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Abstract

A quantization procedure for the Yang-Mills equations for the Minkowski space $\mathbf{R}^{1,3}$ is carried out in such a way that field maps satisfying Wightman axioms of Constructive Quantum Field Theory can be obtained. Moreover, by removing the infrared and ultraviolet cutoffs, the spectrum of the corresponding (non-local) QCD Hamilton operator is proven to be positive and bounded away from zero, except for the case of the vacuum state, which has vanishing energy level. The whole construction is invariant for all gauge transformations preserving the Coulomb gauge. As expected from QED, if the coupling constant converges to zero, then so does the mass gap. This is the case for the running coupling constant leading to asymptotic freedom.

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Contents

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1 Introduction

Yang-Mills fields, which are also called gauge fields, are used in modern physics to describe physical fields that play the role of carriers of an interaction (cf. [EoM02]). Thus, the electromagnetic field in electrodynamics, the field of vector bosons, carriers of the weak interaction in the Weinberg-Salam theory of electrically weak interactions, and finally, the gluon field, the carrier of the strong interaction, are described by Yang-Mills fields. The gravitational field can also be interpreted as a Yang-Mills field (see [DP75]).

The idea of a connection as a field was first developed by H. Weyl (1917), who also attempted to describe the electromagnetic field in terms of a connection. In 1954, C.N. Yang and R.L. Mills (cf. [MY54]) suggested that the space of intrinsic degrees of freedom of elementary particles (for example, the isotropic space describing the two degrees of freedom of a nucleon that correspond to its two pure states, proton and neutron) depends on the points of space-time, and the intrinsic spaces corresponding to different points are not canonically isomorphic.

In geometrical terms, the suggestion of Yang and Mills was that the space of intrinsic degrees of freedom is a vector bundle over space-time that does not have a canonical trivialization, and physical fields are described by cross-sections of this bundle. To describe the differential evolution equation of a field, one has to define a connection in the bundle, that is, a trivialization of the bundle along the curves in the base. Such a connection with a fixed holonomy group describes a physical field, usually called a Yang-Mills field. The equations for a free Yang-Mills field can be deduced from a variational principle. They are a natural non-linear generalization of Maxwell's equations (cf.[Br03]).

Field theory does not give the complete picture. Since the early part of the 20th century, it has been understood that the description of nature at the subatomic scale requires quantum mechanics, where classical observables correspond to typically non commuting self-adjoint operators on a Hilbert space, and classic notions as "the trajectory of a particle" do not apply. Since fields interact with particles, it became clear by the late 1920s that an internally coherent account of nature must incorporate quantum concepts for fields as well as for particles. Under this approach components of fields at different points in space-time become non-commuting operators.

The most important Quantum Field Theories describing elementary particle physics are gauge theories formulated in terms of a principal fibre bundle over the Minkowskian space-time with particular choices of the structure group. They are depicted in Table 1.

As shown in [JW04], in order for Quantum Chromodynamics to completely explain the observed world of strong interactions, the theory must imply:

- **Mass gap:** There must exist some positive constant η such that the excitation of the vacuum state has energy at least η. This would explain why the nuclear force is strong but short-ranged, by providing the mathematical evidence that the corresponding exchange particle, the gluon, has non vanishing rest mass.

- **Quark confinement:** The physical particle states corresponding to proton, neutron and pion must be SU(3)-invariant. This would explain why individual quarks are never observed.

- **Chiral symmetry breaking:** In the limit for vanishing quark-bare masses the vacuum is in-

Gauge Theory	**Fundamental Forces**	**Structure Group**
Quantum Electrodynamics (QED)	Electromagnetism	$U(1)$
Electroweak Theory (Glashow-Salam-Weinberg)	Electromagnetism and weak force	$\mathrm{SU}(2) \times U(1)$
Quantum Chromodynamics (QCD)	Strong force and electromagnetism	SU(3)
Standard Model	Strong, weak forces and electromagnetism	$\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)$
Georgi-Glashow Grand Unified Theory (GUT1)	Strong, weak forces and electromagnetism	SU(5)
Fritzsch-Georgi-Minkowski Grand Unified Theory (GUT2)	Strong, weak forces and electromagnetism	SO(10)
Grand Unified Theory (GUT3)	Strong, weak forces and electromagnetism	SU(8)
Grand Unified Theory (GUT4)	Strong, weak forces and electromagnetism	O(16)

Table 1: Gauge Theories

variant under a certain subgroup of the full symmetry group acting on the quark fields. This is required in order to account for the "current algebra" theory of soft pions.

The Seventh CMI-Millenium prize problem is the following conjecture.

Conjecture 1. For any compact simple Lie group G there exists a non-trivial Yang-Mills theory on the Minkowskian $\mathbf{R}^{1,3}$, whose quantization satisfies Wightman axiomatic properties of Constructive Quantum Field Theory and has a mass gap $\eta > 0$.

The conjecture is explained in [JW04] and commented in [Do04] and in [Fa05]. To our knowledge this conjecture is unproved.

The first rigorous program of study of this problem is the one by Balaban ([Ba84], [Ba84Bis], [Ba85], [Ba85Bis], [Ba85Tris], [Ba85Quater], [Ba87], [Ba88], [Ba88Bis], [Ba89] and [Ba89Bis]). This program defines a sequence of block-spin transformations for the pure Yang-Mills theory in a finite volume on the lattice, a toroidal Euclidean space-time, and shows that, as the lattice spacing tends to 0 and these transformations are iterated many times, the resulting effective action on the unit lattice remains bounded. From this result the existence of an ultraviolet limit for gauge invariant observables such as "smoothed Wilson loops" should follow, at least through a compactness argument using a subsequence of approximations; but the limit is not necessarily unique. From this point of view, one must verify the existence of an ultraviolet limit of appropriate expectations of gauge-invariant observables as the lattice spacing tends to zero (ultraviolet cutoff removal) and the volume tends to infinity (infrared cutoff removal). This program is applied for the Φ^4_3 model by Dimock in the very readable [Di13], [Di13Bis], [Di14].

Magnen, Rivasseau and Sénéor provide in [MRS93] the basis for a rigorous construction of the Schwinger functions of the pure $SU(2)$ Yang-Mills field theory in four dimensions (in the trivial topological sector) with a fixed infrared cutoff and removed ultraviolet cutoff, in a regularized axial gauge. They check the validity of the construction by showing that Slavnov identities (which express infinitesimal gauge invariance) do hold non-perturbatively. They do not analyze the spectral properties of the Hamilton operator.

Very recently, regularity structures have been successfully by Chandra-Chevyrev-Hairer-Shen ([CCHS22]) to obtain a stochastic quantization of Yang–Mills–Higgs in 3D. Previously, regularity structures, which had been pioneered by Hairer in [Ha14], have been applied by Gubinelli-Hofmanová ([GuHo19]) to extend a stochastic quantization of the Euclidean Φ^4_3 quantum field theory from the torus T^3 to $\mathbf{R}^3$. It would be interesting to see this approach applied to the Yang-Mills problem in $(3+1)$D.

The main contributions of this paper are:

- the construction of a rigorous quantum Yang-Mills theory over the whole Minkowski space satisfying the Osterwalder-Schrader axioms of Constructive Quantum Field Theory,
- the proof that the quantum hamiltonian for QCD has a spectral gap.

This paper is organized as follows. Section 2 presents the classical Yang-Mills equations and their Hamiltonian formulation for the Minkowskian $\mathbf{R}^{1,3}$. Section 3 depicts the axioms of Constructive Quantum Field Theory and may be skipped by the acquainted reader. In Sections 4 the Yang-Mills Equations are quantized, the Osterwalder-Schrader and hence the Wightman axioms are verified, and the existence of a positive mass gap proven. More in detail, the construction of the Yang-Mills Quantum Field Theory in $3+1$ spacetime dimensions and the proof that the corresponding Hamiton operator possesses a spectral gap is done by passing to the corresponding Euclidean QFT and progresses through the following steps:

1. Impose an infrared and an ultraviolet cutoff.
2. Construct a background "quasi-free" QFT as the solution to an SDE via the Fokker-Planck equation using the part of the Euclidean action of the form $-\Delta + B \cdot \nabla$. In contrast to the free QFT this part contains a non local term.
3. Use the Feynman-Kač-Nelson formula to add in the interaction.

4. Remove the infrared cutoff and prove the Osterwalder-Schrader axioms for any fixed ultraviolet cutoff
5. Remove the ultraviolet cutoff and prove the Osterwalder-Schrader axioms for the limit of the ultraviolet cutoff parameter tending to infinity
6. Verify the gauge invariance.
7. Show the existence of the mass gap.

Even more in detail, by verifying the Osterwalder-Schrader axioms and the reconstruction theorem for quantum mechanics, the QFT Hamiltonian for the continuum theory, constructed via quantization of the classical Hamiltonian, is proved to be a selfadjoint operator on the Hilbert Space of L^2-Hida distributions. The Hamilton operator is non-local and the proof of its selfadjointness for a particular probability measure requires the construction of the infinitesimal generator of a stochastic process taking values in a L^2-space over the physical space, and the extension of the Feynman-Kač-Nelson formula for the L^2-Hida distributions.

The QFT Hamiltonian has a continuous spectrum, which can be expressed as the direct limit of the continuous spectrum of another selfadjoint operator, the QFT Hamiltonian with ultraviolet cutoff, when the cutoff parameter tends to infinity. For both operators strictly positive lower bounds of the spectra can be inferred. These depend on the bare coupling constant for both Hamiltonians and on the cutoff magnitude for the Hamiltonian with cutoff. The gap for the cutoff case "survives" then in the continuum limit for a strictly positive bare coupling constant. As expected from QED, if the coupling constant converges to zero, then so does the mass gap.

We remark, that, due to our particular choice of the ultraviolet cutoff differing substantially from the one chosen by Magnen-Rivasseau-Sénéor in [MRS93], we do not observe divergences arising for the ultraviolet cutoff parameter tending to infinity, and, hence, no dependence of the bare coupling constant on the cutoff parameter and no renormalization have to be introduced in our basic construction at this stage. Note that in the Φ_3^4 model renormalization is needed, but concerns other parameters than the bare coupling constant (see [GJ73], [FO76] and [MS76]). Later, in order to obtain asymptotic freedom we will study the running of the coupling constant with respect to the energy scale, showing that the running mass gap vanishes in the asymptotic freedom limit, because the running coupling constant does, too.

The whole construction is proved to be invariant for those gauge transforms which preserve the Coulomb gauge. Section 5 concludes.

2 Yang-Mills Connections

2.1 Definitions, Existence and Uniqueness

A Yang-Mills connection is a connection in a principal fibre bundle over a (pseudo-)Riemannian manifold whose curvature satisfies the harmonicity condition, i.e. the Yang-Mills equation.

Definition 2.1 (**Yang-Mills Connection**). Let P be a principal G-fibre bundle over a pseudoriemannian m-dimensional manifold (M, h), and let V be the vector bundle associated with P and $\mathbf{R}^K$, induced by the representation $\rho : G \to \mathrm{GL}(\mathbf{R}^K)$, where $K := \dim(G)$. A connection on the principal fibre bundle P is a Lie-algebra $\mathfrak{g}$ valued 1-form ω on P, such that the following properties hold:

(i) Let $A \in \mathfrak{g}$ and A^* the vector field on P defined by

$$A^*_p := \frac{d}{dt}\bigg|_{t:=0} (p \exp(tA)). \tag{1}$$

Then, $\omega(A^*_p) = A$.

(ii) For $g \in G$ let

$$\begin{aligned} \mathrm{Ad}_g : G \to G, h \mapsto \mathrm{Ad}_g(h) := L_g \circ R_{g^{-1}}(h) = ghg^{-1} \\ \mathrm{ad}_g : \mathfrak{g} \to \mathfrak{g}, A \mapsto \mathrm{Ad}_g(A) := \frac{d}{dt}\bigg|_{t:=0} (g \exp(tA) g^{-1}) \end{aligned} \tag{2}$$

be the adjoint isomorphism and the adjoint representation, respectively.
Then, $R^*_g \omega = \mathrm{ad}_{g^{-1}} \omega$.

The connection ω on P defines a connection ∇ for the vector bundle V, i.e. an operator acting on the space of cross sections of V. The vector bundle connection ∇ can be extended to an operator $d : \Gamma(\bigwedge^p(M) \bigotimes V) \to \Gamma(\bigwedge^{p+1}(M) \bigotimes V)$, by the formula

$$d^\nabla(\eta \otimes v) := d\eta \otimes v + (-1)^p \eta \otimes \nabla v. \tag{3}$$

The operator $\delta^\nabla : \Gamma(\bigwedge^{p+1}(M) \bigotimes V) \to \Gamma(\bigwedge^p(M) \bigotimes V)$, defined as the formal adjoint to d, is equal to

$$\delta^\nabla \eta = (-1)^{p+1} * d^\nabla *, \tag{4}$$

where $*$ denotes the Hodge-star operator on the pseudoriemannian manifold M.

A connection ω in a principal fibre bundle P is called a Yang-Mills field if the curvature $F :=$

$d\omega + \omega \wedge \omega$, considered as a 2-form with values in the Lie algebra $\mathfrak{g}$, satisfies the Yang-Mills equations

$$\delta^{\nabla} F = 0, \tag{5}$$

or, equivalently,

$$\delta^{\nabla} R^{\nabla} = 0, \tag{6}$$

where $R^{\nabla}(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ denotes the curvature of the vector bundle V, and is a 2-form with values in V.

