Monadic vs adjoint decomposition

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MONADIC VS ADJOINT DECOMPOSITION

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Abstract. It is known that the so-called monadic decomposition, applied to the adjunction connecting the category of bialgebras to the category of vector spaces via the tensor and the primitive functors, returns the usual adjunction between bialgebras and (restricted) Lie algebras. Moreover, in this framework, the notions of augmented monad and combinatorial rank play a central role. In order to set these results into a wider context, we are led to substitute the monadic decomposition by what we call the adjoint decomposition. This construction has the advantage of reducing the computational complexity when compared to the first one. We connect the two decompositions by means of an embedding and we investigate its properties by using a relative version of Grothendieck fibration. As an application, in this wider setting, by using the notion of augmented monad, we introduce a notion of combinatorial rank that, among other things, is expected to give some hints on the length of the monadic decomposition.

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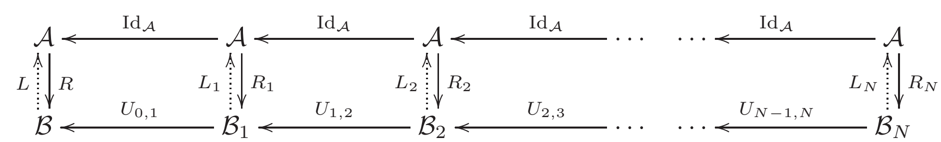
Introduction

Let $\mathcal{A}$ be a category with all coequalizers and let $L \dashv R : \mathcal{A} \to \mathcal{B}$ be an adjunction with unit $\eta : \mathrm{Id} \to RL$ and counit $\epsilon : LR \to \mathrm{Id}$. Consider the Eilenberg-Moore category $\mathcal{B}_1$ of algebras over the monad $(RL, R\epsilon L, \eta)$. Then the comparison functor $R_1 : \mathcal{A} \to \mathcal{B}_1$ has a left adjoint L_1, with unit $\eta_1 : \mathrm{Id} \to R_1L_1$ and counit $\epsilon_1 : L_1R_1 \to \mathrm{Id}$, and we can compute the Eilenberg-Moore category $\mathcal{B}_2$ of algebras over the monad $(R_1L_1, R_1\epsilon_1L_1, \eta_1)$. Going on this way we obtain a tower



where $U_{n,n+1}$ denotes the forgetful functor and $U_{n,n+1} \circ R_{n+1} = R_n$. If this process stops exactly after N steps, meaning that N is the smallest positive integer such that $U_{N,N+1}$ is a category

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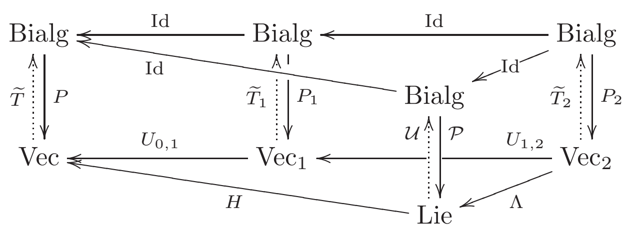
isomorphism, then R is said to have a *monadic decomposition of monadic length N*. For relevant outcomes of this notion we refer to [AGTM, AGM2, AM2]. We just mention here how our interest in this construction stems from the case when (L, R) is the adjunction $(\widetilde{T}, P)$, where P is the functor that associates to any bialgebra, over a base field $\Bbbk$, its space of primitive elements and its left adjoint $\widetilde{T}$ associates to a vector space V its tensor algebra TV endowed with the usual bialgebra structure in which the elements of V are primitive. One of the outcomes of the papers quoted above is the existence of an equivalence Λ between the category Vec_2 and the category Lie of either Lie algebras, if $\mathrm{char}(\Bbbk) = 0$, or restricted Lie algebras, if $\mathrm{char}(\Bbbk) > 0$. Moreover one has $\Lambda \circ P_2 = \mathcal{P}$ and $H \circ \Lambda = U_{0,1}U_{1,2}$. Here $(\mathcal{U}, \mathcal{P})$ is the usual adjunction between Bialg and Lie given by the (restricted) universal enveloping algebra functor and the primitive functor.

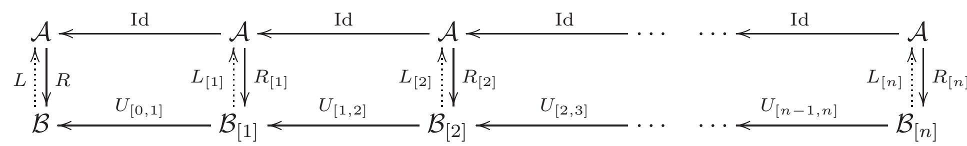


The starting functor P comes out to have monadic decomposition of monadic length at most 2 and this reflects the fact that the functor $\mathcal{U}$ is fully faithful, or equivalently the unit $\mathrm{Id} \to \mathcal{PU}$ of the adjunction $(\mathcal{U}, \mathcal{P})$ is invertible, which is part of the so-called Milnor-Moore theorem. Thus if the input (L, R) of the monadic decomposition procedure is the adjunction $(\widetilde{T}, P)$, then the corresponding output, when the iteration stops, is the adjunction $(\mathcal{U}, \mathcal{P})$ up to equivalence.

We point out that unit $\widetilde{\eta}$ of the adjunction $(\widetilde{T}, P)$ splits via a suitable natural retraction $\gamma : P\widetilde{T} \to \mathrm{Id}$, i.e. an augmentation for the associated monad, that allows to define a functor $\Gamma_1 : \mathrm{Vec} \to \mathrm{Vec}_1, V \mapsto (V, \gamma V)$. The composite functor $S_1 := \widetilde{T}_1\Gamma_1 : \mathrm{Vec} \to \mathrm{Bialg}$ associates to a vector space V the tensor bialgebra $\widetilde{T}V$ factored out by the ideal generated by its homogeneous primitive elements of degree at least two. If $\widetilde{\eta}_1$ denotes the unit of $(\widetilde{T}_1, P_1)$, it comes out that $\widetilde{\eta}_1\Gamma_1 V$ is invertible for every V (this is equivalent to ask that V has *combinatorial rank* at most one) and this plays a central role in proving that the iteration stops after two steps.

It is natural to wonder what happens to monadic decomposition if we substitute the category of vector spaces over $\Bbbk$ and the category of bialgebras over $\Bbbk$ by an arbitrary braided monoidal category $\mathcal{M}$ and the category $\mathrm{Bialg}(\mathcal{M})$ of bialgebras in $\mathcal{M}$ respectively, once we made the proper assumptions on $\mathcal{M}$ to have an analogue of the adjunction $(\widetilde{T}, P)$. Partial results have been obtained in [AM2] giving rise to the notion of Milnor-Moore Category. It is worth to notice that, to the best of our knowledge, even in the more restrictive case when $\mathcal{M}$ is a symmetric monoidal category it is an open problem whether the monadic length is still at most 2.

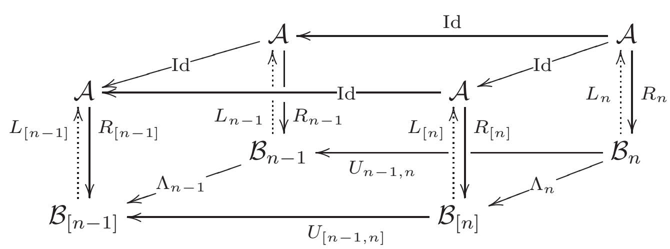
In order to look at the problem from a more general perspective, unconstrained by the particular features of the examples considered above, we think one has to investigate the stationarity of monadic decomposition at the level of an arbitrary adjunction (L, R). The notions of augmented monad and of combinatorial rank, mentioned above, are expected to play a relevant role in the picture. Moreover, since the procedure may, in principle, stop at some level higher than 2, the functor Γ_1, arising from the augmentation, should be extended to some functor $\Gamma_n : \mathcal{B} \to \mathcal{B}_n$. A first attempt to define such a functor shows how it is inconvenient to prove that the candidate object $\Gamma_n B$ belongs to $\mathcal{B}_n$ for every B in $\mathcal{B}$. This is due to the fact that to test if an object belongs to this category several equalities have to be checked. The first aim of this paper is to reduce drastically the number of equalities to verify by replacing the category $\mathcal{B}_n$ by a new category $\mathcal{B}_{[n]}$. More precisely, we construct a kind of monadic decomposition that we call an *adjoint decomposition* as follows,



where $(L_{[n]}, R_{[n]}, \epsilon_{[n]}, \eta_{[n]})$ is a suitable adjunction. Denote by $U_{[a,b]} : \mathcal{B}_{[b]} \to \mathcal{B}_{[a]}$ the composite functor $U_{[a,a+1]} \circ U_{[a+1,a+2]} \circ \cdots \circ U_{[b-2,b-1]} \circ U_{[b-1,b]}$ for all $a \leq b$. An object in $\mathcal{B}_{[n]}$ is a pair $B_{[n]} := (B_{[n-1]}, b_{[n-1]})$ where $B_{[n-1]}$ is an object in $\mathcal{B}_{[n-1]}$ and $b_{[n-1]} : RL_{[n-1]}B_{[n-1]} \to U_{[0,n-1]}B_{[n-1]}$ is a morphism in $\mathcal{B}$. Thus it can be regarded as a datum $B_{[n]} := (B_{[0]}, b_{[0]}, b_{[1]}, \ldots, b_{[n-1]})$, where $b_{[t]} : RL_{[t]}B_{[t]} \to B_{[0]}$ and $B_{[t]} := U_{[t,n]}B_{[n]}$, for each $t \in \{0, \ldots, n-1\}$. A morphism $f_{[n]} : B_{[n]} \to B'_{[n]}$ in $\mathcal{B}_{[n]}$ is a morphism $f_{[n-1]} : B_{[n-1]} \to B'_{[n-1]}$ in $\mathcal{B}_{[n-1]}$ such that $U_{[0,n-1]}f_{[n-1]} \circ b_{[n-1]} = b'_{[n-1]} \circ RL_{[n-1]}f_{[n-1]}$.

For every $n \geq 1$, we can construct a fully faithful functor $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ which satisfies the equalities $\Lambda_n \circ R_n = R_{[n]}$ and $U_{[n-1,n]} \circ \Lambda_n = \Lambda_{n-1} \circ U_{n-1,n}$ i.e. that makes commutative the solid faces of the following diagram.

(1)



Furthermore we have an isomorphism $\lambda_n : L_{[n]}\Lambda_n \to L_n$. By means of a relative version of Grothendieck fibration, we are able to give sufficient conditions for an object in $\mathcal{B}_{[n]}$ to be the image through Λ_n of an object in $\mathcal{B}_n$. As an instance of how this strategy works we construct, under appropriate conditions, involving an augmentation for the monad RL, a family of functors $\Gamma_{[n]} : \mathcal{B} \to \mathcal{B}_{[n]}$, $n \in \mathbb{N}$, that factor through Λ_n returning the desired functor of $\Gamma_n : \mathcal{B} \to \mathcal{B}_n$. These constructions apply to the adjunction $\widetilde{T} \dashv P : \mathrm{Bialg}(\mathcal{M}) \to \mathcal{M}$. In the particular case when $\mathcal{M}$ is the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over a Hopf algebra H, we obtain an explicit description of the functors $S_{[n]} := \widetilde{T}_{[n]}\Gamma_{[n]} \cong \widetilde{T}_n\Gamma_n =: S_n$, which extend the functor S_1 mentioned above. The combinatorial rank of an object V in ${}^H_H\mathcal{YD}$, regarded as a braided vector space through the braiding of ${}^H_H\mathcal{YD}$, is exactly the smallest n such that the canonical projection $S_{[n]}V \to S_{[n+1]}V$ is invertible and in this case $S_{[n]}V$ is isomorphic to the Nichols algebra of V. Since the previous projection makes sense also if we start from a general adjunction $L \dashv R : \mathcal{A} \to \mathcal{B}$ and an object B in $\mathcal{B}$, we are led to a notion of combinatorial rank in this wide setting that, among other things, is expected to give some hints on the length of the monadic decomposition. Finally we propose possible lines of future investigation.

Description of main results and applications. The paper is organized as follows.

In Section 1 we recall the notion of monadic decomposition and the definition of inserter category together with its properties needed in the paper.

In Section 2 we revise the notion of Adjoint triangle introduced by Dubuc. In Proposition 2.5, we give a procedure to associate a new adjoint triangle to a given one. By means of this result, we construct iteratively the adjoint decomposition.

In Section 3 we compare monadic and adjoint decompositions. More explicitly, we construct a fully faithful functor $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$, which is injective on objects, connecting the two decompositions. This is obtained in Remark 3.7 by applying iteratively Proposition 3.6.

In Section 4, we investigate a relative version of Grothendieck fibrations. As a byproduct, we deduce other properties of the functor Λ_n. In particular, in Theorem 4.19, we prove it is an $\mathsf{M}\left(U_{[n]}\right)$-fibration, where $\mathsf{M}\left(U_{[n]}\right)$ stands for the class of morphisms in $\mathcal{B}_{[n]}$ whose image in $\mathcal{B}$ via the forgetful functor $U_{[n]}:\mathcal{B}_{[n]}\to\mathcal{B}$ are monomorphisms. As a consequence, in Theorem 4.20, we give conditions guaranteeing that an object $B_{[n]}\in\mathcal{B}_{[n]}$ is the image of an object in $\mathcal{B}_n$ through Λ_n. These conditions enable to reduce the number of equalities to check in order to establish that an object lives inside $\mathcal{B}_n$. In Corollary 4.21 we are able to prove that if $L_{[N]}$ is fully faithful for some N, then R has a monadic decomposition of monadic length at most N.

In Section 5 we connect these results to the notion of augmented monad. Explicitly, given a suitable diagram involving two adjunctions (L,R) and (L',R'), in Theorem 5.1, we prove that under certain assumptions, if the monad $R'L'$ is augmented, then so is RL and we can construct a family of functors $\Gamma_{[n]}:\mathcal{B}\to\mathcal{B}_{[n]}$, $n\in\mathbb{N}$. Any object of the form $\Gamma_{[n]}B\in\mathcal{B}_{[n]}$, with $B\in\mathcal{B}$, fulfills the conditions mentioned above and hence it belongs to the image of Λ_n. As a consequence $\Gamma_{[n]}$ factors through Λ_n, see Proposition 5.2.

In Section 6, we study our prototype example for Theorem 5.1 which also explains the relevant role played by the functors $\Gamma_{[n]}$. Given a preadditive braided monoidal category $\mathcal{M}$ having equalizers, denumerable coproducts and coequalizers of reflexive pairs of morphisms and such that all of them are preserved by the tensor products, we construct a diagram, as in Theorem 5.1,

$$\begin{array}{ccc} \mathrm{Bialg}(\mathcal{M}) & \xrightarrow{\mho^{+}} & \mathrm{Alg}^{+}(\mathcal{M}) \\ \widetilde{T}\uparrow\downarrow P & & T^{+}\uparrow\downarrow\Omega^{+} \\ \mathcal{M} & \xrightarrow{\mathrm{Id}} & \mathcal{M} \end{array}$$

where $\mathrm{Bialg}(\mathcal{M})$ is the category of bialgebras in $\mathcal{M}$, $\mathrm{Alg}^{+}(\mathcal{M})$ is the category of augmented algebras in $\mathcal{M}$, $\widetilde{T}$ is the tensor bialgebra functor, P is the primitive functor, T^{+} is essentially the tensor algebra functor and Ω^{+} associates to an augmented algebra (A,ε) the kernel in $\mathcal{M}$ of its augmentation ε. The functor $\mho^{+}$ is just the forgetful functor. By the foregoing we get a family of functors $\Gamma_{[n]}:\mathcal{M}\to\mathcal{M}_{[n]}$, $n\in\mathbb{N}$, that factor through Λ_n, as desired.

In Section 7 we describe explicitly these functors $\Gamma_{[n]}$ in the case when $\mathcal{M}$ is the category ${}_H^H\mathcal{YD}$ of Yetter-Drinfeld modules over a finite-dimensional Hopf algebra H, the particular case of the category Vec of vector spaces being obtained by taking $H=\Bbbk$. Concurrently we are lead to define a possible analogue of the notion of combinatorial rank $\kappa(V,c)$ of a braided vector space (V,c) as defined in [Ar2, Section 5] by mimicking V. K. Kharchenko's definition in [Kh2, Definition 5.4]. We refer to [Kh1] for an overview on the notion of combinatorial rank and its importance. Recall that a braided vector space (V,c) is a vector space V endowed with a braiding $c:V\otimes V\to V\otimes V$. The tensor algebra TV can be endowed with a braided bialgebra structure (this means to have a braided vector space endowed with an algebra and a coalgebra structure suitably compatible with the braiding), arising from the braiding of V, that we denote by $T(V,c)$. If we divide out $T(V,c)$ by the ideal generated by its homogeneous primitive elements of degree at least two we obtain a new braided bialgebra, say $S_{[1]}(V,c)$. We can repeat the same procedure on this braided bialgebra obtaining a new quotient braided bialgebra $S_{[2]}(V,c)$ and go on this way. At the limit this procedure yields the so-called Nichols algebra $B(V,c)$ and the number of steps occurred is exactly $\kappa(V,c)$.

Now it is well-known that under some finiteness conditions, a braided vector space (V,c) can be realized as an object in the category ${}_H^H\mathcal{YD}$ for some Hopf algebra H and c becomes the braiding $c_{V,V}$ of ${}_H^H\mathcal{YD}$ applied to V, see [Scha, 3.2.9]. On the other hand ${}_H^H\mathcal{YD}$ is a braided monoidal category and any bialgebra in it becomes in a natural way a braided bialgebra in the above sense if we forget the Yetter-Drinfeld module structure and we just keep the underlying braiding, algebra and coalgebra structures. In particular $\widetilde{T}V\in\mathrm{Bialg}\left({}_H^H\mathcal{YD}\right)$ becomes the braided tensor algebra $T(V,c)$ mentioned above. Define the functors $S_{[n]}:=\widetilde{T}_{[n]}\Gamma_{[n]}:{}_H^H\mathcal{YD}\to\mathrm{Bialg}\left({}_H^H\mathcal{YD}\right)$. In Example 7.1, we shows that $S_{[n]}V\in\mathrm{Bialg}\left({}_H^H\mathcal{YD}\right)$ becomes the braided bialgebra $S_{[n]}(V,c)$ mentioned above, for each $n\in\mathbb{N}$. As a consequence the combinatorial rank of V, regarded as braided vector space through the braiding $c=c_{V,V}$ of ${}_H^H\mathcal{YD}$ as above, is the smallest n such that the canonical projection $S_{[n]}V\to S_{[n+1]}V$ is invertible, if such an n exists, and in this case we have $S_{[n]}V=\mathcal{B}(V,c)$.

Since, in the setting of Theorem 5.1, we can always define $S_{[n]} := L_{[n]}\Gamma_{[n]} : \mathcal{B} \to \mathcal{A}$, for every $B \in \mathcal{B}$ we are lead to define (see Definition 7.3) the *combinatorial rank* of an object $B \in \mathcal{B}$, with respect to the adjunction (L, R), to be the smallest n such that the canonical projection $S_{[n]}B \to S_{[n+1]}B$ is invertible (see Lemma 5.3), if such an n exists. Thus a concept of combinatorial rank can be introduced and investigated in this very general setting in which there is neither a bialgebra nor a braided vector spaces but just an adjunction (L, R) as in Theorem 5.1. In the case when $\mathcal{M}$ is the category Vec of vector spaces and the adjunction is $(\widetilde{T}, P)$, every object in Vec has combinatorial rank at most one (Example 7.7), but this is not true for an arbitrary $\mathcal{M}$, e.g. the category ${}^H_H\mathcal{YD}$, see Example 7.7. In Theorem 7.5, we prove that, if the adjunction (L_N, R_N) is idempotent for some positive integer N (e.g. R has a monadic decomposition of length N), then every object in the domain of R has combinatorial rank at most N with respect to the adjunction (L, R). As a corollary we obtain that every symmetric MM-category in the sense of [AM2, Definition 7.4] has all objects of combinatorial rank at most one, with respect to the adjunction $(\widetilde{T}, P)$, see Corollary 7.6.

A possible idea for a future investigation is to establish whether the general framework, in which the notion of combinatorial rank is settled here, can give new hints on the existence of some bound for the combinatorial rank of objects in a proper category $\mathcal{B}$ with respect to an adjunction (L, R) (or more specifically in a category $\mathcal{M}$ with respect to the adjunction $(\widetilde{T}, P)$) as it happens in Vec. The fact that all objects in Vec have combinatorial rank at most one constitutes one of the main ingredients in [AGTM] to prove that the monadic decomposition of $P : \mathrm{Bialg}(\mathrm{Vec}) \to \mathrm{Vec}$ has length at most two. A natural question, that we also leave to future investigations, is to determine whether a similar result still holds in the setting of $\mathcal{B}$ as above for the functor R. Such a result would be related to an analogue of the so-called Milnor-Moore theorem, see Remark 7.8. More generally one can ask whether the length of the monadic decomposition of the functor R is upper-bounded in case the combinatorial rank of objects in $\mathcal{B}$ with respect to (L, R) is upper-bounded.

1. Monadic Decomposition and Inserter Category

Throughout this paper $\Bbbk$ will denote a field. All vector spaces and (co)algebras will be defined over $\Bbbk$. The unadorned tensor product $\otimes$ will denote the tensor product over $\Bbbk$ if not stated otherwise. We denote either by $\mathfrak{M}$ or Vec the category of vector spaces.

Definition 1.1. Recall that a *monad* on a category $\mathcal{A}$ is a triple $\mathbb{Q} := (Q, m, u)$, where $Q : \mathcal{A} \to \mathcal{A}$ is a functor, $m : QQ \to Q$ and $u : \mathcal{A} \to Q$ are functorial morphisms satisfying the associativity and the unitality conditions $m \circ mQ = m \circ Qm$ and $m \circ Qu = \mathrm{Id}_Q = m \circ uQ$. An *algebra* over a monad $\mathbb{Q}$ on $\mathcal{A}$ (or simply a $\mathbb{Q}$-*algebra*) is a pair (X, μ) where $X \in \mathcal{A}$ and $\mu : QX \to X$ is a morphism in $\mathcal{A}$ such that $\mu \circ Q\mu = \mu \circ mX$ and $\mu \circ uX = \mathrm{Id}_X$. A *morphism between two* $\mathbb{Q}$-*algebras* (X, μ) and (X', μ') is a morphism $f : X \to X'$ in $\mathcal{A}$ such that $\mu' \circ Qf = f \circ \mu$. We will denote by ${}_{\mathbb{Q}}\mathcal{A}$ the category of $\mathbb{Q}$-algebras and their morphisms. This is the so-called *Eilenberg-Moore category* of the monad $\mathbb{Q}$ (which is sometimes also denoted by $\mathcal{A}^{\mathbb{Q}}$ in the literature). When the multiplication and unit of the monad are clear from the context, we will just write Q instead of $\mathbb{Q}$.

A monad $\mathbb{Q}$ on $\mathcal{A}$ gives rise to an adjunction $(F, U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathcal{A} \to \mathcal{A}$ is the forgetful functor and $F : \mathcal{A} \to {}_{\mathbb{Q}}\mathcal{A}$ is the free functor. Explicitly:

$$U(X, \mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX), \quad Ff := Qf.$$

Note that $UF = Q$. The unit of the adjunction (F, U) is given by the unit $u : \mathcal{A} \to UF = Q$ of the monad $\mathbb{Q}$. The counit $\lambda : FU \to {}_{\mathbb{Q}}\mathcal{A}$ of this adjunction is uniquely determined by the equality $U(\lambda(X, \mu)) = \mu$ for every $(X, \mu) \in {}_{\mathbb{Q}}\mathcal{A}$. It is well-known that the forgetful functor $U : {}_{\mathbb{Q}}\mathcal{A} \to \mathcal{A}$ is faithful and reflects isomorphisms (see e.g. [Bo2, Proposition 4.1.4]).

Let $L \dashv R : \mathcal{A} \to \mathcal{B}$ be an adjunction with unit $\eta : \mathrm{Id}_{\mathcal{B}} \to RL$ and counit $\epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$. Then $(RL, R\epsilon L, \eta)$ is a monad on $\mathcal{B}$ and we can consider the so-called *comparison functor* $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ which is defined by $KX := (RX, R\epsilon X)$ and $Kf := Rf$. Note that ${}_{RL}U \circ K = R$.

Definition 1.2. An adjunction $L \dashv R : \mathcal{A} \to \mathcal{B}$ is called *monadic* (tripleable in Beck's terminology [Be, Definition 3, page 8]) whenever the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ is an equivalence of categories. A functor R is called *monadic* if it has a left adjoint L such that the adjunction (L,R) is monadic, see [Be, Definition 3', page 8].

Definition 1.3. (See [AGTM, Definition 2.7], [AHW, Definition 2.1] and [MS, Definitions 2.10 and 2.14]) Fix a $N\in\mathbb{N}$. We say that a functor R has a *monadic decomposition of monadic length* N whenever there exists a sequence $(R_n)_{n\leq N}$ of functors R_n such that

1) $R_0=R$;

2) for $0\leq n\leq N$, the functor R_n has a left adjoint functor L_n;

3) for $0\leq n\leq N-1$, the functor R_{n+1} is the comparison functor induced by the adjunction (L_n,R_n) with respect to its associated monad;

4) L_N is fully faithful while L_n is not fully faithful for $0\leq n\leq N-1$.

Compare with the construction performed in [Ma, 1.5.5, page 49].

For $R:\mathcal{A}\to\mathcal{B}$, as above we have a diagram

$$\begin{array}{ccccccccc}
\mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} \cdots & \cdots \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} \\
L_0\uparrow\downarrow R_0 & & L_1\uparrow\downarrow R_1 & & L_2\uparrow\downarrow R_2 & & & L_N\uparrow\downarrow R_N \\
\mathcal{B}_0 & \xleftarrow{U_{0,1}} & \mathcal{B}_1 & \xleftarrow{U_{1,2}} & \mathcal{B}_2 & \xleftarrow{U_{2,3}} \cdots & \cdots \xleftarrow{U_{N-1,N}} & \mathcal{B}_N
\end{array} \tag{2}$$

where $\mathcal{B}_0=\mathcal{B}$ and, for $1\leq n\leq N$,

- $\mathcal{B}_n$ is the category of Q_{n-1}-algebras ${}_{Q_{n-1}}\mathcal{B}_{n-1}$, where $Q_{n-1}:=R_{n-1}L_{n-1}$;
- $U_{n-1,n}:\mathcal{B}_n\to\mathcal{B}_{n-1}$ is the forgetful functor ${}_{Q_{n-1}}U$.

We will denote by $\eta_n:\mathrm{Id}_{\mathcal{B}_n}\to R_nL_n$ and $\epsilon_n:L_nR_n\to\mathrm{Id}_{\mathcal{A}}$ the unit and counit of the adjunction (L_n,R_n) respectively for $0\leq n\leq N$. Note that one can introduce the forgetful functor $U_{m,n}:\mathcal{B}_n\to\mathcal{B}_m$ for all $m\leq n$ with $0\leq m,n\leq N$.

We point out that L_N is full and faithful is equivalent to the fact that the forgetful functor $U_{N,N+1}$ is a category isomorphism, see e.g. [AGTM, Remark 2.4].

We refer to [AGTM, Remarks 2.8 and 2.10] for further comments on monadic decompositions.

We now recall the notion of inserter category which will be a crucial tool in the construction of the adjoint decomposition.

Definition 1.4. Let $F,G:\mathcal{A}\to\mathcal{B}$ be functors. The *inserter category* $\langle F|G\rangle$ has objects the pairs (A,α_A) where $A\in\mathcal{A}$ and $\alpha_A:FA\to GA$ is a morphism in $\mathcal{B}$. A morphism $f:(A,\alpha_A)\to(A',\alpha_{A'})$ is a morphism $f:A\to A'$ in $\mathcal{A}$ such that the following diagram commutes

$$\begin{array}{ccc}
FA & \xrightarrow{Ff} & FA' \\
{\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_{A'}} \\
GA & \xrightarrow{Gf} & GA'
\end{array}$$

If we denote by

$$P=P_{\langle F|G\rangle}:\langle F|G\rangle\to\mathcal{A},\qquad (A,\alpha_A)\mapsto A,\qquad f\mapsto f$$

the forgetful functor, then there is a natural transformation

$$\psi:=\psi_{\langle F|G\rangle}:FP\to GP$$

which is defined by $\psi(A,\alpha)=\alpha$ for every $(A,\alpha)\in\langle F\downarrow G\rangle$.

Given functors $F,G,F',G':\mathcal{A}\to\mathcal{B}$ and natural transformations $\phi:F'\to F$ and $\gamma:G\to G'$ we can define the functor

$$\langle\phi|\gamma\rangle:\langle F|G\rangle\to\langle F'|G'\rangle,\quad \left(A,FA\overset{\alpha_A}{\to}GA\right)\mapsto\left(A,F'A\overset{\phi A}{\to}FA\overset{\alpha_A}{\to}GA\overset{\gamma A}{\to}G'A\right),\quad f\mapsto f.$$

Remark 1.5. We point out that $\langle F|G\rangle$ is exactly the inserter $\mathbf{Insert}(F,G)$ in the 2-category Cat, see e.g. [CS, page 157].

Lemma 1.6. *1) Let $F,G:\mathcal{A}\to\mathcal{B}$ be functors and let $Q:\mathcal{Q}\to\mathcal{A}$ be a functor endowed with a natural transformation $q:FQ\to GQ$. Then there is a unique functor $Q[q]:\mathcal{Q}\to\langle F|G\rangle$ such that $P\circ Q[q]=Q$ and $\psi Q[q]=q$. Explicitly $Q[q]X:=(QX,qX)\in\langle F|G\rangle$ for every $X\in\mathcal{Q}$. Clearly any functor $N:\mathcal{Q}\to\langle F|G\rangle$ is of the form $Q[q]$ for $Q=PN$ and $q=\psi N$.*

2) Let $Q[q], K[k] : \mathcal{Q} \to \langle F|G\rangle$ be functors and let $\pi : Q \to K$ be such that $G\pi \circ q = k \circ F\pi$. Then there is a unique natural transformation $\widetilde{\pi} : Q[q] \to K[k]$ such that $P\widetilde{\pi} = \pi$. Clearly any natural transformation $\nu : Q[q] \to K[k]$ is of the form $\widetilde{\pi}$ for $\pi = P\nu$.

3) Let $\phi : F' \to F$ and $\gamma : G \to G'$. Then $\langle \phi|\gamma\rangle \circ Q[q] = Q[\gamma Q \circ q \circ \phi Q]$.

