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Semi-abelian condition for color Hopf algebras ☆

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Recently, in [22], it was shown that the category of cocommutative Hopf algebras over an arbitrary field $\Bbbk$ is semi-abelian. We extend this result to the category of cocommutative color Hopf algebras, i.e. of cocommutative Hopf monoids in the symmetric monoidal category of G-graded vector spaces with G an abelian group, given an arbitrary skew-symmetric bicharacter on G, when G is finitely generated and the characteristic of $\Bbbk$ is different from 2 (not needed if G is finite of odd cardinality). We also prove that this category is action representable and locally algebraically cartesian closed, then algebraically coherent. In particular, these results hold for the category of cocommutative super Hopf algebras by taking $G = \mathbb{Z}_2$. Furthermore, we prove that, under the same assumptions on G and $\Bbbk$, the abelian category of abelian objects in the category of cocommutative color Hopf algebras is given by those cocommutative color Hopf algebras which are also commutative.

Contents

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1. Introduction

The notion of semi-abelian category was introduced by G. Janelidze et al. in [25] in order to capture typical algebraic properties valid for groups, rings and algebras. As it is said in [25], semi-abelian categories provide a good categorical foundation for a meaningful treatment of radical and commutator theory and of (co)homology theory of non-abelian structures. Semi-abelian categories have many nice properties: in particular in these categories it is possible to define and investigate the notions of semi-direct product, internal action, and internal crossed module. Some examples of semi-abelian categories are the categories of groups, Lie algebras, (associative) rings and compact groups. In [21] M. Gran et al. proved that the category of cocommutative Hopf algebras over a field $\Bbbk$, denoted by $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$, is semi-abelian when $\Bbbk$ has characteristic 0. Then, the result was extended to arbitrary characteristic in [22]. Hence it becomes natural to ask if this is true also for the category of cocommutative color Hopf algebras, i.e. of cocommutative Hopf monoids in the category Vec_G of G-graded vector spaces, which we denote by $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, when G is an abelian group. Indeed, we know that, in this case, Vec_G becomes a symmetric monoidal category by using a skew-symmetric bicharacter on G which modifies the braiding of $\mathrm{Vec}_\Bbbk$ given by the usual tensor flip. We show that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian if the abelian group G is finitely generated and the characteristic of the field $\Bbbk$ is not 2 (not needed if G is finite of odd cardinality). This generalizes the result for ordinary cocommutative Hopf algebras since we can recover $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$ by taking G as the trivial group, in which case the symmetric monoidal category Vec_G is exactly $\mathrm{Vec}_\Bbbk$. Furthermore, if we consider $G = \mathbb{Z}_2$, we obtain that the category of cocommutative super Hopf algebras, extensively used in Mathematics and Physics, is semi-abelian if $\mathrm{char}\Bbbk \neq 2$.

The organization of the paper is the following. After calling back some basic notions and results about monoidal categories and (color) Hopf algebras, we prove the completeness and cocompleteness of $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by explicitly showing limits and colimits and the protomodularity of $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by using a categorical result. We also observe that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is locally presentable, which is not guaranteed in general for the category of (cocommutative) Hopf monoids in a symmetric monoidal category. Then, we show the regularity of $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ through the same steps of [22]. In particular, we obtain a generalization of a theorem by K. Newman [30, Theorem 4.1] for cocommutative color Hopf algebras in case $\mathrm{char}\Bbbk \neq 2$ and the abelian group G is finitely generated, by using [27, Theorem 3.10 (3)] about cocommutative super Hopf algebras together with a braided strong monoidal functor from the category Vec_G to the category $\mathrm{Vec}_{\mathbb{Z}_2}$ from [7]. Then, through an equivalent characterization given in [25], we obtain that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian, still in case the abelian group G is finitely generated and $\mathrm{char}\Bbbk \neq 2$. Finally, we also prove that, under the same assumptions on G and $\Bbbk$, the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is action representable and locally algebraically cartesian closed (then algebraically coherent) and that the category of abelian objects in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ consists of those cocommutative color Hopf algebras which are also commutative and then, as a consequence, this category is abelian.

2. Preliminaries

2.1. Monoidal categories

First we recall some basic facts about monoidal categories, which can be found in [31,4]. Let $(\mathcal{M}, \otimes, \mathbf{I}, a, l, r)$ be a monoidal category. We write $(\mathcal{M}, \otimes, \mathbf{I})$ without the constraints a, l and r if these are clear from the context and we usually omit to write a in the computations since it will be clear when it is needed, in order to have slightly more compact formulas. We know that we can consider the category $\mathrm{Mon}(\mathcal{M})$ of monoids in $\mathcal{M}$, whose objects will be denoted as (A, m, u), and the category $\mathrm{Comon}(\mathcal{M})$ of comonoids in $\mathcal{M}$, whose objects will be denoted as (C, Δ, ϵ). Recall that a monoid M' is a *submonoid* of a monoid M, provided there exists a monoid morphism $i : M' \to M$ such that it is a monomorphism in $\mathcal{M}$. Analogously, a comonoid C' is a *subcomonoid* of a comonoid C, provided there exists a comonoid morphism $i : C' \to C$ such that it is a monomorphism in $\mathcal{M}$. In case $(\mathcal{M}, \otimes, \mathbf{I})$ has a braiding c, i.e. for every $X, Y \in \mathcal{M}$ there is an isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ which is natural in X and Y and it satisfies the well-known hexagon identities, the categories $\mathrm{Mon}(\mathcal{M})$ and $\mathrm{Comon}(\mathcal{M})$ become monoidal with the same constraints a, l, r. In this case, given monoids (M_1, m_1, u_1) and (M_2, m_2, u_2) in $\mathcal{M}$, the tensor product $\otimes$ is such that we have $(M_1, m_1, u_1) \otimes (M_2, m_2, u_2) := (M_1 \otimes M_2, m, u)$ where

$$m := (m_1 \otimes m_2) \circ (\mathrm{Id}_{M_1} \otimes c_{M_2,M_1} \otimes \mathrm{Id}_{M_2}) \quad \text{and} \quad u := (u_1 \otimes u_2) \circ r_{\mathbf{I}}^{-1}.$$

The unit object of $\mathrm{Mon}(\mathcal{M})$ is given by $(\mathbf{I}, r_{\mathbf{I}}, \mathrm{Id}_{\mathbf{I}})$. Similarly, given comonoids $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$ in $\mathcal{M}$, $(C_1, \Delta_1, \epsilon_1) \otimes (C_2, \Delta_2, \epsilon_2) := (C_1 \otimes C_2, \Delta, \epsilon)$ is a comonoid where

$$\Delta := (\mathrm{Id}_{C_1} \otimes c_{C_1,C_2} \otimes \mathrm{Id}_{C_2}) \circ (\Delta_1 \otimes \Delta_2) \quad \text{and} \quad \epsilon := r_{\mathbf{I}} \circ (\epsilon_1 \otimes \epsilon_2).$$

The unit object of $\mathrm{Comon}(\mathcal{M})$ is given by $(\mathbf{I}, r_{\mathbf{I}}^{-1}, \mathrm{Id}_{\mathbf{I}})$. When $\mathrm{Mon}(\mathcal{M})$ and $\mathrm{Comon}(\mathcal{M})$ are monoidal we can consider monoids and comonoids in them. Hence we have that

$$\mathrm{Bimon}(\mathcal{M}) \cong \mathrm{Mon}(\mathrm{Comon}(\mathcal{M})) \cong \mathrm{Comon}(\mathrm{Mon}(\mathcal{M})) \tag{1}$$

where $\mathrm{Bimon}(\mathcal{M})$ is the category of bimonoids in $\mathcal{M}$, since for $(B, m, u, \Delta, \epsilon)$ the fact that m and the u are morphisms of comonoids is equivalent to Δ and ϵ being morphisms of monoids (see e.g. [4, Proposition 1.11]), while

$$\mathrm{Mon}(\mathrm{Mon}(\mathcal{M})) \cong \mathrm{Mon}_{\mathrm{c}}(\mathcal{M}) \quad \text{and} \quad \mathrm{Comon}(\mathrm{Comon}(\mathcal{M})) \cong \mathrm{Comon}_{\mathrm{coc}}(\mathcal{M}) \tag{2}$$

which are the category of commutative monoids and of cocommutative comonoids in $\mathcal{M}$, respectively, and this follows from the *Eckmann–Hilton argument*: Δ_C is a morphism of comonoids if and only if C is cocommutative and m_A is a morphism of monoids if and only if A is commutative (see e.g. [4, Section 1.2.7]). We recall that a monoid (A, m, u) is *commutative* if $m = m \circ c_{A,A}$ and a comonoid (C, Δ, ϵ) is *cocommutative* if $c_{C,C} \circ \Delta = \Delta$. Also recall that a bimonoid B' is a *sub-bimonoid* of a bimonoid B, provided there exists a bimonoid morphism $i : B' \to B$ such that it is a monomorphism in $\mathcal{M}$. Given $(C, \Delta, \epsilon) \in \mathrm{Comon}(\mathcal{M})$ and $(A, m, u) \in \mathrm{Mon}(\mathcal{M})$, $\mathrm{Hom}_{\mathcal{M}}(C, A)$ is an (ordinary) monoid with *convolution product* such that, given $f, g : C \to A$ in $\mathcal{M}$, the product is $f * g := m \circ (f \otimes g) \circ \Delta$ and the unit is $u \circ \epsilon$. Hence we can consider the category $\mathrm{Hopf}(\mathcal{M})$ of Hopf monoids in $\mathcal{M}$, whose objects are bimonoids B in $\mathcal{M}$ equipped with a morphism $S : B \to B$ (*antipode*) which is the convolution inverse of Id_B. The monoidal categories $\mathrm{Mon}(\mathcal{M})$ and $\mathrm{Comon}(\mathcal{M})$ may fail to be braided and then the categories $\mathrm{Hopf}(\mathcal{M})$, $\mathrm{Bimon}(\mathcal{M})$, $\mathrm{Mon}_{\mathrm{c}}(\mathcal{M})$ and $\mathrm{Comon}_{\mathrm{coc}}(\mathcal{M})$ may fail to be monoidal but, when the braided category $\mathcal{M}$ is symmetric, i.e. $c_{X,Y}^{-1} = c_{Y,X}$ for every X and Y in $\mathcal{M}$, these categories are all braided and symmetric with the same braiding c and the same constraints a, l, r of $\mathcal{M}$

(see [4, Section 1.2.7]). Indeed, if $\mathcal{M}$ is symmetric, given A and B monoids in $\mathcal{M}$, then $c_{A,B} : A \otimes B \to B \otimes A$ is a morphism of monoids and then $\mathrm{Mon}(\mathcal{M})$ is a symmetric monoidal category and, dually, $\mathrm{Comon}(\mathcal{M})$ is a symmetric monoidal category. Iterating these results and applying (1) and (2), one can deduce that $\mathrm{Bimon}(\mathcal{M})$, $\mathrm{Mon}_{\mathrm{c}}(\mathcal{M})$ and $\mathrm{Comon}_{\mathrm{coc}}(\mathcal{M})$ are symmetric monoidal categories as well. Furthermore, if $\mathcal{M}$ is symmetric, given (B, S_B) and $(B', S_{B'})$ in $\mathrm{Hopf}(\mathcal{M})$ we have that $(B, S_B) \otimes (B', S_{B'}) := (B \otimes B', S_B \otimes S_{B'})$ is in $\mathrm{Hopf}(\mathcal{M})$. The antipode is a bimonoid morphism $S : B \to B^{\mathrm{op,cop}}$ where $(B^{\mathrm{op,cop}}, m^{\mathrm{op}}, u, \Delta^{\mathrm{cop}}, \epsilon)$ is a bimonoid with $m^{\mathrm{op}} = m \circ c_{B,B}$ and $\Delta^{\mathrm{cop}} = c_{B,B} \circ \Delta$, hence it satisfies

$$m \circ c_{B,B} \circ (S \otimes S) = S \circ m \quad \text{and} \quad (S \otimes S) \circ \Delta = c_{B,B} \circ \Delta \circ S$$

and so, if B is commutative, then S is a morphism of monoids while, if B is cocommutative, then S is a morphism of comonoids. Also note that if B is commutative or cocommutative then $S^2 = \mathrm{Id}_B$. Indeed, for instance, if B is cocommutative then

$$m \circ (S \otimes S^2) \circ \Delta = m \circ (\mathrm{Id}_B \otimes S) \circ (S \otimes S) \circ \Delta = m \circ (\mathrm{Id}_B \otimes S) \circ c_{B,B} \circ \Delta \circ S = u \circ \epsilon \circ S = u \circ \epsilon$$

and, analogously, $m \circ (S^2 \otimes S) \circ \Delta = u \circ \epsilon$. Since we use several times these facts in the following and, in particular, the fact that $\mathrm{Comon}_{\mathrm{coc}}(\mathcal{M})$ is a monoidal category is central for our proof of protomodularity, then we will work with a symmetric monoidal category $\mathcal{M}$.

Finally, recall that, given monoidal categories $(\mathcal{M}, \otimes, \mathbf{I}, a, l, r)$ and $(\mathcal{M}', \otimes, \mathbf{I}', a', l', r')$ (where we do not use different notations for $\otimes$ for notation convenience), a monoidal functor $(F, \phi^0, \phi^2) : (\mathcal{M}, \otimes, \mathbf{I}, a, l, r) \to (\mathcal{M}', \otimes, \mathbf{I}', a', l', r')$ consists of a functor $F : \mathcal{M} \to \mathcal{M}'$, a morphism $\phi^2_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$ in $\mathcal{M}'$ for every X, Y in $\mathcal{M}$ which is natural in X and Y and a morphism $\phi^0 : F(\mathbf{I}) \to \mathbf{I}'$ in $\mathcal{M}'$ such that

$$(\mathrm{Id}_{F(X)} \otimes \phi^2_{Y,Z}) \circ \phi^2_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) = a'_{F(X),F(Y),F(Z)} \circ (\phi^2_{X,Y} \otimes \mathrm{Id}_{F(Z)}) \circ \phi^2_{X\otimes Y,Z}$$

and

$$l'_{F(X)} \circ (\phi^0 \otimes \mathrm{Id}_{F(X)}) \circ \phi^2_{\mathbf{I},X} = F(l_X) \quad \text{and} \quad r'_{F(X)} \circ (\mathrm{Id}_{F(X)} \otimes \phi^0) \circ \phi^2_{X,\mathbf{I}} = F(r_X).$$

Furthermore, the functor F is called *strong* if ϕ^0 and $\phi^2_{X,Y}$ are isomorphisms for every X, Y in $\mathcal{M}$ and *strict* if ϕ^0 and $\phi^2_{X,Y}$ are identities for every X, Y in $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{M}'$ are (symmetric) braided with braidings c and c' respectively, F is called (symmetric) *braided* if $c'_{F(X),F(Y)} \circ \phi^2_{X,Y} = \phi^2_{Y,X} \circ F(c_{X,Y})$.

If $\mathcal{M}$ is the category $\mathrm{Vec}_{\Bbbk}$ of vector spaces over a field $\Bbbk$, we have the usual notions of $\Bbbk$-algebras, $\Bbbk$-coalgebras, $\Bbbk$-bialgebras and $\Bbbk$-Hopf algebras, usually denoted without $\Bbbk$. In the following we always omit $\Bbbk$ but it will be understood. For classical results and notions about the theory of Hopf algebras we refer to [35] and [37].

2.2. Semi-abelian categories

Here we recall some definitions needed for the notion of semi-abelian category. For the notions of limits and colimits of a functor, as for other basic notions of category theory, we refer to [10,26].

A finitely complete category $\mathcal{C}$ is *regular* if any arrow of $\mathcal{C}$ factors as a *regular epimorphism* (i.e. the coequalizer of a pair of morphisms of $\mathcal{C}$) followed by a monomorphism and if, moreover, regular epimorphisms are stable under pullbacks along any morphism. A *relation* on an object X of $\mathcal{C}$ is an equivalence class of triples (R, r_1, r_2), where R is an object of $\mathcal{C}$ and $r_1, r_2 : R \to X$ is a pair of jointly monic morphisms of $\mathcal{C}$, and two triples (R, r_1, r_2) and (R', r'_1, r'_2) are identified when they both factor through each other. An *equivalence relation* in $\mathcal{C}$ is a relation R on an object X which is reflexive, symmetric and transitive. A regular category $\mathcal{C}$ is (Barr)-*exact* if any equivalence relation R in $\mathcal{C}$ is *effective*, i.e. it is the kernel pair of some morphism in $\mathcal{C}$. Recall also that a category $\mathcal{C}$ is *protomodular*, in the sense of [11], if it has pullbacks of split epimorphisms along any morphism and all the inverse image functors of the fibration of points reflect isomorphisms. We know that, as it is said for instance in [11, Proposition 3.1.2], if $\mathcal{C}$ is pointed (i.e. it has a zero object) and finitely complete, the protomodularity can be expressed by simply asking that the Split Short Five Lemma holds in $\mathcal{C}$. Finally, a category $\mathcal{C}$ is *semi-abelian* if it is pointed, finitely cocomplete, (Barr)-exact and protomodular. Many details and properties about semi-abelian categories can be found in [11].

3. Color Hopf algebras

In this section we recall what color Hopf algebras are and how they differ from common Hopf algebras. We consider the category Vec_G of G-graded vector spaces over an arbitrary field $\Bbbk$ where G is a group. We add conditions on the group G along the way, to make it clear why these are needed. Objects in Vec_G are vector spaces $V = \bigoplus_{g\in G} V_g$ where V_g is a vector subspace of V for every $g \in G$ and the morphisms in Vec_G are linear maps $f : V \to W$ which preserve gradings, i.e. such that $f(V_g) \subseteq W_g$ for every $g \in G$. We know that this category is monoidal with $\otimes$ the tensor product of $\mathrm{Vec}_\Bbbk$ and unit object $\Bbbk = \bigoplus_{g\in G} \Bbbk_g$, where $\Bbbk_g = \{0\}$ if $g \neq 1_G$ and $\Bbbk_{1_G} = \Bbbk$, with 1_G the identity of G. Indeed, given $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$, we have that $V \otimes W = \bigoplus_{g\in G} (V \otimes W)_g$ where $(V \otimes W)_g = \bigoplus_{a\in G}(V_a \otimes W_{a^{-1}g})$. Also the associativity constraint and left and right unit constraints are the usual ones of $\mathrm{Vec}_\Bbbk$.

Remark 3.1. Recall that the category Vec_G is isomorphic to the category ${}^{\Bbbk G}\mathfrak{M}$ of left comodules over the group algebra $\Bbbk G$ with isomorphism given by $F : \mathrm{Vec}_G \to {}^{\Bbbk G}\mathfrak{M}$, $V = \bigoplus_{g\in G} V_g \mapsto (V, \rho)$ with $\rho(\sum_{g\in G} v_g) = \sum_{g\in G} g \otimes v_g$ and $F(f) = f$ and inverse given by $G : {}^{\Bbbk G}\mathfrak{M} \to \mathrm{Vec}_G$, $(V, \rho) \mapsto V = \bigoplus_{g\in G} V_g$, where $V_g = \{w \in V \mid \rho(w) = g \otimes w\}$ and $G(f) = f$. It is known that ${}^{\Bbbk G}\mathfrak{M}$ is a Grothendieck category, then abelian, since this is true in general for ${}^{C}\mathfrak{M}$ (and $\mathfrak{M}^C$) with C a coalgebra, but it is not always true for a coalgebra over a ring (see e.g. [16, 3.13]). So monomorphisms are exactly the injective maps and epimorphisms the surjective maps in Vec_G. Observe that, given a graded vector space $V = \bigoplus_{g\in G} V_g$ and a vector subspace $V' \subseteq V$, we can always consider the graded vector space $\bigoplus_{g\in G} V' \cap V_g \subseteq V'$. Furthermore, V' is a *graded subspace* of V if it is a graded vector space such that the inclusion $i : V' \to V$ is in Vec_G and this happens if and only if for every $x = \sum_{g\in G} x_g \in V'$, with $x_g \in V_g$, then $x_g \in V'$ for any $g \in G$; in this case V' has the induced grading $V' = \bigoplus_{g\in G} V'_g$, where $V'_g = V' \cap V_g$. Furthermore, we can always consider the graded vector space $\bigoplus_{g\in G} V_g/(V_g \cap V')$ and there is a canonical isomorphism of vector spaces $\bigoplus_{g\in G} \frac{V_g}{V_g\cap V'} \cong \frac{\bigoplus_{g\in G} V_g}{\bigoplus_{g\in G} V_g\cap V'}$ and the latter is V/V' in case V' is a graded subspace of V. In this case, we can also consider $\bigoplus_{g\in G} (V_g + V')/V'$, where $(V_g + V')/V'$ is a vector subspace of V/V' for every $g \in G$. Notice that there is a canonical isomorphism $V_g/(V_g \cap V') \cong (V_g + V')/V'$ as vector spaces for any $g \in G$, and then $\bigoplus_{g\in G} (V_g + V')/V'$ and $\bigoplus_{g\in G} V_g/(V_g \cap V')$ can be identified in Vec_G. Accordingly, when V' is a graded subspace of V, we have that $V/V' = \bigoplus_{g\in G} (V_g + V')/V'$ is in Vec_G and it is called *quotient graded vector space*.

