = Z_{\overline{2k}}\,, \quad \text{for every } k = 1, \dots, n.$$

These assumptions imply that the HKT form of g takes its standard expression

$$\Omega = Z^{12} + Z^{34} + \dots + Z^{(2n-1)(2n)}$$

where $\{Z^1, \dots, Z^{2n}\}$ is the dual coframe to $\{Z_1, \dots, Z_{2n}\}$ and by Z^{ij} we mean $Z^i \wedge Z^j$.

We can write

$$[Z_r, Z_s] = \sum_{k=1}^{2n} B^k_{rs} Z_k\,, \qquad [Z_r, \bar{Z}_s] = \sum_{k=1}^{2n}\left(B^k_{r\bar{s}} Z_k + B^{\bar{k}}_{r\bar{s}} Z_{\bar{k}}\right)\,,$$

for some functions $\{B^k_{rs}, B^k_{r\bar{s}}, B^{\bar{k}}_{r\bar{s}}\}$.

For a basic function φ we have

$$\partial_J\varphi = -J\bar{\partial}\varphi = -J\left(Z_{\bar{1}}(\varphi)Z^{\bar{1}} + Z_{\bar{2}}(\varphi)Z^{\bar{2}}\right) = Z_{\bar{1}}(\varphi)Z^2 - Z_{\bar{2}}(\varphi)Z^1;$$

and

$$\begin{aligned}\partial\partial_J\varphi =&\sum_{k=1}^{2n}(Z_kZ_{\bar 1}(\varphi)Z^{k2}-Z_kZ_{\bar 2}(\varphi)Z^{k1})+\sum_{r<s}(-Z_{\bar 1}(\varphi)B^2_{rs}Z^{rs}+Z_{\bar 2}(\varphi)B^1_{rs}Z^{rs})\\ =&\sum_{k=1}^{2n}(Z_kZ_{\bar 1}(\varphi)Z^{k2}-Z_kZ_{\bar 2}(\varphi)Z^{k1})+\sum_{r<s}(Z_{\bar 2}(\varphi)B^1_{rs}-Z_{\bar 1}(\varphi)B^2_{rs})Z^{rs}\\ =&(Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi))Z^{12}+\sum_{k=3}^{2n}(Z_kZ_{\bar 1}(\varphi)Z^{k2}-Z_kZ_{\bar 2}(\varphi)Z^{k1})+\sum_{r<s}(Z_{\bar 2}(\varphi)B^1_{rs}-Z_{\bar 1}(\varphi)B^2_{rs})Z^{rs}\,,\end{aligned}$$

Since $\mathcal{F}$ is a foliation, $B^1_{rs}=0=B^2_{rs}$ for $2<r<s$, thus

$$\begin{aligned}\partial\partial_J\varphi =&(Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi))Z^{12}+\sum_{k=3}^{2n}\sum_{l=1}^{2n}(B^l_{k\bar 1}Z_l(\varphi)Z^{k2}+B^{\bar l}_{k\bar 1}Z_{\bar l}(\varphi)Z^{k2}-B^l_{k\bar 2}Z_l(\varphi)Z^{k1}-B^{\bar l}_{k\bar 2}Z_{\bar l}(\varphi)Z^{k1})\\ &+(Z_{\bar 2}(\varphi)B^1_{12}-Z_{\bar 1}(\varphi)B^2_{12})Z^{12}+\sum_{s=3}^{2n}(Z_{\bar 2}(\varphi)B^1_{1s}-Z_{\bar 1}(\varphi)B^2_{1s})Z^{1s}+\sum_{s=3}^{2n}(Z_{\bar 2}(\varphi)B^1_{2s}-Z_{\bar 1}(\varphi)B^2_{2s})Z^{2s}\\ =&(Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi)+B^1_{12}Z_{\bar 2}(\varphi)-B^2_{12}Z_{\bar 1}(\varphi))Z^{12}\\ &+\sum_{k=3}^{2n}(B^1_{1k}Z_{\bar 2}(\varphi)-B^2_{1k}Z_{\bar 1}(\varphi)+B^1_{k\bar 2}Z_1(\varphi)+B^{\bar 1}_{k\bar 2}Z_{\bar 1}(\varphi)+B^2_{k\bar 2}Z_2(\varphi)+B^{\bar 2}_{k\bar 2}Z_{\bar 2}(\varphi))Z^{1k}\\ &+\sum_{k=3}^{2n}(B^1_{2k}Z_{\bar 2}(\varphi)-B^2_{2k}Z_{\bar 1}(\varphi)-B^1_{k\bar 1}Z_1(\varphi)-B^{\bar 1}_{k\bar 1}Z_{\bar 1}(\varphi)-B^2_{k\bar 1}Z_2(\varphi)-B^{\bar 2}_{k\bar 1}Z_{\bar 2}(\varphi))Z^{2k}\,.\end{aligned}$$

By setting

$$\begin{aligned}P_k(\nabla\varphi)&=B^1_{1k}Z_{\bar 2}(\varphi)-B^2_{1k}Z_{\bar 1}(\varphi)+B^1_{k\bar 2}Z_1(\varphi)+B^{\bar 1}_{k\bar 2}Z_{\bar 1}(\varphi)+B^2_{k\bar 2}Z_2(\varphi)+B^{\bar 2}_{k\bar 2}Z_{\bar 2}(\varphi)\,,\\ Q_k(\nabla\varphi)&=B^1_{2k}Z_{\bar 2}(\varphi)-B^2_{2k}Z_{\bar 1}(\varphi)-B^1_{k\bar 1}Z_1(\varphi)-B^{\bar 1}_{k\bar 1}Z_{\bar 1}(\varphi)-B^2_{k\bar 1}Z_2(\varphi)-B^{\bar 2}_{k\bar 1}Z_{\bar 2}(\varphi)\,,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n}=&\,1+Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi)+B^1_{12}Z_{\bar 2}(\varphi)-B^2_{12}Z_{\bar 1}(\varphi)\\ &+\sum_{j=3}^{n}(P_{2j}(\nabla\varphi)Q_{2j-1}(\nabla\varphi)-P_{2j-1}(\nabla\varphi)Q_{2j}(\nabla\varphi))\;.\end{aligned}$$

Since the Nijenhuis tensor of J vanishes we have

$$\begin{aligned}0&=[Z_1,Z_{\bar 1}]+J[Z_1,JZ_{\bar 1}]+J[JZ_1,Z_{\bar 1}]-[JZ_1,JZ_{\bar 1}]\\ &=\sum_{k=1}^{2n}(B^k_{1\bar 1}Z_k+B^{\bar k}_{1\bar 1}Z_{\bar k}+B^k_{12}JZ_k+B^{\bar k}_{\bar 2\bar 1}JZ_{\bar k}-B^k_{\bar 2 2}Z_k-B^{\bar k}_{\bar 2 2}Z_{\bar k})\end{aligned}$$

and thus

$$\begin{cases}B^{2k-1}_{1\bar 1}-\overline{B^{2k}_{\bar 2\bar 1}}-B^{2k-1}_{\bar 2 2}=0\,,\\ B^{2k}_{1\bar 1}+\overline{B^{2k-1}_{\bar 2\bar 1}}-B^{2k}_{\bar 2 2}=0\,,\end{cases}\qquad k=1,\dots,n\,.$$

Moreover, since $B^{2k-1}_{1\bar 1},B^{2k}_{1\bar 1},B^{2k-1}_{2\bar 2},B^{2k}_{2\bar 2}$ are purely imaginary, for $k=1$ we deduce

$$\begin{cases}B^{\bar 2}_{\bar 2\bar 1}=B^1_{1\bar 1}+B^1_{2\bar 2}=-B^{\bar 1}_{\bar 1 1}-B^{\bar 1}_{\bar 2 2}=-B^2_{21}\,,\\ B^{\bar 1}_{\bar 2\bar 1}=-B^2_{1\bar 1}-B^2_{2\bar 2}=B^{\bar 2}_{\bar 1 1}+B^{\bar 2}_{\bar 2 2}=-B^1_{21}\,,\end{cases}$$

but then B^2_{21} and B^1_{21} are both real and purely imaginary, yielding

$$\begin{cases} B^1_{21} = 0\,, \\ B^2_{21} = 0\,, \\ B^1_{1\bar 1} + B^1_{2\bar 2} = 0\,, \\ B^2_{1\bar 1} + B^2_{2\bar 2} = 0\,. \end{cases}$$

Writing $X_r = \mathrm{Re}(Z_r)$ and $Y_r = \mathrm{Im}(Z_r)$ for $r = 1, 2$, we see that

$$Z_r Z_{\bar r}(\varphi) = (X_r + iY_r)(X_r - iY_r)(\varphi) = X_r X_r(\varphi) + i[Y_r, X_r](\varphi) + Y_r Y_r(\varphi)$$

and also

$$0 = \sum_{k=1}^{2n} (B^k_{1\bar 1} + B^k_{2\bar 2}) Z_k(\varphi) = ([Z_1, Z_{\bar 1}] + [Z_2, Z_{\bar 2}])(\varphi) = 2i([Y_1, X_1] + [Y_2, X_2])(\varphi)$$

so that

$$Z_1 Z_{\bar 1}(\varphi) + Z_2 Z_{\bar 2}(\varphi) = X_1X_1(\varphi) + Y_1Y_1(\varphi) + X_2X_2(\varphi) + Y_2Y_2(\varphi) = \Delta\varphi\,.$$

Furthermore, from the vanishing of the Nijenhuis tensor it easy to observe that

$$Q_{2j-1}(\nabla\varphi) = -\overline{P_{2j}(\nabla\varphi)}\,, \qquad Q_{2j}(\nabla\varphi) = \overline{P_{2j-1}(\nabla\varphi)}\,.$$

Thus we finally obtain

$$\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} = 1 + \Delta\varphi - \sum_{k=3}^{2n} |P_k(\nabla\varphi)|^2\,.$$

The claim then follows by setting

$$Q(\nabla\varphi, \nabla\varphi) = -\sum_{k=3}^{n} |P_k(\nabla\varphi)|^2\,.$$ □

4.2.2 Proof of Theorem 4.12.

From Lemma 4.13 it follows that under our assumptions for a basic datum F equation (4.37) reduces to

$$\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F\,, \quad \int_M \varphi \mathrm{Vol}_g = 0\,. \tag{4.38}$$

We then focus on this last equation and prove its solvability in the general setting of a compact Riemannian manifold.

C^0-estimate.

In order to prove existence of solutions to (4.38) we need to show some a priori estimates. We mention in passing that since we aim to study equation (4.38) on a compact Riemannian manifold we cannot directly apply the known estimates for the quaternionic Monge-Ampère equation as they rely on the HKT condition. The C^0 bound is obtained by using the Alexandrov-Bakelman-Pucci estimate.

Lemma 4.14. *Let (M, g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ negative semi-definite and $F \in C^0(M, \mathbb{R})$. If $(\varphi, b) \in C^2(M, \mathbb{R}) \times \mathbb{R}_+$ solves (4.38) then there exists a positive constant C depending only on M, g, $\|Q\|_{C^0}$ and F such that*

$$\|\varphi\|_{C^0} \le C\,, \qquad b \le C\,.$$

Proof. First of all we bound the constant b. At a maximum point p of φ we have $\nabla\varphi = 0$ and $\Delta\varphi \le 0$, therefore $b\,\mathrm{e}^{F(p)} - 1 \le 0$ and thus $b \le \mathrm{e}^{-F(p)} \le \|\mathrm{e}^{-F}\|_{C^0}$ so that the constant b is bounded.

Let $x_0 \in M$ be a point where φ achieves its minimum and consider a coordinate chart centered at x_0. Without loss of generality we may identify this chart with a ball $B_1(0) \subseteq \mathbb{R}^m$ of unit radius, where $m = \dim(M)$. Fix $\varepsilon > 0$ and define

$$u(x) = \varphi(x) - \max_M \varphi + \varepsilon |x|^2 \,.$$

Applying Proposition 2.56 to u we have

$$\varepsilon^m \leq c_m \int_{\Gamma_\varepsilon} \det(D^2 u) \,. \tag{4.39}$$

We aim to prove that $D^2 u$ is bounded on Γ_ε. Differentiating u we see that

$$\nabla u = \nabla \varphi + 2\varepsilon x \,, \qquad D^2 u = D^2 \varphi + 2\varepsilon \mathbb{1}_m \,,$$

where $\mathbb{1}_m$ is the identity matrix. As a consequence u satisfies the following equation

$$\Delta u - 2m\varepsilon + Q(\nabla u - 2\varepsilon x, \nabla u - 2\varepsilon x) + 1 = b\,\mathrm{e}^F \,.$$

Set, for instance, $\varepsilon = 1$. Now, since on Γ_1 the Hessian $D^2 u$ is non-negative, by the arithmetic-geometric mean inequality we deduce

$$\begin{aligned}
\det(D^2 u(x)) &\leq \left(\frac{\Delta u(x)}{m}\right)^m \leq \left| Q(\nabla u(x) - 2x, \nabla u(x - 2x) + 2m - 1 + b\,\mathrm{e}^{F(x)} \right|^m \\
&\leq \left(\|Q\|_{C^0} |\nabla u(x) - 2x|^2 + 2m + 1 + b \left|\mathrm{e}^{F(x)}\right| \right)^m \\
&\leq \left(\frac{5}{2} \|Q\|_{C^0} + 2m + 1 + b \left|\mathrm{e}^{F(x)}\right| \right)^m \leq C \,,
\end{aligned}$$

for any $x \in \Gamma_1$, where $C > 0$ is a uniform constant.

Now we observe that

$$u(x) \leq u(0) - \nabla u(x) \cdot (-x) \leq u(0) + \frac{1}{2} \,, \qquad \text{for every } x \in \Gamma_1 \,,$$

which implies

$$\varphi(x) - \max_M \varphi + |x|^2 \leq \varphi(0) - \max_M \varphi + \frac{1}{2} = \min_M \varphi - \max_M \varphi + \frac{1}{2} \,, \qquad \text{for every } x \in \Gamma_1 \,,$$

and therefore

$$\max_M \varphi - \min_M \varphi \leq \max_M \varphi - \varphi(x) + \frac{1}{2} \,, \qquad \text{for every } x \in \Gamma_1 \,.$$

It follows that for every $p \geq 1$ we have

$$\left(\max_M \varphi - \min_M \varphi \right) |\Gamma_1|^{1/p} \leq \left\| \max_M \varphi - \varphi + \frac{1}{2} \right\|_{L^p(\Gamma_1)} \leq \left\| \max_M \varphi - \varphi + \frac{1}{2} \right\|_{L^p(B_1(0))} \,.$$

Combining this with (4.39) and the fact that $\int_M \varphi = 0$, we have

$$\|\varphi\|_{C^0} \leq \max_M \varphi - \min_M \varphi \leq |\Gamma_1|^{-1/p} \left\| \max_M \varphi - \varphi + \frac{1}{2} \right\|_{L^p(B_1(0))} \leq C \left\| \max_M \varphi - \varphi \right\|_{L^p(B_1(0))} \,,$$

for every $p \geq 1$. In conclusion we only need to prove an L^p estimate for $\max_M \varphi - \varphi$ to obtain the desired estimate on φ. Since Q is negative semidefinite we see from the equation that

$$\Delta \varphi \geq b\,\mathrm{e}^F - 1 \geq C\,\mathrm{e}^F - 1 \,,$$

where we used that b is uniformly bounded. This entails that $\Delta(\max_M \varphi - \varphi) \leq 1 - C\,\mathrm{e}^F$, and applying

Theorem 2.57 to $\max_M \varphi - \varphi$ with, $1 \leq p \leq m/(m-2)$, $q = 2m$, $r = 1/2$ and $R = 2$ we infer

$$\left\|\max_M \varphi - \varphi\right\|_{L^p(B_1(0))} \leq C\left(\inf_{B_{1/2}(0)}\left(\max_M \varphi - \varphi\right) + \frac{1}{2}\|1 - C\,\mathrm{e}^F\|_{L^m(B_2(0))}\right) \leq C\,,$$

as required. □

Higher order estimates.

For higher order bounds we need to recall the following two results:

Theorem 4.15 (Theorem 3.1, Chapter 4 [203]). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open subset. Consider a semilinear elliptic equation of the following type*

$$\Delta u + a(x, u, \nabla u) = 0\,,$$

where the function $a(x, u, p)$ is measurable for $x \in \bar{\Omega}$ and arbitrary $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and satisfies

$$(1 + |p|)\sum_{i=1}^{n} |p_i| + |a(x, u, p)| \leq \mu(|u|)(1 + |p|)^m\,,$$

for some $m > 1$ and some non-decreasing continuous function $\mu\colon [0, +\infty) \to \mathbb{R}$. Let $u \in C^2(\Omega)$ be a solution of the given equation, then, for any connected open subset $\Omega' \subset \Omega$ there exists a constant $C > 0$ depending only on $\|u\|_{C^0(\Omega)}$, $\mu(\|u\|_{C^0(\Omega)})$, m and $d(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^1(\Omega')} \leq C\,.$$

Theorem 4.16 (Theorem 6.1, Chapter 4 [203]). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open subset. Consider a semilinear elliptic equation of the following type*

$$\Delta u + a(x, u, \nabla u) = 0\,,$$

where the function $a(x, u, p)$ is measurable for $x \in \bar{\Omega}$ and arbitrary $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and satisfies

$$\|a\|_{C^0(\Omega)} < \mu_1\,,$$

for some constant $\mu_1 < \infty$. Let $u \in C^2(\Omega)$ be a solution of the given equation such that

$$\|\nabla u\|_{C^0(\Omega)} < C\,,$$

then there exists $\alpha \in (0, 1)$ depending only on $\|\nabla u\|_{C^0(\Omega)}$ and μ_1 such that $\nabla u \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$. Moreover, for any connected open subset $\Omega' \subset \Omega$ there exists a constant $C > 0$ depending only on $\|\nabla u\|_{C^0(\Omega)}$, μ_1 and $d(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C\,.$$

We can then establish the higher order a priori estimates for solutions to (4.38).

Lemma 4.17. *Let (M, g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ and $F \in C^0(M, \mathbb{R})$. If $(\varphi, b) \in C^2(M, \mathbb{R}) \times \mathbb{R}_+$ satisfies $\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F$, then there exists a positive constant C depending only on M, g, $\|\varphi\|_{C^0}$, b, $\|Q\|_{C^0}$ and F such that*

$$\|\Delta\varphi\|_{C^0} \leq C\,.$$

Proof. As an application of Theorem 4.15 with $a = Q + 1 - b\,\mathrm{e}^F$, $m = 2$, and $\mu \equiv \|Q\|_{C^0} + b\|\mathrm{e}^F\|_{C^0} + \sqrt{n} + 1$ we have that there exists a constant $C > 0$ such that

$$\|\nabla\varphi\|_{C^0} \leq C\,.$$

Then from the equation we have

$$\|\Delta\varphi\|_{C^0} \leq b\,\mathrm{e}^F + 1 + \|Q(\nabla\varphi, \nabla\varphi)\|_{C^0} \leq C\,,$$

and the claim follows. □

Lemma 4.18. *Let (M, g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ and $F \in C^{k,\beta}(M, \mathbb{R})$. If $(\varphi, b) \in C^2(M, \mathbb{R}) \times \mathbb{R}_+$ solves $\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F$, then $\varphi \in C^{k+2,\alpha}(M, \mathbb{R})$ for some $\alpha \in (0, \beta)$ and there is a constant $C > 0$ depending only on M, g, b, $\|\nabla\varphi\|_{C^0}$, $\|Q\|_{C^0}$ and F such that*

$$\|\varphi\|_{C^{k+2,\alpha}} \leq C\,.$$

Proof. Lemma 4.17 implies $\|\nabla\varphi\|_{C^0} \leq C$ and we can apply Theorem 4.16 choosing the constant $\mu_1 = \|Q\|_{C^0} + b\|\mathrm{e}^F\|_{C^0} + 1$ and deduce that there exist $\alpha \in (0, 1)$ and a constant $C > 0$ such that

$$\|\varphi\|_{C^{1,\alpha}} \leq C\,.$$

Then the equation implies the estimate $\|\Delta\varphi\|_{C^{0,\alpha}} \leq C$, which can be improved to a $C^{2,\alpha}$ estimate for φ using Schauder theory by assuming $\alpha < \beta$. Then $\varphi \in C^{2,\alpha}(M, \mathbb{R})$ and by a bootstrapping argument the claim follows. □

Now, we prove that equation (4.38) is always solvable.

Proposition 4.19. *Let (M, g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ be negative semi-definite and $F \in C^{k,\beta}(M, \mathbb{R})$. Then equation (4.38) admits a solution $(\varphi, b) \in C^{k+2,\alpha}(M, \mathbb{R}) \times \mathbb{R}_+$ for $\alpha \in (0, \beta)$.*

Proof. Let $F \in C^{k,\beta}(M, \mathbb{R})$ and consider the set

$$S := \left\{t' \in [0, 1] \ : \ (*_t) \text{ has a solution } (\varphi_t, b_t) \in C^{4,\alpha}(M) \times \mathbb{R}_+ \text{ for } t \in [0, t']\right\}\,,$$

where

$$\Delta\varphi_t + Q(\nabla\varphi_t, \nabla\varphi_t) = b_t\,\mathrm{e}^{tF} - 1\,, \quad \int_M \varphi_t \mathrm{Vol}_g = 0 \qquad (*_t)$$

and $\alpha \in (0, \beta)$ is fixed.

S in not empty since the pair $(\varphi_0, b_0) = (0, 1)$ solves $(*_t)$ for $t = 0$ and, therefore, $0 \in S$. In order to prove the statement we need to show that S is open and closed in $[0, 1]$.

To show that S is open we apply, as usual, the inverse function theorem between Banach spaces. Let $\hat{t} \in S$ and $(\varphi_{\hat{t}}, b_{\hat{t}})$ be solution of $(*_{\hat{t}})$, let B_1 and B_2 be the Banach spaces

$$B_1 := \left\{\psi \in C^{4,\alpha}(M, \mathbb{R}) \ : \ \int \psi \mathrm{Vol}_g = 0\right\}\,, \quad B_2 := C^{2,\alpha}(M, \mathbb{R})$$

and let $\Psi\colon B_1 \times \mathbb{R}_+ \to B_2$ be the operator

$$\Psi(\psi, a) := \log\left(\frac{\Delta\psi + Q(\nabla\psi, \nabla\psi) + 1}{a}\right)\,.$$

The differential $\Psi_{*|(\varphi_{\hat{t}}, b_{\hat{t}})}\colon B_1 \times \mathbb{R} \to B_2$ is

$$\Psi_{*|(\varphi_{\hat{t}}, b_{\hat{t}})}(\eta, c) = \frac{\Delta\eta + 2Q(\nabla\eta, \nabla\varphi_{\hat{t}})}{b_{\hat{t}}\,\mathrm{e}^{\hat{t}F}} - \frac{c}{b_{\hat{t}}}\,.$$

Since $T\colon \eta \mapsto \Delta\eta + 2Q(\nabla\eta, \nabla\varphi_{\hat{t}})$ is a second order linear elliptic operator without terms of degree zero, by the maximum principle its kernel is the set of constant functions on M. Moreover, since T has the same principal symbol of the Laplacian operator it has index zero. Denoting with T^* the formal adjoint of T we then have

$$\dim\ker(T^*) = \dim\operatorname{coker}(T) = \dim\ker(T) - \operatorname{ind}(T) = 1\,.$$

Let $\rho \in C^{2,\alpha}(M,\mathbb{R})$. Equation

$$\Delta\eta + 2Q(\nabla\eta, \nabla\varphi_{\hat t}) = c\,\mathrm{e}^{\hat t F} + \rho\, b_{\hat t}\,\mathrm{e}^{\hat t F} \tag{4.40}$$

is solvable if and only if its right hand side is orthogonal to $\ker(T^*)$, or equivalently to a generator of $\ker(T^*)$. This can always be accomplished by a suitable choice of the constant c and, therefore, there always exists a solution $\eta \in C^{4,\alpha}(M)$ to (4.40). Moreover the solution η is unique in B_1 because $\int_M \eta \mathrm{Vol}_g = 0$. The differential $\Psi_{*|(\varphi_{\hat t}, b_{\hat t})}$ is then an isomorphism and it follows by the inverse function theorem that the operator Ψ is locally invertible around $(\varphi_{\hat t}, b_{\hat t})$, implying that there exists $\varepsilon > 0$ such that for $t \in [\hat t, \hat t + \varepsilon)$ equation $(*_t)$ can be solved.

Next we prove that S is closed. Let $\{t_j\}$ be a sequence in S converging to some $t \in [0,1]$ and consider the corresponding solutions $(\varphi_j, b_j) = (\varphi_{t_j}, b_{t_j})$ to $(*_{t_j})$. In view of Lemma 4.14 the families $\{\|\varphi_j\|_{C^0}\}$, $\{b_j\}$ are uniformly bounded from above. Moreover, Lemmas 4.17 and 4.18 imply that the family $\{\varphi_j\}$ is uniformly bounded in $C^{k+2,\alpha}$-norm. Consequently, by Ascoli-Arzelà Theorem, up to a subsequence, φ_j converges to some $\varphi_t \in C^{k+2,\alpha}(M,\mathbb{R})$ in $C^{k+2,\alpha}$-norm and b_j converges to some $b_t \in \mathbb{R}$. b_t is in fact positive since from the equation we deduce that the sequence b_j is uniformly bounded from below by a positive quantity. The pair (φ_t, b_t) solves $(*_t)$ and the closedness of S follows. $\square$

Proof of Theorem 4.12.

We are ready to prove Theorem 4.12.

Proof of Theorem 4.12. In view of Lemma 4.13 for every $\varphi \in C^\infty_B(M)$ we have

$$\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} = \Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1\,,$$

where Δ is the Riemannian Laplacian of g and $Q \in \Gamma(T^*M \otimes T^*M)$ is negative semi-definite.

Let $F \in C^\infty_B(M)$. Proposition 4.19 implies that the equation

$$\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F\,, \qquad \int_M \varphi \mathrm{Vol}_g = 0$$

has a solution $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}_+$. We observe that since F is basic, then φ is necessarily basic too. Indeed by setting

$$\Psi(\psi) = \Delta\psi + Q(\nabla\psi, \nabla\psi) + 1$$

we have that for every $X \in \Gamma(\mathcal{F})$ condition $X(F) = 0$ implies

$$0 = X(\Psi(\varphi)) = \Psi_{*|\varphi}(X(\varphi))$$

and since $\Psi_{*|\varphi}$ is a linear elliptic operator without free term, by the maximum principle $X(\varphi)$ must be constant and then necessarily zero. Hence (φ, b) solves the quaternionic Monge-Ampère equation and the claim follows. $\square$

As an explicit example we observe that Theorem 4.12 can be applied for instance to study the quaternionic Monge-Ampère equation on SU(3) endowed with Joyce's hypercomplex structure.

Example 4.20. Recall from Example 2.10 that Lie algebra of SU(3) splits in

$$\mathfrak{su}(3) = \mathfrak{b} \oplus \mathfrak{d} \oplus \mathfrak{f}$$

where

$$[\mathfrak{b}, \mathfrak{d}] = 0\,, \qquad [\mathfrak{b}, \mathfrak{f}] = \mathfrak{f}\,, \qquad [\mathfrak{d}, \mathfrak{f}] = \mathfrak{f}\,, \qquad [\mathfrak{f}, \mathfrak{f}] = \mathfrak{b} \oplus \mathfrak{d}\,, \qquad [\mathfrak{d}, \mathfrak{d}] = \mathfrak{d}\,.$$

In particular $\mathfrak{b} \oplus \mathfrak{d}$ is a subalgebra of $\mathfrak{su}(3)$ and induces a foliation $\mathcal{F}$ on SU(3).

Moreover there exists a basis $(X_1, \ldots, X_8)$ on $\mathfrak{su}(3)$ such that the hypercomplex structure is given by the following relations:

- on $\mathfrak{b} \oplus \mathfrak{d} = \langle X_1, X_2, X_3, X_4 \rangle$ as $IX_1 = X_2,\ IX_3 = X_4,\ JX_1 = X_3,\ JX_2 = -X_4$;
- on $\mathfrak{f} = \langle X_5, X_6, X_7, X_8 \rangle$ as $Iv = [X_2, v],\ Jv = [X_3, v],\ Kv = [X_4, v]$ for every $v \in \mathfrak{f}$.

The metric g such that $(X_1, \ldots, X_8)$ is an orthogonal frame is HKT with respect to (I, J, K). Hence the foliation $\mathcal{F}$ induced by $\mathfrak{b} \oplus \mathfrak{d}$ is (I, J, K)-invariant and all the assumptions of Theorem 4.12 are satisfied. Therefore the quaternionic Calabi-Yau equation on $(\mathrm{SU}(3), I, J, K, g)$ can be solved for every $\mathcal{F}$-basic datum F.

For the sake of the reader we show explicitly how the proof of Lemma 4.13 works in the case of SU(3). We have seen in Example 2.15 that the unitary coframe

$$Z^1 = \frac{1}{2}(X^1 + iX^2)\,, \quad Z^2 = \frac{1}{2}(X^3 + iX^4)\,, \quad Z^3 = -\frac{1}{2}(X^5 + iX^6)\,, \quad Z^4 = \frac{1}{2}(X^7 + iX^8)$$

satisfies

$$\partial Z^1 = 0\,, \qquad \partial Z^2 = -2Z^{12} - 2Z^{34}\,, \qquad \partial Z^3 = -(1+3i)Z^{13}\,, \qquad \partial Z^4 = (3i-1)Z^{14}\,.$$

For a basic function φ we have $Z_1(\varphi) = Z_2(\varphi) = 0$, where $(Z_1, \ldots, Z_4)$ is the dual frame of $(Z^1, \ldots, Z^4)$, and thus we obtain

$$\begin{aligned}
\partial\partial_J\varphi &= -\,\partial J\left(Z_{\bar 3}(\varphi)\, Z^{\bar 3} + Z_{\bar 4}(\varphi)\, Z^{\bar 4}\right) = \partial\left(Z_{\bar 3}(\varphi)\, Z^4 - Z_{\bar 4}(\varphi)\, Z^3\right)\\
&= (Z_3 Z_{\bar 3}(\varphi) + Z_4 Z_{\bar 4}(\varphi))\, Z^{34} - (Z_1 Z_{\bar 4}(\varphi) - (1+3i)Z_{\bar 4}(\varphi))\, Z^{13}\\
&\quad + (Z_1 Z_{\bar 3}(\varphi) + (3i-1)Z_{\bar 3}(\varphi))\, Z^{14} + Z_2 Z_{\bar 3}(\varphi)\, Z^{24} - Z_2 Z_{\bar 4}(\varphi) Z^{23}
\end{aligned}$$

which, using the following brackets (that can be deduced from those of the X_i's listed in Example 2.15):

$$[Z_1, Z_{\bar 4}] = (3i-1)Z_{\bar 4}\,, \qquad [Z_1, Z_{\bar 3}] = -(1+3i)Z_{\bar 3}\,, \qquad [Z_2, Z_{\bar 3}] = 2Z_4\,, \qquad [Z_2, Z_{\bar 4}] = -2Z_3\,,$$

simplifies to

$$\partial\partial_J\varphi = (Z_3 Z_{\bar 3}(\varphi) + Z_4 Z_{\bar 4}(\varphi))\, Z^{34} + 2Z_{\bar 4}(\varphi)\, Z^{13} - 2Z_{\bar 3}(\varphi)\, Z^{14} + 2Z_4(\varphi)\, Z^{24} + 2Z_3(\varphi) Z^{23}\,.$$

Taking into account that the HKT form is $\Omega = Z^{12} + Z^{34}$ we obtain

$$\frac{(\Omega + \partial\partial_J\varphi)^2}{\Omega^2} = 1 + Z_3 Z_{\bar 3}(\varphi) + Z_4 Z_{\bar 4}(\varphi) - 4|Z_3(\varphi)|^2 - 4|Z_4(\varphi)|^2$$

in accordance with Lemma 4.13.

CHAPTER 5

THE QUATERNIONIC CALABI CONJECTURE: A PARABOLIC APPROACH

We consider here the natural generalization of the parabolic Monge-Ampère equation to HKT geometry. We prove that in the compact case the equation has always a short-time solution and when the hypercomplex structure is locally flat and admits a compatible hyperkähler metric, then the equation has a long-time solution whose normalization converges to a solution of the quaternionic Monge-Ampère equation. The result gives an alternative proof of Alesker's theorem 2.53.

The results of this chapter are contained in [33]. We also mention the independent paper [333] where the parabolic quaternionic Monge-Ampère equation is studied and its long-time behaviour is described with techniques different from ours.

5.1 Preliminaries.

The parabolic quaternionic Monge Ampère equation.

Let (M, I, J, K, g) be a compact HKT manifold. Consider the space of smooth *quaternionic* Ω-*plurisubharmonic functions*:

$$\mathcal{H}_\Omega = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \Omega_\varphi := \Omega + \partial\partial_J\varphi > 0\}\,,$$

where the inequality "$\Omega_\varphi > 0$" means that Ω_φ is q-positive and therefore induces a new hyperhermitian metric on M.

In the present chapter we approach the *quaternionic Monge-Ampère equation*

$$(\Omega + \partial\partial_J\varphi)^n = b\,\mathrm{e}^F\,\Omega^n\,, \tag{5.1}$$

via the following geometric flow

$$\varphi_t = \log\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} - F\,, \quad \varphi(x,0) = 0\,, \tag{5.2}$$

where $F \in C^\infty(M,\mathbb{R})$ is the datum and the solution φ is supposed to satisfy $\varphi(\cdot,t) \in \mathcal{H}_\Omega$ for every t and the subscript t denotes the derivative of φ with respect to the variable t. The same dynamic approach was pursued on Kähler manifolds [69], on Hermitian manifolds [144, 275] and on almost Hermitian manifolds [91].

Our main result is the following theorem which provides an alternative proof of Alesker's Theorem 2.53.

Theorem 5.1. *Let (M, I, J, K, g) be a compact HKT manifold with (I, J, K) locally flat and assume that there exists a hyperkähler metric $\hat{g}$ on (M, I, J, K) with corresponding HKT form $\Omega_{\hat{g}}$. Then there exists a long-time solution $\varphi \in C^\infty(M \times \mathbb{R}_+, \mathbb{R})$ to the parabolic quaternionic Monge-Ampère equation* (5.2) *such that*

$$\tilde{\varphi} = \varphi - \frac{\int_M \varphi\, \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}}{\int_M \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}}$$

converges in C^∞-topology to a smooth function $\tilde{\varphi}_\infty \in C^\infty(M, \mathbb{R})$. Moreover if

$$b := \frac{\int_M \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}}{\int_M \mathrm{e}^F \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}}\,,$$

then $(\tilde{\varphi}_\infty, b)$ solves the quaternionic Monge-Ampère equation (5.1).

Now we describe the layout of the proof. Since (5.2) is strongly parabolic, it admits a unique maximal solution $\varphi \in C^\infty(M \times [0, T), \mathbb{R})$.

Step 1. From the equation we directly deduce a uniform C^0 bound on φ_t (Lemma 5.4).

Step 2. The C^0 estimate for solutions of the quaternionic Calabi-Yau equation (5.1) then implies a uniform bound on $\operatorname{osc}\varphi$ (Lemma 5.5).

Step 3. We use the existence of the hyperkähler metric and the local flatness of the hypercomplex structure in order to establish a uniform upper bound on $\Delta_{\hat{g}}\varphi$ (Lemma 5.6).

Step 4. A general result in [90] implies a uniform Hölder estimate on the second derivatives of φ, thus a classical bootstrapping argument using Schauder estimates implies $T = \infty$ and a uniform bound on $|\nabla^k \varphi|$ for $k \geq 1$ (Lemmas 5.7 and 5.8).

Step 5. We prove the convergence of $\tilde{\varphi}$ using an argument due to Phong-Sturm [245] based on an adapted Mabuchi-type functional (Lemma 5.9).

We point out that the local flatness of the hypercomplex structure plays a role in steps 3 and 4, while the existence of a background hyperkähler metric is only used in step 3.

Remark 5.2. Flow (5.2) can be regarded as a geometric flow in Hermitian Geometry. Here we assume that the canonical bundle of (M, I) is trivial and we fix a q-real complex volume form Θ on (M, I). As shown in [18] one has

$$(\Omega + \partial\partial_J \varphi)^n \wedge \bar{\Theta} = i^n(\omega - i\partial\bar{\partial}\varphi)^n \wedge \Phi\,, \quad \Omega^n \wedge \bar{\Theta} = i^n \omega^n \wedge \Phi$$

where ω is the fundamental form of (g, I) and Φ is a real (n, n)-form which is positive in a weak sense. By setting $u = -\varphi$ we can then rewrite (5.2) as

$$u_t = -\log \frac{(\omega + i\partial\bar{\partial}u)^n \wedge \Phi}{\omega^n \wedge \Phi} + F\,, \quad u(0) = 0\,. \tag{5.3}$$

Equation (5.3) reminds the *parabolic k-Hessian flow*

$$u_t = \log \frac{(\chi + i\partial\bar{\partial}u)^k \wedge \alpha^{n-k}}{\alpha^n} + F\,, \quad u(0) = 0 \tag{5.4}$$

studied by Phong and Tô on a complex n-dimensional Hermitian manifold (M, α) in [246], where $1 \leq k \leq n$ and χ is a real k-positive $(1,1)$-form. According to [246] (5.4) has always a long-time solution whose normalization converges in C^∞-topology to a solution of the k-Hessian equation. Equation (5.3) differs from the parabolic n-Hessian flow since the role of α^n is replaced by the form Φ which is positive in a weak sense and the theorem of Phong and Tô cannot be directly applied.

Some useful identities.

We assume that the canonical bundle of (M, I) is holomorphically trivial and we let Θ be a q-positive holomorphic volume form on (M, I). Note that $\Omega^n \wedge \bar{\Theta}$ is a real volume form; indeed, J acts trivially on top forms and thus $\overline{\Omega^n \wedge \bar{\Theta}} = J\Omega^n \wedge J\bar{\Theta} = \Omega^n \wedge \bar{\Theta}$.

The HKT metric induces the *quaternionic Laplacian* operator

$$\Delta_g \varphi := n \frac{\partial \partial_J \varphi \wedge \Omega^{n-1}}{\Omega^n}$$

for $\varphi \in C^\infty(M, \mathbb{R})$. We recall that, from Lemma 2.28 we know that the integral of the quaternionic Laplacian with respect to the volume $\Omega^n \wedge \bar{\Theta}$ is zero and it is a self-adjoint operator with respect to the corresponding L^2-product. Moreover the following formula will be useful:

$$\frac{\partial \eta \wedge \partial_J \psi \wedge \Omega^{n-1}}{\Omega^n} = \frac{1}{2n} g(\partial \eta, \bar{\partial} \psi) \, . \tag{5.5}$$

It is a direct consequence of 2.21.

The basic example of hyperhermitian manifold is given by an open set A of $\mathbb{R}^{4n}$ with the standard hyperhermitian structure

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & \mathbb{1}_n & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & 0 & 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & -\mathbb{1}_n & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \end{pmatrix},$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix. In order to identify $\mathbb{R}^{4n}$ with $\mathbb{H}^n$, the real coordinates on are taken as $(x_0^1, \dots, x_0^n, x_1^1, \dots, x_1^n, x_2^1, \dots, x_2^n, x_3^1, \dots, x_3^n)$ and the quaternionic coordinates are $q^r = x_0^r + i x_1^r + j x_2^r + k x_3^r$.

Let $\mathrm{Hyp}^+(n)$ denote the set of positive-definite hyperhermitian matrices. We recall here some facts following from the discussion carried out in Subsection 2.2.1. Any hyperhermitian Riemannian metric g on (A, I_0, J_0, K_0) defines a smooth map $G \colon A \to \mathrm{Hyp}^+(n)$,

$$G_{rs} := g(\partial_{q^r}, \partial_{q^s}) \, .$$

For instance given a local potential u for the HKT form Ω corresponding to g we have $G = \mathrm{Hess}_{\mathbb{H}} u$, where we recall that for any smooth function $u \colon A \to \mathbb{R}$ the *quaternionic Hessian matrix* is defined as

$$(\mathrm{Hess}_{\mathbb{H}} u)_{\bar{r}s} := \frac{1}{4} u_{\bar{r}s} = \frac{1}{4} \partial_{\bar{q}^r} \partial_{q^s} u \, .$$

Finally, we provide a lemma which will be helpful in the proof of the main theorem.

Lemma 5.3. *Let $U \colon A \to \mathrm{Hyp}^+(n, \mathbb{H})$ be a smooth map and assume that there exists a point $p \in A$ such that $U(p)$ is diagonal. Let $\hat{g}$ be a hyperhermitian metric on A such that the induced matrix $\hat{G}$ is the identity. Then*

$$\Delta_{\hat{g}} \log \det(U) = -\frac{1}{4} \sum_{r,s,t=1}^{n} \sum_{i=0}^{3} \frac{1}{U_{ss}} \frac{1}{U_{tt}} |U_{st, x_i^r}|^2 + \sum_{s=1}^{n} \frac{1}{U_{ss}} \Delta_{\hat{g}} U_{ss}$$

at p, where the subindex "x_i^r" denotes the derivative with respect to the corresponding real coordinate.

Proof. Using Jacobi's formula (1.15) we directly compute

$$\begin{aligned} \partial_{\bar{q}^r} \partial_{q^r} \log \det(U) &= \sum_{i=0}^{3} \partial^2_{x_i^r} \log \det(U) = \sum_{i=0}^{3} \partial_{x_i^r} \mathrm{Re}\, \mathrm{tr} \left(U^{-1} U_{, x_i^r} \right) \\ &= \sum_{i=0}^{3} \mathrm{Re}\, \mathrm{tr} \left(-U^{-1} U_{, x_i^r} U^{-1} U_{, x_i^r} + U^{-1} U_{, x_i^r x_i^r} \right) \end{aligned}$$

and at the point p where U takes a diagonal form

$$\begin{aligned}\Delta_{\hat g}\log\det(U) &= \frac{1}{4}\sum_{r=1}^{n}\partial_{\bar q^r}\partial_{q^r}\log\det(U) = \frac{1}{4}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\mathrm{Re}\left(-U^{ss}U_{st,x_i^r}U^{tt}U_{ts,x_i^r}+U^{ss}U_{ss,x_i^rx_i^r}\right)\\ &= -\frac{1}{4}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{U_{ss}}\frac{1}{U_{tt}}|U_{st,x_i^r}|^2+\sum_{s=1}^{n}\frac{1}{U_{ss}}\Delta_{\hat g}U_{ss}\end{aligned}$$

and the claim follows. □

5.2 Proof of Theorem 5.1.

Let (M,I,J,K,g) be a HKT manifold with HKT form Ω. Every $\varphi\in\mathcal{H}_\Omega$ induces a HKT metric g_φ and a quaternionic Laplacian $\Delta_\varphi:=\Delta_{g_\varphi}$. Consider the operator

$$P\colon\mathcal{H}_\Omega\to C^\infty(M,\mathbb{R})\,,\quad P(\varphi)=\log\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n}-F\,.$$

The first variation of P is

$$P_{*|\varphi}(\psi)=n\frac{\partial\partial_J\psi\wedge(\Omega+\partial\partial_J\varphi)^{n-1}}{(\Omega+\partial\partial_J\varphi)^n}=\Delta_\varphi\psi\,.$$

Since Δ_φ is a strongly elliptic operator, equation (5.2) is always well-posed and it admits a unique maximal solution $\varphi\in C^\infty(M\times[0,T),\mathbb{R})$. Assume further that the canonical bundle of (M,I) is holomorphically trivial and let $\Theta\in\Lambda^{2n,0}(M)$ be a q-real holomorphic volume form. We then denote

$$\tilde\varphi=\varphi-\frac{\int_M\varphi\,\Omega^n\wedge\bar\Theta}{\int_M\Omega^n\wedge\bar\Theta}\,.$$

C^0-estimate.