Proof. 1) For every $X \in \mathcal{Q}$ define $Q[q]X := (QX, qX : FQX \to GQX) \in \langle F|G\rangle$. Given $f : X \to Y$ in $\mathcal{Q}$, by naturality of q we have $qY \circ FQf = GQf \circ qX$ so that f induces a morphism $Q[q]f : Q[q]X \to Q[q]Y$ such that $PQ[q]f = Qf$. Thus the functor $Q[q] : \mathcal{Q} \to \langle F|G\rangle$ is defined. Moreover $\psi Q[q]X = \psi(QX, qX) = qX$ so that $\psi Q[q] = q$. Let us check that $Q[q]$ is unique. Given a functor $N : \mathcal{Q} \to \langle F|G\rangle$ such that $PN = Q$ and $\psi N = q$ we have that $PNX = QX$ so that $NX = (QX, \alpha)$ for some α. Moreover $qX = \psi NX = \psi(QX, \alpha) = \alpha$ and hence $NX = (QX, qX) = Q[q]X$. Moreover, given $f : X \to Y$ in $\mathcal{Q}$, we have $PNf = Qf = PQ[q]f$. Since P is faithful, we deduce $Nf = Q[q]f$ and hence $W = Q[q]$.

2) For $X \in \mathcal{Q}$ we have $G\pi X \circ qX = kX \circ F\pi X$. Since $Q[q]X := (QX, qX)$ and $K[k]X := (KX, kX)$, we get that πX induces $\widetilde{\pi}X : Q[q]X \to K[k]X$ such that $P\widetilde{\pi}X = \pi X$. The naturality of πX induces the one of $\widetilde{\pi}X$ so that we get $\widetilde{\pi} : Q[q] \to K[k]$ such that $P\widetilde{\pi} = \pi$.

3) First we have

$$\begin{aligned} P_{\langle F'|G'\rangle} \circ \langle \phi|\gamma\rangle \circ Q[q] &= P_{\langle F|G\rangle} \circ Q[q] = Q \\ \psi_{\langle F'|G'\rangle}(\langle \phi|\gamma\rangle \circ Q[q]) &= \left(\gamma P_{\langle F|G\rangle} \circ \psi \circ \phi P_{\langle F|G\rangle}\right) Q[q] \\ &= \gamma P_{\langle F|G\rangle} Q[q] \circ \psi Q[q] \circ \phi P_{\langle F|G\rangle} Q[q] = \gamma Q \circ q \circ \phi Q \end{aligned}$$

so that $\langle \phi|\gamma\rangle \circ Q[q] = Q[\gamma Q \circ q \circ \phi Q]$. □

Proposition 1.7. *Consider the forgetful functor $P = P_{\langle F|G\rangle} : \langle F|G\rangle \to \mathcal{A}$. Let $f : (A, a) \to (C, c)$ and $g : (B, b) \to (C, c)$ morphisms in $\langle F|G\rangle$ and let $h : A \to B$ be a morphism in $\mathcal{A}$ such that $Pf = Pg \circ h$. If GPg is a monomorphism, then there is a (unique) morphism $h' : (A, a) \to (B, b)$ such that $Ph' = h$ and $f = g \circ h'$.*

Proof. Consider the following diagram

$$\begin{array}{ccccc} FA & \xrightarrow{Fh} & FB & \xrightarrow{FPg} & FC \\ \downarrow a & & \downarrow b & & \downarrow c \\ GA & \xrightarrow{Gh} & GB & \xrightarrow{GPg} & GC \end{array}$$

Since f is a morphism in $\langle F|G\rangle$, then the external diagram commutes, and since g is a morphism in $\langle F|G\rangle$, so does the right square. Using that GPg is a monomorphism, we deduce that the left square commutes as well i.e. h induces a morphism $h' : (A, a) \to (B, b)$ such that $Ph' = h$. Now $Pf = Pg \circ h = P(g \circ h')$ and P is faithful imply $f = g \circ h'$. Since P is faithful, h' is unique. □

Example 1.8. Recall from [Pi, Definition 2.2.2], that given an endofunctor $F : \mathcal{A} \to \mathcal{A}$, the category of F**-algebras** (not to be confused with an Eilenberg-Moore algebra) is $F\text{-Alg} = \langle F|\mathrm{Id}_{\mathcal{A}}\rangle$.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors and let $\epsilon : F \to G$ be a natural transformation. If $\mathcal{B}$ has coequalizers we can define the functor

$$\mathcal{U}(\epsilon) : \langle F|G\rangle \to \mathcal{B} \tag{3}$$

by the following coequalizer of natural transformations

$$FP \underset{\epsilon P}{\overset{\psi}{\rightrightarrows}} GP \xrightarrow{\pi := \pi(\epsilon)} \mathcal{U}(\epsilon) \tag{4}$$

2. Adjoint Triangles and Adjoint Decomposition

In this section we construct iteratively the category $\mathcal{B}_{[n]}$ and an analogue of the monadic decomposition that will be called the adjoint decomposition. Our first aim is to obtain an analogue of the Eilenberg-Moore category. For this purpose we will use the notion of adjoint triangle.

Definition 2.1. [Du, Definition 1] By an *adjoint triangle*, we mean a diagram of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & \zeta & {\scriptstyle L'}\uparrow\downarrow{\scriptstyle R'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \tag{5}$$

where (L, R, η, ϵ) and $(L', R', \eta', \epsilon')$ are adjunctions and $GR = R'$.

The letter ζ inserted in (5) is the unique natural transformation $\zeta : L'G \to L$ such that $\epsilon \circ \zeta R = \epsilon'$ namely $\zeta := \epsilon' L \circ L'G\eta$. The convention to write this natural transformation inside the respective adjoint triangle will be useful to state our results along the paper. It is easy to check that

$$R'\zeta \circ \eta' G = G\eta. \tag{6}$$

Note that diagram (5) has been drawn as a square to make it more readable, although the two copies of $\mathcal{A}$ on the top can be glued together to give rise, in fact, to a triangle.

Remark 2.2. As a particular case of horizontal composition of adjoint squares (see [Gra, I,6.8]), we can define the (horizontal) composition $\mathbb{T}'' := \mathbb{T}' * \mathbb{T}$ of two adjoint triangles $\mathbb{T}'$ and $\mathbb{T}$ by

$$\mathbb{T}' := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L''}\uparrow\downarrow{\scriptstyle R''} & \theta & {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} \\ \mathcal{B}'' & \xrightarrow{\Theta} & \mathcal{B} \end{array} \qquad \mathbb{T} := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & \zeta & {\scriptstyle L'}\uparrow\downarrow{\scriptstyle R'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \qquad \mathbb{T}'' := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L''}\uparrow\downarrow{\scriptstyle R''} & \zeta''=\theta*\zeta & {\scriptstyle L'}\uparrow\downarrow{\scriptstyle R'} \\ \mathcal{B}'' & \xrightarrow{G''=G\Theta} & \mathcal{B}' \end{array}$$

where $\theta * \zeta := \left(L'G\Theta \overset{\zeta\Theta}{\to} L\Theta \overset{\theta}{\to} L'' \right)$. In fact $(G\Theta)\, R'' = GR = R'$ and

$$\epsilon'' \circ (\theta * \zeta)\, R'' = \epsilon'' \circ \theta R'' \circ \zeta\Theta R'' = \epsilon \circ \zeta R = \epsilon'.$$

To any adjoint triangle $\mathbb{T}$ as in (5) we would like to associate a new adjoint triangle $\mathbb{T}^2$ as follows

$$\mathbb{T} := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & \zeta & {\scriptstyle L'}\uparrow\downarrow{\scriptstyle R'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \quad \rightsquigarrow \quad \mathbb{T}^2 := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L(\mathbb{T})}\uparrow\downarrow{\scriptstyle R(\mathbb{T})} & & {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} \\ \mathrm{I}(\mathbb{T}) & \xrightarrow{P(\mathbb{T})} & \mathcal{B} \end{array} \tag{7}$$

First we have to introduce the category

$$\mathrm{I}(\mathbb{T}) := \langle R'L|G \rangle.$$

For any category $\mathcal{A}$ we can consider the functor

$$D : \mathcal{A} \to \langle \mathrm{Id}_{\mathcal{A}} | \mathrm{Id}_{\mathcal{A}} \rangle, \qquad A \mapsto (A, \mathrm{Id}_A), \qquad h \mapsto h.$$

If $F, G : \mathcal{A} \to \mathcal{B}$, $H : \mathcal{B} \to \mathcal{B}'$ and $K : \mathcal{A}' \to \mathcal{A}$ are functors, we can define

$$S^H : \langle F|G \rangle \to \langle HF|HG \rangle, \qquad (A, \alpha_A : FA \to GA) \mapsto (A, H\alpha_A : HFA \to HGA), \qquad f \mapsto f$$
$$D^K : \langle FK|GK \rangle \to \langle F|G \rangle, \quad (A', \alpha_{A'} : FKA' \to GKA') \mapsto (KA', \alpha_{A'} : FKA' \to GKA'), \quad f \mapsto Kf.$$

Given $\epsilon : F \to G$ we define the functor

$$\mathcal{S}(\epsilon) := \left(\mathcal{A} \overset{D}{\to} \langle \mathrm{Id}_{\mathcal{A}} | \mathrm{Id}_{\mathcal{A}} \rangle \overset{S^G}{\to} \langle G|G \rangle \overset{\langle \epsilon | \mathrm{Id}_{\mathcal{A}} \rangle}{\to} \langle F|G \rangle \right).$$

Explicitly, by the notation of Lemma 1.6, we have

$$\mathcal{S}(\epsilon) = \mathrm{Id}_{\mathcal{A}}[\epsilon] : \mathcal{A} \to \langle F|G \rangle, \qquad A \mapsto (A, \epsilon A), \qquad f \mapsto f.$$

Let us show how $\mathcal{S}(\epsilon)$ relates to the functor $\mathcal{U}(\epsilon)$ of (3) in the particular case when $G = \mathrm{Id}_{\mathcal{A}}$.

Lemma 2.3. *Let $F : \mathcal{A} \to \mathcal{A}$ be a functor and let $\epsilon : F \to \mathrm{Id}_{\mathcal{A}}$ be a natural transformation. Assume $\mathcal{A}$ has coequalizers. Then $\pi(\epsilon)\, \mathcal{S}(\epsilon)$ is invertible and $\left(\mathcal{U}(\epsilon), \mathcal{S}(\epsilon), \eta(\epsilon), (\pi(\epsilon)\, \mathcal{S}(\epsilon))^{-1} \right)$ is an adjunction where $\eta(\epsilon) : \mathrm{Id}_{\langle F | \mathrm{Id}_{\mathcal{A}} \rangle} \to \mathcal{S}(\epsilon)\, \mathcal{U}(\epsilon)$ is uniquely determined by $P\eta(\epsilon) = \pi(\epsilon)$.*

Proof. Set $\pi := \pi(\epsilon)$. Note that $\psi\mathcal{S}(\epsilon) = \epsilon$ and $P \circ \mathcal{S}(\epsilon) = \mathrm{Id}_{\mathcal{A}}$. Thus, if we evaluate the left-hand side coequalizer below on $\mathcal{S}(\epsilon)$, we obtain the right-hand side one.

$$FP \underset{\epsilon P}{\overset{\psi}{\rightrightarrows}} P \xrightarrow{\pi} \mathcal{U}(\epsilon) \qquad F \underset{\epsilon}{\overset{\epsilon}{\rightrightarrows}} \mathrm{Id}_{\mathcal{A}} \xrightarrow{\pi\mathcal{S}(\epsilon)} \mathcal{U}(\epsilon)\mathcal{S}(\epsilon)$$

This means $\pi\mathcal{S}(\epsilon)$ is invertible. Let us check that there is $\eta(\epsilon) : \mathrm{Id}_{\langle F|\mathrm{Id}_{\mathcal{A}}\rangle} \to \mathcal{S}(\epsilon)\mathcal{U}(\epsilon)$ such that $P\eta(\epsilon) = \pi$. We have

$$\pi \circ \psi = \pi \circ \epsilon P = \epsilon\mathcal{U}(\epsilon) \circ F\pi.$$

Since $\mathrm{Id}_{\langle F|\mathrm{Id}_{\mathcal{A}}\rangle} = P[\psi]$ and $\mathcal{S}(\epsilon)\mathcal{U}(\epsilon) = \mathcal{U}(\epsilon)[\psi\mathcal{S}(\epsilon)\mathcal{U}(\epsilon)] = \mathcal{U}(\epsilon)[\epsilon\mathcal{U}(\epsilon)]$, by Lemma 1.6 there is a unique natural transformation $\eta(\epsilon) : \mathrm{Id}_{\langle F|\mathrm{Id}_{\mathcal{A}}\rangle} \to \mathcal{S}(\epsilon)\mathcal{U}(\epsilon)$ such that $P\eta(\epsilon) = \pi$. We have

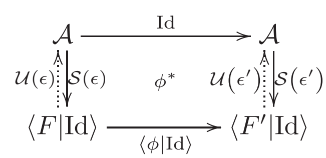
$$P\eta(\epsilon)\mathcal{S}(\epsilon) = \pi\mathcal{S}(\epsilon) = P\mathcal{S}(\epsilon)\pi\mathcal{S}(\epsilon)$$

so that $\eta(\epsilon)\mathcal{S}(\epsilon) = \mathcal{S}(\epsilon)\pi\mathcal{S}(\epsilon)$. Moreover

$$\mathcal{U}(\epsilon)\eta(\epsilon) \circ \pi = \pi\mathcal{S}(\epsilon)\mathcal{U}(\epsilon) \circ P\eta(\epsilon) = \pi\mathcal{S}(\epsilon)\mathcal{U}(\epsilon) \circ \pi$$

and hence $\mathcal{U}(\epsilon)\eta(\epsilon) = \pi\mathcal{S}(\epsilon)\mathcal{U}(\epsilon)$. Therefore $\left(\mathcal{U}(\epsilon), \mathcal{S}(\epsilon), \eta(\epsilon), (\pi(\epsilon)\mathcal{S}(\epsilon))^{-1}\right)$ is an adjunction. □

Lemma 2.4. *Let $F, F' : \mathcal{A} \to \mathcal{A}$ be functors. Let $\epsilon : F \to \mathrm{Id}_{\mathcal{A}}$ and $\phi : F' \to F$ be natural transformations. Set $\epsilon' := \epsilon \circ \phi : F' \to \mathrm{Id}_{\mathcal{A}}$. If $\mathcal{A}$ has coequalizers, we have the adjoint triangle*



Moreover the natural transformation $\phi^* : \mathcal{U}(\epsilon') \circ \langle\phi|\mathrm{Id}\rangle \to \mathcal{U}(\epsilon)$ satisfies

$$\phi^* \circ \pi(\epsilon')\langle\phi|\mathrm{Id}\rangle = \pi(\epsilon)$$

and $\phi^*\mathcal{S}(\epsilon) \circ \pi'\mathcal{S}(\epsilon') = \pi\mathcal{S}(\epsilon)$. In particular $\phi^*\mathcal{S}(\epsilon)$ is invertible.

Proof. By Lemma 2.3, we have that $\mathcal{U}(\epsilon) \dashv \mathcal{S}(\epsilon)$ and $\mathcal{U}(\epsilon') \dashv \mathcal{S}(\epsilon')$.

Set $P := P_{\langle F|\mathrm{Id}_{\mathcal{A}}\rangle}$ and $P' := P_{\langle F'|\mathrm{Id}_{\mathcal{A}}\rangle}$. Set also $\pi = \pi(\epsilon)$ and $\pi' = \pi(\epsilon')$. By Lemma 1.6, we have

$$\langle\phi|\mathrm{Id}\rangle \circ \mathcal{S}(\epsilon) = \langle\phi|\mathrm{Id}\rangle \circ \mathrm{Id}_{\mathcal{A}}[\epsilon] = \mathrm{Id}_{\mathcal{A}}[\epsilon \circ \phi] = \mathrm{Id}_{\mathcal{A}}[\epsilon'] = \mathcal{S}(\epsilon').$$

so that $\langle\phi|\mathrm{Id}\rangle \circ \mathcal{S}(\epsilon) = \mathcal{S}(\epsilon')$ and hence the diagram in the statement is an adjoint triangle.

By definition $\phi^* = (\pi(\epsilon')\mathcal{S}(\epsilon')\mathcal{U}(\epsilon))^{-1} \circ \mathcal{U}(\epsilon')\langle\phi|\mathrm{Id}\rangle\eta(\epsilon)$. Then

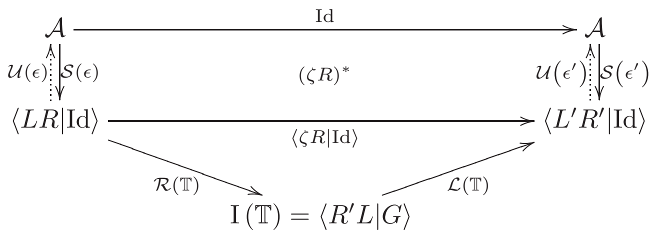
$$\begin{aligned} \phi^* \circ \pi'\langle\phi|\mathrm{Id}\rangle &= (\pi'\mathcal{S}(\epsilon')\mathcal{U}(\epsilon))^{-1} \circ \mathcal{U}(\epsilon')\langle\phi|\mathrm{Id}\rangle\eta(\epsilon) \circ \pi'\langle\phi|\mathrm{Id}\rangle \\ &= (\pi'\mathcal{S}(\epsilon')\mathcal{U}(\epsilon))^{-1} \circ \pi'\langle\phi|\mathrm{Id}\rangle\mathcal{S}(\epsilon)\mathcal{U}(\epsilon) \circ P'\langle\phi|\mathrm{Id}\rangle\eta(\epsilon) \\ &= (\pi'\mathcal{S}(\epsilon')\mathcal{U}(\epsilon))^{-1} \circ \pi'\mathcal{S}(\epsilon')\mathcal{U}(\epsilon) \circ P\eta(\epsilon) = \pi. \end{aligned}$$

Moreover

$$\phi^*\mathcal{S}(\epsilon) \circ \pi'\mathcal{S}(\epsilon') = \phi^*\mathcal{S}(\epsilon) \circ \pi'\langle\phi|\mathrm{Id}\rangle\mathcal{S}(\epsilon) = (\phi^* \circ \pi'\langle\phi|\mathrm{Id}\rangle)\mathcal{S}(\epsilon) = \pi\mathcal{S}(\epsilon).$$

Since $\pi'\mathcal{S}(\epsilon')$ and $\pi\mathcal{S}(\epsilon)$ are invertible, then so is $\phi^*\mathcal{S}(\epsilon)$. □

Given an adjoint triangle as in (5), assume $\mathcal{A}$ has coequalizers and consider the following diagram where we apply Lemma 2.4 to the counit $\epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$ and $\phi := \zeta R : L'R' \to LR$ to get the adjoint triangle with $(\zeta R)^*$ in the middle.



The functors $\mathcal{L}(\mathbb{T})$ and $\mathcal{R}(\mathbb{T})$ appearing in the diagram above are defined as follows

$$\begin{array}{rcl}\mathcal{L}(\mathbb{T}) = LP_{\mathrm{I}(\mathbb{T})}\left[\zeta P_{\mathrm{I}(\mathbb{T})} \circ L'\psi_{\mathrm{I}(\mathbb{T})}\right] & : & \mathrm{I}(\mathbb{T}) := \langle R'L|G\rangle \overset{S^{L'}}{\to} \langle L'R'L|L'G\rangle \overset{\langle \mathrm{Id}_{L'R'L}|\zeta\rangle}{\to} \langle L'R'L|L\rangle \overset{D^L}{\to} \langle L'R'|\mathrm{Id}_{\mathcal{A}}\rangle, \\ (B,b) & \mapsto & (LB, \alpha(B,b) := \zeta B \circ L'b), \qquad h \mapsto Lh\end{array}$$

$$\begin{array}{c}\mathcal{R}(\mathbb{T}) = RP_{\langle LR|\mathrm{Id}_{\mathcal{A}}\rangle}\left[R'\psi_{\langle LR|\mathrm{Id}_{\mathcal{A}}\rangle}\right] \quad : \quad \langle LR|\mathrm{Id}_{\mathcal{A}}\rangle \overset{S^{R'}}{\to} \langle R'LR|R'\rangle = \langle R'LR|GR\rangle \overset{D^R}{\to} \langle R'L|G\rangle = \mathrm{I}(\mathbb{T}), \\ (A, \alpha_A : LRA \to A) \mapsto (RA, R'\alpha_A : R'LRA \to R'A = GRA), \qquad f \mapsto Rf.\end{array}$$

Set

$$\begin{array}{rl}\mathrm{R}(\mathbb{T}) = R[R'\epsilon] = \mathcal{R}(\mathbb{T}) \circ \mathcal{S}(\epsilon): & \mathcal{A} \to \mathrm{I}(\mathbb{T}), \quad A \mapsto (RA, R'\epsilon A), \quad f \mapsto Rf, \\ \mathrm{L}(\mathbb{T}) = \mathcal{U}(\epsilon') \circ \mathcal{L}(\mathbb{T}): & \mathrm{I}(\mathbb{T}) \to \mathcal{A}, \quad (B,b) \mapsto \mathcal{U}(\epsilon')(LB, \zeta B \circ L'b), \quad h \mapsto \mathcal{U}(\epsilon')(Lh).\end{array}$$

Consider also the forgetful functor

$$P(\mathbb{T}) = P_{\mathrm{I}(\mathbb{T})} : \mathrm{I}(\mathbb{T}) \to \mathcal{B}, \quad (B,b) \mapsto B, \quad f \mapsto f$$

and the functor

$$G(\mathbb{T}) := G \circ P(\mathbb{T}), \quad \mathrm{I}(\mathbb{T}) \to \mathcal{B}' : (B,b) \mapsto GB, \quad h \mapsto Gh.$$

Note that,if we set $P' := P_{\langle L'R'|\mathrm{Id}_{\mathcal{A}}\rangle}$, we get

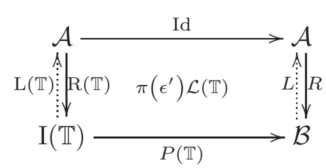
$$P(\mathbb{T})\,\mathrm{R}(\mathbb{T}) = R, \tag{8}$$

$$P'\mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T}) = LR \tag{9}$$

We are now ready to construct the adjoint triangle $\mathbb{T}^2$ announced in (7).

Proposition 2.5. *Assume $\mathcal{A}$ has coequalizers. Given an adjoint triangle $\mathbb{T}$ as in (5), then*

$$\mathbb{T}^2 :=$$



is an adjoint triangle where the adjunction $(\mathrm{L}(\mathbb{T}), \mathrm{R}(\mathbb{T}), \eta(\mathbb{T}), \epsilon(\mathbb{T}))$ *is uniquely determined by the following equalities*

$$P(\mathbb{T})\,\eta(\mathbb{T}) = R\pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \eta P(\mathbb{T}), \tag{10}$$

$$\epsilon(\mathbb{T}) \circ \pi(\epsilon')\,\mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T}) = \epsilon. \tag{11}$$

Proof. Set $\psi' := \psi_{\langle L'R'|\mathrm{Id}_{\mathcal{A}}\rangle}$ and $\pi' := \pi(\epsilon')$ and let us construct $\epsilon(\mathbb{T})$.

One easily checks that $\mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T}) = (LR)[\zeta R \circ L'R'\epsilon]$. Moreover $\mathcal{S}(\epsilon') = \mathrm{Id}_{\mathcal{A}}[\epsilon']$.

Since $\epsilon \circ (\zeta R \circ L'R'\epsilon) = \epsilon' \circ L'R'\epsilon$, by Lemma 1.6, the counit $\epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$ induces $\widetilde{\epsilon} : \mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T}) \to \mathcal{S}(\epsilon')$ such that $P'\widetilde{\epsilon} = \epsilon$. Define

$$\epsilon(\mathbb{T}) := (\pi(\epsilon')\,\mathcal{S}(\epsilon'))^{-1} \circ \mathcal{U}(\epsilon')\,\widetilde{\epsilon} : \mathrm{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T}) \to \mathrm{Id}_{\mathcal{A}}.$$

Now we define $\eta(\mathbb{T}) : \mathrm{Id}_{\mathrm{I}(\mathbb{T})} \to \mathrm{R}(\mathbb{T})\,\mathrm{L}(\mathbb{T})$. One easily checks that $\mathrm{R}(\mathbb{T})\,\mathrm{L}(\mathbb{T}) = (R\mathrm{L}(\mathbb{T}))\,[R'\epsilon\mathrm{L}(\mathbb{T})]$ and $\mathrm{Id}_{\mathrm{I}(\mathbb{T})} = P(\mathbb{T})\left[\psi_{\mathrm{I}(\mathbb{T})}\right]$. Set

$$\alpha := \left(P(\mathbb{T}) \xrightarrow{\eta P(\mathbb{T})} RLP(\mathbb{T}) = RP'\mathcal{L}(\mathbb{T}) \xrightarrow{R\pi\left(\epsilon'\right)\mathcal{L}(\mathbb{T})} R\mathcal{U}(\epsilon')\mathcal{L}(\mathbb{T}) = R\mathrm{L}(\mathbb{T})\right).$$

We compute

$$\begin{aligned}
\epsilon'\mathrm{L}(\mathbb{T}) \circ L'\left(G\alpha \circ \psi_{\mathrm{I}(\mathbb{T})}\right) &= \epsilon'\mathrm{L}(\mathbb{T}) \circ L'G\alpha \circ L'\psi_{\mathrm{I}(\mathbb{T})} \\
&= \epsilon'\mathrm{L}(\mathbb{T}) \circ L'GR\pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ L'G\eta P(\mathbb{T}) \circ L'\psi_{\mathrm{I}(\mathbb{T})} \\
&= \epsilon'\mathrm{L}(\mathbb{T}) \circ L'R'\pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ L'G\eta P(\mathbb{T}) \circ L'\psi_{\mathrm{I}(\mathbb{T})} \\
&= \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \epsilon' LP(\mathbb{T}) \circ L'G\eta P(\mathbb{T}) \circ L'\psi_{\mathrm{I}(\mathbb{T})} \\
&= \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \zeta P(\mathbb{T}) \circ L'\psi_{\mathrm{I}(\mathbb{T})} \\
&\overset{(*)}{=} \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \psi'\mathcal{L}(\mathbb{T}) \\
&\overset{\mathrm{def.}\pi\left(\epsilon'\right)}{=} \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \epsilon' P'\mathcal{L}(\mathbb{T}) \\
&= \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \epsilon' LP(\mathbb{T}) \\
&= \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \epsilon LP(\mathbb{T}) \circ L\eta P(\mathbb{T}) \circ \epsilon' LP(\mathbb{T}) \\
&= \epsilon\mathrm{L}(\mathbb{T}) \circ LR\pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ L\eta P(\mathbb{T}) \circ \epsilon' LP(\mathbb{T}) \\
&= \epsilon\mathrm{L}(\mathbb{T}) \circ L\alpha \circ \epsilon' LP(\mathbb{T}) \\
&= \epsilon'\mathrm{L}(\mathbb{T}) \circ L'R'(\epsilon\mathrm{L}(\mathbb{T}) \circ L\alpha) = \epsilon'\mathrm{L}(\mathbb{T}) \circ L'(R'\epsilon\mathrm{L}(\mathbb{T}) \circ R'L\alpha)
\end{aligned}$$

where (*) follows by the equality $\zeta P(\mathbb{T}) \circ L'\psi_{\mathrm{I}(\mathbb{T})} = \psi'\mathcal{L}(\mathbb{T})$ that can be easily checked.

We have so proved that $\epsilon'\mathrm{L}(\mathbb{T}) \circ L'\left(G\alpha \circ \psi_{\mathrm{I}(\mathbb{T})}\right) = \epsilon'\mathrm{L}(\mathbb{T}) \circ L'(R'\epsilon\mathrm{L}(\mathbb{T}) \circ R'L\alpha)$. By the adjunction this is equivalent to $G\alpha \circ \psi_{\mathrm{I}(\mathbb{T})} = R'\epsilon\mathrm{L}(\mathbb{T}) \circ R'L\alpha$. By Lemma 1.6, the map α induces $\eta(\mathbb{T}) : \mathrm{Id}_{\mathrm{I}(\mathbb{T})} \to \mathrm{R}(\mathbb{T})\,\mathrm{L}(\mathbb{T})$ such that $P(\mathbb{T})\,\eta(\mathbb{T}) = \alpha$. We compute

$$\begin{aligned}
& \epsilon(\mathbb{T})\,\mathrm{L}(\mathbb{T}) \circ \mathrm{L}(\mathbb{T})\,\eta(\mathbb{T}) \circ \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \\
= \quad & (\pi(\epsilon')\,\mathcal{S}(\epsilon')\,\mathrm{L}(\mathbb{T}))^{-1} \circ \mathcal{U}(\epsilon')\,\widetilde{\epsilon}\mathrm{L}(\mathbb{T}) \circ \mathcal{U}(\epsilon')\,\mathcal{L}(\mathbb{T})\,\eta(\mathbb{T}) \circ \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \\
\overset{\mathrm{nat.}\pi\left(\epsilon'\right)}{=} \quad & (\pi(\epsilon')\,\mathcal{S}(\epsilon')\,\mathrm{L}(\mathbb{T}))^{-1} \circ \pi(\epsilon')\,\mathcal{S}(\epsilon')\,\mathrm{L}(\mathbb{T}) \circ P'\widetilde{\epsilon}\mathrm{L}(\mathbb{T}) \circ P'\mathcal{L}(\mathbb{T})\,\eta(\mathbb{T}) \\
= \quad & P'\widetilde{\epsilon}\mathrm{L}(\mathbb{T}) \circ LP(\mathbb{T})\,\eta(\mathbb{T}) = \epsilon\mathrm{L}(\mathbb{T}) \circ L\alpha \\
= \quad & \epsilon\mathrm{L}(\mathbb{T}) \circ LR\pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ L\eta P(\mathbb{T}) \\
= \quad & \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \epsilon LP(\mathbb{T}) \circ L\eta P(\mathbb{T}) = \pi(\epsilon')\,\mathcal{L}(\mathbb{T})
\end{aligned}$$

so that $\epsilon(\mathbb{T})\,\mathrm{L}(\mathbb{T}) \circ \mathrm{L}(\mathbb{T})\,\eta(\mathbb{T}) = \mathrm{Id}_{\mathrm{L}(\mathbb{T})}$.