Remark 2.1 (**Local Representations of Connections on Vector and Principle Fibre Bundles**). The local section $\sigma : U \subset M \to P$ is defines the local representation of the connection given on the open $U \subset M$ by $A := \omega \circ \sigma : U \to \mathfrak{g}$ a Lie-algebra $\mathfrak{g}$ valued 1-form on U, the fields $A_j(x) := A(x)e_j = \sum_{k=1}^{K} A_j^k(x) t_k$ define by means of the tangential map $T_e\rho : \mathfrak{g} \to \mathcal{L}(\mathbf{C}^K)$ of the representation $\rho : G \to \mathrm{GL}(\mathbf{C}^K)$, with fields of endomorphisms $T_e\rho A_1, \ldots, T_e\rho A_m \in \mathcal{L}(V_x)$ for the bundle V. Given a basis of the Lie-algebra $\mathfrak{g}$ denoted by $\{t_1, \ldots, t_K\}$, the endomorphisms $\{w_s := T_e\rho.t_s\}_{s=1,\ldots,K}$ in $\mathcal{L}(\mathbf{C}^K)$ have matrix representations with respect to a local basis $\{v_s(x)\}_{s=1,\ldots,K}$ denoted by $[w_s]_{\{v_s(x)\}}$. Since ρ is a representation, $T_e\rho$ has maximal rank and the endomorphisms are linearly independent. Given a local basis $\{e_j(x)\}_{j=1,\ldots,m}$ for $x \in U \subset M$, the Christoffel symbols of the connection ∇ are locally defined by the equation

$$\nabla_{e_j} v_s = \sum_{r=1}^{K} \Gamma_{j,s}^r v_r, \tag{7}$$

holding true on U, and they ,satisfy the equalities

$$\Gamma_{j,s}^r = \sum_{a=1}^{K} [w_a]_s^r A_j^a. \tag{8}$$

Given a local vector field $v = \sum_{s=1}^{K} f^s v_s$ in $V|_U$ and a local vector field e in $TM|_U$, the connection ∇ has a local representation

$$\nabla_e v = \sum_{s=1}^{K} (df^s(e).v_s + f^s \omega(e).v_s), \tag{9}$$

where ω is an element of $T^*U|_U \bigotimes \mathcal{L}(V|_U)$, i.e. an endomorphism valued $1-$form satisfying

$$\omega(e_j)v_s = \sum_{r=1}^{K} \Gamma_{j,s}^r v_r. \tag{10}$$

Remark 2.2. The curvature 2-form reads in local coordinates as

$$F = \sum_{1\leqslant i<j\leqslant M} \sum_{k=1}^{K} F_{i,j}^{k}\, t_k\, dx_i \wedge dx_j = \frac{1}{2}\sum_{i,j=1}^{M}\sum_{k=1}^{K}\left(\partial_j A_i^k - \partial_i A_j^k - \sum_{a,b=1}^{K} C_{a,b}^{k} A_i^a A_j^b\right) t_k\, dx_i \wedge dx_j, \tag{11}$$

where $C = [C_{a,b}^c]_{a,b,c=1,\dots,K}$ are the *structure constants* of the Lie-algebra $\mathfrak{g}$ corresponding to the basis $\{t_1, \dots, t_K\}$, which means that for any a, b

$$[t_a, t_b] = \sum_{c=1}^{K} C_{a,b}^c t_c. \tag{12}$$

The existence and uniqueness of solutions of the Yang-Mills equations in the Minkowski space have been first established by Segal (cf. [Se78] and [Se79]), who proves that the corresponding Cauchy problem encoding initial regular data has always a unique local and global regular solution. He proves as well that the temporal gauge ($A_0 = 0$) chosen to express the solution does not affect generality, because any solution of the Yang-Mills equation can be carried to one satisfying the temporal gauge. This subject has been undergoing intensive research, improving the original results. For example in [EM82] and in [EM82Bis] the Yang-Mills-Higgs equations, which generalize (5) and are non linear PDEs of order two, have been reformulated in the temporal gauge as a non-linear PDE of order one, satisfying a constraint equation. This PDE can be written as an integral equation solving (always and uniquely, locally and globally) the Cauchy data problem with improved regularity results. Existence, uniqueness and regularity of the Yang-Mills-Higgs equations under the MIT Bag boundary conditions have been investigated in [ScSn94] and [ScSn95].

2.2 Hamiltonian Formulation for the Minkowski Space

The Hamilton function describes the dynamics of a physical system in classical mechanics by means of Hamilton's equations. Therefore, we have to reformulate the Yang-Mills equations in Hamiltonian mechanical terms. We focus our attention on the Minskowski $\mathbf{R}^4$ with the pseudoriemannian structure of special relativity $h = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3$. The coordinate x^0 represents the time t, while x^1, x^2, x^3 are the space coordinates.

We introduce Einstein's summation notation, and adopt the convention that indices for coordinate variables from the greek alphabet vary over $\{0, 1, 2, 3\}$, and those from the latin alphabet vary over the space indices $\{1, 2, 3\}$. For a generic field $F = [F_\mu]_{\mu=0,1,2,3}$ let $\mathbf{F} := [F_i]_{i=1,2,3}$ denote the "space" component. The color indices lie in $\{1,\dots,K\}$. Let

$$\varepsilon^{a,b,c} := \begin{cases} +1 & (\pi \text{ is even}) \\ -1 & (\pi \text{ is odd}) \\ 0 & (\text{two indices are equal}), \end{cases} \tag{13}$$

and any other choice of lower and upper indices, be the Levi-Civita symbol, defined by mean of the permutation $\pi := \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$ in $\mathfrak{S}^3$.

Remark 2.3. If the Lie-group G is simple, then the Lie-Algebra is simple, and the structure constants can be written as

$$C^c_{a,b} = g\varepsilon^c_{a,b}, \tag{14}$$

for a positive constant g called *(bare) coupling constant*, (see f.i. [We05] Chapter 15, Appendix A). The components of the curvature then read

$$F^k_{\mu,\nu} = \frac{1}{2}(\partial_\nu A^k_\mu - \partial_\mu A^k_\nu - g\varepsilon^k_{a,b}A^a_\mu A^b_\nu). \tag{15}$$

We will consider only simple Lie groups. As we will see, it is essential for the existence of a mass gap for the group G to be non-abelian.
The number $C_2(G)$ is defined as

$$\delta^{k,l}C_2(G) = \sum_{a,b=1}^{K} \varepsilon^k_{a,b}\varepsilon^l_{a,b}, \tag{16}$$

which is the quadratic **Casimir operator** in the adjoint representation of the Lie algebra of G.

We need to introduce an appropriate gauge for the connections we are considering.

Definition 2.2 (**Coulomb Gauge**)**.** A connection A over the Minkowski space satisfies the *Coulomb gauge* if and only if

$$\partial_j A^a_j = 0 \tag{17}$$

for all $a = 1,\dots,K$.

Definition 2.3 (**Transverse Projector**)**.** Let $\mathcal{F}$ be the Fourier transform on functions in $L^2(\mathbf{R}^3,\mathbf{R})$. The transverse projector $T : L^2(\mathbf{R}^3,\mathbf{R}^3) \to L^2(\mathbf{R}^3,\mathbf{R}^3)$ is defined as

$$(Tv)_i := \mathcal{F}^{-1}\left(\left[\delta_{i,j} - \frac{p_ip_j}{|p|^2}\right]\mathcal{F}(v_j)\right), \tag{18}$$

and the vector field v decomposes into a sum of a *transversal* ($v^{\perp}$) and a *longitudinal* ($v^{\|}$) component:

$$v_i = v_i^{\perp} + v_i^{\|}, \quad v_i^{\perp} := (Tv)_i, \quad v_i^{\|} := v_i - (Tv)_i. \tag{19}$$

Remark 2.4. The Coulomb gauge condition for the space part of a connection A is equivalent to the vanishing of its longitudinal component:

$$A_i^{a\|}(t,\cdot) = 0 \tag{20}$$

for all $i = 1,2,3$, all $a = 1,\dots K$ and any $t \in \mathbf{R}$. The time part A_0 of the connection A vanishes by definition of Coulomb gauge.

Proposition 2.1. *For a simple Lie-group as structure group let A be a connection over the Minkowskian $\mathbf{R}^4$ satisfying the Coulomb gauge, and assume that $A_i^a(t,\cdot) \in C^{\infty}(\mathbf{R}^3,\mathbf{R}) \cap L^2(\mathbf{R}^3,\mathbf{R})$ for all $i = 1,2,3$, all $a = 1,\dots K$ and any $t \in \mathbf{R}$. The operator L on the real Hilbert space $L^2(\mathbf{R}^3,\mathbf{R}^K)$ defined as*

$$L = L(\mathbf{A};x) = [L^{a,b}(\mathbf{A};x)] := [\delta^{a,b}\Delta_x^{\mathbf{R}^3} + g\varepsilon^{a,c,b}A_k^c(t,x)\partial_k] \tag{21}$$

is essentially self adjoint and elliptic for any time parameter $t \in \mathbf{R}$. Its spectrum lies on the real line, and decomposes into discrete $\mathrm{spec}_d(L)$ and continuous spectrum $\mathrm{spec}_c(L)$. If 0 is an eigenvalue, then it has finite multiplicity, i.e. $\ker(L)$ is always finite dimensional .

The modified Green's function $G = G(\mathbf{A};x,y) = [G^{a,b}(\mathbf{A};x,y)] \in \mathcal{S}'(\mathbf{R}^3,\mathbf{R}^{K\times K})$ for the operator L is the distributional solution to the equation

$$L^{a,b}(\mathbf{A};x)G^{b,d}(\mathbf{A};x,y) = \delta^{a,d}\delta(x-y) - \sum_{n=1}^{N}\psi_n^a(\mathbf{A};x)\psi_n^d(\mathbf{A};y), \tag{22}$$

where $\{\psi_n(\mathbf{A};\cdot)\}_n$ is an o.n. L^2-basis of N-dimensional $\ker(L)$. In equation (22) x is seen as variable, while y is considered as a parameter. This modified Green's function can be written as a Riemann-Stielties integral: For any $\varphi \in \mathcal{S}(\mathbf{R}^3,\mathbf{R}^K) \cap L^2(\mathbf{R}^3,\mathbf{R}^K)$

$$G(\mathbf{A};x,\cdot)(\varphi) = \int_{\lambda\neq 0}\frac{1}{\lambda}d(E_\lambda\varphi)(x), \tag{23}$$

where $(E_\lambda)_{\lambda\in\mathbf{R}}$ is the resolution of the identity corresponding to L.

Remark 2.5. In [Br03] and [Pe78] the modified Green's function is constructed assuming that the operator L has a discrete spectral resolution $(\psi_n(\mathbf{A};\cdot),\lambda_n)_{n\geqslant 0}$ as

$$G(\mathbf{A};x,y) = \sum_{n:\lambda_n\neq 0}\frac{1}{\lambda_n}\psi_n(\mathbf{A};x)\psi_n^{\dagger}(\mathbf{A};y). \tag{24}$$

In particular, we have the symmetry property

$$G(\mathbf{A};x,y)^{\dagger} = G(\mathbf{A};y,x) \tag{25}$$

for all $x, y, \mathbf{A}$ for which the expression is well defined. Since the discontinuity points of the spectral resolution $(E_\lambda)_{\lambda\in\mathbf{R}}$ are the eigenvalues, i. e. the elements of $\text{spec}_d(L)$ (cf. [Ri85], Chapter 9), the solution (23) extends (24) to the general case.

Proof of Proposition 2.1. As long as the connection satisfies the Coulomb gauge condition, the operator L is symmetric and essentially selfadjoint on the appropriate domain, as a direct computation involving integration by parts can show. As its leading symbol is elliptic, the operator L is elliptic and restricted to $[-\frac{R}{2}, +\frac{R}{2}]^3$, under the Dirichlet boundary conditions it has a discrete spectral resolution (cf. [Gi95], Chapter 1.11.3). Every eigenvalue has finite multiplicity. The dimension of the eigenspaces is an integer-valued, continuous, and hence constant function of R. For $R \to +\infty$ the discrete Dirichlet spectrum of L on $[-\frac{R}{2}, +\frac{R}{2}]^3$ clusters in the spectrum of L on $\mathbf{R}^3$, which decomposes into a discrete and a continuous spectrum. Therefore, the eigenvalues must have finite multiplicity and, in particular, $\ker(L)$ is finite dimensional.

Equation (23) gives the modified Green's function as it can be verified by the following computation, which holds true for any $\varphi \in \mathcal{S}(\mathbf{R}^3, \mathbf{R}^K) \cap L^2(\mathbf{R}^3, \mathbf{R}^K)$:

$$\begin{aligned} L(\mathbf{A};x)G(\mathbf{A};x,\cdot)(\varphi) &= L\int_{\lambda\neq 0}\frac{1}{\lambda}d(E_\lambda\varphi)(x) = \int_{\lambda\neq 0}\frac{1}{\lambda}Ld(E_\lambda\varphi)(x) = \\ &= \int_{\mathbf{R}} d(E_\lambda\varphi)(x) - \int_{0^-}^{0^+} d(E_\lambda\varphi)(x) = \phi(x) - P_{\ker(L)}\varphi(x) = \\ &= \delta(x-\cdot)(\varphi) - \sum_{n=1}^{N}\psi_n(\mathbf{A};x)\psi_n^{\dagger}(\mathbf{A};\cdot)(\varphi). \end{aligned} \tag{26}$$

□

The existence of eigenvalues of L depends on the additive perturbation to the Laplacian given by $g\varepsilon^{a,c,b}A_k^c(t,x)\partial_k$. For example, if $g = 0$ or $\mathbf{A} = 0$, the operator L has no eigenvalues and $\text{spec}(L) = \text{spec}_c(L) =]-\infty, 0]$. In general, the spectrum depends on the choice of the connection A. In [BEP78] and in [Pe78] special cases comprising pure gauges and Wu-Yang monopoles are computed explicitly. We are interested in a reformulation of the general solution (23), where the dependence on the connection becomes explicit. Inspired by [ChHa99] we find

Proposition 2.2. *For a simple Lie-group as structure group, if we assume that the coupling constant* $g < 1$, *then a Green's function* $K = K(\mathbf{A};x,y) = [K^{a,b}(\mathbf{A};x,y)]$ *for the operator* L *in Proposition 2.1,* *that is, a distributional solution to the equation*

$$L^{a,b}(\mathbf{A};x)K^{b,d}(\mathbf{A};x,y) = \delta^{a,d}\delta(x-y), \tag{27}$$

is given by the convergent series in $\mathcal{S}'(\mathbf{R}^3, \mathbf{R}^{K\times K})$

$$\begin{aligned}
&K^{b,d}(\mathbf{A};x,y) = \frac{\delta^{b,d}}{4\pi|x-y|} + 2g\varepsilon^{e_1,b,d}\int_{\mathbf{R}^3} d^3u_1 \frac{1}{4\pi|x-u_1|}A_k^{e_1}(u_1)\partial_k\left(\frac{1}{4\pi|u_1-y|}\right)+\\
&-3g^2\varepsilon^{e_1,b,s_1}\varepsilon^{s_1,e_2,d}\int_{\mathbf{R}^3} d^3u_1 \frac{1}{4\pi|x-u_1|}A_k^{e_1}(u_1)\partial_k\int_{\mathbf{R}^3} d^3u_2 \frac{1}{4\pi|u_1-u_2|}A_i^{e_2}(u_2)\partial_i\left(\frac{1}{4\pi|u_2-y|}\right)\\
&+\cdots+\\
&+(-1)^{n-1}(n+1)g^n\varepsilon^{e_1,b,s_1}\varepsilon^{s_1,e_2,s_2}\cdots\varepsilon^{s_{n-1},e_n,d}\int_{\mathbf{R}^3} d^3u_1 \frac{1}{4\pi|x-u_1|}A_k^{e_1}(u_1)\cdot\\
&\cdot\partial_k\int_{\mathbf{R}^3} d^3u_2 \frac{1}{4\pi|u_1-u_2|}A^{e_2}(u_2)\partial_j\ldots\int_{\mathbf{R}^3} d^3u_n \frac{1}{4\pi|u_{n-1}-u_n|}A_l^{e_n}(u_n)\partial_l\left(\frac{1}{4\pi|u_n-y|}\right)+\\
&+\ldots
\end{aligned} \tag{28}$$