We start by proving C^0 bounds for the time derivatives φ_t and $\tilde\varphi_t$ and then use these to prove the C^0 estimate for $\tilde\varphi$. In what follows we denote by C all the uniform constants (which may be different from line to line).

Lemma 5.4. *There exists a uniform constant $C>0$ such that*

$$|\varphi_t(x,t)|\le C\,,\quad|\tilde\varphi_t(x,t)|\le C$$

for every $(x,t)\in M\times[0,T)$.

Proof. Since

$$\frac{\partial}{\partial t}\log\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n}=n\frac{\partial\partial_J\varphi_t\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n}=\Delta_\varphi\varphi_t\,,$$

we have

$$\varphi_{tt}=\Delta_\varphi\varphi_t$$

and the parabolic maximum principle implies the a priori C^0 estimate for φ_t. The estimate on $\tilde\varphi_t$ immediately follows. □

Lemma 5.5. *We have*

$$\max_M\varphi-\min_M\varphi\le C$$

and

$$|\tilde\varphi| \leq C\,,$$

for a uniform constant $C > 0$.

Proof. Since $|\varphi_t|$ is bounded and

$$(\Omega + \partial\partial_J\varphi)^n = \mathrm{e}^{F+\varphi_t}\,\Omega^n\,,$$

for every fixed t, $\varphi(\cdot,t)$ solves the quaternionic Monge-Ampère equation (5.1) with datum $F+\varphi_t$. In view of the C^0 estimate for solutions to the quaternionic Monge-Ampère equation (Theorem 2.52), φ satisfies the bound

$$\max_M \varphi - \min_M \varphi \leq C\,, \tag{5.6}$$

where C depends only on (M,I,J,K,g) and an upper bound of $\max|F+\varphi_t|$. Therefore Lemma 5.4 implies that the constant C in (5.6) may be chosen so that it only depends on (M,I,J,K,g) and an upper bound of $\max|F|$. Now, let $(x,t)\in M\times[0,T)$, since $\tilde\varphi$ is normalized, there exist (y,t) such that $\tilde\varphi(y,t)=0$, and thus we obtain $|\tilde\varphi(x,t)| = |\tilde\varphi(x,t)-\tilde\varphi(y,t)| = |\varphi(x,t)-\varphi(y,t)| \leq C$ and the claim follows. □

C^0-estimate for the Laplacian.

Lemma 5.6. *Assume that* (I,J,K) *is locally flat and that there exists a hyperkähler metric* $\hat g$ *on* (M,I,J,K)*. Then*

$$\Delta_{\hat g}\varphi \leq C\,,$$

for a uniform constant C*.*

Proof. Let

$$Q = 2\sqrt{\mathrm{tr}_{\hat g} g_\varphi} - \varphi\,.$$

Fix $T' < T$ and let (x_0,t_0) be a point where Q achieves its maximum in $M\times[0,T']$. We may assume without loss of generality that $t_0>0$. Since (I,J,K) is locally flat, then in a neighborhood of x_0 we can locally identify M with an open set A of $\mathbb{H}^n$. Let G and $\hat G$ be the hyperhermitian matrices in A induced by g and $\hat g$ respectively. We may further assume that $G = \mathrm{Hess}_{\mathbb{H}} v$ in A, that $\hat G$ is the identity in A and that $U = \mathrm{Hess}_{\mathbb{H}}(v+\varphi)$ is diagonal at x_0. Let $u = v+\varphi$. Then in A we have

$$Q = 2\sqrt{\Delta_{\hat g} u} - \varphi$$

and the flow rewrites as

$$\varphi_t = \log\frac{\det U}{\det G} - F\,. \tag{5.7}$$

Computing at (x_0,t_0), we have

$$\Delta_\varphi Q = \frac{1}{\sqrt{\Delta_{\hat g} u}}\sum_{r=1}^n \frac{1}{u_{r\bar r}}\left(-\frac12\frac{1}{\Delta_{\hat g} u}|\Delta_{\hat g}\, u_r|^2 + \Delta_{\hat g} u_{r\bar r}\right) - \Delta_\varphi\varphi\,.$$

Using (5.7) and applying Lemma 5.3 we infer

$$\begin{aligned}
\partial_t Q &= \frac{1}{\sqrt{\Delta_{\hat g} u}}\Delta_{\hat g}\varphi_t - \varphi_t = \frac{1}{\sqrt{\Delta_{\hat g} u}}\Delta_{\hat g}\left(\log\det(U) - \log\det(G) - F\right) - \varphi_t\\
&= \frac{1}{\sqrt{\Delta_{\hat g} u}}\left(-\frac14\sum_{r,s,t=1}^n\sum_{i=0}^3 \frac{1}{u_{s\bar s}}\frac{1}{u_{t\bar t}}|u_{s\bar t,x_i^r}|^2 + \sum_{r=1}^n\frac{1}{u_{r\bar r}}\Delta_{\hat g} u_{r\bar r} - \Delta_{\hat g}(\log\det(G)+F)\right) - \varphi_t
\end{aligned}$$

which implies

$$\partial_t Q - \Delta_\varphi Q =$$
$$\frac{1}{\sqrt{\Delta_{\hat g} u}}\left(\frac{1}{2\Delta_{\hat g} u}\sum_{r=1}^n \frac{1}{u_{r\bar r}}|\Delta_{\hat g}\, u_r|^2 - \frac14 \sum_{r,s,t=1}^n\sum_{i=0}^3 \frac{1}{u_{s\bar s}}\frac{1}{u_{t\bar t}}|u_{s\bar t, x_i^r}|^2 - \Delta_{\hat g}(\log\det(G)+F)\right) + \Delta_\varphi\varphi - \varphi_t\,.$$

Using the Cauchy-Schwarz inequality and [14, Proposition 3.1] we obtain

$$\begin{aligned}\sum_{r=1}^n \frac{1}{u_{r\bar r}}|\Delta_{\hat g} u_r|^2 &= \sum_{r=1}^n\sum_{i=0}^3 \frac{1}{u_{r\bar r}}(\Delta_{\hat g} u_{x_i^r})^2 = \frac{1}{16}\sum_{r=1}^n\sum_{i=0}^3\frac{1}{u_{r\bar r}}\left(\sum_{s=1}^n \frac{\sqrt{u_{s\bar s}}}{\sqrt{u_{s\bar s}}} u_{s\bar s, x_i^r}\right)^2\\ &\le \frac{1}{16}\sum_{r,s,t=1}^n u_{t\bar t}\sum_{i=0}^3 \frac{1}{u_{r\bar r}}\frac{1}{u_{s\bar s}}(u_{s\bar s, x_i^r})^2 = \frac14 \Delta_{\hat g} u \sum_{r,s=1}^n\sum_{i=0}^3 \frac{1}{u_{r\bar r}}\frac{1}{u_{s\bar s}}(u_{s\bar s,x_i^r})^2\\ &\le \frac12 \Delta_{\hat g} u \sum_{r,s,t=1}^n\sum_{i=0}^3 \frac{1}{u_{s\bar s}}\frac{1}{u_{t\bar t}}|u_{s\bar t, x_i^r}|^2\end{aligned}$$

i.e.

$$\frac{1}{2\Delta_{\hat g} u}\sum_{r=1}^n \frac{1}{u_{r\bar r}}|\Delta_{\hat g} u_r|^2 \le \frac14 \sum_{r,s,t=1}^n\sum_{i=0}^3 \frac{1}{u_{s\bar s}}\frac{1}{u_{t\bar t}}|u_{s\bar t,x_i^r}|^2\,,$$

from which it follows

$$0 \le \partial_t Q - \Delta_\varphi Q \le \Delta_\varphi \varphi - \frac{\Delta_{\hat g}(\log\det(G)+F)}{\sqrt{\Delta_{\hat g} u}} - \varphi_t \le 1 - \Delta_\varphi v - \frac{\Delta_{\hat g}(\log\det(G)+F)}{\sqrt{\Delta_{\hat g} u}} - \varphi_t$$

at (x_0, t_0), where we have used that it is a maximum point as well as the relation

$$\Delta_\varphi \varphi = 1 - \Delta_\varphi v\,.$$

Hence

$$\Delta_\varphi v \le 1 - \frac{\Delta_{\hat g}(\log\det(G)+F)}{\sqrt{\Delta_{\hat g} u}} - \varphi_t$$

at (x_0, t_0). Since $|\varphi_t|$ is uniformly bounded we obtain

$$\Delta_\varphi v(x_0,t_0) \le C + \frac{C}{\sqrt{\sum_{r=1}^n u_{r\bar r}(x_0,t_0)}} \tag{5.8}$$

for a uniform constant C. In terms of u and G equation (5.2) writes as

$$u_t = \log\det(U) - \log\det(G) - F$$

and then

$$u_t(x_0,t_0) = \log\prod_{r=1}^n u_{r\bar r}(x_0,t_0) - \log\det(G(x_0)) - F(x_0)\,.$$

Lemma 5.4 implies that $|u_t|$ is uniformly bounded and we deduce that

$$\frac{1}{C} \le \prod_{r=1}^n u_{r\bar r}(x_0,t_0) \le C\,.$$

Thus in particular by the geometric-arithmetic mean inequality we have $\sum_{r=1}^n u_{r\bar r}(x_0,t_0) \ge C$. Since

$$\Delta_\varphi v(x_0,t_0) = \sum_{r=1}^n \frac{1}{u_{r\bar r}(x_0,t_0)} v_{r\bar r}(x_0)\,,$$

by (5.8) we finally deduce

$$\sum_{r=1}^{n} \frac{1}{u_{r\bar r}(x_0,t_0)} \le C\,.$$

Therefore

$$\Delta_{\hat g} u(x_0,t_0) = \sum_{r=1}^{n} u_{r\bar r}(x_0,t_0) \le \frac{1}{(n-1)!}\left(\sum_{r=1}^{n} \frac{1}{u_{r\bar r}(x_0,t_0)}\right)^{n-1} \prod_{r=1}^{n} u_{r\bar r}(x_0,t_0) \le C\,.$$

It follows

$$2\sqrt{\Delta_{\hat g} u(x,t)} \le C + \varphi(x,t) - \varphi(x_0,t_0) \le C + \operatorname{osc}\varphi \qquad \text{in } M\times[0,T']\,,$$

from which, using Lemma 5.5, we get

$$\Delta_{\hat g} u \le C$$

for a uniform C and the claim is proved. □

Higher order estimates and proof of Theorem 5.1**.**

Lemma 5.7. *Assume that (I,J,K) is locally flat and that there exists a hyperhermitian metric $\hat g$ on (M,I,J,K) such that $\Delta_{\hat g}\varphi \le C$ for a uniform constant C. Then for $0<\alpha<1$ we have*

$$\|\nabla^2\varphi\|_{C^\alpha} \le C$$

for a uniform constant C.

Proof. We prove the result by applying [90, Theorem 5.1]. Note that the real representation $\gamma\colon \mathbb{H}^{n,n} \to \{H\in\mathbb{R}^{4n,4n} : I_0HI_0 = J_0HJ_0 = K_0HK_0 = -H\}$ of quaternionic matrices introduced in Subsection 1.1.2 is monotonic in the sense that when H_1,H_2 are hyperhermitian one has

$$H_1 \le H_2 \Leftrightarrow \gamma(H_1) \le \gamma(H_2)\,,$$

where $H_1\le H_2$ means that all the eigenvalues of H_2-H_1 are non-negative.

Let $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to \{H\in\mathbb{R}^{4n,4n} : I_0HI_0 = J_0HJ_0 = K_0HK_0 = -H\}$ be the projection defined as

$$\mathrm{p}(N) := \frac14 (N - I_0NI_0 - J_0NJ_0 - K_0NK_0)\,.$$

Then for any real valued smooth function f and any hyperhermitian matrix H we have

$$\gamma(\mathrm{Hess}_{\mathbb{H}} f) = 16\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} f)\,, \quad \det(\gamma(H)) = (\det H)^4\,.$$

Thus, once local quaternionic coordinates are fixed, working as in the proof of Lemma 5.6, we can rewrite equation (5.2) as

$$u_t = \frac14 \log\det\left(16\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u)\right) - \log\det(G) - F\,,$$

where $u = v+\varphi$ and v is a HKT potential of Ω. We rewrite the last equation as

$$u_t = P(\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u)) - \log\det(G) - F \tag{5.9}$$

where for $N\in \mathrm{Sym}(4n,\mathbb{R})$ such that $\det N>0$ we set

$$P(N) = \frac14 \log\det(16N)\,.$$

Fix positive constants $C_1<C_2$ and let

$$\mathcal{E} := \left\{N\in\mathrm{Sym}(4n,\mathbb{R}) \,:\, C_1\mathbb{1}_{4n} \le N \le C_2\mathbb{1}_{4n}\right\}\,.$$

Then $\mathcal{E}$ is a compact convex subset of $\mathrm{Sym}(4n,\mathbb{R})$. We observe that P and p satisfy the assumptions in [90, Theorem 5.1]. Indeed

- P is uniformly elliptic in $\mathcal{E}$;
- P is concave in $\mathcal{E}$;
- p is linear;
- if $N \geq 0$, then $\mathrm{p}(N) \geq 0$ and $C^{-1}\|N\| \leq \|\mathrm{p}(N)\| \leq C\|N\|$ for C uniform.

Therefore, if we show that $\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u) \in \mathcal{E}$ for a suitable choice of C_1 and C_2 equation (5.9) belongs to the class of equations considered in [90, Theorem 5.1].

Without loss of generality we can fix $x_0 \in M$ and assume that $\hat{G}$ is the identity at x_0. Our assumption $\Delta_{\hat{g}}\varphi \leq C$ implies

$$\sum_{r=1}^{n} u_{r\bar{r}} \leq C \tag{5.10}$$

at x_0 for a uniform $C > 0$ and thus

$$\mathrm{Hess}_{\mathbb{H}} u \leq C\mathbb{1}_n\,.$$

On the other hand, equation (5.2) writes as

$$u_t = \log\det(\mathrm{Hess}_{\mathbb{H}} u) - \log\det(G) - F\,.$$

Thus by Lemma 5.4

$$\prod_{i=1}^{n} \lambda_i = \det(\mathrm{Hess}_{\mathbb{H}} u) \geq \det(G)\,\mathrm{e}^{-\frac{1}{\kappa}|u_t|+F} \geq C\,,$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $\mathrm{Hess}_{\mathbb{H}} u$ and $C > 0$ is a uniform constant. From (5.10) we also infer $\sum_{i=1}^{n} \lambda_i \leq C$ at x_0 which then implies a uniform lower bound for each λ_i at the point x_0, but such bound does not depend on x_0.

Therefore

$$C_1\mathbb{1}_n \leq \mathrm{Hess}_{\mathbb{H}} u \leq C_2\mathbb{1}_n\,.$$

By applying γ we get

$$C_1\mathbb{1}_{4n} \leq \mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u) \leq C_2\mathbb{1}_{4n}\,.$$

Then we can work as in the proof of [91, Lemma 6.1].

We assume that the domain of u is $B \times [0,T)$ with B diffeomorphic to the unit ball in $\mathbb{R}^{4n}$. If $T < 1$, then Lemma 5.4 implies

$$|u| \leq CT + C \leq C$$

for a uniform C and [90, Theorem 5.1] implies the result. If $T \geq 1$ we define, for any $a \in (0, T-1)$

$$\hat{u}(x,t) := u(x,t+a) - \inf_{B\times[a,a+1)} u(x,t)$$

for all $t \in [0,1)$. We immediately deduce

$$\hat{u}_t = \log\det(\mathrm{Hess}_{\mathbb{H}}\hat{u}) - \log\det(G) - F\,, \qquad \sup_{B\times[0,1)} |\hat{u}(x,t)| \leq C\,.$$

Invoking again [90, Theorem 5.1], chosen $\varepsilon \in (0,\frac{1}{2})$ and $\alpha \in (0,1)$ we have

$$\|\nabla^2 u\|_{C^\alpha(B\times[a+\varepsilon,a+1))} = \|\nabla^2 \hat{u}\|_{C^\alpha(B\times[\varepsilon,1))} \leq C$$

where the constant C depends on ε and α. As a was chosen arbitrarily in $(0, T-1)$ we have

$$\|\nabla^2 u\|_{C^\alpha(B\times[\varepsilon,T))} \leq C\,,$$

and the lemma follows. □

Lemma 5.8. *Assume that there exists* $0 < \alpha < 1$ *such that* $\|\nabla^2\varphi\|_{C^\alpha} \le C$ *for a uniform constant* C*. Then* $T = \infty$ *and for every* $k \ge 1$

$$\|\nabla^k \varphi\|_{C^0} \le C$$

for a uniform constant C*.*

Proof. Our assumptions imply that the spatial derivatives of φ satisfy a uniformly parabolic equation and uniform bounds on $\|\nabla^k\varphi\|_{C^0}$ with $k \ge 1$ follow by Schauder theory and a standard bootstrapping argument.

Now we shall prove the long-time existence. Assume by contradiction that the maximal time interval $[0, T)$ of existence of φ is bounded. Then the achieved estimates and short-time existence would allow us to extend φ past T, which is a contradiction, thus $T = \infty$. □

Lemma 5.9. *Assume* $T = \infty$ *and that* $\|\nabla^k\varphi\|_{C^0}$ *is uniformly bounded for every* $k \ge 1$*. Then*

$$\tilde\varphi := \varphi - \frac{\int_M \varphi\, \Omega^n \wedge \bar\Theta}{\int_M \Omega^n \wedge \bar\Theta}$$

converges in C^∞*-topology to a smooth function* $\tilde\varphi_\infty$*. Moreover if*

$$b := \frac{\int_M \Omega^n \wedge \bar\Theta}{\int_M \mathrm{e}^F \Omega^n \wedge \bar\Theta}\,,$$

then $(\tilde\varphi_\infty, b)$ *solves the quaternionic Monge-Ampère equation* (5.1)*.*

Proof. Let

$$f(t) := \int_M \varphi_t\, \Omega_\varphi^n \wedge \bar\Theta = \int_M \log\frac{\Omega_\varphi^n}{\Omega^n}\, \Omega_\varphi^n \wedge \bar\Theta - \int_M F\, \Omega_\varphi^n \wedge \bar\Theta\,.$$

Using (5.5) we have

$$\begin{aligned} f' &= \int_M \left(\Delta_\varphi\varphi_t + \log\frac{\Omega_\varphi^n}{\Omega^n}\Delta_\varphi\varphi_t - F\Delta_\varphi\varphi_t\right)\Omega_\varphi^n \wedge \bar\Theta \\ &= \int_M \varphi_t\Delta_\varphi\varphi_t\,\Omega_\varphi^n \wedge \bar\Theta = \int_M \varphi_t\partial\partial_J\varphi_t \wedge \Omega_\varphi^{n-1} \wedge \bar\Theta = -\int_M \partial\varphi_t \wedge \partial_J\varphi_t \wedge \Omega_\varphi^{n-1} \wedge \bar\Theta \\ &= -\frac{1}{2n}\int_M |\partial\varphi_t|^2_{g_\varphi}\Omega_\varphi^n \wedge \bar\Theta\,. \end{aligned}$$

Differentiating again we obtain

$$f'' = -\frac{1}{2n}\int_M \tfrac{\partial}{\partial t}|\partial\varphi_t|^2_{g_\varphi}\Omega_\varphi^n \wedge \bar\Theta - \frac{1}{2n}\int_M |\partial\varphi_t|^2_{g_\varphi}\Delta_\varphi\varphi_t\,\Omega_\varphi^n \wedge \bar\Theta\,.$$

Now

$$\frac{\partial}{\partial t}|\partial\varphi_t|^2_{g_\varphi} = -g_\varphi\left(\tfrac{\partial}{\partial t}g_\varphi, \partial\varphi_t \otimes \bar\partial\varphi_t\right) + 2\mathrm{Re}\, g_\varphi(\partial\varphi_{tt}, \bar\partial\varphi_t)\,. \tag{5.11}$$

For the first term of (5.11) Cauchy-Schwarz inequality gives

$$-g_\varphi\left(\tfrac{\partial}{\partial t}g_\varphi, \partial\varphi_t \otimes \bar\partial\varphi_t\right) \le |\tfrac{\partial}{\partial t}g_\varphi|_{g_\varphi}\, |\partial\varphi_t|^2_{g_\varphi} \le C\,|\partial\varphi_t|^2_{g_\varphi}$$

because of how Ω_φ and g_φ are related and the fact that Ω_φ and $\frac{\partial}{\partial t}\Omega_\varphi$ are uniformly bounded in C^k-norm for every k. For the second term of (5.11) using (5.5) again we have

$$\begin{aligned} -\frac{1}{n}\mathrm{Re}\int_M g_\varphi(\partial\Delta_\varphi\varphi_t, \bar\partial\varphi_t)\Omega_\varphi^n \wedge \bar\Theta &= -2\mathrm{Re}\int_M \partial\Delta_\varphi\varphi_t \wedge \partial_J\varphi_t \wedge \Omega_\varphi^{n-1} \wedge \bar\Theta \\ &= 2\mathrm{Re}\int_M \Delta_\varphi\varphi_t\,\partial\partial_J\varphi_t \wedge \Omega_\varphi^{n-1} \wedge \bar\Theta = \frac{2}{n}\int_M (\Delta_\varphi\varphi_t)^2\,\Omega_\varphi^n \wedge \bar\Theta \end{aligned}$$

therefore

$$f'' \geq -C \int_M |\partial \varphi_t|^2_{g_\varphi} \, \Omega^n_\varphi \wedge \bar{\Theta} \, .$$

Thus we have a non increasing smooth function $f : [0, +\infty) \to \mathbb{R}$ which is bounded from below and such that $f''(t) \geq C f'(t)$ for some positive constant C. This implies that $\lim_{t \to +\infty} f'(t) = 0$, i.e.

$$\lim_{t \to \infty} \int_M |\partial \varphi_t|^2_{g_\varphi} \Omega^n_\varphi \wedge \bar{\Theta} = 0 \, . \tag{5.12}$$

Now, $\tilde{\varphi}$ has a uniform C^∞ bound and Ascoli-Arzelà theorem implies that there exists a sequence $\{t_k\} \subseteq \mathbb{R}$, $t_k \to \infty$ such that $\tilde{\varphi}(\cdot, t_k)$ converges to some $\tilde{\varphi}_\infty$ in C^∞-topology. Since

$$\tilde{\varphi}_t = \log \frac{\Omega^n_{\tilde{\varphi}}}{\Omega^n} - F - \frac{\int_M \left(\log \frac{\Omega^n_{\tilde{\varphi}}}{\Omega^n} - F \right) \Omega^n \wedge \bar{\Theta}}{\int_M \Omega^n \wedge \bar{\Theta}} \, ,$$

by (5.12) we get

$$0 = \lim_{t \to \infty} \int_M |\partial \tilde{\varphi}_t|^2_{g_{\tilde{\varphi}}} \Omega^n_{\tilde{\varphi}} \wedge \bar{\Theta} = \int_M \left| \partial \left(\log \frac{\Omega^n_{\tilde{\varphi}_\infty}}{\Omega^n} - F \right) \right|^2_{g_{\tilde{\varphi}}} \Omega^n_{\tilde{\varphi}_\infty} \wedge \bar{\Theta} \, .$$

It follows that

$$\log \frac{\Omega^n_{\tilde{\varphi}_\infty}}{\Omega^n} - F = C$$

for some constant C, so that

$$\Omega^n_{\tilde{\varphi}_\infty} = \mathrm{e}^{F+C} \Omega^n \, .$$

This means that $(\tilde{\varphi}_\infty, \mathrm{e}^C)$ solves the quaternionic Calabi-Yau equation. Finally, we prove that $\lim_{t \to \infty} \tilde{\varphi} = \tilde{\varphi}_\infty$. Assume by contradiction that there exists $\varepsilon > 0$ and a sequence $t_k \to \infty$ such that

$$\|\tilde{\varphi}(\cdot, t_k) - \tilde{\varphi}_\infty\|_{C^\infty} \geq \varepsilon$$

for every t_k. We may assume that $\tilde{\varphi}(\cdot, t_k)$ converges in C^∞-topology to $\tilde{\varphi}'_\infty$. Hence

$$\Omega^n_{\tilde{\varphi}'_\infty} = \mathrm{e}^{F+C'} \Omega^n \, .$$

Since $\tilde{\varphi}_\infty$ and $\tilde{\varphi}'_\infty$ solve the same quaternionic Calabi-Yau equation, from uniqueness follows $\tilde{\varphi}_\infty = \tilde{\varphi}'_\infty$ and the lemma is proved. □

Proof of Theorem 5.1. We put together Lemmas 5.4–5.9 proved in this section. Lemmas 5.4,5.5,5.6 imply that if φ solves (5.2), its quaternionic Laplacian $\Delta_{\hat{g}} \varphi$ with respect to the background hyperkähler metric $\hat{g}$ has a uniform upper bound. Hence Lemmas 5.7 and 5.8 can be applied and (5.2) has a long-time solution φ such that $\|\nabla^k \varphi\|_{C^0}$ is bounded for every $k \geq 1$. Therefore, taking $\Theta = \Omega^n_{\hat{g}}$, Lemma 5.9 implies the last part of the statement. □

CHAPTER 6
MORE GENERAL ELLIPTIC EQUATIONS

Mainly motivated by a conjecture of Alesker and Verbitsky, we study a class of fully non-linear elliptic equations on certain compact hyperhermitian manifolds. By adapting the approach of Székelyhidi [280] to the hypercomplex setting, we prove some a priori estimates for solutions to such equations under the assumption of existence of $\mathcal{C}$-subsolutions. In the estimate of the quaternionic Laplacian, we need to further assume the existence of a flat hyperkähler metric. As an application of our results we prove that the quaternionic analogue of the Hessian equation and Monge-Ampère equation for $(n-1)$-plurisubharmonic functions can always be solved on compact flat hyperkähler manifolds. The results of this chapter have been obtained in [139].

6.1 Overview.

Setting of the problem.

Fix a q-real $(2,0)$ form Ω, a smooth map $\varphi\colon M \to \mathbb{R}$ on a hyperhermitian manifold (M, I, J, K, g) is called *quaternionic* Ω*-plurisubharmonic* if

$$\Omega_\varphi := \Omega + \partial\partial_J\varphi\,,$$

is positive.

Animated by the study of "canonical" HKT metrics, in analogy to the Calabi conjecture [67] proved by Yau in [327], Alesker and Verbitsky proposed in [18] to study the *quaternionic Monge-Ampère equation*:

$$\Omega_\varphi^n = b\,\mathrm{e}^H\Omega_0^n \tag{6.1}$$

on a compact HKT manifold, where $H \in C^\infty(M,\mathbb{R})$ is given, while $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}_+$ is the unknown.

Following the parallelism between Hermitian and hyperhermitian geometry it is quite natural to enlarge the study of the quaternionic Monge-Ampère equation to a general set of fully non-linear elliptic equations on hypercomplex manifolds. Here we adapt the description given by Székelyhidi in [280] to the hypercomplex setting.

On a *locally flat* hypercomplex manifold (M, I, J, K), we can locally regard every q-real $(2,0)$-form Ω on M as a hyperhermitian matrix $(\Omega_{\bar r s})$, i.e. as a $n \times n$ quaternionic matrix lying in $\mathrm{Hyp}(n)$. Moreover, for a smooth real-valued function φ on M, the matrix associated to $\Omega_\varphi = \Omega + \partial\partial_J\varphi$ is $(\Omega^\varphi_{\bar r s}) = (\Omega_{\bar r s} + \frac14 \partial_{\bar q^r}\partial_{q^s}\varphi)$. The matrix $\mathrm{Hess}_{\mathbb{H}}\varphi := (\varphi_{\bar r s}) = (\frac14 \partial_{\bar q^r}\partial_{q^s}\varphi)$ is usually called the *quaternionic Hessian* of φ.

Now we can describe the class of equations we take into account in the present paper.
Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold and let Ω be a fixed q-real

$(2,0)$-form on M (Ω is not necessarily the $(2,0)$-form induced by g). For a smooth real function φ on M let $\Omega_\varphi := \Omega + \partial\partial_J\varphi$ and $A^r_s = g^{\bar{j}r}\Omega^\varphi_{\bar{j}s}$. The matrix (A^r_s) defines a hyperhermitian endomorphism of TM with respect to the metric g, i.e. $A = g^{-1}A^*g$. Note that in general, for quaternionic matrices one does not have (right) eigenvalues in the usual sense, rather conjugacy classes of them. However for hyperhermitian matrices there is a single real eigenvalue in each conjugacy class. Therefore, we consider the function $\lambda\colon \mathrm{Hyp}(n) \to \mathbb{R}^n$ which associates to a matrix A the n-tuple of its eigenvalues $\lambda(A)$.

We can then consider an equation of the following type

$$F(A) = h\,, \tag{6.2}$$

where $h \in C^\infty(M,\mathbb{R})$ is given and $F(A) = f(\lambda(A))$ is a smooth symmetric operator of the eigenvalues of A. Here $f\colon \Gamma \to \mathbb{R}$, where Γ is a proper convex open cone in $\mathbb{R}^n$ with vertex at the origin which is symmetric (i.e. it is invariant under permutations of the λ_i's) and contains the positive orthant

$$\Gamma_n = \{\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mid \lambda_i > 0,\ i = 1,\dots,n\}\,.$$

We further require that $f\colon \Gamma \to \mathbb{R}$ satisfies the following assumptions:

C1) $f_i := \frac{\partial f}{\partial \lambda_i} > 0$ for all $i = 1,\dots,n$ and f is a concave function.

C2) $\sup_{\partial\Gamma} f < \inf_M h$, where $\sup_{\partial\Gamma} f = \sup_{\lambda_0\in\partial\Gamma}\limsup_{\lambda\to\lambda_0} f(\lambda)$.

C3) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$ we have $\lim_{t\to\infty} f(t\lambda) > \sigma$.

Assumption C1 ensures that equation (6.2) is elliptic when φ is Γ-*admissible*, i.e.

$$\lambda\left(g^{\bar{k}r}(\Omega_{\bar{k}s} + \varphi_{\bar{k}s})\right) \in \Gamma\,.$$

Assumption C2 says that the level sets of f never touch the boundary of Γ, which also ensures that (6.2) is non-degenerate and then uniformly elliptic once we have established the C^2 estimate.

An analogue framework was firstly considered by Caffarelli, Nirenberg and Spruck [66] in $\mathbb{R}^n$ and later by Li [210], Urbas [296], Guan [152, 153] and Guan and Jiao [154] on Riemannian manifolds. Székelyhidi [280] studied this framework in Hermitian Geometry for elliptic equations and Phong and Tô [246] for parabolic equations. Székelyhidi's work has been recently generalized in [92, 179] to the almost Hermitian setting.

Statement of the main result.

Our main result is the following:

Theorem 6.1. *Let (M,I,J,K,g) be a compact flat hyperkähler manifold, Ω a q-real $(2,0)$-form, and $\underline{\varphi}$ a $\mathcal{C}$-subsolution of (6.2). Then there exist $\alpha \in (0,1)$ and a constant $C > 0$, depending only on (M,I,J,K,g), Ω, h and $\underline{\varphi}$, such that any Γ-admissible solution φ to (6.2) with $\sup_M \varphi = 0$ satisfies the estimate*

$$\|\varphi\|_{C^{2,\alpha}} \le C\,.$$

In the above statement the notion of subsolution is the following:

Definition 6.2. A function $\varphi \in C^2(M,\mathbb{R})$ is a $\mathcal{C}$-subsolution of (6.2) if for every $x \in M$ the set

$$\left(\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \underline{\varphi}_{\bar{j}s})\right) + \Gamma_n\right) \cap \partial\Gamma^{h(x)}$$

is bounded, where for any $\sigma > \sup_{\partial\Gamma} f$, Γ^σ denotes the convex superlevel set $\Gamma^\sigma = \{\lambda \in \Gamma \mid f(\lambda) > \sigma\}$.

We remark that the assumption of admitting a flat hyperkähler metric in particular implies that (M,I,J,K) is locally flat.

6.2 A priori estimates.

6.2.1 C^0-estimate.

The C^0-estimate for solutions to (6.2) is obtained by adapting [280, Proposition 10] to our setting and by using the ABP method.

As a preliminary step, we prove an L^p-estimate. From here on, we will always denote with C a positive constant that only depends on background data and which may change from line to line.
It will be useful to observe that the domain Γ of f satisfies

$$\Gamma \subseteq \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i > 0 \right\}. \tag{6.3}$$

From (6.3) we have $\operatorname{Re}\operatorname{tr}_g(\Omega_\varphi) > 0$, where $\Omega_\varphi = \Omega + \partial\partial_J\varphi$, which in turn translates into a lower bound for the quaternionic Laplacian of φ:

$$\Delta_g\varphi = \operatorname{Re}\operatorname{tr}_g(\Omega_\varphi) - \operatorname{Re}\operatorname{tr}_g(\Omega) \geq -C\,.$$

The next lemma gives us the desired L^p estimate:

Lemma 6.3. *Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. If φ satisfies*

$$\Delta_g\varphi \geq C' \tag{6.4}$$

for some (not necessarily positive) constant C', then there exist $p, C > 0$, depending only on the background data, such that

$$\|\varphi - \sup_M \varphi\|_{L^p} \leq C\,.$$

Proof. Suppose for simplicity $\sup_M \varphi = 0$. An L^1-bound for φ can be obtained by using the Green operator as in [15]. We give here some details for convenience of the reader. By a quaternionic version of Gauduchon theorem [15, Proposition 2.2], there exists a q-positive $(2n, 0)$-form Θ (which might not be holomorphic) such that $\partial\partial_J\left(\Omega_0^{n-1} \wedge \bar\Theta\right) = 0$. In addition, we may normalize Θ so that $\int_M \Omega_0^n \wedge \bar\Theta = 1$. By [15, Lemma 23], the quaternionic Laplacian admits a non-negative Green function $G(p, q) \geq 0$, namely, for each function u of class C^2 and each point $p \in M$,

$$-\int_{q\in M} G(p, q)\Delta_g u(q)\, \Omega_0^n \wedge \bar\Theta = u(p) - \int_M u\, \Omega_0^n \wedge \bar\Theta\,.$$

Choose a point $p \in M$ such that φ attains its maximum at p. Since we assumed $\sup_M \varphi = 0$ we have

$$\|\varphi\|_{L^1} = \int_M (-\varphi)\, \Omega_0^n \wedge \bar\Theta = -\int_{q\in M} G(p, q)\Delta_g\varphi(q)\, \Omega_0^n \wedge \bar\Theta \leq C \int_{q\in M} G(p, q)\, \Omega_0^n \wedge \bar\Theta \leq C\,.$$

Alternatively an L^p-bound can be obtained by using the weak Harnack inequality as follows. Take an open cover of M made of coordinate balls $B_{2r_i}(x_i)$ such that $\{B_i = B_{r_i}(x_i)\}$ still covers M. Since φ is non-positive and it satisfies the elliptic inequality (6.4), the weak Harnack inequality (Theorem 2.57) implies

$$\|\varphi\|_{L^p(B_i)} = \left(\int_{B_i} (-\varphi)^p\right)^{1/p} \leq C\left(\inf_{B_i}(-\varphi) + 1\right)$$

where $p, C > 0$ depend only on the cover and the background metric. Since $\sup_M \varphi = 0$ there is at least one index j such that $\inf_{B_j}(-\varphi) = -\sup_{B_j}\varphi = 0$, and thus $\|\varphi\|_{L^p(B_j)} \leq C$. This bound can be extended to all balls B_i such that $B_i \cap B_j \neq \emptyset$, indeed the estimate on $\|\varphi\|_{L^p(B_j)}$ yields an upper bound for $\inf_{B_i}(-\varphi)$ as

$$\inf_{B_i}(-\varphi) \leq \inf_{B_i\cap B_j}(-\varphi) \leq \frac{1}{\operatorname{Vol}(B_i \cap B_j)^{1/p}}\|\varphi\|_{L^p(B_i\cap B_j)} \leq \frac{1}{\operatorname{Vol}(B_i \cap B_j)^{1/p}}\|\varphi\|_{L^p(B_j)}\,.$$

We can now reiterate the argument and in a finite number of steps we will have bound $\|\varphi\|_{L^p(B_i)}$ for each i, and thus also $\|\varphi\|_{L^p(M)}$. □

Proposition 6.4. *Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. If $\underline{\varphi}, \varphi$ are a $\mathcal{C}$-subsolution and a solution to (6.2) respectively, with $\sup_M \varphi = 0$, then there is a constant $C > 0$, depending only on the background data and the subsolution $\underline{\varphi}$, such that*

$$\|\varphi\|_{C^0} \leq C\,.$$

Proof. Without loss of generality we may assume that $\underline{\varphi} \equiv 0$, otherwise we could modify Ω to simplify the equation. Since $\sup_M \varphi = 0$, we only need to bound $S = \inf_M \varphi$ from below. For convenience, we may assume $S \leq -1$, otherwise we are done.

Since $\underline{\varphi}$ is a $\mathcal{C}$-subsolution there exist $\delta, R > 0$ such that

$$\left(\lambda\left(g^{\bar{j}r}\Omega_{\bar{j}s}\right) - \delta\mathbf{1} + \Gamma_n\right) \cap \partial\Gamma^{h(x)} \subseteq B_R(0)\,, \qquad \text{at every } x \in M\,, \tag{6.5}$$

where $\mathbf{1} = (1, 1, \dots, 1)$.

Consider quaternionic local coordinates $(q^1, \dots, q^n)$ centered at the point where φ attains its minimum S. We may identify such coordinate neighborhood with the open ball of unit radius $B_1 = B_1(0) \subseteq \mathbb{H}^n$ centered at the origin. Let $v(x) = \varphi(x) + \varepsilon|x|^2$ be defined on B_1 for some small fixed $\varepsilon > 0$. Observe that $\inf_{B_1} v = v(0) = \varphi(0) = S$ and $\inf_{\partial B_1} v \geq v(0) + \varepsilon$. These conditions allow us to apply the ABP method (see Proposition 2.56) to obtain

$$C_0\varepsilon^{4n} \leq \int_P \det(D^2 v)\,, \tag{6.6}$$

where $C_0 > 0$ is a dimensional constant,

$$P = \left\{x \in B_1 \mid |Dv(x)| < \frac{\varepsilon}{2},\, v(y) \geq v(x) + Dv(x)\cdot(y - x) \text{ for all } y \in B_1\right\}\,,$$

and Dv, D^2v are the gradient and the (real) Hessian of v. Note that $P \subseteq \{x \in B_1 \mid D^2v(x) \geq 0\}$ and since convexity implies quaternionic plurisubharmonicity (see e.g. [9]), at any point $x \in P$ we have $\mathrm{Hess}_{\mathbb{H}} v(x) \geq 0$. Therefore $\mathrm{Hess}_{\mathbb{H}}\varphi(x) \geq -\frac{\varepsilon}{2}\mathbb{1}$, where $\mathbb{1}$ is the $n \times n$ identity matrix. Choosing ε small enough depending on g and δ, we have

$$\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \lambda\left(g^{\bar{j}r}\Omega_{\bar{j}s}\right) - \delta\mathbf{1} + \Gamma_n\,, \qquad \text{at every } x \in P\,.$$

On the other hand, equation (6.2) also gives

$$\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \partial\Gamma^{h(x)}\,, \qquad \text{at every } x \in P\,.$$

These two facts, together with (6.5) imply $|\varphi_{\bar{r}s}| \leq C$ on P and thus also $v_{\bar{r}s} \leq C$. Combining a calculation in [43] with [270, Lemma 2], or alternatively using directly a computation in the proof of [15, Proposition 2.1], at any point $x \in P$ we have

$$\det(D^2 v) \leq 2^{4n}\det(\mathrm{Hess}_{\mathbb{H}}(v))^4\,,$$

where, on the right-hand side, “det” denotes the Moore determinant. Therefore, from (6.6) we see that

$$C_0\varepsilon^{4n} \leq C\mathrm{Vol}(P)\,.$$

The definition of P entails that $v(0) \geq v(x) - Dv(x)\cdot x > v(x) - \varepsilon/2$, i.e. $v(x) < S + \varepsilon/2 < 0$ for all $x \in P$. As a consequence for any $p > 0$

$$\|v\|^p_{L^p(M)} \geq \|v\|^p_{L^p(P)} = \int_P (-v)^p \geq \left|S + \frac{\varepsilon}{2}\right|^p \mathrm{Vol}(P)\,.$$

From the previous lemma we know that there is a $p > 0$ such that $\|v\|_{L^p}$ is bounded, therefore also $S = \inf_M \varphi$ must be bounded. □

6.2.2 Laplacian estimate.

This section is devoted to derive a C^0-estimate for the quaternionic Laplacian of solutions to (6.2) in terms of the squared norm of the gradient. This step is the most involved in terms of calculations and it is here that we use our strongest assumptions to have a locally flat hypercomplex structure and a hyperkähler metric compatible with it.

We follow Székelyhidi [280] and Hou-Ma-Wu [176], which in turn is based on an idea of Chou and Wang [89] for the real Hessian equation. Our restrictive assumptions simplify quite a bit the computations.

Preliminary results.