We compute

$$\begin{aligned}
P(\mathbb{T})\,(\mathrm{R}(\mathbb{T})\,\epsilon(\mathbb{T}) \circ \eta(\mathbb{T})\,\mathrm{R}(\mathbb{T})) &= P(\mathbb{T})\,\mathrm{R}(\mathbb{T})\,\epsilon(\mathbb{T}) \circ P(\mathbb{T})\,\eta(\mathbb{T})\,\mathrm{R}(\mathbb{T}) \\
&= R\epsilon(\mathbb{T}) \circ \alpha\mathrm{R}(\mathbb{T}) \\
&= (R\pi(\epsilon')\,\mathcal{S}(\epsilon'))^{-1} \circ R\mathcal{U}(\epsilon')\,\widetilde{\epsilon} \circ R\pi(\epsilon')\,\mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T}) \circ \eta P(\mathbb{T})\,\mathrm{R}(\mathbb{T}) \\
&= (R\pi(\epsilon')\,\mathcal{S}(\epsilon'))^{-1} \circ R\pi(\epsilon')\,\mathcal{S}(\epsilon') \circ RP'\widetilde{\epsilon} \circ \eta R \\
&= R\epsilon \circ \eta R = \mathrm{Id}_R.
\end{aligned}$$

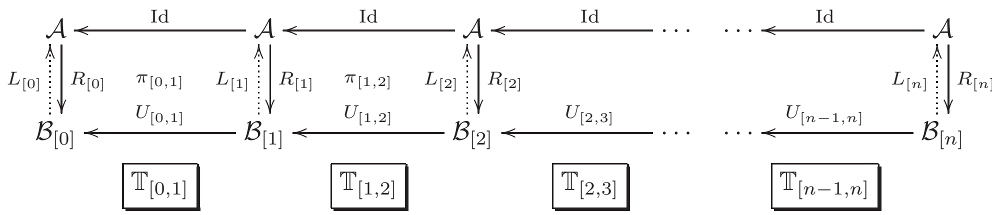
We have so proved that $(\mathrm{L}(\mathbb{T}), \mathrm{R}(\mathbb{T}), \eta(\mathbb{T}), \epsilon(\mathbb{T}))$ is an adjunction. We compute

$$\begin{aligned}
\epsilon\mathrm{L}(\mathbb{T}) \circ LP(\mathbb{T})\,\eta(\mathbb{T}) &= \epsilon\mathrm{L}(\mathbb{T}) \circ L\alpha \\
&= \epsilon\mathrm{L}(\mathbb{T}) \circ LR\pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ L\eta P(\mathbb{T}) \\
&= \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) \circ \epsilon LP(\mathbb{T}) \circ L\eta P(\mathbb{T}) = \pi(\epsilon')\,\mathcal{L}(\mathbb{T}).
\end{aligned}$$

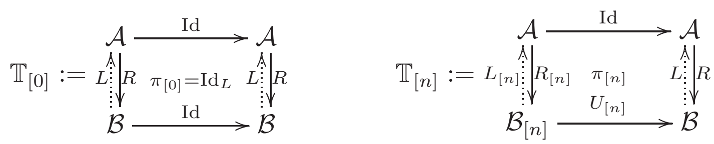
Thus, since $P(\mathbb{T})\,\mathrm{R}(\mathbb{T}) = R$, the diagram in the statement is an adjoint triangle and the natural transformation inside it is the correct one. Note that (11) follows by definition of adjoint triangle and, since $\pi(\epsilon')\,\mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T})$ is an epimorphism, it uniquely determines $\epsilon(\mathbb{T})$. □

Starting from an adjunction (L, R, η, ϵ), with $R : \mathcal{A} \to \mathcal{B}$ and where $\mathcal{A}$ has all coequalizers, we are now able to construct a kind of monadic decomposition that will be called an **adjoint decomposition** as follows, where we set $L_{[0]} := L$, $R_{[0]} := R$, $\eta_{[0]} := \eta$, $\epsilon_{[0]} := \epsilon$ and $\mathcal{B}_{[0]} := \mathcal{B}$.

(12)



In the diagram above we label by $\mathbb{T}_{[0,1]}$ the first adjoint triangle from left, by $\mathbb{T}_{[1,2]}$ the second one and in general by $\mathbb{T}_{[n-1,n]}$ the n-th one. Denote by $\mathbb{T}_{[n]}$ the composition of the first n adjoint triangles. They are constructed iteratively as follows. The adjoint triangle $\mathbb{T}_{[0]}$ is defined as in the following diagram while, for $n > 0$, we set $\mathbb{T}_{[n-1,n]} := \left(\mathbb{T}_{[n-1]}\right)^2$ (see Proposition 2.5) and $\mathbb{T}_{[n]} := \mathbb{T}_{[n-1,n]} * \mathbb{T}_{[n-1]}$.



Explicitly $\mathcal{B}_{[1]} = \mathrm{I}\left(\mathbb{T}_{[0]}\right) = \langle R'L|G\rangle = \langle RL|\mathrm{Id}_{\mathcal{B}}\rangle$, $U_{[1]} = U_{[0,1]} = P\left(\mathbb{T}_{[0]}\right)$ the forgetful functor.

$$\begin{array}{rcl} \mathcal{L}\left(\mathbb{T}_{[0]}\right) & : & \mathcal{B}_{[1]} \to \langle LR|\mathrm{Id}_{\mathcal{A}}\rangle, \quad (B,b) \mapsto (LB, \zeta B \circ Lb) = (LB, Lb), \qquad h \mapsto Lh, \\ R_{[1]} & = & \mathrm{R}\left(\mathbb{T}_{[0]}\right) = \mathcal{R}\left(\mathbb{T}_{[0]}\right) \circ \mathcal{S}(\epsilon) : \mathcal{A} \to \mathrm{I}\left(\mathbb{T}_{[0]}\right), \quad A \mapsto (RA, R\epsilon A), \quad f \mapsto Rf, \\ L_{[1]} & = & \mathrm{L}\left(\mathbb{T}_{[0]}\right) = \mathcal{U}(\epsilon) \circ \mathcal{L}\left(\mathbb{T}_{[0]}\right) : \mathrm{I}\left(\mathbb{T}_{[0]}\right) \to \mathcal{A}, \quad (B,b) \mapsto \mathcal{U}(\epsilon)(LB, Lb), \quad h \mapsto \mathcal{U}(\epsilon)(Lh), \\ \pi_{[1]} & = & \pi_{[0,1]} = \pi(\epsilon)\,\mathcal{L}\left(\mathbb{T}_{[0]}\right). \end{array}$$

The unit $\eta_{[1]} = \eta\left(\mathbb{T}_{[0]}\right)$ and the counit $\epsilon_{[1]} = \epsilon\left(\mathbb{T}_{[0]}\right)$ of the adjunction $\left(L_{[1]}, R_{[1]}\right)$ are uniquely defined by

$$U_{[1]}\eta_{[1]} = R\pi_{[1]} \circ \eta U_{[1]} \qquad \text{and} \qquad \epsilon_{[1]} \circ \pi_{[1]}R_{[1]} = \epsilon = \epsilon_{[0]}. \tag{13}$$

Note that for every $B_{[1]} := (B, b) \in \mathcal{B}_{[1]}$ we have the following coequalizer

$$LRLB \underset{\epsilon LB}{\overset{Lb}{\rightrightarrows}} LB \xrightarrow{\pi_{[1]}B_{[1]}} L_{[1]}B_{[1]} \tag{14}$$

Next $\mathcal{B}_{[2]} = \mathrm{I}\left(\mathbb{T}_{[1]}\right) = \left\langle RL_{[1]}|U_{[1]}\right\rangle$, $U_{[1,2]} := P\left(\mathbb{T}_{[1]}\right)$ and $U_{[2]} = U_{[0,1]} \circ U_{[1,2]}$. Moreover

$$\begin{array}{rcl} \mathcal{L}\left(\mathbb{T}_{[1]}\right) & : & \mathcal{B}_{[2]} \to \langle LR|\mathrm{Id}_{\mathcal{A}}\rangle : \quad \left(B_{[1]}, b_{[1]}\right) \mapsto \left(L_{[1]}B_{[1]}, \pi_{[1]}B_{[1]} \circ Lb_{[1]}\right), \quad h \mapsto L_{[1]}h, \\ R_{[2]} & = & \mathrm{R}\left(\mathbb{T}_{[1]}\right) = \mathcal{R}\left(\mathbb{T}_{[1]}\right) \circ \mathcal{S}\left(\epsilon_{[1]}\right) : \mathcal{A} \to \mathcal{B}_{[2]}, \quad A \mapsto \left(R_{[1]}A, R\epsilon_{[1]}A\right), \quad f \mapsto R_{[1]}f, \\ L_{[2]} & = & \mathrm{L}\left(\mathbb{T}_{[1]}\right) = \mathcal{U}(\epsilon) \circ \mathcal{L}\left(\mathbb{T}_{[1]}\right) : \mathcal{B}_{[2]} \to \mathcal{A}, \\ & & \left(B_{[1]}, b_{[1]}\right) \mapsto \mathcal{U}(\epsilon)\left(L_{[1]}B_{[1]}, \pi_{[1]}B_{[1]} \circ Lb_{[1]}\right), \quad h \mapsto \mathcal{U}(\epsilon)\left(L_{[1]}h\right), \\ \pi_{[1,2]} & = & \pi(\epsilon)\,\mathcal{L}\left(\mathbb{T}_{[1]}\right) \\ \pi_{[2]} & = & \pi_{[1,2]} * \pi_{[1,0]} = \pi_{[1,2]} \circ \pi_{[1]}U_{[1,2]} \end{array}$$

The unit $\eta_{[2]} = \eta\left(\mathbb{T}_{[1]}\right)$ and the counit $\epsilon_{[2]} = \epsilon\left(\mathbb{T}_{[1]}\right)$ of the adjunction $\left(L_{[2]}, R_{[2]}\right)$ are uniquely determined by

$$U_{[1,2]}\eta_{[2]} = R_{[1]}\pi_{[1,2]} \circ \eta_{[1]}U_{[1,2]} \qquad \text{and} \qquad \epsilon_{[2]} \circ \pi_{[1,2]}R_{[2]} = \epsilon_{[1]} \tag{15}$$

Note that for every $B_{[2]} := \left(B_{[1]}, b_{[1]}\right) \in \mathcal{B}_{[2]}$ we have the following coequalizer

$$LRL_{[1]}B_{[1]} \underset{\epsilon L_{[1]}B_{[1]}}{\overset{\pi_{[1]}B_{[1]}\circ Lb_{[1]}}{\rightrightarrows}} L_{[1]}B_{[1]} \xrightarrow{\pi_{[1,2]}B_{[2]}} L_{[2]}B_{[2]}$$

Finally $\mathcal{B}_{[n+1]} = \mathrm{I}\left(\mathbb{T}_{[n]}\right) = \left\langle RL_{[n]} | U_{[n]}\right\rangle, U_{[n,n+1]} = P\left(\mathbb{T}_{[n]}\right)$ and $U_{[n+1]} = U_{[n]} \circ U_{[n,n+1]}$. Moreover

$$\begin{array}{rcl}
\mathcal{L}\left(\mathbb{T}_{[n]}\right) & : & \mathcal{B}_{[n+1]} \to \langle LR|\mathrm{Id}_{\mathcal{A}}\rangle, \quad \left(B_{[n]}, b_{[n]}\right) \mapsto \left(L_{[n]}B_{[n]}, \pi_{[n]}B_{[n]} \circ Lb_{[n]}\right), \qquad h \mapsto L_{[n]}h, \\
R_{[n+1]} & = & \mathrm{R}\left(\mathbb{T}_{[n]}\right) = \mathcal{R}\left(\mathbb{T}_{[n]}\right) \circ \mathcal{S}\left(\epsilon_{[n]}\right) : \mathcal{A} \to \mathcal{B}_{[n+1]}, \quad A \mapsto \left(R_{[n]}A, R\epsilon_{[n]}A\right), \qquad f \mapsto R_{[n]}f, \\
L_{[n+1]} & = & \mathrm{L}\left(\mathbb{T}_{[n]}\right) = \mathcal{U}\left(\epsilon\right) \circ \mathcal{L}\left(\mathbb{T}_{[n]}\right), \\
 & & \mathcal{B}_{[n+1]} \to \mathcal{A} : \left(B_{[n]}, b_{[n]}\right) \mapsto \mathcal{U}\left(\epsilon\right)\left(L_{[n]}B_{[n]}, \pi_{[n]}B_{[n]} \circ Lb_{[n]}\right), \quad h \mapsto \mathcal{U}\left(\epsilon\right)\left(L_{[n]}h\right), \\
\pi_{[n,n+1]} & = & \pi\left(\epsilon\right)\mathcal{L}\left(\mathbb{T}_{[n]}\right) \\
\pi_{[n+1]} & = & \pi_{[n,n+1]} * \pi_{[n]} = \pi_{[n,n+1]} \circ \pi_{[n]}U_{[n,n+1]}
\end{array}$$

The unit $\eta_{[n+1]} = \eta\left(\mathbb{T}_{[n]}\right)$ and the counit $\epsilon_{[n+1]} = \epsilon\left(\mathbb{T}_{[n]}\right)$ of the adjunction $\left(L_{[n+1]}, R_{[n+1]}\right)$ are uniquely determined by

$$U_{[n,n+1]}\eta_{[n+1]} = R_{[n]}\pi_{[n,n+1]} \circ \eta_{[n]}U_{[n,n+1]} \qquad \text{and} \qquad \epsilon_{[n+1]} \circ \pi_{[n,n+1]}R_{[n+1]} = \epsilon_{[n]} \tag{16}$$

Note that for every $B_{[n+1]} := \left(B_{[n]}, b_{[n]}\right) \in \mathcal{B}_{[n+1]}$ we have the following coequalizer

$$LRL_{[n]}B_{[n]} \underset{\epsilon L_{[n]}B_{[n]}}{\overset{\pi_{[n]}B_{[n]}\circ Lb_{[n]}}{\rightrightarrows}} L_{[n]}B_{[n]} \xrightarrow{\pi_{[n,n+1]}B_{[n+1]}} L_{[n+1]}B_{[n+1]} \tag{17}$$

so that

$$\pi_{[n+1]}B_{[n+1]} \circ Lb_{[n]} = \pi_{[n,n+1]}B_{[n+1]} \circ \epsilon L_{[n]}B_{[n]} \tag{18}$$

By composing the functors on the bottom of (12) and the corresponding natural transformations one defines, for $0 \leq t \leq n$,

$$\begin{array}{rcl}
U_{[t,n]} & = & U_{[t,t+1]} \circ U_{[t+1,t+2]} \circ \cdots \circ U_{[n-2,n-1]} \circ U_{[n-1,n]}, \\
\pi_{[t,n]} & = & \pi_{[n-1,n]} * \pi_{[n-2,n-1]} * \cdots * \pi_{[t+1,t+2]} * \pi_{[t,t+1]} \\
 & = & \pi_{[n-1,n]} \circ \pi_{[n-2,n-1]}U_{[n-1,n]} \circ \cdots \circ \pi_{[t,t+1]}U_{[t+1,n]}.
\end{array}$$

Let us give a more explicit description of objects and morphisms in the category $\mathcal{B}_{[n]}$ for $n \in \mathbb{N}$. First $\mathcal{B}_{[0]} = \mathcal{B}$. An object in $\mathcal{B}_{[1]}$ is a pair $B_{[1]} = \left(B, b_{[0]} : RL_{[0]}B \to B\right)$ where $B \in \mathcal{B}, b_{[0]} \in \mathcal{B}$. An object in $\mathcal{B}_{[2]}$ is a pair $B_{[2]} = \left(B_{[1]}, b_{[1]} : RL_{[1]}B_{[1]} \to B\right)$, where $B_{[1]} = \left(B, b_{[0]}\right) \in \mathcal{B}_{[1]}, b_{[1]} \in \mathcal{B}$. Thus we can regard $B_{[2]}$ as the triple $\left(B, b_{[0]} : RL_{[0]}B \to B, b_{[1]} : RL_{[1]}B_{[1]} \to B\right)$. Going on this way, an object in $\mathcal{B}_{[n]}$ has the form $B_{[n]} = \left(B, b_{[0]}, b_{[1]}, \ldots, b_{[n-1]}\right)$ where $b_{[t]} : RL_{[t]}B_{[t]} \to B$ and $B_{[t]} = U_{[t,n]}B_{[n]} = \left(B, b_{[0]}, b_{[1]}, \ldots, b_{[t-1]}\right)$ for each $t \in \{0, \ldots, n-1\}$.

The lower case $n = 0$ can also be included in the notation $B_{[n]} = \left(B, b_{[0]}, b_{[1]}, \ldots, b_{[n-1]}\right)$ by thinking that the $b_{[i]}$'s disappear. A datum such as $\left(b_{[0]}, b_{[1]}, \ldots, b_{[n-1]}\right)$ is called a R-structured sink in the literature.

A morphism $f_{[1]} : B_{[1]} \to B'_{[1]}$ in $\mathcal{B}_{[1]}$ means a morphism $f = U_{[1]}f_{[1]} : B \to B'$ such that

$$\begin{array}{ccc}
RL_{[0]}B & \xrightarrow{RL_{[0]}f} & RL_{[0]}B' \\
{\scriptstyle b_{[0]}}\downarrow & & \downarrow{\scriptstyle b'_{[0]}} \\
B & \xrightarrow{f} & B'
\end{array}$$

For $n>1$, a morphism $f_{[n]}:B_{[n]}\to B'_{[n]}$ in $\mathcal{B}_{[n]}$ is a morphism $f_{[n-1]}=U_{[n-1,n]}f_{[n]}:B_{[n-1]}\to B'_{[n-1]}$ such that

$$\begin{array}{ccc} RL_{[n-1]}B_{[n-1]} & \xrightarrow{RL_{[n-1]}f_{[n-1]}} & RL_{[n-1]}B'_{[n-1]} \\ {\scriptstyle b_{[n-1]}}\downarrow & & \downarrow{\scriptstyle b'_{[n-1]}} \\ B & \xrightarrow{\quad f \quad} & B' \end{array}$$

where $f=U_{[n]}f_{[n]}:B\to B'$.

3. Comparing monadic and adjoint decompositions

Next aim is to connect the monadic and adjoint decompositions by constructing functors $(\Lambda_n)_{n\in\mathbb{N}}$ making commutative the solid faces of diagram (1) for every $n\geq 1$.

To this aim we first prove some technical results needed to obtain Proposition 3.6 which is the main tool to iteratively construct $(\Lambda_n)_{n\in\mathbb{N}}$ in Remark 3.7.

Proposition 3.1. *Assume $\mathcal{A}$ has coequalizers and consider the two adjoint triangles $\mathbb{T}$, $\mathbb{T}'$ and their composition $\mathbb{T}''$ of Remark 2.2. Then there is an adjoint triangle*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad \mathrm{Id} \quad} & \mathcal{A} \\ {\scriptstyle \mathrm{L}(\mathbb{T}'')}\uparrow\downarrow{\scriptstyle \mathrm{R}(\mathbb{T}'')} & \mathcal{U}(\epsilon')\boldsymbol{\theta} & {\scriptstyle \mathrm{L}(\mathbb{T})}\uparrow\downarrow{\scriptstyle \mathrm{R}(\mathbb{T})} \\ \mathrm{I}(\mathbb{T}'') & \xrightarrow{\quad \mathrm{I}(\theta) \quad} & \mathrm{I}(\mathbb{T}) \end{array} \tag{19}$$

The functor $\mathrm{I}(\theta)$ is defined by

$$\mathrm{I}(\theta):\mathrm{I}(\mathbb{T}'')\to\mathrm{I}(\mathbb{T}),\qquad (B'',\mu'')\mapsto(\Theta B'',\mu''\circ R'\theta B''),\qquad f\mapsto\Theta P(\mathbb{T}'')f$$

and it satisfies

$$P(\mathbb{T})\circ\mathrm{I}(\theta)=\Theta\circ P(\mathbb{T}'')\qquad \text{and}\qquad G(\mathbb{T})\circ\mathrm{I}(\theta)=G''\circ P(\mathbb{T}'').$$

The natural transformation $\boldsymbol{\theta}:\mathcal{L}(\mathbb{T})\mathrm{I}(\theta)\to\mathcal{L}(\mathbb{T}'')$ appearing inside the adjoint triangle is defined uniquely by $U'_{[1]}\boldsymbol{\theta}=\theta P(\mathbb{T}'')$, where $U'_{[1]}:\langle L'R'|\mathrm{Id}\rangle\to\mathrm{Id}_{\mathcal{A}}$ is the forgetful functor.

If Θ is faithful, then so is $\mathrm{I}(\theta)$.

If θ is invertible (resp. the identity), then so is $\boldsymbol{\theta}$.

Proof. The functor $\mathrm{I}(\theta)$ can be more properly defined as follows

$$\begin{aligned} \mathrm{I}(\theta) \quad &:= \quad D^{\Theta}\circ\langle R'\theta\mid\mathrm{Id}_{G''}\rangle:\mathrm{I}(\mathbb{T}'')=\langle R'L''|G''\rangle\to\mathrm{I}(\mathbb{T})=\langle R'L|G\rangle, \\ &\left(B'',R'L''B''\overset{\mu''}{\to}G''B''\right)\mapsto\left(\Theta B'',R'L\Theta B''\overset{R'\theta B''}{\to}R'L''B''\overset{\mu''}{\to}G''B''=G\Theta B''\right) \end{aligned}$$

We compute

$$\begin{aligned} P(\mathbb{T})\circ\mathrm{I}(\theta) &= P(\mathbb{T})\circ D^{\Theta}\circ\langle R'\theta\mid\mathrm{Id}_{G''}\rangle=\Theta\circ P_{\langle R'L\Theta|G''\rangle}\circ\langle R'\theta\mid\mathrm{Id}_{G''}\rangle=\Theta\circ P(\mathbb{T}''), \\ G(\mathbb{T})\circ\mathrm{I}(\theta) &= G\circ P(\mathbb{T})\circ\mathrm{I}(\theta)=G\circ\Theta\circ P(\mathbb{T}'')=G''\circ P(\mathbb{T}''). \end{aligned}$$

Let us construct $\boldsymbol{\theta}:\mathcal{L}(\mathbb{T})\mathrm{I}(\theta)\to\mathcal{L}(\mathbb{T}'')$. It is easy to check that

$$\begin{aligned} \mathcal{L}(\mathbb{T})\mathrm{I}(\theta) &= (L\Theta P(\mathbb{T}''))\left[\zeta\Theta P(\mathbb{T}'')\circ L'\psi_{\mathrm{I}(\mathbb{T}'')}\circ L'R'\theta P(\mathbb{T}'')\right], \\ \mathcal{L}(\mathbb{T}'') &= (L''P(\mathbb{T}''))\left[\theta P(\mathbb{T}'')\circ\zeta\Theta P(\mathbb{T}'')\circ L'\psi_{\mathrm{I}(\mathbb{T}'')}\right] \end{aligned}$$

Since

$$\theta P(\mathbb{T}'')\circ\left(\zeta\Theta P(\mathbb{T}'')\circ L'\psi_{\mathrm{I}(\mathbb{T}'')}\circ L'R'\theta P(\mathbb{T}'')\right)=\left(\theta P(\mathbb{T}'')\circ\zeta\Theta P(\mathbb{T}'')\circ L'\psi_{\mathrm{I}(\mathbb{T}'')}\right)\circ L'R'\theta P(\mathbb{T}''),$$

by Lemma 1.6, there is a unique $\boldsymbol{\theta}:\mathcal{L}(\mathbb{T})\mathrm{I}(\theta)\to\mathcal{L}(\mathbb{T}'')$ such that $U'_{[1]}\boldsymbol{\theta}=\theta P(\mathbb{T}'')$.

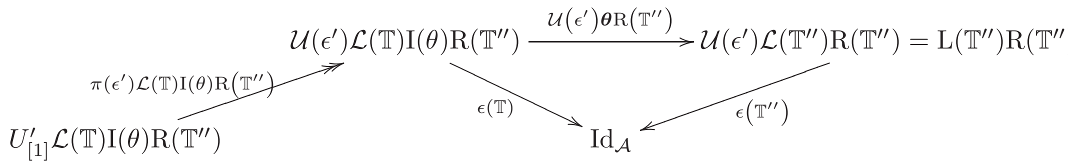
Consider $\mathcal{U}(\epsilon')\boldsymbol{\theta}:\mathcal{U}(\epsilon')\mathcal{L}(\mathbb{T})\mathrm{I}(\theta)\to\mathcal{U}(\epsilon')\mathcal{L}(\mathbb{T}'')$ i.e. $\mathcal{U}(\epsilon')\boldsymbol{\theta}:\mathrm{L}(\mathbb{T})\mathrm{I}(\theta)\to\mathrm{L}(\mathbb{T}'')$. This gives rise the adjoint triangle (19). In fact we have

$$P(\mathbb{T})\circ\mathrm{I}(\theta)\circ\mathrm{R}(\mathbb{T}'')\quad=\quad\Theta\circ P(\mathbb{T}'')\circ\mathrm{R}(\mathbb{T}'')=\Theta\circ R''=R=P(\mathbb{T})\circ\mathrm{R}(\mathbb{T}),$$

$$\begin{aligned}\psi_{\mathrm{I}(\mathbb{T})}\mathrm{I}(\theta)\,\mathrm{R}(\mathbb{T}'') &\overset{(*)}{=} \left(\psi_{\mathrm{I}(\mathbb{T}'')}\circ R'\theta P(\mathbb{T}'')\right)\mathrm{R}(\mathbb{T}'') = \psi_{\mathrm{I}(\mathbb{T}'')}\mathrm{R}(\mathbb{T}'')\circ R'\theta P(\mathbb{T}'')\,\mathrm{R}(\mathbb{T}'')\\ &= R'\epsilon''\circ R'\theta R'' = R'(\epsilon''\circ\theta R'') = R'\epsilon = \psi_{\mathrm{I}(\mathbb{T})}\mathrm{R}(\mathbb{T})\end{aligned}$$

where in $(*)$ we used the equality $\psi_{\mathrm{I}(\mathbb{T})}\mathrm{I}(\theta) = \psi_{\mathrm{I}(\mathbb{T}'')}\circ R'\theta P(\mathbb{T}'')$ which follows from the computation $\psi_{\mathrm{I}(\mathbb{T})}\mathrm{I}(\theta)(B'',\mu'') = \psi_{\mathrm{I}(\mathbb{T})}(\Theta B'',\mu''\circ R'\theta B'') = \mu''\circ R'\theta B'' = \psi_{\mathrm{I}(\mathbb{T}'')}(B'',\mu'')\circ R'\theta P(\mathbb{T}'')(B'',\mu'')$.

By Lemma 1.6, we get $\mathrm{I}(\theta)\circ\mathrm{R}(\mathbb{T}'') = \mathrm{R}(\mathbb{T})$. Consider the morphisms in the following diagram



Then

$$\begin{aligned}\epsilon(\mathbb{T}'')\circ\mathcal{U}(\epsilon')\boldsymbol{\theta}\mathrm{R}(\mathbb{T}'')\circ\pi(\epsilon')\mathcal{L}(\mathbb{T})\,\mathrm{I}(\theta)\,\mathrm{R}(\mathbb{T}'') &= \epsilon(\mathbb{T}'')\circ\pi(\epsilon')\mathcal{L}(\mathbb{T}'')\,\mathrm{R}(\mathbb{T}'')\circ U'_{[1]}\boldsymbol{\theta}\mathrm{R}(\mathbb{T}'')\\ \overset{(11)}{=}\epsilon''\circ\theta P(\mathbb{T}'')\,\mathrm{R}(\mathbb{T}'') &= \epsilon''\circ\theta R'' = \epsilon \overset{(11)}{=} \epsilon(\mathbb{T})\circ\pi(\epsilon')\mathcal{L}(\mathbb{T})\,\mathrm{R}(\mathbb{T})\\ &= \epsilon(\mathbb{T})\circ\pi(\epsilon')\mathcal{L}(\mathbb{T})\,\mathrm{I}(\theta)\,\mathrm{R}(\mathbb{T}'')\end{aligned}$$

so that $\epsilon(\mathbb{T}'')\circ\mathcal{U}(\epsilon')\boldsymbol{\theta}\mathrm{R}(\mathbb{T}'') = \epsilon(\mathbb{T})$ which means that $\mathcal{U}(\epsilon')\boldsymbol{\theta}$ is the correct natural transformation to put inside the adjoint triangle.

If Θ is faithful, from $P(\mathbb{T})\circ\mathrm{I}(\theta) = \Theta\circ P(\mathbb{T}'')$ and the fact that $P(\mathbb{T}'')$ is faithful we get that $\mathrm{I}(\theta)$ is faithful too.

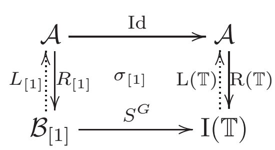
If θ is invertible, from $U_{[1]}\boldsymbol{\theta} = \theta P(\mathbb{T}'')$ and the fact that $U_{[1]}$ reflects isomorphisms, we deduce that $\boldsymbol{\theta}$ is invertible as well.

If θ is the identity, then $L\Theta = L''$ so that, by the foregoing, we get

$$\mathcal{L}(\mathbb{T})\,\mathrm{I}(\theta)(B'',\mu'') = (L\Theta B'',\zeta\Theta B''\circ L'\mu''\circ L'R'\theta_{B''}) = (L''B'',\theta B''\circ\zeta\Theta B''\circ L'\mu'') = \mathcal{L}(\mathbb{T}'')(B'',\mu'').$$

Hence the domain and codomain of $\boldsymbol{\theta}(B'',\mu'')$ are the same. Thus, since $U'_{[1]}\boldsymbol{\theta}(B'',\mu'') = \theta B'' = \mathrm{Id}_{L''B''} = U'_{[1]}\mathrm{Id}_{\mathcal{L}(\mathbb{T}'')(B'',\mu'')}$ and $U'_{[1]}$ is faithful, we obtain $\boldsymbol{\theta}(B'',\mu'') = \mathrm{Id}_{\mathcal{L}(\mathbb{T}'')(B'',\mu'')}$. □

Proposition 3.2. *Assume $\mathcal{A}$ has coequalizers. Given an adjoint triangle $\mathbb{T}$ as in (5), then*



is an adjoint triangle too. If ζR is epimorphism on each component then $\sigma_{[1]}$ is invertible.