Remark 3.2. We recall that, if $f : A \to B$ is in Vec_G, then $\ker(f)$ and $\mathrm{Im}(f)$ are graded subspaces of A and B, respectively. If f is surjective, the grading of $B = f(A)$ is the unique induced by A through f, i.e. $B_g = f(A_g)$ for every $g \in G$.

3.1. Graded (co)algebras

The objects of the categories $\mathrm{Mon}(\mathrm{Vec}_G)$ and $\mathrm{Comon}(\mathrm{Vec}_G)$ are called *G-graded algebras* and *G-graded coalgebras* respectively, which will be referred to as *graded algebras* and *graded coalgebras*, for short. Many details and properties about graded algebras and graded coalgebras can be found in [28,29]. Note that graded algebras and graded coalgebras are often used to denote algebras and coalgebras graded over $\mathbb{N}$, while here gradings will be always over G.

A graded algebra is an algebra (A, m, u) where $A = \bigoplus_{g\in G} A_g$ is a graded vector space such that m and u preserve gradings, i.e. for every $h, k \in G$ we have $A_h A_k \subseteq A_{hk}$ and $u(\Bbbk) \subseteq A_{1_G}$ and a morphism of graded algebras is a morphism of algebras that preserves gradings. Since monomorphisms in Vec_G are exactly the injective maps, a submonoid of a graded algebra (A, m, u), called *graded subalgebra*, is a graded subspace $V \subseteq A$ such that $1_A \in V$ and $m(V \otimes V) \subseteq V$. Then, V is a graded vector space with $V_g = V \cap A_g$ for every $g \in G$, an algebra and

$$V_g V_h = m((V \cap A_g) \otimes (V \cap A_h)) = m((V \otimes V) \cap (A_g \otimes A_h)) \subseteq m(V \otimes V) \cap m(A_g \otimes A_h) \subseteq V \cap A_{gh} = V_{gh}$$

for every $g, h \in G$. Furthermore, if we consider a graded two-sided ideal I of A such that $A/I = \bigoplus_{g\in G} (A_g + I)/I$ is a graded vector space, we know that $(A/I, u_{A/I}, m_{A/I})$ is an algebra with $u_{A/I} = \pi \circ u_A$ and $m_{A/I} \circ (\pi \otimes \pi) = \pi \circ m_A$ where $\pi : A \to A/I$ is the canonical quotient morphism and it is graded since $u_{A/I}$ and $m_{A/I}$ are in Vec_G with π, u_A and m_A in Vec_G; it is called *quotient graded algebra*.

Similarly, a graded coalgebra is a coalgebra (C, Δ, ϵ) where $C = \bigoplus_{g\in G} C_g$ is a graded vector space such that Δ and ϵ preserve gradings, i.e. $\Delta(C_g) \subseteq \bigoplus_{h\in G}(C_h \otimes C_{h^{-1}g})$ and $\epsilon(C_g) \subseteq \delta_{g,1_G}\Bbbk$ for every $g \in G$ and a morphism of graded coalgebras is a morphism of coalgebras that preserves gradings. A subcomonoid of a graded coalgebra (C, Δ, ϵ), called *graded subcoalgebra*, is a graded vector subspace $V \subseteq C$ such that $\Delta(V) \subseteq V \otimes V$ ($\epsilon(V) \subseteq \Bbbk$ is automatic). Then, V is a graded vector space, a coalgebra and

$$\Delta(V_g) = \Delta(V \cap C_g) \subseteq \Delta(V) \cap \Delta(C_g) \subseteq (V \otimes V) \cap (C \otimes C)_g = (V \otimes V)_g,$$

for every $g \in G$, since, from V graded subspace of C, we have that $V \otimes V$ is a graded subspace of $C \otimes C$. If I is a graded two-sided coideal of C, then C/I is a graded vector space and it is a coalgebra with $\Delta_{C/I} \circ \pi = (\pi \otimes \pi) \circ \Delta_C$ and $\epsilon_{C/I} \circ \pi = \epsilon_C$, where $\pi : C \to C/I$ is the canonical quotient morphism. Thus, C/I is a graded coalgebra because $\Delta_{C/I}$ and $\epsilon_{C/I}$ clearly preserve gradings since ϵ_C, Δ_C and π are in Vec_G; it is called *quotient graded coalgebra*.

3.2. Color bialgebras and color Hopf algebras

We are interested in studying Hopf monoids in Vec_G but, in order to do this, first we need that Vec_G is braided. One can give to Vec_G a braiding by using a bicharacter ϕ on G (see for example [7]), i.e. a map $\phi : G \times G \to \Bbbk - \{0\}$ such that

$$\phi(gh, l) = \phi(g, l)\phi(h, l) \text{ and } \phi(g, hl) = \phi(g, h)\phi(g, l) \text{ for every } g, h, l \in G.$$

It follows immediately that $\phi(1_G, g) = \phi(g, 1_G) = 1_\Bbbk$ for all $g \in G$. The monoidal category Vec_G is braided with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ such that $c_{X,Y}(x \otimes y) = \phi(g, h)y \otimes x$ for $x \in X_g$, $y \in Y_h$ and $g, h \in G$, defined on the components of the grading and extended by linearity, for every X and Y in Vec_G. In order to obtain that the braiding is in Vec_G, the group G needs to be *abelian*, as it is said in [7, Section 1.1] or in [5, pag. 193]. Hence, from now on, we will always consider G an abelian group. As we said before, we also need that the category Vec_G is symmetric and then we have to require that ϕ is a *commutation factor* on G that is a skew-symmetric bicharacter on G, i.e. that ϕ satisfies further $\phi(g, h)\phi(h, g) = 1_\Bbbk$ for $g, h \in G$. We will usually work on the components of the grading and all maps will be understood to be extended by linearity. For the braiding we use the same notation of [7] and we write $c(x \otimes y) = \phi(|x|, |y|)y \otimes x$ with

$x \in X$ and $y \in Y$, intending to work on homogeneous components and extend by linearity. Note that, given X, Y and Z in Vec_G, the hexagon relations

$$(\mathrm{Id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{Id}_Z) = c_{X,Y\otimes Z} \quad \text{and} \quad (c_{X,Z} \otimes \mathrm{Id}_Y) \circ (\mathrm{Id}_X \otimes c_{Y,Z}) = c_{X\otimes Y,Z}$$

on elements $x \in X$, $y \in Y$ and $z \in Z$, in terms of ϕ, are exactly

$$\phi(|x|,|y|)\phi(|x|,|z|) = \phi(|x|,|y||z|) = \phi(|x|,|y \otimes z|) \text{ and } \phi(|x|,|z|)\phi(|y|,|z|) = \phi(|x||y|,|z|) = \phi(|x \otimes y|,|z|).$$

Also note that, if X and Y are graded coalgebras, then

$$l_X \circ (\epsilon_Y \otimes \mathrm{Id}_X) \circ c_{X,Y} = l_X \circ c_{X,\Bbbk} \circ (\mathrm{Id}_X \otimes \epsilon_Y) = r_X \circ (\mathrm{Id}_X \otimes \epsilon_Y)$$

and $r_Y \circ (\mathrm{Id}_Y \otimes \epsilon_X) \circ c_{X,Y} = l_Y \circ (\epsilon_X \otimes \mathrm{Id}_Y)$ which on elements $x \in X$ and $y \in Y$ are

$$\phi(|x|,|y|)\epsilon(y)x = x\epsilon(y) \quad \text{and} \quad \phi(|x|,|y|)y\epsilon(x) = \epsilon(x)y. \tag{3}$$

Note that if $y \in Y_g$ with $g \neq 1_G$ then $\epsilon(y) = 0$ and if $y \in Y_{1_G}$ then $\phi(|x|,|y|) = 1_\Bbbk$, so we still have that $\epsilon(y)x = x\epsilon(y)$ (as clearly it must be) but these relations will be useful in the computations. A graded algebra A is *commutative* if $ab = \phi(|a|,|b|)ba$ for every $a, b \in A$ and a graded coalgebra C is *cocommutative* if $x_1 \otimes x_2 = \phi(|x_1|,|x_2|)x_2 \otimes x_1$ for every $x \in C$, where we shall adapt Sweedler notation and write $\Delta(x) = x_1 \otimes x_2$ always assuming homogeneous terms in the sum. Note that, given A and B in $\mathrm{Mon}(\mathrm{Vec}_G)$, the multiplication of $A \otimes B$ is given by $(a\otimes b)\cdot(c\otimes d) = \phi(|b|,|c|)ac\otimes bd$ and, given C and D in $\mathrm{Comon}(\mathrm{Vec}_G)$, the comultiplication of $C \otimes D$ is given by $\Delta_{C\otimes D}(c \otimes d) = \phi(|c_2|,|d_1|)c_1 \otimes d_1 \otimes c_2 \otimes d_2$.

The objects of the categories $\mathrm{Bimon}(\mathrm{Vec}_G)$ and $\mathrm{Hopf}(\mathrm{Vec}_G)$ are called *color bialgebras* and *color Hopf algebras*, respectively. A color bialgebra is a datum $(B, m, u, \Delta, \epsilon)$ where (B, m, u) is a graded algebra, (B, Δ, ϵ) is a graded coalgebra, and the two structures are compatible in the sense that Δ and ϵ are graded algebra morphisms or, equivalently, m and u are graded coalgebra morphisms. Hence $B = \bigoplus_{g\in G} B_g$ is an ordinary algebra and an ordinary coalgebra with m, u, Δ, ϵ which preserve gradings, but the condition of compatibility between the two structures differs from that in $\mathrm{Bialg}_\Bbbk$, only for the part that involves the braiding. So we have that

$$\epsilon(ab) = \epsilon(a)\epsilon(b),\ \epsilon(1_B) = 1_\Bbbk,\ \Delta(1_B) = 1_B \otimes 1_B \text{ and } \Delta(ab) = \phi(|a_2|,|b_1|)a_1b_1 \otimes a_2b_2$$

for every $a, b \in B$. A morphism of color bialgebras is a morphism of algebras and of coalgebras which preserves gradings. Given a color bialgebra B, a sub-bimonoid $B' \subseteq B$, called *color sub-bialgebra*, will be a graded subalgebra which is also a graded subcoalgebra (the compatibility between the two structures is that of B). Furthermore, given a color bialgebra B and a graded bi-ideal I (which is a two-sided ideal and two-sided coideal) we know that B/I is a graded algebra and a graded coalgebra and we show that the compatibility between the two structures is automatically maintained. In fact, given $\pi : B \to B/I$ the canonical quotient morphism, we have that

$$\Delta_{B/I} \circ m_{B/I} \circ (\pi \otimes \pi) = \Delta_{B/I} \circ \pi \circ m_B = (\pi \otimes \pi) \circ \Delta_B \circ m_B$$

and

$$(m_{B/I} \otimes m_{B/I}) \circ (\mathrm{Id}_{B/I} \otimes c_{B/I,B/I} \otimes \mathrm{Id}_{B/I}) \circ (\Delta_{B/I} \otimes \Delta_{B/I}) \circ (\pi \otimes \pi) =$$
$$(m_{B/I} \otimes m_{B/I}) \circ (\mathrm{Id}_{B/I} \otimes c_{B/I,B/I} \otimes \mathrm{Id}_{B/I}) \circ (\pi \otimes \pi \otimes \pi \otimes \pi) \circ (\Delta_B \otimes \Delta_B) =$$

$$(m_{B/I}\otimes m_{B/I})\circ(\pi\otimes\pi\otimes\pi\otimes\pi)\circ(\mathrm{Id}_B\otimes c_{B,B}\otimes\mathrm{Id}_B)\circ(\Delta_B\otimes\Delta_B)=$$
$$(\pi\otimes\pi)\circ(m_B\otimes m_B)\circ(\mathrm{Id}_B\otimes c_{B,B}\otimes\mathrm{Id}_B)\circ(\Delta_B\otimes\Delta_B)=(\pi\otimes\pi)\circ\Delta_B\circ m_B,$$

since c is natural and B is a color bialgebra. Now, since $\pi\otimes\pi$ is surjective, we have that

$$(m_{B/I}\otimes m_{B/I})\circ(\mathrm{Id}_{B/I}\otimes c_{B/I,B/I}\otimes\mathrm{Id}_{B/I})\circ(\Delta_{B/I}\otimes\Delta_{B/I})=\Delta_{B/I}\circ m_{B/I},$$

hence B/I is a color bialgebra, called *quotient color bialgebra*.

Given $(C,\Delta,\epsilon)\in\mathrm{Comon}(\mathrm{Vec}_G)$ and $(A,m,u)\in\mathrm{Mon}(\mathrm{Vec}_G)$, we have the convolution product of two morphisms $f,g:C\to A$ in Vec_G given by $f*g:=m\circ(f\otimes g)\circ\Delta$. A color Hopf algebra is a color bialgebra with a morphism $S:B\to B$ in Vec_G (antipode) such that $S*\mathrm{Id}_B=u\circ\epsilon=\mathrm{Id}_B*S$, thus it is a linear map which preserves gradings such that $b_1S(b_2)=\epsilon(b)1_B=S(b_1)b_2$ for all $b\in B$. A morphism of color Hopf algebras is just a morphism of color bialgebras, since the compatibility with antipodes is automatically guaranteed (see e.g. [4, Proposition 1.16]). Given a color Hopf algebra H, a *color Hopf subalgebra* $H'\subseteq H$ will be a color sub-bialgebra such that $S_H(H')\subseteq H'$. Furthermore, given a graded bi-ideal I such that $S_H(I)\subseteq I$, there is a unique linear map $S_{H/I}:H/I\to H/I$ such that $S_{H/I}\circ\pi=\pi\circ S_H$ which preserves gradings since S_H and π do. This is clearly the antipode of H/I (which is a color bialgebra), in fact as usual

$$\begin{aligned}m_{H/I}\circ(S_{H/I}\otimes\mathrm{Id}_{H/I})\circ\Delta_{H/I}\circ\pi&=m_{H/I}\circ(S_{H/I}\otimes\mathrm{Id}_{H/I})\circ(\pi\otimes\pi)\circ\Delta_H\\&=m_{H/I}\circ(\pi\otimes\pi)\circ(S_H\otimes\mathrm{Id}_H)\circ\Delta_H\\&=\pi\circ m_H\circ(S_H\otimes\mathrm{Id}_H)\circ\Delta_H=\pi\circ u_H\circ\epsilon_H\\&=u_{H/I}\circ\epsilon_{H/I}\circ\pi\end{aligned}$$

and from the surjectivity of π we obtain $m_{H/I}\circ(S_{H/I}\otimes\mathrm{Id}_{H/I})\circ\Delta_{H/I}=u_{H/I}\circ\epsilon_{H/I}$. Analogously for the other equality, so H/I is a color Hopf algebra, called *quotient color Hopf algebra*. Observe that the properties of the antipode S of a color Hopf algebra H on elements $x,y\in H$ are:

$$S(xy)=\phi(|x|,|y|)S(y)S(x),\ \ S(1_B)=1_B\ \ \text{and}\ \ \Delta(S(x))=\phi(|x_1|,|x_2|)S(x_2)\otimes S(x_1),\ \ \epsilon(S(x))=\epsilon(x).$$

If H is commutative then $S(xy)=S(x)S(y)$ and $S^2=\mathrm{Id}_H$ and if H is cocommutative then $\Delta(S(x))=S(x_1)\otimes S(x_2)$ and $S^2=\mathrm{Id}_H$.

Clearly the category $\mathrm{Vec}_{\Bbbk}$ is exactly Vec_G with $G=\{1_G\}$ the trivial group. Hence, motivated by the fact that $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$ is a semi-abelian category ([22, Theorem 2.10]), our question is now to establish whether the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is semi-abelian.

4. Limits, colimits and protomodularity of $\mathbf{Hopf_{coc}(Vec_G)}$

In this section we show that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is pointed, finitely complete, cocomplete and protomodular. Clearly $\Bbbk$ with the trivial grading is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and it is a zero object of the category. In fact, given H in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we have that ϵ is the unique morphism of coalgebras from H to $\Bbbk$ and it is also a morphism of algebras and it preserves gradings. Similarly, u is the unique morphism of algebras from $\Bbbk$ to H, and it is also a morphism of coalgebras and it preserves gradings. Hence $\Bbbk$ is a terminal and initial object in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, so a zero object and $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is pointed. Note that this is true also for $\mathrm{Hopf}(\mathrm{Vec}_G)$ and $\mathrm{Bimon}(\mathrm{Vec}_G)$ while, with the same reasoning, $\Bbbk$ is initial in $\mathrm{Mon}(\mathrm{Vec}_G)$ and terminal in $\mathrm{Comon}(\mathrm{Vec}_G)$.

Now we show the finite completeness of $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, by constructing equalizers and binary products and by using [10, Proposition 2.8.2]. Note that these limits have the same form, as vector spaces, of those of $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$, given for instance in [38] (see also [2]). The constructions given for $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$ fit with this more general context and the naturality of the braiding or the fact that the category is symmetric is often required to check what appears immediate in the $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$ case. Since we have not seen these computations in the literature for $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we give the explicit constructions of these limits, also because they will be used in the following.

Remark 4.1. Recall that, given a color Hopf algebra A and a graded subspace V of A, then V is a color Hopf subalgebra of A if it contains 1_A and it is closed under m_A, Δ_A and S_A. Observe that if A is (co)commutative, clearly also V is (co)commutative.

4.1. Equalizers

Let $f, g : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we can consider

$$K = \{x \in A \mid (\mathrm{Id}_A \otimes f)\Delta(x) = (\mathrm{Id}_A \otimes g)\Delta(x)\} \subseteq A.$$

Observe that, as vector space, $K = \ker((\mathrm{Id}_A \otimes f - \mathrm{Id}_A \otimes g) \circ \Delta)$ and $(\mathrm{Id}_A \otimes f - \mathrm{Id}_A \otimes g) \circ \Delta$ is in Vec_G since Δ, $\mathrm{Id}_A \otimes f$ and $\mathrm{Id}_A \otimes g$ are in Vec_G. Thus, by Remark 3.2, K is a graded subspace of A, i.e. $K = \bigoplus_{g \in G} K_g$ with $K_g = K \cap A_g$. By Remark 4.1 we have that, if it is closed under the operations of A, then it will be automatically in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Clearly $1_A \in K$ and, given $x, y \in K$, then $xy \in K$ since $(\mathrm{Id}_A \otimes f) \circ \Delta$ and $(\mathrm{Id}_A \otimes g) \circ \Delta$ are morphisms of graded algebras. Indeed, given $x, y \in K$, we have that

$$(\mathrm{Id}_A \otimes f)\Delta(xy) = (\mathrm{Id}_A \otimes f)\Delta(x) \cdot (\mathrm{Id}_A \otimes f)\Delta(y) = (\mathrm{Id}_A \otimes g)\Delta(x) \cdot (\mathrm{Id}_A \otimes g)\Delta(y) = (\mathrm{Id}_A \otimes g)\Delta(xy),$$

denoting by $\cdot$ the multiplication $m_{A\otimes B}$, hence K is closed under m_A. Furthermore, since by cocommutativity of A we have that $x_1 \otimes x_2 = \phi(|x_1|, |x_2|)x_2 \otimes x_1$ for every $x \in A$ (and then for every $x \in K$), if $\Delta(x) \in A \otimes K$ then we obtain that

$$K \otimes A \ni c_{A,K}(\phi(|x_1|, |x_2|)x_2 \otimes x_1) = \phi(|x_1|, |x_2|)\phi(|x_2|, |x_1|)x_1 \otimes x_2 = x_1 \otimes x_2,$$

since ϕ is a commutation factor, thus we only have to show $\Delta(K) \subseteq A \otimes K$. But we have that $K = \ker((\mathrm{Id}_A \otimes (f - g)) \circ \Delta)$, thus

$$A \otimes K = \ker(\mathrm{Id}_A \otimes (\mathrm{Id}_A \otimes (f - g)) \circ \Delta),$$

hence we desire to show that $x \in K$ implies $(\mathrm{Id}_A \otimes (\mathrm{Id}_A \otimes (f - g))\Delta)\Delta(x) = 0$. This is equivalent to show that $(\mathrm{Id}_A \otimes \mathrm{Id}_A \otimes f)(\mathrm{Id}_A \otimes \Delta)\Delta(x) = (\mathrm{Id}_A \otimes \mathrm{Id}_A \otimes g)(\mathrm{Id}_A \otimes \Delta)\Delta(x)$, i.e. $x_1 \otimes x_{2_1} \otimes f(x_{2_2}) = x_1 \otimes x_{2_1} \otimes g(x_{2_2})$. Moreover, $x_1 \otimes x_{2_1} \otimes f(x_{2_2}) = x_{1_1} \otimes x_{1_2} \otimes f(x_2)$ and $x_1 \otimes x_{2_1} \otimes g(x_{2_2}) = x_{1_1} \otimes x_{1_2} \otimes g(x_2)$ by coassociativity and so we have to prove $(\Delta \otimes \mathrm{Id}_B)(\mathrm{Id}_A \otimes f)\Delta(x) = (\Delta \otimes \mathrm{Id}_B)(\mathrm{Id}_A \otimes g)\Delta(x)$ and this is true because $x \in K$. Hence K is closed under Δ_A. Furthermore, since A is cocommutative we have $\Delta \circ S_A = (S_A \otimes S_A) \circ \Delta$ and then, given $x \in K$, we obtain

$$(\mathrm{Id}_A \otimes f)\Delta(S_A(x)) = (\mathrm{Id}_A \otimes f)(S_A \otimes S_A)\Delta(x) = (S_A \otimes S_B)(\mathrm{Id}_A \otimes f)\Delta(x) = (S_A \otimes S_B)(\mathrm{Id}_A \otimes g)\Delta(x),$$

which is exactly $(\mathrm{Id}_A \otimes g)\Delta(S_A(x))$. Thus, we have that K is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Now, since the inclusion $i : K \to A$ is in Vec_G and it is a morphism of algebras and coalgebras, we obtain that (K, i) is the equalizer in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ of the pair (f, g). In fact, with $x \in K$, from $x_1 \otimes f(x_2) = x_1 \otimes g(x_2)$ we immediately obtain that $f(x) = g(x)$ and, if $h : C \to A$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is such that $f \circ h = g \circ h$, then the image of h is in K. Indeed, since h is a morphism of coalgebras, we obtain

$$(\mathrm{Id}_A \otimes f) \circ \Delta_A \circ h = (\mathrm{Id}_A \otimes f) \circ (h \otimes h) \circ \Delta_C = (\mathrm{Id}_A \otimes g) \circ (h \otimes h) \circ \Delta_C = (\mathrm{Id}_A \otimes g) \circ \Delta_A \circ h.$$

We denote the equalizer of the pair (f, g) in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by $(\mathrm{Eq}(f, g), i)$.