As declared in the overview, let $F(A) = f(\lambda(A))$ be a symmetric function of the eigenvalues of $A_{rs} = g^{\bar{j}r}\Omega^{\varphi}_{\bar{j}s} = g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})$. We denote the derivatives of F by

$$F^{rs} = \frac{\partial F}{\partial A_{rs}}, \qquad F^{rs,lt} = \frac{\partial^2 F}{\partial A_{rs} \partial A_{lt}}.$$

Let Q_{rs} be the standard quaternionic coordinates on $\mathbb{H}^{n,n}$ and let E^p_{rs} be the real coordinates underlying Q_{rs}, i.e. $Q_{rs} = E^0_{rs} + E^1_{rs}i + E^2_{rs}j + E^3_{rs}k$. We have the following:

Lemma 6.5. *The linearization of F at φ is the operator*

$$L(\psi) = \operatorname{Re} \sum_{r,s=1}^{n} F^{rs} g^{\bar{j}r} \psi_{\bar{j}s}\,.$$

Proof. With respect to the real coordinates E^p_{rs} we decompose a matrix $A \in \mathbb{H}^{n,n}$ as $A^{rs}_p E^p_{rs}$. Define the derivatives $F^{rs}_p := \frac{\partial F}{\partial A^{rs}_p}$ and the matrix $H = (F^{rs})$. For a curve of hyperhermitian matrices A_t with respect to g we have

$$\frac{d}{dt} F(A_t) = \sum_{r,s=}^{n} \sum_{p=0}^{3} F^{rs}_p(A_t)(A'_t)^{rs}_p = \operatorname{Re} F^{rs}(A_t)(A'_t)_{rs}$$

Now, for each $\psi \in C^2(M, \mathbb{R})$ and $t \in (-\varepsilon, \varepsilon)$, let $\varphi(t)$ be a curve of Γ-admissible functions in $C^2(M, \mathbb{R})$ with $\varphi(0) = \varphi$ and $\varphi'(0) = \psi$ and set $A_t = g^{-1}\left(\Omega + \operatorname{Hess}_{\mathbb{H}} \varphi(t)\right)$, then

$$L(\psi) = \frac{d}{dt} F(A_t)\big|_{t=0} = \operatorname{Re} F^{rs}(A_0)(A'_0)_{rs} = \operatorname{Re} \sum_{r,s=1}^{n} F^{rs}(A_0) g^{\bar{j}r} \psi_{\bar{j}s}\,. \qquad \square$$

In order to prove the desired bound we will need the following preliminary lemma.

Lemma 6.6. *Let $\sup_{\partial\Gamma} f < a < b < \sup_{\Gamma} f$ and $\delta, R > 0$. Then there exists a constant $\kappa > 0$ such that for any $\sigma \in [a, b]$, $B \in \operatorname{Hyp}(n, \mathbb{H})$ satisfying*

$$(\lambda(B) - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^{\sigma} \subseteq B_R(0)\,,$$

$A \in \mathrm{Hyp}(n)$ satisfying $\lambda(A) \in \partial\Gamma^\sigma$ and $|\lambda(A)| > R$, we have

$$\text{either} \qquad \mathrm{Re}\, F^{rs}(A)\,(B_{rs} - A_{rs}) > \kappa \sum_{r=1}^{n} F^{rr}(A)\,,$$

$$\text{or} \qquad F^{ss}(A) > \kappa \sum_{r=1}^{n} F^{rr}(A)\,, \qquad \text{for all } s = 1, \dots, n\,.$$

Proof. The lemma follows from the very same argument as [280, Proposition 6] together with the quaternionic analogue of the Schur-Horn theorem (Proposition 1.46). □

The main result of this section is the following:

Proposition 6.7. *Let (M, I, J, K, g) be a compact flat hyperkähler manifold. If $\underline{\varphi}, \varphi$ are a $\mathcal{C}$-subsolution and a solution to (6.2) respectively, then there is a constant $C > 0$, depending only on (M, I, J, K), $\|g\|_{C^2}$, $\|h\|_{C^2}$, $\|\Omega\|_{C^2}$, $\|\varphi\|_{C^0}$ and $\underline{\varphi}$, such that*

$$\|\Delta_g \varphi\|_{C^0} \leq C\left(\|\nabla\varphi\|^2_{C^0} + 1\right)\,.$$

Here ∇ denotes the Obata connection on M.

Perturbation of A.

We observe that at a point where A is diagonal with distinct eigenvalues we have

- $\lambda_i^{rs} := \frac{\partial \lambda_i}{\partial A_{rs}} = \delta_{ir}\delta_{is}$,
- $\lambda_i^{rs,tl} := \frac{\partial^2 \lambda_i}{\partial A_{rs}\partial A_{tl}} = (1-\delta_{ir})\frac{\delta_{is}\delta_{it}\delta_{rl}}{\lambda_i - \lambda_r} + (1-\delta_{it})\frac{\delta_{il}\delta_{ir}\delta_{st}}{\lambda_i - \lambda_t}$

(see e.g. [141, 268]). Furthermore, since $F(A) = f(\lambda(A))$ for f symmetric, then $F^{rs} = \delta_{rs} f_r$, and since f is concave and satisfies $f_i > 0$ (assumption C1 in the overview), then F is concave and $\frac{f_r - f_s}{\lambda_r - \lambda_s} \leq 0$. In particular $f_r \geq f_s$ anytime $\lambda_r \leq \lambda_s$. Finally, we observe that by [280, Lemma 9 (b)] for any fixed $x \in M$ there is a constant $\tau > 0$ depending on $h(x)$ such that

$$\sum_{a=1}^{n} F^{aa}(x) > \tau > 0\,. \tag{6.7}$$

We will mainly be interested in the largest eigenvalue λ_1 of the matrix A around some fixed point x_0. As pointed out by Székelyhidi [280] in order for $\lambda_1 \colon M \to \mathbb{R}$ to define a smooth function at x_0 we need the eigenvalues to be distinct; to be sure of that, we perturb the matrix A.

At any fixed point $x_0 \in M$ we can perturb A in order to have a matrix with distinct eigenvalues. Indeed, fix quaternionic local coordinates around the point x_0 such that, at x_0, A is diagonal and its eigenvalues satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\,; \tag{6.8}$$

take a constant diagonal matrix D whose entries satisfy

$$0 = D_{11} < D_{22} < \cdots < D_{nn}\,.$$

The matrix $\tilde{A} = A - D$ has, at x_0, the eigenvalues

$$\tilde{\lambda}_1 = \lambda_1\,, \qquad \tilde{\lambda}_i = \lambda_i - D_{ii}\,,\ \text{for } i = 2, \dots, n\,,$$

which are distinct by construction.

C^0-estimate for the Laplacian.

We will make use of the linearized operator L defined by $L(u) = 4\mathrm{Re}\sum_{a,b=1}^n F^{ab} g^{\bar c a} u_{\bar c b}$, where $u_{\bar c b} = \frac{1}{4}\partial_{\bar q^c}\partial_{q^b} u$. First of all, we prove the following inequality for $L\left(2\sqrt{\tilde\lambda_1}\right)$.

Lemma 6.8. *With respect to quaternionic local coordinates around x_0 such that $(g_{\bar r s})$ is the identity at x_0 and $(\Omega^\varphi_{\bar r s})$ is diagonal at x_0, we have*

$$L\left(2\sqrt{\tilde\lambda_1}\right) \geq -\frac{F^{aa}|\Omega^\varphi_{\bar 1 1,a}|^2}{2\lambda_1\sqrt{\lambda_1}} - \frac{C\mathcal{F}}{\sqrt{\lambda_1}}\,,$$

where $\mathcal{F} = \sum_{a=1}^n F^{aa}(x_0)$, $\Omega^\varphi_{\bar 1 1,a} = \partial_{q^a}\Omega^\varphi_{\bar 1 1}$ and $C > 0$ is a positive constant depending only on (M,I,J,K), $\|\Omega\|_{C^2}$ and $\|h\|_{C^2}$.

Proof. We have for the perturbed matrix $\tilde A_{rs} = A_{rs} - D_{rr}\delta_{rs} = g^{\bar j r}\Omega^\varphi_{\bar j s} - D_{rr}\delta_{rs}$ at the point x_0 where $(g_{\bar r s})$ is the identity and A (and thus (F^{rs})) is diagonal

$$L\left(2\sqrt{\tilde\lambda_1}\right) = 8\mathrm{Re}\,F^{ab}\left(\sqrt{\tilde\lambda_1}\right)_{\bar a b} = 2F^{aa}\sum_{p=0}^{3}\left(\sqrt{\tilde\lambda_1}\right)_{x^a_p x^a_p} = F^{aa}\sum_{p=0}^{3}\left(\frac{\tilde\lambda_{1,x^a_p x^a_p}}{\sqrt{\lambda_1}} - \frac{\tilde\lambda^2_{1,x^a_p}}{2\lambda_1\sqrt{\lambda_1}}\right), \tag{6.9}$$

where the subscript x^a_p denotes the real derivative with respect to the corresponding coordinate. Using the formulas for the derivatives of the eigenvalues we obtain at x_0

$$\begin{aligned}
\tilde\lambda_{1,x^a_p} &= \tilde\lambda_1^{rs}\tilde A_{rs,x^a_p} = \Omega^\varphi_{\bar 1 1,x^a_p}\\
\tilde\lambda_{1,x^a_p x^a_p} &= \tilde\lambda_1^{rs,lt}\tilde A_{rs,x^a_p}\tilde A_{lt,x^a_p} + \tilde\lambda_1^{rs}\tilde A_{rs,x^a_p x^a_p} = \sum_{r>1}\frac{\tilde A_{r1,x^a_p}\tilde A_{1r,x^a_p} + \tilde A_{1r,x^a_p}\tilde A_{r1,x^a_p}}{\lambda_1 - \tilde\lambda_r} + \Omega^\varphi_{\bar 1 1,x^a_p x^a_p}\\
&= \sum_{r>1}\frac{A_{r1,x^a_p}A_{1r,x^a_p} + A_{1r,x^a_p}A_{r1,x^a_p}}{\lambda_1 - \tilde\lambda_r} + g^{\bar j 1}\Omega^\varphi_{\bar j 1,x^a_p x^a_p} = 2\sum_{r>1}\frac{|\Omega^\varphi_{\bar r 1,x^a_p}|^2}{\lambda_1 - \tilde\lambda_r} + \Omega^\varphi_{\bar 1 1,x^a_p x^a_p}\,,
\end{aligned}$$

where we used that the derivatives of D vanish because it is a constant matrix.

Differentiating the equation $F(A) = h$ twice with respect to x^1_p gives, at the point x_0,

$$\mathrm{Re}\,F^{rs,tl}\Omega^\varphi_{\bar s r,x^1_p}\Omega^\varphi_{\bar l t,x^1_p} + F^{rr}\Omega^\varphi_{\bar r r,x^1_p x^1_p} = h_{x^1_p x^1_p}\,. \tag{6.10}$$

We observe that

$$\sum_{p=0}^{3}\Omega^\varphi_{\bar 1 1,x^a_p x^a_p} = \sum_{p=0}^{3}\left(\Omega_{\bar 1 1,x^a_p x^a_p} + \varphi_{\bar 1 1 x^a_p x^a_p}\right) = 4\Omega_{\bar 1 1,\bar a a} + 4\varphi_{\bar a a \bar 1 1} = 4\Omega_{\bar 1 1,\bar a a} - 4\Omega_{\bar a a,\bar 1 1} + \sum_{p=0}^{3}\Omega^\varphi_{\bar a a,x^1_p x^1_p}$$

and thus, by (6.10) and (6.7)

$$F^{aa}\sum_{p=0}^{3}\tilde\lambda_{1,x^a_p x^a_p} \geq F^{aa}\sum_{p=0}^{3}\Omega^\varphi_{\bar 1 1,x^a_p x^a_p} \geq -\mathrm{Re}\,F^{rs,tl}\sum_{p=0}^{3}\Omega^\varphi_{\bar r s,x^1_p}\Omega^\varphi_{\bar t l,x^1_p} - C\mathcal{F} \geq -C\mathcal{F}$$

where we also used the concavity of F. Finally from (6.9) we have the desired inequality

$$L\left(2\sqrt{\tilde\lambda_1}\right) \geq -\frac{F^{aa}\sum_{p=0}^{3}(\Omega^\varphi_{\bar 1 1,x^a_p})^2}{2\lambda_1\sqrt{\lambda_1}} - \frac{C\mathcal{F}}{\sqrt{\lambda_1}}\,. \qquad \square$$

Proof of Proposition 6.7. We have already seen that the Laplacian is bounded from below, as a consequence of (6.3), therefore it is enough to obtain a bound of the form

$$\frac{\lambda_1}{\|\nabla\varphi\|^2_{C^0} + 1} \leq C\,.$$

Define the function
$$G = 2\sqrt{\tilde\lambda_1} + \alpha(|\nabla\varphi|^2) + \beta(\varphi)\,,$$
where
$$\alpha(t) = -\frac{1}{2}\log\left(1 - \frac{t}{2N}\right)\,, \qquad N = \|\nabla\varphi\|^2_{C^0} + 1\,,$$
$$\beta(t) = -2St + \frac{1}{2}t^2\,, \qquad S > \|\varphi\|_{C^0}\,,\ \text{large constant to be chosen later}\,,$$
and $\tilde\lambda_1$ is, as before, the highest eigenvalue of the perturbed matrix $\tilde A$ around a point x_0, which we choose to be a maximum point of G. The derivative of the functions α and β satisfy
$$\frac{1}{4N} < \alpha'(|\nabla\varphi|^2) < \frac{1}{2N}\,, \qquad \alpha'' = 2(\alpha')^2\,, \tag{6.11}$$
$$S \leq -\beta'(\varphi) \leq 3S\,, \qquad \beta'' = 1\,. \tag{6.12}$$

At x_0 we have $L(G) \leq 0$. Choose quaternionic local coordinates such that $(g_{\bar r s})$ is the identity in the whole neighborhood of x_0 and $(\Omega^\varphi_{\bar r s})$ is diagonal at x_0. This is possible because we are assuming g hyperkähler and flat. Then
$$0 \geq 4\mathrm{Re}\, F^{ab}G_{\bar a b} = 4F^{aa}G_{\bar a a} = F^{aa}\sum_{p=0}^{3} G_{x_p^a x_p^a}\,, \tag{6.13}$$
because F^{ab} is diagonal at x_0. We compute the derivatives of G at x_0:
$$0 = G_{x_p^a} = \left(2\sqrt{\tilde\lambda_1}\right)_{x_p^a} + \alpha'\sum_{r=1}^{n}(\varphi_{\bar r x_p^a}\varphi_r + \varphi_{\bar r}\varphi_{r x_p^a}) + \beta'\varphi_{x_p^a}\,,$$
$$G_{x_p^a x_p^a} = \left(2\sqrt{\tilde\lambda_1}\right)_{x_p^a x_p^a} + \alpha''\left(\sum_{r=1}^{n}(\varphi_{\bar r x_p^a}\varphi_r + \varphi_{\bar r}\varphi_{r x_p^a})\right)^2$$
$$+ \alpha'\sum_{r=1}^{n}(\varphi_{\bar r x_p^a x_p^a}\varphi_r + 2|\varphi_{r x_p^a}|^2 + \varphi_{\bar r}\varphi_{r x_p^a x_p^a}) + \beta''\varphi^2_{x_p^a} + \beta'\varphi_{x_p^a x_p^a}\,.$$
Differentiating the equation $F(A) = h$ yields
$$F^{aa}\Omega^\varphi_{\bar a a, x_p^r} = h_{x_p^r}\,, \qquad \text{at } x_0.$$
Using this, Cauchy-Schwarz inequality and (6.11) we have
$$\begin{aligned}\alpha' F^{aa}\sum_{r=1}^{n}(\varphi_{\bar r\bar a a}\varphi_r + \varphi_{\bar r}\varphi_{r\bar a a}) &= \alpha' F^{aa}\sum_{r=1}^{n}(\varphi_{\bar a a\bar r}\varphi_r + \varphi_{\bar r}\varphi_{\bar a a r})\\ &= \alpha'\sum_{r=1}^{n}\left((h_{\bar r} - F^{aa}\Omega_{\bar a a,\bar r})\varphi_r + \varphi_{\bar r}(h_r - F^{aa}\Omega_{\bar a a, r})\right)\\ &\geq -\frac{C}{N}(N^{1/2} + N^{1/2}\mathcal{F}) \geq -C\mathcal{F}\,,\end{aligned}$$
where we used (6.7) to absorb the constants into $C\mathcal{F}$. Again using (6.11) we also obtain
$$\begin{aligned}2\alpha' F^{aa}\sum_{r=1}^{n}\sum_{p=0}^{3}|\varphi_{r x_p^a}|^2 &\geq \frac{1}{2N}F^{aa}\sum_{r=1}^{n}\sum_{p,q=0}^{3}\varphi^2_{x_q^r x_p^a} \geq \frac{1}{2N}F^{aa}\sum_{p=0}^{3}\varphi^2_{x_p^a x_p^a} = \frac{8}{N}F^{aa}\varphi^2_{\bar a a}\\ &= \frac{8}{N}F^{aa}(\lambda_a - \Omega_{\bar a a})^2 \geq \frac{2}{N}F^{aa}\lambda_a^2 - C\mathcal{F}\,,\end{aligned}$$

where, for the last inequality we used that $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$. Thanks to the last two inequalities, from our main inequality (6.13) we get

$$0 \geq L\left(2\sqrt{\tilde{\lambda}_1}\right) + \alpha'' F^{aa} \sum_{p=0}^{3} \left(2\sum_{r=1}^{n} \mathrm{Re}(\varphi_{\bar{r}x_p^a}\varphi_r)\right)^2 + \beta'' F^{aa}|\varphi_a|^2 + 4\beta' F^{aa}\varphi_{\bar{a}a} + \frac{2F^{aa}\lambda_a^2}{N} - C\mathcal{F}\,. \quad (6.14)$$

By $G_{x_p^a}(x_0) = 0$ we have

$$\begin{aligned} \alpha'' F^{aa}\left(2\sum_{r=1}^{n}\mathrm{Re}(\varphi_{\bar{r}x_p^a}\varphi_r)\right)^2 &= 2F^{aa}\left(\frac{\Omega^{\varphi}_{\bar{1}1,x_p^a}}{\sqrt{\lambda_1}} + \beta'\varphi_{x_p^a}\right)^2 \\ &\geq 2\varepsilon \frac{F^{aa}(\Omega^{\varphi}_{\bar{1}1,x_p^a})^2}{\lambda_1} - \frac{2\varepsilon}{1-\varepsilon}(\beta')^2 F^{aa}\varphi^2_{x_p^a}\,, \end{aligned} \quad (6.15)$$

where we used the inequality $(a+b)^2 \geq \varepsilon a^2 - \frac{\varepsilon}{1-\varepsilon}b^2$, which holds for $\varepsilon \in (0,1)$. Summing (6.15) over p and combining it with Lemma 6.8 we obtain from (6.14)

$$0 \geq \left(4\varepsilon\sqrt{\lambda_1} - 1\right)\frac{F^{aa}|\Omega^{\varphi}_{\bar{1}1,a}|^2}{2\lambda_1\sqrt{\lambda_1}} + \left(\beta'' - \frac{2\varepsilon(\beta')^2}{1-\varepsilon}\right)F^{aa}|\varphi_a|^2 + 4\beta' F^{aa}\varphi_{\bar{a}a} + \frac{2F^{aa}\lambda_a^2}{N} - C\mathcal{F}\,. \quad (6.16)$$

Choosing $\varepsilon = 1/(18S^2+1) < 1$, (6.12) implies

$$\beta'' - \frac{2\varepsilon}{1-\varepsilon}(\beta')^2 \geq 0\,.$$

Furthermore, we can assume without loss of generality $\sqrt{\lambda_1} > \frac{1}{4\varepsilon}$ and deduce

$$\left(4\varepsilon\sqrt{\lambda_1} - 1\right)\frac{F^{aa}|\Omega^{\varphi}_{\bar{1}1,a}|^2}{2\lambda_1\sqrt{\lambda_1}} \geq 0\,.$$

Then we obtain from (6.16)

$$0 \geq 4\beta' F^{aa}\varphi_{\bar{a}a} + \frac{2F^{aa}\lambda_a^2}{N} - C\mathcal{F}\,. \quad (6.17)$$

As before, we can assume $\underline{\varphi} \equiv 0$, otherwise we could choose a suitable background form Ω in order to simplify the equation. Set $B_{rs} = g^{\bar{j}r}\Omega_{\bar{j}s}$ and let $\delta, R > 0$ be such that

$$(\lambda(B) - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^{h(x)} \subseteq B_R(0)\,, \qquad \text{at every } x \in M\,,$$

which exist because of the definition of $\mathcal{C}$-subsolution. Supposing $\lambda_1 > R$ we have $|\lambda(A)| > R$ and we can then apply Lemma 6.6 according to which there exists $\kappa > 0$ such that one of the following two cases occur:

- First case:
$$\mathrm{Re}\,F^{rs}(A)(B_{rs} - A_{rs}) = -\mathrm{Re}\sum_{r,s=1}^{n} F^{rs}(A)g^{\bar{j}r}\varphi_{\bar{j}s} > \kappa\sum_{r=1}^{n}F^{rr}(A)\,,$$
i.e. $-F^{aa}\varphi_{\bar{a}a} > \kappa\mathcal{F}$ at x_0, which for a choice of S large enough implies $4\beta' F^{aa}\varphi_{\bar{a}a} - C\mathcal{F} \geq 0$ allowing us to deduce from (6.17) $0 \geq \frac{2}{N}F^{aa}\lambda_a^2$ which is a contradiction.

- Second case:
$$F^{ss}(A) > \kappa\sum_{r=1}^{n}F^{rr}(A)\,, \qquad \text{for all } s = 1,\dots,n\,,$$
and in particular $F^{11} > \kappa\mathcal{F}$. Therefore $F^{aa}\lambda_a^2 \geq F^{11}\lambda_1^2 \geq \kappa\mathcal{F}\lambda_1^2$. Moreover, we can assume $F^{aa}\lambda_a \leq F^{aa}\lambda_a^2/(12NS)$ for otherwise we would have $\kappa\mathcal{F}\lambda_1^2 < 12NS\mathcal{F}\lambda_1$ and we would conclude.

Then we have

$$4\beta' F^{aa}\varphi_{\bar{a}a} \geq -12SF^{aa}\lambda_a - C\mathcal{F} \geq -\frac{F^{aa}\lambda_a^2}{N} - C\mathcal{F}\,.$$

Substituting this last inequality into (6.17) we get

$$0 \geq \kappa\frac{\lambda_1^2}{N^2} - C\,.$$

This gives the bound we were searching for at the maximum point x_0 of G, but by monotony of the square root such bound holds globally, depending additionally on a bound for $\|\varphi\|_{C^0}$. □

Remark 6.9. Removing the hypothesis that the metric g is hyperkähler one has to deal with its derivatives. Most of the terms are not an issue and can be easily controlled, however those terms that contain the third derivative of φ seem not to be straightforwardly manageable.

Remark 6.10. The function G used in the proof of Proposition 6.7 is basically the same as the one used in [280], however we replaced the logarithm with the square root, a trick which is inspired by the work of Alesker [14]. It seems that using the square root allows to simplify the argument.

Remark 6.11. Under an additional assumption the Laplacian can be controlled linearly by the gradient. Indeed, if we further assume

$$F^{aa}\lambda_a \leq c_0\,, \tag{6.18}$$

which is the case for the quaternionic Monge-Ampère, the quaternionic Hessian, and the quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions, we obtain the following sharper estimate in the second case above, more precisely, from (6.17), $F^{11} > \kappa\mathcal{F}$ and (6.18) we get

$$\begin{aligned}0 &\geq 4\beta' F^{aa}(\lambda_a - 1) + \frac{2F^{11}\lambda_1^2}{N} - C\mathcal{F} \geq 4\beta' F^{aa}\lambda_a + \frac{2\kappa\lambda_1^2}{N}\mathcal{F} + (-4\beta' - C)\,\mathcal{F}\\ &\geq 4\beta' c_0 + \frac{2\kappa\lambda_1^2}{N}\mathcal{F} + (-4\beta' - C)\,\mathcal{F} \geq \frac{2\kappa\lambda_1^2}{N}\mathcal{F} + \left(-4\beta' - C + \frac{4\beta' c_0}{\tau}\right)\mathcal{F},\end{aligned}$$

where we have used $\mathcal{F} \geq \tau > 0$ in the last inequality. Then we have

$$0 \geq 2\kappa\frac{\lambda_1^2}{N} - \left(4\beta' + C - \frac{4\beta' c_0}{\tau}\right)\,,$$

which gives a sharper bound

$$\lambda_1 \leq C(1 + \|\nabla\varphi\|_{C^0})\,.$$

6.2.3 Gradient estimate.

In this section we show that a bound for the gradient of solutions to (6.2) can be obtained by using a Liouville-type theorem. We adapt the approach of Dinew and Kołodziej [103] to our setting. The blow-up argument was introduced in the setting of fully non-linear complex equations by Chen [80, Section 3.2]. See also [108, Proposition 8] for an earlier similar rescaling argument (we also mention its improvement in [255, Proposition 33]).

Blow-up analysis.

We introduce the following:

Definition 6.12. A continuous function $u\colon \mathbb{H}^n \to \mathbb{R}$ is a *(viscosity)* Γ*-subsolution (resp. supersolution)* if for all $\psi\colon \mathbb{H}^n \to \mathbb{R}$ of class C^2 such that $u - \psi$ has a local maximum (resp. minimum) at p, we have $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \bar{\Gamma}$ (resp. $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \mathbb{R}^n \setminus \Gamma$) at p. We say that u is a *(viscosity)* Γ*-solution* if it is both a subsolution and a supersolution.

We show that if the gradient bound for solutions to (6.2) does not hold, we are able to find a bounded $C^{1,\alpha}$ viscosity Γ-solution $u\colon \mathbb{H}^n \to \mathbb{R}$ with bounded gradient and such that $|\nabla u(0)| = 1$. In particular u is non-constant. In the next section we prove a Liouville-type theorem for this kind of functions, thus yielding a contradiction and showing implicitly that the gradient bound holds.

Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. Consider a sequence $(\underline{\varphi}_j)_j$, $(\varphi_j)_j$, $(h_j)_j$ of real smooth functions on M and a sequence $(\Omega_j)_j$ of q-real $(2,0)$-forms on M such that $\underline{\varphi}_j$ are $\mathcal{C}$-subsolutions and φ_j, h_j, Ω_j satisfy

$$\begin{cases} F\left(g^{\bar t r}((\Omega_j)_{\bar t s} + (\varphi_j)_{\bar t s})\right) = h_j\,, \\ \sup_M \varphi_j = 0\,, \\ \|\nabla \varphi_j\|_{C^0} \geq j\,. \end{cases} \tag{6.19}$$

Assume further that $(\underline{\varphi}_j)_j$, $(h_j)_j$ and $(\Omega_j)_j$ are uniformly bounded in C^2-norm.

Set $N_j = \|\nabla\varphi_j\|^2_{C^0}$, $g_j = N_j g$ and let $x_j \in M$ be such that $|\nabla\varphi_j(x_j)|^2 = N_j$ for each $j > 0$. Choose quaternionic local coordinates $(q^1, \dots, q^n)$ around x_j for $|q^i| < N_j^{1/2}$ such that

$$(g_j)_{\bar r s} = \delta_{\bar r s} + O(N_j^{-1}|x|)\,, \qquad (\Omega_j)_{\bar r s} = O(N_j^{-1})\,, \qquad h_j = h_j(x_j) + O(N_j^{-1}|x|)\,.$$

Then $|\nabla\varphi_j(x_j)|^2_{g_j} = 1$ and by Propositions 6.4 and 6.7 we have in this coordinates

$$\|\varphi_j\|_{C^0} \leq C\,, \qquad |\Delta_g \varphi_j|_{g_j} \leq C\,, \qquad \text{on } B_{N_j^{1/2}}(x_j)\,,$$

where $C > 0$ is uniform in j. It follows by [143, Theorem 8.32] that $(\varphi_j)_j$ is uniformly bounded in $C^{1,\alpha}$-norm for any $\alpha \in (0,1)$. Furhermore, letting $j \to \infty$, we see that Ω_j tends to zero, while g_j tends to the standard Euclidean metric and $(\varphi_j)_{\bar r s}$ stays bounded. Therefore

$$\lambda(A_j) = \lambda((\varphi_j)_{\bar r s}) + O(N_j^{-1}|x|)\,, \tag{6.20}$$

where $(A_j)^r_s = g_j^{\bar t r}((\Omega_j)_{\bar t s} + (\varphi_j)_{\bar t s})$.

By Ascoli-Arzelà Theorem we can extract from $(\varphi_j)_j$ a subsequence converging uniformly in $C^{1,\alpha}$ to some $u\colon \mathbb{H}^n \to \mathbb{R}$, moreover, such limiting function satisfies $\|u\|_{C^0} \leq C$, $\|\nabla u\|_{C^0} \leq C$ and $|\nabla u(0)| = 1$. We aim to prove that u is a Γ-solution.

Suppose there exists $\psi \in C^2$, such that $u - \psi$ has a local maximum at some point $p_0 \in \mathbb{H}^n$. By construction of u, for any $\varepsilon > 0$ there are a j large enough, $a \in (-\varepsilon, \varepsilon)$ and a point p_1 with $|p_1 - p_0| < \varepsilon$ such that $\varphi_j - \psi - \varepsilon|x - p_0|^2 + a$ has a local maximum at p_1. As a consequence the quaternionic Hessian of ψ satisfies

$$\mathrm{Hess}_{\mathbb{H}}\psi + \frac{\varepsilon}{2}\mathbb{1} \geq \mathrm{Hess}_{\mathbb{H}}\varphi_j\,, \qquad \text{at } p_1\,,$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. By (6.20), if j is large enough we see that $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \Gamma - \varepsilon\mathbf{1}$ at p_1 and letting $\varepsilon \to 0$ we deduce $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \bar\Gamma$ at p_0 because $p_1 \to p_0$. This shows that u is a viscosity Γ-subsolution.

To see that u is also a Γ-supersolution we proceed similarly. Suppose that $u - \psi$ has a local minimum at $p_0 \in \mathbb{H}^n$, then for any $\varepsilon > 0$ there are j large enough, $a \in (-\varepsilon, \varepsilon)$ and $p_1 \in \mathbb{H}^n$ such that $\varphi_j - \psi + \varepsilon|x - p_0|^2 + a$ has a local minimum at p_1. Hence

$$\mathrm{Hess}_{\mathbb{H}}\psi - \frac{\varepsilon}{2}\mathbb{1} \leq \mathrm{Hess}_{\mathbb{H}}\varphi_j\,, \qquad \text{at } p_1\,.$$

By contradiction, suppose $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi(p_1)) \in \Gamma + \frac{5}{2}\varepsilon\mathbf{1}$, then $\lambda(\mathrm{Hess}_{\mathbb{H}}\varphi_j(p_1)) \in \Gamma + 2\varepsilon\mathbf{1}$ and for j large enough (6.20) we have $\lambda(A_j) \in \Gamma + \varepsilon\mathbf{1}$. By [280, Lemma 9 (a)] it follows that for N_j large enough $\Gamma + N_j\varepsilon\mathbf{1} \subseteq \Gamma^{h_j(p_1)}$ and consequently we deduce

$$N_j\lambda(A_j) \in N_j\Gamma + N_j\varepsilon\mathbf{1} = \Gamma + N_j\varepsilon\mathbf{1} \subseteq \Gamma^{h_j(p_1)}$$

for j sufficiently large. On the other hand, φ_j satisfies (6.19), i.e.

$$N_j\lambda(A_j) = \lambda\left(g^{\bar{t}r}((\Omega_j)_{\bar{t}s} + (\varphi_j)_{\bar{t}s})\right) \in \partial\Gamma^{h_j(p_1)}\,,$$

which gives a contradiction. Therefore $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi(p_1)) \notin \Gamma + \frac{5}{2}\varepsilon\mathbf{1}$ and letting $\varepsilon \to 0$ we finally obtain $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi(p_0)) \notin \Gamma$ and u is a viscosity Γ-solution.

Liouville-type theorem.

As in Székelyhidi [280] we can interpret the notion of being a Γ-subsolution (resp. solution) as that of being a viscosity subsolution (resp. solution) of a suitable equation. Indeed, define the function G_0 on the space of hyperhermitian matrices as the function such that

$$\lambda(A) - G_0(A)\mathbf{1} \in \bar{\Gamma},$$

consider the projection $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to \{H \in \mathbb{R}^{4n,4n} \mid I_0HI_0 = J_0HJ_0 = K_0HK_0 = -H\}$

$$\mathrm{p}(H) = \frac{1}{4}(H - I_0HI_0 - J_0HJ_0 - K_0HK_0)\,,$$

where (I_0, J_0, K_0) is the standard hyperhermitian structure on $\mathbb{R}^{4n}$ written in block form as

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & -\mathbb{1} & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \end{pmatrix}, \qquad (6.21)$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. Then, defining the function G on the space of $4n \times 4n$ symmetric matrices $\mathrm{Sym}(4n, \mathbb{R})$ as $G(H) = G_0(\mathrm{p}(H))$, we have that u is a Γ-subsolution (resp. solution) if and only if it is a viscosity subsolution (resp. solution) of the equation $G(D^2u) = 0$.

Therefore we can take advantage from the known results regarding viscosity subsolutions and solutions (see [65]). In particular we will use the following:

- If $(u_j)_j$ is a sequence of Γ-subsolutions (resp. solutions) converging locally uniformly to u, then u is a Γ-subsolution (resp. solution) as well.
- If u, v are Γ-subsolutions, then $u + v$ is a Γ-subsolution as well.
- A mollification of a Γ-subsolution is again a Γ-subsolution.

We will also need the following comparison result

Lemma 6.13. *If u is a Γ-solution and v a smooth Γ-subsolution on a bounded open set $U \subseteq \mathbb{H}^n$ such that $u = v$ on ∂U, then $u \geq v$ in U.*

Proof. The very same proof of [280, Lemma 17], which is the analogous result in $\mathbb{C}^n$, can be carried out in our hypothesis. □

The next lemma follows from the same argument as [280, Lemmas 18-19]. The additional case when $\Gamma = \Gamma_n$ is quite easy and can be deduced along the same lines.

Lemma 6.14. *Suppose $v\colon \mathbb{H}^n \to \mathbb{R}$ is a Γ-solution which is independent of the last variable q_n. Define*

$$\Gamma' = \begin{cases} \Gamma_{n-1} & \text{if } \Gamma = \Gamma_n\,, \\ \Gamma \cap \{x_n = 0\} & \text{if } \Gamma \neq \Gamma_n\,, \end{cases} \qquad (6.22)$$

then Γ' is a symmetric proper convex open cone in $\mathbb{R}^{n-1}$ containing the cone Γ_{n-1} and the function $w(q_1, \ldots, q_{n-1}) = v(q_1, \ldots, q_{n-1}, 0)$ is a Γ'-solution on $\mathbb{H}^{n-1}$.

We remark that in view of (6.3) every Γ-subsolution is subharmonic.

Proposition 6.15 (Liouville-type Theorem)**.** *A Lipschitz bounded viscosity* Γ*-solution* $u\colon \mathbb{H}^n \to \mathbb{R}$ *with* $\|\nabla u\|_{C^0} \leq C$ *is constant.*

Proof. The result is proved by induction over n. For $n = 1$ the function u is harmonic and the result is well-known.

Assume now that the result holds for $n-1$ and let us prove it for n. By contradiction we suppose that u is not constant and $\inf_M u = 0$, $\sup_M u = 1$. We adopt the notation of [280] and, for any function $v\colon \mathbb{H}^n \to \mathbb{R}$ we write its mollification

$$[v]_r(q) = \int_{q' \in \mathbb{H}^n} v(q + rq')\psi(q')\, \mathrm{dV}\,,$$

where, here and hereafter, dV denotes the standard volume form in $\mathbb{H}^n$ and $\psi\colon \mathbb{H}^n \to \mathbb{R}$ is a smooth mollifier with support in $B_1(0)$ such that $\psi > 0$ in $B_1(0)$ and $\int_{\mathbb{H}^n} \psi\, \mathrm{dV} = 1$. During the proof we will need to regularize u, considering $u^\varepsilon = [u]_\varepsilon$ for a small $\varepsilon > 0$. Following [103] we use Cartan's Lemma to deduce

$$\lim_{r\to\infty} [u^2]_r(q) = \lim_{r\to\infty} [u]_r(q) = 1\,.$$

For $\rho > 0$ and $r > 0$ consider the set

$$U(\rho, r) = \left\{ q \in \mathbb{H}^n \mid 2u(q) \leq [u^2]_r(q) + [u]_\rho(q) - \frac{4}{3} \right\}\,.$$

Suppose there are $\rho > 0$, $\varepsilon_j \to 0$, $q_j \in \mathbb{H}^n$, $r_j \to \infty$ and a unit vector $\xi_j \in \mathbb{H}^n$ such that $q_j \in U(\rho, r_j)$ and

$$\lim_{j\to\infty} \int_{B_{r_j}(q_j)} |\bar\partial_{\xi_j} u^{\varepsilon_j}|^2 \mathrm{dV} = 0\,, \tag{6.23}$$

where for any vector $\xi = (\xi_0^1 + \xi_1^1 i + \xi_2^1 j + \xi_3^1 k, \dots, \xi_0^n + \xi_1^n i + \xi_2^n j + \xi_3^n k) \in \mathbb{H}^n$ and any function $w\colon \mathbb{H}^n \to \mathbb{R}$ we use the notation

$$\bar\partial_\xi w = \sum_{r=1}^n \left(\xi_0^r w_{x_0^r} + \xi_1^r w_{x_1^r} i + \xi_2^r w_{x_2^r} j + \xi_3^r w_{x_3^r} k \right)\,.$$

Composing with rotations and translations, for each j we can take q_j to the origin and assume $\xi_j = q^n/2$, obtaining a sequence $(u_j)_j$ of Γ-solutions satisfying

$$[u_j^2]_{r_j}(0) + [u_j]_\rho(0) - 2u_j(0) \geq \frac{4}{3}\,, \qquad \lim_{j\to\infty} \int_{B_{r_j}(0)} \left| \bar\partial_{\frac{q^n}{2}} u_j^{\varepsilon_j} \right|^2 \mathrm{dV} = 0\,. \tag{6.24}$$

Since u has bounded gradient, by the Ascoli-Arzelà Theorem, up to a subsequence, $(u_j)_j$ converges locally uniformly to some $v\colon \mathbb{H}^n \to \mathbb{R}$ which must be again a Γ-solution with bounded gradient. Also $u_j^{\varepsilon_j}$ converges to v locally uniformly and working as in [103] we infer that v does not depend on the last variable q^n.

Indeed, if v were not constant along lines with fixed $q' = (q^1, \dots, q^{n-1})$, there would be $a, b \in \mathbb{H}$ and a positive $c \in \mathbb{R}$ such that $v(q_0', a) - v(q_0', b) > 2c$. Since the gradient of v is bounded from above, we could choose δ small enough such that

$$\inf \{v(q', q^n) \mid |q' - q_0'| < \delta,\, |q^n - a| < \delta\} - \sup \{v(q', q^n) \mid |q' - q_0'| < \delta,\, |q^n - b| < \delta\} > c\,.$$

Let $\xi \in \mathbb{H}^n$ be the unit vector with last entry $(b-a)/|b-a|$ and all others zero. Let γ be the segment joining $(q', a'), (q', b') \in \mathbb{H}^n$, where $b' - a' = b - a$, $|q' - q_0'| < \delta$, $|a' - a| < \delta$, $|b' - b| < \delta$, then we would have

$$\left| \int_\gamma \bar\partial_\xi v\, d\xi \right| = |v(q', b') - v(q', a')| > c\,.$$

Cauchy-Schwarz inequality would now give

$$c^2 < \left|\int_\gamma \bar\partial_\xi v\, d\xi\right|^2 \leq \left(\int_\gamma |\bar\partial_\xi v|^2 d\xi\right)\left(\int_\gamma d\xi\right) = |b-a| \int_\gamma |\bar\partial_\xi v|^2 d\xi\,.$$

Let I_1, I_2, I_3 be intervals of length δ all perpendicular to each other and to $[a,b]$ in the q^n-space. Using Fubini's theorem over the set $B(q_0', \delta) \times [a,b] \times I_1 \times I_2 \times I_3$ we would find a strictly positive lower bound for the integral of $|\bar\partial_{q^n/2} v|^2 \mathrm{dV}$. But this would contradict the uniform convergence as the u_j's satisfy (6.24). Therefore v does not depend on the last variable.

The function $w(q^1, \dots, q^{n-1}) = v(q^1, \dots, q^{n-1}, 0)$ is then a Γ'-solution, thanks to Lemma 6.14, where Γ' is the cone defined in (6.22). By the induction hypothesis w is constant and then so is v. But by Cartan's Lemma this contradicts the first of (6.24) because

$$\frac{4}{3} \leq \lim_{j\to\infty} \left([u_j^2]_{r_j}(0) + [u_j]_\rho(0) - 2u_j(0)\right) = 1 + [v]_\rho(0) - 2v(0) = 1 - v(0) \leq 1$$

as v inherits from u the property that $0 \leq v \leq 1$.

This means that (6.23) cannot hold, in particular for all $\rho > 0$, there exists $c_\rho > 0$ such that if $r > c_\rho$, for each $q \in U(\rho, r)$, $\varepsilon < c_\rho^{-1}$ and unit vector $\xi \in \mathbb{H}^n$ we must have

$$\int_{B_r(q)} |\partial_\xi u^\varepsilon|^2 \mathrm{dV} > c_\rho\,. \tag{6.25}$$

Define

$$U'(\rho, r) = \left\{ q \in \mathbb{H}^n \mid 2u(q) < [u^2]_r(q) + [u]_\rho(q) - \frac{4}{3}\right\} \subseteq U(\rho, r)\,.$$

We may choose the origin so that $u(0) < 1/12$, and $\rho > 0$ and $r > c$ big enough to have $[u]_\rho(0) > 3/4$ and $[u^2]_r(0) > 3/4$ which can be done by Cartan's Lemma. It follows that $0 \in U'(\rho, r)$.

Since $\partial_{\bar q^i}\partial_{q^j}(u^\varepsilon)^2 = 2u^\varepsilon u^\varepsilon_{\bar i j} + 2u^\varepsilon_{\bar i} u^\varepsilon_j$, proceeding similarly as in [280] we can use (6.25) to prove that there exists a constant $\delta > 0$ small enough to guarantee that $[(u^\varepsilon)^2]_r - \delta|q|^2$ is a Γ-subsolution over $U'(\rho, r)$. By local uniform convergence also $[u^2]_r - \delta|q|^2$ is a Γ-subsolution. Finally consider

$$U''(\rho, r) = \left\{ q \in \mathbb{H}^n \mid 2u(q) < [u^2]_r(q) - \delta|q|^2 + [u]_\rho(q) - \frac{4}{3}\right\} \subseteq U'(\rho, r)$$

and observe that since $0 \leq u \leq 1$ this set is bounded. The fact that u is a Γ-solution and yet $[u^2]_r(q) - \delta|q|^2 + [u]_\rho(q) - \frac{4}{3}$ is a smooth Γ-subsolution contradicts the comparison principle of Lemma 6.13. We conclude that u must be constant. □

6.2.4 $C^{2,\alpha}$-estimate.