Proof. Recall that $\mathcal{B}_{[1]} = \langle RL \mid \mathrm{Id}\rangle$. Note that

$$S^G R_{[1]}A = S^G(RA,R\epsilon A) = (RA,GR\epsilon A) = (RA,R'\epsilon A) = \mathrm{R}(\mathbb{T})\,A$$

and since $S^G R_{[1]}$ and $\mathrm{R}(\mathbb{T})$ coincide also on morphisms we get they are equal. Hence we have an adjoint triangle as in the statement where $\sigma_{[1]} := \epsilon(\mathbb{T})\,L_{[1]}\circ\mathrm{L}(\mathbb{T})\,S^G\eta_{[1]}$. Call $\mathbb{T}'$ the diagram in the statement and let $\mathbb{T}^2$ be the diagram of Proposition 2.5. Since $P(\mathbb{T})\circ S^G = U_{[1]}$ we get that $\mathbb{T}' * \mathbb{T}^2 = \mathbb{T}_{[1]}$. Thus

$$\pi_{[1]} = \sigma_{[1]} * \pi(\epsilon')\,\mathcal{L}(\mathbb{T}) = \sigma_{[1]}\circ\pi(\epsilon')\,\mathcal{L}(\mathbb{T})\,S^G.$$

It is easy to check that $U_{[1]}\circ\mathcal{L}(\mathbb{T})\circ S^G = L\circ U_{[1]}$. Moreover, by definition of $\mathcal{L}(\mathbb{T})$ one gets

$$\psi_{\langle L'R'|\mathrm{Id}\rangle}\mathcal{L}(\mathbb{T}) = \zeta P(\mathbb{T})\circ L'\psi_{\mathrm{I}(\mathbb{T})}.$$

In particular

$$\psi_{\langle L'R'|\mathrm{Id}\rangle}\mathcal{L}(\mathbb{T})\,S^G = \zeta P(\mathbb{T})\,S^G\circ L'\psi_{\mathrm{I}(\mathbb{T})}S^G = \zeta U_{[1]}\circ L'G\psi_{\langle RL|\mathrm{Id}\rangle} \overset{\text{nat.}\zeta}{=} L\psi_{\langle RL|\mathrm{Id}\rangle}\circ\zeta RLU_{[1]}$$

so that the following diagram of coequalizers serially commutes

$$\begin{array}{ccccc} L'R'P_{\langle L'R'|\mathrm{Id}\rangle}\mathcal{L}\left(\mathbb{T}\right)S^{G} & \overset{\psi_{\langle L'R'|\mathrm{Id}\rangle}\mathcal{L}(\mathbb{T})S^{G}}{\underset{\epsilon' P_{\langle L'R'|\mathrm{Id}\rangle}\mathcal{L}(\mathbb{T})S^{G}}{\rightrightarrows}} & P_{\langle L'R'|\mathrm{Id}\rangle}\mathcal{L}\left(\mathbb{T}\right)S^{G} & \xrightarrow{\pi\left(\epsilon'\right)\mathcal{L}(\mathbb{T})S^{G}} & \mathcal{U}\left(\epsilon'\right)\mathcal{L}\left(\mathbb{T}\right)S^{G}=\mathrm{L}\left(\mathbb{T}\right)S^{G} \\ \Big\downarrow{\scriptstyle \zeta RLU_{[1]}} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \sigma_{[1]}} \\ LRLU_{[1]} & \overset{L\psi_{\langle RL|\mathrm{Id}\rangle}}{\underset{\epsilon LU_{[1]}}{\rightrightarrows}} & LU_{[1]} & \xrightarrow{\pi_{[1]}} & L_{[1]} \end{array}$$

If ζR is epimorphism on each component it is then easy to check that $\pi\left(\epsilon'\right)\mathcal{L}\left(\mathbb{T}\right)S^{G}$ is a coequalizer for the pair $\left(L\psi_{\langle RL|\mathrm{Id}\rangle},\epsilon LU_{[1]}\right)$ and hence $\sigma_{[1]}$ is invertible.

□

Proposition 3.3. *Assume $\mathcal{A}$ has coequalizers. Consider the two adjoint triangles $\mathbb{T}$, $\mathbb{T}'$ and their composition $\mathbb{T}''$ of Remark 2.2. Then we can define a new adjoint triangle*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L''_{[1]}}\uparrow\downarrow{\scriptstyle R''_{[1]}} & \theta_{[1]} & {\scriptstyle \mathrm{L}(\mathbb{T})}\uparrow\downarrow{\scriptstyle \mathrm{R}(\mathbb{T})} \\ \mathcal{B}''_{[1]} & \xrightarrow{\Theta_{[1]}} & \mathrm{I}(\mathbb{T}) \end{array} \tag{20}$$

where

$$\Theta_{[1]}:\mathcal{B}''_{[1]}\to\mathrm{I}\left(\mathbb{T}\right),\qquad \left(V'',\mu''\right)\mapsto\left(\Theta V'',G''\mu''\circ R'\theta V''\right),\qquad f\mapsto\Theta U''_{[1]}f,$$

and such that

$$P\left(\mathbb{T}\right)\Theta_{[1]}=\Theta U''_{[1]}\qquad \textit{and}\qquad G\left(\mathbb{T}\right)\Theta_{[1]}=G''U''_{[1]}.$$

1) *If Θ is faithful, then so is $\Theta_{[1]}$.*
2) *If θ is invertible and any component of ζR is an epimorphism, then $\theta_{[1]}$ is invertible.*

Proof. By composing the two adjoint triangles obtained in Proposition 3.1 and Proposition 3.2, the latter applied to $\mathbb{T}''$, i.e.

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L''_{[1]}}\uparrow\downarrow{\scriptstyle R''_{[1]}} & \sigma''_{[1]} & {\scriptstyle \mathrm{L}(\mathbb{T}'')}\uparrow\downarrow{\scriptstyle \mathrm{R}(\mathbb{T}'')} & \mathcal{U}\left(\epsilon'\right)\boldsymbol{\theta} & {\scriptstyle \mathrm{L}(\mathbb{T})}\uparrow\downarrow{\scriptstyle \mathrm{R}(\mathbb{T})} \\ \mathcal{B}''_{[1]}=\langle R''L''\mid\mathrm{Id}\rangle & \xrightarrow{S^{G''}} & \langle G''R''L''\mid G''\rangle=\mathrm{I}(\mathbb{T}'') & \xrightarrow{\mathrm{I}(\theta)} & \mathrm{I}(\mathbb{T}) \end{array}$$

we obtain the triangle (20) with $\theta_{[1]}:=\sigma''_{[1]}*\mathcal{U}\left(\epsilon'\right)\boldsymbol{\theta}=\sigma''_{[1]}\circ\mathcal{U}\left(\epsilon'\right)\boldsymbol{\theta}S^{G''}$ and $\Theta_{[1]}=\mathrm{I}\left(\theta\right)\circ S^{G''}$. Explicitly for every $\left(V'',\mu''\right)\in\mathcal{B}''_{[1]}$ we have

$$\Theta_{[1]}\left(V'',\mu''\right)=\mathrm{I}\left(\theta\right)S^{G''}\left(V'',\mu''\right)=\mathrm{I}\left(\theta\right)\left(V'',G''\mu''\right)=\left(\Theta V'',G''\mu''\circ R'\theta V''\right)$$

and for every morphism $f\in\mathcal{B}''_{[1]}$ we have $\Theta_{[1]}f=\mathrm{I}\left(\theta\right)S^{G''}f=\Theta U''_{[1]}f$.

If Θ is faithful, then, by Proposition 3.1, so is $\mathrm{I}\left(\theta\right)$. Since $S^{G''}$ acts as the identity on morphisms it is faithful too and we get that $\Theta_{[1]}$ is faithful as a composition of faithful functors.

Assume θ is invertible and that any component of ζR is an epimorphism. By Proposition 3.1, $\boldsymbol{\theta}$ is invertible. Now

$$\zeta''R''=\left(\theta*\zeta\right)R''=\theta R''\circ\zeta\Theta R''=\theta R''\circ\zeta R$$

which is an epimorphism on each component. Thus, by Proposition 3.2, we get that $\sigma''_{[1]}$ is invertible. Hence $\theta_{[1]}$ is invertible as a composition of invertible natural transformations. □

Notation 3.4. *By applying Proposition 3.3 in the particular case when $G=\mathrm{Id}$, $L'=L,R'=R$, we get the functor*

$$\Theta'_{[1]}:\mathcal{B}''_{[1]}\to\mathcal{B}_{[1]},\qquad \left(V'',\mu''\right)\mapsto\left(\Theta V'',\Theta\mu''\circ R\theta V''\right),\qquad f\mapsto\Theta U''_{[1]}f.$$

This functor will be used in the following section.

Recall from Definition 1.3 that given an adjunction $L \dashv R : \mathcal{A} \to \mathcal{B}$ we can consider the category $\mathcal{B}_1$ of RL-algebras over $\mathcal{B}$ and the corresponding forgetful functor $U_{0,1} : \mathcal{B}_1 \to \mathcal{B}$. The functor $R = R_0$ induces the comparison functor $R_1 : \mathcal{A} \to \mathcal{B}_1$.

Proposition 3.5. *Assume $\mathcal{A}$ has coequalizers. Consider an adjunction $L \dashv R : \mathcal{A} \to \mathcal{B}$. Then the inclusion functor $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_{[1]}$ gives rise to an adjoint triangle*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L_1} \uparrow \downarrow {\scriptstyle R_1} & \lambda_1 = \mathrm{Id}_{L_1} & {\scriptstyle L_{[1]}} \uparrow \downarrow {\scriptstyle R_{[1]}} \\ \mathcal{B}_1 & \xrightarrow{\Lambda_1} & \mathcal{B}_{[1]} = \langle RL \mid \mathrm{Id} \rangle \end{array}$$

such that $U_{[1]} \circ \Lambda_1 = U_{0,1}$ and $L_{[1]} \circ \Lambda_1 = L_1$

Proof. Clearly $\mathcal{B}_1$ is a full subcategory of $\mathcal{B}_{[1]}$ through Λ_1 and one has $\Lambda_1 \circ R_1 = R_{[1]}$. By construction of L_1 and $L_{[1]}$, for every (B, b) in $\mathcal{B}_1$, we have that $L_{[1]}\Lambda_1(B, b) = L_{[1]}(B, b) = \mathcal{U}(\epsilon)(LB, Lb)$ which is the coequalizer of the pair $(Lb, \epsilon LB)$ namely $L_1(B, b)$. Similarly $L_{[1]} \circ \Lambda_1$ and L_1 agree on morphisms and hence $L_1 = L_{[1]} \circ \Lambda_1$. Moreover the components of $\pi_{[1]}\Lambda_1 = \pi_1 : LU_1 \to L_1$ are the universal morphisms defining the coequalizer $L_1(B, b)$. Note that $L_{[1]}R_{[1]} = L_{[1]}\Lambda_1 R_1 = L_1 R_1$ and

$$\epsilon_{[1]} \circ \pi_1 R_1 = \epsilon_{[1]} \circ \pi_{[1]}\Lambda_1 R_1 = \epsilon_{[1]} \circ \pi_{[1]} R_{[1]} \overset{(13)}{=} \epsilon = \epsilon_1 \circ \pi_1 R_1$$

so that $\epsilon_{[1]} = \epsilon_1$. Note that the last equality, in the above displayed formula, is just the definition of the counit ϵ_1 of (L_1, R_1). As a consequence $\lambda_1 = \epsilon_{[1]} L_1 \circ L_{[1]}\Lambda_1 \eta_1 = \epsilon_1 L_1 \circ L_1 \eta_1 = \mathrm{Id}_{L_1}$. □

Proposition 3.6. *Assume $\mathcal{A}$ has coequalizers. Consider the two adjoint triangles $\mathbb{T}$, $\mathbb{T}'$ and their composition $\mathbb{T}''$ of Remark 2.2. Then we can define a new adjoint triangle*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L''_1} \uparrow \downarrow {\scriptstyle R''_1} & \theta_1 & {\scriptstyle \mathrm{L}(\mathbb{T})} \uparrow \downarrow {\scriptstyle \mathrm{R}(\mathbb{T})} \\ \mathcal{B}''_1 & \xrightarrow{\Theta_1} & \mathrm{I}(\mathbb{T}) \end{array}$$

where

$$\Theta_1 : \mathcal{B}''_1 \to \mathrm{I}(\mathbb{T}), \qquad (V'', \mu'') \mapsto (\Theta V'', G''\mu'' \circ R'\theta V''), \qquad f \mapsto \Theta U''_{0,1} f,$$

is such that

$$P(\mathbb{T})\,\Theta_1 = \Theta U''_{0,1} \qquad \text{and} \qquad G(\mathbb{T})\,\Theta_1 = G'' U''_{0,1}.$$

1) *If Θ is faithful, then so is Θ_1.*
2) *Assume that θ is invertible and that any component of ζR is an epimorphism.*
   Then θ_1 is invertible. Moreover if Θ is full, then so is Θ_1 and, if Θ is injective on objects, then so is Θ_1.

Proof. Compose the adjoint triangles of Proposition 3.5 (applied to the adjunction (L'', R'')) and Proposition 3.3

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ {\scriptstyle L''_1} \uparrow \downarrow {\scriptstyle R''_1} & \lambda_1 = \mathrm{Id}_{L''_1} & {\scriptstyle L''_{[1]}} \uparrow \downarrow {\scriptstyle R''_{[1]}} & \theta_{[1]} & {\scriptstyle \mathrm{L}(\mathbb{T})} \uparrow \downarrow {\scriptstyle \mathrm{R}(\mathbb{T})} \\ \mathcal{B}''_1 & \xrightarrow{\Lambda_1} & \mathcal{B}''_{[1]} = \langle R''L'' \mid \mathrm{Id} \rangle & \xrightarrow{\Theta_{[1]}} & \mathrm{I}(\mathbb{T}) \end{array}$$

to get the adjoint triangle in the present statement. Thus $\Theta_1 = \Theta_{[1]} \circ \Lambda_1$ and $\theta_1 = \lambda_1 * \theta_{[1]} = \lambda_1 \circ \theta_{[1]}\Lambda_1 = \theta_{[1]}\Lambda_1$.

If Θ is faithful, then, by Proposition 3.3, so is $\Theta_{[1]}$. Since the inclusion functor Λ_1 is faithful it is then clear that Θ_1 is faithful too as a composition of faithful functors.

Assume that θ is invertible and that any component of ζR is an epimorphism. Still by Proposition 3.3, we deduce that $\theta_{[1]}$ is invertible. Thus $\theta_1 = \theta_{[1]}\Lambda_1$ is invertible too.

It remains to prove that if Θ is full, then so is Θ_1. Let $\xi : \Theta_1 (V'', \mu_{V''}) \to \Theta_1 (W'', \mu_{W''})$ be a morphism in $\mathrm{I}(\mathbb{T}) = \langle R'L|G\rangle$. This means to have a morphism $h := P(\mathbb{T})\xi : \Theta V'' \to \Theta W''$ such that

$$Gh \circ (G''\mu_{V''} \circ R'\theta V'') = (G''\mu_{W''} \circ R'\theta W'') \circ R'Lh.$$

Since $G'' = G\Theta$, $R = \Theta R''$ and $R' = GR$ we can rewrite this equality as

$$G(h \circ \Theta\mu_{V''} \circ \Theta R''\theta V'') = G(\Theta\mu_{W''} \circ \Theta R''\theta W'' \circ \Theta R''Lh).$$

Since Θ is full, there is a morphism $g : V'' \to W''$ such that $h = \Theta g$ so that, using $G'' = G\Theta$, we can further rewrite

$$G''(g \circ \mu_{V''} \circ R''\theta V'') = G''(\mu_{W''} \circ R''\theta W'' \circ R''L\Theta g).$$

By naturality of θ we have $\mu_{W''} \circ R''\theta W'' \circ R''L\Theta g = \mu_{W''} \circ R''L''g \circ R''\theta V''$ so that, since θ is invertible, we obtain

$$G''(g \circ \mu_{V''}) = G''(\mu_{W''} \circ R''L''g)$$

and hence

$$\begin{array}{rl} & L''(g \circ \mu_{V''}) \circ \zeta''R''L''V'' \overset{\text{nat. } \zeta''}{=} \zeta''W'' \circ L'G''(g \circ \mu_{V''}) \\ = & \zeta''W'' \circ L'G''(\mu_{W''} \circ R''L''g) \overset{\text{nat. } \zeta''}{=} L''(\mu_{W''} \circ R''L''g) \circ \zeta''R''L''V''. \end{array}$$

Now $\zeta'' = \theta * \zeta = \theta \circ \zeta\Theta$ so that $\zeta''R'' = \theta R'' \circ \zeta\Theta R'' = \theta R'' \circ \zeta R$ which is an epimorphism on each component. Thus we arrive at

$$L''(g \circ \mu_{V''}) = L''(\mu_{W''} \circ R''L''g).$$

Using this equality we compute

$$\begin{array}{rcl} g \circ \mu_{V''} & = & \mu_{W''} \circ \eta''W'' \circ g \circ \mu_{V''} = \mu_{W''} \circ R''L''(g \circ \mu_{V''}) \circ \eta''R''L''V'' \\ & = & \mu_{W''} \circ R''L''(\mu_{W''} \circ R''L''g) \circ \eta''R''L''V'' \\ & = & \mu_{W''} \circ \eta''W'' \circ \mu_{W''} \circ R''L''g = \mu_{W''} \circ R''L''g \end{array}$$

This means there is a morphism $g_1 : (V'', \mu_{V''}) \to (W'', \mu_{W''})$ such that $U''_{0,1}g_1 = g$. We have

$$P(\mathbb{T})\Theta_1 g_1 = \Theta U''_{0,1}g_1 = \Theta g = h = P(\mathbb{T})\xi.$$

Since $P(\mathbb{T})$ is faithful, we deduce that $\Theta_1 g_1 = \xi$. Thus Θ_1 is full.

Assume that Θ is injective on objects and $\Theta_1(V'', \mu_{V''}) = \Theta_1(W'', \mu_{W''})$. Then we can apply the above argument for $\xi := \mathrm{Id}$. In this case $h := P(\mathbb{T})\xi = \mathrm{Id} : \Theta V'' \to \Theta W''$. The fact that Θ is injective on objects tells that $V'' = W''$ so that we write $h = \Theta g$ for $g = \mathrm{Id}$ (and the above assumption that Θ is full can be dropped out). As above we arrive at $g \circ \mu_{V''} = \mu_{W''} \circ R''L''g$ i.e. $\mu_{V''} = \mu_{W''}$. We have so proved that $(V'', \mu_{V''}) = (W'', \mu_{W''})$ and hence Θ_1 is injective on objects. □

Remark 3.7. Consider an adjunction (L, R, η, ϵ) and assume $\mathcal{A}$ has coequalizers.

Apply Proposition 3.5 to obtain the adjoint triangle $\mathbf{\Lambda}_{[1]}$

$$\mathbf{\Lambda}_{[1]} := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ L_1 \uparrow\downarrow R_1 & \lambda_1 = \mathrm{Id}_{L_1} & L_{[1]} \uparrow\downarrow R_{[1]} \\ \mathcal{B}_1 & \xrightarrow{\Lambda_1} & \mathcal{B}_{[1]} \end{array} \qquad \mathbf{\Lambda}_{[2]} := \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\ L_2 \uparrow\downarrow R_2 & \lambda_2 := (\lambda_1)_1 & L_{[2]} \uparrow\downarrow R_{[2]} \\ \mathcal{B}_2 & \xrightarrow{\Lambda_2 := (\Lambda_1)_1} & \mathcal{B}_{[2]} \end{array}$$

where $U_{[0,1]}\Lambda_1 = U_{0,1}$ and Λ_1 is fully faithful and injective on objects. Recall that any component of $\pi_{[1]}R_{[1]}$ is an epimorphism.

Hence all the conditions in Proposition 3.6 are verified for $\mathbb{T}' = \mathbf{\Lambda}_{[1]}$ and $\mathbb{T} = \mathbb{T}_{[1]}$ and we obtain the adjoint triangle $\mathbf{\Lambda}_{[2]}$ where $\lambda_2 : L_{[2]}\Lambda_2 \to L_2$ is invertible. Moreover $U_{[1,2]}\Lambda_2 = \Lambda_1 U_{1,2}$ and $U_{[2]}\Lambda_2 = U_{[0,1]}\Lambda_1 U_{1,2}$ i.e. $U_{[0,2]}\Lambda_2 = U_{0,1}U_{1,2} = U_{0,2}$. Furthermore Λ_2 is fully faithful and injective on objects. Recall that any component of $\pi_{[2]}R_{[2]}$ is an epimorphism.

Going on this way we construct iteratively $(\Lambda_n)_{n\in\mathbb{N}}$ such that $\Lambda_0 := \mathrm{Id}$ and $U_{[n-1,n]}\Lambda_n = \Lambda_{n-1}U_{n-1,n}$, for every $n \geq 1$. Moreover $\lambda_n : L_{[n]}\Lambda_n \to L_n$ is invertible, $U_{[0,n]}\Lambda_n = U_{0,n}$ for every $n \in \mathbb{N}$ and Λ_n is fully faithful and injective on objects.

Remark 3.8. Note that, by construction Λ_n is defined as follows

$$\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]},\ \left(V_{n-1},\mu_{n-1}\right) \mapsto \left(\Lambda_{n-1}V_{n-1}, U_{[n-1]}\Lambda_{n-1}\mu_{n-1}\circ R\lambda_{n-1}V_{n-1}\right),\ f \mapsto \Lambda_{n-1}U_{n-1,n}f$$

i.e.

$$\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}, \qquad \left(V_{n-1},\mu_{n-1}\right) \mapsto \left(\Lambda_{n-1}V_{n-1}, U_{n-1}\mu_{n-1}\circ R\lambda_{n-1}V_{n-1}\right), \qquad f \mapsto \Lambda_{n-1}U_{n-1,n}f.$$

Since $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ is fully faithful, we get that $\mathcal{B}_n$ is equivalent to the essential image of Λ_n. Later on we will look for handy criteria for an object in $\mathcal{B}_{[n]}$ to belong to the image of Λ_n.

4. Relative Grothendieck fibrations

In order to deduce properties of the functors Λ_n, a relative version of the notion of Grothendieck fibration is needed. We collect here its definition and properties.

Definition 4.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and let M be a class of morphisms in $\mathcal{B}$.

We say that a morphism $f \in \mathcal{A}$ is **cartesian** (with respect to F) over a morphism $f' \in \mathcal{B}$ whenever $Ff = f'$ and given $g \in \mathcal{A}$ and $h \in \mathcal{B}$ such that $Ff \circ h = Fg$, then there exists a unique morphism $k \in \mathcal{A}$ such that $Fk = h$ and $f \circ k = g$, [Bo2, Definition 8.1.2].

$$\begin{array}{ccc} & & FZ \\ & \swarrow^{h} & \downarrow{\scriptstyle Fg} \\ FX & \xrightarrow{Ff} & FY \end{array} \qquad\qquad \begin{array}{ccc} & & Z \\ & \swarrow^{k} & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

We say that F is an M-**fibration** if every morphism $f' : B \to FA$ in M there is $f : A' \to A$ which is cartesian over f'. When M is the class of all morphisms in $\mathcal{B}$ we recover the notion of fibration, see [Bo2, Definition 8.1.3].

Remark 4.2. A morphism $f : X \to Y$ is cartesian over Ff if and only if following diagram is a pullback for every object Z, where the vertical maps are obtained by evaluating F.

$$\begin{array}{ccc} \mathrm{Hom}(Z,X) & \xrightarrow{\mathrm{Hom}(Z,f)} & \mathrm{Hom}(Z,Y) \\ {\scriptstyle F_{Z,X}}\downarrow & & \downarrow{\scriptstyle F_{Z,Y}} \\ \mathrm{Hom}(FZ,FX) & \xrightarrow{\mathrm{Hom}(FZ,Ff)} & \mathrm{Hom}(FZ,FY) \end{array}$$

In fact the map $\mathrm{Hom}(Z,X) \to \mathrm{Hom}(Z,Y) \times_{\mathrm{Hom}(FZ,FY)} \mathrm{Hom}(FZ,FX) : k \mapsto (f \circ k, Fk)$ into the pullback becomes bijective. This fact is well-known, see e.g. [St, Definition 4.32.1].

Lemma 4.3 (Cf. [Vi, Proposition 3.4(ii)]). *Being cartesian is transitive.*

Proof. Since the vertical composition of pullbacks is a pullback ([Bo1, Proposition 2.5.9]), it follows from Remark 4.2. □

Recall that an **isofibration** (called *transportable functor* in [Gro, Corollaire 4.4]) is a functor $F : \mathcal{A} \to \mathcal{B}$ such that for any object $A \in \mathcal{A}$ and any isomorphism $f' : B \to FA$, there exists an isomorphism $f : A' \to A$ such that $Ff = f'$. A **discrete isofibration** is an isofibration such that f is unique (see [LP, page 13]).

It is known that every fibration is an isofibration. Let us prove a relative version of this result.

Proposition 4.4. *Let* Iso *be the class of all isomorphisms in* $\mathcal{B}$. *Then* $F : \mathcal{A} \to \mathcal{B}$ *is an* Iso-*fibration if and only if it is an isofibration.*

Proof. ($\Rightarrow$) Let $f' : B \to FA$ be any isomorphism. Then $f' \in \mathsf{Iso}$. Since F is an Iso-fibration we have that there is $f : A' \to A$ which is cartesian over f'. In particular $Ff = f'$ and $FA' = B$. From $Ff \circ (f')^{-1} = F\mathrm{Id}_A$ we deduce that there exists a unique morphism $k : A \to A'$ such that $Fk = (f')^{-1}$ and $f \circ k = \mathrm{Id}_A$. Similarly, from $Ff \circ \mathrm{Id}_B = Ff$, we get a unique morphism $\lambda : A' \to A'$

such that $F\lambda = \mathrm{Id}_B$ and $f \circ \lambda = f$. Since $F(k \circ f) = Fk \circ Ff = (f')^{-1} \circ f' = \mathrm{Id}$ and $f \circ (k \circ f) = f$, we get $\lambda = k \circ f$. On the other hand since $F\mathrm{Id}_{A'} = \mathrm{Id}_B$ and $f \circ \mathrm{Id}_{A'} = f$ we also have $\lambda = \mathrm{Id}_{A'}$. Hence $k \circ f = \mathrm{Id}$. We have so proved that f is an isomorphism. Thus F is an isofibration.

($\Leftarrow$) Let $f' : B \to FA$ be in Iso. Then it is an isomorphism. Since F is an isofibration there is there exists an isomorphism $f : A' \to A$ such that $Ff = f'$. Let $g \in \mathcal{A}$ and $h \in \mathcal{B}$ such that $Ff \circ h = Fg$. Then we can take $k = f^{-1} \circ g$ to get $Fk = F(f^{-1}) \circ F(g) = F(f^{-1}) \circ Ff \circ h = h$ and $f \circ k = g$. On the other hand any morphism k such that $Fk = h$ and $f \circ k = g$, from the latter equality is $f^{-1} \circ g$. We have so proved that f is cartesian over f'. Hence F is an M-fibration. □

Corollary 4.5. *If $\mathsf{M} \supseteq \mathsf{Iso}$ and $F : \mathcal{A} \to \mathcal{B}$ is an M-fibration then F is an isofibration.*

Proof. Clearly from $\mathsf{M} \supseteq \mathsf{Iso}$ we deduce that F is M-fibration implies F is an Iso-fibration. By Proposition 4.4, F is an isofibration. □

Remark 4.6. If F is an isofibration which is faithful and injective on objects then F is a discrete isofibration. In fact, if there is another $t : A'' \to A$ such that $Ft = f'$, then $FA'' = B = FA'$ so that $A = A'$. Moreover $Ft = f' = Ff$ so that $t = f$. Hence f is unique.

Definition 4.7. Given a functor $F : \mathcal{A} \to \mathcal{B}$, if we define the **image** of F, denoted by $\mathrm{Im}(F)$, as the class of objects in $\mathcal{B}$ of the form FA for some $A \in \mathcal{A}$ together with the class of morphisms in $\mathcal{B}$ of the form Ff for some $f \in \mathcal{A}$, it is not necessarily true that $\mathrm{Im}(F)$ is a subcategory of $\mathcal{B}$ in general. However this holds in some particular cases e.g. when F is either injective on objects, see e.g. [Mi, page 62], or full, see Lemma 4.8 below.

In general we can consider the following categories.

- $\mathrm{Eim}(F)$, the **essential image** of F, i.e. the full subcategory of $\mathcal{B}$ whose objects are those isomorphic to FA for some $A \in \mathcal{A}$.
- $\mathrm{Im}'(F)$, i.e. the full subcategory of $\mathcal{B}$ whose objects are of the form FA for some $A \in \mathcal{A}$.

Clearly $\mathrm{Im}(F) \subseteq \mathrm{Im}'(F) \subseteq \mathrm{Eim}(F)$ hold always.

Lemma 4.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.*

1) *If F is an isofibration then* $\mathrm{Eim}(F) \subseteq \mathrm{Im}'(F)$.
2) *If F is full then* $\mathrm{Im}(F)$ *is a category and* $\mathrm{Im}'(F) = \mathrm{Im}(F)$.

Proof. 1). Given an object B in $\mathrm{Eim}(F)$ then $B \in \mathcal{B}$ is endowed with an isomorphism $f' : B \to FA$ for some $A \in \mathcal{A}$. Since F is an isofibration we get an isomorphism $f : A' \to A$ such that $Ff = f'$. In particular $FA' = B$ whence $B \in \mathrm{Im}(F)$. Since $\mathrm{Eim}(F)$ and $\mathrm{Im}'(F)$ ar both full subcategories of $\mathcal{B}$ we get $\mathrm{Eim}(F) \subseteq \mathrm{Im}'(F)$.

2). Let $g : FA' \to FA$ be a morphism in $\mathrm{Im}'(F)$. Since F is full, there is $f : A' \to A$ such that $g = Ff$. Then g is a morphism in $\mathrm{Im}(F)$. As a consequence, the composition in $\mathrm{Im}'(F)$ of two morphisms in $\mathrm{Im}(F)$ lies in $\mathrm{Im}(F)$ too. Since $\mathrm{Im}'(F)$ and $\mathrm{Im}(F)$ have the same objects we get $\mathrm{Im}'(F) \subseteq \mathrm{Im}(F)$ and hence the equality. □

Lemma 4.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Then F is an M-fibration if and only if for every morphism $f' : B \to FA$ in M there is $A' \in \mathcal{A}$ such that $FA' = B$.*

Proof. Assume that for every morphism $f' : B \to FA$ in M there is $A' \in \mathcal{A}$ such that $FA' = B$.

Since F is fully faithful there is $f : A' \to A$ such that $Ff = f'$. In order to prove that $f : A' \to A$ is cartesian over f', let $g : C \to A$ in $\mathcal{A}$ and $h : FC \to FA'$ in $\mathcal{B}$ such that $Ff \circ h = Fg$. Since F is fully faithful there exists a unique morphism $k \in \mathcal{A}$ such that $Fk = h$ and from $Ff \circ h = Fg$ and faithfulness of F we conclude that $f \circ k = g$ as desired. □

Remark 4.10. Proposition 1.7 states that a morphism $g \in \langle F|G \rangle$ is cartesian over Pg whenever GPg is a monomorphism. In other words any morphism $g \in \mathsf{M}(GP)$ is cartesian over Pg where

$$\mathsf{M}(F) = \{f \in \mathcal{A} \mid Ff \text{ is a monomorphism}\}.$$

Proposition 4.11. *For every $n \in \mathbb{N}$, every morphism $g \in \mathsf{M}(U_{[n]})$ is cartesian over $U_{[n]}g$.*

Proof. We proceed by induction on $n \in \mathbb{N}$. The first step is trivially true since $U_{[0]} = \mathrm{Id}_{\mathcal{B}}$.

Let $n \geq 1$ and assume the statement true for $n-1$. Let g be a morphism in $\mathsf{M}\left(U_{[n]}\right)$. Since $U_{[n]} = U_{[n-1]}U_{[n-1,n]}$ we get that $U_{[n-1,n]}g \in \mathsf{M}\left(U_{[n-1]}\right)$. By inductive hypothesis we have that $U_{[n-1,n]}g$ is cartesian over $U_{[n-1]}U_{[n-1,n]}g = U_{[n]}g$. By Lemma 4.3, it remains to prove that g is cartesian over $U_{[n-1,n]}g$ in order to conclude. To this aim observe that $U_{[n-1,n]} = P\left(\mathbb{T}_{[n-1]}\right)$. Thus, by Remark 4.10 applied to $P = P\left(\mathbb{T}_{[n-1]}\right), F = RL_{[n-1]}, G = U_{[n-1]}$, we get that g is cartesian over $U_{[n-1,n]}g$ as $g \in \mathsf{M}\left(U_{[n-1]}U_{[n-1,n]}\right)$. □

The following result provides a characterization in term of suitable pullbacks for a morphism to be cartesian with respect to an adjoint functor. It will not be used in the sequel but we think it might be of some intrinsic interest.