4.2. Binary products

If we take A, B in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ we can consider $(A \otimes B, \pi_A, \pi_B)$ where

$$\pi_A := r_A \circ (\mathrm{Id}_A \otimes \epsilon_B) \;\text{ and }\; \pi_B := l_B \circ (\epsilon_A \otimes \mathrm{Id}_B).$$

In particular, $(A \otimes B, m, u, \Delta, \epsilon, S)$ is a cocommutative color Hopf algebra since $\mathrm{Hopf}(\mathrm{Vec}_G)$ and $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G)$ have a monoidal structure with Vec_G symmetric and we recall that $m = (m_A \otimes m_B) \circ (\mathrm{Id}_A \otimes c_{B,A} \otimes \mathrm{Id}_B)$, $u = (u_A \otimes u_B) \circ r_{\Bbbk}^{-1}$, $\Delta = (\mathrm{Id}_A \otimes c_{A,B} \otimes \mathrm{Id}_B) \circ (\Delta_A \otimes \Delta_B)$, $\epsilon = r_{\Bbbk} \circ (\epsilon_A \otimes \epsilon_B)$ and $S = S_A \otimes S_B$. Furthermore, π_A and π_B are algebra morphisms and coalgebra morphisms and they preserve gradings, since this is true for r_A, l_B and ϵ_A, ϵ_B and then they are morphisms in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. We only have to prove that, for every H in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we have a bijection between the set of morphisms in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ from H to $A \otimes B$ and the cartesian product of the set of morphisms from H to A and that of morphisms from H to B in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Given a map $f : H \to A \otimes B$ we can consider the pair $(\pi_A \circ f, \pi_B \circ f)$ and given a pair (g, h), with $g : H \to A$ and $h : H \to B$, we can consider the morphism $(g \otimes h) \circ \Delta_H$; this map will be the diagonal morphism of the pair (g, h), usually denoted by $\langle g, h\rangle$. It is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ since Δ_H is a morphism of coalgebras with H cocommutative (and only in this case). Hence it is clear that this construction is specific for the cocommutative case. Clearly, given $g : H \to A$ and $h : H \to B$, we have

$$\pi_A \circ (g \otimes h) \circ \Delta_H = r_A \circ (\mathrm{Id}_A \otimes \epsilon_B) \circ (g \otimes h) \circ \Delta_H = r_A \circ (g \otimes \mathrm{Id}_{\Bbbk}) \circ (\mathrm{Id}_H \otimes \epsilon_H) \circ \Delta_H = g \circ r_H \circ (\mathrm{Id}_H \otimes \epsilon_H) \circ \Delta_H = g$$

and, analogously, $\pi_B \circ (g \otimes h) \circ \Delta_H = h$. On the other hand, given $f : H \to A \otimes B$, we have that

$$((\pi_A \circ f) \otimes (\pi_B \circ f)) \circ \Delta_H = (\pi_A \otimes \pi_B) \circ (f \otimes f) \circ \Delta_H = (\pi_A \otimes \pi_B) \circ \Delta_{A \otimes B} \circ f,$$

by using the fact that f is a morphism of coalgebras. Now we show that $(\pi_A \otimes \pi_B) \circ \Delta_{A \otimes B} = \mathrm{Id}_{A \otimes B}$. Indeed, we can compute

$$\begin{aligned}(\pi_A \otimes \pi_B) \circ \Delta_{A \otimes B} &= (\pi_A \otimes \pi_B) \circ (\mathrm{Id}_A \otimes c_{A,B} \otimes \mathrm{Id}_B) \circ (\Delta_A \otimes \Delta_B) \\ &= (r_A \otimes l_B) \circ (\mathrm{Id}_A \otimes \epsilon_B \otimes \epsilon_A \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_A \otimes c_{A,B} \otimes \mathrm{Id}_B) \circ (\Delta_A \otimes \Delta_B) \\ &= (r_A \otimes l_B) \circ (\mathrm{Id}_A \otimes c_{\Bbbk,\Bbbk} \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_A \otimes \epsilon_A \otimes \epsilon_B \otimes \mathrm{Id}_B) \circ (\Delta_A \otimes \Delta_B) \\ &= (r_A \otimes l_B) \circ (\mathrm{Id}_A \otimes \epsilon_A \otimes \epsilon_B \otimes \mathrm{Id}_B) \circ (\Delta_A \otimes \Delta_B) = \mathrm{Id}_A \otimes \mathrm{Id}_B = \mathrm{Id}_{A \otimes B}\end{aligned}$$

where we only use the naturality of c and the fact that $c_{\Bbbk,\Bbbk} = \mathrm{Id}_{\Bbbk,\Bbbk}$. Hence $(A \otimes B, \pi_A, \pi_B)$ is the binary product of A and B in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and we denote the object part of the categorical product of A and B by $A \times B$.

We have obtained that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is finitely complete and now we show the cocompleteness. To this aim we prove, more generally, the cocompleteness of $\mathrm{Hopf}(\mathrm{Vec}_G)$ by constructing coequalizers and arbitrary coproducts and that the colimits are the same in the cocommutative case. As for limits, also colimits have the same form as vector spaces of those in $\mathrm{Hopf}_{\Bbbk}$, which are given for instance in [3].

Remark 4.2. The fact that colimits are the same in the cocommutative case should not surprise us. In fact we recall that, given a symmetric monoidal category $\mathcal{M}$, the forgetful functor $U_a : \mathrm{Mon}(\mathcal{M}) \to \mathcal{M}$ creates limits and the forgetful functor $U_c : \mathrm{Comon}(\mathcal{M}) \to \mathcal{M}$ creates colimits and then $\mathrm{Mon}(\mathcal{M})$ is closed under limits in $\mathcal{M}$ as $\mathrm{Comon}(\mathcal{M})$ is closed under colimits in $\mathcal{M}$ (see e.g. [31, Fact 10], [32, Fact 4]). Hence also $\mathrm{Mon}_c(\mathcal{M})$ is closed under limits in $\mathrm{Mon}(\mathcal{M})$ and $\mathrm{Comon}_{\mathrm{coc}}(\mathcal{M})$ is closed under colimits in $\mathrm{Comon}(\mathcal{M})$. Furthermore, $\mathrm{Bimon}_{\mathrm{coc}}(\mathcal{M}) = \mathrm{Comon}(\mathrm{Comon}(\mathrm{Mon}(\mathcal{M})))$, so $\mathrm{Bimon}_{\mathrm{coc}}(\mathcal{M})$ is closed under colimits in $\mathrm{Bimon}(\mathcal{M})$. We will see that colimits in $\mathrm{Hopf}(\mathrm{Vec}_G)$ are the same of those in $\mathrm{Bimon}(\mathrm{Vec}_G)$ and then, clearly, $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is closed under colimits in $\mathrm{Hopf}(\mathrm{Vec}_G)$. Observe also that $\mathrm{Bimon}(\mathrm{Vec}_G)$=$\mathrm{Comon}(\mathrm{Mon}(\mathrm{Vec}_G))$ is closed under colimits in $\mathrm{Mon}(\mathrm{Vec}_G)$ and then colimits in $\mathrm{Hopf}(\mathrm{Vec}_G)$ will derive from those of $\mathrm{Mon}(\mathrm{Vec}_G)$. However, we show all the details in the sequel.

Remark 4.3. Recall that, given a color Hopf algebra H and a graded bi-ideal I such that $S(I) \subseteq I$, then H/I is a color Hopf algebra. Observe also that if H is (co)commutative then also H/I is (co)commutative. Indeed, for instance, if H is cocommutative, by naturality of c we have that

$$c_{H/I,H/I} \circ \Delta_{H/I} \circ \pi = c_{H/I,H/I} \circ (\pi \otimes \pi) \circ \Delta_H = (\pi \otimes \pi) \circ c_{H,H} \circ \Delta_H = (\pi \otimes \pi) \circ \Delta_H = \Delta_{H/I} \circ \pi$$

and then, since $\pi : H \to H/I$ is surjective, we obtain $c_{H/I,H/I} \circ \Delta_{H/I} = \Delta_{H/I}$.

4.3. Coequalizers

Let $f, g : A \to B$ in $\mathrm{Hopf}(\mathrm{Vec}_G)$, we can consider $I = B((f-g)(A))B$, the two-sided ideal of B generated by the graded subspace of B given by $(f-g)(A) := \{f(a) - g(a) \mid a \in A\}$, which is graded by Remark 3.2, since $I = m_B(m_B \otimes \mathrm{Id}_B)(B \otimes (f-g)(A) \otimes B)$. Thus, in order to prove that B/I is a color Hopf algebra, we only have to check that I is a two-sided coideal and that $S(I) \subseteq I$, by Remark 4.3. Given $a \in A$, since f and g are morphisms of coalgebras, we obtain

$$\begin{aligned}
\Delta(f(a) - g(a)) &= \Delta(f(a)) - \Delta(g(a)) = f(a_1) \otimes f(a_2) - g(a_1) \otimes g(a_2) \\
&= f(a_1) \otimes f(a_2) - g(a_1) \otimes f(a_2) + g(a_1) \otimes f(a_2) - g(a_1) \otimes g(a_2) \\
&= (f(a_1) - g(a_1)) \otimes f(a_2) + g(a_1) \otimes (f(a_2) - g(a_2)).
\end{aligned}$$

Hence we have that $\Delta((f-g)(A)) \subseteq (f-g)(A) \otimes B + B \otimes (f-g)(A)$ and from this, using that Δ is a morphism of algebras and that B is a color bialgebra, we have that

$$\begin{aligned}
\Delta_B(B((f-g)(A))) &= \Delta_B(m_B(B \otimes (f-g)(A))) \\
&= (m_B \otimes m_B)(\mathrm{Id}_B \otimes c_{B,B} \otimes \mathrm{Id}_B)(\Delta_B \otimes \Delta_B)(B \otimes (f-g)(A)) \\
&\subseteq (m_B \otimes m_B)(\mathrm{Id}_B \otimes c_{B,B} \otimes \mathrm{Id}_B)(B \otimes B \otimes (f-g)(A) \otimes B) \\
&+ (m_B \otimes m_B)(\mathrm{Id}_B \otimes c_{B,B} \otimes \mathrm{Id}_B)(B \otimes B \otimes B \otimes (f-g)(A)) \\
&\subseteq B((f-g)(A)) \otimes B + B \otimes B((f-g)(A))
\end{aligned}$$

and then $\Delta(I) \subseteq I \otimes B + B \otimes I$. Furthermore, $\epsilon(I) = 0$ since ϵ is a morphism of algebras and thus I is a two-sided coideal. Furthermore, we have that

$$\begin{aligned}
S_B(I) &= S_B m_B(m_B \otimes \mathrm{Id}_B)(B \otimes (f-g)(A) \otimes B) \\
&= m_B c_{B,B}(S_B \otimes S_B)(m_B \otimes \mathrm{Id}_B)(B \otimes (f-g)(A) \otimes B) \\
&= m_B c_{B,B}(m_B \otimes \mathrm{Id}_B)(c_{B,B} \otimes \mathrm{Id}_B)(S_B \otimes S_B \otimes S_B)(B \otimes (f-g)(A) \otimes B) \\
&\subseteq m_B c_{B,B}(m_B \otimes \mathrm{Id}_B)(c_{B,B} \otimes \mathrm{Id}_B)(B \otimes (f-g)(S_A(A)) \otimes B) \\
&\subseteq m_B c_{B,B}(m_B \otimes \mathrm{Id}_B)(c_{B,B} \otimes \mathrm{Id}_B)(B \otimes (f-g)(A) \otimes B) \subseteq I.
\end{aligned}$$

Hence B/I is a color Hopf algebra and $\pi : B \to B/I$ is in $\mathrm{Hopf}(\mathrm{Vec}_G)$ such that clearly $\pi \circ f = \pi \circ g$. Now, given $h : B \to H$ in $\mathrm{Hopf}(\mathrm{Vec}_G)$ such that $h \circ f = h \circ g$, we have that $I \subseteq \ker(h)$ and then there exists a unique morphism of coalgebras $h' : B/I \to H$ such that $h' \circ \pi = h$ which is also of algebras and it preserves gradings since this is true for π and h, hence it is the unique morphism in $\mathrm{Hopf}(\mathrm{Vec}_G)$ such that $h' \circ \pi = h$. Thus, $(B/I, \pi)$ is the coequalizer in $\mathrm{Hopf}(\mathrm{Vec}_G)$ of the pair (f, g), which we denote by $(\mathrm{Coeq}(f, g), \pi)$. Observe here that, clearly, this is also the coequalizer for f, g in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ since if B is cocommutative also B/I is cocommutative, as said in Remark 4.3.

Remark 4.4. We know that, given $V = \bigoplus_{g\in G} V_g$ in Vec_G and $T^p(V) = V \otimes \cdots \otimes V$ p-times with $p \in \mathbb{N}$ (where $T^0(V) = \Bbbk$), we have that $T^p(V)$ is graded with $T^p(V)_g = \bigoplus_{h_1\cdots h_p=g} V_{h_1} \otimes \cdots \otimes V_{h_p}$ for every $g \in G$ and then we have

$$T(V) = \bigoplus_{p\in\mathbb{N}} T^p(V) = \bigoplus_{p\in\mathbb{N}} \bigoplus_{g\in G} T^p(V)_g = \bigoplus_{g\in G} \bigoplus_{p\in\mathbb{N}} T^p(V)_g = \bigoplus_{g\in G} T(V)_g,$$

so $T(V)$ is graded as vector space. But $T(V)$ is also an algebra and it is graded since $1_\Bbbk \in \Bbbk_{1_G} = T^0(V)_{1_G} \subseteq T(V)_{1_G}$ and given $x \in T(V)_g$ and $y \in T(V)_h$ with $g, h \in G$ we have that $xy \in T(V)_{gh}$. In fact, given an element $x = x_{h_1} \otimes \cdots \otimes x_{h_p} \in T^p(V)_g$ with $h_1 \cdots h_p = g$ and an element $y = y_{k_1} \otimes \cdots \otimes y_{k_s} \in T^s(V)_h$ with $k_1 \cdots k_s = h$, we have that $xy = x_{h_1} \otimes \cdots \otimes x_{h_p} \otimes y_{k_1} \otimes \cdots \otimes y_{k_s} \in T^{p+s}(V)_{gh}$ since $h_1 \cdots h_p \cdot k_1 \cdots k_s = gh$. Note also that the canonical inclusion $i : V \to T(V)$ preserves gradings since $i(V_g) = V_g = T^1(V)_g \subseteq T(V)_g$.

4.4. Coproducts

Let $\{H_l\}_{l\in I}$ be a family of color Hopf algebras, we can take $T(\bigoplus_{l\in I} H_l)/L$ where L is the two-sided ideal in $T(\bigoplus_{l\in I} H_l)$ generated by the linear span of the set

$$J := \{i(j_l(x_l y_l)) - i(j_l(x_l))i(j_l(y_l)), 1_{T(\bigoplus H_l)} - i(j_l(1_{H_l})) \mid x_l, y_l \in H_l,\ l \in I\},$$

where $j_t : H_t \to \bigoplus_{l\in I} H_l$ sends v to the element with v as t-component, the only one not trivial and $i : \bigoplus_{l\in I} H_l \to T(\bigoplus_{l\in I} H_l)$ is the canonical inclusion. Now, since $H_l = \bigoplus_{g\in G} H_{l,g}$ for every $l \in I$, we have that $\bigoplus_{l\in I} H_l = \bigoplus_{l\in I} \bigoplus_{g\in G} H_{l,g} = \bigoplus_{g\in G} \bigoplus_{l\in I} H_{l,g}$, then $\bigoplus_{l\in I} H_l$ is in Vec_G and so $T(\bigoplus_{l\in I} H_l)$ is a graded algebra by Remark 4.4. But now, clearly, L is graded since it is generated by homogeneous elements; indeed i, j_l, m_l and $m_{T(\bigoplus H_l)}$ are in Vec_G, for every $l \in I$. Thus, also $T(\bigoplus_{l\in I} H_l)/L$ is a graded algebra. For all $l \in I$ define $q_l := \nu \circ i \circ j_l$, where $\nu : T(\bigoplus_{l\in I} H_l) \to T(\bigoplus_{l\in I} H_l)/L$ is the canonical quotient morphism. Then, q_l is a morphism of algebras for every $l \in I$ by the relations of J and since ν is an algebra morphism and it preserves gradings since this is true for the three maps. Now, given a graded algebra C and graded algebra morphisms $g_l : H_l \to C$ for $l \in I$, there exists a unique linear map $k : \bigoplus_{i\in I} H_i \to C$ such that $k \circ j_l = g_l$ for every $l \in I$ by the universal property of the coproduct of vector spaces and k also preserves gradings since j_l and g_l do (it is the universal property of the coproduct in Vec_G). By the universal property of the tensor algebra, there is a unique algebra morphism $s : T(\bigoplus_{l\in I} H_l) \to C$ such that $s \circ i = k$ and s also preserves gradings since i and k do. Finally, we have

$$s(i(j_l(x_l y_l))) = k(j_l(x_l y_l)) = g_l(x_l y_l) = g_l(x_l)g_l(y_l) = s(i(j_l(x_l)))s(i(j_l(y_l)))$$

and $s(i(j_l(1_{H_l}))) = k(j_l(1_{H_l})) = g_l(1_{H_l}) = 1_C = s(1_{T(\bigoplus H_l)})$ since g_l and s are algebra morphisms, for every $l \in I$. Hence $L \subseteq \ker(s)$ and then there exists a unique algebra morphism $p : T(\bigoplus_{l\in I} H_l)/L \to C$ such that $p \circ \nu = s$ which preserves gradings since s and ν do. We have that $p \circ q_l = g_l$ and this morphism p is the unique in $\mathrm{Mon}(\mathrm{Vec}_G)$ such that $p \circ q_l = g_l$ for every $l \in I$. Indeed, if there is a morphism $\tilde{p} : T(\bigoplus_{l\in I} H_l)/L \to C$ such that $\tilde{p} \circ q_l = g_l$ for every $l \in I$, then from $(\tilde{p} \circ \nu \circ i) \circ j_l = g_l$ we obtain $(\tilde{p} \circ \nu) \circ i = k$, so $\tilde{p} \circ \nu = s$

and hence $p = \tilde{p}$. We have shown that $(T(\bigoplus_{l\in I} H_l)/L, (q_l)_{l\in I})$ is the coproduct of the family $\{H_l\}_{l\in I}$ in $\mathrm{Mon}(\mathrm{Vec}_G)$, and we denote $T(\bigoplus_{l\in I} H_l)/L$ by $\coprod_{l\in I} H_l$.