Note that x *is the variable and* y *a parameter.*

Proof. The series (28) converges because of the integrability of the connection A and the fact that $g<1$. Recall that $\frac{1}{|x-y|}\in L^1_{loc}(\mathbf{R}^3)\subset\mathcal{S}'(\mathbf{R}^3)$ for any fixed $y\in\mathbf{R}^3$. We now check that it represents a Green's function for L:

$$L(\mathbf{A};x)^{a,b}K^{b,d}(\mathbf{A};x,y) = \delta^{a,d}\delta(x-y) + \lim_{n\to+\infty}\mathrm{Rest}_n, \tag{29}$$

where, after having evaluated a "telescopic sum", the remainder part reads

$$\begin{aligned}
\mathrm{Rest}_n =&(-1)^{n-1}(n+1)g^{n+1}\varepsilon^{e_1,b,s_1}\varepsilon^{s_1,e_2,s_2}\cdots\varepsilon^{s_{n-1},e_n,d}\varepsilon^{a,c,b}A_k^c(x)\int_{\mathbf{R}^3} d^3u_1 \frac{-x_k}{4\pi|x-u_1|^3}A_k^{e_1}(u_1)\cdot\\
&\cdot\partial_k\int_{\mathbf{R}^3} d^3u_2 \frac{1}{4\pi|u_1-u_2|}\ldots\int_{\mathbf{R}^3} d^3u_n \frac{1}{4\pi|u_{n-1}-u_n|}A_l^{e_n}(u_n)\partial_l\left(\frac{1}{4\pi|u_n-y|}\right).
\end{aligned} \tag{30}$$

Because of the integrability of the connection, there exists a constant $C>0$ such that for any $\varphi\in\mathcal{S}(\mathbf{R}^3,\mathbf{R})$

$$|\mathrm{Rest}_n(\varphi)| \leqslant Cg^{n+1}\|\varphi\|_{L^2(\mathbf{R}^3,\mathbf{R})}\to 0 \quad (n\to+\infty), \tag{31}$$

and the proposition follows. □

Remark 2.6. For the Minkowskian $\mathbf{R}^4$ the assumption of a (dimensionless) bare coupling constant $g<1$ is well posed: for the Yang-Mills theory one is basically allowed to choose any value $g>0$ by appropriate rescaling of the energy scale (see [SaSc10]). Moreover, we will later have to analyze the case where $g\to0^+$.

Corollary 2.3. *Under the same assumptions as Proposition 2.2 the distribution*

$$G(\mathbf{A};x,y) = \frac{1}{2}(K(\mathbf{A};x,y) + K(\mathbf{A};x,y)^\dagger) - \frac{1}{2}\sum_{n:\lambda_n=0}\left(\psi_n(\mathbf{A};x)\psi_n^\dagger(\mathbf{A};y) + \psi_n(\mathbf{A};y)\psi_n^\dagger(\mathbf{A};x)\right) \quad (32)$$

is a symmetric modified Green's function for the operator L.

After this preparation we can turn to the Hamiltonian formulation of Yang-Mills' equations, following the results in [Br03] and [Pe78], which just need to be adapted for the generic case that L has a mixture of discrete and continuous spectral resolution. Note that

$$\mathcal{S}(\mathbf{R}^3, \mathbf{R}^{K\times K}) \subset L^2(\mathbf{R}^3, \mathbf{R}^{K\times K}, d^3x) \subset \mathcal{S}'(\mathbf{R}^3, \mathbf{R}^{K\times K}) \quad (33)$$

is a rigged Hilbert space and L a selfadjoint operator with a complete set of generalized eigenvectors (cf. Appendix A).

Theorem 2.4. . *For a simple Lie group as structure group and for canonical variables satisfying the Coulomb gauge condition, the Yang-Mills equations for the Minkowskian $\mathbf{R}^4$ can be written as Hamilton equations*

$$\begin{cases} \frac{d\mathbf{E}}{dt} & = -\frac{\partial H}{\partial \mathbf{A}}(\mathbf{A},\mathbf{E}) \\ \\ \frac{d\mathbf{A}}{dt} & = +\frac{\partial H}{\partial \mathbf{E}}(\mathbf{A},\mathbf{E}) \end{cases} \quad (34)$$

for the following choices:

- ***Position variable:*** $\mathbf{A} = [A_i^a(t,x)]_{\substack{a=1,\dots,K \\ i=1,2,3}}$ *also termed potentials* ,
- ***Momentum variable:*** $\mathbf{E} = [E_i^a(t,x)]_{\substack{a=1,\dots,K \\ i=1,2,3}}$, *whose entries are termed chromoelectric fields,*
- ***Hamilton function:*** *defined as a function of $\mathbf{A}$ and $\mathbf{E}$ as*

$$H = H(\mathbf{A},\mathbf{E}) := \frac{1}{2}\int_{\mathbf{R}^3} d^3x\left(E_i^a(t,x)^2 + B_i^a(t,x)^2 + f^a(t,x))\Delta f^a(t,x) + 2\rho^c(t,x)A_0^c(t,x)\right), \quad (35)$$

where $\mathbf{B} = [B_i^a]$, whose entries are termed chromomagnetic fields, is the matrix valued-function defined as

$$B_i^a := \frac{1}{4}\varepsilon_i^{j,k}\left(\partial_j A_k^a - \partial_k A_j^a + g\varepsilon_{b,c}^a A_j^b A_k^c\right), \quad (36)$$

and $\rho = [\rho^a(t,x)]$, termed charge density, is the vector valued function defined as

$$\rho^a := g\varepsilon^{a,b,c}E_i^b A_i^c, \quad (37)$$

and where

$$f^a(t,x) := -\int_{\mathbf{R}^3} d^3y\, G^{a,b}(\mathbf{A};x,y)\rho^b(t,y) = -G^{a,b}(\mathbf{A};x,\cdot)(\rho^b(t,\cdot))$$
$$A_0^a(t,x) := \int_{\mathbf{R}^3} d^3y\, G^{a,b}(\mathbf{A};x,y)\Delta f^b(t,y) = G^{a,b}(\mathbf{A};x,\cdot)(\Delta f^b(t,\cdot)) \tag{38}$$

for the modified Green's function $G(\mathbf{A};x,y)$ for the operator $L(\mathbf{A};x)$.

We consider position $\mathbf{A}$ and momentum variable $\mathbf{E}$ as elements of $\mathcal{S}(\mathbf{R}^3,\mathbf{C}^{K\times 3})$ depending on the time parameter t, so that the RHSs of equations (38) are well defined distributions applied to test functions.

Remark 2.7. As shown in [Pe78] the ambiguities discussed by Gribov in [Gr78] concerning the gauge fixing (see also [Si78] and [He97]) can be traced precisely to the existence of zero eigenfunctions of the operator L.

Corollary 2.5. *The Hamilton function (35) for the Yang-Mills equations can be written as*

$$H = H_I + H_{II} + V, \tag{39}$$

where

$$H_I = \frac{1}{2}\int_{\mathbf{R}^3} d^3x\, E_i^a(t,x)^2$$

$$H_{II} = \frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, \partial_i G^{a,b}(\mathbf{A};x,y)\varepsilon^{b,c,d} A_k^d(t,y) E_k^c(t,y)\right]^2 \tag{40}$$

$$V = \frac{1}{16}\int_{\mathbf{R}^3} d^3x\, \varepsilon_i^{j,k}\varepsilon_i^{p,q}[(\partial_j A_k^a - \partial_k A_j^a + g\varepsilon^{a,b,c}A_j^b A_k^c)(\partial_p A_q^a - \partial_q A_p^a + g\varepsilon^{a,b,c}A_p^b A_q^c)](t,x).$$

Proof. The expressions for the functions H_I and V are obtained by a straightforward calculation. For the function H_{II} some more work is needed. Inserting (38) in the last addendum of (35) we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^3} d^3x \left(\frac{1}{2} f^a(t,x)\Delta f^a(t,x) + \rho^c(t,x)A_0^c(t,x)\right) = \\
&= \int_{\mathbf{R}^3} d^3x \left[\frac{1}{2}\left(-\int_{\mathbf{R}^3} d^3y\, G^{a,b}(\mathbf{A};x,y)\rho^b(t,y)\right)\Delta\left(-\int_{\mathbf{R}^3} d^3\bar{y}\, G^{a,b}(\mathbf{A};x,\bar{y})\rho^b(t,\bar{y})\right) + \right.\\
&\qquad \left. + \rho^c(t,x)\int_{\mathbf{R}^3} d^3y\, G^{c,b}(\mathbf{A};x,y)\int_{\mathbf{R}^3} d^3\bar{y}\, \Delta G^{b,d}(\mathbf{A};y,\bar{y})\rho^d(t,\bar{y})\right] = \\
&= \int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y \int_{\mathbf{R}^3} d^3\bar{y} \left[\frac{1}{2}\left(G^{a,b}(\mathbf{A};x,y)\rho^b(t,y)\Delta G^{a,d}(\mathbf{A};x,\bar{y})\rho^d(t,\bar{y})\right) + \right.\\
&\qquad \left. - \rho^c(t,x)G^{c,b}(\mathbf{A};x,y)\Delta G^{b,d}(\mathbf{A};y,\bar{y})\rho^d(t,\bar{y})\right] = \\
&= \int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y \int_{\mathbf{R}^3} d^3\bar{y} \left[\frac{1}{2}\left(G^{a,b}(\mathbf{A};x,y)\Delta G^{a,d}(\mathbf{A};x,\bar{y})\rho^b(t,y)\rho^d(t,\bar{y})\right) + \right.\\
&\qquad \left. - G^{a,b}(\mathbf{A};y,x)\Delta G^{a,d}(\mathbf{A};x,\bar{y})\rho^b(t,y)\rho^d(t,\bar{y})\right] = \\
&= \frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y \int_{\mathbf{R}^3} d^3\bar{y} \left[\partial_i G^{a,b}(\mathbf{A};x,y)\rho^b(t,y)\right]\left[\partial_i G^{a,d}(\mathbf{A};x,\bar{y})\rho^d(t,\bar{y})\right] = \\
&= \frac{1}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, \partial_i G^{a,b}(\mathbf{A};x,y)\rho^b(t,y)\right]^2,
\end{aligned}
\tag{41}
$$

where in the last transformation formula we have utilized integration by parts in the variables x_1, x_2, x_3, and the fact that $G(\mathbf{A};x,y) = G(\mathbf{A};y,x)$. Inserting expression (37) for the charge density completes the proof. □

3 Axioms of Constructive Quantum Field Theory

3.1 Wightman Axioms

In 1956 Wightman first stated the axioms needed for CQFT in his seminal work [Wig56], which remained not very widely spread in the scientific community till 1964, when the first edition of [SW10] appeared. We will list the axioms in the slight refinement of [BLOT89] and [DD10].

Definition 3.1 (Wightman Axioms). A scalar (respectively vectorial-spinorial) quantum field theory consists of a separable Hilbert space $\mathcal{E}$, whose elements are called states, a unitary representation U of the Poincaré group $\mathcal{P}$ in $\mathcal{E}$, an operator valued distribution Φ (respectively $\Phi_1, \dots, \Phi_d$) on $\mathcal{S}(\mathbf{R}^4)$ with values in the unbounded operators of $\mathcal{E}$, and a dense subspace $\mathcal{D} \subset \mathcal{E}$ such that the following properties hold:

(W1) Relativistic invariance of states: The representation $U : \mathcal{P} \to \mathcal{U}(\mathcal{E})$ is strongly continuous.

(W2) Spectral condition: Let P_0, P_1, P_2, P_3 be the infinitesimal generators of the one-parameter groups $t \mapsto U(te_\mu, I)$ for $\mu = 0, 1, 2, 3$. The operators P_0 and $P_0^2 - P_1^2 - P_2^2 - P_3^2$ are positive. This is equivalent to the spectral measure $E.$ on $\mathbf{R}^4$ corresponding to the restricted representation $\mathbf{R}^4 \ni a \mapsto U(a, I)$ having support in the positive light cone (cf. [RS75], Chapter IX.8).

(W3) Existence and uniqueness of the vacuum: There exists a unique state $\Omega_0 \in \mathcal{D} \subset \mathcal{E}$ such that $U(a, I)\Omega_0 = \Omega_0$ for all $a \in \mathbf{R}^4$.

(W4) Invariant domains for fields: The maps $\Phi : \mathcal{S}(\mathbf{R}^4) \to \mathcal{O}(\mathcal{E})$, and, respectively $\Phi_1, \ldots, \Phi_d : \mathcal{S}(\mathbf{R}^4) \to \mathcal{O}(\mathcal{E})$, from the Schwartz space of test functions to (possibly) unbounded selfadjoint operators on the Hilbert space, satisfy following properties

(a) For all $\varphi \in \mathcal{S}(\mathbf{R}^4)$ and all field maps, the domain of definitions $\mathcal{D}(\Phi(\varphi))$, $\mathcal{D}(\Phi(\varphi)^*)$, and respectively $\mathcal{D}(\Phi_j(\varphi))$, $\mathcal{D}(\Phi_j(\varphi)^*)$, all contain $\mathcal{D}$ and the restrictions of all operators to $\mathcal{D}$ agree.

(b) $\Phi(\varphi)(\mathcal{D}) \subset \mathcal{D}$, and, respectively $\Phi_j(\varphi)(\mathcal{D}) \subset \mathcal{D}$.