The main theorem follows once we obtain the $C^{2,\alpha}$-estimate. We obtain the desired bound in two ways, by using an analogue of Evans-Krylov theory as developed in Tosatti-Wang-Weinkove-Yang [286] and by adapting the argument of Alesker [14] for the treatment of the quaternionic Monge-Ampère equation.

Proposition 6.16. *Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. If φ is a solution to (6.2) such that $\|\varphi\|_{C^0}$ and $\Delta_g\varphi$ are bounded from above, then there is $\alpha \in (0,1)$ and a constant $C > 0$, depending only on the background data such that*

$$\|\varphi\|_{C^{2,\alpha}} \leq C\,.$$

First proof of the $C^{2,\alpha}$-estimate.

Let $V = \{H \in \mathbb{R}^{4n,4n} \mid I_0 H I_0 = J_0 H J_0 = K_0 H K_0 = -H\}$, where (I_0, J_0, K_0) is the standard hypercomplex structure on $\mathbb{R}^{4n}$ as in (6.21). Consider the real representation of quaternionic matrices $\gamma\colon \mathbb{H}^{n,n} \to V$, defined in Subsection 1.1.2. The map γ is an isomorphism of real algebras and $\gamma(\mathrm{Hyp}(n)) = V \cap \mathrm{Sym}(4n, \mathbb{R})$. Let $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to V$ be the projection

$$\mathrm{p}(H) := \frac{1}{4}(H - I_0 H I_0 - J_0 H J_0 - K_0 H K_0)\,.$$

If we take on $\mathbb{H}^n$ the real coordinates $(x_0^1, \dots, x_0^n, x_1^1, \dots, x_1^n, x_2^1, \dots, x_2^n, x_3^1, \dots, x_3^n)$ underlying the quaternionic coordinates $(q^1, \dots, q^n)$, for a C^2 function $u\colon \mathbb{H}^n \to \mathbb{R}$ we have

$$\gamma(\mathrm{Hess}_{\mathbb{H}} u) = 16\mathrm{p}(D^2 u)\,.$$

For any point $x_0 \in M$, take a quaternionic coordinate chart centered at x_0 and assume that the domain of the chart contains $B_1(0)$. For any $H \in \mathrm{Sym}(4n, \mathbb{R})$ we have $\gamma^{-1}(p(H)) \in \mathrm{Hyp}(n)$, therefore

$$\tilde H_{rs}(x) = g^{\bar j r}(x)(\gamma^{-1}(p(H)))_{\bar j s}\,, \qquad x \in B_1(0)\,,$$

is hyperhermitian with respect to g.

Define the set

$$\mathcal{E} = \left\{ H \in \mathrm{Sym}(4n, \mathbb{R}) \mid \lambda(\tilde H(0)) \in \bar\Gamma^\sigma \cap \overline{B_{2R}(0)} \right\}\,,$$

where σ and R are chosen below. $\mathcal{E}$ is compact and also convex by convexity of Γ. Possibly shrinking $B_1(0)$ to a smaller radius $r \in (0,1)$ we may assume that if H lies in a sufficiently close neighborhood U of $\mathcal{E}$, then $\lambda(\tilde H(x)) \in \bar\Gamma^\sigma \cap \overline{B_{4R}(0)}$ for any $x \in B_1(0)$.

The bound $\Delta_g \varphi \le C$ implies that σ and R can be chosen so that

$$\lambda\left(g^{\bar j r}\left(\Omega_{\bar j s} + \varphi_{\bar j s}\right)\right) \in \bar\Gamma^\sigma \cap \overline{B_R(0)}\,, \qquad \text{on } B_1(0)\,.$$

Therefore, by continuity of g, and possibly shrinking $B_1(0)$ again, for each $x \in B_1(0)$ we have

$$\gamma(\Omega_{\bar r s}(x)) + 16\mathrm{p}(D^2\varphi(x)) = \gamma\left(\Omega_{\bar r s}(x) + \varphi_{\bar r s}(x)\right) \in \mathcal{E}\,.$$

This discussion and our assumptions on f show that we can apply the main Theorem of [286] with

- $P\colon \mathrm{Sym}(4n, \mathbb{R}) \times B_1(0) \to \mathbb{R}$ defined as $P(H, x) = f(\lambda(\tilde H(x)))$ for $H \in U$, and extended smoothly to all of $\mathrm{Sym}(4n, \mathbb{R}) \times B_1(0)$ (in [286] is called F);
- $S\colon B_1(0) \to \mathrm{Sym}(4n, \mathbb{R})$ defined as $S(x) = \gamma(\Omega_{\bar r s}(x))$;
- $T\colon \mathrm{Sym}(4n, \mathbb{R}) \times B_1(0) \to \mathrm{Sym}(4n, \mathbb{R})$ defined as $T(H, x) = 16\mathrm{p}(H)$.

And since $\|\varphi\|_{C^0} \le C$ we obtain the desired bound $\|\varphi\|_{C^{2,\alpha}} \le C$ for some $\alpha \in (0,1)$.

Second proof of the $C^{2,\alpha}$-estimate.

Since M is locally flat, we only need to prove the following interior $C^{2,\alpha}$ estimate for $w = \varphi + u$, where $u \in C^\infty_{\mathrm{loc}}(M, \mathbb{R})$ is a local potential for Ω.

Now, $w \in C^4(\mathcal{O})$ satisfies

$$F(w_{\bar r s}) = h\,,$$

where $\mathcal{O} \subset \mathbb{H}^n$ is an arbitrary open subset and $h \in C^\infty(\mathcal{O})$. Let $\mathcal{O}' \subset \mathcal{O}$ be a relatively compact open subset. We shall prove that there exist a constant $\alpha \in (0,1)$ depending only on n, h, $\|w\|_{C^0(\mathcal{O})}$, $\|\Delta w\|_{C^0(\mathcal{O})}$ and a constant C depending in addition on $\mathrm{dist}(\mathcal{O}, \mathcal{O}')$ such that

$$\|w\|_{C^{2,\alpha}(\mathcal{O})} \le C\,.$$

There is a difference with respect to the argument of Alesker [14]: the quaternionic Monge-Ampère operator can be written in the divergence form, while this might not be true for more general fully non-linear equations. To overcome this issue we will need a more general version of the weak Harnack inequality for second order uniformly elliptic operators.

Let W be the quaternionic Hessian $(w_{\bar{r}s})$ and define a second order linear operator $\mathcal{D}$ by

$$\mathcal{D}v = \operatorname{Re} F^{rs}(W) v_{\bar{r}s} .$$

Notice that every $n \times n$ hyperhermitian matrix defines a hyperhermitian semilinear form on $\mathbb{H}^n$. Hence it also determines a symmetric bilinear form on $\mathbb{R}^{4n}$. Let $(a_{ij}) \in \mathrm{Sym}(4n, \mathbb{R})$ be the realization of $(F^{rs}(W))$. Then we can rewrite $\mathcal{D}v$ in the following form

$$\mathcal{D}v = \sum_{r,s=1}^{4n} a_{rs} D_r D_s v ,$$

Since F is uniformly elliptic on Γ, the operator $\mathcal{D}$ is uniformly elliptic as well.

Let $R > 0$ be such that the open ball B_{2R} of radius $2R$ centered at a point $z_0 \in \mathcal{O}'$ is contained in $\mathcal{O}$. For an arbitrary unitary vector $\xi \in \mathbb{H}^n$, we let Δ_ξ denote the Laplacian on any translate of the quaternionic line spanned by ξ. By virtue of concavity of F, for any unitary vector $\xi \in \mathbb{H}^n$, we have

$$\operatorname{Re} F^{rs}(W) \Delta_\xi (w_{\bar{r}s}) \geq \Delta_\xi h . \tag{6.26}$$

Consider the function

$$\hat{w} = \sup_{B_{2R}} \Delta_\xi w - \Delta_\xi w .$$

it follows from (6.26) that $\mathcal{D}\hat{w} \leq -\Delta_\xi h$, where we used the fact $\Delta_\xi(w_{\bar{r}s}) = (\Delta_\xi w)_{\bar{r}s}$.

Then, applying the weak Harnack inequality (Theorem 2.57), there exists a positive constant C depending on n, $\|h\|_{C^2(\mathcal{O})}$ and $\|\Delta u\|_{C^0(\mathcal{O})}$ such that

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} \hat{w} \leq C \left(\inf_{B_R} \hat{w} + R \right) .$$

Equivalently, we have

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} \left(\sup_{B_{2R}} \Delta_\xi w - \Delta_\xi w \right) \leq C \left(\sup_{B_{2R}} \Delta_\xi w - \sup_{B_R} \Delta_\xi w + R \right) . \tag{6.27}$$

Since F is concave on Γ for any pair of A, $B \in \mathrm{Hyp}(n, \mathbb{H})$, we have

$$F(B) - F(A) \leq \operatorname{Re} F^{rs}(A)(B_{rs} - A_{rs}) .$$

Choosing $A = W(y)$ and $B = W(x)$ for $x, y \in B_{2R}$, it follows that

$$\operatorname{Re} F^{rs}(W(y))(w_{\bar{r}s}(y) - w_{\bar{r}s}(x)) \leq F(W(y)) - F(W(x)) = h(y) - h(x) \leq C\|y - x\| \tag{6.28}$$

for some positive constant C depending on $\|h\|_{C^1(\mathcal{O})}$.

Now we need the following lemma from matrix theory, which is well-known in the settings of $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ (see e.g. [143, 42, 14]).

Lemma 6.17. [14, Lemma 4.9]. *Let $\lambda, \Lambda \in \mathbb{R}$ satisfy $0 < \lambda < \Lambda < +\infty$. There exist a uniform constant N, unit vectors $\xi_1, \cdots, \xi_N \in \mathbb{H}^n$ and positive numbers $\lambda_* < \Lambda_* < +\infty$, depending only on n, λ, Λ such that any $A \in \mathrm{Hyp}(n, \mathbb{H})$ with eigenvalues lying in the interval $[\lambda, \Lambda]$ can be written in the form*

$$A = \sum_{k=1}^{N} \beta_k \xi_k^* \otimes \xi_k , \qquad i.e.\ A_{rs} = \sum_{k=1}^{N} \beta_k \bar{\xi}_{kr} \xi_{ks} ,$$

for some $\beta_k \in [\lambda_*, \Lambda_*]$.

We apply the previous lemma with $A = (F^{rs}(W))$, obtaining immediately

$$\begin{aligned}\operatorname{Re} F^{rs}(W(y))(w_{\bar{r}s}(y) - w_{\bar{r}s}(x)) &= \sum_{k=1}^{N} \beta_k(y)\bar{\xi}_{kr}\xi_{ks}(w_{\bar{r}s}(y) - w_{\bar{r}s}(x)) \\ &= \sum_{k=1}^{N} \beta_k(y)(\Delta_{\xi_k} w(y) - \Delta_{\xi_k} w(x))\end{aligned}$$

for some functions $\beta_k(y) \in [\lambda_*, \Lambda_*]$. By (6.28), we then have

$$\sum_{k=1}^{N} \beta_k(y)(\Delta_{\xi_k} w(y) - \Delta_{\xi_k} w(x)) \le C\|y - x\| \qquad \text{for } x, y \in B_{2R}\,. \tag{6.29}$$

Let us denote

$$M_{k,tR} = \sup_{B_{tR}} \Delta_{\xi_k} w\,, \qquad m_{k,tR} = \inf_{B_{tR}} \Delta_{\xi_k} w\,, \qquad \eta(tR) = \sum_{k=1}^{N} (M_{k,tR} - m_{k,tR})\,,$$

for $t = 1, 2$.

Summing up (6.27) over ξ_k for $k \neq l$ yields

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} \sum_{k \neq l} \left(M_{k,2R} - \Delta_{\xi_k} w\right) \le C(\eta(2R) - \eta(R) + R)\,. \tag{6.30}$$

Choosing a point $x \in B_{2R}$ at which the infimum $m_{l,2R}$ is attained, by (6.29) we also know that

$$\Delta_{\xi_l} w(y) - m_{l,2R} \le \frac{1}{\lambda_*} \left(CR + \Lambda_* \sum_{k \neq l} (M_{k,2R} - \Delta_{\xi_k} w) \right) \tag{6.31}$$

Integrating (6.31) on B_R and using (6.30) yields

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} (\Delta_{\xi_l} w - m_{l,2R}) \le C(\eta(2R) - \eta(R) + R)\,.$$

Using (6.27) again, we then obtain

$$\begin{aligned}\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} (\Delta_{\xi_l} w - m_{l,2R}) &\ge \frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} (\Delta_{\xi_l} w - M_{l,2R}) + M_{l,2R} - m_{l,2R} \\ &\ge M_{l,2R} - m_{l,2R} - C(M_{l,2R} - M_{l,R} + R) \\ &\ge C(M_{l,R} - m_{l,R}) - (C-1)(M_{l,2R} - m_{l,2R}) - CR\,,\end{aligned}$$

since $m_{k,tR}$ is non-increasing with respect to t. Inserting this last inequality into (6.30) we get

$$\eta(2R) - \eta(R) \ge C(M_{l,R} - m_{l,R}) - (C-1)(M_{l,2R} - m_{l,2R}) - CR\,,$$

and summing up over l,

$$\eta(R) \le (1 - 1/C)\eta(2R) + CR.$$

Now applying [143, Lemma 8.23] the proof is complete.

6.3 Proof of Theorem 6.1 and consequences.

In this subsection we prove Theorem 6.1 and obtain some interesting corollaries.

Proof of Theorem 6.1. Let (M, I, J, K, g) be a compact flat hyperkähler manifold, $\underline{\varphi}, \varphi \colon M \to \mathbb{R}$ be a $\mathcal{C}$-subsolution and a solution to (6.2) respectively, with $\sup_M \varphi = 0$. By Proposition 6.4 we deduce $\|\varphi\|_{C^0} \leq C$. Proposition 6.7 now implies $\|\Delta_g \varphi\|_{C^0} \leq C(\|\nabla\varphi\|^2_{C^0} + 1)$. The blow-up argument together with the Liouville-type Theorem 6.15 yield a gradient bound for φ. Therefore $\|\Delta_g \varphi\|_{C^0} \leq C$ and we can deduce from Proposition 6.16 the desired $C^{2,\alpha}$-estimate $\|\varphi\|_{C^{2,\alpha}} \leq C$, where the constant $C > 0$ only depends on the background data, including $\underline{\varphi}$. □

Quaternionic Hessian equation.

As an application of Theorem 6.1 we first have the solvability of the quaternionic Hessian equation on hyperhermitian manifolds admitting a flat hyperkähler metric.

Let $(M, I, J, K, g, \Omega_0)$ be a compact hyperhermitian manifold where Ω_0 is the $(2,0)$-form induced by g, fix $1 \leq k \leq n$ and let Ω be a q-real $(2,0)$-form which is k-positive in the sense that

$$\frac{\Omega^i \wedge \Omega_0^{n-i}}{\Omega_0^n} > 0 \qquad \text{for every } i = 1, \dots, k\,. \tag{6.32}$$

Let $\mathcal{H}^k_{\Omega_0}$ be the set of smooth functions φ such that Ω_φ is a k-positive q-real $(2,0)$-form. Then the *quaternionic Hessian equation* is defined as

$$\frac{\Omega_\varphi^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = b\,\mathrm{e}^H, \qquad \varphi \in \mathcal{H}^k_{\Omega_0}\,, \tag{6.33}$$

where $H \in C^\infty(M, \mathbb{R})$ is the datum and $(\varphi, b) \in \mathcal{H}^k_{\Omega_0} \times \mathbb{R}_+$ is the unknown. The constant b is uniquely determined by

$$b = \frac{\int_M \Omega_\varphi^k \wedge \Omega_0^{n-k} \wedge \bar{\Omega}_0^n}{\int_M \mathrm{e}^H \Omega_0^n \wedge \bar{\Omega}_0^n}\,.$$

Equation (6.33) reduces to the quaternionic Monge-Ampère equation for $k = n$ and to the classical Poisson equation for $k = 1$. Moreover equation (6.33) is the analogue of the real and complex Hessian equations in the quaternionic setting.

The Hessian equation on manifolds has been first investigated by Li [210] and Urbas [296] in the Riemannian case (see also the survey of Wang [320]). Later some partial results have been obtained in the Kähler setting by Hou [175], Jbilou [189] and Kokarev [198] independently. The solution in its full generality on compact Kähler manifolds came by Dinew and Kołodziej [103] building on the estimate of Hou, Ma and Wu [176]. The equation has also been solved on compact Hermitian and almost Hermitian manifolds (see [288, 93] for the case $k = n$ and [329, 92] for the general case).

Applying Theorem 6.1 we solve equation (6.33) on compact flat hyperkähler manifolds:

Theorem 6.18. *Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold and Ω a q-real k-positive $(2,0)$-form. Then the quaternionic Hessian equation*

$$\frac{\Omega_\varphi^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = b\,\mathrm{e}^H\,, \qquad \int_M \varphi\,\Omega_0^n \wedge \bar{\Omega}_0^n = 0\,, \qquad \varphi \in \mathcal{H}^k_{\Omega_0}\,,$$

has a unique smooth solution $(\varphi, b) \in \mathcal{H}^k_{\Omega_0} \times \mathbb{R}_+$ for every $H \in C^\infty(M, \mathbb{R})$.

For the quaternionic Hessian equation as the cone Γ we consider the k-positive cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \dots, \sigma_k(\lambda) > 0\}\,,$$

where $1 \leq k \leq n$ and σ_r is the r-th elementary symmetric function

$$\sigma_r(\lambda) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}\,, \qquad \text{for all } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n\,.$$

Observe that by Lemma 2.25 on a locally flat hyperhermitian manifold (M, I, J, K, g) a q-real $(2,0)$-form Ω is k-positive in the sense that it satisfies (6.32) if and only if $\lambda(g^{\bar{j}r}\Omega_{\bar{j}s}) \in \Gamma_k$.

Moreover, for every $(\lambda_1, \dots, \lambda_n) \in \Gamma_k$ we clearly have

$$\lim_{t\to\infty} \sigma_k(\lambda_1, \dots, \lambda_{n-1}, t) = \infty$$

and by [280, Remark 8] any Γ_k-admissible function is a $\mathcal{C}$-subsolution. Hence for the quaternionic Hessian equation we easily have existence of a $\mathcal{C}$-subsolution.

Proof of Theorem 6.18. On Γ_k we define $f = \log \sigma_k$, in order to rewrite the quaternionic Hessian equation as

$$f\left(\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right)\right) = h\,,$$

for some positive $h \in C^\infty(M, \mathbb{R})$ depending on H. The function f satisfies conditions C1–C3 stated in the overview (see e.g. [268]).

We apply the method of continuity. Let $H_0 \in C^\infty(M, \mathbb{R})$ be the function such that

$$\frac{\Omega^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = \mathrm{e}^{H_0}$$

and consider the t-dependent family of equations

$$\frac{\Omega^k_{\varphi_t} \wedge \Omega_0^{n-k}}{\Omega_0^n} = b_t\, \mathrm{e}^{tH+(1-t)H_0}, \quad \varphi_t \in \mathcal{H}^k_{\Omega_0}\,, \quad t \in [0,1]\,. \tag{$*_t$}$$

Let

$$S = \left\{t \in [0,1] \mid (*_t) \text{ has a solution } (\varphi_t, b_t) \in C^{2,\beta}(M, \mathbb{R}) \times \mathbb{R}_+\right\}\,.$$

By our choice of H_0, the pair $(\varphi, b) = (0,1)$ solves $(*_0)$, hence the set S is non-empty.

Since we assumed Ω to be k-positive $\underline{\varphi} \equiv 0$ is Γ_k-admissible and therefore a $\mathcal{C}$-subsolution. Closedness of S now follows from the $C^{2,\alpha}$-estimate of Theorem 6.1, a standard bootstrapping argument and the Ascoli-Arzelà Theorem.

Finally, in order to show that S is open, take $t' \in S$ and let $(\varphi_{t'}, b_{t'})$ be the corresponding solution to $(*_{t'})$. Consider the Banach spaces

$$B_1 := \left\{\psi \in C^{2,\beta}(M, \mathbb{R}) \mid \psi \in \mathcal{H}^k_{\Omega_0}\,, \int_M \psi\, \Omega_0^n \wedge \bar{\Omega}_0^n = 0\right\}\,, \qquad B_2 := C^{0,\beta}(M, \mathbb{R})\,,$$

and the linearization of the operator

$$B_1 \times \mathbb{R}_+ \to B_2\,, \qquad (\psi, a) \mapsto \log \frac{\Omega^k_\psi \wedge \Omega_0^{n-k}}{\Omega_0^n} - \log(a)$$

at $(\varphi_{t'}, b_{t'})$, which is

$$L\colon T_{\varphi_{t'}} B_1 \times \mathbb{R} \to B_2\,, \qquad L(\rho, c) = k\frac{\partial\partial_J \rho \wedge \Omega^{k-1}_{\varphi_{t'}} \wedge \Omega_0^{n-k}}{b_{t'}\, \mathrm{e}^{t'H+(1-t')H_0}\Omega_0^n} - \frac{c}{b_{t'}} =: L'(\rho) - \frac{c}{b_{t'}}\,,$$

where

$$T_{\varphi_{t'}} B_1 = \left\{\rho \in C^{2,\beta}(M, \mathbb{R}) \mid \int_M \rho\, \Omega_0^n \wedge \bar{\Omega}_0^n = 0\right\}\,.$$

By the maximum principle the kernel of the operator L' over $C^{2,\beta}(M, \mathbb{R})$ is the set of constant functions. Moreover the principal symbol of L' is self-adjoint and therefore L' has index zero, which implies that its formal adjoint $(L')^*$ has one-dimensional kernel as well. In order to show that L is surjective, let $\zeta \in C^{0,\beta}(M, \mathbb{R})$ and choose $c \in \mathbb{R}$ such that $\zeta + c/b_{t'}$ is orthogonal to $\ker((L')^*)$. By the Fredholm alternative there exists $\rho \in B_1$ such that

$$L'(\rho) = \zeta + c/b_{t'}$$

and the surjectivity of L follows.

By the inverse function theorem between Banach spaces S is open. This proves the existence of a solution to the quaternionic Hessian equation.

Finally we show uniqueness. Suppose $(\varphi_1, b_1), (\varphi_2, b_2)$ are both solutions and assume $b_1 \geq b_2$; then

$$(\Omega^k_{\varphi_1} - \Omega^k_{\varphi_2}) \wedge \Omega_0^{n-k} \geq 0\,,$$

which can be rewritten as

$$\partial\partial_J(\varphi_1 - \varphi_2) \wedge \left(\sum_{i=0}^{k-1} \Omega_{\varphi_1}^{k-i-1} \wedge \Omega^i_{\varphi_2} \right) \wedge \Omega_0^{n-k} \geq 0\,.$$

Since

$$\varphi \mapsto \frac{\partial\partial_J\varphi \wedge \left(\sum_{i=0}^{k-1} \Omega_{\varphi_1}^{k-i-1} \wedge \Omega^i_{\varphi_2} \right) \wedge \Omega_0^{n-k}}{\Omega_0^n}$$

is a second order linear elliptic operator without free term, by the maximum principle we deduce $\varphi_1 = \varphi_2$ and thus also $b_1 = b_2$. □

From Theorem 6.18 we recover as a special case the result of Alesker 2.53, where the quaternionic Monge-Ampère equation is solved on compact flat hyperkähler manifolds. We note that during the proof of Theorem 6.1 the a priori estimates, except for the C^2-estimate, are obtained without assuming anything about the closure of Ω_0 and this suggests that it is worth studying the quaternionic Hessian equation on non-HKT hyperhermitian manifolds.

Quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions.

Our second application is the quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions. Let $(M, I, J, K, g, \Omega_0)$ be a compact hyperhermitian manifold and Ω_1 a positive q-real $(2,0)$-form. We say that a C^2 function φ on M is $(n-1)$-quaternionic plurisubharmonic with respect to Ω_1 and Ω_0 if the $(2,0)$-form $\Omega_1 + \frac{1}{n-1}[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi]$ is q-positive, where Δ_g is the quaternionic Laplacian with respect to g. We also refer to Harvey and Lawson [165, 166] for more general notions of plurisubharmonicity. The *quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions* is written as

$$\left(\Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big]\right)^n = b\,\mathrm{e}^H\Omega_0^n\,, \qquad \Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big] > 0\,. \tag{6.34}$$

Here the constant b is uniquely determined by

$$b = \frac{\int_M \left(\Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big]\right)^n \wedge \bar\Omega_0^n}{\int_M \mathrm{e}^H\Omega_0^n \wedge \bar\Omega_0^n}\,.$$

Equation (6.34) is the analogue of the complex Monge-Ampère equation for $(n-1)$-plurisubharmonic functions, which originally arose from superstring theory in the works of Fu, Wang and Wu [131, 132], and was then solved by Tosatti and Weinkove [290, 292] (see also [92, 179]).

Theorem 6.19. *Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold and Ω_1 a q-real positive $(2,0)$-form. Then there is a unique solution $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ to the equation*

$$\begin{cases} \left(\Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big]\right)^n = b\,\mathrm{e}^H\Omega_0^n\,, \\ \Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big] > 0\,, \quad \sup_M\varphi = 0\,, \end{cases} \tag{6.35}$$

for every given $H \in C^\infty(M,\mathbb{R})$.

Proof of Theorem 6.19. Similarly as discussed in [280], let T be the linear map given by

$$T(\lambda) = \big(T(\lambda)_1, \ldots, T(\lambda)_n\big)\,, \qquad T(\lambda)_k = \frac{1}{n-1}\sum_{i\neq k}\lambda_i\,,$$

for every $\lambda \in \mathbb{R}^n$ and define

$$f = \log\sigma_n(T), \qquad \Gamma = T^{-1}(\Gamma_n)\,.$$

It is straightforward to verify that the above setting satisfies the assumptions C1–C3 in the introduction. Let

$$\Omega := \mathrm{Re}\left(g^{\bar{j}s}(\Omega_1)_{\bar{j}s}\right)\Omega_0 - (n-1)\Omega_1\,.$$

Thus, equation (6.34) can be written as

$$f(\lambda) = H + \log b\,, \qquad \lambda = \lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \Gamma\,.$$

Then, Theorem 6.19 can be proved by a similar argument of Theorem 6.18, we give some details here.

We consider the following family of equations for $t \in [0,1]$:

$$\begin{cases} \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi_t)\Omega_0 - \partial\partial_J\varphi_t\right]\right)^n = \mathrm{e}^{tH+(1-t)H_0+c_t}\Omega_0^n\,, \\ \Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi_t)\Omega_0 - \partial\partial_J\varphi_t\right] > 0\,, \quad \sup_M\varphi_t = 0\,, \end{cases} \tag{$(*)_t$}$$

where $H_0 = \log\frac{\Omega_1^n}{\Omega_0^n}$ and $c_t : [0,1] \to \mathbb{R}$ is a path from $c_0 = 0$ to $c_1 = \log b$. Let us define

$$S = \{t \in [0,1] \mid \text{ there exists a pair } (\varphi_t, c_t) \in C^\infty(M,\mathbb{R}) \times \mathbb{R} \text{ solving } (*)_t\ \}\,.$$

Note that $(\varphi_0, c_0) = (0,0)$ solves $(*)_0$ and hence $S \neq \emptyset$. To prove the existence of solutions to (6.35), it suffices to show that S is both closed and open.

Step 1. S is closed. We first show that $\{c_t\}$ is uniformly bounded. Suppose φ_t achieves its maximum at the point $p_t \in M$, then the maximum principle yields that $\partial\partial_J\varphi_t$ is non-positive at p_t. Combining this with $(*)_t$, we obtain the upper bound for c_t:

$$c_t \leq (-tH + H_0)(p_t) \leq C\,,$$

for some C depending only on H, Ω_1 and Ω. The lower bound of c_t can be obtained similarly.

Observe that the positivity of Ω_1 implies that $\underline{\varphi} \equiv 0$ is a $\mathcal{C}$-subsolution of $(*)_t$. Then C^∞ a priori estimates of φ_t follow from Theorem 6.1. Combining this with the Arzelà-Ascoli theorem, we conclude that S is closed.

Step 2. S is open. Suppose there exists a pair $(\varphi_{\hat{t}}, c_{\hat{t}})$ satisfies $(*)_{\hat{t}}$. We shall prove that when t is close to $\hat{t}$, there exists a pair $(\varphi_t, c_t) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}$ solving $(*)_t$.

First of all, let Θ be a q-positive holomorphic $(2n,0)$-form with respect to I. For every function $\psi : M \to \mathbb{R}$ of class C^2, we define

$$L_{\hat{\varphi}}(\psi) := \frac{n}{n-1}\frac{\left((\Delta_g\psi)\Omega_0 - \partial\partial_J\psi\right) \wedge \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\hat{\varphi})\Omega_0 - \partial\partial_J\hat{\varphi}\right]\right)^{n-1}}{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\hat{\varphi})\Omega_0 - \partial\partial_J\hat{\varphi}\right]\right)^n}\,.$$

Since the operator $L_{\hat{\varphi}}$ is second order elliptic its symbol is self-adjoint, and therefore the index is zero. Then the classical maximum principle yields that

$$\ker(L_{\hat{\varphi}}) = \{\mathrm{const}\}\,. \tag{6.36}$$

Denote by $L^*_{\hat{\varphi}}$ the L^2-adjoint operator of $L_{\hat{\varphi}}$ with respect to the volume form

$$\mathrm{dvol} = \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\hat{\varphi})\Omega_0 - \partial\partial_J\hat{\varphi}\right]\right)^n \wedge \bar{\Theta}\,.$$

By the index theorem, we know there is a non-negative function ζ such that

$$\ker(L^*_{\hat{\varphi}}) = \mathrm{Span}\{\zeta\}\,. \tag{6.37}$$

It follows from the strong maximum principle that $\zeta > 0$. Up to a constant, we may and do assume

$$\int_M \zeta \,\mathrm{dvol} = 1.$$

Define a Banach space

$$B_1 := \left\{\varphi \in C^{2,\alpha}(M,\mathbb{R}) \mid \lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \Gamma, \int_M \varphi\,\zeta\,\mathrm{dvol} = 0\right\}.$$

It is easy to verify that the tangent space of B_1 at $\hat{\varphi}$ is given by

$$T_{\hat{\varphi}}B_1 = \left\{\psi \in C^{2,\alpha}(M,\mathbb{R}) \mid \int_M \psi\,\zeta\,\mathrm{dvol} = 0\right\}\,.$$

Let us consider the map

$$\tilde{H}(\varphi, c) = \log \frac{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\right]\right)^n}{\Omega_0^n} - c\,,$$

which maps $B_1 \times \mathbb{R}$ to $C^{0,\alpha}(M,\mathbb{R})$. The linearized operator of $\tilde{H}$ at $(\hat{\varphi}, \hat{t})$ is given by

$$L_{\hat{\varphi}} - c : T_{\hat{\varphi}}B_1 \times \mathbb{R} \to C^{0,\alpha}(M,\mathbb{R})\,. \tag{6.38}$$

On the one hand, for any real-valued $h \in C^{0,\alpha}(M,\mathbb{R})$, there exists a unique real constant c such that

$$\int_M (h + c)\zeta\,\mathrm{dvol} = 0.$$

By (6.37) and Fredholm theorem, there exists a real function ψ on M such that $L_{\hat{\varphi}}(\psi) - c = h$. Hence, the map $L_{\hat{\varphi}} - c$ is surjective. On the other hand, let (ψ_1, c_1) be a solution of $L_{\hat{\varphi}}(\psi) - c = 0$. By (6.37) and Fredholm theorem again, we get $c_1 = 0$. Using (6.36) and (6.38), we also obtain $\psi_1 = 0$. Therefore, $L_{\hat{\varphi}} - c$ is injective.

As a consequence, we conclude that $L_{\hat{\varphi}} - c$ is bijective. By the implicit function theorem, we know that when $|t - \hat{t}|$ is small enough, there exists a pair (φ_t, c_t) satisfying

$$\tilde{H}(\varphi_t, c_t) = tH + (1-t)H_0.$$

In the general case, when we assume M is a compact manifold which admits a flat hyperkähler metric g compatible with the underlying hypercomplex structure, we may take $\Theta = \Omega^n$ and apply the previous procedure to show existence of solutions to (6.35).

Uniqueness can be obtained with a very similar technique as in Theorem 6.18, therefore we omit the proof here. □

From Theorem 6.19 we can also obtain Calabi-Yau–type Theorems for quaternionic balanced, quaternionic Gauduchon and quaternionic strongly Gauduchon metrics. We refer the reader to [207, Table 2] for the relevant definitions, which are entirely analogous to the complex case.

Corollary 6.20. *Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold and take a quaternionic balanced (resp. quaternionic Gauduchon, quaternionic strongly Gauduchon) metric with induced $(2,0)$-form Ω_2. Then there is a unique positive constant b' and a unique quaternionic balanced (resp. quaternionic Gauduchon, quaternionic strongly Gauduchon) metric with induced $(2,0)$-form $\tilde{\Omega}$, such that*

$$\tilde{\Omega}^{n-1} = \Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}\,,$$

for some $\varphi \in C^\infty(M, \mathbb{R})$*, and which solves*

$$\tilde{\Omega}^n = b' \, \mathrm{e}^{H'} \Omega_0^n \,,$$

for any given $H' \in C^\infty(M, \mathbb{R})$.

Before we move on to the proof of Corollary 6.20 we need to lay down some preliminaries in linear algebra in order to mimic the proof of [290, Corollary 1.3]. Let $(M, I, J, K, g, \Omega_0)$ be a compact hyperhermitian manifold. Let $(z^1, \dots, z^{2n})$ be holomorphic coordinates with respect to I and denote $\Lambda^{p,0}(M)$ the space of $(p,0)$-forms with respect to I. Consider the pointwise inner product $\langle \cdot, \cdot \rangle_g$ defined by

$$\langle \alpha, \beta \rangle_g = \frac{1}{p!} g^{r_1 \bar{s}_1} \cdots g^{r_p \bar{s}_p} \alpha_{r_1 \cdots r_p} \overline{\beta_{s_1 \cdots s_p}} \,, \qquad \text{for every } \alpha, \beta \in \Lambda^{p,0}_I(M) \,,$$

where any $(p,0)$-form α is locally written as $\alpha = \frac{1}{p!} \alpha_{r_1 \cdots r_p} dz^{r_1} \wedge \cdots \wedge dz^{r_p}$ and $(g^{r\bar{s}})$ is the inverse of the Hermitian matrix $(g_{r\bar{s}})$ induced by the I-Hermitian metric g.

We will need the following Hodge star-type operator $*\colon \Lambda^{p,0}(M) \to \Lambda^{2n-p,0}(M)$, defined by the relation

$$\alpha \wedge *\beta = \frac{1}{n!} \langle \alpha, \beta \rangle_g \Omega_0^n \,, \qquad \text{for } \alpha, \beta \in \Lambda^{p,0}_I(M) \,.$$

We fix a point $x_0 \in M$ and take holomorphic coordinates $(z^1, \dots, z^{2n})$ with respect to I such that $(g_{r\bar{s}})$ is the identity at x_0, then we may compute

$$*(dz^{2i-1} \wedge dz^{2i}) = dz^1 \wedge \cdots \wedge \widehat{dz^{2i-1}} \wedge \widehat{dz^{2i}} \wedge \cdots \wedge dz^{2n} \,. \tag{6.39}$$

Observe that the Hodge operator sends q-real $(2,0)$-forms to q-real $(2n-2,0)$-forms and vice versa. Recall that, when the hypercomplex structure is locally flat, to any q-real $(2,0)$-form Ω is associated a hyperhermitian matrix $(\Omega_{\bar{r}s})$, thus, we may define the determinant of Ω as the Moore determinant of $(\Omega_{\bar{r}s})$. This definition naturally extends to any q-real $(2n-2,0)$-form Φ by setting $\det(\Phi) = \frac{1}{(n-1)!} \det(*\Phi)$. In particular, for any q-real $\Omega \in \Lambda^{2,0}(M)$, we have

$$\det(\Omega^{n-1}) = \det(\Omega)^{n-1} \,, \tag{6.40}$$

which can be checked by taking coordinates in which $(\Omega_{\bar{r}s})$ is diagonal at a given point and using (6.39). For any pair of q-real $\chi, \Omega \in \Lambda^{2,0}_I(M)$, we also have

$$\frac{\chi^n}{\Omega^n} = \frac{\det(\chi)}{\det(\Omega)} = \frac{\det(*\chi)}{\det(*\Omega)} \,. \tag{6.41}$$

A q-real $(2n-2,0)$-form Φ is said to be q-positive if $\Phi \wedge \Omega > 0$ for all q-positive $(2,0)$-forms Ω. We observe that the Hodge star maps q-positive $(2,0)$-forms to q-positive $(2n-2,0)$-forms and conversely. On a locally flat hyperhermitian manifold the $(n-1)^{\text{th}}$ power $\Omega \mapsto \Omega^{n-1}$ is a bijective correspondence between the cone of q-positive $(2,0)$-forms and the cone of q-positive $(2n-2,0)$-forms. The proof of this fact is just a matter of linear algebra and it is entirely analogous to the argument in [225, pp. 279-280], therefore we omit it.

Proof of Corollary 6.20. For starters, we claim

$$\frac{1}{(n-1)!} * \left(\partial \partial_J \varphi \wedge \Omega_0^{n-2} \right) = \frac{1}{n-1} \left[(\Delta_g \varphi) \Omega_0 - \partial \partial_J \varphi \right] \,, \tag{6.42}$$

for any arbitrary function $\varphi \in C^2(M, \mathbb{R})$. It is enough to prove that for every $W \in \Lambda^{2n-2,0}_I(M)$, we have

$$\partial \partial_J \varphi \wedge \frac{\Omega_0^{n-2}}{(n-2)!} \wedge (*W) = (\Delta_g \varphi) W \wedge \Omega_0 - W \wedge \partial \partial_J \varphi.$$

Let $Z = dz^1 \wedge \cdots \wedge dz^{2n}$ for simplicity and fix a point $x_0 \in M$ where Ω_0 takes the standard form

$$\Omega_0 = \sum_{i=1}^{n} dz^{2i-1} \wedge dz^{2i} \,.$$

Without loss of generality, we may assume $W = \widehat{dz^1} \wedge \widehat{dz^2} \wedge dz^3 \wedge \cdots \wedge dz^{2n}$. It is easy to see that

$$W \wedge \Omega_0 = Z \,, \qquad W \wedge \partial\partial_J \varphi = (\varphi_{1\bar{1}} + \varphi_{2\bar{2}})Z \,.$$

As $*W = dz^1 \wedge dz^2$, we obtain

$$\begin{aligned}
\partial\partial_J\varphi \wedge \frac{\Omega_0^{n-2}}{(n-2)!} \wedge (*W) =& \partial\partial_J\varphi \wedge \frac{\Omega_0^{n-2}}{(n-2)!} dz^1 \wedge dz^2 \\
=& \partial\partial_J\varphi \wedge \sum_{i>1} dz^1 \wedge dz^2 \wedge \cdots \widehat{dz^{2i-1}} \wedge \widehat{dz^{2i}} \wedge \cdots \wedge dz^{2n} \\
=& \sum_{i>1} (\varphi_{2i-1\overline{2i-1}} + \varphi_{2i\overline{2i}})Z = (\Delta_g\varphi)Z - (\varphi_{1\bar{1}} + \varphi_{2\bar{2}})Z \\
=& (\Delta_g\varphi) W \wedge \Omega_0 - W \wedge \partial\partial_J\varphi \,,
\end{aligned}$$

as claimed.

From (6.41) and (6.42), it follows that

$$\begin{aligned}
\frac{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega - \partial\partial_J\varphi\right]\right)^n}{\Omega_0^n} &= \frac{\det\left(*\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega - \partial\partial_J\varphi\right]\right)\right)}{\det(*\Omega_0)} \\
&= \frac{\det\left(\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}\right)}{\det(\Omega_0^{n-1})} \,.
\end{aligned}$$

This implies that given a positive $(2,0)$-form Ω_1 and a smooth function H on M, the pair $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ is a solution to (6.35) if and only if it solves

$$\begin{cases} \det(\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}) = b\, \mathrm{e}^H \det(\Omega_0^{n-1}) \,, \\ \Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2} > 0 \,, \quad \sup_M \varphi = 0 \,, \end{cases} \tag{6.43}$$

where Ω_2 is uniquely defined by

$$\Omega_1 = \frac{1}{(n-1)!} * \Omega_2^{n-1} \,,$$

because the $(n-1)^{\text{th}}$ power is a bijection between the spaces of positive $(2,0)$-forms and positive $(2n-2,0)$-forms.

Now, let $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ be the solution to (6.35), or equivalently (6.43), with datum $H = (n-1)H'$. Define $\tilde{\Omega}$ as the unique $(n-1)^{\text{th}}$ root of $\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}$. Then it is clear that if Ω_2 is the $(2,0)$-form induced by a quaternionic balanced (resp. quaternionic Gauduchon, quaternionic strongly Gauduchon) metric, then so is $\tilde{\Omega}$. Finally, set $b' = b^{1/(n-1)}$, then using (6.40) we conclude

$$\frac{\tilde{\Omega}^n}{\Omega_0^n} = \left(\frac{\det(\tilde{\Omega}^{n-1})}{\det(\Omega_0^{n-1})}\right)^{\frac{1}{n-1}} = \left(\frac{\det\left(\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}\right)}{\det(\Omega_0^{n-1})}\right)^{\frac{1}{n-1}} = \left(b\,\mathrm{e}^H\right)^{\frac{1}{n-1}} = b'\,\mathrm{e}^{H'} \,. \qquad \square$$

CHAPTER 7

MORE GENERAL PARABOLIC EQUATIONS

After Yau's solution [327] of the Calabi conjecture, Cao [69] was able to provide a parabolic proof, using what is now called the Kähler-Ricci flow. Ever since then, it is now a well-established practice to design parabolic geometric flows as an alternative way to solve fully non-linear elliptic equations (see e.g. [33, 91, 119, 120, 144, 145, 257, 265, 275, 277, 276, 334]).

Following this line of thoughts, the investigation of the previous chapter about fully non-linear elliptic equations on hyperhermitian manifolds is hereby extended to the parabolic setting. Here we develop the corresponding parabolic theory in the same spirit as Phong-Tô [246].

The treatment of this chapter is based on [140].

7.1 Overview.

Setting of the problem.