Now, since H_l is a color bialgebra for every $l \in I$, we can show that $\coprod_{l\in I} H_l$ is a color bialgebra and that it is the coproduct of the family $\{H_l\}_{l\in I}$ in $\mathrm{Bimon}(\mathrm{Vec}_G)$. The comultiplication and the counit are given by the unique graded algebra morphisms such that the following diagrams commute

$$\begin{array}{ccc} H_l & \xrightarrow{q_l} & \coprod_{i\in I} H_i \\ \downarrow{\scriptstyle \Delta_l} & & \downarrow{\scriptstyle \Delta} \\ H_l \otimes H_l & \xrightarrow[q_l\otimes q_l]{} & \coprod_{i\in I} H_i \otimes \coprod_{i\in I} H_i \end{array} \qquad \begin{array}{ccc} H_l & \xrightarrow{q_l} & \coprod_{i\in I} H_i \\ & {\scriptstyle \epsilon_l}\searrow & \downarrow{\scriptstyle \epsilon} \\ & & \Bbbk \end{array} \tag{4}$$

by the universal property of the coproduct in $\mathrm{Mon}(\mathrm{Vec}_G)$. Thus, we already have the compatibility and, if we prove that Δ is coassociative and counitary, we will have that Δ and ϵ make $\coprod_{i\in I} H_i$ a color bialgebra so that the two commutative diagrams (4) will prove that q_l is a coalgebra morphism for every $l \in I$ and then a color bialgebra morphism. In order to obtain $(\mathrm{Id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{Id})\circ\Delta$ it is sufficient to show that $(\mathrm{Id}\otimes\Delta)\circ\Delta\circ q_l = (\Delta\otimes\mathrm{Id})\circ\Delta\circ q_l$ by using the universal property, since $(\mathrm{Id}\otimes\Delta)\circ\Delta$ and $(\Delta\otimes\mathrm{Id})\circ\Delta$ are both graded algebra morphisms and, for the same argument, if we show $l\circ(\epsilon\otimes\mathrm{Id})\circ\Delta\circ q_l = \mathrm{Id}\circ q_l$ we obtain that $l\circ(\epsilon\otimes\mathrm{Id})\circ\Delta = \mathrm{Id}$, where $l : \Bbbk\otimes\coprod_{i\in I} H_i \to \coprod_{i\in I} H_i$ is the canonical isomorphism. So, having in mind the two diagrams in (4) and the fact that H_l is a coalgebra for $l \in I$, we obtain that

$$\begin{aligned}(\mathrm{Id}\otimes\Delta)\circ\Delta\circ q_l =& (\mathrm{Id}\otimes\Delta)\circ(q_l\otimes q_l)\circ\Delta_l = (q_l\otimes q_l\otimes q_l)\circ(\mathrm{Id}\otimes\Delta_l)\circ\Delta_l \\ =& (q_l\otimes q_l\otimes q_l)\circ(\Delta_l\otimes\mathrm{Id})\circ\Delta_l = (\Delta\otimes\mathrm{Id})\circ(q_l\otimes q_l)\circ\Delta_l \\ =& (\Delta\otimes\mathrm{Id})\circ\Delta\circ q_l\end{aligned}$$

and

$$\begin{aligned} l\circ(\epsilon\otimes\mathrm{Id})\circ\Delta\circ q_l =& l\circ(\epsilon\otimes\mathrm{Id})\circ(q_l\otimes q_l)\circ\Delta_l = l\circ(\epsilon_l\otimes q_l)\circ\Delta_l \\ =& l\circ(\mathrm{Id}\otimes q_l)\circ(\epsilon_l\otimes\mathrm{Id})\circ\Delta_l = q_l\circ l_{H_l}\circ(\epsilon_l\otimes\mathrm{Id})\circ\Delta_l = \mathrm{Id}\circ q_l.\end{aligned}$$

Similarly, $r\circ(\mathrm{Id}\otimes\epsilon)\circ\Delta = \mathrm{Id}$ where $r : \coprod_{i\in I} H_i\otimes\Bbbk \to \coprod_{i\in I} H_i$ is the canonical isomorphism. Hence $\coprod_{l\in I} H_l$ is a color bialgebra and q_l is a color bialgebra morphism for every $l \in I$. Now, given a color bialgebra C and color bialgebra morphisms $g_l : H_l \to C$, we have a unique graded algebra morphism $p : \coprod_{i\in I} H_i \to C$ such that $p\circ q_l = g_l$ for every $l \in I$ by the universal property of the coproduct in $\mathrm{Mon}(\mathrm{Vec}_G)$. We show that p is also a coalgebra morphism in order to obtain that $(\coprod_{l\in I} H_l, (q_l)_{l\in I})$ is the coproduct in $\mathrm{Bimon}(\mathrm{Vec}_G)$ of the family $\{H_l\}_{l\in I}$. By the argument used above, it is enough to show that $(p\otimes p)\circ\Delta\circ q_l = \Delta_C\circ p\circ q_l$ and $\epsilon_C\circ p\circ q_l = \epsilon\circ q_l$. So, since g_l is a coalgebra morphism for every $l \in I$, we have that

$$(p\otimes p)\circ\Delta\circ q_l = (p\otimes p)\circ(q_l\otimes q_l)\circ\Delta_l = (g_l\otimes g_l)\circ\Delta_l = \Delta_C\circ g_l = \Delta_C\circ p\circ q_l$$

and $\epsilon_C\circ p\circ q_l = \epsilon_C\circ g_l = \epsilon_l = \epsilon\circ q_l$.

Now we let $H := \coprod_{i\in I} H_i$. Every H_l has an antipode $S_l : H_l \to H_l$ which is a color bialgebra morphism from H_l to $H_l^{\mathrm{op,cop}}$ where $x \cdot_{\mathrm{op}} y := mc_{H_l,H_l}(x\otimes y) = \phi(|x|,|y|)yx$ and $\Delta^{\mathrm{cop}}(x) := c_{H_l,H_l}\Delta(x) = \phi(|x_1|,|x_2|)x_2\otimes x_1$, for every $x, y \in H_l$. Since q_l is a color bialgebra morphism from $H_l^{\mathrm{op,cop}}$ to $H^{\mathrm{op,cop}}$, the universal property of the coproduct in $\mathrm{Bimon}(\mathrm{Vec}_G)$ yields a unique color bialgebra morphism $S : H \to H^{\mathrm{op,cop}}$ such that the following diagram commutes for all $l \in I$.

$$\begin{array}{ccc} H_l & \xrightarrow{q_l} & H \\ {\scriptstyle S_l}\downarrow & & \downarrow{\scriptstyle S} \\ H_l^{\mathrm{op,cop}} & \xrightarrow[q_l]{} & H^{\mathrm{op,cop}} \end{array}$$

If we prove that S is the antipode of H, then H is a color Hopf algebra and q_l is a morphism of color Hopf algebras for every $l \in I$. Furthermore, given C a color Hopf algebra and $g_l : H_l \to C$ a color Hopf algebra morphism for every $l \in I$, there is a unique color bialgebra morphism $t : H \to C$ (a posteriori the unique color Hopf algebra morphism) such that $t \circ q_l = g_l$ for every $l \in I$. Hence, in this case, $(H, (q_l)_{l\in I})$ is the coproduct in $\mathrm{Hopf}(\mathrm{Vec}_G)$ of the family of color Hopf algebras $\{H_l\}_{l\in I}$. Thus, in order to conclude, we prove that $m \circ (\mathrm{Id} \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{Id}) \circ \Delta = u \circ \epsilon$. Since $S : H \to H^{\mathrm{op,cop}}$ is a color bialgebra morphism we only need to prove these equalities on the generators of H as a graded algebra. Indeed, let h, k be generators in H for which the relations hold, we obtain

$$\begin{aligned} m(\mathrm{Id}_H \otimes S)\Delta(hk) &= m(\mathrm{Id}_H \otimes S)(\phi(|h_2|, |k_1|)h_1k_1 \otimes h_2k_2) = \phi(|h_2|, |k_1|)h_1k_1S(h_2k_2) \\ &= \phi(|h_2|, |k_1|)\phi(|h_2|, |k_2|)h_1k_1S(k_2)S(h_2) = \phi(|h_2|, |k|)h_1\epsilon(k)S(h_2) \\ &= h_1S(h_2)\epsilon(k) = \epsilon(h)\epsilon(k)1_H = \epsilon(hk)1_H = u\epsilon(hk) \end{aligned}$$

and, similarly, $m(S \otimes \mathrm{Id})\Delta(hk) = u\epsilon(hk)$, so the relations hold for hk and thus for all the elements in H. So, having in mind that $H := T(\bigoplus_{l\in I} H_l)/L$, we only need to prove the relations for the elements $\bar{x} = i(x) + L \in H$ with $x \in \bigoplus_{l\in I} H_l$ whereas the tensor algebra $T(\bigoplus_{l\in I} H_l)$ is the free algebra on $\bigoplus_{l\in I} H_l$. Moreover, since elements $x \in \bigoplus_{l\in I} H_l$ are such that $x_l = 0$ for every $l \in I$ except for a finite number, by linearity it is enough to show that the relations hold for every $x_l \in H_l$ with $l \in I$. Using the commutativity of the three diagrams before, the fact that H_l is a color Hopf algebra and that q_l is an algebra morphism for $l \in I$, we obtain

$$\begin{aligned} m \circ (\mathrm{Id} \otimes S) \circ \Delta \circ q_l &= m \circ (\mathrm{Id} \otimes S) \circ (q_l \otimes q_l) \circ \Delta_l = m \circ (q_l \otimes q_l) \circ (\mathrm{Id} \otimes S_l) \circ \Delta_l \\ &= q_l \circ m_l \circ (\mathrm{Id} \otimes S_l) \circ \Delta_l = q_l \circ u_l \circ \epsilon_l = u \circ \epsilon_l = u \circ \epsilon \circ q_l, \end{aligned}$$

hence $m \circ (\mathrm{Id} \otimes S) \circ \Delta = u \circ \epsilon$. In the same way it can be shown that $m \circ (S \otimes \mathrm{Id}) \circ \Delta = u \circ \epsilon$. Thus, S is the antipode of H and then $(H, (q_l)_{l\in I})$ is the coproduct of the family $\{H_l\}_{l\in I}$ in $\mathrm{Hopf}(\mathrm{Vec}_G)$. It is clear that if we consider H_l a cocommutative color Hopf algebra for every $l \in I$ then $(H, (q_l)_{l\in I})$ will be the coproduct in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, since H is cocommutative. In fact, since Vec_G is symmetric, c is a braiding for $\mathrm{Mon}(\mathrm{Vec}_G)$ and, in particular, $c_{H,H}$ is a morphism of graded algebras, so the same is true for $c_{H,H} \circ \Delta$. Thus, from

$$c_{H,H} \circ \Delta \circ q_l = c_{H,H} \circ (q_l \otimes q_l) \circ \Delta_l = (q_l \otimes q_l) \circ c_{H_l,H_l} \circ \Delta_l = (q_l \otimes q_l) \circ \Delta_l = \Delta \circ q_l$$

we obtain that $c_{H,H} \circ \Delta = \Delta$ by the universal property of the coproduct.

The arguments above show that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ (and also $\mathrm{Hopf}(\mathrm{Vec}_G)$) is cocomplete. Note that, even if only finite cocompleteness is required in the definition of a semi-abelian category, the fact that this category has all small colimits will be used to obtain that it is semi-abelian, through an equivalent characterization.

Remark 4.5. Note that in [33, Proposition 4.1.1] it has been proven that $\mathrm{Hopf}(\mathcal{M})$, $\mathrm{Hopf}_{\mathrm{coc}}(\mathcal{M})$ and $\mathrm{Hopf}_{\mathrm{c}}(\mathcal{M})$ are always accessible categories for every symmetric monoidal category $\mathcal{M}$. Hence we have that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is accessible and then, since we have shown that it is cocomplete, we obtain that it is complete and locally presentable. In fact we know that, as it is said in [1, Corollary 2.47], a category is locally presentable if and only if it is accessible and complete if and only if it is accessible and cocomplete. Observe that, while accessibility is always true for the category of Hopf monoids in a symmetric monoidal category, this is not the same for local presentability. As it is said in [31, Propositions 49,52,53] this is true when the forgetful functor $U_a : \mathrm{Mon}(\mathcal{M}) \to \mathcal{M}$ is an extremally monadic functor or when the forgetful functor $U_c : \mathrm{Comon}(\mathcal{M}) \to \mathcal{M}$ is an extremally comonadic functor, since in these cases we have that the category $\mathrm{Hopf}(\mathcal{M})$ is closed under colimits and limits in $\mathrm{Bimon}(\mathcal{M})$, respectively.

4.5. Protomodularity

Recall that if $\mathcal{M}$ is a category with binary products, i.e. there exists the binary product $A \times B$ for every object A and B in $\mathcal{M}$, and with terminal object $\mathbf{I}$, the monoidal category $(\mathcal{M}, \times, \mathbf{I})$ is called *cartesian* and the category of internal groups in $\mathcal{M}$, denoted by $\mathrm{Grp}(\mathcal{M})$, has objects which are monoids (G, m, u) in $\mathcal{M}$ equipped with a morphism $i : G \to G$ in $\mathcal{M}$ (called inversion) such that $m \circ \langle \mathrm{Id}_G, i\rangle = u \circ t_G = m \circ \langle i, \mathrm{Id}_G\rangle$, where t_G is the unique morphism from G to $\mathbf{I}$ and $\langle \mathrm{Id}_G, i\rangle$, $\langle i, \mathrm{Id}_G\rangle$ are the diagonal morphisms.

In [14, Proposition 3.24] it is proved that, given a cartesian monoidal category $\mathcal{M}$ with finite limits, then the category $\mathrm{Grp}(\mathcal{M})$ is protomodular. Note that the same terminal object, equalizers and binary products given before say that $\mathrm{Comon_{coc}}(\mathrm{Vec}_G)$ is finitely complete. This category is also cartesian since its unit object $\Bbbk$ is the terminal object and the tensor product is the binary product and then we have that $\mathrm{Grp}(\mathrm{Comon_{coc}}(\mathrm{Vec_G}))$ is protomodular. Furthermore, as it is said for instance in [33, Remark 3.3], for every symmetric monoidal category $\mathcal{M}$ we have that $\mathrm{Hopf_{coc}}(\mathcal{M})$=$\mathrm{Grp}(\mathrm{Comon_{coc}}(\mathcal{M}))$ and then $\mathrm{Hopf_{coc}}(\mathrm{Vec}_G) = \mathrm{Grp}(\mathrm{Comon_{coc}}(\mathrm{Vec_G}))$. Indeed, we have

$$\mathrm{Mon}(\mathrm{Comon_{coc}}(\mathrm{Vec}_G)) = \mathrm{Mon}(\mathrm{Comon}(\mathrm{Comon}(\mathrm{Vec}_G))) = \mathrm{Comon}(\mathrm{Comon}(\mathrm{Mon}(\mathrm{Vec}_G))) = \mathrm{Bimon_{coc}}(\mathrm{Vec}_G)$$

so monoids in $\mathrm{Comon_{coc}}(\mathrm{Vec}_G)$ are given by cocommutative color bialgebras. Hence an object in $\mathrm{Grp}(\mathrm{Comon_{coc}}(\mathrm{Vec}_G))$ is a cocommutative color bialgebra $(B, m, u, \Delta, \epsilon)$ equipped with a morphism $i : B \to B$ in $\mathrm{Comon_{coc}}(\mathrm{Vec}_G)$ such that $m \circ \langle \mathrm{Id}_B, i\rangle = u \circ t_B = m \circ \langle i, \mathrm{Id}_B\rangle$. But we have seen before that $t_B = \epsilon$ and $\langle \mathrm{Id}_B, i\rangle = (\mathrm{Id}_B \otimes i) \circ \Delta$, $\langle i, \mathrm{Id}_B\rangle = (i \otimes \mathrm{Id}_B) \circ \Delta$, thus we obtain

$$m \circ (\mathrm{Id}_B \otimes i) \circ \Delta = u \circ \epsilon = m \circ (i \otimes \mathrm{Id}_B) \circ \Delta,$$

so i is the antipode of B. Hence we have that $\mathrm{Grp}(\mathrm{Comon_{coc}}(\mathrm{Vec}_G))$ is exactly the category $\mathrm{Hopf_{coc}}(\mathrm{Vec}_G)$. Thus, we have that $\mathrm{Hopf_{coc}}(\mathrm{Vec}_G)$ is protomodular. Recall that here the fact that Vec_G is symmetric ensures that $\mathrm{Comon_{coc}}(\mathrm{Vec}_G)$ is monoidal.

5. Regularity of $\mathbf{Hopf_{coc}}(\mathbf{Vec}_G)$

The most delicate point in the proof of the semi-abelianness of the category $\mathrm{Hopf_{coc}}(\mathrm{Vec}_G)$ is the regularity, as in the case of $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$. Following [22] the regularity will be shown through the following characterization:

Lemma 5.1. *Let $\mathcal{C}$ be a finitely complete category. Then, $\mathcal{C}$ is a regular category if and only if*

(1) any arrow in $\mathcal{C}$ factors as a regular epimorphism followed by a monomorphism;
(2) given any regular epimorphism $f : A \to B$ in $\mathcal{C}$ and any object E in $\mathcal{C}$, the induced arrow $\mathrm{Id}_E \times f : E \times A \to E \times B$ is a regular epimorphism;
(3) regular epimorphisms are stable under pullbacks along split monomorphisms.

Since the zero morphism in $\mathrm{Hopf_{coc}}(\mathrm{Vec}_G)$ between A and B is $u_B \circ \epsilon_A$, the categorical kernel of $f : A \to B$ in $\mathrm{Hopf_{coc}}(\mathrm{Vec}_G)$, i.e. the equalizer of the pair $(f, u_B \circ \epsilon_A)$, is given by $(\mathrm{Hker}(f), j : \mathrm{Hker}(f) \to A)$ with

$$\mathrm{Hker}(f) = \{x \in A \mid (\mathrm{Id}_A \otimes f)\Delta(x) = (\mathrm{Id}_A \otimes u_B\epsilon_A)\Delta(x)\} = \{x \in A \mid x_1 \otimes f(x_2) = x \otimes 1_B\}$$

and j the canonical inclusion. The categorical cokernel of f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, i.e. the coequalizer of the pair $(f, u_B \circ \epsilon_A)$, is given by $(B/I, \pi : B \to B/I)$ where

$$I = Bf(A)^+B$$

and π is the canonical quotient morphism and where, for any coalgebra C, we write $C^+ = \{x \in C \mid \epsilon(x) = 0\}$. First note that for any coalgebra morphism $f : C \to D$, $f(C^+) = f(C)^+$. Indeed, $y \in f(C)^+$ if and only if $y = f(x)$ for some $x \in C$ and $0 = \epsilon_D(y) = \epsilon_D(f(x)) = \epsilon_C(x)$ if and only if $y \in f(C^+)$. Now I is the two-sided ideal of B generated by the set $\{f(a) - u_B\epsilon_A(a) \mid a \in A\} = \{f(a) - \epsilon_A(a)1_B \mid a \in A\}$. But we have that $(f - u_B \circ \epsilon_A)(A) = f(A^+) = f(A)^+$ since if $x = f(a) - \epsilon_A(a)1_B$ we have $x = f(a - \epsilon_A(a)1_A)$ with $\epsilon_A(a - \epsilon_A(a)1_A) = 0$ so that $(f - u_B \circ \epsilon_A)(A) \subseteq f(A^+)$ and the other inclusion is trivial. Now, given $f : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we can consider the categorical cokernel of its categorical kernel in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ that is given, as a map, by $p : A \to A/A(\mathrm{Hker}(f))^+A$. Since j is the kernel of f we have that $f \circ j = u_B \circ \epsilon_{\mathrm{Hker}(f)} = f \circ u_A \circ \epsilon_{\mathrm{Hker}(f)}$, thus, by the universal property of the cokernel, there exists a unique morphism i in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ such that $f = i \circ p$.

$$\begin{array}{ccccc} \mathrm{Hker}(f) & \overset{j}{\hookrightarrow} & A & \overset{p}{\longrightarrow} & \frac{A}{A(\mathrm{Hker}(f))^+A} \\ & & & \underset{f}{\searrow} & \Big\downarrow i \\ & & & & B \end{array}$$

If we show that i is a monomorphism we obtain the decomposition regular epimorphism-monomorphism of f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. In the case of $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$, Newman's Theorem [30, Theorem 4.1] tells us that for a cocommutative Hopf algebra H there is a bijective correspondence between the set of Hopf subalgebras of H and that of left ideals which are also two-sided coideals of H: given a Hopf subalgebra K of H and a left ideal, two-sided coideal I of H the two maps are

$$K \mapsto HK^+ \text{ and } I \mapsto H^{\mathrm{co}\frac{H}{I}} := \{x \in H \mid (\mathrm{Id}_H \otimes \pi)\Delta_H(x) = x \otimes \pi(1_H)\}$$

which is also $\{x \in H \mid (\mathrm{Id}_H \otimes \pi)\Delta_H(x) = (\mathrm{Id}_H \otimes \pi u_H\epsilon_H)\Delta_H(x)\}$, where $\pi : H \to H/I$ is the canonical quotient morphism and this result is used in [22] to deduce that the vector space $\ker(f)$ is exactly $A(\mathrm{Hker}(f))^+A$ and then that the morphism i of the previous factorization is injective and so a monomorphism. We would like to obtain the same fact in the graded case.