PROPOSITION 4.12. *Let (L,R), with $R : \mathcal{A} \to \mathcal{B}$, be an adjunction with unit η and counit ϵ.*

(1) A morphism f is cartesian with respect to L over Lf if and only if the following diagram is a pullback.

(21)
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \eta X}\downarrow & \lrcorner & \downarrow{\scriptstyle \eta Y} \\ RLX & \xrightarrow{\quad RLf \quad} & RLY \end{array}$$

(2) A morphism f is cartesian with respect to R over Rf if and only if $\epsilon Z \perp f$ (that is ϵZ and f are orthogonal) for every $Z \in \mathcal{A}$ i.e. any commutative diagram as follows admits a unique diagonal filler k making both triangles commute.

(22)
$$\begin{array}{ccc} LRZ & \xrightarrow{\quad \epsilon Z \quad} & Z \\ {\scriptstyle u}\downarrow & \overset{k}{\swarrow} & \downarrow{\scriptstyle v} \\ X & \xrightarrow[\quad f \quad]{} & Y \end{array}$$

Proof. (1). Assume that $f : X \to Y$ is cartesian over Lf and let us prove that (21) is a pullback. Let $u : Z \to Y$ and $v : Z \to RLX$ be such that $\eta Y \circ u = RLf \circ v$. We have

$$Lf \circ \epsilon LX \circ Lv = \epsilon LY \circ LRLf \circ Lv = \epsilon LY \circ L\left(RLf \circ v\right) = \epsilon LY \circ L\left(\eta Y \circ u\right) = \epsilon LY \circ L\eta Y \circ Lu = Lu$$

so that the following diagram commutes.

$$\begin{array}{ccc} & & LZ \\ & \overset{\epsilon LX \circ Lv}{\swarrow} & \downarrow{\scriptstyle Lu} \\ LX & \xrightarrow[\quad Lf \quad]{} & LRLY \end{array}$$

Since f is cartesian over Lf, there is a unique $k : Z \to X$ such that $Lk = \epsilon LX \circ Lv$ and $f \circ k = u$. The condition $Lk = \epsilon LX \circ Lv$, via the adjunction isomorphism, is equivalent to $RLk \circ \eta Z = R\left(\epsilon LX \circ Lv\right) \circ \eta Z$ i.e. $\eta X \circ k = R\epsilon LX \circ RLv \circ \eta Z$ i.e. $\eta X \circ k = R\epsilon LX \circ \eta RLX \circ v$ i.e. $\eta X \circ k = v$. Thus there is a unique $k : Z \to X$ such that $\eta X \circ k = v$ and $f \circ k = u$. In other words (21) is a pullback.

Conversely assume that (21) is a pullback and let us prove that $f : X \to Y$ is cartesian over Lf. Let $u : Z \to Y$ and $h : LZ \to LX$ be such that $Lf \circ h = Lu$. Set $v := Rh \circ \eta Z$. Then

$$RLf \circ v = RLf \circ Rh \circ \eta Z = RLu \circ \eta Z = \eta Y \circ u.$$

By the universal property of the pullback there is a unique morphism $k : Z \to X$ such that $\eta X \circ k = v$ and $f \circ k = u$. The condition $\eta X \circ k = v$, via the adjunction isomorphism, is equivalent to $\epsilon LX \circ L\left(\eta X \circ k\right) = \epsilon LX \circ Lv$ i.e. $\epsilon LX \circ L\eta X \circ Lk = \epsilon LX \circ L\left(Rh \circ \eta Z\right)$ i.e. $Lk = \epsilon LX \circ LRh \circ L\eta Z = h \circ \epsilon LZ \circ L\eta Z = h$. Thus there is a unique morphism $k : Z \to X$ such that $Lk = h$ and $f \circ k = u$. In other words f is cartesian over Lf.

(2). Assume that $f : X \to Y$ is cartesian over Rf and let us prove that $\epsilon Z \perp f$. Consider a commutative square as in (22). Set $h := Ru \circ \eta RZ$. Then $Rf \circ h = Rf \circ Ru \circ \eta RZ = Rv \circ R\epsilon Z \circ \eta RZ = Rv$. Since f is cartesian over Rf, there is a unique $k : Z \to X$ such that $Rk = h$ and $f \circ k = v$. By the adjunction, the condition $Rk = h$ is equivalent to $\epsilon X \circ LRk = \epsilon X \circ Lh$ i.e. $k \circ \epsilon Z = \epsilon X \circ L(Ru \circ \eta RZ)$ i.e. $k \circ \epsilon Z = u \circ \epsilon LRZ \circ L\eta RZ = u$. Thus there is a unique $k : Z \to X$ such that $k \circ \epsilon Z = u$ and $f \circ k = v$. Hence ϵZ and f are orthogonal.

Conversely suppose $\epsilon Z \perp f$ and let us prove f is cartesian over Rf. Let $v : Z \to Y$ and $h : RZ \to RX$ be such that $Rf \circ h = Rv$. By the adjunction the later equality is equivalent to $\epsilon Y \circ L(Rf \circ h) = \epsilon Y \circ LRv$ i.e. $f \circ \epsilon X \circ Lh = v \circ \epsilon Z$. Thus, if we set $u := \epsilon X \circ Lh$, by the orthogonality there is a unique morphism $k : Z \to X$ such that $k \circ \epsilon Z = u$ and $f \circ k = v$. By the adjunction the condition $k \circ \epsilon Z = u$ is equivalent to $R(k \circ \epsilon Z) \circ \eta RZ = Ru \circ \eta RZ$ i.e. $Rk \circ R\epsilon Z \circ \eta RZ = R(\epsilon X \circ Lh) \circ \eta RZ$ i.e. $Rk = R\epsilon X \circ RLh \circ \eta RZ = R\epsilon X \circ \eta RX \circ h = h$. Thus there is a unique $k : Z \to X$ such that $k \circ \epsilon Z = u$ and $Rk = h$ i.e. f is cartesian over Rf.

A somewhat faster proof could be obtained by applying Remark 4.2.

If $F = L$ or $F = R$ we can rewrite the corresponding pullback by means of the adjunction obtaining respectively the diagrams

$$\begin{array}{ccc} \mathrm{Hom}(Z,X) & \xrightarrow{\mathrm{Hom}(Z,f)} & \mathrm{Hom}(Z,Y) \\ {\scriptstyle \mathrm{Hom}(Z,\eta X)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(Z,\eta Y)} \\ \mathrm{Hom}(Z,RLX) & \xrightarrow{\mathrm{Hom}(Z,RLf)} & \mathrm{Hom}(Z,RLY) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}(Z,X) & \xrightarrow{\mathrm{Hom}(Z,f)} & \mathrm{Hom}(Z,Y) \\ {\scriptstyle \mathrm{Hom}(\epsilon Z,X)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\epsilon Z,Y)} \\ \mathrm{Hom}(LRZ,X) & \xrightarrow{\mathrm{Hom}(LRZ,f)} & \mathrm{Hom}(LRZ,Y) \end{array}$$

The fact that the left-hand side diagram is a pullback means that (21) is a pullback, while the fact that the right-hand side diagram is a pullback means that $\epsilon Z \perp f$. □

Next lemma will be used in order to prove Theorem 4.18 that is our main tool to get Theorem 4.19 where the embedding Λ_n is shown to be an $\mathsf{M}(U_{[n]})$-fibration.

LEMMA 4.13. *The functor $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_{[1]}$ of Remark 3.7 is an $\mathsf{M}(U_{[1]})$-fibration.*

Proof. Let $f_{[1]} : B_{[1]} \to \Lambda_1 C_1$ be a morphism in $\mathsf{M}(U_{[1]})$ i.e. such that $U_{[1]} f_{[1]}$ is a monomorphism. Since Λ_1 is fully faithful, in order to conclude, by Lemma 4.9, it suffices to prove that there is $B_1 \in \mathcal{B}_1$ such that $\Lambda_1 B_1 = B_{[1]}$.

Write $B_{[1]} = (B, b : RLB \to B)$, $C_1 = (C, c : RLC \to C)$ and note that $C_{[1]} := \Lambda_1 C_1 = (C, c : RLC \to C)$ this time regarded as an object in $\mathcal{B}_{[1]}$. Set $f := U_{[1]} f_{[1]} : B \to C$ and consider the following diagrams.

$$\begin{array}{ccccc} RLRLB & \overset{RLb}{\underset{R\epsilon LB}{\rightrightarrows}} & RLB & \xrightarrow{b} & B \\ {\scriptstyle RLRLf}\downarrow & & \downarrow{\scriptstyle RLf} & & \downarrow{\scriptstyle f} \\ RLRLC & \overset{RLc}{\underset{R\epsilon LC}{\rightrightarrows}} & RLC & \xrightarrow{c} & C \end{array} \qquad \begin{array}{ccccc} B & \xrightarrow{\eta B} & RLB & \xrightarrow{b} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle RLf} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{\eta C} & RLC & \xrightarrow{c} & C \end{array}$$

The left-hand side one serially commutes since f induces the morphism $f_{[1]}$ and by naturality of ϵ. Since $C_1 \in \mathcal{B}_1$, we also have that $c \circ RLc = c \circ R\epsilon LC$. Since f is a monomorphism, we deduce that $b \circ RLb = b \circ R\epsilon LB$. A similar argument as above, but applied on the right-hand side diagram, shows that $b \circ \eta B = \mathrm{Id}_B$. This means that $B_1 := (B, b : RLB \to B) \in \mathcal{B}_1$. By definition of Λ_1, we have that $\Lambda_1 B_1 = B_{[1]}$. □

The following lemma will be central in order to prove Propositions 4.15 and Proposition 4.16.

LEMMA 4.14. *1) Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and let $f \in \mathsf{M}(F)$ be cartesian over Ff. Then f is a monomorphism.*

2) Let $F : \mathcal{A} \to \mathcal{B}$ be an $\mathsf{M}(G)$-fibration. Then any morphism $f \in \mathsf{M}(F) \cap \mathsf{M}(GF)$ factors as $f = g \circ k$, where g is a monomorphism and $Fk = \mathrm{Id}$.

Proof. 1) Let $f : A \to A'$ in $\mathsf{M}(F)$ be cartesian over Ff. By definition of $\mathsf{M}(F)$ we have that Ff is a monomorphism. Let $a, b : A'' \to A$ be such that $f \circ a = f \circ b$. Then $Ff \circ Fa = Ff \circ Fb$. Since Ff is a monomorphism, we get $Fa = Fb$. Call h this morphism and $g := f \circ a$. Then $Ff \circ h = Fg$. Since f is cartesian over Ff, there exists a unique morphism $k \in \mathcal{A}$ such that $Fk = h$ and $f \circ k = g$. Hence $a = k = b$.

2) Let $f : A \to A'$ be in $\mathsf{M}(F) \cap \mathsf{M}(GF)$. Then Ff and GFf are both monomorphisms.

In particular $f' := Ff : FA \to FA'$ is in $\mathsf{M}(G)$ so that, by definition of $\mathsf{M}(G)$-fibration, there is $g : E \to A'$ which is cartesian over f'. Since $Fg \circ \mathrm{Id}_{FA} = Ff$, there is a unique $k : A \to E$ such that $Fk = \mathrm{Id}_{FA}$ and $g \circ k = f$.

Moreover $g \in \mathsf{M}(F)$ is cartesian over Fg so that, by 1), we get that g is a monomorphism. □

We are now going to prove Propositions 4.15, Proposition 4.16 and Theorem 4.17. These results will be used to obtain Theorem 4.18.

Proposition 4.15. *In the setting of Proposition 3.2, assume that any morphism $g \in \mathsf{M}(G)$ is cartesian over Gg. Then for every morphism $f' : (B, \beta) \to S^G C_{[1]}$ in $\mathsf{M}(GP(\mathbb{T}))$ there is a morphism $f : B_{[1]} \to C_{[1]}$ in $\mathsf{M}\left(GU_{[1]}\right)$ which is cartesian with respect to $S^G : \mathcal{B}_{[1]} = \langle RL \mid \mathrm{Id} \rangle \to \langle GRL \mid G \rangle = \mathrm{I}(\mathbb{T})$ over f'. In particular S^G is an $\mathsf{M}(G \circ P(\mathbb{T}))$-fibration.*

Proof. Let $f' : (B, \beta) \to S^G C_{[1]}$ be a morphism in $\mathsf{M}(GP(\mathbb{T}))$, in particular it is a morphism in $\mathrm{I}(\mathbb{T})$. Write $C_{[1]} = (C, c : RLC \to C)$ so that $S^G C_{[1]} = (C, Gc)$. The fact that f' is a morphism $\mathrm{I}(\mathbb{T})$ means that the following left-hand side diagram commutes

$$(23)\qquad \begin{array}{ccc} GRLB & \xrightarrow{GRLP(\mathbb{T})f'} & GRLC \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle Gc} \\ GB & \xrightarrow{GP(\mathbb{T})f'} & GC \end{array} \qquad\qquad \begin{array}{ccc} RLB & \xrightarrow{RLP(\mathbb{T})f'} & RLC \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ B & \xrightarrow{P(\mathbb{T})f'} & C \end{array}$$

Since $f' \in \mathsf{M}(GP(\mathbb{T}))$, we get $P(\mathbb{T}) f' \in \mathsf{M}(G)$. By hypothesis $P(\mathbb{T}) f'$ is cartesian with respect to G over $GP(\mathbb{T}) f'$. As a consequence the diagram on the left above implies there is a unique morphism $b : RLB \to B$ such that $Gb = \beta$ and the right-hand side diagram in (23) commutes.

Set $B_{[1]} = (B, b) \in \mathcal{B}_{[1]}$. Then the last diagram means that there is a unique morphism $f : B_{[1]} \to C_{[1]}$ such that $U_{[1]} f = P(\mathbb{T}) f'$. Hence $GU_{[1]} f = GP(\mathbb{T}) f'$ is a monomorphism so that $f \in \mathsf{M}\left(GU_{[1]}\right)$.

Let us check that f is cartesian with respect to S^G over f'.

Note that $S^G B_{[1]} = (B, Gb) = (B, \beta)$ so that $S^G f$ has the same domain and codomain of $f' : (B, \beta) \to S^G C_{[1]}$. Thus we get the equality $S^G f = f'$ by the following computation

$$P(\mathbb{T}) S^G f = U_{[1]} f = P(\mathbb{T}) f'$$

and the fact that $P(\mathbb{T})$ is faithful. Consider $g_{[1]}$ and h as in the following left-hand side diagram.

$$\begin{array}{ccc} & & S^G D_{[1]} \\ & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle S^G g_{[1]}} \\ S^G B_{[1]} & \xrightarrow{S^G f} & S^G C_{[1]} \end{array} \qquad\qquad \begin{array}{ccc} & & U_{[1]} D_{[1]} \\ & \swarrow{\scriptstyle P(\mathbb{T})h} & \downarrow{\scriptstyle U_{[1]} g_{[1]}} \\ U_{[1]} B_{[1]} & \xrightarrow{U_{[1]} f} & U_{[1]} C_{[1]} \end{array}$$

By applying $P(\mathbb{T})$ we get the right-hand side diagram above. We know that $U_{[1]} f = P(\mathbb{T}) f' \in \mathsf{M}(G)$. Then, by hypothesis $U_{[1]} f$ is then cartesian with respect to G over $GU_{[1]} f$. Thus, by Lemma 4.14, we get that $U_{[1]} f$ is a monomorphism. Thus $f \in \mathsf{M}\left(U_{[1]}\right)$. By Proposition 4.11, we get that f is cartesian over $U_{[1]} f$. As a consequence, the right-hand side diagram above implies there is a unique morphism $d_{[1]} : D_{[1]} \to B_{[1]}$ such that $U_{[1]} d_{[1]} = P(\mathbb{T}) h$ and $f \circ d_{[1]} = g_{[1]}$. Note that

$$P(\mathbb{T}) S^G d_{[1]} = U_{[1]} d_{[1]} = P(\mathbb{T}) h$$

and hence $S^G d_{[1]} = h$. It remains to prove the uniqueness of $d_{[1]}$. If there is another $k_{[1]} : D_{[1]} \to B_{[1]}$ such that $S^G k_{[1]} = h$ and $f \circ k_{[1]} = g_{[1]}$. Then

$$U_{[1]} k_{[1]} = P(\mathbb{T}) S^G k_{[1]} = P(\mathbb{T}) h = U_{[1]} d_{[1]}$$

so that $k_{[1]} = d_{[1]}$ as $U_{[1]}$ is faithful. □

Consider now the functor $\Theta'_{[1]} : \mathcal{B}''_{[1]} \to \mathcal{B}_{[1]}$ introduced in Notation 3.4.

Proposition 4.16. *Consider two adjoint triangles $\mathbb{T}$ and $\mathbb{T}'$ as in Remark 2.2. Assume that*

- *any morphism $g \in \mathsf{M}(G)$ is cartesian over Gg;*

- $\Theta : \mathcal{B}'' \to \mathcal{B}$ *is an* $\mathsf{M}\left(G\right)$*-fibration,*
- θ *is invertible.*

Then for every morphism $f : B_{[1]} \to \Theta'_{[1]} C''_{[1]}$ *in* $\mathsf{M}\left(GU_{[1]}\right)$ *there is* $f''_{[1]} : B''_{[1]} \to C''_{[1]}$ *in* $\mathsf{M}\left(U''_{[1]}\right)$ *which is cartesian with respect to* $\Theta'_{[1]}$ *over* f*. In particular* $\Theta'_{[1]} : \mathcal{B}''_{[1]} \to \mathcal{B}_{[1]}$ *is an* $\mathsf{M}\left(GU_{[1]}\right)$*-fibration.*

Proof. Let $f : B_{[1]} \to \Theta'_{[1]} C''_{[1]}$ be a morphism in $\mathsf{M}\left(GU_{[1]}\right)$. Write $B_{[1]} = (B, b : RLB \to B)$ and $C''_{[1]} = (C'', c'' : R''L''C'' \to C'')$. By definition of $\Theta'_{[1]}$, we have

$$\Theta'_{[1]} C''_{[1]} = \Theta'_{[1]} \left(C'', c''\right) = \left(\Theta C'', \Theta c'' \circ R\theta C''\right).$$

The fact that $f \in \mathcal{B}_{[1]}$ means that the first diagram in (24) commutes.
(24)

$$\begin{array}{ccc} RLB & \xrightarrow{RLU_{[1]}f} & RL\Theta C'' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \Theta\left(c''\circ R''\theta C''\right)} \\ B & \xrightarrow[U_{[1]}f]{} & \Theta C'' \end{array} \qquad \begin{array}{ccc} \Theta R''L\Theta B'' & \xrightarrow{\Theta R''L\Theta f''} & \Theta R''L\Theta C'' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \Theta\left(c''\circ R''\theta C''\right)} \\ \Theta B'' & \xrightarrow[\Theta f'']{} & \Theta C'' \end{array} \qquad \begin{array}{ccc} R''L\Theta B'' & \xrightarrow{R''L\Theta f''} & R''L\Theta C'' \\ {\scriptstyle \tau''}\downarrow & & \downarrow{\scriptstyle c''\circ R''\theta C''} \\ B'' & \xrightarrow[f'']{} & C'' \end{array}$$

Since $f \in \mathsf{M}\left(GU_{[1]}\right)$, we have that $U_{[1]}f \in \mathsf{M}\left(G\right)$. Since $\Theta : \mathcal{B}'' \to \mathcal{B}$ is an $\mathsf{M}\left(G\right)$-fibration, there is a morphism $f'' : B'' \to C''$ which is cartesian (with respect to Θ) over $U_{[1]}f$. In particular $\Theta B'' = B$ and $\Theta f'' = U_{[1]}f$. Note that $R = \Theta R''$ so that the first diagram in (24) rewrites as the second one therein. Since f'' is cartesian (with respect to Θ) over $U_{[1]}f = \Theta f''$, this diagram implies there is a unique morphism $\tau'' : R''LB \to B''$ such that $\Theta\tau'' = b$ and the third diagram in (24) commutes.

Set $b'' := \tau'' \circ \left(R''\theta B''\right)^{-1}$ and $B''_{[1]} := (B'', b'')$ Then

$$f'' \circ b'' \circ R''\theta B'' = f'' \circ \tau'' = c'' \circ R''\theta C'' \circ R''L\Theta f'' = c'' \circ R''L''f'' \circ R''\theta B''$$

and hence $f''\circ b'' = c''\circ R''L''f''$. As a consequence f'' induces a morphism $f''_{[1]} : (B'', b'') \to (C'', c'')$ such that $U''_{[1]} f''_{[1]} = f''$. We compute

$$U_{[1]}\Theta'_{[1]} f''_{[1]} = \Theta U''_{[1]} f''_{[1]} = \Theta f'' = U_{[1]} f$$

Since

$$\Theta'_{[1]} B''_{[1]} = \left(\Theta B'', \Theta b'' \circ R\theta B''\right) = \left(\Theta B'', \Theta\left(b'' \circ R''\theta B''\right)\right) = \left(\Theta B'', \Theta\tau''\right) = (B, b) = B_{[1]}$$

we have that $\Theta'_{[1]} f''_{[1]}$ and f have the same domain (and codomain). Since $U_{[1]}$ is faithful, we get $\Theta'_{[1]} f''_{[1]} = f$.

Since $U_{[1]}\Theta'_{[1]} f''_{[1]} = U_{[1]} f \in \mathsf{M}\left(G\right)$, by hypothesis $U_{[1]}\Theta'_{[1]} f''_{[1]}$ is cartesian over $GU_{[1]}\Theta'_{[1]} f''_{[1]}$. By Lemma 4.14, we deduce that $U_{[1]}\Theta'_{[1]} f''_{[1]}$ is a monomorphism. Since $U_{[1]}\Theta'_{[1]} = \Theta U''_{[1]}$, we get that $U''_{[1]} f''_{[1]} \in \mathsf{M}\left(\Theta\right)$. The latter morphism is f'' which is cartesian (with respect to Θ) over $\Theta f''$. Again, by Lemma 4.14, we deduce that $f'' = U''_{[1]} f''_{[1]}$ is a monomorphism i.e. $f''_{[1]} : B''_{[1]} \to C''_{[1]}$ in $\mathsf{M}\left(U''_{[1]}\right)$ as desired.

Let us check that $f''_{[1]} : B''_{[1]} \to C''_{[1]}$ is cartesian with respect to $\Theta'_{[1]}$ over f.

Let $g''_{[1]} : D''_{[1]} \to C''_{[1]}$ in $\mathcal{B}''_{[1]}$ and $h : \Theta'_{[1]} D''_{[1]} \to B_{[1]}$ in $\mathcal{B}_{[1]}$ be such that $\Theta'_{[1]} f''_{[1]} \circ h = \Theta'_{[1]} g''_{[1]}$. By applying on both sides $U_{[1]}$, we get $U_{[1]}\Theta'_{[1]} f''_{[1]} \circ U_{[1]} h = U_{[1]}\Theta'_{[1]} g''_{[1]}$ i.e. $\Theta f'' \circ U_{[1]} h = \Theta U''_{[1]} g''_{[1]}$. Since $f'' : B'' \to C''$ is cartesian over $U_{[1]} f = \Theta f''$, we get that there is a unique morphism $k'' : D'' \to B''$ in $\mathcal{B}''$ such that $U_{[1]} h = \Theta k''$ and $f'' \circ k'' = U''_{[1]} g''_{[1]}$. Let us check that k'' induces a morphism $k''_{[1]} : D''_{[1]} \to B''_{[1]}$ such that $\Theta'_{[1]} k_{[1]} = h$. Write $D''_{[1]} = (D'', d'' : R''L''D'' \to D'')$. Knowing that $f'' \circ b'' = c'' \circ R''L''f''$ and that $g''_{[1]} : D''_{[1]} \to C''_{[1]}$ belongs to $\mathcal{B}''_{[1]}$, we obtain

$$f'' \circ b'' \circ R''L''k'' = c'' \circ R''L''f'' \circ R''L''k'' = c'' \circ R''L''U''_{[1]} g''_{[1]} = U''_{[1]} g''_{[1]} \circ d'' = f'' \circ k'' \circ d''.$$

Since we proved that f'' *is* a monomorphism, we get that $b''\circ R''L''k''=k''\circ d''$ i.e. that k induces a morphism $k''_{[1]}:D''_{[1]}\to B''_{[1]}$ such that $U''_{[1]}k''_{[1]}=k''$. We have

$$U_{[1]}\Theta'_{[1]}k''_{[1]}=\Theta U''_{[1]}k''_{[1]}=\Theta k''=U_{[1]}h.$$

Since $\Theta'_{[1]}B''_{[1]}=B_{[1]}$ we have that $\Theta'_{[1]}k''_{[1]}$ and h have the same domain and codomain. Since $U_{[1]}$ is faithful, we obtain that $\Theta_{[1]}k''_{[1]}=h$. Moreover

$$U''_{[1]}\left(f''_{[1]}\circ k''_{[1]}\right)=U''_{[1]}f''_{[1]}\circ U''_{[1]}k''_{[1]}=f''\circ k''=U''_{[1]}g''_{[1]}.$$

Since $U''_{[1]}$ is faithful, we get $f''_{[1]}\circ k''_{[1]}=g''_{[1]}$. Moreover $k''_{[1]}$ is unique since $U''_{[1]}$ is faithful and f'' is a monomorphism. We have so proved that $f''_{[1]}:B''_{[1]}\to C''_{[1]}$ is cartesian over f. □

THEOREM 4.17. *In the setting of Proposition 3.6, assume that*

- *any morphism* $g\in\mathsf{M}(G)$ *is cartesian over* Gg;
- $\Theta:\mathcal{B}''\to\mathcal{B}$ *is an* $\mathsf{M}(G)$*-fibration,*
- θ *is invertible.*

Then for every morphism $f':(B,\beta)\to\Theta_{[1]}C''_{[1]}$ *in* $\mathsf{M}(G\circ P(\mathbb{T}))$ *there is* $f_{[1]}:B''_{[1]}\to C''_{[1]}$ *in* $\mathsf{M}\left(U''_{[1]}\right)$ *which is cartesian with respect to* $\Theta_{[1]}$ *over* f'. *In particular* $\Theta_{[1]}:\mathcal{B}''_{[1]}\to\mathrm{I}(\mathbb{T})$ *is an* $\mathsf{M}(G\circ P(\mathbb{T}))$*-fibration.*

Proof. First note that $\Theta_{[1]}=S^G\Theta'_{[1]}$ as they coincide on morphisms and for every $C''_{[1]}=(C'',c'':R''L''C''\to C'')\in\mathcal{B}''_{[1]}$, we have

$$\begin{aligned}\Theta_{[1]}C''_{[1]} &= \Theta_{[1]}(C'',c'')=(\Theta C'',G''c''\circ R'\theta C'')=(\Theta C'',G\Theta c''\circ GR\theta C'')\\ &= S^G(\Theta C'',\Theta c''\circ R\theta C'')=S^G\Theta'_{[1]}C''_{[1]}.\end{aligned}$$

Let $f':(B,\beta)\to\Theta_{[1]}C''_{[1]}$ in $\mathsf{M}(G\circ P(\mathbb{T}))$. Since $\Theta_{[1]}C''_{[1]}=S^G\Theta'_{[1]}C''_{[1]}$, by Proposition 4.15, there is a morphism $f:B_{[1]}\to\Theta'_{[1]}C''_{[1]}$ in $\mathsf{M}\left(GU_{[1]}\right)$ which is cartesian with respect to S^G over f'. By Proposition 4.16, there is $f''_{[1]}:B''_{[1]}\to C''_{[1]}$ in $\mathsf{M}\left(U''_{[1]}\right)$ which is cartesian with respect to $\Theta'_{[1]}$ over f. By Lemma 4.3, the morphism $f''_{[1]}$ is cartesian with respect to $\Theta_{[1]}=S^G\Theta'_{[1]}$ over f'. □

THEOREM 4.18. *In the setting of Proposition 3.6, assume that*

- *any morphism* $g\in\mathsf{M}(G)$ *is cartesian over* Gg;
- $\Theta:\mathcal{B}''\to\mathcal{B}$ *is an* $\mathsf{M}(G)$*-fibration,*
- θ *is invertible.*

Then $\Theta_1:\mathcal{B}''_1\to\mathrm{I}(\mathbb{T})$ *is an* $\mathsf{M}(G\circ P(\mathbb{T}))$*-fibration.*

Proof. Let $f':(B,\beta:R'LB\to GB)\to\Theta_1C''_1$ be a morphism in $\mathsf{M}(G\circ P(\mathbb{T}))$ i.e. a morphism in $\mathrm{I}(\mathbb{T})$ such that $GP(\mathbb{T})f':GB\to GP(\mathbb{T})\Theta_1C''_1=G\Theta C''$ is a monomorphism, where $C''_1=(C'',c'':R''L''C''\to C'')$. Since $\Theta_1=\Theta_{[1]}\Lambda_1$, by Theorem 4.17, there is a morphism $f_{[1]}:(B'',b'')\to(C'',c'')$ in $\mathsf{M}\left(U''_{[1]}\right)$ which is cartesian with respect to $\Theta_{[1]}$ over f'. In particular $\Theta_{[1]}(B'',b'')=(B,\beta)$ and $\Theta_{[1]}f_{[1]}=f'$. Since $f_{[1]}\in\mathsf{M}\left(U''_{[1]}\right)$, by Lemma 4.13, there is $f_1:B''_1\to C''_1$ which is cartesian with respect to $\Lambda_1:\mathcal{B}_1\to\mathcal{B}_{[1]}$ over $f_{[1]}$. We compute

$$\Theta_1f_1=\Theta_{[1]}\Lambda_1f_1=\Theta_{[1]}f_{[1]}=f'.$$

Since f_1 is cartesian with respect to Λ_1 over $f_{[1]}$ and $f_{[1]}$ is cartesian with respect to $\Theta_{[1]}$ over f', by Lemma 4.3, f_1 is cartesian with respect to $\Theta_1=\Theta_{[1]}\Lambda_1$ over f'. □

THEOREM 4.19. *Let* $n\in\mathbb{N}$. *The functor* $\Lambda_n:\mathcal{B}_n\to\mathcal{B}_{[n]}$ *of Remark 3.7 is an* $\mathsf{M}\left(U_{[n]}\right)$*-fibration. Moreover it is a discrete isofibration and* $\mathrm{Eim}(\Lambda_n)=\mathrm{Im}(\Lambda_n)=\mathrm{Im}'(\Lambda_n)$ *(see Definition 4.7).*

Proof. We proceed by induction on $n \in \mathbb{N}$. The first step is trivially true since $\Lambda_0 = \mathrm{Id}_{\mathcal{B}} = U_{[0]}$.