Let $(M, I, J, K, g, \Omega_0)$ be a compact locally flat hyperhermitian manifold where Ω_0 is the $(2,0)$-form induced by g, i.e. $\Omega_0 = g(J\cdot,\cdot) + ig(K\cdot,\cdot)$. The assumption of local flatness allows us to represent locally in quaternionic coordinates every q-real $(2,0)$-form Ω by a hyperhermitian matrix $(\Omega_{\bar{r}s})$. Fix one such form Ω, which does not need to coincide with Ω_0. For a smooth real function φ on M the $(2,0)$-form $\partial\partial_J\varphi$ is q-real. Then we may associate a hyperhermitian matrix to the form

$$\Omega_\varphi := \Omega + \partial\partial_J\varphi$$

let us denote it by $(\Omega^{\varphi}_{\bar{r}s})$. Set $A^r_s[\varphi] = g^{\bar{j}r}\Omega^{\varphi}_{\bar{j}s}$. The matrix $(A^r_s[\varphi])$ defines a hyperhermitian endomorphism of TM with respect to the metric g and this makes it meaningful to speak about the n-tuple of its eigenvalues $\lambda(A[\varphi])$.

The class of parabolic equations that we take into account here is the following:

$$\partial_t\varphi = F(A[\varphi]) - h\,, \qquad \varphi(x,0) = \varphi_0\,, \qquad t \in [0,\infty)\,, \tag{7.1}$$

where $h \in C^\infty(M,\mathbb{R})$ is the datum and $F(A[\varphi]) = f(\lambda(A[\varphi]))$ is a smooth symmetric operator of the eigenvalues of $A[\varphi]$ satisfying certain assumptions. Here Γ satisfies the same assumptions of the previous chapter, we repeat them for convenience of the reader. Γ is a proper convex open cone in $\mathbb{R}^n$ with vertex at the origin, containing the positive orthant

$$\Gamma_n = \{\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mid \lambda_i > 0,\, i = 1,\dots,n\}\,,$$

Γ is symmetric and $f\colon \Gamma \to \mathbb{R}$ satisfies the following assumptions:

C1) $f_i := \frac{\partial f}{\partial \lambda_i} > 0$ for all $i = 1,\dots,n$ and f is a concave function.

C2) $\sup_{\partial\Gamma} f < \inf_M h$, where $\sup_{\partial\Gamma} f = \sup_{\lambda_0\in\partial\Gamma} \limsup_{\lambda\to\lambda_0} f(\lambda)$.

C3) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$ we have $\lim_{t\to\infty} f(t\lambda) > \sigma$.

Assumption C1 implies parabolicity of equation (7.1) over the space of Γ-admissible functions, where a function $\varphi \in C^{2,1}(M \times [0,T))$ is Γ-*admissible* if

$$\lambda(A[\varphi]) \in \Gamma\,, \qquad \text{for all } (x,t) \in M \times [0,T)\,.$$

In particular, from standard parabolic theory, equation (7.1) admits a unique maximal smooth solution. Assumption C2 guarantees that the level sets of f do not intersect the boundary of Γ, this yields non-degeneracy of (7.1) and entails uniform parabolicity, once we obtain the $C^{2,1}$ estimate. We also remark that the assumptions on Γ imply the inclusion

$$\Gamma \subseteq \left\{ (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0 \right\}. \tag{7.2}$$

We now project Γ onto a new cone in $\mathbb{R}^{n-1}$:

$$\Gamma_\infty = \{\lambda' = (\lambda_1,\cdots,\lambda_{n-1}) \in \mathbb{R}^{n-1} \mid \text{there exists } \lambda_n \in \mathbb{R} \text{ such that } (\lambda',\lambda_n) \in \Gamma\}\,.$$

Therefore, for every $\lambda' \in \Gamma_\infty$, there exists a constant s_0 such that for each $s \geq s_0$, we have $(\lambda', s) \in \Gamma$. Let $f_\infty(\lambda') = \lim_{s\to\infty} f(\lambda', s)$. It is an observation of Trudinger [293] that, since f is concave on Γ, there is a dichotomy:

(i) Either f_∞ is unbounded at any point in Γ_∞ and we will refer to this case by saying that f is *unbounded* over Γ;

(ii) Or f_∞ is bounded on Γ_∞ and we will simply say that f is *bounded* over Γ.

Statement of the main results.

Before stating our main results, we need to recall the terminology of parabolic $\mathcal{C}$-subsolutions introduced in [246].

Definition 7.1. We say that a function $\underline{\varphi} \in C^{2,1}(M \times [0,T))$ is a **parabolic $\mathcal{C}$-subsolution** for equation (7.1) if there exist uniform constants $\delta, R > 0$, such that on $M \times [0,T)$,

$$f(\lambda(A[\underline{\varphi}]) + \mu) - \partial_t \underline{\varphi} + \tau = h,\ \ \mu + \delta \mathbf{1} \in \Gamma_n \text{ and } \tau > -\delta \tag{7.3}$$

implies that $|\mu| + |\tau| < R$, where $\mathbf{1} = (1, 1, \dots, 1)$.

In the unbounded case, as we shall show, any Γ-admissible function is a parabolic $\mathcal{C}$-subsolution, and we have the following result:

Theorem 7.2. *Suppose f is unbounded on Γ. Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold. Then for any Γ-admissible initial datum φ_0, the solution φ to* (7.1) *exists for all time.*

Moreover, if we let

$$\tilde\varphi = \varphi - \frac{\int_M \varphi\, \Omega_0^n \wedge \bar\Omega_0^n}{\int_M \Omega_0^n \wedge \bar\Omega_0^n}\,, \tag{7.4}$$

then $\tilde\varphi$ converges smoothly to some function $\tilde\varphi_\infty \in C^\infty(M,\mathbb{R})$ as $t \to \infty$, and there exists a constant $b \in \mathbb{R}$ such that

$$F(A[\tilde\varphi_\infty]) = h + b\,. \tag{7.5}$$

In the bounded case we observe that, unfortunately, Γ-admissible functions might not be $\mathcal{C}$-subsolutions. Compared to Theorem 7.2 the main result in the bounded case looks a little bit more artificial, as it requires some additional assumptions.

Theorem 7.3. *Suppose f is bounded on Γ. Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold. For any Γ-admissible initial datum φ_0, let $\varphi \in C^\infty(M \times [0, T), \mathbb{R})$ be the maximal solution of flow* (7.1)*. Assume further that*

(i) either it holds

$$\partial_t \underline{\varphi} \geq \sup_M \left(F(A[\varphi_0]) - h\right) ; \tag{7.6}$$

(ii) or there exists a non-increasing function Φ of class C^1 on $\mathbb{R}$ such that

$$\begin{cases} \sup_M \left(\varphi(\cdot, t) - \underline{\varphi}(\cdot, t) - \Phi(t)\right) \geq 0\,, \\ \sup_M \left(\varphi(\cdot, t) - \Phi(t)\right) \leq -C \inf_M \left(\varphi(\cdot, t) - \Phi(t)\right) + C \end{cases} \tag{7.7}$$

for all $t \in (0, T)$ and a time-independent positive constant C. Then $T = \infty$, i.e. the solution φ exists for all times, and the normalization $\tilde{\varphi}$ converges smoothly to a function $\tilde{\varphi}_\infty \in C^\infty(M, \mathbb{R})$ as $t \to \infty$, which solves (7.5) *for some $b \in \mathbb{R}$.*

Let (M, I, J, K, g) be a locally flat hyperhermitian manifold and Ω a q-real $(2, 0)$-form. Equation (7.1) is expressed in terms of the matrix

$$A^r_s[\varphi] = g^{\bar{j}r} \Omega^\varphi_{\bar{j}s} = g^{\bar{j}r} \left(\Omega_{\bar{j}s} + \varphi_{\bar{j}s}\right)$$

where $(\varphi_{\bar{j}s})$ denotes the hyperhermitian matrix associated to $\partial\partial_J \varphi$. With respect to quaternionic local coordinates $(q^1, \ldots, q^n)$ it is well-known that

$$\varphi_{\bar{r}s} = \frac{1}{4} \partial_{\bar{q}^r} \partial_{q^s} \varphi =: \mathrm{Hess}_{\mathbb{H}} \varphi\,,$$

Now we briefly discuss the notion of $\mathcal{C}$-subsolution. Székelyhidi introduced it in [280] for elliptic equations. His definition is also shown to be a relaxation of that given by Guan [153]. As for the parabolic case, Guan, Shi and Sui [155] worked on Riemannian manifolds with the classical notion of a subsolution, while Phong and Tô provided in [246] the extension to the parabolic case of Székelyhidi's definition. Of course, as we shall see in a moment with a characterization of $\mathcal{C}$-subsolutions, what happens in hyperhermitian geometry is entirely parallel to the Hermitian case. Thus, Definition 7.1 is the right extension of Definition 6.2 for the elliptic case. We shall refer to $\mathcal{C}$-subsolutions in this last sense as *elliptic* ones.

Lemma 7.4. *Let $\underline{\varphi} \in C^{2,1}(M \times [0, +\infty))$ be such that $\|\underline{\varphi}\|_{C^{2,1}} < +\infty$. Then $\underline{\varphi}$ is a parabolic $\mathcal{C}$-subsolution if and only if there exists a uniform constant $\rho > 0$ such that*

$$\lim_{s \to \infty} f(\lambda[\underline{\varphi}(x, t)] + s e_i) - \partial_t \underline{\varphi}(x, t) > \rho + h(x)$$

for each $i = 1, \ldots, n$, where e_i is the i^{th} standard basis vector of $\mathbb{R}^n$. In particular when $\underline{\varphi}$ is time-independent it is a $\mathcal{C}$-subsolution in the parabolic sense if and only if it is such in the elliptic sense.

Proof. The proof can be reproduced almost verbatim from [246, Lemma 8]. □

This lemma in particular implies that when f is unbounded over Γ, every Γ-admissible function is a parabolic $\mathcal{C}$-subsolution.

We conclude this section by fixing some notations. Unless otherwise stated we shall always denote by φ, $\tilde{\varphi}$ and $\underline{\varphi}$ the maximal solution to flow (7.1) with initial datum φ_0, its normalization as in (7.4) and a parabolic $\mathcal{C}$-subsolution in the sense of Definition 7.1, respectively. All these functions are assumed to be defined over $M \times [0, T)$, where (M, I, J, K, g) is a compact locally flat hyperhermitian manifold and T is the maximal time of existence of φ.

From here on, we will always denote with C a positive constant that only depends on background data (not on time!), including the initial datum φ_0. Occasionally we might say that C is *uniform*, to stress that it is time-independent. As it is customary, the constant C may change value from line to line.

7.2 A priori estimates and long-time existence.

7.2.1 C^0 estimates.

In this section we achieve estimates of order zero for the solution φ and its normalization $\tilde{\varphi}$. We start by bounding their time derivatives, then, in order to treat the bounded case we need an additional inequality proved in Lemma 7.6. Such lemma follows as an application of the parabolic version of the Alexandroff-Bakelman-Pucci (ABP) inequality due to Tso [295, Proposition 2.1] by adapting the argument of Phong-Tô [246, Lemma 1].

Bounds on $\partial_t\varphi$ and $\partial_t\tilde{\varphi}$.

Lemma 7.5. *We have*

$$\inf_M \big(F(A[\varphi_0]) - h\big) \leq \partial_t\varphi \leq \sup_M \big(F(A[\varphi_0]) - h\big) \tag{7.8}$$

and

$$|\partial_t\tilde{\varphi}| \leq C\,, \tag{7.9}$$

for a uniform constant $C > 0$ depending only on h and the initial datum φ_0.

Proof. Differentiating the flow (7.1) along ∂_t we see that $\partial_t\varphi$ satisfies the following heat type equation

$$\partial_t\big(\partial_t\varphi\big) = \frac{1}{4}\mathrm{Re}\left(F^{rs}\partial_{\bar{q}^r}\partial_{q^s}(\partial_t\varphi)\right), \tag{7.10}$$

where $F^{rs} := \frac{\partial F}{\partial A_{rs}}$. By the parabolic maximum principle for (7.10), we know that $\partial_t\varphi$ hits its extremum at $t = 0$. Thus,

$$\inf_{M\times\{0\}} \partial_t\varphi \leq \partial_t\varphi \leq \sup_{M\times\{0\}} \partial_t\varphi\,, \qquad \partial_t\varphi(\cdot,0) = F(A[\varphi_0]) - h$$

and we then obtain (7.8). The bound (7.9) on $|\partial_t\tilde{\varphi}|$ follows immediately. □

We remark that a direct consequence of the previous lemma is the following short-time estimate:

$$|\varphi| \leq C\delta\,, \qquad \text{on } M\times[0,\delta]\,. \tag{7.11}$$

Intermediate bounds.

Lemma 7.6. *If there exists a non-increasing function $\Phi \in C^1([0,T),\mathbb{R})$ satisfying*

$$\sup_M \big(\varphi(\cdot,t) - \underline{\varphi}(\cdot,t) - \Phi(t)\big) \geq 0\,,$$

then there exists a constant $C > 0$, depending only on $\Omega, g, \underline{\varphi}, \|\varphi_0\|_{C^0}$ such that

$$\varphi(x,t) - \underline{\varphi}(x,t) - \Phi(t) \geq -C \qquad \textit{for all } (x,t) \in M\times[0,T)\,.$$

Proof. First, observe that the requirement $\Phi' \leq 0$ implies that $\underline{\varphi} + \Phi$ is still a parabolic $\mathcal{C}$-subsolution of (7.1), therefore, as long as the involved constants do not depend on the time derivative of $\underline{\varphi}$, we may assume $\Phi \equiv 0$.

Choose $\delta \in (0,1)$ and $R > 0$ such that (7.3) holds for the subsolution $\underline{\varphi}$. By (7.11), it suffices to estimate $v = \varphi - \underline{\varphi}$ on $M \times [\delta, T)$. Fix an arbitrary $T' < T$ and assume v achieves its minimum S at a point $(x_0, t_0) \in M \times [\delta, T']$, i.e.,

$$S = v(x_0, t_0) = \min_{M \times [\delta, T']} v\,.$$

Now we are reduced to prove that if $\sup_M v \geq 0$ for all $t \in [\delta, T']$, then S is bounded from below by a constant depending only on $\Omega, g, \underline{\varphi}, \|\varphi_0\|$ and independent of T'.

Consider quaternionic local coordinates $(q^1, \ldots, q^n)$ centered at the point x_0. We may identify such coordinate neighborhood with the open ball of unit radius $B_1 = B_1(0) \subseteq \mathbb{H}^n$ centered at the origin. Let

$$w(x,t) = v(x,t) + \frac{\delta^2}{4}|x|^2 + (t - t_0)^2,$$

be a function defined on $\mathcal{B} = B_1 \times [t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}]$. Observe that $\inf_{\mathcal{B}} w = w(0,t_0) = v(0,t_0) = S$ and $\inf_{\partial\mathcal{B}} w \geq w(0,t_0) + \frac{\delta^2}{4}$. These conditions allow us to apply the parabolic ABP method of Tso [295, Proposition 2.1] to obtain

$$C_0 \delta^{8n+2} \leq \int_P |\partial_t w| \det(D^2 w)\,, \tag{7.12}$$

where $C_0 > 0$ is a dimensional constant,

$$P = \left\{ (x,t) \in \mathcal{B} \;\middle|\; \begin{array}{c} w(x,t) \leq S + \dfrac{\delta^2}{4}, \quad |Dw(x,t)| < \dfrac{\delta^2}{8}, \\ w(y,s) \geq w(x,t) + Dw(x,t)\cdot(y-x),\ \forall y \in B_1, \quad s \leq t \end{array} \right\}$$

is the parabolic contact set of w on $\mathcal{B}$ and Dw, $D^2 w$ are the gradient and the (real) Hessian of w on M with respect to the variable x.

Claim: both $|\partial_t w|$ and $\det(D^2 w)$ are bounded on P.

Let $\tau = -\partial_t \varphi + \partial_t \underline{\varphi} = -\partial_t v$ and $\mu = \lambda(A[\varphi]) - \lambda(A[\underline{\varphi}])$. Observe that $D^2 w \geq 0$ and $\partial_t w \leq 0$ on P. Thus,

$$\tau = -\partial_t w + 2(t - t_0) \geq -\delta\,, \qquad \mu + \delta \mathbf{1} \in \Gamma_n\,.$$

Now by Definition 7.1 we conclude that $|\tau| + |\mu| \leq R$, then $|\partial_t w| \leq R$ and $\mathrm{Hess}_{\mathbb{H}} w$ is a bounded matrix. But then we are done as we have

$$\det(D^2 w) \leq 2^{4n} \det(\mathrm{Hess}_{\mathbb{H}} w)^4 \quad \text{on } P\,,$$

This confirms the claim.

With this claim at hand, by (7.12) we have

$$C_0 \delta^{8n+2} \leq C \mathrm{Vol}(P)\,. \tag{7.13}$$

From (7.2) we readily obtain $\mathrm{Re}\,\mathrm{tr}_g(\Omega_\varphi) > 0$, where $\Omega_\varphi = \Omega + \partial\partial_J \varphi$, which in turn yields a uniform lower bound for the quaternionic Laplacian of φ:

$$\Delta_g \varphi = \mathrm{Re}\,\mathrm{tr}_g(\Omega_\varphi) - \mathrm{Re}\,\mathrm{tr}_g(\Omega) \geq -C\,.$$

This also gives a uniform lower bound for $\Delta_g v$. By Lemma 6.3 there exist uniform $p, C > 0$, depending only on the background data such that

$$\|v - \sup_M v\|_{L^p(M)} \leq C. \tag{7.14}$$

The definition of P and our assumption that $\sup_M v \geq 0$ on $[0,T)$ yield

$$v - \sup_M v \leq v \leq w < S + \frac{\delta^2}{4} \quad \text{on } P,$$

We may further assume $S+\frac{\delta^2}{4}<0$, otherwise we are done. As a consequence for any $p>0$

$$\left|S+\frac{\delta^2}{4}\right|^p \mathrm{Vol}(P) \le \int_P |v-\sup_M v|^p dxdt \le \int_{[t_0-\frac{\delta}{2},t_0+\frac{\delta}{2}]} \|v-\sup_M v\|^p_{L^p(M)} dt \le C\delta\,,$$

where we have used (7.14). This, together with (7.13), gives the uniform lower bound of S we were after. □

Bounds on φ and $\tilde{\varphi}$.

As it often happens for solutions to flows, we only manage to control the oscillation and not the full C^0 norm. On the other hand, once the oscillation is under control, we immediately achieve the C^0-estimate for the normalization of the solution.

Proposition 7.7. *Let f be either bounded or unbounded. In case f is bounded on Γ assume that it satisfies either one of the two conditions expressed in Theorem 7.3. Then there exists a uniform constant $C>0$, depending only on the background data such that*

$$\mathrm{osc}_M\varphi(\cdot,t) := \sup_M \varphi(\cdot,t) - \inf_M \varphi(\cdot,t) \le C\,, \tag{7.15}$$

and

$$\|\tilde{\varphi}\|_{C^0} \le C\,. \tag{7.16}$$

Proof. First, we observe that (7.16) follows from (7.15). Indeed, by the normalization of $\tilde{\varphi}$, for any $(x,t)\in M\times[0,T)$ we can find $y(x)\in M$ such that $\tilde{\varphi}(y(x),t)=0$, therefore

$$\|\tilde{\varphi}\|_{C^0} = \sup_{(x,t)\in M} \left|\tilde{\varphi}(x,t)-\tilde{\varphi}(y(x),t)\right| = \sup_{(x,t)\in M} \left|\varphi(x,t)-\varphi(y(x),t)\right| \le \mathrm{osc}_M\varphi(\cdot,t)\,.$$

We will prove (7.15) by rewriting the flow (7.1) as

$$F(A[\varphi]) = h+\partial_t\varphi\,, \tag{7.17}$$

and interpreting it for every fixed time as an elliptic equation with datum $h+\partial_t\varphi$. We split the argument into two cases according as f is bounded or unbounded.

- Case 1. f is unbounded on Γ. In this case any Γ-admissible function is a parabolic $\mathcal{C}$-subsolution, therefore we can take the initial datum φ_0 as such. Since φ_0 is time-independent, it can be regarded as an elliptic $\mathcal{C}$-subsolution. Furthermore, by Lemma 7.5 we know that the right-hand side of (7.17) is uniformly bounded, therefore we may apply Proposition 6.4 to obtain (7.15).
- Case 2. f is bounded on Γ. We consider two subcases. Assume that condition (i) of Theorem 7.2 holds, then (7.6) and Lemma 7.5 imply that $\partial_t\underline{\varphi}\ge\partial_t\varphi$, this entails that $\underline{\varphi}$ is a $\mathcal{C}$-subsolution of (7.17) in the elliptic sense. Again (7.15) follows from Proposition 6.4. If, instead, condition (ii) of Theorem 7.2 is satisfied, then there exists $\Phi\in C^\infty([0,T),\mathbb{R})$ with $\Phi'\le 0$ satisfying (7.7) and we can readily apply Lemma 7.6 to conclude. □

7.2.2 Laplacian estimate.

Here we adopt the technique of [89, 176] which allows to find a Laplacian bound in terms of the squared norm of the gradient.

Before we tackle the proof, we recall the following preliminary lemma given in Phong-Tô [246, Lemma 3], which was inspired by the elliptic version of [280, Proposition 6]. We will use the following derivatives of F

$$F^{rs} := \frac{\partial F}{\partial A_{rs}}\,, \qquad F^{rs,lt} := \frac{\partial^2 F}{\partial A_{rs}\partial A_{lt}}\,.$$

Lemma 7.8. *Let δ, R be uniform constants such that on $M\times[0,T)$, if $(\mu,\tau)\in\mathbb{R}^n\times\mathbb{R}$ satisfy* (7.3)*, then $|\mu|+|\tau|<R$. There exists a uniform constant $\kappa>0$ depending on δ and R such that if $|\lambda(A[\varphi])-\lambda(A[\underline{\varphi}])|>R$, we have*

$$\text{either}\qquad \operatorname{Re}F^{rs}(A[\varphi])\left(A_{rs}[\underline{\varphi}]-A_{rs}[\varphi]\right)-(\partial_t\underline{\varphi}-\partial_t\varphi)>\kappa\sum_{r=1}^n F^{rr}(A[\varphi])\,,$$

$$\text{or}\qquad F^{ss}(A[\varphi])>\kappa\sum_{r=1}^n F^{rr}(A[\varphi])\,,\qquad \text{for all } s=1,\dots,n\,.$$

Proof. Since the quaternionic analogue of the Schur-Horn theorem holds (Proposition 1.46) the proof of the lemma can be adapted from [246, Lemma 3]. □

Proposition 7.9. *Suppose (M,I,J,K,g) is a compact flat hyperkähler manifold. Then there is a constant $C>0$, depending only on (M,I,J,K), $\|g\|_{C^2}$, $\|h\|_{C^2}$, $\|\Omega\|_{C^2}$, $\|\underline{\varphi}\|_{C^{2,1}}$, $\|\partial_t\varphi\|_{C^0}$ and $\|\tilde\varphi\|_{C^0}$, such that*

$$\|\Delta_g\varphi\|_{C^0}\le C\left(\|\nabla\varphi\|^2_{C^0}+1\right)\,.$$

Proof. By (7.2) we already know that the quaternionic Laplacian is uniformly bounded from below, therefore it is enough to obtain a bound of the form

$$\frac{\lambda_1}{\|\nabla\varphi\|^2_{C^0}+1}\le C\,,$$

where λ_1 is the largest eigenvalue of $A[\varphi]$. Let $T'<T$, all computations will be performed in quaternionic local coordinates around some fixed point $p_0=(x_0,t_0)\in M\times[0,T']$ which we will specify in a moment. As pointed out in Section 6.2.2 in order for $\lambda_1\colon M\to\mathbb{R}$ to define a smooth function at p_0 we need the eigenvalues to be distinct; to be sure of that, we perturb the matrix A as follows. Using the assumption that g is a flat hyperkähler metric we may take quaternionic coordinates such that $(g_{\bar r s})$ is the identity in the whole neighborhood of p_0 and $(\Omega^\varphi_{\bar r s})$ is diagonal at p_0. In particular $A[\varphi]$ is diagonal with ordered eigenvalues $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$. Let D be a constant diagonal matrix with entries satisfying $0=D_{11}<D_{22}<\cdots<D_{nn}$. The matrix $\tilde A=A[\varphi]-D$ has distinct eigenvalues $\tilde\lambda_r$ by construction, and its largest eigenvalue $\tilde\lambda_1$ coincides with λ_1 at p_0.

Choose $p_0\in M\times[0,T']$ to be a maximum point of the function

$$\hat G=2\sqrt{\lambda_1}+\alpha(|\nabla\varphi|^2)+\beta(\tilde v)$$

where

$$\alpha(s)=-\frac12\log\left(1-\frac{s}{2N}\right)\,,\qquad N=\|\nabla\varphi\|^2_{C^0}+1\,,$$
$$\beta(s)=-2Ss+\frac12 s^2\,,\qquad S>\|\tilde v\|_{C^0}\,,\ \text{large constant to be chosen later}\,,$$

and $\tilde v$ is the normalization of $v=\varphi-\underline{\varphi}$. As said, to avoid smoothness issues we shall not work with λ_1. Therefore, in a small neighborhood of p_0, instead of working with $\hat G$ we consider the function

$$G=2\sqrt{\tilde\lambda_1}+\alpha(|\nabla\varphi|^2)+\beta(\tilde v)\,.$$

It will be useful to observe that

$$\frac{1}{4N}<\alpha'(|\nabla\varphi|^2)<\frac{1}{2N}\,,\qquad \alpha''=2(\alpha')^2\,,\tag{7.18}$$
$$S\le-\beta'(\tilde v)\le 3S\,,\qquad \beta''=1\,.\tag{7.19}$$

We also remark that, as in [246], at the point p_0 there exists a constant $\tau>0$ depending on $\|h\|_{C^0}$

and $\|\partial_t\varphi\|_{C^0}$ such that

$$\mathcal{F} := \sum_{a=1}^{n} F^{aa}(A[\varphi]) > \tau\,.$$

This will be useful to absorb some constants during our computations.

The linearized operator L is defined by

$$L(u) = 4\sum_{a,b=1}^{n} F^{ab} g^{\bar{c}a} u_{\bar{c}b} - 4\partial_t u\,,$$

where $u_{\bar{a}b} = \frac{1}{4}\partial_{\bar{q}^a}\partial_{q^b}u$. In particular, at p_0 the linearized operator has the simpler expression $L(u) = 4(F^{aa}u_{\bar{a}a} - \partial_t u)$.

At the maximum point p_0 we have $L(G) \le 0$ i.e.

$$0 \ge L\left(2\sqrt{\tilde\lambda_1}\right) + L\left(\alpha(|\nabla\varphi|^2)\right) + L\left(\beta(\tilde v)\right)\,. \tag{7.20}$$

Bound for $L(2\sqrt{\tilde\lambda_1})$.

We claim that

$$L\left(2\sqrt{\tilde\lambda_1}\right) \ge -\frac{F^{aa}|\Omega^{\varphi}_{\bar{1}1,a}|^2}{2\lambda_1\sqrt{\lambda_1}} - \frac{C\mathcal{F}}{\sqrt{\lambda_1}}\,, \tag{7.21}$$

where $\Omega^{\varphi}_{\bar{1}1,a} = \partial_{q^a}\Omega^{\varphi}_{\bar{1}1}$ and $C>0$ is a positive uniform constant.

We clearly have

$$\begin{aligned} L\left(2\sqrt{\tilde\lambda_1}\right) &= 8F^{aa}\left(\sqrt{\tilde\lambda_1}\right)_{\bar{a}a} - 8\partial_t\left(\sqrt{\tilde\lambda_1}\right) = 2F^{aa}\sum_{p=0}^{3}\left(\sqrt{\tilde\lambda_1}\right)_{x_p^a x_p^a} - 8\partial_t\left(\sqrt{\tilde\lambda_1}\right) \\ &= \frac{1}{\sqrt{\lambda_1}}\left(F^{aa}\sum_{p=1}^{3}\tilde\lambda_{1,x_p^a x_p^a} - 4\partial_t\tilde\lambda_1\right) - F^{aa}\sum_{p=0}^{3}\frac{|\lambda_{1,x_p^a}|^2}{2\lambda_1\sqrt{\lambda_1}}\,, \end{aligned} \tag{7.22}$$

where the subscripts x_p^a denote the real derivative with respect to the corresponding real coordinates underlying the chosen quaternionic local coordinates. Using the formulas for the derivatives of the eigenvalues (see Section 6.2.2) and the fact that D is a constant matrix we obtain at p_0

$$\begin{aligned} \tilde\lambda_{1,x_p^a} &= \tilde\lambda_1^{rs}\tilde{A}_{rs,x_p^a} = \Omega^{\varphi}_{\bar{1}1,x_p^a} \\ \tilde\lambda_{1,x_p^a x_p^a} &= \tilde\lambda_1^{rs,lt}\tilde{A}_{rs,x_p^a}\tilde{A}_{lt,x_p^a} + \tilde\lambda_1^{rs}\tilde{A}_{rs,x_p^a x_p^a} = 2\sum_{r>1}\frac{|\Omega^{\varphi}_{\bar{r}1,x_p^a}|^2}{\lambda_1 - \tilde\lambda_r} + \Omega^{\varphi}_{\bar{1}1,x_p^a x_p^a}\,. \end{aligned}$$

Observe that

$$\sum_{p=0}^{3}\Omega^{\varphi}_{\bar{1}1,x_p^a x_p^a} = \sum_{p=0}^{3}\left(\Omega_{\bar{1}1,x_p^a x_p^a} + \varphi_{\bar{1}1x_p^a x_p^a}\right) = 4\Omega_{\bar{1}1,\bar{a}a} + 4\varphi_{\bar{a}a\bar{1}1} = 4\Omega_{\bar{1}1,\bar{a}a} - 4\Omega_{\bar{a}a,\bar{1}1} + \sum_{p=0}^{3}\Omega^{\varphi}_{\bar{a}a,x_p^1 x_p^1}$$

which implies

$$F^{aa}\tilde\lambda_{1,x_p^a x_p^a} \ge F^{aa}\sum_{p=0}^{3}\Omega^{\varphi}_{\bar{a}a,x_p^1 x_p^1} - C\mathcal{F}\,.$$

Differentiating the equation $\partial_t\varphi = F(A[\varphi]) - h$ twice with respect to x_p^1 gives, at p_0,

$$F^{rs,tl}\Omega^{\varphi}_{\bar{r}s,x_p^1}\Omega^{\varphi}_{\bar{t}l,x_p^1} + F^{aa}\Omega^{\varphi}_{\bar{a}a,x_p^1 x_p^1} = h_{x_p^1 x_p^1} + \partial_t(\varphi_{x_p^1 x_p^1})\,.$$

by this and the concavity of F

$$F^{aa}\sum_{p=0}^{3}\tilde\lambda_{1,x_p^a x_p^a}-4\partial_t\tilde\lambda_1\geq\sum_{p=0}^{3}\Big(F^{aa}\Omega^{\varphi}_{\bar aa,x_p^1x_p^1}-\partial_t(\varphi_{x_p^1x_p^1})\Big)-C\mathcal F\geq -C\mathcal F\,. \tag{7.23}$$

Substituting (7.23) into (7.22) we obtained the claimed inequality (7.21).

Bound for $L\left(\alpha(|\nabla\varphi|^2)\right)$.

First of all we compute

$$\begin{aligned} L\left(\alpha(|\nabla\varphi|^2)\right)=&\,\alpha''F^{aa}\sum_{p=0}^{3}\left(\sum_{r=1}^{n}(\varphi_{\bar r x_p^a}\varphi_r+\varphi_{\bar r}\varphi_{r x_p^a})\right)^2\\ &+\alpha' F^{aa}\sum_{p=0}^{3}\sum_{r=1}^{n}\Big(\varphi_{\bar r x_p^a x_p^a}\varphi_r+2|\varphi_{r x_p^a}|^2+\varphi_{\bar r}\varphi_{r x_p^a x_p^a}\Big)\\ &-\alpha'\sum_{r=1}^{n}\Big(\partial_t(\varphi_{\bar r})\varphi_r+\varphi_{\bar r}\partial_t(\varphi_r)\Big)\,. \end{aligned}\tag{7.24}$$

Differentiating the equation $\partial_t\varphi=F(A[\varphi])-h$ yields

$$\partial_t(\varphi_{x_p^r})=F^{aa}\Omega^{\varphi}_{\bar aa,x_p^r}-h_{x_p^r}\,,\qquad \text{at } p_0\,.$$

Together with Cauchy-Schwarz inequality and (7.18) this yields

$$\begin{aligned} \alpha' F^{aa}\sum_{r=1}^{n}(\varphi_{\bar r\bar aa}\varphi_r+\varphi_{\bar r}\varphi_{r\bar aa})-\alpha'\sum_{r=1}^{n}\Big(\partial_t(\varphi_{\bar r})\varphi_r+\varphi_{\bar r}\partial_t(\varphi_r)\Big)&\\ =\alpha'\sum_{r=1}^{n}\left((h_{\bar r}-F^{aa}\Omega_{\bar aa,\bar r})\varphi_r+\varphi_{\bar r}(h_r-F^{aa}\Omega_{\bar aa,r})\right)&\\ \geq -\frac{C}{N}(N^{1/2}+N^{1/2}\mathcal F)\geq -C\mathcal F\,,& \end{aligned}$$

Moreover, we have

$$2\alpha' F^{aa}\sum_{r=1}^{n}\sum_{p=0}^{3}|\varphi_{r x_p^a}|^2\geq\frac{1}{2N}F^{aa}\sum_{p=0}^{3}\varphi^2_{x_p^a x_p^a}=\frac{8}{N}F^{aa}\varphi^2_{\bar aa}=\frac{8}{N}F^{aa}(\lambda_a-\Omega_{\bar aa})^2\geq\frac{4}{N}F^{aa}\lambda_a^2-C\mathcal F\,.$$

Combining the last two inequalities with (7.24) we get

$$L\left(\alpha(|\nabla\varphi|^2)\right)\geq\alpha''F^{aa}\sum_{p=0}^{3}\left(\sum_{r=1}^{n}(\varphi_{\bar r x_p^a}\varphi_r+\varphi_{\bar r}\varphi_{r x_p^a})\right)^2+\frac{4}{N}F^{aa}\lambda_a^2-C\mathcal F\,. \tag{7.25}$$

Conclusion of the proof.

In view of (7.21) and (7.25), the main inequality (7.20) becomes

$$0\geq\alpha''F^{aa}\sum_{p=0}^{3}\left(2\sum_{r=1}^{n}\mathrm{Re}(\varphi_{\bar r x_p^a}\varphi_r)\right)^2-\frac{F^{aa}|\Omega^{\varphi}_{\bar 11,a}|^2}{2\lambda_1\sqrt{\lambda_1}}+\frac{4F^{aa}\lambda_a^2}{N}+L\left(\beta(\tilde v)\right)-C\mathcal F \tag{7.26}$$

Since p_0 is a maximum point for G we have

$$0 = G_{x_p^a} = \frac{\Omega^{\varphi}_{\bar{1}1,x_p^a}}{\sqrt{\lambda_1}} + 2\alpha' \sum_{r=1}^n \mathrm{Re}(\varphi_{\bar{r}x_p^a}\varphi_r) + \beta' \tilde{v}_{x_p^a}$$

and therefore, by (7.18)

$$\begin{aligned} \alpha'' F^{aa} \left(2\sum_{r=1}^n \mathrm{Re}(\varphi_{\bar{r}x_p^a}\varphi_r)\right)^2 &= 2F^{aa}\left(\frac{\Omega^{\varphi}_{\bar{1}1,x_p^a}}{\sqrt{\lambda_1}} + \beta'\tilde{v}_{x_p^a}\right)^2 \\ &\geq 2\varepsilon \frac{F^{aa}(\Omega^{\varphi}_{\bar{1}1,x_p^a})^2}{\lambda_1} - \frac{2\varepsilon}{1-\varepsilon}(\beta')^2 F^{aa}\tilde{v}^2_{x_p^a}\,, \end{aligned} \tag{7.27}$$

where we used the inequality $(a+b)^2 \geq \varepsilon a^2 - \frac{\varepsilon}{1-\varepsilon} b^2$, which holds for $\varepsilon \in (0,1)$. Assuming without loss of generality that $\sqrt{\lambda_1} > \frac{1}{4\varepsilon}$ we get

$$\left(4\varepsilon\sqrt{\lambda_1} - 1\right) \frac{F^{aa}|\Omega^{\varphi}_{\bar{1}1,a}|^2}{2\lambda_1\sqrt{\lambda_1}} \geq 0\,. \tag{7.28}$$

Putting together (7.27), (7.28) and the calculation

$$L\left(\beta(\tilde{v})\right) = \beta'' F^{aa}|\tilde{v}_a|^2 + 4\beta' F^{aa}\tilde{v}_{\bar{a}a} - 4\beta'\partial_t\tilde{v}$$

(7.26) simplifies to

$$0 \geq \frac{4F^{aa}\lambda_a^2}{N} + \left(\beta'' - \frac{2\varepsilon}{1-\varepsilon}(\beta')^2\right) F^{aa}|\tilde{v}_a|^2 + 4\beta'\left(F^{aa}\tilde{v}_{\bar{a}a} - \partial_t\tilde{v}\right) - C\mathcal{F}\,.$$

If we choose $\varepsilon = 1/(18S^2+1) < 1$, then (7.19) yields

$$\beta'' - \frac{2\varepsilon}{1-\varepsilon}(\beta')^2 \geq 0\,,$$

therefore we finally arrive at

$$0 \geq \frac{4F^{aa}\lambda_a^2}{N} + 4\beta'\left(F^{aa}\tilde{v}_{\bar{a}a} - \partial_t\tilde{v}\right) - C\mathcal{F}\,. \tag{7.29}$$

Supposing $\lambda_1 > R$ we have $|\lambda(A[\varphi])| > R$ and we can then apply Lemma 7.8 according to which there exists $\kappa > 0$ such that one of the following two cases occur:

- Case 1:

  $$\mathrm{Re}\, F^{rs}(A[\varphi])\left(A_{rs}[\underline{\varphi}] - A_{rs}[\varphi]\right) - (\partial_t\underline{\varphi} - \partial_t\varphi) > \kappa \sum_{r=1}^n F^{rr}(A[\varphi])\,,$$

  i.e. $-F^{aa}v_{\bar{a}a} + \partial_t v > \kappa\mathcal{F}$ at p_0, where we recall that $v = \varphi - \underline{\varphi}$. This immediately gives

  $$F^{aa}\tilde{v}_{\bar{a}a} - \partial_t\tilde{v} < -C\mathcal{F}$$

  where C depends on $\|\partial_t v\|_{C^0}$. Choosing S so large as to have $\beta'\left(F^{aa}\tilde{v}_{\bar{a}a} - \partial_t\tilde{v}\right) \geq C\mathcal{F}$ we deduce from (7.29) the inequality $0 \geq \frac{4}{N}F^{aa}\lambda_a^2$ which is a contradiction, hence this case cannot occur.

- Case 2:

  $$F^{ss}(A[\varphi]) > \kappa \sum_{r=1}^n F^{rr}(A[\varphi])\,, \qquad \text{for all } s = 1, \dots, n\,,$$

  which in particular gives $F^{11} > \kappa\mathcal{F}$ and thus $F^{aa}\lambda_a^2 \geq F^{11}\lambda_1^2 \geq \kappa\mathcal{F}\lambda_1^2$. We may assume $F^{aa}\lambda_a \leq F^{aa}\lambda_a^2/(6NS)$ because if this were not true we would have $\kappa\mathcal{F}\lambda_1^2 < 6NS\mathcal{F}\lambda_1$ and we would conclude. Then we have

$$4\beta'\left(F^{aa}\tilde{v}_{\bar{a}a}-\partial_t\tilde{v}\right)\geq -12SF^{aa}\varphi_{\bar{a}a}-C\mathcal{F}\geq -\frac{2F^{aa}\lambda_a^2}{N}-C\mathcal{F}\,,$$

This last inequality and (7.29) finally give

$$0\geq 2\kappa\frac{\lambda_1^2}{N^2}-C\,,$$

as was to be shown.

The desired bound is valid at the maximum point x_0 of G, and then also globally. □

Remark 7.10. As in the elliptic case treated in the previous chapter, this is the only step of the proof of our main results that uses the assumption that the metric g is hyperkähler.

7.2.3 Gradient estimate.

The bound find in the previous section is well-suited for the so-called *blow-up analysis*. This technique coupled with a Liouville–type theorem allows to find a non-explicit gradient bound and consequently, also a Laplacian bound.

Proposition 7.11. *Suppose there is a uniform constant C such that*

$$\|\Delta_g\varphi\|_{C^0}\leq C\left(\|\nabla\varphi\|^2_{C^0}+1\right)\,,$$

then there is a uniform bound

$$\|\nabla\varphi\|_{C^0}\leq C\,.$$

Proof. The proof is entirely analogous to the one for the elliptic case; we shall only give an overview. Fix $T'\leq T$ and suppose by contradiction that the gradient bound does not hold. Then we can find a sequence $(h_j)_j$ of real smooth functions, a sequence $(\Omega_j)_j$ of q-real $(2,0)$-forms on M and sequences $(\varphi_j)_j$, $(\underline{\varphi}_j)_j$ of solutions and parabolic $\mathcal{C}$-subsolutions of the equation

$$\begin{cases} F\left(g^{\bar{t}r}((\Omega_j)_{\bar{t}s}+(\varphi_j)_{\bar{t}s})\right)=h_j+\partial_t\varphi_j\,,\\ \sup_{M\times[0,T']}\varphi_j=0\,,\\ |\nabla\varphi_j|\geq j\,. \end{cases}$$

Assume further that $(\underline{\varphi}_j)_j$, $(h_j)_j$ and $(\Omega_j)_j$ are uniformly bounded in C^2-norm.