(c) For any $\psi \in \mathcal{D}$ fixed, the maps $\varphi \mapsto \Phi(\varphi)\psi$, and, respectively $\varphi \mapsto \Phi_j(\varphi)\psi$, are linear.

(W5) Regularity of fields: For all $\psi_1, \psi_2 \in \mathcal{D}$, the map $\varphi \mapsto (\psi_1, \Phi(\varphi)\psi_2)$, and, respectively the maps $\varphi \mapsto (\psi_1, \Phi_j(\varphi)\psi_2)$, are tempered distributions, i.e. elements of $\mathcal{S}'(\mathbf{R}^4)$.

(W6) Poincaré invariance: For all $(a, \Lambda) \in \mathcal{P}$, $\varphi \in \mathcal{S}(\mathbf{R}^4)$, and $\psi \in \mathcal{D}$, the inclusion $U(a, \Lambda)\mathcal{D} \subset \mathcal{D}$ must hold and

(Scalar field case): The following equation must hold for all $\Lambda \in \mathrm{O}(1, 3)$

$$U(a, \Lambda)\Phi(\varphi)U(a, \Lambda)^{-1}\psi = \Phi((a, \Lambda)\varphi)\psi. \tag{42}$$

(Vectorial/Spinorial field case): There exists a representation of $\mathrm{SL}(2, \mathbf{C})$ on $\mathbf{C}^d$ denoted by ρ, and satisfying $\rho(-I) = I$ or $\rho(I) = I$, such that for all $\Lambda \in \mathrm{SO}^+(1, 3)$ and $\Phi := [\Phi_1, \ldots, \Phi_d]^\dagger$

$$U(a, \Lambda)\Phi(\varphi)U(a, \Lambda)^{-1}\psi = \rho(s^{-1}(\Lambda))\Phi((a, \Lambda)\varphi)\psi, \tag{43}$$

where s denotes the spinor map $s : \mathrm{SL}(2, \mathbf{C}) \to \mathrm{SO}^+(1, 3)$ defined as below. The vector spaces of Hermitian matrices $\mathfrak{H}$ in $\mathbf{C}^2$ and $\mathbf{R}^4$ are isomorphically mapped by

$$\begin{aligned} X : \mathbf{R}^4 &\longrightarrow \mathfrak{H} \\ x = (x^0, x^1, x^2, x^3) &\mapsto X(x) := \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}. \end{aligned} \tag{44}$$

The group $\mathrm{SL}(2,\mathbf{C})$ acts on $\mathfrak{H}$ by

$$\begin{aligned} \mathrm{SL}(2,\mathbf{C}) \times \mathfrak{H} &\longrightarrow \mathfrak{H} \\ (P, X) &\mapsto P.X := PXP^*. \end{aligned} \tag{45}$$

The spinor map is defined as

$$\begin{aligned} s : \mathrm{SL}(2,\mathbf{C}) &\longrightarrow \mathrm{SO}^+(1,3) \\ P &\mapsto s(P) : x \mapsto s(P)x := X^{-1}(PX(x)P^*). \end{aligned} \tag{46}$$

(W7) Microscopic causality or local commutativity: Let $\varphi, \chi \in \mathcal{S}(\mathbf{R}^4)$, whose supports are space-like separated, i.e. $\phi(x)\chi(y) = 0$, if $x - y$ is not in the positive light cone. Then,

(Scalar field case): The images of the test functions by the map field must commute

$$[\Phi(\varphi), \Phi(\chi)] = 0. \tag{47}$$

(Vectorial/Spinorial field case): For any field maps $j, i = 1, \dots, d$ either the commutations

$$[\Phi_j(\varphi), \Phi_i(\chi)] = 0, \quad [\Phi_j^*(\varphi), \Phi_i(\chi)] = 0, \tag{48}$$

or the anticommutations

$$[\Phi_j(\varphi), \Phi_i(\chi)]_+ = 0, \quad [\Phi_j^*(\varphi), \Phi_i(\chi)]_+ = 0 \tag{49}$$

hold.

(W8) Cyclicity of the vacuum:

(Scalar field case): The set

$$\mathcal{D}_0 := \{\Phi(\varphi_1) \cdots \Phi(\varphi_n)\Omega_0 \,|\, \varphi_j \in \mathcal{S}(\mathbf{R}^4), n \in \mathbf{N}_0\} \tag{50}$$

is dense in $\mathcal{E}$.

(Vectorial/Spinorial field case): The set

$$\mathcal{D}_0 := \{\Psi(\varphi_1) \cdots \Psi(\varphi_n)\Omega_0 \,|\, \varphi_j \in \mathcal{S}(\mathbf{R}^4), \Psi \in \{\Phi_1, \dots, \Phi_d, \Phi_1^*, \dots, \Phi_d^*\}, n \in \mathbf{N}_0\} \tag{51}$$

is dense in $\mathcal{E}$.

3.2 Gaussian Random Processes

The Hilbert spaces utilized in quantum field theory are realized as L^2 spaces over the tempered distributions, the latter seen as probability space. This construction turns out to be isomorphic to that of the Fock space. We follow chapter 1 of [Sim15], chapter 5 of [BHL11] and appendix $A.4$ of [GJ87]. For a general overview see Kuo ([Ku96]), and the framework for functional integration developed by Cartier/DeWitt-Morette as in [Ca97], [La04] and [CDM10]. Although Feynman's integral can be provided a rigorous foundation by mean of functional integration, we prefer to use the Feynman-Kač approach, since we will be working with Euclidean fields.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and $\mathcal{M}(\Omega, \mathcal{A}) := \{f : \Omega \to \mathbf{R} \,|\, f \text{ measurable}\}$ the algebra of random variables. A random variable f on Ω has a probability distribution function $\mu_f(U) := \mu(f^{-1}(U))$ defined on the measurable sets U of $\mathbf{R}$, and a characteristic function

$$S_f(t) := \int_{\mathbf{R}} e^{itx} d\mu_f(x), \tag{52}$$

defined as the Fourier inverse transform of the probability density $\rho_f := \mu'_f$. A real valued random variable f has mean 0 and variance $a \geqslant 0$ if and only if $S_f(t) = e^{-\frac{a}{2}t^2}$. A well known result (see f.i. [RS75]) is the following

Theorem 3.1 (**Bochner**). *A function $S : \mathbf{R} \to \mathbf{C}$ is the characteristic function of a random variable $f : \Omega \to \mathbf{R}$ if and only if the following conditions are satisfied:*

(i) $S(0) = 1$.

(ii) $t \mapsto S(t)$ *is continuous.*

(iii) $\{t_i\}_{i=1,\dots,n} \subset \mathbf{R}, \{z_i\}_{i=1,\dots,n} \subset \mathbf{C} \Rightarrow \sum_{i,j=1}^n z_i \bar{z}_j S(t_i - t_j) \geqslant 0$.

The construction of Bochner's theorem can be lifted to generating functionals of random variables over the space of tempered distributions.

Definition 3.2. A random process indexed by a real vector space V is a linear map $\Phi : V \to \mathcal{M}(\Omega, \mathcal{A})$. It is termed Gaussian random process if

(i) $\Phi(v)$ is a Gaussian random variable for all $v \in V$.

(ii) $\{\Phi(v) \,|\, v \in V\}$ is full, i.e. $\mathcal{A}$ is the smallest σ-algebra for which this is a family of measurable functions.

Theorem 3.2. *Let $\mathcal{H}$ be a real Hilbert space. Up to isomorphism there exists exactly one Gaussian random process indexed by $\mathcal{H}$ such that*

$$(\Phi(v), \Phi(w))_{L^2(\Omega, d\mu)} := \int_\Omega \Phi(v)\overline{\Phi(w)} = \frac{1}{2}(v, w)_{\mathcal{H}}. \tag{53}$$

Proof. See Theorem I.9 (page 20) in [Sim15] or Lemma 5.4 (page 258) and Proposition 5.6 (page 260) in [BHL11]. □

Given a real Hilbert space there are different but isomorphic models for the probability space $(\Omega, \mathcal{A}, \mu)$ admitting a Gaussian process. We choose the tempered distributions $\mathcal{S}'(\mathbf{R}^N)$ as model space. Remark that any fixed test function $f \in \mathcal{S}(\mathbf{R}^N)$ and variable $\Phi \in \mathcal{S}'(\mathbf{R}^N)$, the expression $\Phi(f)$ defines a random variable over $\mathcal{S}'(\mathbf{R}^N)$. As in appendix $A.6$ of [GJ87] Bochner's theorem generalizes to

Theorem 3.3 (Milnos). *Let $S : \mathcal{S}(\mathbf{R}^N) \to \mathbf{C}$ be a function. There exists a probability measure μ satisfying*

$$S(f) = \int_{\mathcal{S}'(\mathbf{R}^N)} e^{\imath\Phi(f)} d\mu(\Phi), \tag{54}$$

for all $f \in \mathcal{S}(\mathbf{R}^N)$, if and only if

(i) $S(0) = 1$.

(ii) $f \mapsto S(f)$ *is continuous in the* $\mathcal{S}(\mathbf{R}^N) \to \mathbf{C}$ *topology.*

(iii) $\{f_i\}_{i=1,\dots,n} \subset \mathcal{S}(\mathbf{R}^N), \{z_i\}_{i=1,\dots,n} \subset \mathbf{C} \Rightarrow \sum_{i,j=1}^n z_i \bar{z}_j S(f_i - f_j) \geqslant 0$.

Remark 3.1. The σ-algebra $\mathcal{A}$ is generated by the cylinder sets in $\mathcal{S}'(\mathbf{R}^N)$, i.e. subsets of the tempered distribution space of the form

$$\left\{\Phi \in \mathcal{S}'(\mathbf{R}^N) \mid (\Phi(f_1), \dots \Phi(f_n)) \in U\right\}, \tag{55}$$

where U is a fixed Borel set in $\mathbf{R}^n$, and $f_1, \dots, f_n$ fixed test functions in $\mathcal{S}(\mathbf{R}^n)$ for a $n \in \mathbf{N}_0$.

Remark 3.2. Minlos's theorem holds true for topological vector spaces, stating that a cylindrical measure on the dual of a nuclear space is a Radon measure if its Fourier transform is continuous, see ([Sc73]).

We utilize Minlos' theorem to define Gaussian measures on the tempered distributions. Let c be a positive semidefinite quadratic form on $\mathcal{S}(\mathbf{R}^N)$. Applying Theorem 3.3 to the functional $S(f) :=$

$e^{-\frac{1}{2}c(f,f)}$ we can construct a measure μ on $\mathcal{S}'(\mathbf{R}^N)$ such that

$$\int_{\mathcal{S}'(\mathbf{R}^N)} e^{\imath\Phi(f)} d\mu(\Phi) = e^{-\frac{1}{2}c(f,f)} \tag{56}$$

Therefore, $\Phi(f)$ is a Gaussian random variable with variance $c(f,f)$, because for all $t \in \mathbf{R}$

$$\int_{\mathcal{S}'(\mathbf{R}^N)} e^{it\Phi(f)} d\mu(\Phi) = e^{-\frac{t^2}{2}c(f,f)} \tag{57}$$

holds true. If $\mathcal{H}$ is a real Hilbert space such that the embedding $\mathcal{S} \hookrightarrow \mathcal{H}$ is continuous and dense, then the inner product in $\mathcal{H}$ restricts to a positive definite bilinear form on $\mathcal{S}$, which can be written as $c(f,g) = (Cf,g)_{\mathcal{H}}$ for all $f,g \in \mathcal{S}$ for a positive definite operator C on $\mathcal{H}$ with domain of definition $\mathcal{S}$.

Definition 3.3 (**Gaussian measures**)**.** A measure μ on $\mathcal{S}'(\mathbf{R}^N)$ defined by Milnos's theorem satisfying

$$\int_{\mathcal{S}'(\mathbf{R}^N)} e^{\imath\Phi(f)} d\mu(\Phi) = e^{-\frac{1}{2}(Cf,f)_{\mathcal{H}}}, \tag{58}$$

where C is a positive semidefinite operator C on $\mathcal{H}$ with domain of definition $\mathcal{S}$, termed **covariance operator**, is called **Gaussian**.

The operation of derivation can be defined for functionals of random variables as well.

Definition 3.4 (**Functional Derivative**)**.** Let $F \in L^2(\mathcal{S}'(\mathbf{R}^N),\mu)$. Its **functional directional derivative** in the direction $\Upsilon \in \mathcal{S}'(\mathbf{R}^N)$ is defined as Gâteaux derivative as

$$\frac{\delta F}{\delta \Phi}(\Phi).\Upsilon := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} F(\Phi + \epsilon\Upsilon). \tag{59}$$

The special case $\Upsilon = \delta(\cdot - x)$ for $x \in \mathbf{R}^N$ arises often and is thus given a special notation

$$\frac{\delta}{\delta\Phi(x)}F(\Phi) := \frac{\delta F}{\delta\Phi}(\Phi).\delta(\cdot - x). \tag{60}$$

3.2.1 Fock Space

To extend a quantum mechanical model accounting for a fixed number of particles to one accounting for an arbitrary number, the following procedure is required.