Remark 5.2. Recall that given a graded algebra $A = \bigoplus_{g \in G} A_g$, i.e. an object in $\mathrm{Mon}(\mathrm{Vec}_G)$, we can consider the category ${}_A\mathrm{Vec}_G$, whose objects are graded vector spaces $V = \bigoplus_{g \in G} V_g$ that are also left A-modules such that the left A-action $\mu : A \otimes V \to V$ is in Vec_G and then $\mu(A_g \otimes V_h) \subseteq V_{gh}$ for every $g, h \in G$ and whose morphisms are linear maps which preserve gradings and are also left A-linear. If A is in $\mathrm{Bimon}(\mathrm{Vec}_G)$, i.e. it is a color bialgebra, then the category ${}_A\mathrm{Vec}_G$ is monoidal with the same tensor product, unit object and constraints of Vec_G and then of $\mathrm{Vec}_\Bbbk$. Here the unit object $\Bbbk$ has left A-action such that $a \cdot k = \epsilon(a)k$ for $a \in A$ and $k \in \Bbbk$ and, given V and W in ${}_A\mathrm{Vec}_G$, $V \otimes W$ has left A-action given by $a \cdot (v \otimes w) = \phi(|a_2|, |v|)a_1 \cdot v \otimes a_2 \cdot w$, for $a \in A$, $v \in V$ and $w \in W$. With quotient color left A-module coalgebras we mean quotient objects in $\mathrm{Comon}({}_A\mathrm{Vec}_G)$, thus quotient graded vector spaces which are left A-modules with left A-action in Vec_G, which are also coalgebras with Δ and ϵ in ${}_A\mathrm{Vec}_G$; in particular, as coalgebras, they are quotients of a graded coalgebra with a graded two-sided coideal.

Given A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ we define in Vec_G the morphism

$$\xi_A : A \otimes A \to A,\ a \otimes x \mapsto \phi(|a_2|, |x|)a_1 x S(a_2),$$

i.e. $\xi_A = m_A \circ (m_A \otimes S_A) \circ (\mathrm{Id}_A \otimes c_{A,A}) \circ (\Delta_A \otimes \mathrm{Id}_A)$, with c the braiding of Vec_G. By analogy with Theorem 5.6, we say that a color Hopf subalgebra $K \subseteq A$ is **normal** if $\xi_A(A \otimes K) \subseteq K$. First we show some properties of the map ξ_A.

Remark 5.3. If $f : A \to B$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ then $f(A)$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. In fact, by Remark 3.2, we know that $f(A)$ is a graded subspace of B and, as in the usual case, it contains $1_B = f(1_A)$ and it is closed under m_B ($m_B \circ (f \otimes f) = f \circ m_A$) since f is an algebra morphism. It is closed under Δ_B since f is a coalgebra morphism ($\Delta_B \circ f = (f \otimes f) \circ \Delta_A$) and under the antipode S_B since $S_B \circ f = f \circ S_A$. So $f(A)$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by Remark 4.1. Similarly, $\ker(f)$, which is graded by Remark 3.2, is a two-sided ideal of A (since f is an algebra morphism), a two-sided coideal of A (since f is a coalgebra morphism) and it is closed under S_A, so that $A/\ker(f)$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by Remark 4.3.

Lemma 5.4. *Let A and B in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Then, the following properties hold:*

1) ξ_A is a morphism of coalgebras.
2) Given $p : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ then $\xi_B \circ (p \otimes p) = p \circ \xi_A$. As a consequence, when p is surjective and D a normal color Hopf subalgebra of A, then $p(D)$ is a normal color Hopf subalgebra of B.
3) A is commutative if and only if $\xi_A = l_A \circ (\epsilon_A \otimes \mathrm{Id}_A)$. As a consequence, if A is commutative, any color Hopf subalgebra K of A is a normal subalgebra.

Proof. In order to prove 1) we have to show that $\Delta_A \circ \xi_A = (\xi_A \otimes \xi_A) \circ \Delta_{A \otimes A}$ where $\Delta_{A\otimes A} = (\mathrm{Id}_A \otimes c_{A,A} \otimes \mathrm{Id}_A) \circ (\Delta_A \otimes \Delta_A)$ and $\epsilon_A \circ \xi_A = \epsilon_{A \otimes A}$ where $\epsilon_{A\otimes A} = r_{\Bbbk} \circ (\epsilon_A \otimes \epsilon_A)$. Given $a, x \in A$, since Δ is a morphism of graded algebras and A is cocommutative, we have that

$$\begin{aligned}
&\Delta_A \xi_A(a \otimes x)\\
&= \Delta_A(\phi(|a_2|, |x|)a_1 x S(a_2)) = \phi(|a_2|, |x|)(m_A \otimes m_A)(\mathrm{Id} \otimes c_{A,A} \otimes \mathrm{Id})(\Delta_A(a_1 x) \otimes \Delta_A(S(a_2)))\\
&= \phi(|a_{2_1}||a_{2_2}|, |x_1||x_2|)(m_A \otimes m_A)(\mathrm{Id} \otimes c_{A,A} \otimes \mathrm{Id})(\phi(|a_{1_2}|, |x_1|)a_{1_1}x_1 \otimes a_{1_2}x_2 \otimes S(a_{2_1}) \otimes S(a_{2_2}))\\
&= \phi(|a_{2_1}||a_{2_2}|, |x_1||x_2|)\phi(|a_{1_2}|, |x_1|)\phi(|a_{1_2}||x_2|, |a_{2_1}|)a_{1_1}x_1 S(a_{2_1}) \otimes a_{1_2}x_2 S(a_{2_2})\\
&= \phi(|a_{2_1}|, |x_1|)\phi(|a_{2_2}|, |x_1|)\phi(|a_{2_2}|, |x_2|)\phi(|a_{1_2}|, |x_1|)\phi(|a_{1_2}|, |a_{2_1}|)a_{1_1}x_1 S(a_{2_1}) \otimes a_{1_2}x_2 S(a_{2_2})\\
&\overset{(*)}{=} \phi(|a_2|, |x_1|)\phi(|a_{2_2}|, |x_2|)\phi(|a_{1_2}|, |x_1|)a_{1_1}x_1 S(a_{1_2}) \otimes a_{2_1}x_2 S(a_{2_2})\\
&= (\xi_A \otimes \xi_A)(\phi(|a_2|, |x_1|)a_1 \otimes x_1 \otimes a_2 \otimes x_2) = (\xi_A \otimes \xi_A)\Delta_{A\otimes A}(a \otimes x),
\end{aligned}$$

where $(*)$ follows since A is cocommutative and then Δ_A is of graded coalgebras, i.e. $(\Delta_A \otimes \Delta_A) \circ \Delta_A = \Delta_{A\otimes A} \circ \Delta_A = (\mathrm{Id}_A \otimes c_{A,A} \otimes \mathrm{Id}_A) \circ (\Delta_A \otimes \Delta_A) \circ \Delta_A$, which on $a \in A$ is $a_{1_1} \otimes a_{1_2} \otimes a_{2_1} \otimes a_{2_2} = \phi(|a_{1_2}|, |a_{2_1}|)a_{1_1} \otimes a_{2_1} \otimes a_{1_2} \otimes a_{2_2}$. Furthermore, we have that

$$\begin{aligned}
\epsilon_A(\xi_A(a \otimes x)) &= \epsilon_A(\phi(|a_2|, |x|)a_1 x S(a_2)) = \phi(|a_2|, |x|)\epsilon_A(a_1)\epsilon_A(x)\epsilon_A(S(a_2))\\
&= \epsilon_A(a_1)\epsilon_A(a_2)\epsilon_A(x) = \epsilon_A(a)\epsilon_A(x) = \epsilon_{A\otimes A}(a \otimes x).
\end{aligned}$$

Now, since p is a morphism of algebras and of coalgebras, we have that

$$\begin{aligned}
\xi_B(p(a) \otimes p(x)) &= \phi(|p(a_2)|, |p(x)|)p(a_1)p(x)S(p(a_2)) = \phi(|a_2|, |x|)p(a_1 x)p(S(a_2))\\
&= p(\phi(|a_2|, |x|)a_1 x S(a_2)) = p(\xi_A(a \otimes x)).
\end{aligned}$$

Suppose that p is surjective and that D is a normal color Hopf subalgebra of A. We already know that $p(D)$ is a color Hopf subalgebra of B by Remark 5.3, we have to show that it is normal, i.e. that, given $b \in B$ and $d \in D$, then $\xi_B(b \otimes p(d)) \in p(D)$. By surjectivity of p there exists $a \in A$ such that $p(a) = b$. Hence, since D is normal, we obtain

$$\xi_B(b \otimes p(d)) = \xi_B(p \otimes p)(a \otimes d) = p\xi_A(a \otimes d) \in p(D),$$

thus $p(D)$ is normal in B and then also 2) is proved. Finally, if we suppose that A is commutative clearly we obtain that $\xi_A(a \otimes x) = \phi(|a_2|, |x|)a_1 x S(a_2) = a_1 S(a_2)x = \epsilon(a)x$, so $\xi_A(A \otimes K) \subseteq K$ for every $K \subseteq A$ color Hopf subalgebra, while if $\phi(|a_2|, |x|)a_1 x S(a_2) = \epsilon(a)x$ then we have that

$$x\epsilon(a) = \phi(|x|, |a|)\epsilon(a)x = \phi(|x|, |a_1|)\phi(|x|, |a_2|)\phi(|a_2|, |x|)a_1 x S(a_2) = \phi(|x|, |a_1|)a_1 x S(a_2),$$

hence

$$x \otimes a = x \otimes \epsilon(a_1)a_2 = x\epsilon(a_1) \otimes a_2 = \phi(|x|, |a_{1_1}|)a_{1_1} x S(a_{1_2}) \otimes a_2 = \phi(|x|, |a_1|)a_1 x S(a_{2_1}) \otimes a_{2_2}.$$

Thus, we have

$$xa = \phi(|x|, |a_1|)a_1 x S(a_{2_1})a_{2_2} = \phi(|x|, |a_1|)a_1 x\epsilon(a_2) = \phi(|x|, |a_1|)\phi(|x|, |a_2|)a_1\epsilon(a_2)x = \phi(|x|, |a|)ax,$$

so $xa = \phi(|x|, |a|)ax$ and then A is commutative and also 3) is shown. □

Lemma 5.5. *Let $i : B \to A$ be an inclusion of a subalgebra B of A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. If i is the categorical kernel of some morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then B is a normal color Hopf subalgebra of A.*

Proof. Suppose that $B = \mathrm{Hker}(f)$ for some morphism $f : A \to C$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. We already know that B is a color Hopf subalgebra of A, we have to prove that it is normal, i.e. that, given $x \in \mathrm{Hker}(f)$ and $a \in A$, then $\xi_A(a \otimes x) \in \mathrm{Hker}(f)$, i.e. $(\mathrm{Id} \otimes f)\Delta_A\xi_A(a \otimes x) = \xi_A(a \otimes x) \otimes 1_C$. But $f(x) = \epsilon(x)1_C$ with $x \in \mathrm{Hker}(f)$ and, since f is a morphism of graded algebras, we obtain

$$\begin{aligned} f\xi_A(a \otimes x) &= \phi(|a_2|, |x|)f(a_1)f(x)f(S(a_2)) = \phi(|a_2|, |x|)f(a_1)\epsilon(x)f(S(a_2)) = f(a_1)f(S(a_2))\epsilon(x) \\ &= f(a_1 S(a_2))\epsilon(x) = f(\epsilon(a)1_A)\epsilon(x) = \epsilon(a)\epsilon(x)1_C = u_C\epsilon_{A\otimes A}(a \otimes x). \end{aligned}$$

As a consequence, by using 1) of Lemma 5.4, we have that

$$\begin{aligned} (\mathrm{Id} \otimes f)\Delta_A\xi_A(a \otimes x) &= (\mathrm{Id} \otimes f)(\xi_A \otimes \xi_A)\Delta_{A\otimes A}(a \otimes x) = (\xi_A \otimes u_C)(\mathrm{Id} \otimes \mathrm{Id} \otimes \epsilon_{A\otimes A})\Delta_{A\otimes A}(a \otimes x) \\ &= (\xi_A \otimes u_C)(a \otimes x \otimes 1_{\Bbbk}) = \xi_A(a \otimes x) \otimes 1_C, \end{aligned}$$

thus $\mathrm{Hker}(f)$ is normal in A. □

A generalization of Newman's Theorem for the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ of cocommutative super Hopf algebras is proved by A. Masuoka in the case $\mathrm{char}\Bbbk \neq 2$. The result is the following:

Theorem 5.6 *(cf. [27, Theorem 3.10 (3)]). Let H be a cocommutative super Hopf algebra. Then, the super Hopf subalgebras $K \subseteq H$ and the quotient super left H-module coalgebras Q of H are in 1-1 correspondence, and the mutually inverse bijections are given by*

$$\phi_H : K \mapsto H/HK^+ \quad \textit{and} \quad \psi_H : Q \mapsto {}^{\mathrm{co}Q}H = H^{\mathrm{co}Q} = \{x \in H \mid (\mathrm{Id}_H \otimes \pi)\Delta_H(x) = x \otimes \pi(1_H)\},$$

where $\pi : H \to Q$ denotes the quotient. This restricts to a 1-1 correspondence between those super Hopf subalgebras K which are normal, i.e. that satisfy $(-1)^{|h_2||x|}h_1 x S(h_2) \in K$ for every $h \in H$ and $x \in K$, and the quotient super Hopf algebras.

We used the notations ϕ_H and ψ_H for the bijections in analogy with those given for Newman's Theorem in [22]. Observe that last statement in Theorem 5.6 is a generalization of the equivalence between (1) and (2) of [22, Corollary 2.3]. Here we immediately obtain a complete generalization of [22, Corollary 2.3] for cocommutative super Hopf algebras.

Corollary 5.7. *For a super Hopf subalgebra $B \subseteq A$ of a cocommutative super Hopf algebra A, the following conditions are equivalent:*

(1) B is a normal super Hopf subalgebra;
(2) A/AB^+ is a quotient super Hopf algebra;
(3) the inclusion morphism $B \to A$ is the categorical kernel of some morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$.

Proof. We already know that (1) and (2) are equivalent by Theorem 5.6.

(2) $\Longrightarrow$ (3). Since A/AB^+ is a quotient super Hopf algebra, the canonical quotient morphism $\pi : A \to A/AB^+$ is a morphism of cocommutative super Hopf algebras and then clearly $A^{\mathrm{co}\frac{A}{AB^+}}$ is exactly $\mathrm{Hker}(\pi)$ since $x \otimes \pi(1_A) = x \otimes 1_{A/AB^+}$, for $x \in A$. Now, using Theorem 5.6, we obtain

$$\mathrm{Hker}(\pi) = A^{\mathrm{co}\frac{A}{AB^+}} = \psi_A(A/AB^+) = \psi_A(\phi_A(B)) = B.$$

Hence (B, j) is the kernel of π in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$, where $j : B \to A$ is the canonical inclusion. We already know that (3) $\Longrightarrow$ (1) by Lemma 5.5 and then we are done. □

We will obtain a generalization of Theorem 5.6 and of Corollary 5.7 for $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ that will be used to prove that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is regular and semi-abelian.

5.1. From color Hopf algebras to super Hopf algebras

In order to use Theorem 5.6 we are interested in obtaining a braided strong monoidal functor from the category Vec_G to the category $\mathrm{Vec}_{\mathbb{Z}_2}$. In this subsection G and L will denote arbitrary abelian groups.

Remark 5.8. As it is said in [19, Example 2.5.2], given $f : G \to L$ a morphism of groups, any G-graded vector space is naturally L-graded (by pushforward of grading) and we have a natural strict monoidal functor $(F, \phi^0, \phi^2) : \mathrm{Vec}_G \to \mathrm{Vec}_L$ (also denoted by f_*). The functor $F : \mathrm{Vec}_G \to \mathrm{Vec}_L$ is defined, given $V = \bigoplus_{g\in G} V_g$ and f in Vec_G, such that

$$F(\bigoplus_{g\in G} V_g) = \bigoplus_{f(g)\in L} V_{f(g)} \quad \text{where} \quad V_{f(g)} = \bigoplus_{g'\in f^{-1}f(g)} V_{g'} \quad \text{and} \quad F(f) = f,$$

so $V_{f(g)}$ is the direct sum of all the $V_{g'}$'s such that $f(g')$ is the same element $f(g)$ in L and then $F(V)$ is still V as vector space but with a grading over L in which $V_l = \{0\}$ if $l \notin \mathrm{Im}(f)$. Observe that $F(\Bbbk) = \Bbbk$ with $\Bbbk_{1_L} = \Bbbk$ and $\Bbbk_l = 0$ if $l \neq 1_L$, so that one can define $\phi^0 := \mathrm{Id}_{\Bbbk}$ and, given V, W in Vec_G, we have

$$F(V \otimes W) = F(\bigoplus_{g\in G} (V \otimes W)_g) = \bigoplus_{f(g)\in L} (V \otimes W)_{f(g)} = \bigoplus_{f(g)\in L} V_{f(g)} \otimes \bigoplus_{f(g)\in L} W_{f(g)}$$

which is $F(V) \otimes F(W)$, so $F(V \otimes W)$ and $F(V) \otimes F(W)$ are the same L-graded vector space and then one can define $\phi^2_{V,W} := \mathrm{Id}$ for every V, W in Vec_G. Clearly this remark is true also for groups G and L that are not necessarily abelian, in which case Vec_G and Vec_L are not braided.

In [7, Remark 1.2] it is said how to obtain a braided strong monoidal functor from Vec_G to Vec_L when G and L are finite abelian groups and it is not difficult to see that this works also in the case G and L are not necessarily finite. We recall here how to do it. Clearly, if we define $\phi^2_{V,W} := \mathrm{Id}$ for every V and W in Vec_G, we can not obtain in general a braided monoidal functor from Vec_G to Vec_L since the braiding of Vec_G and that of Vec_L are different. Thus, we define $\phi^0 := \mathrm{Id}_\Bbbk$ but we modify the morphisms $\phi^2_{V,W}$ that we want to be isomorphisms in Vec_L in order to have a strong monoidal functor and we recall that $F(V \otimes W) = F(V) \otimes F(W)$ in Vec_L. Given a map $\gamma : G \times G \to \Bbbk - \{0\}$, one can define, for every V and W in Vec_G, isomorphisms in Vec_G given by $f_{V,W} : V \otimes W \to V \otimes W$, $v \otimes w \mapsto \gamma(g,h) v \otimes w$, for $v \in V_g$ and $w \in W_h$ and $g, h \in G$, defined on the components of the grading and extended by linearity. We define $\phi^2_{V,W} := F(f_{V,W}) = f_{V,W}$, which are isomorphisms in Vec_L for every V and W in Vec_G. In order to obtain a monoidal functor we need that γ is a 2-*cocycle* on G, i.e. that it satisfies

$$\gamma(gh,k)\gamma(g,h) = \gamma(g,hk)\gamma(h,k) \quad \text{for every} \quad g,h,k \in G. \tag{5}$$

We immediately have that $\gamma(g, 1_G) = \gamma(1_G, g) = \gamma(1_G, 1_G)$ for every $g \in G$ and γ is said to be normalized if $\gamma(1_G, 1_G) = 1_\Bbbk$. Observe also that a bicharacter ϕ on G is automatically a normalized 2-cocycle on G since

$$\phi(gh,k)\phi(g,h) = \phi(g,k)\phi(h,k)\phi(g,h) = \phi(g,hk)\phi(h,k) \quad \text{for every} \quad g,h,k \in G$$

and $\phi(g, 1_G) = \phi(1_G, g) = 1_\Bbbk$ for every $g \in G$. We have the following result:

Lemma 5.9 *(cf. [7, Remark 1.2]). Let $f : G \to L$ be a morphism of abelian groups, $F : \mathrm{Vec}_G \to \mathrm{Vec}_L$ defined as in Remark 5.8, $\phi^0 := \mathrm{Id}_\Bbbk$ and $\phi^2_{V,W}$ defined as above for every V and W in Vec_G. Then, (F, ϕ^0, ϕ^2) is a strong monoidal functor if and only if γ is a normalized 2-cocycle on G. Furthermore, if ϕ and θ are bicharacters over G and L respectively (which give the respective braidings over Vec_G and Vec_L), then (F, ϕ^0, ϕ^2) is braided if and only if*

$$\phi(g,h) = \theta(f(g), f(h)) \frac{\gamma(g,h)}{\gamma(h,g)} \quad \textit{for every} \quad g,h \in G. \tag{6}$$

Remark 5.10. Consider the map $\bar{u} : G \to \Bbbk - \{0\}$ given by $g \mapsto \phi(g,g)$. Note that, since ϕ is a commutation factor, we obtain that

$$\bar{u}(gg') = \phi(gg', gg') = \phi(g,g)\phi(g,g')\phi(g',g)\phi(g',g') = \phi(g,g)\phi(g',g') = \bar{u}(g)\bar{u}(g')$$

and then $\bar{u}$ is a morphism of groups. But we also have that, again since ϕ is a commutation factor, $\phi(g,g)\phi(g,g) = 1_\Bbbk$, i.e. $\phi(g,g)^2 = 1_\Bbbk$ for every $g \in G$, hence $\phi(g,g) \in \{\pm 1_\Bbbk\}$. Thus, $\bar{u}(G) \subseteq \{\pm 1_\Bbbk\}$ and then we have that, with ϕ a commutation factor, $\bar{u} : G \to \{\pm 1_\Bbbk\}$ is a morphism of groups. If we consider the subgroup

$$H := \ker(\bar{u}) = \{g \in G \mid \bar{u}(g) = 1_\Bbbk\} = \{g \in G \mid \phi(g,g) = 1_\Bbbk\}$$

there are two possibilities: $H = G$ or $\{\pm 1_\Bbbk\} \cong G/H$ and then $|G/H| = 2$. In the second case, if G is finite (and so also H is finite), then $|G| = 2 \cdot |H|$. In the finite case, if the cardinality of G is odd we always have $G = H$ and, with G non trivial, if $H = \{1_G\}$ then $G = \mathbb{Z}_2$. Thus, if the cardinality of G is bigger than 2,

H can not be trivial and hence it never happens that $\phi(h,h) = -1_{\Bbbk}$ for every $h \neq 1_G$. One can find, for example in [6, Section 3.2], a complete classification of a nondegenerate skew-symmetric bicharacter defined on a finite abelian group.