For each j assume that $|\nabla\varphi_j|^2$ achieves its maximum N_j at $(x_j,t_j)\in M\times[0,T']$. Set $g_j=N_jg$ and choose quaternionic local coordinates $(q^1,\dots,q^n)$ around x_j for $|q^i|<N_j^{1/2}$ for every $i=1,\cdots,n$ such that

$$(g_j)_{\bar{r}s}=\delta_{\bar{r}s}+O(N_j^{-1}|x|)\,,\qquad (\Omega_j)_{\bar{r}s}=O(N_j^{-1})\,,\qquad h_j=h_j(x_j)+O(N_j^{-1}|x|)\,. \tag{7.30}$$

We may assume $\lim_{j\to\infty}(x_j,t_j)=(0,T')$. Clearly $|\nabla\varphi_j(x_j,t_j)|^2_{g_j}=1$ and by Propositions 7.7 and 7.9 we have in this coordinates

$$\|\varphi_j(\cdot,t_j)\|_{C^0}\leq C\,,\qquad |\Delta_g\varphi_j(\cdot,t_j)|_{g_j}\leq C\,,\qquad \text{on } B_{N_j^{1/2}}(x_j)\,,$$

where $C>0$ is uniform in j and does not depend on T'. By (7.30) we then have, for $j\to\infty$

$$\lambda\left(g_j^{\bar{k}r}\left((\Omega_j)_{\bar{k}s}+(\varphi_j(\cdot,t_j))_{\bar{k}s}\right)\right)=\lambda\big((\varphi_j(\cdot,t_j))_{\bar{r}s}\big)+O(N_j^{-1}|x|)\,. \tag{7.31}$$

By [143, Theorem 8.32] and Arzelà-Ascoli Theorem, for any $\alpha \in (0,1)$, the sequence $(\varphi_j(\cdot, t_j))_j$ admits subsequence uniformly convergent in $C^{1,\alpha}$ to some $u\colon \mathbb{H}^n \to \mathbb{R}$. The function u satisfies $\|u\|_{C^0} + \|\nabla u\|_{C^0} \leq C$ and $|\nabla u(0)| = 1$, in particular u is non-constant. However, using (7.31) and proceeding similarly to Section 6.2.3 one can easily show that u is a Γ-solution in the sense of Definition 6.12. The proof is now concluded by applying the Liouville-type theorem (Proposition 6.15) for this kind of functions, which contradicts the fact that u is non-constant. □

7.2.4 Higher order estimates and long-time existence.

Here we improve the Laplacian estimate to a Hölder estimate of the quaternionic Hessian of φ. We do so in two ways as in Section 6.2.4, which can both be seen as analogues of the Evans-Krylov theory. By bootstrapping we then obtain estimates of any order on the solution of (7.1) and thus also long-time existence.

Proposition 7.12. *For each $k > 0$, there exists a uniform constant C_k depending on the allowed data, k, $\|\nabla\varphi\|_{C^0}$ and an upper bound for $\Delta_g\varphi$ such that*

$$\|\nabla^k\varphi\|_{C^0} \leq C_k\,. \tag{7.32}$$

where ∇ is the Levi-Civita connection with respect to g. Moreover we have long-time existence for φ, i.e. $T = \infty$.

Proof. Assume (7.32) and suppose $T < \infty$. It follows from (7.8) that there exists a uniform constant C such that

$$|\varphi| \leq T \sup_{M\times[0,T)} |\partial_t\varphi| \leq CT\,, \quad \text{on } M\times[0,T)\,.$$

By this, (7.32) and short-time existence, one can extend the flow to $[0, T+\varepsilon_0)$ for some $\varepsilon_0 > 0$, which yields a contradiction. The interested reader can find more details about this standard discussion in the proof of [285, Theorem 3.1] (see also in [48, 323] and references therein).

We showed that it is enough to prove (7.32). And we claim that (7.32), follows once we have proved a Hölder bound for $\mathrm{Hess}_{\mathbb{H}}\varphi$ of the form

$$\|\,\mathrm{Hess}_{\mathbb{H}}\varphi\|_{C^{0,\alpha}(M\times[\varepsilon,T))} \leq C_\varepsilon \tag{7.33}$$

where $\varepsilon \in (0,T)$ and C_ε is a uniform constant depending only on the initial data and ε. Indeed, given the Hölder bound (7.33) for the matrix $\mathrm{Hess}_{\mathbb{H}}\varphi$ and the second order estimate for φ, we can differentiate the flow (7.1) and then bootstrap using the Schauder estimates in order to obtain the uniform bound

$$\|\nabla^k\varphi\|_{C^{0,\alpha}(M\times[\varepsilon,T))} \leq C_{\varepsilon,k}\,, \quad \text{for any } k>0\,,$$

where $C_{\varepsilon,k}$ depends on ε and k. But since by standard parabolic theory the solution φ is uniquely determined by the initial and background data, we also have a uniform bound

$$\|\nabla^k\varphi\|_{C^{0,\alpha}(M\times[0,\varepsilon))} \leq C_{\varepsilon,k}\,, \quad \text{for any } k>0\,.$$

The estimate (7.33) is standard, we prove it as a separate proposition below. □

Proposition 7.13. *For each $\varepsilon \in (0,T)$ there exists $\alpha \in (0,1)$ and a uniform constant $C_\varepsilon > 0$ depending only on the allowed data, ε, $\|\partial_t\varphi\|_{C^0}$, and an upper bound for $\Delta_g\varphi$ such that*

$$\|\,\mathrm{Hess}_{\mathbb{H}}\varphi\|_{C^{0,\alpha}(M\times[\varepsilon,T))} \leq C_\varepsilon\,.$$

We shall present two proofs of the proposition above.

First proof of Proposition 7.13.

This first proof uses a general result due to Chu [91, Theorem 5.1], which is the parabolic counterpart of the main result in [286].

Let (I_0, J_0, K_0) be the standard hyperhermitian structure on $\mathbb{R}^{4n}$, i.e.

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & -\mathbb{1} & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \end{pmatrix}.$$

Set $V = \{H \in \mathbb{R}^{4n,4n} \mid I_0 H I_0 = J_0 H J_0 = K_0 H K_0 = -H\}$. We will use the isomorphism of real algebras $\gamma\colon \mathbb{H}^{n,n} \to V$, and the projection $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to V$ defined as

$$\gamma(A + iB + jC + kD) := \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix},$$

and

$$\mathrm{p}(H) := \frac{1}{4}(H - I_0 H I_0 - J_0 H J_0 - K_0 H K_0)\,.$$

We have the identity $\gamma(\mathrm{Hess}_{\mathbb{H}} u) = 16\mathrm{p}(D^2 u)$ for any function $u\colon \mathbb{H}^n \to \mathbb{R}$ of class C^2.

Now, since the manifold is locally flat, we can work in a coordinate chart, which we identify with $B = B_1(0)$. The condition $\Delta_g \varphi \le C$ implies that there exist σ and R such that

$$\lambda(A[\varphi]) \in \bar{\Gamma}^\sigma \cap \overline{B_R(0)}\,, \qquad \text{on } B\,,$$

where $\Gamma^\sigma = \{\lambda \in \Gamma \mid f(\lambda) > \sigma\}$. Therefore, setting

$$\mathcal{E} = \left\{ H \in \mathrm{Sym}(4n, \mathbb{R}) \mid \lambda\left(g^{\bar{j}r}(0)(\gamma^{-1}(p(H)))_{\bar{j}s}\right) \in \bar{\Gamma}^\sigma \cap \overline{B_{2R}(0)} \right\},$$

and shrinking the radius of B, if necessary, we can assume

$$\gamma\left(\Omega_{\bar{r}s}(x) + \varphi_{\bar{r}s}(x)\right) \in \mathcal{E}\,.$$

and also that there is neighborhood U of $\mathcal{E}$ such that

$$\lambda\left(g^{\bar{j}r}(x)(\gamma^{-1}(p(H)))_{\bar{j}s}\right) \in \bar{\Gamma}^\sigma \cap \overline{B_{4R}(0)}\,, \qquad \text{for any } H \in U \text{ and } x \in B\,.$$

Define the following operators

- $P\colon \mathrm{Sym}(4n, \mathbb{R}) \times B \to \mathbb{R}$ defined as $P(H, x) = f(\lambda(g^{\bar{j}r}(x)(\gamma^{-1}(p(H)))_{\bar{j}s}))$ for $H \in U$, and extended smoothly to all of $\mathrm{Sym}(4n, \mathbb{R}) \times B$;
- $S\colon B \to \mathrm{Sym}(4n, \mathbb{R})$ defined as $S(x) = \gamma(\Omega_{\bar{r}s}(x))$;
- $T\colon \mathrm{Sym}(4n, \mathbb{R}) \times B \to \mathrm{Sym}(4n, \mathbb{R})$ defined as $T(H, x) = 16\mathrm{p}(H)$.

Then the setup is convenient for an application of [91, Theorem 5.1]. We shall proceed as in Lemma 5.7 to conclude. We consider two cases:

1. $T < 1$. We immediately obtain a C^0 bound for φ by Lemma 7.5, as

$$|\varphi| \le T \sup_{M \times [0,T)} |\partial_t \varphi| \le CT \le C.$$

   With this bound [91, Theorem 5.1] can be applied and we conclude.

2. $T \geq 1$. In this case we consider, for any $a \in (0, T-1)$ an auxiliary function

$$\varphi_a(x,t) := \varphi(x, t+a) - \inf_{B\times[a,a+1)} \varphi\,, \qquad t \in [0,1).$$

Such a function satisfies

$$\partial_t \varphi_a = F(A[\varphi_a]) - h$$

hence, by previous results, it satisfies a uniform Laplacian bound and by Proposition 7.7 it also satisfies $\|\varphi_a\|_{C^0} \leq \mathrm{osc}_M \varphi(\cdot, t) \leq C$. Applying again [91, Theorem 5.1] to φ_a we deduce that for any fixed $\varepsilon \in (0, \frac{1}{2})$ we have

$$\|\nabla^2 \varphi\|_{C^\alpha(B\times[a+\varepsilon,a+1))} \leq \|\nabla^2 \varphi_a\|_{C^\alpha(B\times[\varepsilon,1))} \leq C\,,$$

where C is a uniform constant that depends on ε and α. Since $a \in (0, T-1)$ is arbitrary we obtain the desired estimate

$$\|\nabla^2 \varphi\|_{C^\alpha(B\times[\varepsilon,1))} \leq C\,.$$

Second proof of Proposition 7.13.

The second proof is more classical in flavour and represents an adaptation of Alesker's $C^{2,\alpha}$ estimate for the quaternionic Monge-Ampère equation obtained in [14].

Again the proof is local, since M is locally flat. Let $\mathcal{O} \subset \mathbb{H}^n$ be an arbitrary open subset. For each $\alpha \in (0,1)$, on $\mathcal{O}_T := \mathcal{O} \times [0,T)$, we define

$$[\varphi]_{\alpha,(x,t)} := \sup_{(y,s)\in\mathcal{O}_T\setminus(x,t)} \frac{|\varphi(y,s) - \varphi(x,t)|}{(|y-x| + \sqrt{|s-t|})^\alpha}\,, \qquad [\varphi]_{\alpha,\mathcal{O}_T} := \sup_{(x,t)\in\mathcal{O}_T} [\varphi]_{\alpha,(x,t)}\,.$$

The metric g can be locally represented by a potential w on $\mathcal{O}$, possibly shrinking $\mathcal{O}$ if necessary, in other words $g = \mathrm{Hess}_{\mathbb{H}} w$. Let us denote $u = w + \varphi$ and $U = \mathrm{Hess}_{\mathbb{H}} u$. By concavity of F, and the mean value theorem, for all $(x, t_1), (y, t_2) \in \mathcal{O} \times [0,T)$, we have

$$\begin{aligned} \mathrm{Re}\, F^{rs}(y,t_2)(u_{\bar r s}(x,t_1) - u_{\bar r s}(y,t_2)) &\geq \partial_t\varphi(x,t_1) - \partial_t\varphi(y,t_2) - h(x) + h(y) \\ &\geq \partial_t u(x,t_1) - \partial_t u(y,t_2) - C\|x-y\|\,, \end{aligned} \tag{7.34}$$

for some constant C depending on $\|h\|_{C^1}$.

At this point we recall the following algebraic lemma by Alesker [14, Lemma 4.9].

Lemma 7.14. *Let $\lambda, \Lambda \in \mathbb{R}$ satisfy $0 < \lambda < \Lambda < +\infty$. There exist a uniform constant N, unit vectors $\xi_1, \cdots, \xi_N \in \mathbb{H}^n$ and positive numbers $\lambda_* < \Lambda_* < +\infty$, depending only on n, λ, Λ such that any hyperhermitian matrix $A \in \mathbb{H}^{n,n}$ with eigenvalues lying in the interval $[\lambda, \Lambda]$ can be written as*

$$A = \sum_{k=1}^N \beta_k \xi_k^* \otimes \xi_k\,, \qquad \textit{i.e.} \qquad A_{rs} = \sum_{k=1}^N \beta_k \bar\xi_{kr}\xi_{ks}\,,$$

for some $\beta_k \in [\lambda_*, \Lambda_*]$.

Applying the lemma to $A = (F^{rs}(U))$, immediately yields

$$\begin{aligned} \mathrm{Re}\, F^{rs}(U(y))(u_{\bar r s}(y) - u_{\bar r s}(x)) &= \mathrm{Re} \sum_{k=1}^N \beta_k(y)\bar\xi_{kr}\xi_{ks}(u_{\bar r s}(y) - u_{\bar r s}(x)) \\ &= \sum_{k=1}^N \beta_k(y)\left(\Delta_{\xi_k} u(y) - \Delta_{\xi_k} u(x)\right) \end{aligned}$$

for some functions $\beta_k(y) \in [\lambda_*, \Lambda_*]$, where, for any unit vector $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{H}^n$, we denoted by

Δ_ξ the Laplacian on any translate of the quaternionic line spanned by ξ, i.e.

$$\operatorname{Re}\operatorname{tr}((\xi^*\otimes\xi)(u_{\bar{r}s})) = \operatorname{Re}\operatorname{tr}(\xi^*(u_{\bar{r}s})\xi) = \Delta_\xi u\,.$$

For convenience, let us set $\beta_0(y)\equiv 1$ and $\Delta_{\xi^0} = -\partial_t$. Then, from (7.34) we obtain

$$\sum_{k=0}^{N}\beta_k\big(\Delta_{\xi^k}u(y,t_2) - \Delta_{\xi^k}u(x,t_1)\big) \le C\|x-y\|\,. \tag{7.35}$$

Lemma 7.15. *For any $k = 0,1,\cdots,N$,*

$$\partial_t\Delta_{\xi^k}u \le \operatorname{Re}F^{rs}\big(\Delta_{\xi^k}u_{\bar{r}s}\big) + \Delta_{\xi^k}h\,.$$

Proof. For $k=0$. Applying ∂_t to (7.1), we get

$$\partial_t\big(\partial_t u\big) = \operatorname{Re}F^{rs}\partial_t\big(u_{\bar{r}s}\big)$$

and the lemma follows.

For other $k\ge 1$, write $\xi^k = (\xi^k_1,\cdots,\xi^k_n)$. Differentiating (7.1) along ξ^k_p twice and taking sum over the index p, gives

$$\partial_t\Delta_{\xi^k}u = \operatorname{Re}F^{rs}\big(\Delta_{\xi^k}u_{\bar{r}s}\big) + \operatorname{Re}\sum_{p=1}^{n}F^{rs,tl}u_{\bar{r}s\xi^k_p}u_{\bar{t}l\xi^k_p} - \Delta_{\xi^k}h \le \operatorname{Re}F^{rs}\big(\Delta_{\xi^k}u_{\bar{r}s}\big) - \Delta_{\xi^k}h\,,$$

by the concavity of F. Then the lemma follows. □

Fix $\hat{t}\in[\varepsilon,T)$, and $r\in(0,1)$ such that $10r^2\le\hat{t}$. Define

$$\begin{aligned} P_r &= \big\{(x,t)\in\mathcal{O}_T : \|x\|\le r, \hat{t}-5r^2\le t\le \hat{t}-4r^2\big\},\\ Q_r &= \big\{(x,t)\in\mathcal{O}_T : \|x\|\le r, \hat{t}-r^2\le t\le\hat{t}\big\}. \end{aligned}$$

For every $k=0,1,\cdots,N$, let us denote

$$M_{k,r} = \sup_{Q_r}\Delta_{\xi^k}u,\quad m_{k,r} = \inf_{Q_r}\Delta_{\xi^k}u,\quad \eta(r) = \sum_{k=1}^{N}(M_{k,r}-m_{k,r}).$$

To prove Proposition 7.13, it suffices to find a constant C (depending only on ε), $r_0>0$ and $0<\delta<1$ such that

$$\eta(r)\le Cr^\delta,\quad \text{for all } r<r_0.$$

Let us define an operator $\mathcal{D} = \frac{1}{4}\operatorname{Re}F^{rs}(U)\partial_{\bar{q}^r}\partial_{q^s}$. Let $(a_{ij})\in\operatorname{Sym}(4n,\mathbb{R})$ be the realization of $(F^{rs}(U))$. Then we can rewrite $\mathcal{D}$ as

$$\mathcal{D} = \sum_{s,t=1}^{4n}a_{st}D_sD_t, \tag{7.36}$$

Since F is uniformly elliptic on Γ, then $(a_{st})\in\operatorname{Sym}(4n,\mathbb{R})$ satisfies the uniform elliptic estimate $\lambda\|\xi\|^2\le\sum_{s,t}a_{st}\xi_s\xi_t\le\Lambda\|\xi\|^2$ for some $0<\lambda<\Lambda<\infty$ and any $\xi\in\mathbb{R}^{4n}$.

The following weak parabolic Harnack inequality is well-known.

Lemma 7.16. [212, Theorem 7.37]. *If $v\in W^{2,1}_{2n+1}$ is a nonnegative function and satisfies*

$$-\frac{\partial v}{\partial t} + \sum_{s,t}a_{st}D_sD_tv \le h' \quad \text{on } Q_{4r},$$

where h' is a bounded function and the matrix (a_{st}) is as in (7.36). Then there exist positive constants

C, p depending on n, λ, Λ such that

$$\frac{1}{r^{4n+2}}\left(\int_{P_r} v^p\right)^{\frac{1}{p}} \leq C\left(\inf_{B_r} v + r^{\frac{4n}{4n+1}}\|h'\|_{L^{2n+1}}\right). \tag{7.37}$$

For each $k = 0, 1, \cdots, N$, let us denote $v_k := M_{k,2r} - \Delta_{\xi^k} u$. Then $v_k \in W^{2,1}_{2n+1}$ is a non-negative function and since $\Delta_{\xi^k} u_{\bar{r}s} = (\Delta_{\xi^k} u)_{\bar{r}s}$ on $\mathcal{O}_T$ it satisfies

$$-\partial_t v_k + \operatorname{Re} F^{rs}(v_k)_{\bar{r}s} \leq h'$$

for a bounded function h'. Then by Lemmas 7.15 and 7.16,

$$\frac{1}{r^{4n+2}}\Big(\int_{P_r}(M_{k,2r} - \Delta_{\xi^k} u)^p\Big)^{\frac{1}{p}} \leq C\big(M_{k,2r} - M_{k,r} + r^{\frac{4n}{4n+1}}\big), \tag{7.38}$$

On the other hand, let $(x, t_1), (y, t_2) \in Q_{2r}$, it then follows from (7.35) that

$$\beta_k\big(\Delta_{\xi^k} u(y, t_2) - \Delta_{\xi^k} u(x, t_1)\big) \leq Cr + \sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq k}} \beta_\gamma\big(\Delta_{\xi^\gamma} u(x, t_1) - \Delta_{\xi^\gamma} u(y, t_2)\big).$$

For each $\varepsilon > 0$, pick a point $(x, t_1) \in Q_{2r}$ such that $m_{k,2r} \leq \Delta_{\xi^k} u(x, t_1) + \varepsilon$. As a consequence, after dividing the inequality above by β_k, we obtain

$$\Delta_{\xi^k} u(y, t_2) - m_{k,2r} \leq Cr + C\sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq k}}\left(M_{\gamma,2r} - \Delta_{\xi^\gamma} u(y, t_2)\right),$$

by arbitrariness of ε. Integrating for (y, t_2) over P_r, and using the fundamental inequality $\|a + b\|_p \leq \|a\|_p + \|b\|_p$ for every $p > 1$, yields

$$\begin{aligned}
\frac{1}{r^{4n+2}}\Big(\int_{P_r}\left(\Delta_{\xi^k} u(y, t_2) - m_{k,2r}\right)^p\Big)^{\frac{1}{p}} &\leq \frac{C}{r^{4n+2}}\Big(\int_{P_r}\Big[r + \sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq k}}\left(M_{\gamma,2r} - \Delta_{\xi^\gamma} u(y, t_2)\right)\Big]^p\Big)^{\frac{1}{p}} \\
&\leq Cr + \frac{C}{r^{4n+2}}\sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq k}}\left(\int_{P_r}[M_{\gamma,2r} - \Delta_{\xi^\gamma} u(y, t_2)]^p\right)^{\frac{1}{p}} \\
&\overset{(7.38)}{\leq} C\sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq k}}(M_{\gamma,2r} - M_{\gamma,r}) + Cr^{\frac{4n}{4n+1}},
\end{aligned} \tag{7.39}$$

where we have used the fact $0 < r < 1$ in the last inequality. In light of (7.38) and (7.39), and again the triangle inequality $\|a + b\|_p \leq \|a\|_p + \|b\|_p$, we obtain

$$\begin{aligned}
M_{k,2r} - m_{k,2r} &\leq \frac{C}{r^{4n+2}}\Big(\int_{P_r}(M_{k,2r} - \Delta_{\xi^k} u)^p\Big)^{\frac{1}{p}} + \frac{C}{r^{4n+2}}\Big(\int_{P_r}(\Delta_{\xi^k} u - m_{k,2r})^p\Big)^{\frac{1}{p}} \\
&\leq C\sum_{\gamma=0}^{N}(M_{\gamma,2r} - M_{\gamma,r}) + Cr^{\frac{4n}{4n+1}}.
\end{aligned}$$

Summing over k we deduce

$$\eta(2r) \leq C\sum_{\gamma=0}^{N}(M_{\gamma,2r} - M_{\gamma,r}) + Cr^{\frac{4n}{4n+1}}.$$

By definition, $m_{\cdot,s}$ is non-increasing in s, whence

$$\eta(2r) \leq C \sum_{\gamma=0}^{N} \big((M_{\gamma,2r} - m_{\gamma,2r}) - M_{\gamma,r} + m_{\gamma,r}\big) + Cr^{\frac{4n}{4n+1}} = C\big(\eta(2r) - \eta(r)\big) + Cr^{\frac{4n}{4n+1}} .$$

Equivalently,

$$\eta(r) \leq \Big(1 - \frac{1}{C}\Big)\eta(2r) + Cr^{\frac{4n}{4n+1}} .$$

Applying a standard iteration technique (see [143, Chapter 8] for more details), we finally infer that there exists a dimensional constant $\delta \in (0,1)$ such that $\eta(r) \leq Cr^{\delta}$ as we wanted to show. This completes the proof of Proposition 7.13.

7.3 Convergence of the flow.

Li-Yau type inequality.

Now we consider the following Li-Yau [211] type equation

$$(\mathcal{L} - \partial_t)\psi = 0\,, \quad \psi > 0\,, \tag{7.40}$$

where $\mathcal{L} = \frac{1}{4}\mathrm{Re}\, F^{ik}\partial_{\bar{q}_i}\partial_{q_k}$.

If we let

$$\partial_k \Phi := \sum_{p=0}^{3} \Phi_{x_p^k}\, \bar{e}_p, \qquad \partial_{\bar{k}}\Phi := \sum_{p=0}^{3} e_p \Phi_{x_p^k}\,,$$

where $\Phi_{x_p^k} := \frac{\partial \Phi}{\partial x_p^k}$, and $\bar{e}_p$ denotes the quaternionic conjugate of the quaternionic unit e_p for every p. Then we can rewrite $\mathcal{L}$ as

$$\mathcal{L}\Phi = \frac{1}{4}\mathrm{Re}\, F^{ik}\partial_{\bar{q}_i}\partial_{q_k}\Phi = F^{ik}_{pq}\, \Phi_{x_p^k x_q^i}\,,$$

where $F^{ik}_{pq} := \frac{1}{4}\mathrm{Re}\,\{F^{ik} e_q \bar{e}_p\}$ for simplicity.

Let B be a constant so large that $\psi = \partial_t\varphi + B$ is a solution to (7.40). We consider the quantity

$$H = t(|\partial v|^2 - \alpha\partial_t v)\,, \qquad v = \log\psi\,,$$

where $\alpha \in (1,2)$ is a constant and

$$|\partial v|^2 = \frac{1}{4}\mathrm{Re}\, F^{jl} v_j v_{\bar{l}} = F^{jl}_{rs}\, v_{x_r^l} v_{x_s^j}\,.$$

Lemma 7.17. *There exists a constant $C > 0$ such that*

$$(\mathcal{L} - \partial_t)H \geq \frac{t}{4n}\big(|\partial v|^2 - \partial_t v\big)^2 - 2\langle \partial v, \partial H\rangle - \big(|\partial v|^2 - \alpha\partial_t v\big) - tC|\partial v|^2 - Ct\,, \tag{7.41}$$

where $\langle\cdot,\cdot\rangle$ is the inner product defined by $\langle \partial f, \partial g\rangle = \frac{1}{4}\mathrm{Re}\, F^{ik} f_i g_{\bar{k}} = F^{ik}_{pq}\, f_{x_p^k} g_{x_q^i}$.

Proof. The proof is local. For each $z \in M$, we can find quaternionic coordinates $q_1, \dots, q_n$ on a local chart around z. Assume $f \in C^2(M, \mathbb{R})$, let $f_j = \frac{\partial f}{\partial q_j}$ be the ordinary quaternionic derivative. Plugging $\psi = e^v$ into (7.40) we have

$$\mathcal{L}v - \partial_t v = -|\partial v|^2\,, \tag{7.42}$$

giving

$$H = -t\mathcal{L}v - t(\alpha - 1)\partial_t v\,, \tag{7.43}$$

and thus also

$$t\partial_t\big(\mathcal{L}v\big) = \frac{1}{t}H - \partial_t H - t(\alpha - 1)\partial_t^2 v\,. \tag{7.44}$$

By a straightforward computation we get

$$\begin{aligned}-\partial_t H &= -\big(|\partial v|^2-\alpha\partial_t v\big)-2t\big\langle\partial v,\partial\partial_t v\big\rangle+t\alpha\partial_t^2 v-t\partial_t(F^{ik}_{pq})v_{x^k_p}v_{x^i_q}\,,\\ \mathcal{L}H &= t\mathcal{L}(|\partial v|^2)-t\alpha\mathcal{L}(\partial_t v)\,.\end{aligned}\tag{7.45}$$

First we deal with the term $\mathcal{L}(|\partial v|^2)$. For convenience, let us define

$$\mathcal{V}=F^{ik}_{pq}F^{jl}_{rs}v_{x^l_r x^k_p}v_{x^j_s x^i_q}\,,\qquad \mathcal{W}=F^{ik}_{pq}F^{jl}_{rs}v_{x^l_r x^i_q}v_{x^j_s x^k_p}\,.$$

By a direct calculation, we get

$$\begin{aligned}\mathcal{L}(|\partial v|^2)=&\,\mathcal{V}+\mathcal{W}+\mathcal{L}(F^{jl}_{rs})v_{x^l_r}v_{x^j_s}+F^{ik}_{pq}(F^{jl}_{rs})_{x^k_p}v_{x^l_r}v_{x^j_s x^i_q}+F^{ik}_{pq}(F^{jl}_{rs})_{x^k_p}v_{x^l_r x^i_q}v_{x^j_s}\\ &+F^{ik}_{pq}(F^{jl}_{rs})_{x^i_q}v_{x^l_r x^k_p}v_{x^j_s}+F^{ik}_{pq}(F^{jl}_{rs})_{x^i_q}v_{x^l_r x^k_p}v_{x^j_s}+F^{jl}_{rs}\mathcal{L}(v_{x^l_r})v_{x^j_s}+F^{jl}_{rs}v_{x^l_r}\mathcal{L}(v_{x^j_s})\,.\end{aligned}$$

Note that φ has uniformly bounded C^k norms for every $k>0$ by Proposition 7.12. Hence, analogously to the (almost) Hermitian case [91, 144], we deduce

$$|\mathcal{L}(F^{jl}_{rs})v_{x^l_r}v_{x^j_s}|\leq C|\partial v|^2\,.\tag{7.46}$$

For each $0<\varepsilon<1$, we have that

$$\begin{aligned}\big|F^{ik}_{pq}(F^{jl}_{rs})_{x^k_p}v_{x^l_r}v_{x^j_s x^i_q}\big|+\big|F^{ik}_{pq}(F^{jl}_{rs})_{x^k_p}v_{x^l_r x^i_q}v_{x^j_s}\big|+\big|F^{ik}_{pq}(F^{jl}_{rs})_{x^i_q}v_{x^l_r x^k_p}v_{x^j_s}\big|+\big|F^{ik}_{pq}(F^{jl}_{rs})_{x^i_q}v_{x^l_r x^k_p}v_{x^j_s}\big|\\ \leq\frac{C}{\varepsilon}|\partial v|^2+2\varepsilon\mathcal{W}+2\varepsilon\mathcal{V}\,.\end{aligned}\tag{7.47}$$

Observe that $(\mathcal{L}v)_{x^j_s}-\mathcal{L}(v_{x^j_s})=(F^{ik}_{pq}v_{x^k_p x^i_q})_{x^j_s}-F^{ik}_{pq}v_{x^j_s x^k_p x^i_q}=(F^{ik}_{pq})_{x^j_s}v_{x^k_p x^i_q}$. It follows that

$$\begin{aligned}F^{jl}_{rs}\mathcal{L}(v_{x^l_r})v_{x^j_s}+F^{jl}_{rs}v_{x^l_r}\mathcal{L}(v_{x^j_s})-2\big\langle\partial v,\partial\mathcal{L}v\big\rangle&\\ =&F^{jl}_{rs}v_{x^j_s}\big(\mathcal{L}(v_{x^l_r})-(\mathcal{L}v)_{x^l_r}\big)+F^{jl}_{rs}v_{x^l_r}\big(\mathcal{L}(v_{x^j_s})-(\mathcal{L}v)_{x^j_s}\big)\\ =&-F^{jl}_{rs}v_{x^j_s}(F^{ik}_{pq})_{x^l_r}v_{x^k_p x^i_q}-F^{jl}_{rs}v_{x^l_r}(F^{ik}_{pq})_{x^j_s}v_{x^k_p x^i_q}\\ \geq&-\frac{C}{\varepsilon}|\partial v|^2-\varepsilon\mathcal{V}-\varepsilon\mathcal{W}\,.\end{aligned}\tag{7.48}$$

On the other hand,

$$\begin{aligned}2t\big\langle\partial v,\partial\mathcal{L}v\big\rangle\overset{(7.43)}{=}&-2\big\langle\partial v,\partial H\big\rangle-2t(\alpha-1)\langle\partial v,\partial\partial_t v\rangle\\ \overset{(7.45)}{=}&-2\big\langle\partial v,\partial H\big\rangle-(\alpha-1)\partial_t H+\frac{\alpha-1}{t}H-t\alpha(\alpha-1)\partial_t^2 v\\ &-t(\alpha-1)\partial_t(F^{ik}_{pq})v_{x^k_p}v_{x^i_q}\\ \geq&-2\big\langle\partial v,\partial H\big\rangle-(\alpha-1)\partial_t H+\frac{\alpha-1}{t}H-t\alpha(\alpha-1)\partial_t^2 v-Ct|\partial v|^2\,.\end{aligned}\tag{7.49}$$

It follows from (7.48) and (7.49) that

$$\begin{aligned}t\Big(F^{jl}_{rs}\mathcal{L}(v_{x^l_r})v_{x^j_s}+F^{jl}_{rs}v_{x^l_r}\mathcal{L}(v_{x^j_s})\Big)\geq&-2\big\langle\partial v,\partial H\big\rangle-(\alpha-1)\partial_t H+\frac{\alpha-1}{t}H\\ &-t\alpha(\alpha-1)\partial_t^2 v-Ct|\partial v|^2-\frac{Ct}{\varepsilon}|\partial v|^2-t\varepsilon\mathcal{V}-t\varepsilon\mathcal{W}\,.\end{aligned}\tag{7.50}$$

Now, we treat the second term of $\mathcal{L}H$ in (7.45). Using the Cauchy-Schwarz inequality, at z, we deduce

$$\begin{aligned}-t\alpha\mathcal{L}(\partial_t v) &= -t\alpha\partial_t(\mathcal{L}v) + t\alpha\partial_t(F^{ik}_{pq})v_{x^k_p x^i_q}\\ &\overset{(7.44)}{=} -\frac{\alpha}{t}H + \alpha\partial_t H + t\alpha(\alpha-1)\partial_t^2 v + t\alpha\partial_t(F^{ik}_{pq})v_{x^k_p x^i_q}\\ &\geq -\frac{\alpha}{t}H + \alpha\partial_t H + t\alpha(\alpha-1)\partial_t^2 v - \frac{Ct}{\varepsilon} - t\varepsilon\mathcal{V}\,,\end{aligned} \tag{7.51}$$

where in the last inequality we have used the fact that $-CF^{ik}_{pq} \leq \partial_t(F^{ik}_{pq}) \leq CF^{ik}_{pq}$ for a uniform constant C, which is implied by Proposition 7.12.

Plugging (7.46), (7.47), (7.50) and (7.51) into (7.45), we get

$$\begin{aligned}\mathcal{L}H \geq\,& t\mathcal{W} + t\mathcal{V} - Ct|\partial v|^2 - t\Big(\frac{C}{\varepsilon}|\partial v|^2 + 2\varepsilon\mathcal{V} + 2\varepsilon\mathcal{W}\Big) - 2\langle\partial v, \partial H\rangle - (\alpha-1)\partial_t H\\ &+ \frac{\alpha-1}{t}H - t\alpha(\alpha-1)\partial_t^2 v - Ct|\partial v|^2 - \frac{Ct}{\varepsilon}|\partial v|^2 - t\varepsilon(\mathcal{V}+\mathcal{W})\\ &- \frac{\alpha}{t}H + \alpha\partial_t H + t\alpha(\alpha-1)\partial_t^2 v - \frac{Ct}{\varepsilon} - t\varepsilon\mathcal{V}\\ \geq\,& t(1-4\varepsilon)\mathcal{V} + t(1-3\varepsilon)\mathcal{W} - 2Ct\Big(1+\frac{1}{\varepsilon}\Big)|\partial v|^2 + \partial_t H - \frac{1}{t}H - 2\langle\partial v, \partial H\rangle - \frac{Ct}{\varepsilon}\,.\end{aligned}$$

Thus, if we choose $\frac{1}{16} \leq \varepsilon \leq \frac{1}{8}$,

$$(\mathcal{L}-\partial_t)H \geq \frac{t}{2}\mathcal{V} - Ct|\partial v|^2 - \big(|\partial v|^2 - \alpha\partial_t v\big) - 2\langle\partial v, \partial H\rangle - Ct\,. \tag{7.52}$$

Applying the arithmetic-geometric mean inequality, and by (7.42),

$$\mathcal{V} \geq \frac{1}{n}(\mathcal{L}v)^2 = \frac{1}{n}\big(\partial_t v - |\partial v|^2\big)^2\,.$$

Plugging it into (7.52), we infer that

$$(\mathcal{L}-\partial_t)H \geq \frac{t}{2n}\big(\partial_t v - |\partial v|^2\big)^2 - Ct|\partial v|^2 - \big(|\partial v|^2 - \alpha\partial_t v\big) - 2\langle\partial v, \partial H\rangle - Ct\,.$$

By the arbitrariness of z, this proves (7.41). ☐

Using the parabolic maximum principle, we can prove the following lemma.

Lemma 7.18. *On $M\times(0,T)$, we have*

$$|\partial v|^2 - \alpha\partial_t v \leq \frac{8n\alpha^2}{t} + \sqrt{8n\alpha^2\Big(C + \frac{nC^2\alpha^2}{2(\alpha-1)^2}\Big)}\,.$$

Proof. Let us fix an arbitrary time $t_0 \in (0,T)$. Suppose $H(x,t)$ (as in (7.43)) achieves its maximum at the point $(\hat{q},\hat{t}) \in M\times[0,t_0]$. We may assume $\hat{t} > 0$, otherwise $|\partial v|^2 - \alpha\partial_t v \leq 0$ on $M\times[0,t_0]$ and we are done. It follows that

$$H(\hat{q},\hat{t}) \geq H(\hat{q},0) = 0\,.$$

Using the maximum principle at $(\hat{q},\hat{t})$, we deduce $(\mathcal{L}-\partial_t)H \leq 0$ and $\partial H = 0$. Substituting this into (7.41) yields

$$\frac{\hat{t}^2}{4n}\big(|\partial v|^2 - \partial_t v\big)^2 - C\hat{t}^2|\partial v|^2 - H \leq C\hat{t}^2\,. \tag{7.53}$$

Notice that at $(\hat{q},\hat{t})$,

$$\begin{aligned}\hat{t}^2\left(|\partial v|^2-\partial_t v\right)^2 &= \frac{\hat{t}^2}{\alpha^2}\left(|\partial v|^2-\alpha\partial_t v+(\alpha-1)|\partial v|^2\right)^2\\ &= \frac{H^2}{\alpha^2}+\left(\frac{\alpha-1}{\alpha}\right)^2\hat{t}^2|\partial v|^4+\frac{2(\alpha-1)\hat{t}H}{\alpha^2}|\partial v|^2\\ &\geq \frac{H^2}{\alpha^2}+\left(\frac{\alpha-1}{\alpha}\right)^2\hat{t}^2|\partial v|^4\,,\end{aligned} \tag{7.54}$$

where we have used the fact that H is nonnegative at $(\hat{q},\hat{t})$. Using the elementary inequality $ax^2+bx\geq -\frac{b^2}{4a}$, we get

$$\frac{1}{4n}\left(\frac{\alpha-1}{\alpha}\right)^2\hat{t}^2|\partial v|^4-\hat{t}^2C|\partial v|^2\geq -\frac{nC^2\alpha^2}{2(\alpha-1)^2}\hat{t}^2\,. \tag{7.55}$$

Plugging (7.54) and (7.55) into (7.53) gives

$$\frac{H^2}{4n\alpha^2}\leq H+C\hat{t}^2+\frac{nC^2\alpha^2}{2(\alpha-1)^2}\hat{t}^2\,;$$

from which we can deduce

$$H(\hat{q},\hat{t})\leq 8n\alpha^2+\sqrt{8n\alpha^2\left(C+\frac{nC^2\alpha^2}{2(\alpha-1)^2}\right)}\,\hat{t}\,.$$

Hence, at each point $q\in M$,

$$H(q,t_0)\leq H(\hat{q},\hat{t})\leq 8n\alpha^2+\sqrt{8n\alpha^2\left(C+\frac{nC^2\alpha^2}{2(\alpha-1)^2}\right)}\,t_0\,.$$

Consequently, at (q,t_0),

$$|\partial v|^2-\alpha\partial_t v\leq \frac{8n\alpha^2}{t_0}+\sqrt{8n\alpha^2\left(C+\frac{nC^2\alpha^2}{2(\alpha-1)^2}\right)}\,.$$

Then the lemma follows by arbitrariness of t_0. □

Parabolic Harnack inequality.

Let $\psi=\partial_t\varphi+B$ for a large constant B such that $\psi>0$ on M. By (7.10) we know

$$\mathcal{L}\psi-\partial_t\psi=0\,. \tag{7.56}$$

With the results of the previous subsection we can prove the following useful parabolic Harnack inequality:

Proposition 7.19. *Let $0<t_1<t_2<T$. Then there exist constants C_i $(i=1,2,3)$ depending only on (M,I,J,K), Ω and f such that*

$$\sup_M\psi(\cdot,t_1)\leq\inf_M\psi(\cdot,t_2)\left(\frac{t_2}{t_1}\right)^{C_1}\exp\left(\frac{C_2}{t_2-t_1}+C_3(t_2-t_1)\right)\,. \tag{7.57}$$

Proof. With Lemmas 7.17 and 7.18, we can apply the procedure of [91, 144] verbatim. □

Convergence of the parabolic flow.

Proposition 7.20. *Suppose $T = \infty$, $\mathrm{osc}_M \varphi(\cdot, t) \leq C$ and $\|\nabla^k \varphi\|_{C^0} \leq C$ for any $k > 0$, where $C > 0$ is a uniform constant. Then the normalization $\tilde{\varphi}$ converges in C^∞ topology to a smooth function $\tilde{\varphi}_\infty$ that satisfies*

$$F(A[\tilde{\varphi}_\infty]) = h + b\,,$$

for some constant $b \in \mathbb{R}$.

Proof. Set $\psi = \partial_t \varphi + B$ for a large constant B such that $\psi > 0$. For each $m \in \mathbb{N}$, we define

$$\check{\psi}_m(x,t) := \sup_M \psi(\cdot, m-1) - \psi(x, m-1+t)\,;$$

$$\hat{\psi}_m(x,t) := \psi(x, m-1+t) - \inf_M \psi(\cdot, m-1)\,.$$

It is straightforward to verify that

$$(\partial_t - \mathcal{L})\psi = (\partial_t - \mathcal{L})\hat{\psi}_m = (\partial_t - \mathcal{L})\check{\psi}_m = 0\,.$$

Applying the parabolic Harnack inequality (7.57), this yields

$$\sup_M \hat{\psi}_m(\cdot, t_1) \leq C \inf_M \hat{\psi}_m(\cdot, t_2)\,, \qquad \sup_M \check{\psi}_m(\cdot, t_1) \leq C \inf_M \check{\psi}_m(\cdot, t_2)\,.$$

Choosing $t_1 = \frac{1}{2}$, $t_2 = 1$ we get

$$\begin{aligned} &\sup_M \psi\left(\cdot, m - \frac{1}{2}\right) - \inf_M \psi(\cdot, m-1) \leq C\left(\inf_M \psi(\cdot, m) - \inf_M \psi(\cdot, m-1)\right), \\ &\sup_M \psi(\cdot, m-1) - \inf_M \psi\left(\cdot, m - \frac{1}{2}\right) \leq C\left(\sup_M \psi(\cdot, m-1) - \sup_M \psi(\cdot, m)\right). \end{aligned} \tag{7.58}$$

In light of (7.58), if we set

$$\theta(t) = \sup_M \psi(\cdot, t) - \inf_M \psi(\cdot, t)$$

for the oscillation, then we have

$$\theta(m-1) + \theta\left(m - \frac{1}{2}\right) \leq C\big(\theta(m-1) - \theta(m)\big)\,,$$

which implies that $\theta(m) \leq e^{-\delta}\theta(m-1)$, where $\delta := -\log(1 - \frac{1}{C}) > 0$, and by induction

$$\theta(t) \leq C e^{-\delta t}\,.$$

Since we have $\int_M \partial_t \tilde{\varphi} = 0$, by the mean value theorem, there exists a point $x_t \in M$ such that $\partial_t \tilde{\varphi}(x_t, t) = 0$. Therefore,

$$\left|\partial_t \tilde{\varphi}(x,t)\right| = \left|\partial_t \tilde{\varphi}(x,t) - \partial_t \tilde{\varphi}(x_t, t)\right| \leq \mathrm{osc}_M \partial_t \tilde{\varphi}(\cdot, t) = \mathrm{osc}_M \partial_t \varphi(\cdot, t) = \theta(t) \leq C e^{-\delta t}\,,$$

which yields that $\tilde{\varphi} + \frac{C}{\delta} e^{-\delta t}$ (resp. $\tilde{\varphi} - \frac{C}{\delta} e^{-\delta t}$) is non-increasing (resp. non-decreasing) with respect to t. It then follows from the uniform bounds on φ that $\tilde{\varphi}$ is uniformly bounded in C^∞ topology, therefore there is a sequence of times $t_j \to \infty$ such that $\tilde{\varphi}(\cdot, t_j)$ converges smoothly to some smooth function $\tilde{\varphi}_\infty$ and it is fairly standard to show that actually $\lim_{t\to\infty} \tilde{\varphi} = \tilde{\varphi}_\infty$ in the C^∞ topology.