Let $n \geq 1$ and assume the statement true for $n-1$. Apply Theorem 4.18 to $\Theta := \Lambda_{n-1}, G := U_{[n-1]}, \mathbb{T} = \mathbb{T}_{[n-1]}, \mathbb{T}' = \mathbf{\Lambda}_{[n-1]}$ by noting that $\Lambda_n = (\Lambda_{n-1})_1$, that Θ fulfills the required conditions by inductive hypothesis, G also fulfills them by Proposition 4.11 and $\theta = \lambda_{n-1}$ is invertible by Remark 3.7. Thus Λ_n is an $\mathsf{M}\left(U_{[n]}\right)$-fibration. i Any isomorphism in $\mathcal{B}_{[n]}$ belongs trivially to $\mathsf{M}\left(U_{[n]}\right)$. Moreover, by Remark 3.7, we know that Λ_n is (fully) faithful and injective on objects. Thus we can apply Corollary 4.5 and Remark 4.6 to obtain that Λ_n is a discrete isofibration. From Lemma 4.8 and the fact that Λ_n is full, we get that $\mathrm{Eim}(\Lambda_n) \subseteq \mathrm{Im}'(\Lambda_n) = \mathrm{Im}(\Lambda_n)$. We know that $\mathrm{Im}(\Lambda_n) \subseteq \mathrm{Eim}(\Lambda_n)$ holds always. □

The following result gives conditions for an object in $\mathcal{B}_{[n]}$ to be image via Λ_n of an object in $\mathcal{B}_n$.

THEOREM 4.20. *Fix $n \in \mathbb{N}$ consider the functors $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ and $U_{[n]} : \mathcal{B}_{[n]} \to \mathcal{B}$.*

1) *For every morphism $B_{[n]} \to \Lambda_n C_n$ in $\mathsf{M}\left(U_{[n]}\right)$ we have $B_{[n]} \in \mathrm{Im}(\Lambda_n)$.*
2) *Let $B_{[n]} \in \mathcal{B}_{[n]}$ be such that $\eta_{[n]} B_{[n]}$ is in $\mathsf{M}\left(U_{[n]}\right)$. Then $B_{[n]} \in \mathrm{Im}(\Lambda_n)$.*
3) *Let $\left(B_{[n]}, b_{[n]}\right) \in \mathrm{Im}(\Lambda_{n+1})$. Then $\eta_{[n]} B_{[n]}$ is in $\mathsf{M}\left(U_{[n]}\right)$.*

Proof. 1) By Theorem 4.19, the functor $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ is an $\mathsf{M}\left(U_{[n]}\right)$-fibration. Thus, for every morphism $B_{[n]} \to \Lambda_n C_n$ in $\mathsf{M}\left(U_{[n]}\right)$ there is $B_n \in \mathcal{B}_n$ such that $\Lambda_n B_n = B_{[n]}$.

2) Since $\eta_{[n]} B_{[n]}$ is a morphism $B_{[n]} \to R_{[n]} L_{[n]} B_{[n]} = \Lambda_n R_n L_{[n]} B_{[n]}$, we conclude by 1).

3) Since $\left(B_{[n]}, b_{[n]}\right) \in \mathrm{Im}(\Lambda_{n+1})$, there is $B_{n+1} = (B_n, \mu_n : R_n L_n B_n \to B_n) \in \mathcal{B}_{n+1}$ such that $\left(B_{[n]}, b_{[n]}\right) = \Lambda_{n+1} B_{n+1}$. Then $\mu_n \circ \eta_n B_n = \mathrm{Id}_{B_n}$ and

$$B_{[n]} = U_{[n,n+1]}(B_{[n]}, b_{[n]}) = U_{[n,n+1]} \Lambda_{n+1} B_{n+1} = \Lambda_n U_{n,n+1} B_{n+1} = \Lambda_n B_n.$$

Since λ_n is the natural transformation inside the adjoint triangle $\mathbf{\Lambda}_{[n]}$, see Remark 3.7, we have $\lambda_n = \epsilon_{[n]} L_n \circ L_{[n]} \Lambda_n \eta_n$ so that

$$\begin{aligned} R_{[n]} \lambda_n \circ \eta_{[n]} \Lambda_n &= R_{[n]} \epsilon_{[n]} L_n \circ R_{[n]} L_{[n]} \Lambda_n \eta_n \circ \eta_{[n]} \Lambda_n \\ &= R_{[n]} \epsilon_{[n]} L_n \circ \eta_{[n]} \Lambda_n R_n L_n \circ \Lambda_n \eta_n = R_{[n]} \epsilon_{[n]} L_n \circ \eta_{[n]} R_{[n]} L_n \circ \Lambda_n \eta_n = \Lambda_n \eta_n. \end{aligned}$$

As a consequence we obtain

$$\Lambda_n \mu_n \circ R_{[n]} \lambda_n B_n \circ \eta_{[n]} \Lambda_n B_n = \Lambda_n \mu_n \circ \Lambda_n \eta_n B_n = \mathrm{Id}_{\Lambda_n B_n}.$$

In particular $\eta_{[n]} B_{[n]} = \eta_{[n]} \Lambda_n B_n$ is in $\mathsf{M}\left(U_{[n]}\right)$. □

COROLLARY 4.21. *Fix $n \in \mathbb{N}$. If the functor $L_{[n]}$ is fully faithful, then so is L_n and $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$ is a category isomorphism. In particular R has a monadic decomposition of monadic length at most n.*

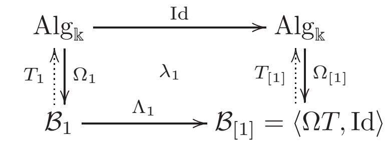
Proof. We have the isomorphism $\lambda_n : L_{[n]} \circ \Lambda_n \to L_n$. Thus if $L_{[n]}$ is fully faithful we get that L_n is fully faithful being isomorphic to a composition of fully faithful functors. By the dual version of [Bo1, Proposition 3.4.1], we have that $\eta_{[n]}$ is invertible and hence $\eta_{[n]} B_{[n]}$ is in $\mathsf{M}\left(U_{[n]}\right)$ for every $B_{[n]} \in \mathcal{B}_{[n]}$. By Theorem 4.20, $B_{[n]} \in \mathrm{Im}(\Lambda_n)$. Thus Λ_n is surjective on objects. We already know that Λ_n is injective on objects, see Remark 3.7, thus it is bijective on objects. Since we know it is also fully faithful, we deduce that it is an isomorphism. □

COROLLARY 4.22. *Consider an adjunction (L, R) such that $L_{[1]}$ and L_1 exist. If $R_{[1]}$ is an equivalence of categories then R is monadic. Moreover $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_{[1]}$ is a category isomorphism.*

Proof. If $R_{[1]}$ is an equivalence of categories then $L_{[1]}$ is an equivalence of categories and hence, by Corollary 4.21, L_1 is fully faithful and $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_{[1]}$ is a category isomorphism. Since $R_{[1]} = \Lambda_1 \circ R_1$ we get that R_1 is an equivalence of categories. Equivalently R is monadic. □

EXAMPLE 4.23. Let us show that the converse of Corollary 4.22 is not true, in general. Let $\mathcal{B} = \mathrm{Vec}_{\Bbbk}$. As a starting adjunction consider (T, Ω) where $T : \mathcal{B} \to \mathrm{Alg}_{\Bbbk}$ is the tensor algebra functor and $\Omega : \mathrm{Alg}_{\Bbbk} \to \mathcal{B}$ is the forgetful functor. It is well-known that Ω is strictly monadic i.e. the comparison functor $\Omega_1 : \mathrm{Alg}_{\Bbbk} \to \mathcal{B}_1$ is a category isomorphism, see [AM2, Theorem A.6]. Given $B \in \mathcal{B}$, consider the zero map $b : \Omega T B \to B$. Then $(B, b) \in \langle \Omega T | \mathrm{Id} \rangle = \mathcal{B}_{[1]}$ but $(B, b) \notin \mathrm{Im}(\Lambda_1)$

since $b \circ \eta B \neq \mathrm{Id}_B$, where η is the unit of the adjunction (T, Ω). Thus Λ_1 is not surjective whence not even a category isomorphism. By Corollary 4.22, we conclude that $\Omega_{[1]}$ is not an equivalence.



We have so proved that R is monadic although $R_{[1]}$ is not an equivalence for $R = \Omega$.

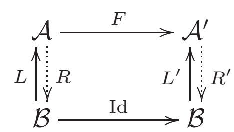
5. Connection to augmented monads

As an application of Theorem 4.20, in this section we show how to construct some functors $\Gamma_n : \mathcal{B} \to \mathcal{B}_{[n]}$ that factor through $\Lambda_n : \mathcal{B}_n \to \mathcal{B}_{[n]}$. The existence of Γ_n is related to the notion of augmented monad.

Recall that an **augmentation** for a monad $(M, m : MM \to M, u : \mathrm{Id} \to M)$ is a natural transformation $\gamma : M \to \mathrm{Id}$ such that $\gamma \circ u = \mathrm{Id}$ and $\gamma\gamma = \gamma \circ m$. We will also say that the monad M is **augmented** via the morphism $\gamma : M \to \mathrm{Id}$.

We will mainly focus on the existence of an augmentation for the monad $(RL, R\epsilon L, \eta)$ associated to a given adjunction (L, R). We point out that such a monad has an augmentation if and only if the left adjoint L is h-separable, see [AM3, Corollary 2.7].

Theorem 5.1. *Consider a diagram*



where (L, R, η, ϵ) and $(L', R', \eta', \epsilon')$ are adjunctions such that $F \circ L = L'$. Define $\xi : R \to R'F$ by

$$R \stackrel{\eta' R}{\to} R'L'R = R'FLR \stackrel{R'F\epsilon}{\to} R'F. \tag{25}$$

Then $\xi L : RL \to R'L'$ is a morphism of monads such that

$$\epsilon' F \circ L'\xi = F\epsilon. \tag{26}$$

Assume that:

1) *$\mathcal{A}$ has all coequalizers and that F preserves them;*
2) *R' preserves coequalizers of pairs (fe, f) where f is composition of regular epimorphisms and e is an idempotent morphism;*
3) *R' preserves regular epimorphisms;*
4) *the monad $R'L'$ has an augmentation $\gamma' : R'L' \to \mathrm{Id}_{\mathcal{B}}$.*

Then the monad RL is augmented via $\gamma := \gamma' \circ \xi L : RL \to \mathrm{Id}_{\mathcal{B}}$. For every $n \in \mathbb{N}$, there are a functor $\Gamma_{[n]} : \mathcal{B} \to \mathcal{B}_{[n]}$ and a natural transformation $\gamma_{[n]} : RL_{[n]}\Gamma_{[n]} \to \mathrm{Id}_{\mathcal{B}}$, such that $\Gamma_{[0]} := \mathrm{Id}_{\mathcal{B}}, \gamma_{[0]} := \gamma$ and, for $n \geq 0$,

$$\Gamma_{[n+1]}B = \left(\Gamma_{[n]}B, \gamma_{[n]}B\right) \in \mathcal{B}_{[n+1]}, \qquad \gamma_{[n]} \circ U_{[n]}\eta_{[n]}\Gamma_{[n]} = \mathrm{Id}_{\mathrm{Id}_{\mathcal{B}}}, \qquad \gamma_{[n+1]} \circ R\pi_{[n,n+1]}\Gamma_{[n+1]} = \gamma_{[n]}.$$

Moreover $U_{[n,n+1]} \circ \Gamma_{[n+1]} = \Gamma_{[n]}$.

Proof. First we have

$$\epsilon' F \circ L'\xi = \epsilon' F \circ L'R'F\epsilon \circ L'\eta' R = F\epsilon \circ \epsilon' L'R \circ L'\eta' R = F\epsilon$$

so that (26) holds true. It is easy to check that $\xi L : RL \to R'FL = R'L'$ is a morphism of monads.

Since $\xi L : RL \to R'L'$ is a morphism of monads and $R'L'$ is augmented via $\gamma' : R'L' \to \mathrm{Id}_{\mathcal{B}}$, we get that $\gamma := \gamma' \circ \xi L : RL \to \mathrm{Id}_{\mathcal{B}}$ is an augmentation for RL.

We set $S_{[n]} := L_{[n]}\Gamma_{[n]}$ and we define iteratively $\Gamma_{[n]}, \gamma'_{[n]} : R'FS_{[n]} \to \mathrm{Id}_{\mathcal{B}}$ and

$$\gamma_{[n]} := \gamma'_{[n]} \circ \xi S_{[n]} : RS_{[n]} \to \mathrm{Id}_{\mathcal{B}}$$

such that

$$U_{[n]}\circ\Gamma_{[n]}=\mathrm{Id}_{\mathcal{B}},\quad \gamma_{[n]}\circ U_{[n]}\eta_{[n]}\Gamma_{[n]}=\mathrm{Id}_{\mathrm{Id}_{\mathcal{B}}}\qquad\text{and}\qquad \gamma'_{[n]}\circ R'F\pi_{[n]}\Gamma_{[n]}=\gamma' \tag{27}$$

as follows.

For $n=0$, we set $\Gamma_{[0]}:=\mathrm{Id}_{\mathcal{B}},\gamma'_{[0]}:=\gamma',\gamma_{[0]}:=\gamma$ as required.

Let $n\geq 0$. Suppose that $\Gamma_{[n]},\gamma'_{[n]}$ such that (27) hold are given and let us construct $\Gamma_{[n+1]},\gamma'_{[n+1]}$ such that $\gamma_{[n+1]}\circ U_{[n+1]}\eta_{[n+1]}\Gamma_{[n+1]}=\mathrm{Id}$ and $\gamma'_{[n+1]}\circ R'F\pi_{[n+1]}\Gamma_{[n+1]}=\gamma'$.

Since $\mathcal{B}_{[n+1]}=\left\langle RL_{[n]}|U_{[n]}\right\rangle$ we can apply Lemma 1.6, taking $Q=\Gamma_{[n]}$ and $q=\gamma_{[n]}$, to construct a unique functor $\Gamma_{[n+1]}=\widetilde{\Gamma_{[n]}}:\mathcal{B}\to\mathcal{B}_{[n+1]}$ such that $U_{[n,n+1]}\circ\Gamma_{[n+1]}=\Gamma_{[n]}$ and $\psi\Gamma_{[n+1]}=\gamma_{[n]}$. Explicitly $\Gamma_{[n+1]}B=\left(\Gamma_{[n]}B,\gamma_{[n]}B\right)$ as desired.

For $B\in\mathcal{B}$ consider the coequalizer (17) taking $B_{[n+1]}:=\Gamma_{[n+1]}B=\left(\Gamma_{[n]}B,\gamma_{[n]}B\right)$:

$$LRS_{[n]}B\underset{\epsilon S_{[n]}B}{\overset{\pi_{[n]}\Gamma_{[n]}B\circ L\gamma_{[n]}B}{\rightrightarrows}}S_{[n]}B\xrightarrow{\pi_{[n,n+1]}\Gamma_{[n+1]}B}S_{[n+1]}B \tag{28}$$

Set $e_{[n]}:=U_{[n]}\eta_{[n]}\Gamma_{[n]}\circ\gamma_{[n]}$. Then $e_{[n]}$ is an idempotent natural transformation. Moreover, since $\mathbb{T}_{[n]}$ is an adjoint triangle, we have $\epsilon=\epsilon_{[n]}\circ\pi_{[n]}R_{[n]}$ so that

$$\begin{aligned}\epsilon S_{[n]}\circ Le_{[n]} &= \epsilon_{[n]}S_{[n]}\circ\pi_{[n]}R_{[n]}S_{[n]}\circ LU_{[n]}\eta_{[n]}\Gamma_{[n]}\circ L\gamma_{[n]}\\ &= \epsilon_{[n]}S_{[n]}\circ L_{[n]}\eta_{[n]}\Gamma_{[n]}\circ\pi_{[n]}\Gamma_{[n]}\circ L\gamma_{[n]}=\pi_{[n]}\Gamma_{[n]}\circ L\gamma_{[n]}\end{aligned}$$

and hence $\left(\pi_{[n]}\Gamma_{[n]}B\circ L\gamma_{[n]}B,\epsilon S_{[n]}B\right)=\left(\epsilon S_{[n]}B\circ Le_{[n]}B,\epsilon S_{[n]}B\right)$. Moreover

$$\epsilon S_{[n]}B=\epsilon_{[n]}S_{[n]}B\circ\pi_{[n]}R_{[n]}S_{[n]}B$$

is a composition of regular epimorphisms as $\epsilon_{[n]}S_{[n]}B$ is the coequalizer of the parallel pair of morphisms $\left(L_{[n]}R_{[n]}\epsilon_{[n]}S_{[n]}B,\epsilon_{[n]}L_{[n]}R_{[n]}S_{[n]}B\right)$ (a split coequalizer, as $S_{[n]}=L_{[n]}\Gamma_{[n]}$) and

$$\pi_{[n]}R_{[n]}S_{[n]}B=\pi_{[0,n]}R_{[n]}S_{[n]}B=\pi_{[n-1,n]}R_{[n]}S_{[n]}B\circ\pi_{[n-2,n-1]}R_{[n-1]}S_{[n]}B\circ\cdots\circ\pi_{[0,1]}R_{[1]}S_{[n]}B$$

Since, by hypothesis, F preserves coequalizers, we get that $\left(FS_{[n+1]}B,F\pi_{[n,n+1]}\Gamma_{[n+1]}B\right)$ is the coequalizer of $\left(F\left(\pi_{[n]}\Gamma_{[n]}B\circ L\gamma_{[n]}B\right),F\epsilon S_{[n]}B\right)=\left(F\epsilon S_{[n]}B\circ FLe_{[n]}B,F\epsilon S_{[n]}B\right)$ where $FLe_{[n]}B$ is still idempotent and $F\epsilon S_{[n]}B$ is still a composition of regular epimorphisms.

By the hypothesis, the latter coequalizer is preserved by R'. Thus we get the coequalizer

$$R'FLRS_{[n]}B\underset{R'F\epsilon S_{[n]}B}{\overset{R'F\pi_{[n]}\Gamma_{[n]}B\circ R'FL\gamma_{[n]}B}{\rightrightarrows}}R'FS_{[n]}B\xrightarrow{R'F\pi_{[n,n+1]}\Gamma_{[n+1]}B}R'FS_{[n+1]}B$$

Let us check that $\gamma'_{[n]}B:R'FS_{[n]}B\to B$ together with the parallel pair above is a fork i.e.

$$\gamma'_{[n]}\circ R'F\pi_{[n]}\Gamma_{[n]}\circ R'FL\gamma_{[n]}=\gamma'_{[n]}\circ R'F\epsilon S_{[n]}. \tag{29}$$

To this aim we first compute

$$\begin{aligned}&\gamma'_{[n]}\circ R'F\pi_{[n]}\Gamma_{[n]}\circ R'FL\gamma'_{[n]}\circ R'FLR'F\pi_{[n]}\Gamma_{[n]}\\ \overset{(27)}{=}\;&\gamma'\circ R'FL\gamma'=\gamma'\gamma'\\ =\;&\gamma'\circ R'\epsilon'L'\\ \overset{(27)}{=}\;&\gamma'_{[n]}\circ R'F\pi_{[n]}\Gamma_{[n]}\circ R'\epsilon'L'\\ =\;&\gamma'_{[n]}\circ R'F\pi_{[n]}\Gamma_{[n]}\circ R'\epsilon'FL\\ \overset{\text{nat. }\epsilon'}{=}\;&\gamma'_{[n]}\circ R'\epsilon'FS_{[n]}\circ R'L'R'F\pi_{[n]}\Gamma_{[n]}\\ =\;&\gamma'_{[n]}\circ R'\epsilon'FS_{[n]}\circ R'FLR'F\pi_{[n]}\Gamma_{[n]}\end{aligned}$$

Since R', F and L preserve regular epimorphisms and $\pi_{[n]}\Gamma_{[n]}$ is a regular epimorphism, we get that $R'FLR'F\pi_{[n]}\Gamma_{[n]}$ is a regular epimorphism and hence

$$\gamma'_{[n]} \circ R'F\pi_{[n]}\Gamma_{[n]} \circ R'FL\gamma'_{[n]} = \gamma'_{[n]} \circ R'\epsilon' FS_{[n]}.$$

Coming back to the equality (29), we compute

$$\begin{aligned}
\gamma'_{[n]} \circ R'F\pi_{[n]}\Gamma_{[n]} \circ R'FL\gamma_{[n]} &= \gamma'_{[n]} \circ R'F\pi_{[n]}\Gamma_{[n]} \circ R'FL\gamma'_{[n]} \circ R'FL\xi S_{[n]} \\
&= \gamma'_{[n]} \circ R'\epsilon' FS_{[n]} \circ R'FL\xi S_{[n]} \\
&= \gamma'_{[n]} \circ R'\epsilon' FS_{[n]} \circ R'L'\xi S_{[n]} \overset{(26)}{=} \gamma'_{[n]} \circ R'F\epsilon S_{[n]}
\end{aligned}$$

Thus $\gamma'_{[n]}$ together with the parallel pair $(R'F\pi_{[n]}\Gamma_{[n]} \circ R'FL\gamma_{[n]}, R'F\epsilon S_{[n]})$ is a fork and hence, the universal property of the above coequalizer yields a unique natural transformation $\gamma'_{[n+1]} : R'FS_{[n+1]} \to \mathrm{Id}$ such that

$$\gamma'_{[n+1]} \circ R'F\pi_{[n,n+1]}\Gamma_{[n+1]} = \gamma'_{[n]}$$

We compute

$$\begin{aligned}
\gamma'_{[n+1]} \circ R'F\pi_{[n+1]}\Gamma_{[n+1]} &= \gamma'_{[n+1]} \circ R'F\left(\pi_{[n,n+1]} \circ \pi_{[n]}U_{[n,n+1]}\right)\Gamma_{[n+1]} \\
&= \gamma'_{[n+1]} \circ R'F\pi_{[n,n+1]}\Gamma_{[n+1]} \circ R'F\pi_{[n]}U_{[n,n+1]}\Gamma_{[n+1]} \\
&= \gamma'_{[n]} \circ R'F\pi_{[n]}\Gamma_{[n]} \overset{(27)}{=} \gamma'.
\end{aligned}$$

We also have

$$\begin{aligned}
\gamma_{[n+1]} \circ U_{[n+1]}\eta_{[n+1]}\Gamma_{[n+1]} &= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ U_{[n+1]}\eta_{[n+1]}\Gamma_{[n+1]} \\
&\overset{(16)}{=} \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ U_{[n]}\left(R_{[n]}\pi_{[n,n+1]} \circ \eta_{[n]}U_{[n,n+1]}\right)\Gamma_{[n+1]} \\
&= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ U_{[n]}R_{[n]}\pi_{[n,n+1]}\Gamma_{[n+1]} \circ U_{[n]}\eta_{[n]}U_{[n,n+1]}\Gamma_{[n+1]} \\
&= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ R\pi_{[n,n+1]}\Gamma_{[n+1]} \circ U_{[n]}\eta_{[n]}\Gamma_{[n]} \\
&= \gamma'_{[n+1]} \circ R'F\pi_{[n,n+1]}\Gamma_{[n+1]} \circ \xi S_{[n]} \circ U_{[n]}\eta_{[n]}\Gamma_{[n]} \\
&= \gamma'_{[n]} \circ \xi S_{[n]} \circ U_{[n]}\eta_{[n]}\Gamma_{[n]} = \gamma_{[n]} \circ U_{[n]}\eta_{[n]}\Gamma_{[n]} \overset{(27)}{=} \mathrm{Id}.
\end{aligned}$$

Finally

$$\begin{aligned}
\gamma_{[n+1]} \circ R\pi_{[n,n+1]}\Gamma_{[n+1]} &= \gamma'_{[n+1]} \circ \xi S_{[n+1]} \circ R\pi_{[n,n+1]}\Gamma_{[n+1]} \\
&= \gamma'_{[n+1]} \circ R'F\pi_{[n,n+1]}\Gamma_{[n+1]} \circ \xi S_{[n]} = \gamma'_{[n]} \circ \xi S_{[n]} = \gamma_{[n]}.
\end{aligned}$$

□

Proposition 5.2. *The functor $\Gamma_{[n]} : \mathcal{B} \to \mathcal{B}_{[n]}$ induces a functor $\Gamma_n : \mathcal{B} \to \mathcal{B}_n$ such that $\Lambda_n \circ \Gamma_n = \Gamma_{[n]}$ and $U_{n,n+1} \circ \Gamma_{n+1} = \Gamma_n$. Moreover there is $\gamma_n : R_nL_n\Gamma_n \to \Gamma_n$ such that $\Gamma_{n+1}B = (\Gamma_nB, \gamma_nB)$, for all $B \in \mathcal{B}$, and $U_n\gamma_n \circ R\lambda_n\Gamma_n = \gamma_{[n]}$. Note that $L_{[n]}\Gamma_{[n]} = L_{[n]}\Lambda_n\Gamma_n \overset{\lambda_n\Gamma_n}{\to} L_n\Gamma_n$ is invertible.*

Proof. The condition $\gamma_{[n]} \circ U_{[n]}\eta_{[n]}\Gamma_{[n]} = \mathrm{Id}$, given in Theorem 5.1, implies that $\eta_{[n]}\Gamma_{[n]}B \in \mathsf{M}\left(U_{[n]}\right)$ for every $B \in \mathcal{B}$. By Theorem 4.20, we have that $\Gamma_{[n]}B \in \mathrm{Im}\left(\Lambda_n\right) = \mathrm{Im}'\left(\Lambda_n\right)$. Thus $\mathrm{Im}'\left(\Gamma_{[n]}\right) \subseteq \mathrm{Im}'\left(\Lambda_n\right)$. Since Λ_n is fully faithful and injective on objects, by [AM2, Lemma 1.12], there is a functor $\Gamma_n : \mathcal{B} \to \mathcal{B}_n$ such that $\Lambda_n \circ \Gamma_n = \Gamma_{[n]}$. We compute

$$\Lambda_n \circ U_{n,n+1} \circ \Gamma_{n+1} = U_{[n,n+1]} \circ \Lambda_{n+1} \circ \Gamma_{n+1} = U_{[n,n+1]} \circ \Gamma_{[n+1]} = \Gamma_{[n]} = \Lambda_n \circ \Gamma_n.$$

Since Λ_n is faithful and injective on objects, we get $U_{n,n+1} \circ \Gamma_{n+1} = \Gamma_n$. Moreover since $\Gamma_{n+1}B \in \mathcal{B}_{n+1}$ and $U_{n,n+1}\Gamma_{n+1}B = \Gamma_nB$, there is $\gamma_nB : R_nL_n\Gamma_nB \to \Gamma_nB$ such that $\Gamma_{n+1}B = (\Gamma_nB, \gamma_nB)$. From $\Lambda_{n+1} \circ \Gamma_{n+1} = \Gamma_{[n+1]}$, we get

$$(\Gamma_{[n]}B, \gamma_{[n]}) = \Gamma_{[n+1]}B = \Lambda_{n+1}\Gamma_{n+1}B = \Lambda_{n+1}\left(\Gamma_nB, \gamma_nB\right) = \left(\Lambda_n\Gamma_nB, U_n\gamma_nB \circ R\lambda_n\Gamma_nB\right)$$

and hence $U_n\gamma_n \circ R\lambda_n\Gamma_n = \gamma_{[n]}$. The last part follows by Remark 3.7. □

Lemma 5.3. *In the setting of Theorem 5.1, define* $S_{[n]} := L_{[n]}\Gamma_{[n]} : \mathcal{B} \to \mathcal{A}$. *Given* $B \in \mathcal{B}$, *a morphism* $f : S_{[n]}B \to A$ *in* $\mathcal{A}$ *together with the pair* $\left(\pi_{[n]}\Gamma_{[n]}B \circ L\gamma_{[n]}B, \epsilon S_{[n]}B\right)$ *is a fork if and only if* Rf *together with the pair* $\left(e_{[n]}B, \mathrm{Id}_{RS_{[n]}B}\right)$ *is a fork, where* $e_{[n]} := U_{[n]}\eta_{[n]}\Gamma_{[n]} \circ \gamma_{[n]}$.

As a consequence, $\pi_{[n,n+1]}\Gamma_{[n+1]}B : S_{[n]}B \to S_{[n+1]}B$ *is invertible if and only if either* $\gamma_{[n]}B$ *or* $\eta_{[n]}\Gamma_{[n]}B$ *is invertible. If* $\pi_{[n,n+1]}\Gamma_{[n+1]}B$ *is invertible so is* $\pi_{[m,m+1]}\Gamma_{[m+1]}B$ *for all* $m \geq n$.

Proof. In the proof of Theorem 5.1, we have seen that $\pi_{[n]}\Gamma_{[n]} \circ L\gamma_{[n]} = \epsilon L_{[n]}\Gamma_{[n]} \circ Le_{[n]} = \epsilon S_{[n]} \circ Le_{[n]}$ where $e_{[n]} := U_{[n]}\eta_{[n]}\Gamma_{[n]} \circ \gamma_{[n]}$. As a consequence, $f : S_{[n]}B \to A$ together with the pair $\left(\pi_{[n]}\Gamma_{[n]}B \circ L\gamma_{[n]}B, \epsilon S_{[n]}B\right)$ is a fork if and only if $f \circ \epsilon S_{[n]}B \circ Le_{[n]}B = f \circ \epsilon S_{[n]}B$ if and only if $Rf \circ R\epsilon S_{[n]}B \circ RLe_{[n]}B \circ \eta RS_{[n]}B = Rf \circ R\epsilon S_{[n]}B \circ \eta RS_{[n]}B$ if and only if $Rf \circ R\epsilon S_{[n]}B \circ \eta RS_{[n]}B \circ e_{[n]}B = Rf$ if and only if $Rf \circ e_{[n]}B = Rf$ if and only if Rf together with the pair $\left(e_{[n]}B, \mathrm{Id}_{RS_{[n]}B}\right)$ is a fork. Since $\pi_{[n,n+1]}\Gamma_{[n+1]}B$ is the coequalizer of the pair $\left(\pi_{[n]}\Gamma_{[n]}B \circ L\gamma_{[n]}B, \epsilon S_{[n]}B\right)$, we get that $\pi_{[n,n+1]}\Gamma_{[n+1]}B$ is invertible if and only if $\pi_{[n]}\Gamma_{[n]}B \circ L\gamma_{[n]}B = \epsilon S_{[n]}B$ if and only if $f = \mathrm{Id}$ together with the pair $\left(\pi_{[n]}\Gamma_{[n]}B \circ L\gamma_{[n]}B, \epsilon S_{[n]}B\right)$ is a fork. By the foregoing this is equivalent to $Rf \circ e_{[n]}B = Rf$ that is $e_{[n]}B = \mathrm{Id}$ i.e. $U_{[0,n]}\eta_{[n]}\Gamma_{[n]}B \circ \gamma_{[n]}B = \mathrm{Id}$.

Since $\gamma_{[n]}B \circ U_{[n]}\eta_{[n]}\Gamma_{[n]}B = \mathrm{Id}$, we conclude that $\pi_{[n,n+1]}\Gamma_{[n+1]}B$ is invertible if and only if either $\gamma_{[n]}B$ or $U_{[n]}\eta_{[n]}\Gamma_{[n]}B$ is invertible. Since $U_{[n]}$ reflects isomorphism, we have that $U_{[n]}\eta_{[n]}\Gamma_{[n]}B$ is invertible if and only if $\eta_{[n]}\Gamma_{[n]}B$ is invertible.