In [7, Section 1.5] it is said how to obtain a braided strong monoidal functor from Vec_G to $\mathrm{Vec}_{\mathbb{Z}_2}$, in case G is a finite abelian group; this also works when G is finitely generated abelian and here we give the proof for completeness. If we consider $\mathbb{Z}_2 := \{0,1\}$, by Remark 5.10 we already have a morphism of groups $\bar{u} : G \to \mathbb{Z}_2$, where $\bar{u}(g) = 0$ if $g \in G$ is such that $\phi(g,g) = 1_{\Bbbk}$ and $\bar{u}(g) = 1$ if $\phi(g,g) = -1_{\Bbbk}$. We need a normalized 2-cocycle γ on G such that (6) is satisfied for ϕ and η where $\eta : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \{\pm 1_{\Bbbk}\} \subseteq \Bbbk - \{0\}$, $(x,y) \mapsto (-1)^{x \cdot y}$ is the commutation factor on $\mathbb{Z}_2$.

Proposition 5.11. *If G is a finitely generated abelian group then we have a braided strong monoidal functor $(F, \phi^0, \phi^2) : \mathrm{Vec}_G \to \mathrm{Vec}_{\mathbb{Z}_2}$.*

Proof. Define $\kappa : G \times G \to \{\pm 1_{\Bbbk}\}$ such that $\kappa(g,h) = 1_{\Bbbk}$ if $\phi(g,g) = 1_{\Bbbk}$ or $\phi(h,h) = 1_{\Bbbk}$ and $\kappa(g,h) = -1_{\Bbbk}$ if $\phi(g,g) = \phi(h,h) = -1_{\Bbbk}$, for every $g,h \in G$. By definition of κ, which is clearly a commutation factor, $\phi\kappa : G \times G \to \Bbbk - \{0\}, (g,h) \mapsto \phi(g,h)\kappa(g,h)$ is a commutation factor on G. Observe that $\phi\kappa(g,g) = 1_{\Bbbk}$ for every $g \in G$. If we consider $\hat{G} := \mathrm{Hom}_{\mathrm{Grp}}(G, \Bbbk - \{0\})$, the character group of G, and we define the group morphism $\chi : G \to \hat{G}$, $g \mapsto \chi_g$ by setting $\chi_g(h) = \phi\kappa(g,h)$ for $g,h \in G$, we have that $\phi\kappa$ induces a non-degenerate bicharacter ϕ' on the group $G' := G/\ker(\chi)$ such that $\phi'(\bar{g},\bar{g}) = 1_{\Bbbk}$ for every $\bar{g} \in G'$. In fact, we can define $\phi'(\bar{g},\bar{g'}) = \phi\kappa(g,g')$ for every $\bar{g},\bar{g'} \in G'$, which is well defined since, for $g,g' \in G$ and $h,h' \in \ker(\chi)$, we have that

$$\phi\kappa(h,g') = \phi\kappa(h,h') = 1_{\Bbbk} \quad \text{and} \quad 1_{\Bbbk} = \phi\kappa(g,h')\phi\kappa(h',g) = \phi\kappa(g,h')$$

and then $\phi\kappa(gh, g'h') = \phi\kappa(g,g')$. Clearly ϕ' is a commutation factor and $\phi'(\bar{g},\bar{g}) = 1_{\Bbbk}$ for every $\bar{g} \in G'$. Since G is a finitely generated abelian group the same is true for G'. Thus, by [36, Lemma 2], since ϕ' is a commutation factor on G' such that $\phi'(\bar{g},\bar{g}) = 1_{\Bbbk}$ for every $\bar{g} \in G'$, there exists a bicharacter $\gamma' : G' \times G' \to \Bbbk - \{0\}$ on G' (so, in particular, a normalized 2-cocycle on G') such that

$$\phi'(\bar{g},\bar{h}) = \frac{\gamma'(\bar{g},\bar{h})}{\gamma'(\bar{h},\bar{g})} \quad \text{for every} \quad \bar{g},\bar{h} \in G'.$$

If we call $p : G \to G'$ the canonical projection, we have that $\phi' \circ (p \times p) = \phi\kappa$ and we can consider the bicharacter (so a normalized 2-cocycle) on G given by $\gamma := \gamma' \circ (p \times p)$. Clearly

$$\phi\kappa(g,h) = \phi'(\bar{g},\bar{h}) = \frac{\gamma'(\bar{g},\bar{h})}{\gamma'(\bar{h},\bar{g})} = \frac{\gamma(g,h)}{\gamma(h,g)} \quad \text{for every } g,h \in G.$$

But now, given η the bicharacter on $\mathbb{Z}_2$ of before, we have that $\kappa = \eta \circ (\bar{u} \times \bar{u})$, i.e. $\kappa(g,h) = \eta(\bar{u}(g), \bar{u}(h))$ for every $g,h \in G$. Thus, we obtain that

$$\frac{\gamma(g,h)}{\gamma(h,g)} = \phi\kappa(g,h) = \phi(g,h)\kappa(g,h) = \phi(g,h)\eta(\bar{u}(g),\bar{u}(h)) \quad \text{for every} \quad g,h \in G$$

and then that

$$\phi(g,h) = \eta(\bar{u}(g),\bar{u}(h))\frac{\gamma(g,h)}{\gamma(h,g)} \quad \text{for every} \quad g,h \in G.$$

Hence, by Lemma 5.9, we have a braided strong monoidal functor $(F, \phi^2, \phi^0) : \mathrm{Vec}_G \to \mathrm{Vec}_{\mathbb{Z}_2}$. $\square$

Thus, from now on, we suppose that the abelian group G is *finitely generated*. We know that a braided strong monoidal functor preserves Hopf monoids (see [4, Propositions 3.46, 3.50]), thus, via (F, ϕ^0, ϕ^2), every color Hopf algebra becomes a super Hopf algebra and every morphism of color Hopf algebras becomes a morphism of super Hopf algebras (we already know that it is automatically in $\mathrm{Vec}_{\mathbb{Z}_2}$, but it will be also a morphism of algebras and of coalgebras with respect to new products and new coproducts). Given a color Hopf algebra $(H := \bigoplus_{g\in G} H_g, m, u, \Delta, \epsilon, S)$, the super Hopf algebra will be given by

$$\begin{aligned}&(F(H), F(m)\circ(\phi^2_{H,H})^{-1}, F(u)\circ(\phi^0)^{-1}, \phi^2_{H,H}\circ F(\Delta), \phi^0\circ F(\epsilon), F(S))\\ &= (H_0\oplus H_1, m\circ(\phi^2_{H,H})^{-1}, u, \phi^2_{H,H}\circ\Delta, \epsilon, S)\end{aligned}$$

since $F(f) = f$ for f in Vec_G and ϕ^0 is the identity, where $H_0 = \bigoplus_{g\in G_0} H_g$ and $H_1 = \bigoplus_{g\in G_1} H_g$ by setting $G_0 := \{g \in G \mid \phi(g, g) = 1_{\Bbbk}\}$ and $G_1 := \{g \in G \mid \phi(g, g) = -1_{\Bbbk}\}$.

Lemma 5.12. *Given a faithful braided strong monoidal functor $F : \mathcal{M} \to \mathcal{M}'$, then A is a (co)commutative (co)monoid in $\mathcal{M}$ if and only if $F(A)$ is a (co)commutative (co)monoid in $\mathcal{M}'$.*

Proof. Since F is braided we have that

$$m_{F(A)}\circ c'_{F(A),F(A)} = F(m_A)\circ(\phi^2_{A,A})^{-1}\circ c'_{F(A),F(A)} = F(m_A\circ c_{A,A})\circ(\phi^2_{A,A})^{-1}$$

so, if A is a commutative monoid in $\mathcal{M}$, then $F(A)$ is a commutative monoid in $\mathcal{M}'$ and

$$c'_{F(A),F(A)}\circ\Delta_{F(A)} = c'_{F(A),F(A)}\circ\phi^2_{A,A}\circ F(\Delta_A) = \phi^2_{A,A}\circ F(c_{A,A}\circ\Delta_A)$$

so, if A is a cocommutative comonoid in $\mathcal{M}$, then $F(A)$ is a cocommutative comonoid in $\mathcal{M}'$. If $F(A)$ is a (co)commutative (co)monoid in $\mathcal{M}'$ then, from the previous two computations, we obtain that A is a (co)commutative (co)monoid in $\mathcal{M}$ by using that F is faithful. □

Corollary 5.13. *Given A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, it is (co)commutative if and only if $F(A)$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ is (co)commutative.*

In the following we will often refer to the functor F restricted to the category of cocommutative color Hopf algebras, still calling it F.

Remark 5.14. In order to avoid confusion, in this remark we denote Vec^{ϕ}_G by indicating the bicharacter associated to the braiding at the top. As it is said in [7, Section 1.5], the normalized 2-cocycle $\gamma : G\times G \to \Bbbk - \{0\}$ induces an equivalence of braided monoidal categories from $\mathrm{Vec}^{\phi\kappa}_G$ to Vec^{t}_G, where $t : G\times G \to \Bbbk - \{0\}$ is the trivial bicharacter such that $t(g, h) = 1_{\Bbbk}$ for every $g, h \in G$. Indeed, if we consider the morphism of groups Id_G and γ, we have

$$\phi\kappa(g, h) = t(g, h)\frac{\gamma(g, h)}{\gamma(h, g)} \text{ for every } g, h \in G$$

and then a braided strong monoidal functor $\mathrm{Vec}^{\phi\kappa}_G \to \mathrm{Vec}^{t}_G$ by Lemma 5.9. But, clearly, we can consider the normalized 2-cocycle $\gamma^{-1} : G \times G \to \Bbbk - \{0\}$, $(g, h) \to \gamma(g, h)^{-1}$ and then

$$t(g, h) = \phi\kappa(g, h)\frac{\gamma(h, g)}{\gamma(g, h)} = \phi\kappa(g, h)\frac{\gamma^{-1}(g, h)}{\gamma^{-1}(h, g)} \text{ for every } g, h \in G,$$

so that we have a braided strong monoidal functor $\mathrm{Vec}_G^t \to \mathrm{Vec}_G^{\phi\kappa}$, again by Lemma 5.9. These two functors give an equivalence of (symmetric) braided monoidal categories between $\mathrm{Vec}_G^{\phi\kappa}$ and Vec_G^t and now there are two possibilities. If $\phi(g,g) = 1_{\Bbbk}$ for every $g \in G$, then clearly $\kappa(g,h) = 1_{\Bbbk}$ for every $g,h \in G$ and then $\phi\kappa = \phi$, so that we have an equivalence of symmetric monoidal categories between Vec_G^ϕ and Vec_G^t. Indeed, observe that in this case $G_0 = G$ and then, given V in Vec_G^ϕ, we have $V = V_0$, $V_1 = 0$ and $\eta(0,0) = 1_{\Bbbk}$. The objects of $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G^t)$, the category of G-graded cocommutative Hopf algebras, are ordinary cocommutative Hopf algebras graded over G as vector spaces with $m, u, \Delta, \epsilon, S$ which preserve gradings (thus G-graded algebras and coalgebras) and the morphisms are algebra and coalgebra morphisms which preserve gradings. In particular, from a cocommutative color Hopf algebra we can obtain an ordinary cocommutative Hopf algebra and vice versa. Otherwise if $\phi(g,g) = -1_{\Bbbk}$ for some $g \in G$, we can return to the braided strong monoidal functor $(F, \phi^0, \phi^2) : \mathrm{Vec}_G^\phi \to \mathrm{Vec}_{\mathbb{Z}_2}^\eta$ of before and, given H and f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G^\phi)$, we obtain $F(H)$ and $F(f)$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G^\eta)$, the category of G-graded cocommutative super Hopf algebras, whose objects are G-graded algebras and coalgebras (since also $\phi^2_{H,H}$ is in Vec_G), then also $\mathbb{Z}_2$-graded algebras and coalgebras by considering the new grading, with respect to which, considering the braiding of super vector spaces, they are cocommutative super Hopf algebras and morphisms are algebra and coalgebra morphisms which preserve the G-gradings (and then those over $\mathbb{Z}_2$).

Hence every H in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ can be seen as a cocommutative super Hopf algebra. If $\phi(g,g) = 1_{\Bbbk}$ for every $g \in G$ we have that this actually is an ordinary cocommutative Hopf algebra and Newman's Theorem holds true; this always happens if we have a finite group G of odd cardinality by Remark 5.10, for example. If $\phi(g,g) = -1_{\Bbbk}$ for some $g \in G$ we can use the more general Theorem 5.6 for cocommutative super Hopf algebras, where $\mathrm{char}\Bbbk \neq 2$ is needed, which allows us to deal with the more general case.

5.2. Generalized Newman's theorem for color Hopf algebras

We can now generalize Theorem 5.6 and Corollary 5.7 to the case of cocommutative color Hopf algebras by using the functor $F : \mathrm{Vec}_G \to \mathrm{Vec}_{\mathbb{Z}_2}$, in case $\mathrm{char}\Bbbk \neq 2$ and G is a finitely generated abelian group.

Lemma 5.15. *The forgetful functor $K : \mathrm{Vec}_G \to \mathrm{Vec}_{\Bbbk}$ is injective on subobjects and on quotients of the same object. As a consequence, the same holds true if K is restricted to the categories* $\mathrm{Mon}(\mathrm{Vec}_G)$, $\mathrm{Comon}(\mathrm{Vec}_G)$, $\mathrm{Bimon}(\mathrm{Vec}_G)$ *and* $\mathrm{Hopf}(\mathrm{Vec}_G)$.

Proof. Given A in Vec_G and B, C graded subspaces of A, then $B_g = B \cap A_g$ and $C_g = C \cap A_g$ for every $g \in G$. Thus, if $K(B) = K(C)$, i.e. B and C are the same vector space, then they must be the same object in Vec_G. Furthermore, if we consider A in $\mathrm{Mon}(\mathrm{Vec}_G)$, $\mathrm{Comon}(\mathrm{Vec}_G)$, $\mathrm{Bimon}(\mathrm{Vec}_G)$ or $\mathrm{Hopf}(\mathrm{Vec}_G)$ and B, C subobjects of A in these categories, we have that B and C are the same object in these categories if and only if they are the same object in Vec_G because their operations are the restrictions of those of A and then this happens if $K(B) = K(C)$. Moreover, given A/B and A/C in Vec_G such that $K(A/B) = K(A/C)$, i.e. A/B and A/C are the same vector space, then $B = 0_{A/B} = 0_{A/C} = C$. As a consequence, $(A/B)_g = (A_g + B)/B = (A_g + C)/C = (A/C)_g$ for every $g \in G$ and then A/B and A/C are the same object in Vec_G. The same result holds true when A/B and A/C are in $\mathrm{Mon}(\mathrm{Vec}_G)$, $\mathrm{Comon}(\mathrm{Vec}_G)$, $\mathrm{Bimon}(\mathrm{Vec}_G)$ or $\mathrm{Hopf}(\mathrm{Vec}_G)$, since A/B and A/C are the same object in these categories if and only if they are the same object in Vec_G because their operations are induced by those of A through the canonical projection. □

Lemma 5.16. *The functor $F : \mathrm{Vec}_G \to \mathrm{Vec}_{\mathbb{Z}_2}$ preserves and reflects submonoids and subcomonoids, then sub-bimonoids and Hopf submonoids.*

Proof. We consider the case of Hopf submonoids which includes all the others in itself. We already know that if C is a color Hopf subalgebra of A, i.e. the inclusion $i : C \to A$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then $F(i)$:

$F(C) \to F(A)$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$, i.e. $F(C)$ is a super Hopf subalgebra of $F(A)$, so we show the other direction, assuming A in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and $C \subseteq A$ a graded subspace such that $F(C)$ is a super Hopf subalgebra of $F(A)$. But now we only have to observe that

$$m_{F(A)}(F(C) \otimes F(C)) = F(m_A)(\phi^2_{A,A})^{-1}(F(C) \otimes F(C)) = F(m_A)(F(C \otimes C)) = F(m_A(C \otimes C)),$$
$$\Delta_{F(A)}(F(C)) = \phi^2_{A,A}F(\Delta_A)(F(C)) = \phi^2_{A,A}F(\Delta_A(C))$$

and also $u_{F(A)}(\Bbbk) = F(u_A(\Bbbk))$, $S_{F(A)}(F(C)) = F(S_A(C))$ so that, since $F(C)$ is a super Hopf subalgebra of $F(A)$ (i.e. $F(C)$ is closed under the operations of $F(A)$), then C is a color Hopf subalgebra of A (i.e. C is closed under the operations of A), since F does not change the structure of vector space. □

Remark 5.17. Observe that, given $\pi : A \to A/I$ in Vec_G, then $F(\pi) : F(A) \to F(A/I)$ is in $\mathrm{Vec}_{\mathbb{Z}_2}$ and it is still surjective, then by Remark 3.2 $F(A/I)$ has the unique grading induced by $F(A)$ through the surjection, i.e. $F(A/I)_i = F(\pi)(F(A)_i) = (F(A)_i + I)/I = (F(A)_i + F(I))/F(I)$ for $i = 0, 1$ and this is exactly the grading in $\mathrm{Vec}_{\mathbb{Z}_2}$ of the quotient of $F(A)$ with its super subspace $F(I)$. Thus, $F(\pi) : F(A) \to F(A/I)$ is the quotient $F(A) \to F(A)/F(I)$ in $\mathrm{Vec}_{\mathbb{Z}_2}$. Also note that, if $\pi : A \to A/I$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then $F(\pi)$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ and $F(A/I)$ has the unique structure in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ induced by $F(A)$.