Definition 3.5 (**Second Quantization**)**.** Let $\mathcal{H}$ be the Hilbert space whose unit sphere corresponds to the possible pure quantum states of the system with a fixed number of particles. The **Fock space** $\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$ where, $\mathcal{H}^{(0)} := \mathbf{C}$ and $\mathcal{H}^{(n)} := \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ (n times tensor product) is the vector space representing the states of a quantum system with a variable number of particles. The vector $\Omega_0 := (1, 0 \dots) \in \mathcal{F}(\mathcal{H})$ is called the **vacuum vector**. Given $\psi \in \mathcal{F}(\mathcal{H})$, we write $\psi^{(n)}$ for the orthogonal projection of ψ onto $\mathcal{H}^{(n)}$. The set F_0 consisting of those ψ such that $\psi^{(n)} = 0$ for all sufficiently large n is a dense subspace of the Fock space, called the **space of finite particles**. The **symmetrization** and anti-symmetrization operators

$$\begin{aligned} S_n(\psi_1 \otimes \dots \otimes \psi_n) &:= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}^n} \psi_{\sigma(1)} \otimes \dots \otimes \sigma_{\sigma(n)} \\ A_n(\psi_1 \otimes \dots \otimes \psi_n) &:= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}^n} (-1)^{\mathrm{sgn}(\sigma)} \psi_{\sigma(1)} \otimes \dots \otimes \sigma_{\sigma(n)} \end{aligned} \tag{61}$$

extend by linearity to $\mathcal{H}^{(n)}$ and are projections. The state space for n fermions is defined as $\mathcal{H}_a^{(n)} := A_n(\mathcal{H}^n)$ and that for n bosons as $\mathcal{H}_s^{(n)} := S_n(\mathcal{H}^n)$. The **Fermionic Fock space** is defined as

$$\mathcal{F}_a(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_a^{(n)}, \tag{62}$$

and the **Bosonic Fock space** as

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_s^{(n)}. \tag{63}$$

A *unitary* operator $U : \mathcal{H} \to \mathcal{H}$ can be uniquely extended to a *unitary* operator $\Gamma(U) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ as

$$\Gamma(U)|_{\mathcal{H}^{(n)}} := \bigotimes_{j=1}^{n} U. \tag{64}$$

A *selfadjoint* operator A on $\mathcal{H}$ with dense subspace $\mathcal{D}(A) \subset \mathcal{H}$ can be uniquely extended to a *selfadjoint* operator $d\Gamma(A)$ on $\mathcal{F}(\mathcal{H})$ as closure of the essentialy selfadjoint operator

$$d\Gamma(A)|_{\mathcal{D}(d\Gamma A) \cap \mathcal{H}^{(n)}} := \bigoplus_{j=1}^{n} \mathbb{1} \otimes \dots \mathbb{1} \otimes \underbrace{A}_{j} \otimes \mathbb{1} \dots \mathbb{1}, \tag{65}$$

where

$$\mathcal{D}(d\Gamma(A)) := \left\{ \psi \in F_0 \,\middle|\, \psi^{(n)} \in \bigotimes_{j=1}^{n} \mathcal{D}(A) \text{ for each } n \right\}. \tag{66}$$

The operator $d\Gamma(A)$ is called the **second quantization** of A.

It is easy to prove that the spectrum of the second quantization can be inferred from the spectrum of the first.

Proposition 3.4. *Let A be a selfadjoint operator with a discrete spectral resolution, i.e. $A\varphi_j = \lambda_j \varphi_j$, where $\{\lambda_j\}_{j \geqslant 0} \subset \mathbf{R}$ and $\{\varphi_j\}_{j \geqslant 0}$ is a o.n.B in $\mathcal{H}$. Then, $d\Gamma(A)|_{\mathcal{H}^{(n)}}$ has a discrete spectral resolution* *given by*

$$d\Gamma(A)|_{\mathcal{H}^{(n)}}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_n} = \left(\sum_{j=1}^{n}\lambda_{i_j}\right)\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_n} \quad (i_j \geqslant 0, 1\leqslant j\leqslant n). \tag{67}$$

If A is a selfadjoint operator with continuous spectrum $\mathrm{spec}_c(A)$*, then* $d\Gamma(A)|_{\mathcal{H}^{(n)}}$ *has a continuous spectrum given by*

$$\mathrm{spec}_c\left(d\Gamma(A)|_{\mathcal{H}^{(n)}}\right) = \left\{\sum_{j=1}^{n}\lambda_j \,\middle|\, \lambda_j \in \mathrm{spec}_c(A) \text{ for } 1\leqslant j\leqslant n\right\}. \tag{68}$$

Definition 3.6 (**Segal Quantization**)**.** Let $f \in \mathcal{H}$ be fixed. For vectors in $\mathcal{H}^{(n)}$ of the form $\eta = \psi_1\otimes\psi_2\otimes\cdots\otimes\psi_n$ we define a map $b^-(f):\mathcal{H}^{(n)}\to\mathcal{H}^{(n-1)}$ by

$$b^-(f)\eta := (f,\psi_1)\psi_2\otimes\cdots\otimes\psi_n. \tag{69}$$

The expression $b^-(f)$ extends by linearity to a bounded operator on $\mathcal{F}(\mathcal{H})$. The operator $N := d\Gamma(I)$ is termed **number operator** and

$$a^-(f) := \sqrt{N+1}\, b^-(f) \tag{70}$$

is called the **annihilation operator** on $\mathcal{F}_s(\mathcal{H})$. Its adjoint, $a^-(f)^*$ is called the **creation operator**. Finally, the real linear (but not complex linear) operator

$$\Phi_S(f) := \frac{1}{\sqrt{2}}\left(a^-(f) + a^-(f)^*\right) \tag{71}$$

is termed **Segal field operator**, and the map

$$\begin{aligned}\Phi_S : \mathcal{H} &\to \mathcal{O}(\mathcal{F}_s(\mathcal{H}))\\ f &\mapsto \Phi_S(f)\end{aligned} \tag{72}$$

the **Segal quantization** over $\mathcal{H}$.

Theorem 3.5. *Let $\mathcal{H}$ be a complex Hilbert space and Φ_S the corresponding Segal quantization. Then:*

(a) *For each $f\in\mathcal{H}$ the operator $\Phi_S(f)$ is essentially selfadjoint on F_0.*

(b) *The vacuum Ω_0 is in the domain of all finite products $\Phi_S(f_1)\Phi_S(f_2)\ldots\Phi_S(f_n)$ and the linear span of $\{\Phi_S(f_1)\Phi_S(f_2)\ldots\Phi_S(f_n)\Omega_0 \,|\, f_i\in\mathcal{H}, n\geqslant 0\}$ is dense in $\mathcal{F}_s(\mathcal{H})$.*

(c) *For each $\psi_0 \in F_0$ and $f, g \in \mathcal{H}$*

$$\begin{aligned} \Phi_S(f)\Phi_S(g)\psi - \Phi_S(g)\Phi_S(f)\psi &= \imath Im(f,g)\psi \\ \exp\left(\imath\Phi_S(f+g)\right) &= \exp\left(-\frac{\imath}{2} Im(f,g)\right) \exp\left(\imath\Phi_S(f)\right) \exp\left(\imath\Phi_S(g)\right). \end{aligned} \tag{73}$$

(d) *If $f_n \to f$ in $\mathcal{H}$, then:*

$$\begin{aligned} \Phi_S(f_n) &\to \Phi_S(f) \\ \exp\left(\imath\Phi_S(f_n)\right) &\to \exp\left(\imath\Phi_S(f)\right) \end{aligned} \tag{74}$$

(e) *For every unitary operator U on $\mathcal{H}$, $\Gamma(U) : \mathcal{D}(\overline{\Phi_S(f)}) \to \mathcal{D}(\overline{\Phi_S(Uf)})$ and for $\psi \in \mathcal{D}(\overline{\Phi_S(Uf)})$ and for all $f \in \mathcal{H}$*

$$\Gamma(U)\overline{\Phi_S(f)}\Gamma(U)^{-1}\psi = \overline{\Phi_S(Uf)}\psi. \tag{75}$$

Proof. See Theorem X.41 in [RS75]. □

3.2.2 Wick Products and Wiener-Itô-Segal Isomorphism

Definition 3.7 (**Wick Products of Random Variables**)**.** Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $f : \Omega \to \mathbf{R}$ a random variable with finite moments. Then, for $n \in \mathbf{N}_0$, the random variable $: f^n :$, termed **nth Wick power** of f is defined recursively by

$$\begin{aligned} &: f^0 := 1, \\ &\frac{\partial}{\partial f} : f^n := n : f^{n-1} : \quad \text{and} \quad \mathbb{E}^\mu[: f^n :] = 0 \quad \text{for } n \geqslant 1. \end{aligned} \tag{76}$$

Let $f_1, \ldots, f_k : \Omega \to \mathbf{R}$ be random variables with finite moments. The Wick product $: f_1^{n_1} \ldots f_k^{n_k} :$ is defined recursively in $n = n_1 + \cdots + n_k$ by

$$\begin{aligned} &: f_1^0 \ldots f_k^0 := 1, \\ &\frac{\partial}{\partial f_i} : f_1^{n_1} \ldots f_k^{n_k} := n_i : f_1^{n_1} \ldots f_i^{n_i - 1} \ldots f_k^{n_k} : \quad \text{and} \quad \mathbb{E}^\mu[: f_1^{n_1} \ldots f_k^{n_k} :] = 0 \quad \text{for } n \geqslant 1. \end{aligned} \tag{77}$$

Definition 3.8 (**Wick Products of Segal Fields**)**.** Let $\mathcal{H}$ be a Hilbert space, and for any $f \in \mathcal{H}$ let $\Phi_S(f)$ be the Segal field on the bosonic Fock space $\mathcal{F}_s(\mathcal{H})$. Then, for $f, f_1, \ldots, f_k \in \mathcal{H}$ the **Wick**

product $:\Phi_S(f_1)\dots\Phi_S(f_k):$ is defined recursively by

$$
\begin{aligned}
&:\Phi_S(f): := \Phi_S(f),\\
&:\Phi_S(f)\prod_{j=1}^{k}\Phi_S(f_j): := \Phi_S(f):\prod_{j=1}^{k}\Phi_S(f_j): -\frac{1}{2}\sum_{j=1}^{k}(f,f_j)_{\mathcal{H}}):\prod_{i\neq j}\Phi_S(f_j):
\end{aligned}
\tag{78}
$$

Proposition 3.6 (**Wiener-Itô-Segal Isomorphism**). *If the probability measure μ on $\mathcal{S}'(\mathbf{R}^N)$ is Gaussian, then the space $\mathcal{F}_s(\mathcal{H})$ is isomorphic to $L^2(\mathcal{S}'(\mathbf{R}^N), d\mu)$ and the isomorphism $\theta_W : \mathcal{F}_s(\mathcal{H}) \to L^2(\mathcal{S}'(\mathbf{R}^N), d\mu)$, termed Wiener-Itô-Segal isomorphism, satisfies the following properties:*

(i) $\theta_W\Omega_0 = 1$,

(ii) $\theta_W\mathcal{H}_s^{(n)} = L^2_n(\mathcal{S}'(\mathbf{R}^N), d\mu)$,

(iii) $\theta_W\Phi_S(f)\theta_W^{-1}\Phi = \Phi(f)$ *for all* $\Phi \in \mathcal{S}'(\mathbf{R}^N)$ *and* $f \in \mathcal{S}(\mathbf{R}^N)$,

where $\Omega_0 = (1, 0, 0, \dots) \in \mathcal{F}_s(\mathcal{H})$ and $L^2_n(\mathcal{S}'(\mathbf{R}^N), d\mu)$ is the closure of all linear combinations of Wick products of random variables over $\mathcal{S}'(\mathbf{R}^N)$ up to order n.

Proof. See Proposition 5.7 in [BHL11]. □

3.3 Osterwalder-Schrader Axioms

Osterwalder and Schrader (see [OS73], [OS73Bis]) utilized the Wick rotation technique to pass from the Minkowskian to the Euclidean space and formulate axioms equivalent to Wightman in terms of Euclidean Green functions. In [OS75] they defined free Bose and Fermi fields and proved a Feynman-Kač formula for boson-fermion models.

The dynamics of the quantized system satisfying Wightman axioms is given by the Schrödinger, or, equivalently by the heat equation, where an H unbounded, selfadjoint Hamilton operator H on a physical Hilbert space $\mathcal{H}$ appears. The Hamilton operator can be extracted from the Osterwalder-Schrader axioms and ideally coincides with an operator obtained by a quantization procedure of the Hamiltonfunction in the classical description of the physical system.

Following chapter 6 of [GJ87] we introduce the

Definition 3.9 (**Osterwalder-Schrader Axioms**). The primitive of a quantum field model is a Borel probability measure $d\mu$ on $\mathcal{S}'(\mathbf{R}^N)$, whose inverse Fourier transform gives the generating functional

$$S(f) := \int_{\mathcal{S}'(\mathbf{R}^N)} e^{\imath\Phi(f)}d\mu(\Phi), \tag{79}$$

where $f \in \mathcal{S}(\mathbf{R}^N)$. Formally, we write $\Phi(f) = \int_{\mathbf{R}^N} \Phi(x) f(x) d^N x$.

(OS0) Analyticity: The functional $S(f)$ is entire analytic. For every finite set of test functions $f_j \in \mathcal{S}(\mathbf{R}^N)$, $j = 1, \ldots, n$ and complex numbers $z := (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$, the function

$$z \mapsto S\left(\sum_{j=1}^{n} z_j f_j\right) \tag{80}$$

is entire on $\mathbf{C}^n$.

(OS1) Regularity: For some $p \in [1, 2]$, some constant c and for all $f \in \mathcal{S}(\mathbf{R}^N)$

$$|S(f)| \leqslant e^{c(\|f\|_{L^1(\mathbf{R}^N)} + \|f\|^p_{L^p(\mathbf{R}^N)})}. \tag{81}$$

(OS2) Euclidean Invariance: The functional $S(f)$ is invariant under Euclidean symmetries E of $\mathbf{R}^N$. This means that for any translation, rotation and reflection, for all $f \in \mathcal{S}(\mathbf{R}^N)$

$$S(Ef) = S(f), \tag{82}$$

where $Ef(x) := f(E^{-1}(x))$.

(OS3) Reflection Positivity: Let

$$\mathcal{L} := \left\{ \psi \,\middle|\, \psi(\Phi) = \sum_{j=1}^{n} c_j e^{\Phi(f_j)}, c_j \in \mathbf{C}, f_j \in \mathcal{S}(\mathbf{R}^N), j = 1, \ldots, n \right\} \tag{83}$$

be the algebra of exponential functionals on tempered distributions and

$$\mathcal{L}_+ := \left\{ \psi \,\middle|\, \psi(\Phi) = \sum_{j=1}^{n} c_j e^{\Phi(f_j)}, c_j \in \mathbf{C}, f_j \in \mathcal{S}(\mathbf{R}^N), \operatorname{supp}(f_j) \subset \{(t, x) \in \mathbf{R}^N \mid t > 0\} \right\} \tag{84}$$

the subalgebra of exponential functionals whose defining functions are supported in the positive time half space. Euclidean transforms E on $\mathbf{R}^N$ act on $\mathcal{S}'(\mathbf{R}^N)$ via

$$(E\psi)(\Phi) := \psi(E\Phi) \text{ and } E\Phi(f) := \Phi(Ef), \tag{85}$$

for all $\psi \in \mathcal{L}$, $\Phi \in \mathcal{S}'(\mathbf{R}^N)$, $f \in \mathcal{S}(\mathbf{R}^N)$.