If $\pi_{[n,n+1]}\Gamma_{[n+1]}B$ is invertible, then $\gamma_{[n]}B$ is invertible. Since $\gamma_{[n+1]}B \circ R\pi_{[n,n+1]}\Gamma_{[n+1]}B = \gamma_{[n]}B$, we obtain that $\gamma_{[n+1]}B$ is invertible. As a consequence $\pi_{[n+1,n+2]}\Gamma_{[n+2]}B$ is invertible. Going on this way, we obtain that $\pi_{[m,m+1]}\Gamma_{[m+1]}B$ is invertible for all $m \geq n$. □

6. Example on monoidal categories

Given a category $\mathcal{A}$ and an object $X \in \mathcal{A}$ we denote by $\mathcal{A}/X$ the correspondent slice category consisting of pairs $(A, tA : A \to X)$ and where a morphism $f : (A, tA) \to (B, tB)$ is a morphism $f : A \to B$ such that $tB \circ f = tA$.

Let $\mathcal{B}$ be a category with pullbacks. It is known that any adjunction (L, R) with unit η and counit ϵ and an object $\mathbf{1} \in \mathcal{B}$ induces an adjunction $(L/\mathbf{1}, R/\mathbf{1})$ as in the following left-hand side diagram where $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are the obvious forgetful functors and $U_{\mathcal{A}} \circ L/\mathbf{1} = L \circ U_{\mathcal{B}}$.

$$\begin{array}{ccc} \mathcal{A}/L\mathbf{1} & \xrightarrow{U_{\mathcal{A}}} & \mathcal{A} \\ {\scriptstyle L/\mathbf{1}}\uparrow\ \downarrow{\scriptstyle R/\mathbf{1}} & & {\scriptstyle L}\uparrow\ \downarrow{\scriptstyle R} \\ \mathcal{B}/\mathbf{1} & \xrightarrow{U_{\mathcal{B}}} & \mathcal{B} \end{array} \qquad \begin{array}{ccc} KA & \xrightarrow{tKA} & \mathbf{1} \\ {\scriptstyle kA}\downarrow & \lrcorner & \downarrow{\scriptstyle \eta\mathbf{1}} \\ RA & \xrightarrow{RtA} & RL\mathbf{1} \end{array}$$

Explicitly $(L/\mathbf{1})(B, tB : B \to \mathbf{1}) := (LB, LtB)$ and $(L/\mathbf{1})f = Lf$. The functor $R/\mathbf{1}$ associates to an object $(A, tA : A \to L\mathbf{1})$ the pair (KA, tKA) given by the pullback in the right-hand side diagram above.

Given a morphism $f : (A, tA : A \to L\mathbf{1}) \to (A', tA' : A' \to L\mathbf{1})$ then $(R/\mathbf{1})f : (KA, tKA) \to (KA', tKA')$ is defined by the universal property of the pullback as the unique morphism such that

$$kA' \circ U_{\mathcal{B}}(R/\mathbf{1})f = RU_{\mathcal{A}}f \circ kA. \tag{30}$$

The unit $\eta/\mathbf{1}$ and counit $\epsilon/\mathbf{1}$ of the adjunction are uniquely defined by the following equalities

$$kLB \circ U_{\mathcal{B}}(\eta/\mathbf{1})(B, tB) = \eta B \qquad U_{\mathcal{A}}(\epsilon/\mathbf{1})(A, tA) = \epsilon A \circ LkA.$$

Remark 6.1. As mentioned, the construction above is well-known. It can be recovered as follows. For every morphism $f : X \to Y$ in a category $\mathcal{C}$ with pullbacks consider the functor $\mathcal{C}/X \xrightarrow{f_*} \mathcal{C}/Y$ defined on objects by $(C, g) := (C, f \circ g)$ and as the identity on morphisms. It is well-known that this functor has a right adjoint f^* given by pullbacks along f in the underlying category (see e.g. [Schu, 16.8.5]). Now note that the functor $L/\mathbf{1} : \mathcal{B}/\mathbf{1} \to \mathcal{A}/L\mathbf{1}$ can be written as the composition

$$\mathcal{B}/\mathbf{1} \overset{(\eta\mathbf{1})_*}{\to} \mathcal{B}/RL\mathbf{1} \overset{L/RL\mathbf{1}}{\to} \mathcal{A}/LRL\mathbf{1} \overset{(\epsilon L\mathbf{1})^*}{\to} \mathcal{A}/L\mathbf{1}.$$

By [Schu, 16.8.7] the functor $R/L\mathbf{1} : \mathcal{A}/L\mathbf{1} \to \mathcal{B}/RL\mathbf{1} : (A, a) \mapsto (RA, Ra), \alpha \mapsto R\alpha$ is a right adjoint for the composition $(\epsilon L\mathbf{1})^* \circ L/RL\mathbf{1}$. As a consequence we get that a right adjoint of $L/\mathbf{1}$ is given by the composition $(\eta\mathbf{1})^* \circ R/L\mathbf{1}$ which is exactly the functor $R/\mathbf{1}$ defined above.

Lemma 6.2. *Let $\mathcal{C}$ be a category and $\mathbf{1} \in \mathcal{C}$. The forgetful functor $U_{\mathcal{C}} : \mathcal{C}/\mathbf{1} \longrightarrow \mathcal{C}$ creates colimits.*

Proof. Cf. [Bo1, Proposition 2.16.3] or dual version of [MaL, Exercice 1, page 108]. □

Remark 6.3. Since $\mathcal{B}$ is a category with pullbacks, if $\mathbf{1} \in \mathcal{B}$ is a terminal object then $\mathcal{B}$ would be finitely complete by [Bo1, Proposition 2.8.2]. As a consequence $\mathcal{B}/\mathbf{1}$ is finitely complete whenever $\mathcal{B}$ is a category with pullbacks (cf. [Bo1, Proposition 2.16.3]).

In the rest of this section $\mathcal{M}$ is a non-empty preadditive braided monoidal category such that

- $\mathcal{M}$ has equalizers, denumerable coproducts and coequalizers of reflexive pairs of morphisms;
- the tensor products are additive and preserve equalizers, denumerable coproducts and coequalizers of reflexive pairs of morphisms.

We include here a well-known result we need.

Lemma 6.4. *A non-empty preadditive category $\mathcal{C}$ with equalizers has a zero object.*

Proof. For every A, B in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ contains a zero morphism i.e. $\mathcal{C}$ is a pointed category. For any morphism $f : A \to B$ we can compute the equalizer of f and the zero morphism $A \to B$, i.e. the kernel of f. By [AHS][7C(d), page 127], the category $\mathcal{C}$ has a zero object. □

Remark 6.5. Let us show that under the hypotheses above, the category $\mathcal{M}$ is a pre-abelian, see [Pop, pag 24]. First we see it is additive. By Lemma 6.4, the category $\mathcal{M}$, being non-empty and preadditive, admits a zero object, say $\mathbf{0}$. Given two objects X_1, X_2 in $\mathcal{M}$ we can set $X_n := \mathbf{0}$ for all $n \in \mathbb{N}$ with $n > 2$. Then the denumerable coproduct $\coprod_{n\in\mathbb{N}} X_n$, which exists by assumption, is just the coproduct of X_1, X_2. By [Bo2, Proposition 1.2.4 and Definition 1.2.5], the category $\mathcal{M}$ has binary biproducts. Since $\mathcal{M}$ has a zero object, then $\mathcal{M}$ is additive, see e.g. [Bo2, Definition 1.2.6].

By hypotheses $\mathcal{M}$ has all equalizers. Moreover, since $\mathcal{M}$ has binary coproducts and coequalizers of reflexive pairs, then $\mathcal{M}$ has all coequalizers: to check this one has to apply the procedure mentioned in [La, page 20] to replace a pair of morphisms by a reflexive pair with the same coequalizer. Since $\mathcal{M}$ has a zero object, we get that $\mathcal{M}$ has all kernels and cokernels. Thus $\mathcal{M}$ is a preabelian category. We point out that, by [Bo1, Proposition 2.8.2] and its dual form, the category $\mathcal{M}$ is finitely complete and finitely cocomplete (this makes sense since the dual of a preabelian category is preabelian, as observed in [Pop, page 24]).

We point out that, since, by hypothesis, denumerable coproducts and coequalizers of reflexive pairs are preserved by tensor products, all coequalizers are preserved too by [AGM1, Lemma 2.3].

By the assumptions above, we can apply [AM1, Theorem 4.6] to give an explicit description of an adjunction $\widetilde{T} \dashv P : \mathrm{Bialg}(\mathcal{M}) \to \mathcal{M}$. Note that $\mathbf{1}$ is a terminal object in $\mathrm{Bialg}(\mathcal{M})$ so that $\mathbf{0} := P\mathbf{1}$ is a terminal object in $\mathcal{M}$, as right adjoint functors preserve the terminal object. It is indeed a zero object in $\mathcal{M}$ by Lemma 6.4.

As a particular case of the constructions above, consider the following left-hand side diagram

$$\begin{array}{ccc} \mathrm{Alg}(\mathcal{M})/T\mathbf{0} & \xrightarrow{U_{\mathrm{Alg}(\mathcal{M})}} & \mathrm{Alg}(\mathcal{M}) \\ {\scriptstyle T/\mathbf{0}}\uparrow \downarrow {\scriptstyle \Omega/\mathbf{0}} & & {\scriptstyle T}\uparrow \downarrow {\scriptstyle \Omega} \\ \mathcal{M}/\mathbf{0} & \xrightarrow{U_{\mathcal{M}}} & \mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}^+(\mathcal{M}) & \xrightarrow{U := U_{\mathrm{Alg}(\mathcal{M})}} & \mathrm{Alg}(\mathcal{M}) \\ {\scriptstyle T^+}\uparrow \downarrow {\scriptstyle \Omega^+} & & {\scriptstyle T}\uparrow \downarrow {\scriptstyle \Omega} \\ \mathcal{M} & \xrightarrow{\mathrm{Id}} & \mathcal{M} \end{array}$$

Since left adjoint functors preserve the initial object, we get that $T\mathbf{0}$ is initial. By uniqueness of initial object, we get $T\mathbf{0} \cong \mathbf{1}$ as $\mathbf{1}$ is the initial object in $\mathrm{Alg}(\mathcal{M})$. Thus $\mathrm{Alg}(\mathcal{M})/T\mathbf{0}$ is $\mathrm{Alg}(\mathcal{M})/\mathbf{1}$ i.e. the category of augmented algebras that will be denoted by $\mathrm{Alg}^+(\mathcal{M})$. Note also that the functor $U_{\mathcal{M}} : \mathcal{M}/\mathbf{0} \to \mathcal{M}$ is a category isomorphism because $\mathbf{0}$ is terminal in $\mathcal{M}$. In light of these observations we can rewrite the starting diagram as the right-hand side one above where $T^{+} := (T/\mathbf{0}) \circ (U_{\mathcal{M}})^{-1}$ and $\Omega^{+} := U_{\mathcal{M}} \circ (\Omega/\mathbf{0})$. Explicitly

$$\begin{aligned} T^{+}M &= (T/\mathbf{0})\,(U_{\mathcal{M}})^{-1} M = (T/\mathbf{0})\,(M, tM) = (TM, TtM) = (TM, \varepsilon_{TM}), \\ \Omega^{+}(A,\varepsilon) &= U_{\mathcal{M}}\,(\Omega/\mathbf{0})\,(A,\varepsilon) = U_{\mathcal{M}}\,(KA, tKA) = KA \end{aligned}$$

where $\Omega/\mathbf{0}$ associates $(A,\varepsilon) \in \mathrm{Alg}^{+}(\mathcal{M})$ the pair (KA, tKA) defined by the pullback in $\mathcal{M}$

$$\begin{array}{ccc} KA & \xrightarrow{tKA} & \mathbf{0} \\ {\scriptstyle kA}\downarrow & \lrcorner & \downarrow{\scriptstyle \eta\mathbf{0} = i\Omega\mathbf{1}} \\ \Omega A & \xrightarrow{\Omega\varepsilon} & \Omega\mathbf{1} \cong \Omega T\mathbf{0} \end{array} \tag{31}$$

where $iX : \mathbf{0} \to X$ is the unique morphism from the initial object $\mathbf{0}$ and $tX : X \to \mathbf{0}$ the unique one into the terminal object. This means that $(KA, kA) = \mathrm{Ker}(\Omega\varepsilon)$. Hence

$$T^{+}M = (TM, \varepsilon_{TM}), \qquad \left(\Omega^{+}(A,\varepsilon), kA\right) = \mathrm{Ker}(\Omega\varepsilon).$$

Given a morphism $f : (A,\varepsilon) \to (A',\varepsilon')$ then $\Omega^{+}f : \Omega^{+}(A,\varepsilon) \to \Omega^{+}(A',\varepsilon')$ is defined by

$$kA' \circ \Omega^{+}f = \Omega U f \circ kA. \tag{32}$$

The unit $\eta^{+} := (U_{\mathcal{M}})\,(\eta/\mathbf{0})\,(U_{\mathcal{M}})^{-1} : \mathrm{Id} \to \Omega^{+}T^{+}$ and the counit $\epsilon^{+} := (\epsilon/\mathbf{0}) : T^{+}\Omega^{+} \to \mathrm{Id}$ are uniquely determined by the following equalities

$$kTB \circ \eta^{+}B = \eta B \qquad U\epsilon^{+}(A,\varepsilon) = \epsilon A \circ TkA. \tag{33}$$

Next aim is to show that the left-hand side diagram below fits into the setting of Theorem 5.1.

$$\begin{array}{ccccc} \mathrm{Bialg}(\mathcal{M}) & \xrightarrow{\mho^{+}} & \mathrm{Alg}^{+}(\mathcal{M}) & \xrightarrow{U := U_{\mathrm{Alg}(\mathcal{M})}} & \mathrm{Alg}(\mathcal{M}) \\ {\scriptstyle \widetilde{T}}\uparrow\downarrow{\scriptstyle P} & & {\scriptstyle T^{+}}\uparrow\downarrow{\scriptstyle \Omega^{+}} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle \Omega} \\ \mathcal{M} & \xrightarrow{\mathrm{Id}} & \mathcal{M} & \xrightarrow{\mathrm{Id}} & \mathcal{M} \end{array} \tag{34}$$

By construction of $\widetilde{T}$ we have $\mho \circ \widetilde{T} = T$. Here $\mho : \mathrm{Bialg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{M})$ and $\mho^{+}$ are the obvious forgetful functors such that $U \circ \mho^{+} = \mho$ and $\mho^{+} \circ \widetilde{T} = T^{+}$.

Moreover the assumptions guarantee that the category $\mathrm{Alg}(\mathcal{M})$ has coequalizers, see e.g. [AGM1, Proposition 2.5], see also [Por, Theorem 2.3]. Since $\mathrm{Bialg}(\mathcal{M}) = \mathrm{Coalg}(\mathrm{Alg}(\mathcal{M}))$ we can apply [Pa, Proposition 2.5] to $\mathcal{C} := \mathrm{Alg}(\mathcal{M})^{\mathrm{op}}$ to obtain that $\mho^{\mathrm{op}} : \mathrm{Bialg}(\mathcal{M})^{\mathrm{op}} \to \mathrm{Alg}(\mathcal{M})^{\mathrm{op}}$ creates limits, equivalently $\mho : \mathrm{Bialg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{M})$ creates colimits. Since $\mathrm{Alg}(\mathcal{M})$ has coequalizers, we deduce that $\mathrm{Bialg}(\mathcal{M})$ has coequalizers and $\mho$ preserves coequalizers. On the other hand the functor $\Omega : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}$ needs not to preserve coequalizers. Nevertheless Ω preserves the coequalizers of reflexive pairs of morphisms, see e.g. [AM2, Corollary A.10]. It is noteworthy that, since Ω has a left adjoint T, then Ω is strictly monadic (the comparison functor is a category isomorphism), see [AM2, Theorem A.6].

Let $V \in \mathcal{M}$. By construction $\Omega TV = \oplus_{n\in\mathbb{N}} V^{\otimes n}$, see [AM1, Remark 1.2]. Let$\alpha_n V : V^{\otimes n} \to \Omega TV$ denote the canonical inclusion. The unit of the adjunction (T,Ω) is $\eta : \mathrm{Id}_{\mathcal{M}} \to \Omega T$ defined by $\eta V := \alpha_1 V$ while the counit $\epsilon : T\Omega \to \mathrm{Id}$ is uniquely defined by the equality

$$\Omega\epsilon(A,m,u) \circ \alpha_n A = m^{n-1} \text{ for every } n \in \mathbb{N} \tag{35}$$

where $m^{n-1} : A^{\otimes n} \to A$ is the iterated multiplication of an algebra (A,m,u) defined by $m^{-1} = u, m^{0} = \mathrm{Id}_A$ and, for $n \geq 2$, by $m^{n-1} = m \circ \left(m^{n-2} \otimes A\right)$.

Denote by $\widetilde{\eta}, \widetilde{\epsilon}$ the unit and counit of the adjunction $(\widetilde{T}, P)$.

Lemma 6.6. *(*[Por, Theorem 2.3]*) The functor $\Omega : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}$ preserves regular epimorphisms.*

Not that the previous result does not mean that Ω preserves coequalizers.

Lemma 6.7. *1) Let* $e : A \to A$ *and* $f : A \to A'$ *be morphisms in* $\mathcal{M}$ *such that* $A \otimes f, f \otimes A$ *are epimorphisms and* $f \circ e \circ e = f \circ e$*. Then* $(-)^{\otimes 2} : \mathcal{M} \to \mathcal{M}$ *preserves the coequalizer of* (fe, f)*.*

2) Let $a, b : A \to B$ *be a reflexive pair of morphisms in* $\mathcal{M}$ *and let* $g : B \to A'$ *be a morphism in* $\mathcal{M}$ *such that* $A \otimes g, g \otimes A$ *are epimorphisms. Then the functor* $(-)^{\otimes 2} : \mathcal{M} \to \mathcal{M}$ *preserves the coequalizer of* (ga, gb)*.*

3) $\Omega : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}$ *creates the coequalizers of those parallel pairs* (f, g) *such that the functor* $(-)^{\otimes 2} : \mathcal{M} \to \mathcal{M}$ *preserves the coequalizer of* $(\Omega f, \Omega g)$*.*

Proof. Recall that $\mathcal{M}$ has all coequalizers and the tensor products preserve coequalizers.

1) Consider the following left-hand side coequalizer

$$A \underset{f}{\overset{fe}{\rightrightarrows}} A' \xrightarrow{p} V \qquad A \otimes A \underset{f \otimes f}{\overset{fe \otimes fe}{\rightrightarrows}} A' \otimes A' \xrightarrow{p \otimes p} V \otimes V \tag{36}$$

Let us check that the right-hand side one is a coequalizer too.

Let $\zeta : A' \otimes A' \to Z$ be such that $\zeta \circ (fe \otimes fe) = \zeta \circ (f \otimes f)$. Then

$$\begin{aligned} \zeta \circ (fe \otimes A') \circ (A \otimes f) &= \zeta \circ (f \otimes f) \circ (e \otimes A) = \zeta \circ (fe \otimes fe) \circ (e \otimes A) \\ &= \zeta \circ (fe \otimes fe) = \zeta \circ (f \otimes f) = \zeta \circ (f \otimes A') \circ (A \otimes f). \end{aligned}$$

Since $A \otimes f$ is an epimorphism, we deduce that $\zeta \circ (fe \otimes A') = \zeta \circ (f \otimes A')$.

Since the tensor products preserve coequalizers, the following are both coequalizers.

$$A \otimes A' \underset{f \otimes A'}{\overset{fe \otimes A'}{\rightrightarrows}} A' \otimes A' \xrightarrow{p \otimes A'} V \otimes A' \qquad V \otimes A \underset{V \otimes f}{\overset{V \otimes fe}{\rightrightarrows}} V \otimes A' \xrightarrow{V \otimes p} V \otimes V \tag{37}$$

By using the left one there is a morphism $\zeta_1 : V \otimes A' \to Z$ such that $\zeta_1 \circ (p' \otimes A') = \zeta$. We have

$$\begin{aligned} \zeta_1 \circ (V \otimes fe) \circ (pf \otimes A) &= \zeta_1 \circ (p \otimes A') \circ (f \otimes f) \circ (A \otimes e) = \zeta \circ (f \otimes f) \circ (A \otimes e) \\ &= \zeta \circ (fe \otimes fe) \circ (A \otimes e) = \zeta \circ (fe \otimes fe) = \zeta \circ (f \otimes f) \\ &= \zeta_1 \circ (p \otimes A') \circ (f \otimes f) \zeta_1 \circ (V \otimes f) \circ (pf \otimes A). \end{aligned}$$

Now, $f \otimes A$ is an epimorphism by assumption. Moreover $p \otimes A$ is an epimorphism because the tensor products preserve coequalizers. Thus, from the chain of equalities above, we deduce that $\zeta_1 \circ (V \otimes fe) = \zeta_1 \circ (V \otimes f)$. Since the right-hand side diagram in (37) is a coequalizer, there is a morphism $\zeta_2 : V \otimes V \to Z$ such that $\zeta_2 \circ (V \otimes p) = \zeta_1$. Therefore

$$\zeta_2 \circ (p \otimes p) = \zeta_2 \circ (V \otimes p) \circ (p \otimes A') = \zeta_1 \circ (p \otimes A') = \zeta.$$

Note also that $p \otimes p = (V \otimes p) \circ (p \otimes A')$ is an epimorphism. Thus the right-hand side diagram in (36) is a coequalizer.

2) Let $s : A' \to A$ be such that $a \circ s = \mathrm{Id} = b \circ s$. Set $f := g \circ b$ and $e := s \circ a$. Then $f \circ e = g \circ b \circ s \circ a = g \circ a$ so that $(ga, gb) = (fe, f)$ and we can apply 1). We just point out that $f \otimes A = (g \otimes A) \circ (b \otimes A)$ is an epimorphism as a composition of the epimorphism $g \otimes A$ by the split-epimorphism $b \otimes A$. Similarly $A \otimes f$ is an epimorphism.

3) It is straightforward. □

Lemma 6.8. *The forgetful functor* $U : \mathrm{Alg}^{+}(\mathcal{M}) \longrightarrow \mathrm{Alg}(\mathcal{M})$ *creates colimits. Moreover the category* $\mathrm{Alg}^{+}(\mathcal{M})$ *has coequalizers and* U *preserves all coequalizers.*

Proof. By Lemma 6.2, the forgetful functor $U : \mathrm{Alg}^{+}(\mathcal{M}) \longrightarrow \mathrm{Alg}(\mathcal{M})$ creates colimits and since $\mathrm{Alg}(\mathcal{M})$ has coequalizers, we deduce that $\mathrm{Alg}^{+}(\mathcal{M})$ has coequalizers and U preserves them. □

Lemma 6.9. *The forgetful functor* $\mho^{+} : \mathrm{Bialg}(\mathcal{M}) \longrightarrow \mathrm{Alg}^{+}(\mathcal{M})$ *preserves coequalizers.*

Proof. Since $\mho : \mathrm{Bialg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{M})$ creates colimits, we have that $\mho$ preserves all colimits that exist in $\mathrm{Alg}(\mathcal{M})$, see [McL, 11.5, page 106]. Since $\mathrm{Alg}(\mathcal{M})$ has all coequalizers, we get that $\mho$ preserves all coequalizers. Since $\mho = U\mho^{+}$ and, by Lemma 6.8, U creates, whence reflects coequalizers [MaL, Exercice 1, page 150], we get that $\mho^{+}$ preserves coequalizers as desired. □

Corollary 6.10. *$\Omega U:\mathrm{Alg}^{+}(\mathcal{M})\longrightarrow\mathcal{M}$ preserves coequalizers for pairs (fe,f) where $f:A\to A'$ is composition of regular epimorphisms in $\mathrm{Alg}^{+}(\mathcal{M})$ and $e:A\to A$ is a morphism in $\mathrm{Alg}^{+}(\mathcal{M})$ such that $f\circ e\circ e=f\circ e$.*

Proof. Consider in $\mathrm{Alg}^{+}(\mathcal{M})$ the following left-hand side coequalizer.

$$A\overset{f\circ e}{\underset{f}{\rightrightarrows}}A'\overset{p}{\longrightarrow}C\qquad UA\overset{Uf\circ Ue}{\underset{Uf}{\rightrightarrows}}UA'\overset{Up}{\longrightarrow}UC$$

By Lemma 6.8, U preserves coequalizers so that also the right-hand side one is a coequalizer.

Since ΩU preserves the regular epimorphisms (as U preserves coequalizers and Ω preserves regular epimorphisms by Lemma 6.6), we get that ΩUf is composition of regular epimorphisms. Since the tensor products preserves coequalizers, $\Omega UA\otimes\Omega Uf$ and $\Omega Uf\otimes\Omega UA$ are epimorphisms.

Since $\Omega Uf\circ\Omega Ue\circ\Omega Ue=\Omega Uf\circ\Omega Ue$ and $\Omega UA\otimes\Omega Uf,\Omega Uf\otimes\Omega UA$ are epimorphisms, we can apply Lemma 6.7-1) to "f" $=\Omega Uf$ and "e" $=\Omega Ue$ to get that the coequalizer of $(\Omega Uf\circ\Omega Ue,\Omega Uf)$ is preserved by $(-)^{\otimes 2}$ and hence, by Lemma 6.7-3), Ω creates the coequalizer of $(Uf\circ Ue,Uf)$. As a consequence the above right-hand side displayed coequalizer is preserved by Ω. Hence ΩU preserves the starting coequalizer. □

Proposition 6.11. *Let $\nu:F\to G$ and $\tau:G\to F$ be natural transformations such that $\tau\circ\nu=\mathrm{Id}$. Then F preserves those colimits which are preserved by G. Moreover F preserves regular epimorphisms which are preserved by G.*

Proof. The first part follows from [MW, Lemma 1.7]. By the dual argument used therein, we have that τ is the coequalizer of the parallel pair $(\nu\circ\tau,\mathrm{Id})$. Therefore, given a morphism $p:A\to B$ such that Gp is a regular epimorphism, the two rows in the following diagram are coequalizers.

$$\begin{array}{ccccc} GA & \overset{\nu A\circ\tau A}{\underset{\mathrm{Id}_{GA}}{\rightrightarrows}} & GA & \overset{\tau A}{\longrightarrow} & FA\\ {\scriptstyle Gp}\downarrow & & {\scriptstyle Gp}\downarrow & & \downarrow{\scriptstyle Fp}\\ GB & \overset{\nu B\circ\tau B}{\underset{\mathrm{Id}_{GB}}{\rightrightarrows}} & GB & \underset{\tau B}{\longrightarrow} & FB \end{array}$$

Since Gp is an epimorphism, by a well-known result (see e.g. [Ho, Proposition 2.4]), we have that the right square above is a pushout. Hence, by [Bo1, Proposition 4.3.8], we conclude that Fp is a regular epimorphism. □

Consider the natural transformation $\xi:P\to\Omega\mho$ defined, as in Theorem 5.1, by

$$P\overset{\eta P}{\to}\Omega TP=\Omega\mho\widetilde{T}P\overset{\Omega\mho\widetilde{\epsilon}}{\to}\Omega\mho.$$

As in the proof of the above theorem, we have (26) that in local notations becomes

$$\epsilon\mho\circ T\xi=\mho\widetilde{\epsilon}. \tag{38}$$

so that ξ is exactly the natural transformation of [AM1, Theorem 4.6], whose components are the canonical inclusions of the subobject of primitives of a bialgebra B in $\mathcal{M}$ into $\Omega\mho B$ and hence they are regular monomorphisms.

Since $UT^{+}=T$, we can define

$$\zeta:=\left(\Omega^{+}\overset{\eta\Omega^{+}}{\to}\Omega T\Omega^{+}=\Omega UT^{+}\Omega^{+}\overset{\Omega U\epsilon^{+}}{\to}\Omega U\right).$$

Given $(A,\varepsilon)\in\mathrm{Alg}^{+}(\mathcal{M})$, we have $\zeta(A,\varepsilon):\mathrm{Ker}(\Omega\varepsilon)\to\Omega A$.

Remark 6.12. We compute

$$\begin{aligned}\zeta(A,\varepsilon) &= \left(\Omega U\epsilon^{+}\circ\eta\Omega^{+}\right)(A,\varepsilon)=\Omega U\epsilon^{+}(A,\varepsilon)\circ\eta\Omega^{+}(A,\varepsilon)\\ &\overset{(33)}{=} \Omega(\epsilon A\circ TkA)\circ\eta\Omega^{+}(A,\varepsilon)=\Omega\epsilon A\circ\Omega TkA\circ\eta\Omega^{+}(A,\varepsilon)=\Omega\epsilon A\circ\eta\Omega A\circ kA=kA.\end{aligned}$$

where kA is the morphism in diagram (31). Thus

$$\zeta(A,\varepsilon)=kA. \tag{39}$$

Lemma 6.13. *There is a natural transformation* $\tau:\Omega U\rightarrow\Omega^{+}$ *such that* $\tau\circ\zeta=\mathrm{Id}$. *As a consequence* $\Omega^{+}:\mathrm{Alg}^{+}(\mathcal{M})\rightarrow\mathcal{M}$ *preserves coequalizers for pairs* (fe,f) *where* $f:A\rightarrow A'$ *is composition of regular epimorphisms in* $\mathrm{Alg}^{+}(\mathcal{M})$ *and* $e:A\rightarrow A$ *is a morphism in* $\mathrm{Alg}^{+}(\mathcal{M})$ *such that* $f\circ e\circ e=f\circ e$. *Moreover* Ω^{+} *preserves regular epimorphisms.*

Proof. Let $(A,\varepsilon)\in\mathrm{Alg}^{+}(\mathcal{M})$. As observed, the pullback (31) means that $\Omega^{+}(A,\varepsilon)=KA=\mathrm{Ker}(\Omega\varepsilon)$. The canonical inclusion is kA which by (39) equals $\zeta(A,\varepsilon)$. Thus we have the following kernel in $\mathcal{M}$.

$$0\rightarrow\Omega^{+}(A,\varepsilon)\overset{\zeta(A,\varepsilon)}{\longrightarrow}\Omega A\overset{\Omega\varepsilon}{\longrightarrow}\Omega\mathbf{1}=\mathbf{1}$$

Since ε is an algebra morphism, we have $\Omega\varepsilon\circ u_{\Omega A}=\mathrm{Id}$. Hence $\Omega\varepsilon\circ(\mathrm{Id}_{\Omega A}-u_{\Omega A}\circ\Omega\varepsilon)=0$ so that, by the universal property of the kernel we get a unique morphism $\tau(A,\varepsilon):\Omega A\rightarrow\Omega^{+}(A,\varepsilon)$ such that $\zeta(A,\varepsilon)\circ\tau(A,\varepsilon)=\mathrm{Id}_{\Omega A}-u_{\Omega A}\circ\Omega\varepsilon$. Moreover $\tau(A,\varepsilon)\circ\zeta(A,\varepsilon)=\mathrm{Id}_{\Omega^{+}(A,\varepsilon)}$.