Proposition 5.18. *The functor $F : \mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G) \to \mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ preserves and reflects equalizers.*

Proof. If we take $f, g : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and $j : \mathrm{Eq}(f, g) \to A$ the equalizer of the pair (f, g) in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then we can consider $F(j) : F(\mathrm{Eq}(f, g)) \to F(A)$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ and we can show that $F(\mathrm{Eq}(f, g)) = \mathrm{Eq}(F(f), F(g))$. We know that $\mathrm{Eq}(F(f), F(g))$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$ is given by those $x \in F(A)$ such that $(\mathrm{Id}_{F(A)} \otimes F(f))\Delta_{F(A)}(x) = (\mathrm{Id}_{F(A)} \otimes F(g))\Delta_{F(A)}(x)$. But now we have that

$$\begin{aligned}(\mathrm{Id}_{F(A)} \otimes F(f)) \circ \Delta_{F(A)} &= (F(\mathrm{Id}_A) \otimes F(f)) \circ \phi^2_{A,A} \circ F(\Delta_A) = \phi^2_{A,B} \circ F(\mathrm{Id}_A \otimes f) \circ F(\Delta_A) \\ &= \phi^2_{A,B} \circ F((\mathrm{Id}_A \otimes f) \circ \Delta_A)\end{aligned}$$

and, similarly, $(\mathrm{Id}_{F(A)} \otimes F(g)) \circ \Delta_{F(A)} = \phi^2_{A,B} \circ F((\mathrm{Id}_A \otimes g) \circ \Delta_A)$. Then, equivalently, $\mathrm{Eq}(F(f), F(g))$ is composed by elements $x \in F(A)$ such that $F((\mathrm{Id}_A \otimes f)\Delta_A)(x) = F((\mathrm{Id}_A \otimes g)\Delta_A)(x)$, i.e. such that $(\mathrm{Id}_A \otimes f)\Delta_A(x) = (\mathrm{Id}_A \otimes g)\Delta_A(x)$ and these are exactly the elements of $F(\mathrm{Eq}(f, g))$. Hence we have that $(F(\mathrm{Eq}(f, g)), F(j))$ is the equalizer of the pair $(F(f), F(g))$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$, so F preserves equalizers. The fact that F reflects equalizers follows using that F preserves equalizers and that F reflects isomorphisms (see [10, Proposition 2.9.7]). □

Clearly the previous result holds true by considering $F : \mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G) \to \mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$.

Lemma 5.19. *Given a graded algebra H and a color left H-module A, then $F(A)$ is a super left $F(H)$-module.*

Proof. Given the action $\mu : H \otimes A \to A$ in Vec_G, we define $\mu' := F(\mu) \circ (\phi^2_{H,A})^{-1} : F(H) \otimes F(A) \to F(A)$ in $\mathrm{Vec}_{\mathbb{Z}_2}$. Thus, we can compute

$$\begin{aligned}\mu' \circ (m_{F(H)} \otimes \mathrm{Id}_{F(A)}) &= F(\mu) \circ (\phi^2_{H,A})^{-1} \circ (F(m_H) \otimes F(\mathrm{Id}_A)) \circ ((\phi^2_{H,H})^{-1} \otimes \mathrm{Id}_{F(A)}) \\ &= F(\mu) \circ F(m_H \otimes \mathrm{Id}_A) \circ (\phi^2_{H\otimes H,A})^{-1} \circ ((\phi^2_{H,H})^{-1} \otimes \mathrm{Id}_{F(A)}) \\ &= F(\mu \circ (\mathrm{Id}_H \otimes \mu)) \circ ((\phi^2_{H,H} \otimes \mathrm{Id}_{F(A)}) \circ \phi^2_{H\otimes H,A})^{-1} \\ &= F(\mu) \circ F(\mathrm{Id}_H \otimes \mu) \circ (\phi^2_{H,H\otimes A})^{-1} \circ (\mathrm{Id}_{F(H)} \otimes \phi^2_{H,A})^{-1} \\ &= F(\mu) \circ (\phi^2_{H,A})^{-1} \circ (\mathrm{Id}_{F(H)} \otimes F(\mu)) \circ (\mathrm{Id}_{F(H)} \otimes (\phi^2_{H,A})^{-1}) \\ &= \mu' \circ (\mathrm{Id}_{F(H)} \otimes \mu')\end{aligned}$$

and

$$\mu' \circ (u_{F(H)} \otimes \mathrm{Id}_{F(A)}) = F(\mu) \circ (\phi^2_{H,A})^{-1} \circ (F(u_H) \otimes F(\mathrm{Id}_A)) = F(\mu) \circ F(u_H \otimes \mathrm{Id}_A) \circ (\phi^2_{\Bbbk,A})^{-1}$$
$$= F(l_A) \circ (\phi^2_{\Bbbk,A})^{-1} = l_{F(A)}. \quad \square$$

Theorem 5.20. *Let H be a cocommutative color Hopf algebra. Then, the color Hopf subalgebras $K \subseteq H$ and the quotient color left H-module coalgebras Q of H are in 1-1 correspondence. The mutually inverse bijections are given by*

$$\phi_H : K \mapsto H/HK^+ \text{ and } \psi_H : Q \mapsto H^{\mathrm{co}Q}.$$

Proof. First of all we show that the two maps are well-defined. So, given a color Hopf subalgebra $K \subseteq H$, we know that HK^+ is a graded left ideal of H and also a two-sided coideal of H since $HK^+ = \mathrm{Im}(m \circ (\mathrm{Id}_H \otimes i) - l \circ (\mathrm{Id}_H \otimes \epsilon))$, where $i : K \to H$ is the inclusion (see [37, Proposition 1.4.8]). Furthermore, recalling the monoidal structure of ${}_H\mathrm{Vec}_G$ given in Remark 5.2, it is not difficult to see that Δ_{H/HK^+} and ϵ_{H/HK^+} are morphisms of left H-modules. Indeed, given $\pi : H \to H/HK^+$ in $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G)$, for every $a, b \in H$ we have that

$$\begin{aligned}\Delta_{H/HK^+}(h \cdot (a + HK^+)) &= \Delta_{H/HK^+}(ha + HK^+) = (\pi \otimes \pi)\Delta_H(ha)\\ &= \phi(|h_2|, |a_1|)(h_1a_1 + HK^+) \otimes (h_2a_2 + HK^+)\\ &= h \cdot \Delta_{H/HK^+}(a + HK^+)\end{aligned}$$

and

$$\epsilon_{H/HK^+}(h \cdot (a + HK^+)) = \epsilon_{H/HK^+}(ha + HK^+) = \epsilon_H(ha) = \epsilon_H(h)\epsilon_H(a) = h \cdot \epsilon_{H/HK^+}(a + HK^+),$$

thus H/HK^+ is a quotient color left H-module coalgebra of H. Furthermore, let Q be a quotient color left H-module coalgebra of H; we show that

$$H^{\mathrm{co}Q} = \{h \in H \mid (\mathrm{Id}_H \otimes \pi)\Delta(h) = h \otimes \pi(1_H)\} = \{h \in H \mid (\mathrm{Id}_H \otimes \pi)\Delta(h) = (\mathrm{Id}_H \otimes \pi u \epsilon)\Delta(h)\}$$

is a color Hopf subalgebra of H. It is in $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G)$ since it is the equalizer object of the pair $(\pi, \pi \circ u_H \circ \epsilon_H)$ in $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G)$. In addition, by Proposition 5.18, we know that $F(H^{\mathrm{co}Q})$ is the equalizer of the pair $(F(\pi), F(\pi \circ u_H \circ \epsilon_H)) = (F(\pi), F(\pi) \circ u_{F(H)} \circ \epsilon_{F(H)})$ in $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_{\mathbb{Z}_2})$, i.e. $F(H^{\mathrm{co}Q}) = F(H)^{\mathrm{co}F(Q)}$. But now $F(H)$ is a super Hopf algebra and $F(Q)$ is a quotient super coalgebra and a quotient super left $F(H)$-module of $F(H)$ by Lemma 5.19. Furthermore, $\Delta_{F(Q)}$ and $\epsilon_{F(Q)}$ are morphisms of left $F(H)$-modules. Indeed, recalling the hexagon relation

$$\begin{aligned}&\phi^2_{H\otimes Q, H\otimes Q} \circ F(\mathrm{Id}_H \otimes c_{H,Q} \otimes \mathrm{Id}_Q) \circ (\phi^2_{H\otimes H, Q\otimes Q})^{-1}\\ &\quad = ((\phi^2_{H,Q})^{-1} \otimes (\phi^2_{H,Q})^{-1}) \circ (\mathrm{Id}_{F(H)} \otimes c'_{F(H),F(Q)} \otimes \mathrm{Id}_{F(Q)}) \circ (\phi^2_{H,H} \otimes \phi^2_{Q,Q})\end{aligned}$$

which holds true since F is a braided strong monoidal functor, we obtain that

$$\begin{aligned}&(\mu' \otimes \mu') \circ (\mathrm{Id}_{F(H)} \otimes c'_{F(H),F(Q)} \otimes \mathrm{Id}_{F(Q)}) \circ (\Delta_{F(H)} \otimes \mathrm{Id}_{F(Q)} \otimes \mathrm{Id}_{F(Q)}) \circ (\mathrm{Id}_{F(H)} \otimes \Delta_{F(Q)}) =\\ &(F(\mu) \otimes F(\mu)) \circ ((\phi^2_{H,Q})^{-1} \otimes (\phi^2_{H,Q})^{-1}) \circ (\mathrm{Id}_{F(H)} \otimes c'_{F(H),F(Q)} \otimes \mathrm{Id}_{F(Q)}) \circ (\phi^2_{H,H} \otimes \phi^2_{Q,Q}) \circ\\ &\quad (F(\Delta_H) \otimes F(\Delta_Q)) =\\ &(F(\mu) \otimes F(\mu)) \circ \phi^2_{H\otimes Q, H\otimes Q} \circ F(\mathrm{Id}_H \otimes c_{H,Q} \otimes \mathrm{Id}_Q) \circ (\phi^2_{H\otimes H, Q\otimes Q})^{-1} \circ (F(\Delta_H) \otimes F(\Delta_Q)) =\end{aligned}$$

$$\phi^2_{Q,Q} \circ F(\mu \otimes \mu) \circ F(\mathrm{Id}_H \otimes c_{H,Q} \otimes \mathrm{Id}_Q) \circ F(\Delta_H \otimes \Delta_Q) \circ (\phi^2_{H,Q})^{-1} =$$
$$\phi^2_{Q,Q} \circ F(\Delta_Q \circ \mu) \circ (\phi^2_{H,Q})^{-1} = \Delta_{F(Q)} \circ \mu'$$

and

$$\epsilon_{F(Q)} \circ \mu' = F(\epsilon_Q) \circ F(\mu) \circ (\phi^2_{H,Q})^{-1} = F(r_{\Bbbk}) \circ F(\epsilon_H \otimes \epsilon_Q) \circ (\phi^2_{H,Q})^{-1} = r_{\Bbbk} \circ (\epsilon_{F(H)} \otimes \epsilon_{F(Q)}).$$

Thus, $F(Q)$ is a quotient super left $F(H)$-module coalgebra of $F(H)$ and then, by Theorem 5.6, we have that $F(H)^{\mathrm{co}F(Q)} = F(H^{\mathrm{co}Q})$ is a super Hopf subalgebra of $F(H)$; thus $H^{\mathrm{co}Q}$ is a color Hopf subalgebra of H by Lemma 5.16. Thus, the two maps are well-defined and now we prove that they are inverse to each other. So we compute

$$K \mapsto H/HK^+ \mapsto H^{\mathrm{co}\frac{H}{HK^+}}$$

and

$$F(H^{\mathrm{co}\frac{H}{HK^+}}) = F(H)^{\mathrm{co}F(\frac{H}{HK^+})} = F(H)^{\mathrm{co}\frac{F(H)}{F(H)F(K)^+}} = \psi_{F(H)}(\phi_{F(H)}(F(K))) = F(K)$$

since $F(K)$ is a super Hopf subalgebra of $F(H)$ by Lemma 5.16. Thus, since K and $H^{\mathrm{co}\frac{H}{HK^+}}$ are color Hopf subalgebras of H and they are the same vector space by the previous equality because F does not change the structure of vector space, they must be the same object in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by Lemma 5.15. Furthermore, we compute

$$Q \mapsto H^{\mathrm{co}Q} \mapsto H/H(H^{\mathrm{co}Q})^+$$

and

$$F(\frac{H}{H(H^{\mathrm{co}Q})^+}) = \frac{F(H)}{F(H)F(H^{\mathrm{co}Q})^+} = \frac{F(H)}{F(H)(F(H)^{\mathrm{co}F(Q)})^+} = \phi_{F(H)}(\psi_{F(H)}(F(Q))) = F(Q)$$

since $F(Q)$ is a quotient super left $F(H)$-module coalgebra of $F(H)$. We know that Q and $H/H(H^{\mathrm{co}Q})^+$ are quotient color left H-module coalgebras of H and so, since they are the same vector space by the previous equality, they must be the same quotient color left H-module coalgebra of H by Lemma 5.15. □

We have adopted the same notations used in Theorem 5.6 for the bijections ϕ_H and ψ_H. The bijective correspondence restricts to a 1-1 correspondence between normal color Hopf subalgebras and quotient color Hopf algebras as it is shown in the following result. Thus, we extend Theorem 5.6 and Corollary 5.7 to the case of cocommutative color Hopf algebras.

Corollary 5.21. *For a color Hopf subalgebra $B \subseteq A$ of a cocommutative color Hopf algebra A, the following conditions are equivalent:*

(1) B is a normal color Hopf subalgebra, i.e. $\phi(|a_2|, |b|)a_1 b S(a_2) \in B$ for every $a \in A$ and $b \in B$;
(2) A/AB^+ is a quotient color Hopf algebra;
(3) the inclusion morphism $B \to A$ is the categorical kernel of some morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

Proof. (1) $\Longrightarrow$ (2). Let B be a normal color Hopf subalgebra of A and consider the quotient color left A-module coalgebra A/AB^+. In order to show that this is a quotient color Hopf algebra we have to prove that AB^+ is a right ideal of A and that it is closed under the antipode of A. First we show that it is a right ideal. For any $a, a' \in A$ and $b \in B^+$, since $(S \otimes S) \circ \Delta = \Delta \circ S$ with A cocommutative, we obtain that

$$\begin{aligned}(ab)a' &= ab\epsilon(a'_1)a'_2 = \phi(|b|,|a'_1|)a\epsilon(a'_1)ba'_2 = \phi(|b|,|a'_1|)aa'_{1_1}S(a'_{1_2})bS(S(a'_2))\\ &= \phi(|b|,|a'_1|)\phi(|b|,|S(a'_{2_1})|)aa'_1S(a'_{2_1})bS(S(a'_{2_2})) = \phi(|b|,|a'_1|)\phi(|b|,|S(a'_2)_1|)aa'_1S(a'_2)_1bS(S(a'_2)_2)\\ &= \phi(|b|,|a'|)\phi(|S(a'_2)_2|,|b|)aa'_1S(a'_2)_1bS(S(a'_2)_2) = \phi(|b|,|a'|)aa'_1\xi_A(S(a'_2)\otimes b) \in AB^+.\end{aligned}$$

To see that AB^+ is stable under the antipode, note that $S(ab) = \phi(|a|,|b|)S(b)S(a)$. From $\epsilon(S(b)) = \epsilon(b) = 0$ with $b \in B^+$ it follows that $S(b) \in B^+ \subseteq AB^+$ and therefore $\phi(|a|,|b|)S(b)S(a) \in AB^+$ since we have just shown that AB^+ is a right ideal.

(2) $\implies$ (3). Since A/AB^+ is a quotient color Hopf algebra and then $\pi : A \to A/AB^+$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we have that $A^{\mathrm{co}\frac{A}{AB^+}} = \{a \in A \mid a_1 \otimes \pi(a_2) = a \otimes 1_{A/AB^+}\} = \mathrm{Hker}(\pi)$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. But now $A^{\mathrm{co}\frac{A}{AB^+}} = \psi_A(\phi_A(B)) = B$ by Theorem 5.20 and then $B = \mathrm{Hker}(\pi)$. Hence (B, j) is the kernel of π in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, where $j : B \to A$ is the canonical inclusion.

By Lemma 5.5 we already know that (3) $\implies$ (1) holds true and then we are done. □

Now we can prove (1) and (2) of Lemma 5.1. So let $f : A \to B$ be a morphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and consider the factorization $f = i \circ p$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ obtained by taking $p : A \to A/A(\mathrm{Hker}(f))^+A$ as the cokernel of the kernel of f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. We have that $A/\ker(f)$ is a quotient color Hopf algebra by Remark 5.3. Thus, since $\pi : A \to A/\ker(f)$ is in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we have that $A^{\mathrm{co}\frac{A}{\ker(f)}} = \mathrm{Hker}(\pi)$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Furthermore, there exists a unique map $\bar{f} : A/\ker(f) \to B,\ a + \ker(f) \mapsto f(a)$ in Vec_G, and then in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, such that $\bar{f} \circ \pi = f$ and $\bar{f}$ is injective. Thus, $\mathrm{Hker}(\pi) = \mathrm{Hker}(f)$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and so we obtain

$$A/\ker(f) = \phi_A(\psi_A(A/\ker(f))) = \phi_A(A^{\mathrm{co}\frac{A}{\ker(f)}}) = \phi_A(\mathrm{Hker}(f)) = A/A(\mathrm{Hker}(f))^+.$$

Then, $\ker(f) = A(\mathrm{Hker}(f))^+$ and so also $\ker(f) = A(\mathrm{Hker}(f))^+A$, hence $p = \pi$ and $i = \bar{f}$ is injective and then a monomorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Observe that, clearly, $\mathrm{Hker}(f)^+ \subseteq \ker(f)$ since with $x \in \mathrm{Hker}(f)$ we have that $f(x) = \epsilon(x)1_B$. Thus, we have obtained a decomposition $f = i \circ p$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ with i a monomorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and p a regular epimorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, so (1) of Lemma 5.1 is proved.

Lemma 5.22. *In* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ *regular epimorphisms are exactly the surjective morphisms and monomorphisms are exactly the injective morphisms.*

Proof. A regular epimorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is surjective since in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ every coequalizer is a projection and, if f in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is surjective, by its decomposition $f = i \circ p$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ we obtain that i is surjective and then it is an isomorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, hence f is a cokernel, so a regular epimorphism. Furthermore, in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ an injective map is clearly a monomorphism and also the vice versa is true. Indeed, given $f : A \to B$ a monomorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and $j : \mathrm{Hker}(f) \to A$ the categorical kernel of f, from $f \circ j = f \circ u_A \circ \epsilon_A \circ j$ we obtain that $j = u_A \circ \epsilon_A \circ j$, i.e., if $a \in \mathrm{Hker}(f)$ then $a = \epsilon_A(a)1_A$, hence $a \in \Bbbk 1_A$. Clearly every $x \in \Bbbk 1_A$ is in $\mathrm{Hker}(f)$, so $\mathrm{Hker}(f) = \Bbbk 1_A$ and then $\mathrm{Hker}(f)^+ = 0$. Hence, since $\ker(f) = A(\mathrm{Hker}(f))^+A$, we have that $\ker(f) = 0$, thus f is injective. □

Now condition (2) of Lemma 5.1 is easily satisfied. In $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ binary products are tensor products, so $E \times A = E \otimes A$ and $E \times B = E \otimes B$ and the induced arrow $\mathrm{Id}_E \times f$ is $\mathrm{Id}_E \otimes f$. Indeed, we have

$$l_B \circ (\epsilon_E \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_E \otimes f) = l_B \circ (\mathrm{Id}_\Bbbk \otimes f) \circ (\epsilon_E \otimes \mathrm{Id}_A) = f \circ l_A \circ (\epsilon_E \otimes \mathrm{Id}_A)$$

and $r_E \circ (\mathrm{Id}_E \otimes \epsilon_B) \circ (\mathrm{Id}_E \otimes f) = \mathrm{Id}_E \circ r_E \circ (\mathrm{Id}_E \otimes \epsilon_A)$. So, given a regular epimorphism $f : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, this is surjective, then $\mathrm{Id}_E \otimes f$ is surjective, hence a regular epimorphism by Lemma 5.22.

Now, in order to obtain the regularity of $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we show (3) of Lemma 5.1 by proving the stability of surjective maps along injective ones under pullbacks.