We assume that the time reflection

$$(t, x) \mapsto \theta(t, x) := (-t, x) \tag{86}$$

Path space	$\mathcal{S}'(\mathbf{R}^N)$
Configuration space	$\mathcal{S}'(\mathbf{R}^{N-1})$
Measure on path space	$d\mu$
Measure on configuration space	$d\nu = d\mu\|_{t:=0}$
Path space for quantum operators	$\mathcal{E} := L^2(\mathcal{S}'(\mathbf{R}^N), d\mu)$
Physical Hilbert space	$\mathcal{H} \cong L^2(\mathcal{S}'(\mathbf{R}^{N-1}), d\nu)$

Table 2: Quantum field theory from Osterwalder-Schrader axioms

satisfies

$$\int_{\mathcal{S}'(\mathbf{R}^N)} \theta\Psi(\Phi)\bar{\Psi}(\Phi)d\mu(\Phi) \geqslant 0. \tag{87}$$

for all $\Psi \in \mathcal{L}_+$.

(OS4) Ergodicity: The Euclidean time translation subgroup $\{T(t)\}_{t\geqslant 0}$ acts ergodically on the measure space $(\mathcal{S}'(\mathbf{R}^N), d\mu)$, i.e.

$$\lim_{t\mapsto+\infty} \frac{1}{t}\int_0^t T(s)\Psi(\Phi)T(s)^{-1}ds = \int_{\mathcal{S}'(\mathbf{R}^N)} \Psi(\Phi)d\mu(\Phi), \tag{88}$$

for all $\Psi \in L^1(\mathcal{S}'(\mathbf{R}^N), d\mu)$.

From the Osterwalder-Schrader axioms we can reconstruct quantum field theory as described by the Wightman axioms and derive the quantum mechanical dynamics. Table 2 depicts the formal scheme of this construction. The proper definition of the quantum mechanical Hilbert space relies on the reflection positivity axiom. The exponential functionals $\mathcal{L}$ are dense in $\mathcal{E} := L^2(\mathcal{S}'(\mathbf{R}^N), d\mu)$. Let $\mathcal{E}_+$ be the closure of $\mathcal{L}_+$ in $\mathcal{E}$, termed as **positive time subspace** of $\mathcal{E}$, and define the bilinear form on $\mathcal{E}_+ \times \mathcal{E}_+$

$$b(\Psi, \Upsilon) := \int_{\mathcal{S}'(\mathbf{R}^N)} \Psi(\Phi)\overline{\Upsilon(\Phi)}d\mu(\Phi). \tag{89}$$

By $(OS3)$ the bilinear form b is positive semidefinite. Let

$$\mathcal{N} := \{\psi \in \mathcal{E}_+ \mid b(\Psi, \Psi) = 0\} \tag{90}$$

the subspace of vectors for which the bilinear form degenerates, and define the quantum mechanical Hilbert space as

$$\mathcal{H} := \overline{\mathcal{E}_+/\mathcal{N}}, \tag{91}$$

with hermitian scalar product defined by (89) independently of the representative of the equivalence class. We denote the canonical embedding of the positive time subspace unto the quantum mechanical Hilbert space as

$$\begin{aligned} \wedge : &\mathcal{E}_+ \to \mathcal{H} \\ &\Psi \mapsto \Psi^\wedge := \Psi + \mathcal{N}, \end{aligned} \tag{92}$$

and transfer operators S acting on $\mathcal{E}_+$ to operators $S^\wedge$ acting on $\mathcal{H}$ by

$$S^\wedge \Psi^\wedge := (S\Psi)^\wedge, \tag{93}$$

which is well defined for all $\Psi \in \mathcal{E}_+$, if

$$S : \mathcal{D}(S) \cap \mathcal{E}_+ \to \mathcal{E}_+ \text{ and } S : \mathcal{D}(S) \cap \mathcal{N} \to \mathcal{N}. \tag{94}$$

Theorem 3.7 (Reconstruction of quantum mechanics)**.** *If the probability measure μ on $\mathcal{S}'(\mathbf{R}^N)$ satisfies the reflection and time translation invariance axiom (OS2), and the reflection positivity axiom (OS3), then for all $t \geqslant 0$ the time translation $T(t)$ satisfies (94) and*

$$T(t)^\wedge = e^{-tH}, \tag{95}$$

where $H = H^* \geqslant 0$ is a (possibly unbounded) selfadjoint operator on $\mathcal{H}$ with ground state $\Omega_0 := 1^\wedge$, i.e. $H\Omega_0 = 0$.

Proof. See Theorem 6.13 in [GJ87]. □

The reflection positivity axiom is not easy to verify. The following proposition provides a useful criterion.

Proposition 3.8. *The probability measure μ on $\mathcal{S}'(\mathbf{R}^N)$ satisfies the reflection positivity axiom (OS3) if and only if the matrix $M := [S(\theta(f_i) - f_j)]$ has positive eigenvalues for all choices of $(f_i)_{i=1,\ldots,n} \subset$*

$\mathcal{S}(\mathbf{R}^N)$ *with support in the time positive half space.*

Proof. See Corollary 3.4.4 in [GJ87]. □

Gaussian measures play a prominent role in quantum field theory, because they originate free fields.

Definition 3.10. A linear operator C defined on $\mathbf{R}^N$ satisfies the reflection positivity property if and only if

$$(\theta f, Cf)_{L^2(\mathbf{R}^N, d^N x)} \geqslant 0, \tag{96}$$

for all $f \in \mathbf{R}^N$ supported at positive times.

Theorem 3.9. *A Gaussian measure on the space of tempered distributions satisfies reflection positivity if and only if its covariance operator does.*

Proof. See Theorem 6.22 in [GJ87]. □

Proposition 3.10. *The covariance operator* $C := (-\Delta_{\mathbf{R}^N} + m^2)^{-1}$ *is reflection positive for all* $m^2 > 0$.

Proof. See Proposition 6.2.5 in [GJ87]. □

Finally, we highlight the equivalence of the Osterwalder-Schrader axioms with Wightman's ones.

Theorem 3.11. *Wightman's axioms (W1)-(W8) are equivalent to Osterwalder-Scharder axioms (OS0)-(OS4).*

Proof. See Theorem II.12 and Theorem II.13 in [Sim15] or Theorem 6.1.5 and Chapter 19 in [GJ87]. □

4 Quantization of Yang-Mills Equations and Positive Mass Gap

There are several methods of designing a quantum theory for non-Abelian gauge fields. The Hamiltonian formulation is the approach used in the original work by Yang-Mills ([MY54]), which was later abandoned in favour of an alternative method based on Feynman path integrals ([FP67]). When it became clear that the Faddeev-Popov method must be incomplete beyond perturbation theory, Hamiltonian formulation enjoyed a partial renaissance. More recent, didactically accessible, examples of the Hamiltonian approach in the physical literature can be found in [Sch08].

In the first section of this chapter we construct a quantized Yang-Mills theory in dimension $3+1$ and in the second we compute spectral lower bounds for the Hamilton operator. In the third we summarize our findings and prove the main result, Theorem 4.26.

4.1 Quantization

In the Yang-Mills 3 + 1 dimensional set up, in order to account for functionals on transversal fields as required by the Coulomb gauge, we introduce the configuration space $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ and the path space $\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})$. These are the duals in the sense of nuclear spaces of the test functions satisfying the transversal condition:

$$\begin{aligned}
L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x) &:= \{\, \mathbf{A} \in L^2(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x) \big|\ \mathbf{A}^{\parallel} = 0\},\\
\mathcal{S}_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}) &:= \mathcal{S}(\mathbf{R}^3, \mathbf{C}^{K\times 3}) \cap L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x)\\
\mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3}) &:= \{f \in \mathcal{S}(\mathbf{R}^4, \mathbf{C}^{K\times 3}) \,|\, f(t,\cdot) \in L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x) \text{ for all } t \in \mathbf{R}\}
\end{aligned} \tag{97}$$

Note that we do not need to bother about the time component of the connection, because it vanishes in the Coulomb gauge. The tempered distributions are defined

$$\begin{aligned}
\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}) &:= \left\{\mathbf{A} : \mathcal{S}_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}) \to \mathbf{C}^{K\times 3} \text{ linear and continuous}\right\}\\
\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3}) &:= \left\{\mathbf{A} : \mathcal{S}_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 4}) \to \mathbf{C}^{K\times 3} \text{ linear and continuous}\right\},
\end{aligned} \tag{98}$$

and $\mathbf{A} \in \mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ is called **regular** if there exists $\mathbf{a} = \mathbf{a}(t,x) \in L^2_{\text{loc}}(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ such that for all $f \in \mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$

$$\mathbf{A}(f) = \int_{\mathbf{R}^4} d^4(t,x) a(t,x).f(t,x), \tag{99}$$

where the dot denotes the pointwise multiplication.

We define the Hilbert space $\mathcal{E} := L^2(\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3}), d\mu)$ and the physical Hilbert space as $\mathcal{H} := L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu)$ for appropriate probability measures μ and ν.
We introduce the Hamilton operator originated by the quantization of the Hamiltonian formulation of Yang-Mills equations. This operator is the infinitesimal generator of a time inhomogeneous Itô's diffusion, whose probability density solves the heat kernel equation. We construct a probability measure on the tempered distributions using this Itô's process as integrator and utilizing the Feynman-Kač formula. We prove that the necessary Osterwalder-Schrader axioms for the Hamilton operator to be selfadjoint on the probability space of the time zero tempered distributions are fulfilled. Then, we verify that the Wightman axioms are satisfied.

When we try to introduce the canonical quantization for the Hamilton function H in (40), we face the problem for H_{II} and V that the classical fields $\mathbf{A}(t,x)$ cannot be directly interpreted as multiplication operators in $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu)$, because the multiplication of distributions cannot be properly defined. To circumvent this issue, following the idea expressed f.i. in [Sim05] (page 257) and applied to Nelson's model in [BHL11] (page 297) in the case of ultraviolet divergence, we introduce a cutoff test function to regularize fields.

Definition 4.1 (**Regularization and Ultraviolet Cutoff**). Let us consider a test function $\varphi \in \mathcal{S}(\mathbf{R}^4, \mathbf{R}^1)$ such that $\varphi_t(x) := \varphi(t, x) \in \mathcal{S}(\mathbf{R}^3, \mathbf{R}^1)$ satisfies for all t

$$\begin{aligned}
&\varphi_t(x) > 0 \text{ for all } x \in \mathbf{R}^3 \\
&\int_{\mathbf{R}^3} \varphi_t(x) = 1 \\
&\text{supp}(\varphi_t) \subset\subset \mathbf{R}^3.
\end{aligned} \tag{100}$$

The test function $\varphi^\Lambda \in \mathcal{S}(\mathbf{R}^4, \mathbf{R}^1)$ defined as $\varphi^\Lambda(t, x) := \Lambda\varphi_t(\Lambda x)$ is called **mollifier** for the ultraviolet cutoff level $\Lambda \geqslant 0$.

Note that for all $t \in \mathbf{R}$ in the limit we have $\mathcal{S}'-\lim_{\Lambda\to+\infty} \varphi_t^\Lambda = \delta \in \mathcal{S}'(\mathbf{R}^3, \mathbf{R}^1)$ and that $\mathbf{A}(\varphi_t^\Lambda(\cdot - x)I^{K\times 3})$ is a polynomially bounded function in $x \in \mathbf{R}^3$. By introducing the notation

$$\mathbf{A}(\varphi_t^\Lambda(\cdot - x)) := \mathbf{A}(\varphi_t^\Lambda(\cdot - x)I^{K\times 3}), \tag{101}$$

Theorem 2.4 can be now quantized as follows.

Proposition 4.1. *Let $H = H(A, E)$ be the Hamilton function of the Hamilton equations equivalent to the Yang-Mills equations as in Theorem 2.4 for a simple Lie-group as structure group. Let us consider a $\Lambda \geqslant 0$ and the mollifier function $\varphi^\Lambda \in \mathcal{S}(\mathbf{R}^4, \mathbf{R}^1)$. For a probability measure ν on $\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})$ the canonical quantization of the position variable A, of the momentum variable E and of the Hamilton function H means the following substitution:*

$$\begin{aligned}
\mathbf{A} \in C^\infty(\mathbf{R}^3, \mathbf{C}^{K\times 3}) &\longrightarrow \mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot)) \in \mathcal{O}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}^{K\times 3}, d\nu)) \\
\mathbf{E} \in C^\infty(\mathbf{R}^3, \mathbf{C}^{K\times 3}) &\longrightarrow \frac{1}{\imath}\frac{\delta}{\delta \mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot))} \in \mathcal{O}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}^{K\times 3}, d\nu)) \\
H \in C^\infty(\mathbf{R}^{2(K\times 3)}, \mathbf{R}) &\longrightarrow H^\Lambda := H\left(\mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot)), \frac{1}{\imath}\frac{\delta}{\delta \mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot))}\right) \in \mathcal{O}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu)),
\end{aligned} \tag{102}$$

where $\mathcal{O}(\cdot)$ denotes the set of all linear operators over the corresponding underlying vector space. The cutoff Hamilton operator H^Λ for the quantized Yang-Mills equations reads

$$H^{\Lambda,g} = H_I^\Lambda + H_{II}^{\Lambda,g} + V^{\Lambda,g} - V_0^{\Lambda,g}, \tag{103}$$

where

$$(H_I^\Lambda\Psi)(\mathbf{A}) = -\frac{1}{2}\int_{\mathbf{R}^3} d^3x \left[\frac{\delta}{\delta A_i^a(\varphi_t^\Lambda(\cdot - x))}\right]^2 \Psi(\mathbf{A})$$

$$(H_{II}^{\Lambda,g}\Psi)(\mathbf{A}) = -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y \partial_i G^{a,b}(\mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot)); x, y)\varepsilon^{b,c,d} A_k^d(\varphi_t^\Lambda(\cdot - y))\frac{\delta}{\delta A_k^c(\varphi_t^\Lambda(\cdot - y))}\right]^2 \Psi(\mathbf{A})$$

$$(V^{\Lambda,g}\Psi)(\mathbf{A}) = \int_{\mathbf{R}^3} d^3x\, V^{\Lambda,g}(t,x,\mathbf{A})\Psi(\mathbf{A}),$$

$$\begin{aligned} V^{\Lambda,g}(t,x,\mathbf{A}) := \frac{1}{16}\varepsilon_i^{j,k}\varepsilon_i^{p,q}\{&[\partial_j A_k^a(\varphi_t^\Lambda(\cdot - x)) - \partial_k A_j^a(\varphi_t^\Lambda(\cdot - x)) + g\varepsilon^{a,b,c}A_j^b(\varphi_t^\Lambda(\cdot - x))A_k^c(\varphi_t^\Lambda(\cdot - x))]\cdot \\ &\cdot [\partial_p A_q^a(\varphi_t^\Lambda(\cdot - x)) - \partial_q A_p^a(\varphi_t^\Lambda(\cdot - x)) + g\varepsilon^{a,b,c}A_p^b(\varphi_t^\Lambda(\cdot - x))A_q^c(\varphi_t^\Lambda(\cdot - x))]\}, \end{aligned}$$

$V_0^{\Lambda,g}$: *a real constant which will be chosen later,*

(104)

with the domain of definition

$$\mathcal{D}(H^{\Lambda,g}) := \left\{\Psi \in L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu) \,\middle|\, H^\Lambda\Psi \in L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu)\right\}. \tag{105}$$

Note that the time t is considered as a parameter in the expressions (102) and (104).