Finally, the limiting function $\tilde{\varphi}_\infty$ satisfies

$$0 = \lim_{t\to\infty} \partial_t \tilde{\varphi}(\cdot, t) = \lim_{t\to\infty} \left(F(A[\tilde{\varphi}]) - h - \frac{\int_M \partial_t \varphi \, \Omega_0^n \wedge \bar{\Omega}_0^n}{\int_M \Omega_0^n \wedge \bar{\Omega}_0^n} \right) = F(A[\tilde{\varphi}_\infty]) - h - b\,,$$

where we set

$$b=\lim_{t\to\infty}\frac{\int_M \partial_t\varphi\,\Omega_0^n\wedge\bar\Omega_0^n}{\int_M \Omega_0^n\wedge\bar\Omega_0^n}\,. \qquad \square$$

7.4 Proof of Theorems 7.2, 7.3 and consequences.

We are ready to complete the proofs of Theorems 7.2 and 7.3.

Proof of Theorem 7.2. Let (M,I,J,K,g) be a compact flat hyperkähler manifold, $\varphi,\tilde\varphi\colon M\to\mathbb{R}$ be the solution to (7.1) and its normalization (defined in (7.4)). The initial datum φ_0 is assumed Γ-admissible and, since f is unbounded, every Γ-admissible function is automatically a parabolic $\mathcal{C}$-subsolution. Hence we may apply Proposition 7.7 and deduce $\mathrm{osc}_M\varphi(\cdot,t)\le C$ and $\|\tilde\varphi\|_{C^0}\le C$. This bounds allow to obtain from Propositions 7.9 and 7.11 a uniform constant C such that $\Delta_g\varphi\le C$. Applying now Proposition 7.12 we infer long-time existence of φ and uniform bounds on its derivatives of any order. Finally, Proposition 7.20 yields smooth convergence of the normalization $\tilde\varphi$ to some function $\tilde\varphi_\infty$ which is a solution of (7.5), i.e.

$$F(A[\tilde\varphi_\infty])=h+b$$

for a suitable constant $b\in\mathbb{R}$. □

Proof of Theorem 7.3. The proof is quite similar to the one of Theorem 7.2. Indeed, suppose f is bounded on Γ and assume that it satisfies either one of the two conditions expressed in the statement of Theorem 7.3, we are still able to apply Proposition 7.7 and deduce $\mathrm{osc}_M\varphi(\cdot,t)\le C$ and $\|\tilde\varphi\|_{C^0}\le C$. Now we can employ the arguments in the proof of Theorem 7.2 to complete the proof. □

Quaternionic Hessian flow.

We shall present two of the many possible applications provided by Theorem 7.2, namely we show the convergence of the *quaternionic Hessian flow* and of the $(n-1)$*-quaternionic plurisubharmonic flow* on compact flat hyperkähler manifolds. Let us start with the former.

Let (M,I,J,K,g,Ω_0) be a compact hyperhermitian manifold. Let $1\le k\le n$ and fix a q-real k-positive $(2,0)$-form Ω, that is

$$\frac{\Omega^i\wedge\Omega_0^{n-i}}{\Omega_0^n}>0\qquad\text{for every } i=1,\dots,k\,.$$

Then the *quaternionic Hessian flow* can be written as

$$\partial_t\varphi=\log\frac{\Omega_\varphi^k\wedge\Omega_0^{n-k}}{\Omega_0^n}-H\,,\qquad \varphi\in\mathcal{H}^k_{\Omega_0}\,,\tag{7.59}$$

where $H\in C^\infty(M,\mathbb{R})$ is the datum and $\mathcal{H}^k_{\Omega_0}$ is the space of smooth functions φ such that Ω_φ is a k-positive q-real $(2,0)$-form.

Theorem 7.21. *Let (M,I,J,K,g,Ω_0) be a compact flat hyperkähler manifold and Ω a q-real k-positive $(2,0)$-form. Then for any smooth initial datum $\varphi_0\in\mathcal{H}^k_{\Omega_0}$,*

1. *the solution φ to (7.59) exists for all time;*
2. *the normalization $\tilde\varphi$ (defined as in (7.4)) converges smoothly as $t\to\infty$ to a function $\tilde\varphi_\infty\in\mathcal{H}^k_{\Omega_0}$, and there exists a constant $b\in\mathbb{R}$ such that*

$$\frac{\Omega^k_{\tilde\varphi_\infty}\wedge\Omega_0^{n-k}}{\Omega_0^n}=b\,\mathrm{e}^H\,.\tag{7.60}$$

We remark that the constant b in (7.60) is uniquely determined by

$$b = \frac{\int_M \Omega^k_{\tilde\varphi_\infty} \wedge \Omega_0^{n-k} \wedge \bar\Omega_0^n}{\int_M \mathrm{e}^H \Omega_0^n \wedge \bar\Omega_0^n} .$$

Flow (7.59) provides the quaternionic counterpart of the complex Hessian flow (see e.g. [257]). For $k = 1$ equation (7.59) is the parabolic Poisson equation, while for $k = n$ it becomes the parabolic quaternionic Monge-Ampère equation. Thus, Theorem 7.21 generalizes the main result of Chapter 5.

Proof of Theorem 7.21. The result follows as a simple application of Theorem 7.2 once we choose $f = \log \sigma_k$ defined over the cone

$$\Gamma = \Gamma_k := \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \dots, \sigma_k(\lambda) > 0\} ,$$

where σ_r is the r-th elementary symmetric function

$$\sigma_r(\lambda) = \sum_{1 \le i_1 < \cdots < i_r \le n} \lambda_{i_1} \cdots \lambda_{i_r} , \qquad \text{for all } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n .$$

Indeed, on a locally flat hyperhermitian manifold a C^2 function u lies in $\mathcal{H}^k_{\Omega_0}$ if and only if it is Γ_k-admissible. The function f satisfies our structural assumptions C1–C3 (see e.g. [268]) and it is straightforward to check that it is unbounded over Γ_k. Finally, with this setup, the quaternionic Hessian flow (7.59) becomes $\partial_t \varphi = f(\lambda(A[\varphi])) - H$, as desired. □

$(n-1)$-quaternionic plurisubharmonic flow.

Our second aforementioned application is the $(n-1)$-quaternionic plurisubharmonic flow. Let $(M, I, J, K, g, \Omega_0)$ be a compact hyperhermitian manifold and Ω_1 be a positive q-real $(2,0)$-form. Denote with Δ_g the quaternionic Laplacian with respect to g. The $(n-1)$-*quaternionic plurisubharmonic flow* is encoded in the following parabolic equation:

$$\partial_t \varphi = \log \frac{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g \varphi)\Omega_0 - \partial\partial_J \varphi\right]\right)^n}{\Omega_0^n} - H , \qquad \varphi \in \mathrm{QPSH}_{n-1}(M, \Omega_1, \Omega_0) , \tag{7.61}$$

where $\mathrm{QPSH}_{n-1}(M, \Omega_1, \Omega_0)$ denotes the space of functions φ that are $(n-1)$-quaternionic plurisubharmonic with respect to Ω_1 and Ω_0, i.e. $\Omega_1 + \frac{1}{n-1}\left[(\Delta_g \varphi)\Omega_0 - \partial\partial_J \varphi\right] > 0$.

Theorem 7.22. *Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold and Ω_1 a q-real positive $(2,0)$-form. Then for any smooth initial datum $\varphi_0 \in \mathrm{QPSH}_{n-1}(M, \Omega_1, \Omega_0)$,*

1. *the solution φ to (7.61) exists for all time;*
2. *the normalization $\tilde\varphi$ of φ (defined as in (7.4)) converges smoothly as $t \to \infty$ to a function $\tilde\varphi_\infty \in \mathrm{QPSH}_{n-1}(M, \Omega_1, \Omega_0)$, and there exists a constant $b \in \mathbb{R}$ such that*

$$\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g \tilde\varphi_\infty)\Omega_0 - \partial\partial_J \tilde\varphi_\infty\right]\right)^n = b\, \mathrm{e}^H \Omega_0^n . \tag{7.62}$$

The constant b in (7.62) is uniquely determined by

$$b = \frac{\int_M \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g \tilde\varphi_\infty)\Omega_0 - \partial\partial_J \tilde\varphi_\infty\right]\right)^n \wedge \bar\Omega_0^n}{\int_M \mathrm{e}^H \Omega_0^n \wedge \bar\Omega_0^n} .$$

The complex version of flow (7.61) was studied by Gill [145] as a parabolic approach to the complex Monge-Ampère equation for $(n-1)$-plurisubharmonic functions. As proven in the previous chapter, the solvability of (7.62) leads to Calabi-Yau–type theorems for quaternionic balanced, quaternionic

Gauduchon, and quaternionic strongly Gauduchon metrics. Therefore, convergence of flow (7.61) results to be an interesting tool in the search of special metrics.

Proof of Theorem 7.22. Define

$$f = \log \sigma_n(T)\,, \qquad \Gamma = T^{-1}(\Gamma_n)\,,$$

where $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the linear map defined by

$$T(\lambda) = \big(T(\lambda)_1, \dots, T(\lambda)_n\big)\,, \qquad T(\lambda)_k = \frac{1}{n-1}\sum_{i\neq k} \lambda_i\,, \qquad \text{for every } \lambda \in \mathbb{R}^n\,.$$

An easy verification shows that assumptions C1–C3 are satisfied and that f is unbounded over Γ. Setting

$$\Omega := \mathrm{Re}\left(g^{\bar{j}s}(\Omega_1)_{\bar{j}s}\right)\Omega_0 - (n-1)\Omega_1\,,$$

one can easily see that $u \in C^2(M,\mathbb{R})$ lies in $\mathrm{QPSH}_{n-1}(M,\Omega_1,\Omega_0)$ if and only if $\lambda(A[u]) \in \Gamma$, where $A[u] = g^{\bar{j}r}(\Omega_{\bar{j}s} + u_{\bar{j}s})$. We can then rewrite the $(n-1)$-quaternionic plurisubharmonic flow (7.61) as $\partial_t\varphi = f(\lambda(A[\varphi])) - H$ and apply Theorem 7.2 to conclude. ☐

Within the bounded case various equations can be included, for instance, parabolic quaternionic Hessian quotient equations, parabolic quaternionic mixed Hessian equations. We limit ourselves to prove the following general result.

Theorem 7.23. *Suppose f is bounded on Γ. Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold. If there exists a Γ-admissible function φ_0 and a $\mathcal{C}$-subsolution of the equation*

$$F(A[\varphi]) = h$$

in the sense of Definition 6.2. Then there exists a smooth solution of the equation

$$F(A[\varphi]) = h + b$$

for some constant $b \in \mathbb{R}$.

Proof. Let $\underline{\varphi}$ be an elliptic $\mathcal{C}$-subsolution of the equation $F(A[\varphi]) = h$, which we have shown that can be seen as a time-independent parabolic $\mathcal{C}$-subsolution of our flow (7.1). Consider flow (7.1) with a Γ-admissible initial datum φ_0, then condition (7.6) of Theorem 7.3 is trivially verified, and this concludes the proof. ☐

CHAPTER 8

THE DEGENERATE QUATERNIONIC MONGE-AMPÈRE EQUATION

The next natural step in the study of the conjecture is the case when the manifold is locally flat, but does not admit any compatible hyperkähler metric. This is the object of the present paper where we prove that in the locally flat case the quaternionic Monge-Ampère equation can be always solved at least in a weak sense. So far the approach for solving the conjecture has consisted in adapting Yau's proof of the Calabi conjecture to the quaternionic case, hence the strategy so far has always been purely PDE-theoretic. The main difficulty in this direction is the proof of the a priori estimates. In the present chapter we consider an alternative method and try to tackle the problem with the different perspective of pluripotential theory. We solve the quaternionic Monge-Ampère equation in a weak sense under the assumption of local flatness with a variational approach via the Ding functional in the same spirit of the paper of Berman, Boucksom, Guedj and Zeriahi [39]. The same method was used by Wan in [313] to study the quaternionic Monge-Ampère equation.

The variational approach has also been implemented to other settings (see e.g. [2, 38, 217, 218]), in particular we highlight that it has been successfully applied by Wan [313] for quaternionic Monge-Ampère equations on a domain of $\mathbb{H}^n$. In order to establish analogue results and characterize the range of the quaternionic Monge-Ampère operator we will first need to set up the ground by introducing quaternionic pluripotential theory on HKT manifolds.

The work initiated by Bedford and Taylor [31, 32] provides perhaps one of the most remarkable and powerful way of investigating the nature of Monge-Ampère operators. The first to implement this framework and obtain weak solutions of the complex Monge-Ampère equation was Kołodziej in the seminal paper [199]. With this pluripotential point of view, here we will study the equation

$$(\Omega + \partial\partial_J \varphi)^n = \mu \tag{8.1}$$

where the right-hand side is no longer a smooth volume form but merely a positive Radon measure satisfying the necessary condition $\mu(M) = \mathrm{Vol}(M)$. Before we do this, we will need to establish a few fact in quaternionic pluripotential theory over HKT manifolds, relating the notion of pluripolarity with some quaternionic capacities in the same spirit of [156]. This will allow to extend the definition of the Monge-Ampère operator on a certain class of unbounded functions. Within this class, we will show that equation (8.1) can be solved if and only if the right hand side is a non-pluripolar measure.

8.1 Quaternionic plurisubharmonic functions.

In this section we recall the definition and some properties of quaternionic plurisubharmonic functions (qpsh for short) on open subset of $\mathbb{H}^n$ and on hypercomplex manifolds.

The local case.

The definition of quaternionic plurisubharmonic functions on domains of $\mathbb{H}^n$ was introduced by Alesker in [9] and the theory of these functions was developed in [11, 12, 13, 17, 310, 311, 312, 313, 315, 316, 317, 318].

Definition 8.1. Let $A \subseteq \mathbb{H}^n$ be an open domain. A function $\varphi\colon A \to [-\infty, \infty)$ is quaternionic plurisubharmonic (qpsh for short) if it is upper semi-continuous, $\varphi \not\equiv -\infty$ and it is either subharmonic or constant $-\infty$ on each affine right quaternionic line, i.e. for every $x \in A$ and $v \in \mathbb{H}$ the function $y \mapsto \varphi(x + vy)$ is either subharmonic in the usual sense or constant $-\infty$.

As noted by Alesker [9] qpsh functions are subharmonic and for $n = 1$ the two notions coincide. In view of this fact qpsh functions inherits all the nice properties of subharmonic functions, for instance, they are L^1_{loc}. Even if, for many aspects, the theory of qpsh functions is analogue to the one of plurisubharmonic functions in the complex space, there are also some differences: for instance complex plurisubharmonic functions are always in L^p_{loc} for any $p \geq 1$, while Sroka showed in [270] that qpsh functions are only in L^p_{loc} for $p < 2$ and such exponent is optimal. Moreover, another remarkable difference is that bounded plurisubharmonic functions in $\mathbb{C}^n$ are necessarily constant, while bounded qpsh functions on $\mathbb{H}^n$ need not be. For example, for $n = 1$ we know that qpsh functions can be regarded as subharmonic functions in $\mathbb{R}^4$ and it is well-known that there exist bounded non-constant subharmonic functions in $\mathbb{R}^4$.

The following proposition collects the essential properties of qpsh functions (see e.g. [313]):

Proposition 8.2. *Let $A \subseteq \mathbb{H}^n$ be an open domain. The following properties hold.*

- *The set of quaternionic plurisubharmonic functions on A forms a convex cone.*
- *If (φ_j) is a decreasing sequence of qpsh functions then $\lim_j \varphi_j$ is either qpsh or constant $-\infty$.*
- *If $(\varphi_j)_{j\in J}$ is a family of qpsh functions such that $u = \sup_{j\in J} u_j$ is locally bounded from above, then also the upper semi-continuous regularization $u^* = \limsup_{A\ni y\to x} u(y)$ is qpsh.*

Using convolution one can also regularize qpsh functions:

Proposition 8.3. *If φ is quaternionic plurisubharmonic and $\varphi_\varepsilon = \varphi * \chi_\varepsilon$ is the standard regularization on $A_\varepsilon = \{x \in A \mid \mathrm{dist}(x, \partial A) > \varepsilon\}$, then $\varphi_\varepsilon \in C^\infty(M, \mathbb{R})$ is qpsh and decreases to φ as $\varepsilon \to 0$.*

Closed and positive currents.

In order to introduce the definition of qpsh functions on manifolds we recall some basic facts about currents on hypercomplex manifolds. Let (M, I, J, K) be a hypercomplex manifold of dimension $4n$. The space $\mathcal{D}^{p,q}(M)$ of (p, q)-currents on M is by definition the topological dual to $\Lambda^{2n-p,2n-q}(M)$. For instance, any (p, q)-form η naturally defines a (p, q)-current T_η given by integration:

$$T_\eta(\alpha) := \int_M \eta \wedge \alpha\,.$$

The actions of I, J, K naturally extend to currents, for example $J\colon \mathcal{D}^{p,q}(M) \to \mathcal{D}^{q,p}(M)$ acts on $T \in \mathcal{D}^{p,q}(M)$ in the following way:

$$(JT)(\alpha) = T(J\alpha)$$

for any compactly supported $\alpha \in \Lambda^{2n-p,2n-q}(M)$. Similarly, the operators $\partial, \partial_J \colon \mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}$ are extended to (p, q)-currents by duality as follows

$$(\partial T)(\alpha) = (-1)^{p+q+1} T(\partial\alpha)\,, \qquad (\partial_J T)(\alpha) = (-1)^{p+q+1} T(\partial_J \alpha)\,.$$

for any compactly supported $\alpha \in \Lambda^{2n-p,2n-q}(M)$.

Definition 8.4. A $(2p, 0)$-current T is called

- **q-real** if $JT = \bar{T}$, where $\bar{T}(\alpha) := \overline{T(\bar{\alpha})}$ for any $\alpha \in \Lambda^{2n-2p,2n}(M)$;

- **q-positive** if it is real and additionally $T(\alpha) \geq 0$ for any q-semipositive $\alpha \in \Lambda^{2n-2p,2n}(M)$;
- **∂-closed** (resp. **∂_J-closed**) if $\partial T = 0$ (resp. $\partial_J T = 0$).
- **closed** if it is both ∂ and ∂_J-closed.

Notice that according to our definition a current is *closed* if it is closed with respect to ∂ and ∂_J and not necessarily with respect d. This notion of closure is more useful in HKT geometry (since d-closed currents never show up) and agrees with the definition given in [316]. Any q-real current is ∂-closed if and only if it is ∂_J-closed. Moreover, by [316, Proposition 3.4] any q-positive current can be regarded as a differential form with Radon measures as its coefficients.

We remark here that Wan and Wang use two anticommuting operators originally defined in [316], which they denote d_0, d_1 and satisfy $d_0^2 = d_1^2 = 0$. However, it was proved by Sroka in [270, Proposition 1] that, after a suitable choice in the definition of these operators, we have

$$d_0 = 2\partial_J\,, \qquad d_1 = -2\partial\,.$$

Since the operators ∂, ∂_J have an intrinsic meaning on any hypercomplex manifold, it is more convenient to use those operators instead of d_0, d_1. For this reason, in what follows, we shall phrase all results in terms of ∂, ∂_J.

The definition of qpsh functions on a domain of $\mathbb{H}^n$ can be characterized in terms of positivity of the current $\partial\partial_J\varphi$.

Proposition 8.5. *If φ is quaternionic plurisubharmonic then $\partial\partial_J\varphi$ is a closed q-positive current. Conversely, if $\varphi \in L^1_{\mathrm{loc}}(A)$ is such that the closed current $\partial\partial_J\varphi$ is q-positive then there exists a quaternionic plurisubharmonic function ψ on A such that $\varphi = \psi$ almost everywhere.*

Proof. The fact that $\partial\partial_J\varphi$ is a q-positive current for any qpsh function φ was proved in [316, Proposition 3.7]. Conversely, let $\varphi \in L^1_{\mathrm{loc}}(A)$ be such that $\partial\partial_J\varphi \geq 0$ in the sense of currents. Take a regularization $\varphi_\varepsilon = \varphi * \chi_\varepsilon$. Since $\partial\partial_J\varphi_\varepsilon = (\partial\partial_J\varphi) * \chi_\varepsilon \geq 0$, and since φ_ε is C^∞ we see that φ_ε is qpsh. Hence, for $\varepsilon \to 0$ the regularization φ_ε decreases to a qpsh function ψ, but since $\varphi_\varepsilon \to \varphi$ in L^1_{loc} we must have $\varphi = \psi$ almost everywhere. ☐

The global case.

The above discussion motivates the following:

Definition 8.6. A function $\varphi \in L^1(M, [-\infty, \infty))$ on a hypercomplex manifold (M, I, J, K) is called **quaternionic plurisubharmonic** (qpsh for short) if $\partial\partial_J\varphi \geq 0$ in the sense of currents.

A function $\varphi \in L^1(M, [-\infty, \infty))$ on a HKT manifold (M, I, J, K, Ω) is called **Ω-quaternionic plurisubharmonic** (Ω-qpsh for short) if $\Omega + \partial\partial_J\varphi \geq 0$ in the sense of currents.

Note that in the compact case the maximum principle implies that every qpsh function is constant.

Remark 8.7. If (M, I, J, K) is *locally flat*, i.e. if it is locally isomorphic to a domain of $\mathbb{H}^n$, a function $\varphi\colon M \to [-\infty, \infty)$ is qpsh if and only if for any point $x \in M$ there is a local chart $\psi\colon U \to \psi(U) \subseteq \mathbb{H}^n$ around x such that $\varphi \circ \psi^{-1}$ is qpsh on $\psi(U)$ in the sense of Definition 8.1.

Remark 8.8. Given a HKT manifold (M, I, J, K, Ω), the HKT form can be always locally written as $\partial\partial_J v$. The function v is called a *local potential* of Ω. A function $\varphi\colon M \to [-\infty, \infty)$ is Ω-qpsh if and only if locally $v + \varphi$ is qpsh with respect to each local potential of Ω. In particular if (M, I, J, K, Ω) is locally flat Proposition 8.5 implies that any qpsh function on (M, I, J, K, Ω) is upper semi-continuous.

Given a HKT manifold (M, I, J, K, Ω), we denote by

$$\mathrm{QPSH}(M, \Omega) := \{\varphi \in L^1(M, [-\infty, \infty)) \mid \Omega + \partial\partial_J\varphi \geq 0\}$$

the space of Ω-qpsh functions on (M, I, J, K, Ω). With the natural L^1-topology $\mathrm{QPSH}(M, \Omega)$ is a closed convex cone of $L^1(M)$ which is closed under taking maximums.

Proposition 8.2 can be directly generalized to Ω-qpsh functions on compact locally flat HKT manifolds.

Proposition 8.9. *Let (M, I, J, K, Ω) be a compact locally flat HKT manifold:*

1. *If (φ_j) is a sequence in $\mathrm{QPSH}(M, \Omega)$ uniformly bounded from above, then either converges uniformly to $-\infty$ or it has a convergent subsequence in $L^1(M)$.*
2. *If (φ_j) is a decreasing sequence in $\mathrm{QPSH}(M, \Omega)$ then it either $\lim_j \varphi_j \in \mathrm{QPSH}(M, \Omega)$ or $\lim_j \varphi_j \equiv -\infty$.*
3. *If $(\varphi_j)_{j\in J}$ is a family in $\mathrm{QPSH}(M, \Omega)$ such that $\varphi = \sup_{j\in J} \varphi_j$ is locally bounded from above, then the upper semi-continuous regularization φ^* is Ω-qpsh.*
4. *Hartogs' Lemma: If (φ_j) is a sequence in $\mathrm{QPSH}(M, \Omega)$ such that $\varphi_j \to \varphi$ in $L^1(M)$, then $\varphi = \varphi^*$ almost everywhere for a unique $\varphi^* \in \mathrm{QPSH}(M, \Omega)$. Moreover $\lim_{j\to\infty} \sup_M \varphi_j = \sup_M \varphi^*$.*

The following compactness result will be very useful in the sequel.

Lemma 8.10. *Let (M, I, J, K, Ω) be a compact locally flat HKT manifold. The set $\{\varphi \in \mathrm{QPSH}(M, \Omega) \mid \sup_M \varphi = 0\}$ is compact in $\mathrm{QPSH}(M, \Omega)$. Furthermore, if μ is a positive Radon measure such that $\mathrm{QPSH}(M, \Omega) \subseteq L^1(\mu)$ then the subset $\{\varphi \in \mathrm{QPSH}(M, \Omega) \mid \int_M \varphi \, d\mu = 0\}$ is relatively compact. In particular there exists C such that for any $\varphi \in \mathrm{QPSH}(M, \Omega)$,*

$$-C + \sup_M \varphi \leq \int_M \varphi \, d\mu \leq \sup_M \varphi.$$

Here we recall that a *Radon measure* is a Borel measure which is inner regular and outer regular.

Proof of Lemma 8.10. By Hartogs' lemma (Proposition 8.9(4)) $\{\varphi \in \mathrm{QPSH}(M, \Omega) \mid \sup_M \varphi = 0\}$ is closed and by Proposition 8.9(1) it is relatively compact.

Let (φ_j) be a sequence in $\mathrm{QPSH}(M, \Omega)$ such that $\int_M \varphi_j \, d\mu = 0$ and let $\psi_j := \varphi_j - \sup_M \varphi_j$. There is a convergent subsequence $\psi_{j_k} \to \psi \in L^1(M)$. Assume that μ is smooth. Then we have convergence $\psi_{j_k}\mu \to \psi\mu$ in the weak sense of measures. Consequently $\int_M \psi_{j_k} \, d\mu \to \int_M \psi \, d\mu > -\infty$ showing that $\|\psi_j\|_{L^1(\mu)}$ is bounded. But

$$\|\psi_j\|_{L^1(\mu)} = \int_M \varphi_j \, d\mu - \int_M \sup_M \varphi_j \, d\mu = -\mu(M) \sup_M \varphi_j$$

implying that the sequence φ_j is uniformly bounded from above. Therefore Proposition 8.9(1) implies the lemma.

If μ is not smooth, it is enough to prove that $\|\psi_j\|_{L^1(\mu)}$ is uniformly bounded for every j. Suppose by contradiction that $\int_M \psi_j \, d\mu \to -\infty$, then, up to a subsequence we may assume $\int_M \psi_j \, d\mu \leq -2^j$. Set $\psi = \sum_{j=1}^{\infty} 2^{-j} \psi_j$. By the first part of the proof the L^1 norm of ψ_j with respect to a smooth positive Radon measure is uniformly bounded. This entails that $\psi \not\equiv -\infty$ and since the function ψ is the limit of a decreasing sequence of functions in $\mathrm{QPSH}(M, \Omega)$ it is itself a function in $\mathrm{QPSH}(M, \Omega)$. However, we get a contradiction, because by the monotone convergence theorem we should have $\int_M \psi \, d\mu = \sum_{j=1}^{\infty} 2^{-j} \int_M \psi_j \, d\mu = -\infty$. □

For what regards regularization we remark that on manifolds convolution with radial smoothing kernels may destroy plurisubharmonicity. In the complex case one can still find approximations via regular qpsh functions, as showed by Błocki and Kołodziej [45], but their argument breaks down in the quaternionic case, even under the assumption of local flatness. However, if the starting qpsh function is continuous, then it can be approximated with smooth Ω-qpsh functions.

Proposition 8.11. *Let $\varphi \in \mathrm{QPSH}(M, \Omega) \cap C(M, \mathbb{R})$, then, for any positive function g on M there exists $\psi \in \mathrm{QPSH}(M, \Omega) \cap C^\infty(M, \mathbb{R})$ such that $|\psi - \varphi| \leq g$.*

Proof. One could repeat the argument of Richberg [249] for (complex) plurisubharmonic functions (see [100, Ch. 1 §5.E]). Alternatively "the local to global" approach by Greene-Wu [150] can be easily applied. □

8.2 The quaternionic Monge-Ampère operator.

In this section we introduce the quaternionic Monge-Ampère operator on HKT manifolds for bounded Ω-qpsh functions (we refer to [316] for the definition on a domain of $\mathbb{H}^n$).

8.2.1 Definition and first properties of the quaternionic Monge-Ampère operator.

Chern-Levine-Nirenberg inequality.

Let (M, I, J, K, Ω) be a compact locally flat HKT manifold. The *quaternionic Monge-Ampère operator* is defined on $\mathrm{QPSH}(M,\Omega) \cap C^2(M,\mathbb{R})$ as

$$\mathrm{MA}_\varphi := (\Omega + \partial\partial_J\varphi)^n \in \Lambda^{2n,0}(M)\,.$$

In order to extend the definition of MA to $\mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ we make use of currents since for every $\varphi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$, $\partial\partial_J\varphi \in \mathcal{D}^{2,0}(M)$. Even if in general the wedge product of two currents is not defined, for a $\varphi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ and $k \in \mathbb{N}$, we can define $(\Omega + \partial\partial_J\varphi)^k$ as follows:
Firstly, given a $T \in \mathcal{D}^{p,q}(M)$ and $\eta \in \Lambda^{r,s}(M)$ on (M, I) it is defined their wedge product as the $(p+r, q+s)$-current acting on $\alpha \in \Lambda^{2n-p-r,2n-q-s}(M)$ as

$$(T \wedge \eta)(\alpha) = T(\eta \wedge \alpha)\,;$$

then for $\varphi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ and a closed q-positive $T \in \mathcal{D}^{2p,0}(M)$, it is defined the product φT and, consequently, $\partial\partial_J\varphi \wedge T$ via the relation $\partial\partial_J\varphi \wedge T := \partial\partial_J(\varphi T)$. Since $\partial\partial_J\varphi \wedge T$ is closed and q-positive we can proceed inductively and define $(\Omega + \partial\partial_J\varphi)^k$.

Similarly, given $\varphi, \psi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ and T closed and q-positive, we can define $\partial\varphi \wedge \partial_J\varphi \wedge T$, which is again closed and positive, and $\partial\varphi \wedge \partial_J\psi \wedge T$ via the identities

$$\begin{aligned}
&\partial\varphi \wedge \partial_J\varphi \wedge T := \frac{1}{2}\partial\partial_J(\varphi^2) \wedge T - \varphi\partial\partial_J\varphi \wedge T\,;\\
&2\partial\varphi \wedge \partial_J\psi \wedge T := \partial(\varphi+\psi) \wedge \partial_J(\varphi+\psi) \wedge T - \partial\varphi \wedge \partial_J\varphi \wedge T - \partial\psi \wedge \partial_J\psi \wedge T\,.
\end{aligned}$$

Definition 8.12. The operator $\mathrm{MA}\colon \mathrm{QPSH}(M,\Omega) \cap L^\infty(M) \to \mathcal{D}^{2n,0}(M)$ given by

$$\mathrm{MA}_\varphi := (\Omega + \partial\partial_J\varphi)^n$$

is called the *quaternionic Monge-Ampère operator*.

The Monge-Ampère operator satisfies the following continuity properties analogous to the ones proved by Bedford and Taylor [31] in the complex case:

- given a decreasing sequence (φ_j) in $\mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ with limit φ, then MA_{φ_j} converges to MA_φ as $j \to \infty$;
- given an increasing sequence (φ_j) in $\mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ which is locally bounded and converging almost everywhere to φ, then MA_{φ_j} converges to MA_φ as $j \to \infty$.

Since we are assuming the manifold to be locally flat the continuity properties can be deduced directly from the local theory [316, 317].

Next we focus on the Chern-Levine-Nirenberg inequality on HKT manifolds, since the inequality has an important role in complex pluripotential theory [83]. In the quaternionic context analogue inequalities are proved in [9, 13, 17, 316] for qpsh functions. In the present paper we need a quaternionic Chern-Levine-Nirenberg inequality in the global setting in the same spirit of [157], where the inequality is proved on compact Kähler manifolds.

Let (M, I, J, K, Ω) be compact HKT manifold. We further assume that the canonical bundle of (M, I) is holomorphically trivial and let Θ be the complex volume form on (M, I) such that

$$\mathrm{Vol}(M) := \int_M \Omega^n \wedge \bar{\Theta} = 1\,.$$

In order to simplify the notation, given a form Ψ of type $(2n, 0)$ on (M, I) we simply write $\int_M \Psi$ instead of $\int_M \Psi \wedge \bar{\Theta}$ since the complex volume form $\bar{\Theta}$ is always fixed (this notation is often adopted in HKT geometry). Coherently, given a $(2n, 0)$-current T we have $T = \mu\, \Omega^n$ for some measure μ and we may define

$$\int_M T := \mu(M)\,.$$

We also introduce the following notation: for $q \geq 1$, a closed q-positive $(2p, 0)$-current T on (M, I) and a $\varphi \in \mathrm{QPSH}(M, \Omega)$, we write $\varphi \in L^q(T)$ if

$$\|\varphi\|_{L^q(T)} := \left(\int_M |\varphi|^q T \wedge \Omega^{n-p}\right)^{1/q} < +\infty\,,$$

Notice that if we consider $T = \Omega^n$, then $L^q(M) = L^q(\Omega^n)$.

Theorem 8.13 (Chern-Levine-Nirenberg inequality)**.** *Let T be a closed q-positive $(2p, 0)$-current and $\varphi \in \mathrm{QPSH}(M, \Omega) \cap L^\infty(M)$. If $\psi \in \mathrm{QPSH}(M, \Omega) \cap L^1(T)$, then $\psi \in L^1(\Omega_\varphi \wedge T)$ and*

$$\|\psi\|_{L^1(\Omega_\varphi \wedge T)} \leq \|\psi\|_{L^1(T)} + \left(2 \sup_M \psi + \sup_M \varphi - \inf_M \varphi\right) \|T\|\,,$$

where $\|T\| = \int_M T \wedge \Omega^{n-p}$.

Proof. Clearly

$$\|\Omega_\varphi \wedge T\| = \int_M \Omega_\varphi \wedge T \wedge \Omega^{n-p-1} = \int_M T \wedge \Omega^{n-p} = \|T\| \tag{8.2}$$

by Stokes theorem. Now, set $\hat{\psi} = \psi - \sup_M \psi \leq 0$, then the triangle inequality and (8.2) yield

$$\|\psi\|_{L^1(\Omega_\varphi \wedge T)} \leq \|\hat{\psi}\|_{L^1(\Omega_\varphi \wedge T)} + \sup_M \psi \|\Omega_\varphi \wedge T\| = \|\hat{\psi}\|_{L^1(\Omega_\varphi \wedge T)} + \sup_M \psi \|T\|\,. \tag{8.3}$$

Replacing φ with $\varphi - \inf_M \varphi$ we may assume $\varphi \geq 0$, thus using again Stokes' theorem

$$\begin{aligned}\|\hat{\psi}\|_{L^1(\Omega_\varphi \wedge T)} &= \int_M (-\hat{\psi}) \Omega_\varphi \wedge T \wedge \Omega^{n-p-1} = \|\hat{\psi}\|_{L^1(T)} + \int_M (-\hat{\psi}) \partial\partial_J \varphi \wedge T \wedge \Omega^{n-p-1} \\ &= \|\hat{\psi}\|_{L^1(T)} - \int_M \varphi\, \partial\partial_J \psi \wedge T \wedge \Omega^{n-p-1}\,.\end{aligned}$$

Since $\varphi T \wedge \Omega^{n-p-1} \geq 0$ and $\Omega + \partial\partial_J \psi \geq 0$ we deduce

$$\|\hat{\psi}\|_{L^1(\Omega_\varphi \wedge T)} \leq \|\hat{\psi}\|_{L^1(T)} + \int_M \varphi\, T \wedge \Omega^{n-p} \leq \|\psi\|_{L^1(T)} + \left(\sup_M \psi + \sup_M \varphi\right) \|T\|\,,$$

which, combined with (8.3) gives the desired inequality. ☐

Corollary 8.14. *If $\varphi \in \mathrm{QPSH}(M, \Omega)$ is such that $0 \leq \varphi \leq 1$ and $\psi \in \mathrm{QPSH}(M, \Omega)$, then*

$$0 \leq \|\psi\|_{L^1(\mathrm{MA}_\varphi)} \leq \|\psi\|_{L^1(M)} + n\left(1 + 2 \sup_M \psi\right)\,.$$

Proof. The result follows from the Chern-Levine-Nirenberg inequality by a simple induction. ☐

Maximum and comparison principle.

We conclude this subsection proving two key results in pluripotential theory: the maximum and the comparison principle. Here we assume that (M, I, J, K, Ω) is also locally flat and, given a subset U of M, we denote by $\mathbf{1}_U$ the characteristic function of U.

Proposition 8.15 (Maximum principle)**.** *Let* $\varphi, \psi \in \mathrm{QPSH}(M, \Omega) \cap L^\infty(M)$*, then*

$$\mathbf{1}_{\{\varphi>\psi\}}\,\mathrm{MA}_\varphi = \mathbf{1}_{\{\varphi>\psi\}}\,\mathrm{MA}_{\max\{\varphi,\psi\}}\,.$$

Proof. Since the nature of the result is local and the manifold is locally flat, it is enough to prove the statement on a open domain $A \subseteq \mathbb{H}^n$. In such a case we have

$$\mathbf{1}_{\{\varphi>\psi\}}(\partial\partial_J\varphi)^k \wedge T = \mathbf{1}_{\{\varphi>\psi\}}(\partial\partial_J \max\{\varphi,\psi\})^k \wedge T$$

in the sense of Borel measures, where T is a closed q-positive $(2n-2k, 0)$-current and $\varphi, \psi \in L^\infty_{\mathrm{loc}}(A)$ are qpsh functions.

If φ is continuous the statement is easy, as $\{\varphi > \psi\}$ is an open subset of A. If φ is not continuous we can take a sequence φ_j of continuous bounded qpsh functions decreasing to φ. Then we have

$$\mathbf{1}_{\{\varphi_j>\psi\}}(\partial\partial_J \max\{\varphi_j,\psi\})^k \wedge T = \mathbf{1}_{\{\varphi_j>\psi\}}(\partial\partial_J\varphi_j)^k \wedge T$$

Let $u_j = (\varphi_j - \psi)^+$ and $u = (\varphi - \psi)^+$ and observe that u_j decreases to u. Therefore, by continuity of the Monge-Ampère operator

$$u(\partial\partial_J \max\{\varphi,\psi\})^k \wedge T = \lim_{j\to\infty} u_j(\partial\partial_J \max\{\varphi_j,\psi\})^k \wedge T = \lim_{j\to\infty} u_j(\partial\partial_J\varphi_j)^k \wedge T = u(\partial\partial_J\varphi)^k \wedge T$$

in the sense of Borel measures. Since $1/(u+\varepsilon)$ is bounded for every $\varepsilon > 0$ we have

$$\frac{u}{u+\varepsilon}(\partial\partial_J \max\{\varphi,\psi\})^k \wedge T = \frac{u}{u+\varepsilon}(\partial\partial_J\varphi)^k \wedge T$$

and this allows to conclude letting ε decrease to 0 because $u/(u+\varepsilon)$ increases to $\mathbf{1}_{\{\varphi>\psi\}}$. □

Proposition 8.16. *Let* $\varphi, \psi \in \mathrm{QPSH}(M, \Omega) \cap L^\infty(M)$*, then*

$$\int_{\{\varphi>\psi\}} \mathrm{MA}_\varphi \le \int_{\{\varphi>\psi\}} \mathrm{MA}_\psi\,.$$

Proof. From the maximum principle we have

$$\int_{\{\varphi>\psi\}} \mathrm{MA}_\varphi = \int_{\{\varphi>\psi\}} \mathrm{MA}_{\max\{\varphi,\psi\}} = 1 - \int_{\{\varphi\le\psi\}} \mathrm{MA}_{\max\{\varphi,\psi\}} \le 1 - \int_{\{\varphi<\psi\}} \mathrm{MA}_\psi = \int_{\{\varphi\ge\psi\}} \mathrm{MA}_\psi\,.$$

The desired inequality follows by replacing ψ with $\psi + \varepsilon$, so that

$$\int_{\{\varphi>\psi+\varepsilon\}} \mathrm{MA}_\varphi \le \int_{\{\varphi\ge\psi+\varepsilon\}} \mathrm{MA}_\psi \le \int_{\{\varphi>\psi\}} \mathrm{MA}_\psi\,,$$

and letting $\varepsilon \searrow 0$. □

8.2.2 Capacities and pluripolarity.

In this section we study some notions of capacity and pluripolar sets on HKT manifolds (we refer to [315] for the theory in domains of $\mathbb{H}^n$). In complex pluripotential theory, the notion of relative capacity was introduced by Bedford-Taylor [32] and then generalized by Kołodziej on Kähler manifolds in [200].

In the whole section we consider a compact locally flat HKT manifold (M, I, J, K, Ω) such that the canonical bundle of (M, I) is holomorphically trivial and let Θ be the holomorphic volume form on

(M, I) such that

$$\int_M \Omega^n \wedge \bar{\Theta} = 1\,.$$

Quaternionic capacities.

For any Borel subset $E \subseteq M$ we define the *quaternionic Monge-Ampère capacity*

$$\mathrm{Cap}_\Omega(E) := \sup\left\{\int_E \mathrm{MA}_\varphi \mid \varphi \in \mathrm{QPSH}(M,\Omega)\,,\ 0 \leq \varphi \leq 1\right\}\,, \tag{8.4}$$

according to the definition of MA_φ given in the previous section. We extend the definition of Cap_Ω to arbitrary subsets $E \subseteq M$ by

$$\mathrm{Cap}_\Omega(E) := \sup\{\mathrm{Cap}_\Omega(K) \mid K \subseteq E \text{ is compact}\}$$

Corollary 8.14 implies that the Monge-Ampère capacity of every set in M is finite.