If we have a morphism $p : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and we consider $C \subseteq B$ a color Hopf subalgebra of B, then the subspace $p^{-1}(C)$ of A defined as in [24] by

$$p^{-1}(C) = \{x \in A |\ (p \otimes \mathrm{Id}_A)\Delta(x) \in C \otimes A\}$$

is a color Hopf subalgebra of A. Indeed, observe that

$$p^{-1}(C) = ((p \otimes \mathrm{Id}_A)\Delta)^{-1}(C \otimes A) = ((p \otimes \mathrm{Id}_A)\Delta)^{-1}(\bigoplus_{g\in G}(C \otimes A)_g) = \bigoplus_{g\in G}((p \otimes \mathrm{Id}_A)\Delta)^{-1}((C \otimes A)_g)$$

since the morphisms preserve gradings. Hence $p^{-1}(C) = \bigoplus_{g\in G} P_g$ is a graded vector space where

$$P_g = \{x \in A_g \mid (p \otimes \mathrm{Id}_A)\Delta(x) \in (C \otimes A)_g\} = ((p \otimes \mathrm{Id}_A)\Delta)^{-1}((C \otimes A)_g).$$

By Remark 4.1, we only have to show that $p^{-1}(C)$ is closed under Δ_A, m_A and S_A and that it contains 1_A. Clearly it contains 1_A and it is closed under m_A since $(p \otimes \mathrm{Id}_A) \circ \Delta$ is in $\mathrm{Mon}(\mathrm{Vec}_G)$: indeed, for any $x, y \in p^{-1}(C)$,

$$(p \otimes \mathrm{Id}_A)\Delta(xy) = m_{C\otimes A}((p \otimes \mathrm{Id}_A)\Delta \otimes (p \otimes \mathrm{Id}_A)\Delta)(x \otimes y) \in C \otimes A.$$

It is also easy to check that it is closed under the antipode since, in the cocommutative case, we have that $\Delta(S(x)) = (S \otimes S)\Delta(x)$, so with $x \in p^{-1}(C)$ we have

$$(p \otimes \mathrm{Id}_A)\Delta(S_A(x)) = (p \otimes \mathrm{Id}_A)(S_A \otimes S_A)\Delta(x) = (S_B \otimes S_A)(p \otimes \mathrm{Id}_A)\Delta(x) \in S_B(C) \otimes S_A(A) \subseteq C \otimes A.$$

Finally, we have to show that $\Delta(p^{-1}(C)) \subseteq p^{-1}(C) \otimes p^{-1}(C)$ and, since A is cocommutative, we only have to prove that $\Delta(p^{-1}(C)) \subseteq p^{-1}(C) \otimes A$. But now, given $x \in p^{-1}(C)$, we have that

$$(p \otimes \mathrm{Id}_A)\Delta(x_1) \otimes x_2 = p(x_1) \otimes x_{2_1} \otimes x_{2_2} = (\mathrm{Id}_B \otimes \Delta)(p \otimes \mathrm{Id}_A)\Delta(x) \in (\mathrm{Id}_B \otimes \Delta)(C \otimes A) \subseteq C \otimes A \otimes A$$

and then $\Delta(x) \in p^{-1}(C) \otimes A$, so that $p^{-1}(C)$ is closed under Δ_A and hence it is a color Hopf subalgebra of A. Now we show some results which generalize those given in [22] for the case of $\mathrm{Hopf}_{\Bbbk,\mathrm{coc}}$. The following Lemma 5.23 and Lemma 5.24 correspond to [22, Lemma 2.5] and [22, Lemma 2.6], respectively, and they have the same proof, which we report for the sake of completeness and in order to show that there are no problems with the respective generalizations.

Lemma 5.23. *Given a morphism $p : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we have the following facts:*

1) For all color Hopf subalgebras $C \subseteq B$, $p(p^{-1}(C)) \subseteq C$.
2) For all color Hopf subalgebras $D \subseteq A$, $D \subseteq p^{-1}(p(D))$.
3) For all color Hopf subalgebras $C \subseteq B$, then $C = p(p^{-1}(C))$ if and only if $C = p(D)$, for some $D \subseteq A$ color Hopf subalgebra.

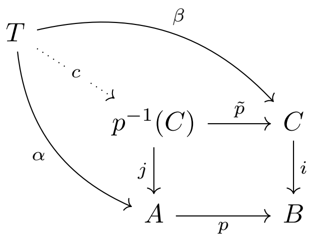
Proof. If $x \in p^{-1}(C)$, i.e. $p(x_1) \otimes x_2 \in C \otimes A$, then $p(x) \in C$, so 1) is shown. Recall that if D is a color Hopf subalgebra of A then $p(D)$ is a color Hopf subalgebra of B by Remark 5.3. If $d \in D$ we have that $(p\otimes\mathrm{Id}_A)\Delta(d) \in p(D)\otimes D \subseteq p(D)\otimes A$ and then also 2) is proved. Finally, if $C = p(p^{-1}(C))$ clearly we can take $D = p^{-1}(C)$ while, if $C = p(D)$ for some color Hopf subalgebra D of A, then $D \subseteq p^{-1}(p(D)) = p^{-1}(C)$ by 2) and, by applying p, one gets $C = p(D) \subseteq p(p^{-1}(C))$. Since $p(p^{-1}(C)) \subseteq C$ by 1), we have $C = p(p^{-1}(C))$ and we obtain 3). □

Lemma 5.24. *Given $p : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and an inclusion $i : C \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, then the diagram*

$$\begin{array}{ccc} p^{-1}(C) & \xrightarrow{\tilde{p}} & C \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ A & \xrightarrow[p]{} & B \end{array}$$

is a pullback in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, where j is the inclusion and $\tilde{p}$ is the restriction of p to $p^{-1}(C)$.

Proof. By 1) of Lemma 5.23 the diagram is commutative. To check the universal property, consider two morphisms $\alpha : T \to A$ and $\beta : T \to C$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ such that $p \circ \alpha = i \circ \beta$ and let us show that $\alpha(T) \subseteq p^{-1}(C)$. Then, taken $c : T \to p^{-1}(C)$ as α with codomain $p^{-1}(C)$ we have $j \circ c = \alpha$ and $i \circ \tilde{p} \circ c = p \circ j \circ c = p \circ \alpha = i \circ \beta$, hence $\tilde{p} \circ c = \beta$ since i is injective. This c is unique, since we must have $j \circ c = \alpha$.



Thus, we show that $\alpha(T) \subseteq p^{-1}(C)$. Given $t \in T$, since α is a morphism of coalgebras, we have

$$(p \otimes \mathrm{Id}_A)\Delta_A(\alpha(t)) = (p \otimes \mathrm{Id}_A)(\alpha \otimes \alpha)\Delta_T(t) = (i \otimes \mathrm{Id}_A)(\beta \otimes \alpha)\Delta_T(t) \in C \otimes A,$$

then the diagram is a pullback. □

Proposition 5.25. *Consider a surjective morphism $p : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and an inclusion $i : C \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Then, the morphism $\tilde{p}$ in the pullback of Lemma 5.24 is also surjective.*

Proof. If we compute the pullback of the pair (p, i) in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ we obtain $(p^{-1}(C), j, \tilde{p})$ as in Lemma 5.24 and we want to show that $\tilde{p}$ is surjective if p is surjective. Since $\tilde{p}$ is just given by the restriction of p, we have that $\tilde{p}$ is surjective if and only if $C = p(p^{-1}(C))$ and this is equivalent, with C a color Hopf subalgebra of B, to prove that $C = p(D)$ for some color Hopf subalgebra D of A by 3) of Lemma 5.23. We know that $\pi : B \to B/BC^+$ is a quotient color left B-module coalgebra and, since p is a morphism of color Hopf algebras, we have that $\pi \circ p$ is a morphism of color left A-module coalgebras, so that $A/\ker(\pi \circ p)$ is a quotient color left A-module coalgebra. We set $D := A^{\mathrm{co}\frac{A}{\ker(\pi \circ p)}} = \psi_A(A/\ker(\pi \circ p))$, which is a color Hopf subalgebra of A by Theorem 5.20. Then, we obtain

$$A/AD^+ = \phi_A(D) = \phi_A(\psi_A(A/\ker(\pi \circ p))) = A/\ker(\pi \circ p)$$

by Theorem 5.20, hence $AD^+ = \ker(\pi \circ p)$. Thus, since p is a surjective morphism of algebras, we obtain

$$Bp(D)^+ = p(A)p(D^+) = p(AD^+) = p(\ker(\pi \circ p)) = \ker(\pi) = BC^+$$

and then

$$\phi_B(C) = B/BC^+ = B/Bp(D)^+ = \phi_B(p(D))$$

so that, by applying ψ_B and by using Theorem 5.20 again, we obtain that $C = p(D)$. □

We have shown the stability of surjective morphisms (i.e. regular epimorphisms by Lemma 5.22) along inclusions under pullbacks in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. But every injective morphism $f : C \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ can be decomposed as $i \circ \phi$ with ϕ an isomorphism between C and $f(C)$ and i the inclusion of $f(C)$ into B. Now, if we consider the pullback of p along f, we have that, since the inner right square in the following diagram is also a pullback by Lemma 5.24, also the left square is a pullback by [10, Proposition 2.5.9].

$$\begin{array}{ccccc} A \times_B C & \xrightarrow{\tilde{\phi}} & p^{-1}(f(C)) & \xrightarrow{j} & A \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[\phi]{} & f(C) & \xrightarrow[i]{} & B \end{array}$$

Then, since ϕ is an isomorphism so is $\tilde{\phi}$ and from $\tilde{p} \circ \tilde{\phi} = \phi \circ \alpha$ we obtain that $\alpha = \phi^{-1} \circ \tilde{p} \circ \tilde{\phi}$. But now, since $\tilde{p}$ is surjective by Proposition 5.25, then also α is surjective, thus regular epimorphisms are stable under pullbacks along injective morphisms (i.e. monomorphisms by Lemma 5.22) in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and (3) of Lemma 5.1 is proved. We have obtained the following result:

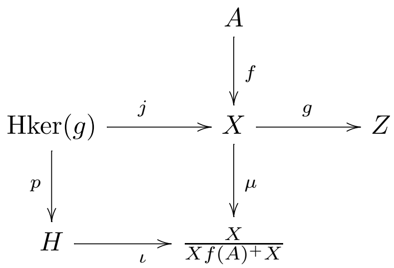
Proposition 5.26. *If G is a finitely generated abelian group and $\mathrm{char}\Bbbk \neq 2$ then the category* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ *is regular.*

6. Semi-abelian condition for $\mathbf{Hopf_{coc}(Vec_{\mathit{G}})}$

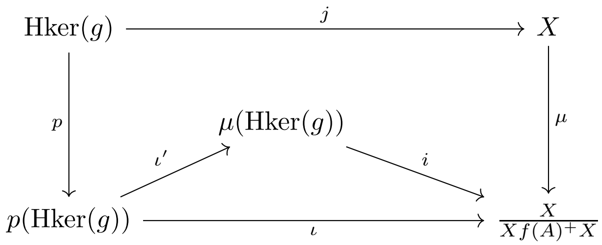
Let G be a finitely generated abelian group and $\mathrm{char}\Bbbk \neq 2$. In [25, 3.7] an equivalent characterization for semi-abelian categories is given. It is required that $\mathcal{C}$ satisfies the following properties:

1) $\mathcal{C}$ has binary products and coproducts and a zero object;
2) $\mathcal{C}$ has pullbacks of (split) monomorphisms;
3) $\mathcal{C}$ has cokernels of kernels and every morphism with zero kernel is a monomorphism;
4) the Split Short Five Lemma holds true in $\mathcal{C}$;
5) cokernels are stable under pullback;
6) images of kernels along cokernels are kernels.

For the second part of 3) we observe that, since the categorical kernel of a morphism $f : A \to B$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is given by the inclusion $i : \mathrm{Hker}(f) \to A$, if this is the zero morphism $u_A \circ \epsilon_{\mathrm{Hker}(f)}$ then, given $x \in \mathrm{Hker}(f)$, we have $x = \epsilon(x)1_A$ and again $\mathrm{Hker}(f) = \Bbbk 1_A$, so $\mathrm{Hker}(f)^+ = 0$. Hence, since we know that the vector space $\ker(f)$ is $A(\mathrm{Hker}(f))^+A$, then f is injective or, equivalently, a monomorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ by Lemma 5.22. Since we have shown that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is pointed, finitely complete (also complete by Remark 4.5), cocomplete, protomodular and regular, properties 1)-5) follow (recall that, with $\mathcal{C}$ a pointed and finitely complete category, 4) is equivalent to the protomodularity of $\mathcal{C}$) and then it only remains to prove that the image of a kernel along a cokernel is a kernel. Precisely, we want to show that, given $j : \mathrm{Hker}(g) \to X$ a kernel of a mophism $g : X \to Z$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and $\mu : X \to X/Xf(A)^+X$ a cokernel of a morphism $f : A \to X$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, there exist a morphism $p : \mathrm{Hker}(g) \to H$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and a kernel $\iota : H \to X/Xf(A)^+X$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ such that the following diagram commutes.



Now, we know that the morphism $\mu \circ j$ has a factorization regular epimorphism-monomorphism in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ since this category is regular, i.e. there exist a regular epimorphism p and a monomorphism ι in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ such that $\mu \circ j = \iota \circ p$. But it is not true in general that every monomorphism is a kernel and then we do not have that ι is a kernel automatically. We know that p is surjective and ι is injective by Lemma 5.22, then we have $\iota = i \circ \iota'$ where i is an inclusion and ι' is an isomorphism between $p(\mathrm{Hker}(g))$ and $\iota(p(\mathrm{Hker}(g))) = \mu(j(\mathrm{Hker}(g))) = \mu(\mathrm{Hker}(g))$.



By Corollary 5.21 we have that $\mathrm{Hker}(g)$ is a normal color Hopf subalgebra of X and then $\mu(\mathrm{Hker}(g))$ is a normal color Hopf subalgebra of $X/Xf(A)^+X$ by 2) of Lemma 5.4, since μ is surjective. So the inclusion i is a kernel again by Corollary 5.21 and, since ι' is an isomorphism, also ι is a kernel in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ and we are done. Finally, we have obtained the following result:

Theorem 6.1. *The category* $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ *is semi-abelian, if G is a finitely generated abelian group and* $\mathrm{char}\Bbbk \neq 2$.

6.1. Some consequences

Let G be a finitely generated abelian group and $\mathrm{char}\Bbbk \neq 2$. We can go further and prove something else about $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, which we know is semi-abelian. Recall from [11, Proposition 5.1.2] that every semi-abelian category is a Mal'tsev category, so $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is an exact Mal'tsev category with coequalizers and zero object, thus we know by [20, Corollary 4.2] that the category of abelian objects in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, which we denote by $\mathrm{Ab}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$, is abelian. Hence we want to determine the category $\mathrm{Ab}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ and we use the characterization given in [9, Theorem 6.9] (see also [13, Proposition 9]), which states that an object C in a semi-abelian category $\mathcal{C}$ is abelian if and only if its diagonal $\langle \mathrm{Id}_C, \mathrm{Id}_C \rangle : C \to C \times C$ is a normal monomorphism, i.e. it is the kernel of some morphism in $\mathcal{C}$. But we have that $C \times C = C \otimes C$ and $\langle \mathrm{Id}_C, \mathrm{Id}_C \rangle = (\mathrm{Id}_C \otimes \mathrm{Id}_C) \circ \Delta_C = \Delta_C$ in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, hence $\mathrm{Ab}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ is given by those cocommutative color Hopf algebras whose comultiplication is a kernel in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$.

Theorem 6.2. $\mathrm{Ab}(\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G))$ *is the category of commutative and cocommutative color Hopf algebras. In particular, this category is abelian.*

Proof. Given C in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$, we know that Δ is injective and then, if we write $\Delta = i \circ \Delta'$ with $\Delta' : C \to \mathrm{Im}(\Delta)$ an isomorphism and $i : \mathrm{Im}(\Delta) \to C \otimes C$ the inclusion, we have that Δ is a kernel in

$\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ if and only if i is a kernel in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$. Furthermore, i is a kernel in $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ if and only if $\mathrm{Im}(\Delta)$ is a normal color Hopf subalgebra of $C \otimes C$ by Corollary 5.21. If C is a commutative color Hopf algebra, then also $C \otimes C$ is a commutative color Hopf algebra and then $\mathrm{Im}(\Delta)$, which is a color Hopf subalgebra of $C \otimes C$ by Remark 5.3, is normal by 3) of Lemma 5.4. On the other hand, suppose that $\mathrm{Im}(\Delta)$ is a normal color Hopf subalgebra of $C \otimes C$, i.e. that we have $\xi_{C\otimes C}(C \otimes C \otimes \mathrm{Im}(\Delta)) \subseteq \mathrm{Im}(\Delta)$. For every $a, b, c \in C$ we have that

$$\begin{aligned}
\xi_{C\otimes C}(c \otimes 1_C \otimes a \otimes b) &= m_{C\otimes C}(m_{C\otimes C} \otimes S_C \otimes S_C)(\mathrm{Id} \otimes c_{C\otimes C,C\otimes C})(\Delta_{C\otimes C} \otimes \mathrm{Id})(c \otimes 1_C \otimes a \otimes b) \\
&= m_{C\otimes C}(m_{C\otimes C} \otimes S_C \otimes S_C)(\mathrm{Id} \otimes c_{C\otimes C,C\otimes C})(\phi(|c_2|, 1_G)c_1 \otimes 1_C \otimes c_2 \otimes 1_C \otimes a \otimes b) \\
&= m_{C\otimes C}(m_{C\otimes C} \otimes S_C \otimes S_C)(\phi(|c_2 \otimes 1_C|, |a \otimes b|)c_1 \otimes 1_C \otimes a \otimes b \otimes c_2 \otimes 1_C) \\
&= m_{C\otimes C}(\phi(|c_2|, |a|)\phi(|c_2|, |b|)\phi(1_G, |a|)c_1 a \otimes b \otimes S_C(c_2) \otimes S_C(1_C)) \\
&= \phi(|c_2|, |a|)\phi(|c_2|, |b|)\phi(|b|, |c_2|)c_1 a S_C(c_2) \otimes b = \phi(|c_2|, |a|)c_1 a S_C(c_2) \otimes b \\
&= (\xi_C \otimes \mathrm{Id}_C)(c \otimes a \otimes b).
\end{aligned}$$

Hence, for every $a, c \in C$, we have that

$$\mathrm{Im}(\Delta) \ni \xi_{C\otimes C}(c \otimes 1_C \otimes \Delta(a)) = (\xi_C \otimes \mathrm{Id}_C)(\mathrm{Id}_C \otimes \Delta)(c \otimes a)$$

and then there exists $x \in C$ such that

$$\Delta(x) = (\xi_C \otimes \mathrm{Id}_C)(\mathrm{Id}_C \otimes \Delta)(c \otimes a).$$

Thus, we obtain

$$\begin{aligned}
x &= l_C(\epsilon_C \otimes \mathrm{Id}_C)\Delta(x) = l_C(\epsilon_C \otimes \mathrm{Id}_C)(\xi_C \otimes \mathrm{Id}_C)(\mathrm{Id}_C \otimes \Delta)(c \otimes a) \\
&= l_C(\epsilon_{C\otimes C} \otimes \mathrm{Id}_C)(\mathrm{Id}_C \otimes \Delta)(c \otimes a) = l_C(\epsilon_C \otimes \mathrm{Id}_C)(c \otimes a)
\end{aligned}$$

but we also have that

$$x = r_C(\mathrm{Id}_C \otimes \epsilon_C)\Delta(x) = r_C(\mathrm{Id}_C \otimes \epsilon_C)(\xi_C \otimes \mathrm{Id}_C)(\mathrm{Id}_C \otimes \Delta)(c \otimes a) = \xi_C(c \otimes a_1)\epsilon_C(a_2) = \xi_C(c \otimes a)$$

and then

$$\xi_C(c \otimes a) = l_C(\epsilon_C \otimes \mathrm{Id}_C)(c \otimes a) \text{ for every } a, c \in C.$$

Hence C is a commutative color Hopf algebra by 3) of Lemma 5.4. □

The notion of semi-abelian category was introduced to capture typical algebraic properties of groups but it was noted that there are many significant aspects of groups which are not captured in this more general context, then reinforcements of this notion were born. We recall that a category with finite limits $\mathcal{C}$ is called *algebraically coherent* if for each morphism $f : X \to Y$ in $\mathcal{C}$ the change-of-base functor $f^* : \mathrm{Pt}_Y(\mathcal{C}) \to \mathrm{Pt}_X(\mathcal{C})$ is coherent, i.e. it preserves finite limits and jointly strongly epimorphic pairs (see [18, Definition 3.1]). In [18, Theorems 6.18 and 6.24] it is shown that semi-abelian categories which are algebraically coherent satisfy both the condition (SH) and (NH) and are peri-abelian and strongly protomodular, thus they have significantly stronger properties than general semi-abelian categories. So it is interesting to understand if the category $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is algebraically coherent, still with G a finitely generated abelian group and $\mathrm{char}\Bbbk \neq 2$. We recall from [15] that a finitely complete category $\mathcal{C}$ is said to be *locally algebraically cartesian closed* when, for every $f : X \to Y$ in $\mathcal{C}$, the change-of-base functor $f^* : \mathrm{Pt}_Y(\mathcal{C}) \to \mathrm{Pt}_X(\mathcal{C})$ is a left adjoint and that if $\mathcal{C}$ is locally algebraically cartesian closed then it is algebraically coherent by [18, Theorem 4.5]. We conclude with the following result:

Proposition 6.3. *The category of cocommutative color Hopf algebras is action representable and locally algebraically cartesian closed.*

Proof. By [34, Proposition 3.2] (see also [8]), the category $\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is cartesian closed since Vec_G is a symmetric monoidally closed category (see e.g. [17]). Thus, since $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G) = \mathrm{Grp}(\mathrm{Comon}_{\mathrm{coc}}(\mathrm{Vec}_G))$, we have that $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is locally algebraically cartesian closed by [23, Proposition 5.3]. Furthermore, the category of internal groups in a cartesian closed category is always action representable, provided it is semi-abelian, as it is shown in [12, Theorem 4.4] and then $\mathrm{Hopf}_{\mathrm{coc}}(\mathrm{Vec}_G)$ is also action representable. □

CRediT authorship contribution statement

Andrea Sciandra: Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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