It remains to check that $\tau(A,\varepsilon)$ is natural in (A,ε). To this aim, first let $f:(A,\varepsilon_{A})\rightarrow(B,\varepsilon_{B})$ be a morphism in $\mathrm{Alg}^{+}(\mathcal{M})$ and compute

$$\begin{aligned}\zeta(B,\varepsilon_{B})\circ\tau(B,\varepsilon_{B})\circ\Omega Uf &= (\mathrm{Id}_{\Omega B}-u_{\Omega B}\circ\Omega\varepsilon_{B})\circ\Omega Uf=\Omega Uf-u_{\Omega B}\circ\Omega\varepsilon_{B}\circ\Omega Uf\\ &= \Omega Uf-u_{\Omega B}\circ\Omega(\varepsilon_{B}\circ Uf)=\Omega Uf-u_{\Omega B}\circ\Omega\varepsilon_{A}\\ &= \Omega Uf-\Omega Uf\circ u_{\Omega A}\circ\Omega\varepsilon_{A}=\Omega Uf\circ(\mathrm{Id}_{\Omega A}-u_{\Omega A}\circ\Omega\varepsilon_{A})\\ &= \Omega Uf\circ\zeta(A,\varepsilon_{A})\circ\tau(A,\varepsilon)=\Omega Uf\circ kA\circ\tau(A,\varepsilon)\\ &\overset{(32)}{=} kA'\circ\Omega^{+}f\circ\tau(A,\varepsilon)=\zeta(B,\varepsilon_{B})\circ\Omega^{+}f\circ\tau(A,\varepsilon).\end{aligned}$$

Since $\zeta(B,\varepsilon_{B})$ is a monomorphism we deduce $\tau(B,\varepsilon_{B})\circ\Omega Uf=\Omega^{+}f\circ\tau(A,\varepsilon)$ which means that τ is natural. Thus $\zeta:\Omega^{+}\rightarrow\Omega U$ cosplits via $\tau:\Omega U\rightarrow\Omega^{+}$ i.e. $\tau\circ\zeta=\mathrm{Id}$.

Now, by Lemma 6.6, the functor $\Omega:\mathrm{Alg}(\mathcal{M})\rightarrow\mathcal{M}$ preserves regular epimorphisms.

By Lemma 6.8, the forgetful functor $U:\mathrm{Alg}^{+}(\mathcal{M})\longrightarrow\mathrm{Alg}(\mathcal{M})$ creates colimits and preserves all coequalizers. As a consequence $\Omega U:\mathrm{Alg}^{+}(\mathcal{M})\longrightarrow\mathcal{M}$ preserves regular epimorphisms. Hence by Proposition 6.11 also Ω^{+} preserves regular epimorphisms. By Corollary 6.10, the functor $\Omega U:\mathrm{Alg}^{+}(\mathcal{M})\longrightarrow\mathcal{M}$ preserves coequalizers for pairs (fe,f) where $f:A\rightarrow A'$ is composition of regular epimorphisms in $\mathrm{Alg}^{+}(\mathcal{M})$ and $e:A\rightarrow A$ is a morphism in $\mathrm{Alg}^{+}(\mathcal{M})$ such that $f\circ e\circ e=f\circ e$. By Proposition 6.11, the functor Ω^{+} preserves the same type of coequalizers. □

Next aim is to show that the functor $T^{+}:\mathcal{M}\rightarrow\mathrm{Alg}^{+}(\mathcal{M})$ is h-separable. First note that there is a unique morphism $\omega V:\Omega TV\rightarrow V$ such that

$$\omega V\circ\alpha_{n}V=\delta_{n,1}\mathrm{Id}_{V}. \tag{40}$$

Given $f:V\rightarrow W$ a morphism in $\mathcal{M}$, we get for every $n\in\mathbb{N}$,

$$\omega W\circ\Omega Tf\circ\alpha_{n}V=\omega W\circ\alpha_{n}W\circ f^{\otimes n}=\delta_{n,1}f^{\otimes n}=\delta_{n,1}f=f\circ\omega V\circ\alpha_{n}V$$

so that $\omega W\circ\Omega Tf=f\circ\omega V$ which means that $\omega:=(\omega V)_{V\in\mathcal{M}}$ is a natural transformation $\omega:\Omega T\rightarrow\mathrm{Id}_{\mathcal{M}}$.

Lemma 6.14. *The natural transformation* ω *fulfills* $\omega\circ\eta=\mathrm{Id}$ *and*

$$\omega\omega\circ\Omega T\zeta T^{+}=\omega\circ\Omega\epsilon T\circ\Omega T\zeta T^{+}. \tag{41}$$

Proof. In [AM3, Lemma 5.2] we prove that $\omega\omega\circ\Omega T\zeta'\widetilde{T}=\omega\circ\Omega\epsilon T\circ\Omega T\zeta'\widetilde{T}$ where $\zeta':E\rightarrow\Omega\mho$ is a natural transformation whose domain is the functor $E:\mathrm{Bialg}(\mathcal{M})\rightarrow\mathcal{M}$ assigning to each bialgebra A the kernel $\left(EA,\zeta'A:EA\rightarrow\Omega\mho A\right)$ in $\mathcal{M}$ of its counit $\Omega\varepsilon_{\mho A}$, where here $\varepsilon_{\mho A}$ is regarded as an algebra map. Then, for every $M\in\mathcal{M}$, we have

$$\left(E\widetilde{T}M,\zeta'\widetilde{T}M\right)=\mathrm{Ker}\left(\Omega\varepsilon_{\mho\widetilde{T}M}\right)=\mathrm{Ker}\left(\Omega\varepsilon_{TM}\right)=\left(\Omega^{+}\left(TM,\varepsilon_{TM}\right),kTM\right)\overset{(39)}{=}\left(\Omega^{+}T^{+}M,\zeta T^{+}M\right).$$

Moreover, given $f:M\rightarrow N$, since, by (39) we have $kTM=\zeta T^{+}M$, we obtain

$$\begin{aligned}\zeta T^{+}N\circ\Omega^{+}T^{+}f &= kTN\circ\Omega^{+}T^{+}f=\Omega UT^{+}f\circ kTM=\Omega Tf\circ\zeta T^{+}M\\ &= \Omega\mho\widetilde{T}f\circ\zeta'\widetilde{T}M=\zeta'\widetilde{T}N\circ E\widetilde{T}f=\zeta T^{+}N\circ E\widetilde{T}f\end{aligned}$$

so that $\Omega^+T^+f = E\widetilde{T}f$. As a consequence $E\widetilde{T} = \Omega^+T^+$ and $\zeta'\widetilde{T} = \zeta T^+$. If we substitute $\zeta'\widetilde{T}$ by ζT^+ in the starting equality we obtain the desired one. $\square$

The following result shows that T^+ is h-separable

Lemma 6.15. *$\zeta T^+ : \Omega^+T^+ \to \Omega T$ is a monad morphism between the monads associated to (T^+, Ω^+) and (T, Ω). Moreover $\omega^+ := \omega \circ \zeta T^+ : \Omega^+T^+ \to \mathrm{Id}$ is an augmentation for the monad associated to the adjunction (T^+, Ω^+). Equivalently T^+ is h-separable.*

Proof. The monads to consider are $(\Omega^+T^+, \Omega^+\epsilon^+T^+, \eta^+)$ and $(\Omega T, \Omega\epsilon T, \eta)$. We compute

$$\zeta T^+ \circ \Omega^+\epsilon^+T^+ \overset{\text{nat. } \zeta}{=} \Omega U\epsilon^+T^+ \circ \zeta T^+\Omega^+T^+ \overset{(33),(39)}{=} \Omega\epsilon UT^+ \circ \Omega T\zeta T^+ \circ \zeta T^+\Omega^+T^+ = \Omega\epsilon T \circ \zeta T^+\zeta T^+$$

$$\zeta T^+ \circ \eta^+ \overset{(39)}{=} kUT^+ \circ \eta^+ = kT \circ \eta^+ \overset{(33)}{=} \eta.$$

We have so proved that $\zeta T^+ : \Omega^+T^+ \to \Omega T$ is a morphism of monads. We compute

$$\begin{aligned} \omega^+\omega^+ &= \omega\omega \circ \zeta T^+\zeta T^+ = \omega\omega \circ \Omega T\zeta T^+ \circ \zeta T^+\Omega^+T^+ \overset{(41)}{=} \omega \circ \Omega\epsilon T \circ \Omega T\zeta T^+ \circ \zeta T^+\Omega^+T^+, \\ &= \omega \circ \Omega\epsilon T \circ \zeta T^+\zeta T^+ = \omega \circ \zeta T^+ \circ \Omega^+\epsilon^+T^+ = \omega^+ \circ \Omega^+\epsilon^+T^+. \end{aligned}$$

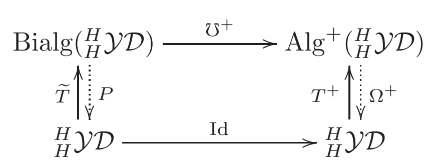
Moreover $\omega^+ \circ \eta^+ = \omega \circ \zeta T^+ \circ \eta^+ = \omega \circ \eta = \mathrm{Id}$. Thus ω^+ is an augmentation for the monad $(\Omega^+T^+, \Omega^+\epsilon^+T^+, \eta^+)$. By [AM3, Corollary 2.7], this means that T^+ is h-separable. $\square$

As a consequence of the results above, Theorem 5.1 applies to the leftmost diagram in (34).

7. Conclusions

In this section we collect some fallouts of Theorem 5.1. We describe explicitly the functor $\Gamma_{[n]}$ in case of Yetter-Drinfeld modules and in particular of vector spaces. We infer an analogue of the notion of combinatorial rank and we propose possible lines of future investigation on the subject.

Example 7.1. Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$. We want to apply the results of the previous sections in the case when $\mathcal{M}$ is the category ${}^H_H\mathcal{YD}$ of (left-left) Yetter-Drinfeld modules over H. This category is braided as the antipode of H is invertible. Moreover ${}^H_H\mathcal{YD}$ satisfies all the requirements of Section 6. The related diagram rewrites as follows and fulfills the assumptions of Theorem 5.1.



Let $V \in {}^H_H\mathcal{YD}$ and $(A, \varepsilon) \in \mathrm{Alg}^+\left({}^H_H\mathcal{YD}\right)$. The object $\widetilde{T}V$ is the usual tensor algebra TV that becomes the tensor algebra in ${}^H_H\mathcal{YD}$, as V belongs to ${}^H_H\mathcal{YD}$, and that is endowed with a braided bialgebra structure by means of its universal property and the braiding of V. By definition

$$T^+V = (TV, \varepsilon_{TV}), \qquad \left(\Omega^+(A, \varepsilon), kA\right) = \mathrm{Ker}(\Omega\varepsilon).$$

Thus $\Omega^+(A, \varepsilon)$ is nothing but the augmentation ideal A^+ regarded as an object in ${}^H_H\mathcal{YD}$ being the kernel of ε which is a morphism in this category. The monad Ω^+T^+ is augmented via the morphism $\omega^+ : \Omega^+T^+ \to \mathrm{Id}$ of Lemma 6.15. Note that, for any $V \in {}^H_H\mathcal{YD}$, the map $\omega^+V : \Omega^+T^+V \to V$ is just the restriction to $\Omega^+T^+V = (TV)^+$ of the canonical projection $\omega V : \Omega TV \to V$ onto V (note that ωV is not in general an augmentation for ΩT because T is not h-separable [AM3, Corollary 2.7 and Remark 5.3]).

By Theorem 5.1, also the monad $P\widetilde{T}$ is augmented via $\gamma := \omega^+ \circ \xi\widetilde{T} : P\widetilde{T} \to \mathrm{Id}$. Explicitly γV is the restriction of ωV to the Yetter-Drinfeld submodule of primitive elements of $\widetilde{T}V$. For every $n \in \mathbb{N}$, there are a functor $\Gamma_{[n]} : {}^H_H\mathcal{YD} \to {}^H_H\mathcal{YD}_{[n]}$ and a natural transformation $\gamma_{[n]} : P\widetilde{T}_{[n]}\Gamma_{[n]} \to$

Id, such that $\Gamma_{[0]} := \mathrm{Id}, \gamma_{[0]} := \gamma$ and, for $n \geq 0, \Gamma_{[n+1]}V = \left(\Gamma_{[n]}V, \gamma_{[n]}V\right) \in {}_{H}^{H}\mathcal{YD}_{[n+1]}$ and $\gamma_{[n]} \circ U_{[n]}\widetilde{\eta}_{[n]}\Gamma_{[n]} = \mathrm{Id}$. Let us describe explicitly the functor

$$S_{[n]} := \widetilde{T}_{[n]}\Gamma_{[n]} : {}_{H}^{H}\mathcal{YD} \to \mathrm{Bialg}\left({}_{H}^{H}\mathcal{YD}\right).$$

For $n = 0$ we have $S_{[0]} := \widetilde{T}_{[0]}\Gamma_{[0]} = \widetilde{T}$. Moreover $S_{[n+1]}V$ is given by the coequalizer, see (28):

$$\widetilde{T}PS_{[n]}V \underset{\widetilde{\epsilon}S_{[n]}V}{\overset{\pi_{[n]}\Gamma_{[n]}V \circ \widetilde{T}\gamma_{[n]}V}{\rightrightarrows}} S_{[n]}V \xrightarrow{\pi_{[n,n+1]}\Gamma_{[n+1]}V} S_{[n+1]}V$$

By Lemma 5.3, a bialgebra map $f : S_{[n]}V \to B$ in ${}_{H}^{H}\mathcal{YD}$ together with the above parallel pair is a fork if and only if $Pf \circ e_{[n]}V = Pf$, where $e_{[n]} := U_{[n]}\widetilde{\eta}_{[n]}\Gamma_{[n]} \circ \gamma_{[n]}$. Since $Pf : PS_{[n]}V \to PB$ is just the restriction of f to the primitive elements, we get that $S_{[n+1]}V$ is obtained by factoring out $S_{[n]}V$ by its two-sided ideal generated by $\mathrm{Im}\left(\mathrm{Id} - e_{[n]}V\right)$. Since $e_{[n]}V$ is idempotent, we have that $\mathrm{Im}\left(\mathrm{Id} - e_{[n]}V\right) = \mathrm{Ker}\left(e_{[n]}V\right)$. By definition of $e_{[n]}V$ and since $U_{[n]}\widetilde{\eta}_{[n]}\Gamma_{[n]}V$ is split-injective, its retraction being $\gamma_{[n]}V$, we get $\mathrm{Im}\left(\mathrm{Id} - e_{[n]}V\right) = \mathrm{Ker}\left(\gamma_{[n]}V\right)$. Hence $S_{[n+1]}V = \frac{S_{[n]}V}{\left\langle \mathrm{Ker}\left(\gamma_{[n]}V\right)\right\rangle}$.

In order to give explicitly $\mathrm{Ker}\left(\gamma_{[n]}V\right)$ aind to get a complete description of the functors $\Gamma_{[n]}$, let us take a closer look at $\gamma_{[n]}$. By construction (see Theorem 5.1), we have that $\gamma_{[n]} := \omega_{[n]}^{+} \circ \xi S_{[n]} : PS_{[n]} \to \mathrm{Id}$ where $\omega_{[n]}^{+} : \Omega^{+}\mho^{+}S_{[n]} \to \mathrm{Id}$ (denoted by $\gamma_{[n]}'$ in the quoted theorem as it stems from $\gamma' = \omega^{+}$) is defined iteratively by the following equality $\omega_{[n+1]}^{+} \circ \Omega^{+}\mho^{+}\pi_{[n,n+1]}\Gamma_{[n+1]} = \omega_{[n]}^{+}$. Since $\Omega^{+}\mho^{+}\pi_{[n,n+1]}\Gamma_{[n+1]}$ is surjective, we get that $\omega_{[n]}^{+}V : \Omega^{+}\mho^{+}S_{[n]}V = \left(S_{[n]}V\right)^{+} \to V$ is just the projection onto V passed to the quotient. Since $\gamma_{[n]} := \omega_{[n]}^{+} \circ \xi S_{[n]}$ we get that $\gamma_{[n]}V : PS_{[n]}V \to V$ is still the projection onto V. As a consequence $\mathrm{Ker}\left(\gamma_{[n]}V\right)$ is spanned by the homogeneous elements of $PS_{[n]}V$ of degree at least two.

Note that, if we forget the structure of Yetter-Drinfeld module and we just keep the underlying braided bialgebra structure, the braided bialgebra $S_{[n]}V$ is exactly what in [Ar2, Definition 3.10] was denoted by $S^{[n]}(B)$ for $B := \widetilde{T}V$. As a consequence, the direct limit of the direct system

$$\widetilde{T}V \to S_{[1]}V \to S_{[2]}V \to \cdots$$

is the Nichols algebra $\mathcal{B}(V, c)$ ([Ar2, Corollary 3.17 and Remark 5.4]), where $c : V \otimes V \to V \otimes V$ is the braiding of V in ${}_{H}^{H}\mathcal{YD}$.

Remark 7.2. In the previous example $S_{[n+1]}V$ is obtained by factoring out $S_{[n]}V$ by the two-sided ideal generated by the primitive elements in $S_{[n]}V$ of degree at least two. Following [Ar2, Definition 4.1 and Section 5], we get that the combinatorial rank of V, regarded as braided vector space through the braiding c of ${}_{H}^{H}\mathcal{YD}$ as above, is the smallest n such that $\pi_{[n,n+1]}\Gamma_{[n+1]}V : S_{[n]}V \to S_{[n+1]}V$ is invertible, if such an n exists. In this case obviously $S_{[n]}V = \mathcal{B}(V, c)$.

Since, in the setting of Theorem 5.1, we can always define $S_{[n]} := L_{[n]}\Gamma_{[n]} : \mathcal{B} \to \mathcal{A}$, for every $B \in \mathcal{B}$ we are lead to the following definition.

Definition 7.3. In the setting of Theorem 5.1, consider the functor $S_{[n]} := L_{[n]}\Gamma_{[n]} : \mathcal{B} \to \mathcal{A}$. We define the **combinatorial rank** of an object $B \in \mathcal{B}$ (with respect to the adjunction (L, R)) to be the smallest n such that $\pi_{[n,n+1]}\Gamma_{[n+1]}B : S_{[n]}B \to S_{[n+1]}B$ is invertible, if such an n exists.

Remark 7.4. Thus a concept of combinatorial rank can be introduced and investigated in this very general setting in which there are neither bialgebras nor braided vector spaces but just an adjunction (L, R) as in Theorem 5.1. Note that, by Lemma 5.3, the morphism $\pi_{[n,n+1]}\Gamma_{[n+1]}B$ is invertible if and only if either $\gamma_{[n]}B$ or $\eta_{[n]}\Gamma_{[n]}B$ is invertible.

As we will see below, a case of interest is the one in which all objects in $\mathcal{B}$ have combinatorial rank at most one, equivalently $\eta_{[1]}\Gamma_{[1]} : \Gamma_{[1]} \to R_{[1]}S_{[1]}$ is invertible. Since, to this aim, only the functor $\Gamma_{[1]}$ is needed, we can even more relax our assumptions by taking just an adjunction (L, R)

with an augmentation $\gamma : RL \to \mathrm{Id}$ for the associated monad, avoiding the setting of Theorem 5.1 and define directly $\Gamma_{[1]}$ by $\Gamma_{[1]}B := (B, \gamma B)$.

Theorem 7.5. *In the setting of Theorem 5.1, if the adjunction (L_N, R_N) is idempotent for some $N \in \mathbb{N}$, then every object in $\mathcal{B}$ has combinatorial rank at most N with respect to the adjunction (L, R). In particular the length of the monadic decomposition of $R : \mathcal{A} \to \mathcal{B}$ is an upper bound for the combinatorial rank of objects in $\mathcal{B}$ with respect to the adjunction (L, R).*

Proof. The fact that the adjunction (L_N, R_N) is idempotent is equivalent to require that $\eta_N U_{N,N+1}$ is an isomorphism. By Proposition5.2, we have that $\Gamma_{[N]} = \Lambda_N \Gamma_N$ and $U_{N,N+1} \circ \Gamma_{N+1} = \Gamma_N$. Thus $\eta_N \Gamma_N = \eta_N U_{N,N+1} \Gamma_{N+1}$ is an isomorphism. As in the proof of Theorem 4.20, we get $R_{[N]} \lambda_N \circ \eta_{[N]} \Lambda_N = \Lambda_N \eta_N$. In particular we get $R_{[N]} \lambda_N \Gamma_N \circ \eta_{[N]} \Lambda_N \Gamma_N = \Lambda_N \eta_N \Gamma_N$ i.e. $R_{[N]} \lambda_N \Gamma_N \circ \eta_{[N]} \Gamma_{[N]} = \Lambda_N \eta_N \Gamma_N$. Since $\eta_N \Gamma_N$ and λ_N are invertible, we get that $\eta_{[N]} \Gamma_{[N]}$ is invertible. By the foregoing, every object in $\mathcal{B}$ has combinatorial rank at most N.

If R has a monadic decomposition of monadic length N, then L_N is fully faithful i.e. η_N is invertible. Thus, in particular, $\eta_N U_{N,N+1}$ is an isomorphism and hence (L_N, R_N) is idempotent. As a consequence every object in $\mathcal{B}$ has combinatorial rank at most N. □

Corollary 7.6. *Let $\mathcal{M}$ be a symmetric MM-category in the sense of* [AM2, Definition 7.4]. *Then every object in $\mathcal{M}$ has combinatorial rank at most one with respect to the adjunction $(\widetilde{T}, P)$.*

Proof. By hypothesis all the requirements of Section 6 are satisfied so that the adjunction $(\widetilde{T}, P)$ is in the setting of Theorem 5.1. By [AM2, Theorem 7.2] the adjunction $(\widetilde{T}_1, P_1)$ is idempotent. We conclude by Theorem 7.5. □

As a consequence all the symmetric MM-categories given in [AM2, Section 9] have objects with combinatorial rank at most one.

Example 7.7. Consider the particular case when $\mathcal{M}$ is the category Vec of vector spaces over a field $\Bbbk$. Since Vec is just ${}^H_H\mathcal{YD}$ in case H is the trivial Hopf algebra $\Bbbk$, this is a particular case of Example 7.1. The diagram above can be more easily written as follows

$$\begin{array}{ccc} \mathrm{Bialg} & \xrightarrow{\mho^+} & \mathrm{Alg}^+ \\ {\scriptstyle \widetilde{T}}\uparrow\downarrow{\scriptstyle P} & & {\scriptstyle T^+}\uparrow\downarrow{\scriptstyle \Omega^+} \\ \mathrm{Vec} & \xrightarrow{\mathrm{Id}} & \mathrm{Vec} \end{array}$$

As above we can define $S_{[n]} := \widetilde{T}_{[n]} \Gamma_{[n]} : \mathrm{Vec} \to \mathrm{Bialg}$. Thus $S_{[0]} := \widetilde{T}$ and $S_{[n+1]}V = \frac{S_{[n]}V}{\langle \mathrm{Ker}\left(\gamma_{[n]}V\right) \rangle}$ is obtained by factoring out $S_{[n]}V$ by the two-sided ideal generated by the homogeneous primitive elements in $S_{[n]}V$ of degree at least two. Note that the procedure we used to compute $S_{[1]}V = \frac{\widetilde{T}V}{\langle \mathrm{Ker}(\gamma V) \rangle}$ is essentially the same used to compute $L_1 V_1$ in the proof of [AGTM, Theorem 3.4].

By [Ar2, Definition 6.8 and Theorem 6.13], if *char* $(\Bbbk) = 0$, and [Ar3, Example 3.13], if *char* $(\Bbbk) = p$, we get that V, regarded as a braided vector space via the braiding $c : V \otimes V \to V \otimes V : x \otimes y \mapsto y \otimes x$ of Vec, has combinatorial rank at most one. Thus Vec is an example of braided monoidal category where every object has combinatorial rank at most one with respect to the adjunction $(\widetilde{T}, P)$.

By the foregoing $S_{[1]}V$ coincides with the Nichols algebra $\mathcal{B}(V, c)$ and all the maps $\pi_{[1,2]} \Gamma_{[2]} V : S_{[1]}V \to S_{[2]}V$, $\gamma_{[1]}V : PS_{[1]}V \to V$ and $U_{[1]} \eta_{[1]} \Gamma_{[1]} V : V \to RS_{[1]}V$ are invertible. By Lemma 5.3, we have that $\pi_{[n,n+1]} \Gamma_{[n+1]} B$ is invertible for all $n \geq 1$ and hence $\gamma_{[n]}V$ is invertible for all $n \geq 1$.

In Example 7.7 we observed that $\gamma_{[1]}V : PS_{[1]}V \to V$ (equivalently $U_{[1]} \eta_{[1]} \Gamma_{[1]} V : V \to RS_{[1]}V$) is an isomorphism for $\mathcal{M} = \mathrm{Vec}$. This fact may fail to be true if we change $\mathcal{M}$. For instance, let us come back to the category ${}^H_H\mathcal{YD}$. By the foregoing we have $S_{[1]}V = \frac{\widetilde{T}V}{\langle \mathrm{Ker}(\gamma V) \rangle}$ where $\gamma V : P\widetilde{T}V \to V$ is the projection on degree one and hence $\mathrm{Ker}(\gamma V)$ are the elements of $P\widetilde{T}V$ of degree at least two. In order to see that the projection $\gamma_{[1]}V : PS_{[1]}V \to V$ and the injection $U_{[1]} \eta_{[1]} \Gamma_{[1]} V : V \to RS_{[1]}V$ need not to be invertible we refer to [Ar2, Section 7] where examples of braided vector spaces of combinatorial rank greater than two, arising as object in ${}_{H}^{H}\mathcal{YD}$ and braided via the braiding of ${}_{H}^{H}\mathcal{YD}$, are given.

It would be of interest to determine which conditions on H guarantee that $U_{[1]}\eta_{[1]}\Gamma_{[1]}V : V \to RS_{[1]}V$ is always invertible for every $V \in {}_{H}^{H}\mathcal{YD}$, equivalently any object in ${}_{H}^{H}\mathcal{YD}$ has combinatorial rank at most one.

Remark 7.8. In [AGTM, Theorem 3.4] we showed that the functor P in case $\mathcal{M} = \mathrm{Vec}$ admits a monadic decomposition of length at most two, represented in the following diagram.

$$\begin{array}{ccccc} \mathrm{Bialg} & \xleftarrow{\mathrm{Id}} & \mathrm{Bialg} & \xleftarrow{\mathrm{Id}} & \mathrm{Bialg} \\ \widetilde{T}\uparrow\downarrow P & & \widetilde{T}_1\uparrow\downarrow P_1 & & \widetilde{T}_2\uparrow\downarrow P_2 \\ \mathrm{Vec} & \xleftarrow{U_{0,1}} & \mathrm{Vec}_1 & \xleftarrow{U_{1,2}} & \mathrm{Vec}_2 \end{array}$$

This result was obtained by proving first that the adjunction $(\widetilde{T}_1, P_1)$ is idempotent or equivalently that $\widetilde{\eta}_1 U_{1,2}$ is an isomorphism. Note that, by [AGTM, Proposition 2.3], we can take $\widetilde{T}_2 := \widetilde{T}_1 U_{1,2}$, $U_{1,2}\widetilde{\eta}_2 = \widetilde{\eta}_1 U_{1,2}$ and $\widetilde{\epsilon}_2 = \widetilde{\epsilon}_1$. We have seen in [AM2, Theorems 7.2 and 8.1] and [AGM2, Theorem 3.3] that the category Vec_2 is equivalent to the category Lie of (restricted) Lie algebras over $\Bbbk$ and that the adjunction $(\widetilde{T}_2, P_2)$ plays the role of the usual adjunction, between the categories Bialg and Lie, given by the (restricted) universal enveloping algebra functor and the primitive functor. The fact that the monadic decomposition has length at most two means that the unit $\widetilde{\eta}_2 : \mathrm{Id} \to P_2\widetilde{T}_2$ is invertible. In view of the identifications we mentioned, this is the counterpart of half of the Milnor–Moore theorem [MM, Theorems 5.18(1) and 6.11(1)]. Now, given $V_2 := (V, \mu, \mu_1) \in \mathrm{Vec}_2$, with $\mu : P\widetilde{T}V \to V$, $V_1 := (V, \mu)$ and $\mu_1 : P_1\widetilde{T}_1 V_1 \to V_1$, one has $\mu_1 \circ \widetilde{\eta}_1 = \mathrm{Id}$ and hence $\mu_1 = (\widetilde{\eta}_1 V_1)^{-1}$ (note that $\widetilde{\eta}_1 V_1 = \widetilde{\eta}_1 U_{1,2} V_2$ is invertible). Moreover $\widetilde{T}_2 V_2 = \widetilde{T}_1 U_{1,2} V_2 = \widetilde{T}_1 V_1$. Following the proof of [AGTM, Theorem 3.4], we can compute explicitly $\widetilde{T}_1 V_1$ as $\frac{\widetilde{T}V}{\langle z - \mu(z) \mid z \in EV\rangle}$, where EV denotes the subspace of $P\widetilde{T}V$ spanned by element of homogeneous degree greater than one, and hence we obtain that $\widetilde{T}_1 V_1 = U(V, c, \mu)$ in the sense of [Ar1, Definition 3.5], where $c : V \otimes V \to V \otimes V : x \otimes y \mapsto y \otimes x$ is the braiding of Vec.

Note that, in the same quoted definition, it is set $S(V, c) := U(V, c, 0) = \frac{\widetilde{T}V}{\langle z \mid z \in EV\rangle}$. Clearly $S(V, c)$ coincides with $S_{[1]}V$ of Example 7.7. In [Ar1, Corollary 5.5] it is proved that $PU(V, c, \mu) \cong V$ using the fact that $PS(V, c) \cong V$. In view of the above identifications, the latter isomorphism means that $U_{[1]}\eta_{[1]}\Gamma_{[1]}V : V \to PL_{[1]}\Gamma_{[1]}V = PS_{[1]}V$ is invertible and we already observed that this is another way to say that V has combinatorial rank at most one (the primitive elements in $PS_{[1]}V$ are concentrated in degree one). On the other hand, the first isomorphism implies that $U_1\widetilde{\eta}_1 V_1 : V \to P\widetilde{T}_1 V_1$ is invertible for any $V_2 \in \mathrm{Vec}_2$. Equivalently $U_1\widetilde{\eta}_1 U_{1,2}$ is invertible which is the same as requiring that $\widetilde{\eta}_1 U_{1,2}$ is invertible i.e. the condition, recalled above, saying that the adjunction $(\widetilde{T}_1, P_1)$ is idempotent. Summing up, using that any object in Vec has combinatorial rank at most one, we can prove that $(\widetilde{T}_1, P_1)$ is idempotent and hence that P has monadic decomposition of length at most two.

As mentioned, we can consider an adjunction (L, R) whose associated monad is augmented. If every object in $\mathcal{B}$ has combinatorial rank at most one, it is natural to wonder if, also in this wider setting, it is true that (L_1, R_1) is idempotent and hence R has monadic decomposition of length at most two. In this way the adjunction (L_2, R_2) would be involved in an analogue of the Milnor–Moore theorem in the above sense. More generally one can ask whether (L_N, R_N) is idempotent in case the combinatorial rank of objects in $\mathcal{B}$ for an adjunction (L, R) as in Theorem 5.1 is at most $N \in \mathbb{N}$. This would provide an inverse of Theorem 7.5.

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