Lemma 8.17. *The following properties hold:*

1. *If $E_1 \subseteq E_2 \subseteq M$ are Borel subsets, then*

$$\mathrm{Vol}(E_1) \leq \mathrm{Cap}_\Omega(E_1) \leq \mathrm{Cap}_\Omega(E_2) \leq \mathrm{Cap}_\Omega(M) = \mathrm{Vol}(M) = 1\,.$$

2. *If $\{E_j\}$ si a family of Borel subsets of M, then $\mathrm{Cap}_\Omega(\bigcup E_j) \leq \sum \mathrm{Cap}_\Omega(E_j)$. Moreover, if $E_j \subseteq E_{j+1}$, then $\mathrm{Cap}_\Omega(\bigcup E_j) = \lim_{j\to\infty} \mathrm{Cap}_\Omega(E_j)$.*

3. *For all $a \geq 1$, $\mathrm{Cap}_\Omega \leq \mathrm{Cap}_{a\Omega} \leq a^n \mathrm{Cap}_\Omega$. In particular if Ω' is another HKT form, there exists $b \geq 1$ such that $b^{-1}\mathrm{Cap}_\Omega \leq \mathrm{Cap}_{\Omega'} \leq b\mathrm{Cap}_\Omega$.*

Proof. Assertions (1) and (2) are straightforward, while (3) can be proved as follows:
For any $a \geq 1$, since $\Omega \leq a\Omega$, then $\mathrm{QPSH}(M,\Omega) \subseteq \mathrm{QPSH}(M, a\Omega)$ hence $\mathrm{Cap}_\Omega \leq \mathrm{Cap}_{a\Omega}$. Now, for any $\varphi \in \mathrm{QPSH}(M, a\Omega)$ such that $0 \leq \varphi \leq 1$ we also have $\varphi/a \in \mathrm{QPSH}(M,\Omega)$ with $0 \leq \varphi/a \leq 1/a \leq 1$ and $(a\Omega + \partial\partial_J\varphi)^n = a^n(\Omega + \partial\partial_J(\varphi/a))^n$. Therefore $\mathrm{Cap}_{a\Omega} \leq a^n\mathrm{Cap}_\Omega$. In particular if Ω' is another HKT form there exists $a \geq 1$ such that $a^{-1}\Omega \leq \Omega' \leq a\Omega$ implying the desired inequality between Cap_Ω and $\mathrm{Cap}_{\Omega'}$ with $b = a^n$. □

In [315] Wan and Kang introduced the following notion of *relative Monge-Ampère capacity* on domain of $\mathbb{H}^n$

$$C(E, A) := \sup\left\{\int_E (\partial\partial_J\varphi)^n \mid \varphi \text{ qpsh in } A\,,\ 0 \leq \varphi \leq 1\right\}\,,$$

where $E \subseteq A$ is a Borel set. Since we are assuming (M, I, J, K) locally flat, $C(E, A)$ induces a capacity on M. More precisely, let $U_1, \dots, U_N$ be a finite open cover of M made by quaternionic strictly pseudoconvex open subsets $U_j = \{x \in M \mid \rho_j(x) < 0\}$, where ρ_j is a smooth strictly quaternionic plurisubharmonic function defined on a neighborhood of $\bar{U}_j$. Choose another open cover $V_1, \dots, V_N$ such that $V_j \subseteq U_j$ and define

$$C(E) := \sum_{j=1}^N C(E \cap V_j, U_j)\,,$$

for every Borel subset $E \subseteq M$. In the next lemma we observe that the two capacities are comparable. This is analogue to what happens in Kähler geometry [200].

Lemma 8.18. *There exists a constant $\lambda \geq 1$ such that for every Borel subset $E \subseteq M$*

$$\lambda^{-1}C(E) \leq \mathrm{Cap}_\Omega(E) \leq \lambda C(E)\,.$$

Proof. From [315] we have

$$C(E \cap V_j, U_j) = \int_{E \cap V_j} (\partial \partial_J v_j^*)^n$$

where we set

$$v_j := \sup\{\varphi \text{ qpsh in } U_j \mid u \le 0, u|_{E \cap V_j} \le -1\}\,.$$

Observe that v_j^* is qpsh by Proposition 8.9 and satisfies $-1 \le v_j^* \le 0$ as well as $v_j^* = 0$ on ∂U_j. The lemma can now be proved with the same ideas used in [200]. □

Definition 8.19. The **quaternionic polar** (shortly **q-polar**) set of a qpsh function φ is the set $\{\varphi = -\infty\}$. A subset $P \subseteq M$ is called **(locally) quaternionic pluripolar** (shortly **q-pluripolar**) if it is (locally) contained in some q-polar set.

Let us define the outer capacity Cap^*_Ω associated to Cap_Ω as

$$\mathrm{Cap}^*_\Omega(E) := \inf\{\mathrm{Cap}_\Omega(U) \mid U \text{ is open and } E \subseteq U\}\,,$$

similarly, one can define the outer capacity C^* associated to the Wan-Wang capacity.

Corollary 8.20. *For a subset $P \subseteq M$ following are equivalent:*

- *P is locally q-pluripolar;*
- *P is negligible, i.e. of the form $\{\varphi < \varphi^*\}$ for some upper envelope $\varphi = \sup_{j\in J} \varphi_j$ of functions $(\varphi_j)_{j\in J}$ in $\mathrm{QPSH}(M,\Omega)$ locally bounded from above;*
- *$C^*(P) = 0$;*
- $\mathrm{Cap}^*_\Omega(P) = 0$.

Proof. This immediately follows from the local result [315, Theorem 1.2]. □

The aim of this section is to prove that a set is locally q-pluripolar if and only if it is $\mathrm{QPSH}(M,\Omega)$-polar, i.e. it is contained in the $-\infty$-locus of some Ω-qpsh function. The easy implication of this fact, i.e. that $\mathrm{QPSH}(M,\Omega)$-polar sets are locally q-pluripolar follows from the following lemma.

Lemma 8.21. *If $\varphi \in \mathrm{QPSH}(M,\Omega)$ is such that $\varphi \le 0$, then for any $t > 0$*

$$\mathrm{Cap}_\Omega(\{\varphi < -t\}) \le \frac{1}{t}\left(\|\varphi\|_{L^1(M)} + n\right)\,.$$

Proof. Pick $\psi \in \mathrm{QPSH}(M,\Omega)$ such that $0 \le \psi \le 1$. From Chebyshev inequality and Corollary 8.14 we infer

$$\mathrm{MA}_\psi(\{\varphi < -t\}) \le \frac{1}{t}\int_{\{\varphi<-t\}} (-\varphi)\Omega_\psi^n \le \frac{1}{t}\|\varphi\|_{L^1(\mathrm{MA}_\psi)} \le \frac{1}{t}\left(\|\varphi\|_{L^1(M)} + n\right)$$

which gives the lemma once we take the supremum over all Ω-qpsh ψ such that $0 \le \psi \le 1$. □

Extremal functions and Josefson's Theorem.

A fundamental tool to understand the Monge-Ampère capacity is that of extremal functions. The natural extremal functions to take here into account are the relative and the global ones. The definition of the relative extremal function is by now classical, as for the global extremal function, we follow Guedj-Zeriahi [156], which were inspired by Siciak's extremal function, defined and studied in [258, 259, 260, 328].

Definition 8.22. For any Borel subset $E \subseteq M$ we define the *relative (quaternionic) extremal function* v_E and the *global (quaternionic) extremal function* V_E of E as

$$v_{E,\Omega}(x) := \sup\{\varphi(x) \mid \varphi \in \mathrm{QPSH}(M,\Omega)\,,\ \varphi \le 0\,,\ \varphi|_E \le -1\}\,,$$
$$V_{E,\Omega}(x) := \sup\{\varphi(x) \mid \varphi \in \mathrm{QPSH}(M,\Omega)\,,\ \varphi|_E \le 0\}\,.$$

Sometimes we will use the same name for the upper semi-continuous regularizations $v^*_{E,\Omega}$ and $V^*_{E,\Omega}$. When no confusion occurs we will drop the reference to the HKT form in the subscript. The function v^*_E is Ω-qpsh by Proposition 8.9; furthermore, it satisfies $-1 \leq v^*_E \leq 0$. We also observe that if $E_1 \subseteq E_2$ then $v^*_{E_1} \geq v^*_{E_2}$.

Proposition 8.23. *Let E be a Borel subset of M.*

1. *If $U_E := \{x \in M \mid v^*_E(x) < 0\}$ is non-empty, then*
$$\mathrm{MA}_{v^*_E} = 0\,, \qquad \text{in } U_E \setminus \bar{E}.$$
2. *If V^*_E is bounded, it satisfies*
$$\mathrm{MA}_{V^*_E} = 0\,, \qquad \text{in } M \setminus \bar{E}\,.$$

Proof. Both assertions are a consequence of the fact that we can solve the Dirichlet problem in sufficiently small balls (see [315, Lemma 3.2]). Indeed, if V^*_E is bounded it is Ω-qpsh, and one can apply a standard balayage procedure as in [156, Proposition 4.1, Theorem 5.2(2)]. □

The global extremal function characterizes $\mathrm{QPSH}(M,\Omega)$-polar sets in the following sense.

Proposition 8.24. *A Borel subset $E \subseteq M$ is $\mathrm{QPSH}(M,\Omega)$-polar if and only if $V^*_E \equiv +\infty$ if and only if $\sup_M V^*_E = +\infty$.*

Proof. If $V^*_E \equiv +\infty$ then in particular $\sup_M V^*_E = +\infty$ and by Choquet's lemma, we can find an increasing sequence $\varphi_j \in \mathrm{QPSH}(M,\Omega)$ such that $\varphi_j = 0$ on E, $V^*_E = (\lim_{j\to\infty} \varphi_j)^*$ and $\sup_M \varphi_j \geq 2^j$. Set $\psi := \sum_{j=1}^{\infty} 2^{-j}(\varphi_j - \sup_M \varphi_j)$. Then clearly $E \subseteq \{\psi = -\infty\}$ and ψ is either identically $-\infty$ or Ω-qpsh, as a decreasing limit of Ω-qpsh functions. By Lemma 8.10 for any smooth volume form μ there exists $C > 0$ such that $\int_M (\varphi_j - \sup_M \varphi_j) d\mu \geq -C$, therefore $\int_M \psi d\mu \geq -C$ implying that ψ must be Ω-qpsh.

Suppose now that $E \subseteq \{\psi = -\infty\}$ for some $\psi \in \mathrm{QPSH}(M,\Omega)$. For all $c \in \mathbb{R}$ we have $V_E \geq \psi + c$, therefore we have $V_E = +\infty$ outside $\{\psi = -\infty\}$, and since such set has measure zero, we get $V^*_E \equiv +\infty$ on the whole M. □

We are ready to relate the quaternionic Monge-Ampère capacity to the relative extremal function.

Proposition 8.25. *For any subset $E \subseteq M$ we have*
$$\mathrm{Cap}^*_\Omega(E) = \int_M (-v^*_E)\,\mathrm{MA}_{v^*_E}\,.$$

Furthermore, Cap^*_Ω is an outer regular Choquet capacity. More precisely if E_j is an increasing sequence of subsets of M and K_j is a decreasing sequence of compact subsets of M then
$$\mathrm{Cap}^*_\Omega\left(\bigcup_{j=0}^{\infty} E_j\right) = \lim_{j\to\infty} \mathrm{Cap}^*_\Omega(E_j)\,, \quad \mathrm{Cap}^*_\Omega\left(\bigcap_{j=0}^{\infty} K_j\right) = \lim_{j\to\infty} \mathrm{Cap}^*_\Omega(K_j)\,.$$

In particular all Borel subsets E of M are capacitable, i.e. $\mathrm{Cap}^*_\Omega(E) = \mathrm{Cap}_\Omega(E)$.

Proof. The proposition can be obtained by adapting the argument of [156, Theorem 4.2]. □

We can now prove the quaternionic analog of Josefson's theorem. Such a result in the local context is proved in [315, Theorem 1.1] following ideas of Bedford-Taylor [32] in the complex setting. The original result shows equivalence of being locally (complex) pluripolar and globally (complex) pluripolar in $\mathbb{C}^n$ and was proved by Josefson [191] with fairly involved techniques. The same result on compact Kähler manifolds is due to Guedj-Zeriahi [156, Theorem 7.2]. Here, we follow the proof of Lu-Nguyen [218, Theorem 4.10].

Theorem 8.26. *Then $P \subseteq M$ is locally q-pluripolar if and only if it is $\mathrm{QPSH}(M,\Omega)$-polar.*

Proof. We already have as a consequence of Lemma 8.21 that a QPSH(M,Ω)-polar set is locally q-pluripolar. Conversely, let $P \subseteq M$ be locally q-pluripolar, by Proposition 8.24 it is enough to prove that $V_P^* \equiv +\infty$. Assume by contradiction that V_P^* is bounded, then by Proposition 8.23(2) we deduce that V_P^* is non-constant. Set $m = \sup_M V_P^*$, then it is easy to check that $v := (V_P^* - m)/m$ is the relative extremal function $v^*_{P,\Omega/m}$ of P with respect to Ω/m. Since $\mathrm{Cap}^*_{\Omega/m}(P) = 0$, from Proposition 8.25 we infer $\int_{\{v<0\}} \mathrm{MA}_v = 0$ which implies $v \equiv 0$ i.e. $V_P^* \equiv m$ which contradicts the fact that V_P^* is non-constant. □

8.3 Variational approach to the quaternionic Calabi conjecture.

Let (M, I, J, K, Ω) be a locally flat compact HKT manifold with holomorphically trivial canonical bundle and let Θ be the holomorphic q-real q-positive volume form such that

$$\int_M \Omega^n \wedge \bar{\Theta} = 1\,.$$

The definition of the Monge-Ampère operator MA_φ can be extended to some unbounded Ω-qpsh functions as follows.

The finite energy class.

For $\varphi \in \mathrm{QPSH}(M,\Omega)$, we denote by $(\varphi_j)_{j\in\mathbb{N}}$, the *canonical approximation*, where $\varphi_j := \max\{\varphi, -j\} \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$. For each $j \in \mathbb{N}$ it is defined MA_{φ_j} and $\mathbf{1}_{\{\varphi>-j\}}\,\mathrm{MA}_{\varphi_j}$ gives a sequence of Borel measures. The idea is to define MA_φ as the weak limit of $\mathbf{1}_{\{\varphi>-j\}}\,\mathrm{MA}_{\varphi_j}$.

Proposition 8.27. *The sequence $\mathbf{1}_{\{\varphi>-j\}}\,\mathrm{MA}_{\varphi_j}$ is increasing and converges weakly to a positive Borel measure μ_φ such that $\mu_\varphi(M) \leq 1$.*

Proof. Proposition 8.15 implies

$$\mathbf{1}_{\{\varphi_j>-k\}}\,\mathrm{MA}_{\varphi_j} = \mathbf{1}_{\{\varphi_j>-k\}}\,\mathrm{MA}_{\max\{\varphi_j,-k\}}\,;$$

hence, for $j \geq k$ we obtain

$$\mathbf{1}_{\{\varphi>-j\}}\,\mathrm{MA}_{\varphi_j} \geq \mathbf{1}_{\{\varphi>-k\}}\,\mathrm{MA}_{\varphi_j} = \mathbf{1}_{\{\varphi>-k\}}\,\mathrm{MA}_{\varphi_k}\,.$$

This shows that the sequence $\mu_j := \mathbf{1}_{\{\varphi>-j\}}\,\mathrm{MA}_{\varphi_j}$ is increasing. Integrating by parts it is straightforward to see that the total mass $\mu_j(M)$ is bounded from above by $\mathrm{MA}_{\varphi_j}(M) = \mathrm{Vol}(M) = 1$. Thus, we can define

$$\mu_\varphi := \lim_{j\to\infty} \mu_j = \lim_{j\to\infty} \mathbf{1}_{\{\varphi>-j\}}\,\mathrm{MA}_{\varphi_j}\,,$$

which is itself a positive Borel measure with total mass bounded by 1. □

Definition 8.28. An Ω-qpsh function φ *has finite energy* if $\mu_\varphi(M) = 1$. For an Ω-qpsh function φ with finite energy the Monge-Ampère operator MA_φ is defined as μ_φ and we set

$$\mathcal{E}(M,\Omega) := \{\varphi \in \mathrm{QPSH}(M,\Omega) \mid \mu_\varphi(M) = 1\}\,.$$

The next Proposition can be proved exactly as in the Kähler case [157, Section 1] by replacing the role of the Kähler form with the HKT form Ω and the role of $\bar\partial$ with ∂_J.

Proposition 8.29. *The following facts are true:*

1. *$\varphi \in \mathcal{E}(M,\Omega)$ if and only if $\mathrm{MA}_{\varphi_j}(\{\varphi \leq -j\}) \to 0$, as $j \to \infty$, if and only if $\mathrm{MA}_\varphi(B) = \lim_{j\to\infty} \mathrm{MA}_{\varphi_j}(B)$ for all Borel subsets $B \subseteq M$. In particular the Monge-Ampère operator does not charge pluripolar sets.*

2. (*Comparison Principle*) *If* $\varphi, \psi \in \mathcal{E}(M,\Omega)$, *then* $\int_{\{\varphi>\psi\}} \mathrm{MA}_\varphi \leq \int_{\{\varphi>\psi\}} \mathrm{MA}_\psi$;
3. (*Maximum principle*) *If* $\varphi \in \mathcal{E}(M,\Omega)$ *and* $\psi \in \mathrm{QPSH}(M,\Omega)$, *then* $\mathbf{1}_{\{\varphi>\psi\}} \mathrm{MA}_\varphi = \mathbf{1}_{\{\varphi>\psi\}} \mathrm{MA}_{\max\{\varphi,\psi\}}$.
4. (*Continuity of the Monge-Ampère operator*) *Let* (φ_j) *be a sequence in* $\mathcal{E}(M,\Omega)$ *decreasing to* $\varphi \in \mathcal{E}(M,\Omega)$. *Then* $\mathrm{MA}_{\varphi_j} \to \mathrm{MA}_\varphi$, *as* $j \to \infty$.
5. *If* $\varphi, \psi \in \mathcal{E}(M,\Omega)$ *are such that* $\mathrm{MA}_\varphi \geq \mu$ *and* $\mathrm{MA}_\psi \geq \mu$ *for some positive Borel measure* μ *on* M, *then also* $\mathrm{MA}_{\max\{\varphi,\psi\}} \geq \mu$.

Next we introduce the following subclass of Ω-qpsh functions with finite energy:

$$\mathcal{E}^1(M,\Omega) := \{\varphi \in \mathcal{E}(M,\Omega) \mid \varphi \in L^1(\mathrm{MA}_\varphi)\}.$$

The subclass $\mathcal{E}^1(M,\Omega)$ is not affected by translations, meaning that $\varphi \in \mathcal{E}^1(M,\Omega)$ if and only if $\varphi + c \in \mathcal{E}^1(M,\Omega)$ for any constant $c \in \mathbb{R}$.

The energy functional.

We also define the *quaternionic Monge-Ampère energy functional* $E\colon \mathrm{QPSH}(M,\Omega) \cap L^\infty(M) \to \mathbb{R}$ as

$$E(\varphi) := \frac{1}{(n+1)} \sum_{j=0}^{n} \int_M \varphi\,(\Omega + \partial\partial_J\varphi)^j \wedge \Omega^{n-j}.$$

The definition of the energy is extended to $\mathrm{QPSH}(M,\Omega)$ by setting

$$E(\varphi) := \{\inf E(\psi) \mid \varphi \leq \psi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)\},$$

this is coherent with the monotonicity of E proved below in Proposition 8.30.

Proposition 8.30. *The energy is non-decreasing and concave, furthermore for any non-positive* $\varphi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ *we have*

$$\int_M \varphi\,\mathrm{MA}_\varphi \leq E(\varphi) \leq \frac{1}{(n+1)} \int_M \varphi\,\mathrm{MA}_\varphi.$$

Moreover, E *is upper semi-continuous in the* L^1*-topology and is continuous along decreasing sequences.*

Proof. Let $\varphi, \psi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ and assume $\varphi \leq \psi$. Set $\varphi_t = (1-t)\varphi + t\psi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$. From straightforward computations we obtain

$$\frac{d}{dt} E(\varphi_t) = \int_M \dot\varphi_t\,\mathrm{MA}_{\varphi_t} \geq 0, \tag{8.5}$$

$$\frac{d^2}{dt^2} E(\varphi_t) = -n \int_M \partial\dot\varphi_t \wedge \partial_J\dot\varphi_t \wedge (\Omega + \partial\partial_J\varphi_t)^{n-1} \leq 0,$$

showing that the energy is non-decreasing and concave.

Let now $\varphi \in \mathrm{QPSH}(M,\Omega) \cap L^\infty(M)$ be non-positive, then it is clear that $E(\varphi) \leq \frac{1}{n+1} \int_M \varphi\,\mathrm{MA}_\varphi$. The other inequality is implied by the following

$$\begin{aligned}
\int_M \varphi(\Omega + \partial\partial_J\varphi)^{j+1} \wedge \Omega^{n-j-1} &= \int_M \varphi(\Omega + \partial\partial_J\varphi)^j \wedge \Omega^{n-j} + \int_M \varphi\partial\partial_J\varphi \wedge (\Omega + \partial\partial_J\varphi)^j \wedge \Omega^{n-j-1} \\
&= \int_M \varphi(\Omega + \partial\partial_J\varphi)^j \wedge \Omega^{n-j} - \int_M \partial\varphi \wedge \partial_J\varphi \wedge (\Omega + \partial\partial_J\varphi)^j \wedge \Omega^{n-j-1} \\
&\leq \int_M \varphi(\Omega + \partial\partial_J\varphi)^j \wedge \Omega^{n-j}.
\end{aligned}$$

We prove the upper semi-continuity. Take a sequence $\varphi_j \to \varphi$ in L^1. If $\limsup E(\varphi_j) = -\infty$ the proposition is clear, hence we may assume that $E(\varphi_j)$ is uniformly bounded from below. Let ψ_j be the upper semi-continuous regularization of $\sup_{k\geq j}\varphi_k$. Clearly the sequence ψ_j decreases pointwise to φ and $\psi_j \geq \varphi_j$. Let $u \in \mathrm{QPSH}(M,\Omega)\cap L^\infty(M)$ be such that $u \geq \varphi$. Since E is non-decreasing and by continuity of the Monge-Ampère operator we see that

$$E(u) = \lim_{j\to\infty} E(\max\{u,\psi_j\}) \geq \limsup_{j\to\infty} E(\psi_j) \geq \limsup_{j\to\infty} E(\varphi_j)$$

and thus

$$E(\varphi) = \inf\{E(u) \mid u \in \mathrm{QPSH}(M,\Omega)\cap L^\infty(M)\,,\, u\geq\varphi\} \geq \limsup_{j\to\infty} E(\varphi_j)\,,$$

which gives upper semi-continuity. If the sequence is decreasing, by monotonicity of the energy functional we also have $E(\varphi)\leq \liminf_{j\to\infty} E(\varphi_j)$ and thus continuity. □

In particular the previous proposition shows that

$$\mathcal{E}^1(M,\Omega) = \{\varphi\in\mathcal{E}(M,\Omega) \mid E(\varphi) > -\infty\}\,,$$

which motivates the terminology.

The Ding functional.

The remaining part of the chapter, is devoted to the proof of the following Theorem:

Theorem 8.31. *Let (M,I,J,K,Ω) be a compact locally flat HKT manifold such that the canonical bundle of (M,I) is holomorphically trivial and let $\Theta\in\Lambda_I^{2n,0}(M)$ be a q-positive and q-real holomorphic form. Then the quaternionic Monge-Ampère equation has a unique solution $\varphi\in\mathcal{E}(M,\Omega)$.*

Theorem 8.31 is obtained as a consequence of the following more general result:

Proposition 8.32. *The quaternionic Monge-Ampère equation*

$$(\Omega + \partial\partial_J\varphi)^n = \mu$$

can be solved in $\mathcal{E}^1(M,\Omega)$ if and only if $\mathcal{E}^1(M,\Omega)\subseteq L^1(\mu)$.

This is the quaternionic analogue of a result of Guedj and Zeriahi in [157]. One implication is easy:

Proposition 8.33. *For any $\psi\in\mathcal{E}^1(M,\Omega)$*

$$\mathcal{E}^1(M,\Omega)\subseteq L^1(\mathrm{MA}_\psi)\,.$$

Proof. It is enough to prove that every $\varphi\in\mathcal{E}^1(M,\Omega)$ satisfies the following inequality:

$$0\leq \|\varphi\|_{L^1(\mathrm{MA}_\psi)} \leq 2\|\varphi\|_{L^1(\mathrm{MA}_\varphi)} + 2\|\psi\|_{L^1(\mathrm{MA}_\psi)}\,.$$

We may assume $\varphi,\psi\leq 0$.

$$-\int_M \varphi\,\mathrm{MA}_\psi = \int_{-\infty}^0 \mathrm{MA}_\psi(\{\varphi<t\})dt \leq 2\int_{-\infty}^0 \mathrm{MA}_\psi(\{\varphi<2t\})dt\,.$$

By the inclusion $\{\varphi<2t\}\subseteq\{\varphi<\psi+t\}\cup\{\psi<t\}$ we derive

$$-\int_M \varphi\,\mathrm{MA}_\psi \leq 2\int_{-\infty}^0 \mathrm{MA}_\psi(\{\varphi<\psi+t\})dt - 2\int_M \psi\,\mathrm{MA}_\psi\,.$$

Invoking the comparison principle and the inclusion $\{\varphi<\psi+t\}\subseteq\{\varphi<t\}$ we conclude

$$\int_{-\infty}^0 \mathrm{MA}_\psi(\{\varphi<\psi+t\})dt \leq \int_{-\infty}^0 \mathrm{MA}_\varphi(\{\varphi<\psi+t\})dt \leq -\int_M \varphi\,\mathrm{MA}_\varphi$$

as desired. □

The other implication requires more work and will be achieved with a variational technique involving the *Ding functional* $\mathcal{F}\colon \mathcal{E}^1(M,\Omega) \to \mathbb{R}$, defined as

$$\mathcal{F}(\varphi) = E(\varphi) - \int_M \varphi \, d\mu \,,$$

where μ is a given probability measure such that $\mathcal{E}^1(M,\Omega) \subseteq L^1(\mu)$. Since $E(\varphi + c) = E(\varphi) + c$ for $c \in \mathbb{R}$ we also have $\mathcal{F}(\varphi + c) = \mathcal{F}(\varphi)$. Formula (8.5) implies that for a path $\varphi_t\colon [0,1] \to \mathcal{E}^1(M,\Omega)$ we have

$$\frac{d}{dt}\mathcal{F}(\varphi_t) = \int_M \dot{\varphi}_t \, \mathrm{MA}_{\varphi_t} - \int_M \dot{\varphi}_t \, d\mu \,.$$

In particular φ is a critical point for $\mathcal{F}$ if and only if it solves the quaternionic Monge-Ampère equation

$$(\Omega + \partial\partial_J \varphi)^n = \mu \,. \tag{8.6}$$

The concavity of E readily implies that for every positive constant C the set of

$$\mathcal{E}^1_C(M,\Omega) := \{\varphi \in \mathcal{E}^1(M,\Omega) \mid E(\varphi) \geq -C \,,\, \varphi \leq 0\} \subseteq \mathcal{E}^1(M,\Omega)$$

is convex. Moreover $\mathcal{E}^1_C(M,\Omega)$ is compact in L^1-topology since it is closed by the upper semi-continuity of E and it is contained in the set

$$\left\{\varphi \in \mathrm{QPSH}(M,\Omega) \mid -C^{-1} \leq \sup_M \varphi \leq 0\right\}$$

which is compact in view of Hartogs' Lemma (Proposition 8.9(4)).

Proving the main theorem.

The solvability of equation (8.6) is obtained by showing that $\mathcal{F}$ has a maximizer in $\varphi \in \mathcal{E}^1(M,\Omega)$. The strategy to do so is the following:

- Show that $\mathcal{F}$ is upper semi-continuous on $\mathcal{E}^1_C(M,\Omega)$ with respect to the L^1 topology for every fixed constant $C > 0$ under the additional assumption $\mu \leq A\mathrm{Cap}_\Omega$ for some constant $A > 0$.
- Show that $\mathcal{F}$ is proper on $\mathcal{E}^1_C(M,\Omega)$ with respect to E for every fixed $C > 0$.
- From the first two steps follows that $\mathcal{F}$ has a maximizer in $\mathcal{E}^1(M,\Omega)$ whenever $\mu \leq A\mathrm{Cap}_\Omega$ for some constant $A > 0$. However, to achieve this result, one has to take into account the fact that the maximizer is not smooth, implying that it might not be a critical point. This issue is resolved applying the Projection Theorem (Proposition 8.37).
- The additional assumption that $\mu \leq A\mathrm{Cap}_\Omega$ is removed by means of a trick of Cegrell [77] and the proof of Proposition 8.32 is completed.

Lemma 8.34. *Under the assumptions $\mathcal{E}^1(M,\Omega) \subseteq L^1(\mu)$ and $\mu \leq A\mathrm{Cap}_\Omega$ for some constant $A > 0$, the operator $\varphi \mapsto \int_M \varphi \, d\mu$ is continuous over $\mathcal{E}^1_C(M,\Omega)$.*

Proof. Given a sequence (φ_j) in $\mathcal{E}^1_C(M,\Omega)$ which converges in L^1-topology to a $\varphi \in \mathcal{E}^1_C(M,\Omega)$, we show that $\int_M \varphi_j \, d\mu \to \int_M \varphi \, d\mu$. Since $\mu \leq A\mathrm{Cap}_\Omega$ we have

$$\int_M \varphi_j^2 d\mu = 2\int_0^{+\infty} t\mu(\{\varphi_j < -t\})dt \leq 2A\int_0^{+\infty} t\,\mathrm{Cap}_\Omega(\{\varphi_j < -t\})dt \,.$$

By [270, Lemma 3] for any fixed $1 < p < 2$ there exists a constant $C(p,R)$ such that for any Borel subset $B \subseteq M$ we have

$$\mathrm{Vol}(B) \leq C(p,R)\mathrm{Cap}_\Omega(B)^p \,.$$

In view of this fact, analogously to the case of the complex Hessian equation on Kähler manifolds [218, Lemma 6.8], we deduce

$$\sup\left\{\int_0^{+\infty} t\,\mathrm{Cap}_\Omega(\{\varphi<-t\})dt \mid \varphi\in\mathcal{E}^1_C(M,\Omega)\right\}<+\infty$$

and that the sequence $\int_M \varphi_j^2\,d\mu$ is uniformly bounded. Invoking [313, Lemma 4.5] we are done. □

In particular $\mathcal{F}$ is upper semi-continuous on $\mathcal{E}^1_C(M,\Omega)$, whenever $\mu\le A\mathrm{Cap}_\Omega$.

Lemma 8.35. *$\mathcal{F}$ is proper with respect to E. More precisely there is a constant $C>0$ such that*

$$\mathcal{F}(\varphi)\le E(\varphi)-\sup_M\varphi+C\left|E(\varphi)-\sup_M\varphi\right|^{1/2},$$

for all $\varphi\in\mathcal{E}^1(M,\Omega)$

Proof. Let $\varphi\in\mathcal{E}^1(M,\Omega)$, and for simplicity suppose $\sup_M\varphi=0$. Without loss of generality assume $E(\varphi)\le-1$. Set $\varepsilon=|E(\varphi)|^{-1/2}$ so that $\psi=\varepsilon\varphi$ is still Ω-qpsh.

For any $1\le j\le n$ we have

$$\Omega^j_\psi\wedge\Omega^{n-j}=\Omega^n+\sum_{k=1}^{j}\binom{j}{k}\varepsilon^k(\partial\partial_J\varphi)^k\wedge\Omega^{n-k}\le\Omega^n+N\varepsilon\sum_{k=0}^{n}\Omega^k_\varphi\wedge\Omega^{n-k}$$

for some $N\in\mathbb{N}$, therefore

$$E(\psi)=\frac{\varepsilon}{(n+1)}\sum_{j=0}^{n}\int_M\varphi\,\Omega^j_\psi\wedge\Omega^{n-j}\ge\int_M\varphi\,\Omega^n+(n+1)N\varepsilon^2E(\varphi)\ge-C\,,$$

proving that $\psi\in\mathcal{E}^1_C(M,\Omega)$. Since $\mathcal{E}^1_C(M,\Omega)$ is compact and convex, it is easy to show that there is a constant C' such that $\int_M\psi\,d\mu\ge-C'$ for every $\psi\in\mathcal{E}^1_C(M,\Omega)$. Therefore

$$\int_M\varphi\,d\mu=|E(\varphi)|^{1/2}\int_M\psi\,d\mu\ge-C'|E(\varphi)|^{1/2}\,,$$

as desired. □

Let $P(\psi)$ denote the Ω-plurisubharmonic envelope of an upper semi-continuous function

$$P(\psi)(x):=\{\sup\varphi(x)\mid\varphi\in\mathrm{QPSH}(M,\Omega)\,,\varphi\le\psi\}\,.$$

Observe that $P(\psi)$ is upper semi-continuous, as $P(\psi)\le\psi$ implies $P(\psi)^*\le\psi^*=\psi$, but then $P(\psi)^*$ is a competitor in the definition of $P(\psi)$, thus $P(\psi)=P(\psi)^*$.

Lemma 8.36. *For every continuous function $\psi\in C(M,\mathbb{R})$ the Monge-Ampère measure of $P(\psi)$ is supported on $\{P(\psi)=\psi\}$.*

Proof. Since ψ is continuous and $P(\psi)$ is upper semi-continuous the set $\{P(\psi)<\psi\}$ is open. The Lemma follows from a balayage argument performed on small balls inside $\{P(\psi)<\psi\}$. □

One key result to prove the existence of a critical point for the Ding functional is the following Projection Theorem. The idea is due to Berman and Boucksom [37], the proof was later simplified by Lu and Nguyen [218] which we follow closely.

Proposition 8.37 (Projection Theorem)**.** *For every $\varphi\in\mathcal{E}^1(M,\Omega)$ and $v\in C(M,\mathbb{R})$*

$$\frac{d}{dt}E(P(\varphi+tv))\Big|_{t=0}=\int_M v\;\mathrm{MA}_\varphi\,.$$

Proof. First, we show that it is enough to prove that

$$\frac{d}{dt}E(P(\psi+tv))\big|_{t=0} = \int_M v \ \mathrm{MA}_{P(\psi)}\,, \tag{8.7}$$

or equivalently

$$E(P(\psi+v)) - E(P(\psi)) = \int_0^1 \left(\int_M v \ \mathrm{MA}_{P(\psi+tv)}\right) dt$$

for every continuous function ψ on M. Take a sequence φ_j of continuous functions on M that decrease to φ. Such a sequence exists because φ is upper semi-continuous (observe that the φ_j's need not be in $\mathrm{QPSH}(M,\Omega)$). By continuity of the energy and the Monge Ampère operator along decreasing sequences we get

$$E(P(\varphi+v)) - E(P(\varphi)) = \lim_{j\to\infty}\big(E(P(\varphi_j+v)) - E(P(\varphi_j))\big)$$

and

$$\int_0^1 \left(\int_M v \ \mathrm{MA}_{\varphi}\right) dt = \lim_{j\to\infty}\int_0^1 \left(\int_M v \ \mathrm{MA}_{P(\varphi_j)}\right) dt\,.$$

And the desired formula follows from (8.7) and dominated convergence.

It only remains to prove (8.7), where $\psi \in C(M,\mathbb{R})$. Exchanging v with $-v$ it suffices to consider $t>0$. From the concavity of the energy we deduce

$$E(P(\psi+tv)) \le E(P(\psi)) + E'(P(\psi))(P(\psi+tv)-P(\psi))\,,$$
$$E(P(\psi)) \le E(P(\psi+tv)) + E'(P(\psi+tv))(P(\psi)-P(\psi+tv))\,,$$

which, together with (8.5), gives

$$\int_M \frac{P(\psi+tv)-P(\psi)}{t}\,\mathrm{MA}_{P(\psi+tv)} \le \frac{E(P(\psi+tv))-E(P(\psi))}{t} \le \int_M \frac{P(\psi+tv)-\psi}{t}\,\mathrm{MA}_{P(\psi)}\,. \tag{8.8}$$

Using Lemma 8.36 and the inequality $P(\psi+tv)\le\psi+tv$ we obtain from (8.8)

$$\int_M v \ \mathrm{MA}_{P(\psi+tv)} \le \frac{E(P(\psi+tv))-E(P(\psi))}{t} \le \int_M v \ \mathrm{MA}_{P(\psi)}\,. \tag{8.9}$$

Since the projection is uniformly Lipschitz, we see that

$$\sup_M |P(\psi+tv)-P(\psi)| \le t\sup_M |v|$$

so that $P(\psi+tv)\to P(\psi)$ uniformly as $t\to 0^+$. By continuity of the Monge-Ampère operator, taking the limit in (8.9) as $t\to 0^+$ yields (8.7) as desired. □

Theorem 8.38. *If $\mathcal{E}^1(M,\Omega)\subseteq L^1(\mu)$ and $\mu\le A\mathrm{Cap}_\Omega$, then there exists a solution $\varphi\in\mathcal{E}^1(M,\Omega)$ of the quaternionic Monge-Ampère equation such that*

$$\mathcal{F}(\varphi) = \sup_{\mathcal{E}^1(M,\Omega)}\mathcal{F}\,.$$

Proof. By translation invariance and properness of $\mathcal{F}$ there exists $C>0$ large enough to ensure

$$\sup_{\mathcal{E}^1(M,\Omega)}\mathcal{F} = \sup_{\mathcal{E}^1_C(M,\Omega)}\mathcal{F}\,.$$

The upper semi-continuity of $\mathcal{F}$ ensures the existence of a maximizer φ on the compact convex set $\mathcal{E}^1_C(M,\Omega)$. Since φ is not necessarily strictly Ω-qpsh, in general $\varphi+tv$ could fall outside $\mathrm{QPSH}(M,\Omega)$, even for small t, therefore φ may not be a critical point and we cannot conclude immediately. With the Projection Theorem at hand we can conclude along the lines of [39, Theorem 4.1]. □

Theorem 8.39. *Let μ be a probability measure. Then there exists a solution $\varphi \in \mathcal{E}^1(M,\Omega)$ of the quaternionic Monge-Ampère equation*

$$\mathrm{MA}_\varphi = \mu$$

if and only if $\mathcal{E}^1(M,\Omega) \subseteq L^1(\mu)$.

Proof. The strategy of the proof takes advantage from a decomposition trick that goes back to Cegrell [77].

Consider the set $P(M)$ of positive Radon measures on M and its subset

$$P'(M) = \{\nu \in P(M) \mid \nu \le \mathrm{Cap}_\Omega\}$$

which is clearly convex. Moreover, by outer regularity of the quaternionic Monge-Ampère capacity (Proposition 8.25), $P'(M)$ is also a compact subset of $P(M)$. We can then apply Rainwater's generalized Radon-Nikodým decomposition [248] and write $\mu = f\nu + \nu'$, where $\nu \le \mathrm{Cap}_\Omega$ is a positive Radon measure, $0 \le f \in L^1(\nu)$ and ν' is orthogonal to $P'(M)$. Since μ is non-pluripolar we have $\nu' \equiv 0$.

Consider normalizing constants $a_j \ge 1$ decreasing to 1 such that the positive Radon measures $\mu_j := a_j \min\{f,j\}\nu$ still satisfy $\int_M d\mu_j = \mathrm{Vol}(M)$. Clearly $\mu_j \le j a_j \mathrm{Cap}_\Omega$, which allows to apply Theorem 8.38 and find $\varphi_j \in \mathcal{E}^1(M,\Omega)$ such that $\mu_j = \mathrm{MA}_{\varphi_j}$. Without loss of generality we assume $\sup_M \varphi_j = 0$, and, up to a subsequence $\varphi_j \to \varphi$ in L^1 for some $\varphi \in \mathcal{E}^1(M,\Omega)$. Indeed by properness of the Ding functional and the fact that $\mathrm{MA}_{\varphi_j} = a_j \min\{f,j\}\nu \le 2f\nu = 2\mu$ for j large enough, we obtain

$$|E(\varphi_j)| \le \int_M (-\varphi_j)\,\mathrm{MA}_{\varphi_j} \le \int_M (-\varphi_j) d\mu = \mathcal{F}(\varphi_j) - E(\varphi_j) \le C|E(\varphi_j)|^{1/2}$$

which shows that $E(\varphi_j)$ is uniformly bounded, and thus $\varphi \in \mathcal{E}^1(M,\Omega)$ by upper semi-continuity of the energy.

Set $\psi_j = (\sup_{k\ge j}\varphi_k)^*$. The sequence ψ_j decreases to φ and, since for $k \ge j$ we have $\mathrm{MA}_{\varphi_k} = a_k \min\{f,k\}\nu \ge \min\{f,j\}\nu$, we also have $\mathrm{MA}_{\psi_j} \ge \min\{f,j\}\nu$ by Proposition 8.29(5). The continuity of the Monge-Ampère operator then yields

$$\mathrm{MA}_\varphi = \lim_{j\to\infty} \mathrm{MA}_{\psi_j} \ge \lim_{j\to\infty} \min\{f,j\}\nu = \mu$$

but since these two measures have the same total mass they must be equal. □

Theorem 8.40. *Let μ be a probability measure. Then there exists a solution $\varphi \in \mathcal{E}(M,\Omega)$ of the quaternionic Monge-Ampère equation*

$$\mathrm{MA}_\varphi = \mu$$

if and only if μ is non q-pluripolar.

Proof. That Monge-Ampère measures are non q-pluripolar was observed in Proposition 8.29(1). Suppose μ puts no mass on q-pluripolar sets. With the same argument of the previous theorem we have $\mu = f\nu$ for some $0 \le f \in L^1(\nu)$ and $\nu \le \mathrm{Cap}_\Omega$. Furthermore, we may assume there are $\varphi_j \in \mathcal{E}^1(M,\Omega)$ such that $\sup_M \varphi_j = 0$, $\varphi_j \to \varphi$ in L^1 for some $\varphi \in \mathrm{QPSH}(M,\Omega)$ and $\mathrm{MA}_{\varphi_j} = a_j \min\{f,j\}\nu$ where $1 \le a_j \le 2$ are decreasing to 1 and such that $\int_M a_j \min\{f,j\} d\nu = \mathrm{Vol}(M)$.

Using the argument in [157, Theorem 4.6] one can show that $\varphi \in \mathcal{E}(M,\Omega)$. Furthermore, as in the previous theorem, we also have $\mathrm{MA}_\varphi \ge \mu$ and since $\mathrm{MA}_\varphi(M) = \mu(M)$ the two measures must be equal. □

Theorem 8.41. *If $\varphi, \psi \in \mathcal{E}(M,\Omega)$ are such that $\mathrm{MA}_\varphi = \mathrm{MA}_\psi$, then $\varphi - \psi$ is constant.*

Proof. All the ideas in [105] used to prove uniqueness in the Kähler setting can be generalized to our framework. □

At this point one wishes to improve the regularity and show that the weak solutions found with Theorem 8.40 are actually smooth. In the complex case this has been done by Székelyhidi and Tosatti [281] but their proof ultimately relies on Yau's a priori estimates for the complex Monge-Ampère equation, hence we cannot follow this path, unless we have already solved the quaternionic Monge-Ampère equation with the method of continuity.

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