Remark 4.1. The first dot in $\partial_i G^{a,b}(\mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot))$ in the expression for $H_{II}^{\Lambda,g}$ in (104) refers to the application of the distribution $\mathbf{A}$ on the test function i.e. the integration variable if $\mathbf{A}$ is regular. The second is the generic variable sign, since the modified Green function G is a functional of a function in the first argument, as explained in Proposition 2.1.

Proof of Proposition 4.1. It is a straightforward consequence of Corollary 2.5 where we introduce the quantization specified by (102). □

Remark 4.2. Later, we will prove that $\Omega_0 \in \mathcal{D}(H^{\Lambda,g})$ is the **ground state** of $H^{\Lambda,g}$, which is an eigenvector for the eigenvalue 0 with simple multiplicity, unique up to multiplication with a constant.

Remark 4.3. The derivative

$$\frac{\delta}{\delta A_i^a(\varphi_t^\Lambda(\cdot - x))}\Psi(\mathbf{A}) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon:=0} \Psi(\mathbf{A} + \varepsilon\varphi_t^\Lambda(\cdot - x))) \longrightarrow \frac{\delta}{\delta A_i^a(t,x)}\Psi(\mathbf{A})\ (\Lambda \to +\infty) \tag{106}$$

in line with fact that $A_i^a(\varphi_t^\Lambda(\cdot - x))$ tends to $A_i^a(t,x)$, because $\mathcal{S}' - \lim_{\Lambda\to+\infty}\varphi_t^\Lambda(\cdot - x)) = \delta(\cdot - x)$ for all t.

Remark 4.4. The operator H_I^Λ tends for $\Lambda \to +\infty$ to the Laplace operator in infinite dimensions for functionals of the potential fields. The operator V^Λ is a multiplication operator corresponding to the fibrewise multiplication with the square of the connection curvature. Both operators H_I^Λ and V^Λ do not vanish if $g = 0$ and do not contribute to the existence of a mass gap. The operator H_{II}^Λ vanishes if $g = 0$, f.i. when G is an abelian groups, and, as we will see, is responsible for the existence of a mass gap.

Remark 4.5. In the physical literature (see f.i. [Sch08]) the operator $C = C(\mathbf{A}; x, y)$

$$C^{a,b}(\mathbf{A}; x, y) := -\int_{\mathbf{R}^3} d^3\bar{y}G^{a,c}(\mathbf{A}; x, \bar{y})\Delta G^{c,b}(\mathbf{A}; y, \bar{y}) \tag{107}$$

is termed *Coulomb operator* and $G = G(\mathbf{A}; x, y)$ is also called the *Faddeev-Popov operator*. Note that they make sense only after the substitution

$$\mathbf{A}(t, x) \to \mathbf{A}(\varphi_t^\Lambda(\cdot - x)), \tag{108}$$

and passing to the limit $\Lambda \to +\infty$.

A direct computation shows

Proposition 4.2 (**Canonical Commutation Relations**). *The operators $Q_i^j(x)$ and $P_i^j(x)$ defined as*

$$\begin{aligned} Q_i^a(x)\Psi(\mathbf{A}) &:= A_i^a(\varphi_t^\Lambda(x - \cdot))\Psi(\mathbf{A}) \\ P_i^a(x)\Psi(\mathbf{A}) &:= \frac{1}{\imath}\frac{\delta}{\delta A_i^a(\varphi_t^\Lambda(x - \cdot))}\Psi(\mathbf{A}), \end{aligned} \tag{109}$$

on the appropriate domains, are in $\mathcal{O}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^{\Lambda,g}))$ for all i, a, x and all measures $\nu^{\Lambda,g}$. Their commutators satisfy the equations

$$[Q_i^a(x), Q_j^b(y)] = 0 \qquad [P_i^a(x), P_j^b(y)] = 0 \qquad [P_i^a(x), Q_j^b(y)] = \frac{1}{\imath}\delta^{a,b}\delta_{i,j}\delta_{x,y}, \tag{110}$$

for all i, j, a, b, x, y.

4.2 Construction of a Complete Set of Generalized Eigenvectors

Theorem 4.3. *For $\Lambda \geqslant 0$ big enough there exists a probability measure $\nu^{\Lambda,g}$ such that $H^{\Lambda,g}$ is a selfadjoint operator on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu^{\Lambda,g})$ for all real constants $V_0^{\Lambda,g}$ There exists a **ground state** $\Omega_0^{\Lambda,g} \in \mathcal{D}(\mathcal{H}^{\Lambda,g})$ of $H^{\Lambda,g}$ for the eigenvalue 0 for one choice of the real constant $V_0^{\Lambda,g}$:*

$$H^{\Lambda,g}\Omega_0^{\Lambda,g} = 0. \tag{111}$$

The ground state $\Omega_0^{\Lambda,g}$ is an eigenvector with simple multiplicity, and hence it is unique up to multiplication with a constant.

The proof of Theorem 4.3 is an elaborated functional analytic construction of a probability measure making the Hamilton operator selfadjoint and with a unique ground state.

On a finite dimensional vector space an operator possessing a basis of eigenvectors for real eigenvalues is selfadjoint only with respect to the scalar product which makes that basis orthonormal. For an infinite space the situation is similar but more complicated by the presence of generalized eigenvalues which are not proper eigenvalues, i.e. by the continuous spectrum.

Proposition 4.4. *Let ν_0 be the standard Gaussian probability on $\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})$. In the notation of Example A.2*

$$\begin{aligned}
&\mathcal{E}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0)) := \left\{\varphi \in L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0) \mid \|\varphi\|_k < +\infty\right\}\\
&\mathcal{E}'(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0)) : \text{ dual space of } (\mathcal{S}_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3},\mathbf{C},d\nu_0)).
\end{aligned} \tag{112}$$

Then, we have a rigged Hilbert space

$$\mathcal{E}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0)) \subset L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0) \subset \mathcal{E}'(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0)) \tag{113}$$

and the operators H_I^Λ, $H_{II}^{\Lambda,g}$, $V^{\Lambda,g}$ have a complete set of generalized eigenvectors in $\mathcal{E}'(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0))$ for real non-negative generalized eigenvalues. The operator $H^{\Lambda,g}$ is selfadjoint on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0)$ with a non negative spectrum for $V_0^{\Lambda,g}$ sufficiently small and Λ sufficiently big.

To prove this proposition we need to prove some intermediate Lemmata beforehand.

Lemma 4.5. *Let $R > 0$ be a positive constant and let*

$$D := \sum_{k=1}^{N} f_k(A)\partial_{A_k} \tag{114}$$

be a first order PDO on $\mathbf{R}^N$ for given functions $f_1, f_2, \ldots, f_N$ on $\mathbf{R}^N$. If there exists a diffeomeorphism B mapping some compact subset of $\mathbf{R}^N$ (in the "A"-space) to $[-\frac{R}{2}, +\frac{R}{2}]^N$ (in the "B"-space), such that

$$g_j(B) := \sum_{k=1}^{N} f_k(A)\partial_{A_k} B_j \tag{115}$$

depends only on B_j, so that $g_j(B) = g_j(B_j)$, then the Dirichlet eigenvalues of D^2 on $B^{-1}([-\frac{R}{2}, +\frac{R}{2}]^N)$ are

$$-\sum_{j=1}^{N} \frac{\pi^2 k_j^2}{\left[\int_{-\frac{R}{2}}^{+\frac{R}{2}} dB_j\, g_j(B_j)^{-1}\right]^2}, \tag{116}$$

where $k_j \in \mathbf{Z}^*$ for $j = 1, \ldots, N$, provided the integrals in the denominators of (116) exists.

Lemma 4.5 is proved by a direct computation utilizing second order ODE.

Lemma 4.6. *Let us assume that for all $k = 1, \ldots, N$*

$$\int_{-\infty}^{+\infty} dA_k\, f_k(A)^{-1} < +\infty \tag{117}$$

holds uniformly in A. Then, the function $B : \mathbf{R}^N \to \mathbf{R}^N$ defined as

$$B_j := \Phi^{-1}\left(L_j\left[\int_{-\infty}^{A_j} d\bar{A}_j f_j^{-1}(A_1, \ldots, \bar{A}_j, \ldots, A_N) \;+\; K_j(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_N)\right]\right), \tag{118}$$

where

$$\begin{aligned}
K_j(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_N) &:= -\inf_{A_j} \int_{-\infty}^{A_j} d\bar{A}_j f_j^{-1}(A_1, \ldots, \bar{A}_j, \ldots, A_N) > -\infty \\
L_j &:= \left[\sup_A \left[\int_{-\infty}^{A_j} d\bar{A}_j f_j^{-1}(A_1, \ldots, \bar{A}_j, \ldots, A_N) + K_j(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_N)\right]\right]^{-1} < +\infty \\
\Phi(v) &:= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{v} du\, e^{-u^2},
\end{aligned} \tag{119}$$

is a diffeomorphism satisfying the assumptions of Lemma 4.5 for any constant $R > 0$, and the functions g_j read

$$g_j(B_j) = \sqrt{\pi}\left(\sum_{k=1}^{N} L_k\right) e^{B_j^2}. \tag{120}$$

Proof. By definition (115), if

$$f_k \partial_{A_k} B_j = C_{k,j}(B_j) \tag{121}$$

for a function $C_{k,j}$ of one variable, then the function g_j explicitly depends on the variable B_j only. This leads to the differential equation

$$\frac{dB_j}{C_{j,k}(B_j)} = \frac{dA_k}{f_k(A)}, \tag{122}$$

which is fulfilled if

$$\int_{-\infty}^{B_j} \frac{d\bar{B}_j}{C_{j,k}(\bar{B}_j)} = \int_{-\infty}^{A_k} \frac{d\bar{A}_k}{f_k(A_1, \ldots, A_{k-1}, \bar{A}_k, A_k, \ldots, A_N)} + K_k(A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_N), \tag{123}$$

where K_k is a function of A not depending on A_k. The choice

$$C_{j,k}(u) := \sqrt{\pi} L_j e^{u^2} \tag{124}$$

for L_j and K_j defined in (119) leads to the desired result. □

Now we can compute the generalized spectral decomposition of the Hamilton operator $H^{\Lambda,g}$.

Proof of Proposition 4.4. The rigged Hilberst space statement (113) follows from the Kubo-Takenaka construction explained in Example A.2. We will construct generalized eigenvectors for real non negative eigenvalues of $H^{\Lambda,g}$ which decomposes as

$$H^{\Lambda,g} = H_I^{\Lambda} + H_{II}^{\Lambda,g} + V^{\Lambda,g} - V_0^{\Lambda,g}. \tag{125}$$

First, we analyze the operator H_I^{Λ}, which can be written as

$$\begin{aligned} H_I^{\Lambda} &= U^{-1} H_I U \\ H_I &= -\frac{1}{2} \int_{\mathbf{R}^3} d^3x \Delta_{\mathbf{R}^{3K}} \\ U\Psi(\mathbf{A}) &= \Psi(\mathbf{A}(\varphi_t^{\Lambda}(\cdot - \cdot))) \text{ for } \mathbf{A} \in \mathcal{S}'_{\perp}(\mathbf{R}^4, \mathbf{C}^{K\times 3}). \end{aligned} \tag{126}$$

Hence it suffices to construct a generalized eigenvector for H_I and one follows for H_I^g for the same generalized eigenvalue. Let $x \in \mathbf{R}^3$ and $t \in \mathbf{R}$ now be fixed. For any $R > 0$ the Laplace operator Δ on $[-\frac{R}{2}, +\frac{R}{2}]^{3K}$ under Dirichlet boundary conditions has a discrete spectral resolution $(\lambda_k, \psi_k)_{k\geqslant 0}$, where $\lambda_k = -\frac{\pi^2}{R^2}(k+1)$, and $\psi_k = \psi_k(\mathbf{A}) \in C_0^{\infty}([-\frac{R}{2}, +\frac{R}{2}]^{3K}, \mathbf{C})$. We can extend ψ_k outside the cube by setting its value to 0, obtaining an approximated eigenvector for the approximated eigenvalue λ_k, which is in line with the fact that the Laplacian on $L^2(\mathbf{R}^{3K}, \mathbf{C})$ has solely a continuous spectrum, which is $]-\infty, 0]$. The functional on $\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3})$

$$\Psi_k^{\mathbf{A};x_0}(\bar{\mathbf{A}}) := \delta(\bar{\mathbf{A}} - \mathbf{A})\delta(x - x_0)\psi_k(\mathbf{A}) \tag{127}$$

for $x_0 \in \mathbf{R}^3, k \in \mathbf{N}$ and $\mathbf{A} \in \mathbf{R}^{3K}$ is a generalized eigenvector in $\mathcal{E}'(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu_0)$ for the operator H_I on the rigged Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu_0)$ for the generalized eigenvalue $\frac{\pi^2}{R^2}(k+1)$. By varying the generalized eigenvector over x_0, $\mathbf{A}$, k and R, we obtain a complete set of generalized eigenvectors for $\mathcal{E}'(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0))$, because, if for any $\Phi \in \mathcal{E}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0))$

$$\int_{\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})} \Psi_k^{\mathbf{A};x_0}(\bar{\mathbf{A}})\bar{\Phi}(\bar{\mathbf{A}})d\nu_0(\bar{\mathbf{A}}) = 0, \tag{128}$$

that is

$$\delta(x - x_0)\psi_k(\mathbf{A})\bar{\Phi}(\mathbf{A}) = 0, \tag{129}$$

which holds true for all $x_0 \in \mathbf{R}^3$, $\mathbf{A} \in \mathbf{R}^{3K}$ and all $k \geqslant 0$ iff $\Phi = 0$. By Corollary A.3 the operator H_I on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0)$ is selfadjoint with non negative spectrum. The same holds true for H_I^Λ Next, we analyze the operator $H_{II}^{\Lambda,g}$, which can be written as

$$\begin{aligned}
H_{II}^{\Lambda,g} &= U^{-1} H_{II}^g U \\
H_{II} &= -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, D_i^a(\mathbf{A}; x, y)\right]^2 \\
U\Psi(\mathbf{A}) &= \Psi(\mathbf{A}(\varphi_t^\Lambda(\cdot - \cdot))) \text{ for } \mathbf{A} \in \mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3}),
\end{aligned} \tag{130}$$

for the operator $D = D(\mathbf{A}; \mathbf{x}, \mathbf{y})$ defined as

$$D_i^a(\mathbf{A}; x, y) := \partial_i G^{a,b}(\mathbf{A}(t, y); x, y)\varepsilon^{b,c,d} A_k^d(t, y)\frac{\delta}{\delta A_k^c(t, y)}. \tag{131}$$

Hence it suffices to construct a generalized eigenvector for H_{II}^g and one follows for $H_{II}^{\Lambda,g}$ for the same generalized eigenvalue. Let $x_0, y_0 \in \mathbf{R}^3$ now be fixed. We set

$$f_{i,k}^{a,c}(\mathbf{A}; x_0, y_0) := \partial_i G^{a,b}(\mathbf{A}(t, y_0); x_0, y_0)\varepsilon^{b,c,d} A_k^d(t, y_0) \tag{132}$$

and apply Lemma 4.5 and Lemma 4.6. Assuming that for all indices c, k

$$\int_{-\infty}^{+\infty} dA_k^c\, f_{i,k}^{a,c}(\mathbf{A}(t, y_0); x_0, y_0)^{-1} < +\infty \tag{133}$$

uniformly in $\mathbf{A}$, we can find a diffeomeorphism $\mathbf{B} : \mathbf{R}^{3K} \to \mathbf{R}^{3K}$ in the form of formula (118), such that for any $R > 0$ the operator $D_i^a(\mathbf{A}(t, y_0); x_0, y_0)^2$ on $B^{-1}([-\frac{R}{2}, +\frac{R}{2}]^{3K})$ under Dirichlet boundary conditions has a discrete spectral resolution $(\lambda_{i,s}^a(x_0,y_0), \psi_{i,s}^a(\mathbf{A}(t,y_0);x_0,y_0))_{s\geqslant 0}$, where

$$\lambda_{i,s}^a(x_0,y_0) = -\sum_{j=1}^{3}\sum_{c=1}^{K}\frac{\pi^2 k_{j,c,s}^2}{\left[\int_{-\frac{R}{2}}^{+\frac{R}{2}} dB_j^c\, g_{i,j}^{a,c}(B_j^c;x_0,y_0)^{-1}\right]^2}, \tag{134}$$

where $k_{j,c,s}\in\mathbf{Z}^*$ for all indices $s\in\mathbf{N}_0$, $j\in\{1,2,3\}$ and $c\in\{1,\dots,K\}$, and, by Lemma 4.6 we defined

$$g_{i,j}^{a,l}(B_j^l;x_0,y_0) := \left(\sum_{k=1}^{3}\sum_{c=1}^{K} L_{i,k}^{a,c}(x_0,y_0)\right) e^{{B_j^l}^2} \tag{135}$$

for

$$\begin{aligned} L_{i,j}^{a,c}(x_0,y_0) &= \left[\sup_{\mathbf{A}}\left[\int_{-\infty}^{A_j^c} d\bar{A}_j^c\, f_{i,j}^{a,c}(\mathbf{A};x_0,y_0)^{-1} + K_{i,j}^{a,c}(\mathbf{A};x_0,y_0)\right]\right]^{-1} \\ K_{i,j}^{a,c}(\mathbf{A}\,;x_0,y_0) &:= -\inf_{A_j^c}\int_{-\infty}^{A_j^c} d\bar{A}_j^c\, f_{i,j}^{a,c}(\mathbf{A};x_0,y_0)^{-1}. \end{aligned} \tag{136}$$

Note that for any $R>0$ the operator $D_i^a(\mathbf{A}(t,y_0);x_0,y_0)$ on $B^{-1}([-\frac{R}{2},+\frac{R}{2}]^{3K})$ under Dirichlet boundary conditions has a discrete spectral resolution with the same eigenvectors as $D_i^a(\mathbf{A}(t,y_0);x_0,y_0)^2$ but other eigenvalues $(\zeta_{i,s}^a(x_0,y_0), \psi_{i,s}^a(\mathbf{A}(t,y_0);x_0,y_0))_{s\geqslant 0}$, where

$$\zeta_{i,s}^a(x_0,y_0) = -\sum_{j=1}^{3}\sum_{c=1}^{K}\frac{\imath\pi k_{j,c,s}^2}{\int_{-\frac{R}{2}}^{+\frac{R}{2}} dB_j^c\, g_{i,j}^{a,c}(B_j^c;x_0,y_0)^{-1}}, \tag{137}$$

where $k_{j,c,s}\in\mathbf{Z}^*$ for all indices $s\in\mathbf{N}_0$, $j\in\{1,2,3\}$ and $c\in\{1,\dots,K\}$.

Since $B^{-1}([-\frac{R}{2},+\frac{R}{2}]^{3K})\uparrow\mathbf{R}^{3K}$ for $R\uparrow+\infty$, we can extend $\psi_{i,s}^a(\cdot;x_0,y_0)$ outside the cube by setting its value to 0, obtaining an approximated eigenvector for the approximated eigenvalue $\lambda_{i,s}^a(x_0,y_0)$ for the operator $D_i^a(\mathbf{A};x_0,y_0)^2$ on $L^2(\mathbf{R}^{3K},\mathbf{C})$, which means that $\lambda_{i,s}^a(x_0,y_0)\in\mathrm{spec}_c(D_i^a(\mathbf{A};x_0,y_0)^2)$. For fixed i,s and a the functional

$$\Psi_{i,k}^{a,x_0,y_0,\mathbf{A}}(\bar{\mathbf{A}}) := \delta(\bar{\mathbf{A}}-\mathbf{A})\delta(x-x_0)\delta(y-y_0)\delta(\bar{y}-y_0)\psi_{i,k}^a(\mathbf{A};x_0,y_0) \tag{138}$$

is a generalized eigenvector in $\mathcal{E}'(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),d\nu_0)$ for the operator

$$\begin{aligned} H_{i,II}^{a,g} &:= -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x\left[\int_{\mathbf{R}^3} d^3y\, D_i^a(\mathbf{A};x,y)\right]^2 = \\ &= -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x\left[\int_{\mathbf{R}^3} d^3y\, D_i^a(\mathbf{A};x,y)\right]\left[\int_{\mathbf{R}^3} d^3\bar{y}\, D_i^a(\mathbf{A};x,\bar{y})\right] \end{aligned} \tag{139}$$

on the rigged Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3}), d\nu_0)$ for the strictly positive generalized eigenvalue

$$\lambda_{i,s}^{a,g}(x_0, y_0) = g^2 \sum_{j=1}^{3} \sum_{c=1}^{K} \frac{\frac{\pi^2}{2} k_{j,c,s}^2}{\left[\int_{-\frac{R}{2}}^{+\frac{R}{2}} dB_j^c \, g_{i,j}^{a,c}(B_j^c; x_0, y_0)^{-1}\right]^2}. \tag{140}$$

By varying the generalized eigenvector over x_0, y_0, $\mathbf{A}$, k and R, we obtain a complete set of generalized eigenvectors for $\mathcal{E}'(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0))$, because, if for any $\Phi \in \mathcal{E}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0))$

$$\int_{\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})} \Psi_{i,k}^{a,x_0,y_0,\mathbf{A}}(\bar{\mathbf{A}}) \bar{\Phi}(\bar{\mathbf{A}}) d\nu_0(\bar{\mathbf{A}}) = 0, \tag{141}$$

that is

$$\delta(x - x_0)\delta(y - y_0)\psi_{i,k}^a(\mathbf{A}; x_0, y_0) = 0, \tag{142}$$

which holds true for all $x_0, y_0 \in \mathbf{R}^3$, $\mathbf{A} \in \mathbf{R}^{3K}$ and all $k \geqslant 0$ iff $\Phi = 0$. By Corollary A.3 the operators $H_{i,II}^a$ on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0)$ is selfadjoint with non negative spectrum. The same holds true for and thus H_{II}^g and $H_{II}^{\Lambda,g}$ for all $\Lambda \geqslant 0$.

Finally, we analyze the multiplication operator

$$(V^{\Lambda,g}\Psi)(\mathbf{A}) = \int_{\mathbf{R}^3} d^3x \, V^{\Lambda,g}(t, x, \mathbf{A})\Psi(\mathbf{A}),$$

$$\begin{aligned} V^{\Lambda,g}(t, x, \mathbf{A}) := \frac{1}{16}\varepsilon_i^{j,k}\varepsilon_i^{p,q} \{&[\partial_j A_k^a(\varphi_t^\Lambda(\cdot - x)) - \partial_k A_j^a(\varphi_t^\Lambda(\cdot - x)) + g\varepsilon^{a,b,c} A_j^b(\varphi_t^\Lambda(\cdot - x)) A_k^c(\varphi_t^\Lambda(\cdot - x))] \cdot \\ &\cdot [\partial_p A_q^a(\varphi_t^\Lambda(\cdot - x)) - \partial_q A_p^a(\varphi_t^\Lambda(\cdot - x)) + g\varepsilon^{a,b,c} A_p^b(\varphi_t^\Lambda(\cdot - x)) A_q^c(\varphi_t^\Lambda(\cdot - x))]\}. \end{aligned} \tag{143}$$

Let $\mathbf{A} \in L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x)$ now be fixed.

$$\begin{aligned} V^{\Lambda,g}(t, x, \mathbf{A}) \longrightarrow \frac{1}{16}\varepsilon_i^{j,k}\varepsilon_i^{p,q} \{&[\partial_j A_k^a(t, x) - \partial_k A_j^a(t, x) + g\varepsilon^{a,b,c} A_j^b(t, x) A_k^c(t, x)] \cdot \\ &\cdot [\partial_p A_q^a(t, x) - \partial_q A_p^a(t, x) + g\varepsilon^{a,b,c} A_p^b(t, x) A_q^c(t, x)]\} = |R^{\nabla^{\mathbf{A}}}(t, x)|^2 \quad (\Lambda \to +\infty), \end{aligned} \tag{144}$$

where $R^{\nabla^{\mathbf{A}}}$ is the curvature operator associated to the connection $\mathbf{A}$. Any non zero $\psi \in C_0^\infty(\mathbf{R}^{3K}, \mathbf{C})$ is eigenvector of the multiplication operator on $L^2(\mathbf{R}^{3K}, \mathbf{C})$ with the non negative real $V^{\Lambda,g}(t, x, \mathbf{A})$ for Λ big enough. The functional

$$\Psi_k^{\mathbf{A};x_0}(\bar{\mathbf{A}}) := \delta(\bar{\mathbf{A}} - \mathbf{A})\delta(x - x_0)\psi_k(\mathbf{A}), \tag{145}$$

where $(\psi_k)_{k\geqslant 0}$ is an orthonormal basis of $L^2(\mathbf{R}^{3K}, \mathbf{C})$, is a generalized eigenvector in $\mathcal{E}'(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu_0)$

for the operator $V^{\Lambda,g}$ on the rigged Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),d\nu)$ for the generalized eigenvalue $V^{\Lambda,g}(t,x_0,\mathbf{A})$. By varying the generalized eigenvector over $x_0\in\mathbf{R}^3$, $\mathbf{A}\in\mathbf{R}^{3K}$, and $k\geqslant 0$, we obtain a complete set of generalized eigenvectors for $\mathcal{E}'(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0))$, because, if for any $\Phi\in\mathcal{E}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0))$

$$\int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})}\Psi_k^{x_0,\mathbf{A}}(\bar{\mathbf{A}})\bar{\Phi}(\bar{\mathbf{A}})d\nu_0(\bar{\mathbf{A}})=0, \tag{146}$$

that is

$$\delta(x-x_0)\psi_k(\mathbf{A};x_0)=0, \tag{147}$$

which holds true for all $x_0\in\mathbf{R}^3$, $\mathbf{A}\in\mathbf{R}^{3K}$ and all $k\geqslant 0$ iff $\Phi=0$. By Corollary A.3 the operator $V^{\Lambda,g}$ on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu_0)$ is selfadjoint with non negative spectrum if Λ is big enough.

We conclude that $H^{\Lambda,g}$ is a selfadjoint operator as a sum of selfadjoint operators. For Λ big enough, if we choose $V_0^{\Lambda,g}\leqslant\inf\operatorname{spec}((V^{\Lambda,g})$, then the spectrum of $H^{\Lambda,g}$ lies in $[0,+\infty[$. The proof is concluded. □

Next, in view of the proof of Theorem 4.3 we consider the commutative version of Theorem 4 in [Gro72] considering the remarks on page 59 therein.

Theorem 4.7. *Let H_0 be a nonnegative selfadjoint operator on $L^2(X,\mathbf{C},d\nu)$, where $(X,\mathcal{A},\nu)$ is a probability space. Assume*

(i) $\exp(-tH_0)$ is a contraction in $L^p(X,\mathbf{C},d\nu)$ norm for all $t>0$ and all $p\in[1,+\infty]$ and $\exp(-TH_0)$ is a contraction from $L^2(X,\mathbf{C},d\nu)$ to $L^4(X,\mathbf{C},d\nu)$ for some real number $T>0$.

(ii) $\exp(-tH_0)$ is positivity preserving for all $t>0$.

(iii) The null space of H_0, $\ker(H_0)$ is spanned by the identity element of the algebra of bounded measurable functions on $(X,\mathcal{A})$.

Let V be a selfadjoint operator given by the multiplication by some measurable real function v on $L^2(X,\mathbf{C},d\nu)$. Assume

(iv) $v\in L^k(X,\mathbf{C},d\nu)$ for some real number $k>2$ and $\exp(-v)\in L^p(X,\mathbf{C},d\nu)$ is for all $p<+\infty$.

Then:

(a) H_0+V is essentially selfadjoint and its closure H is bounded from below.

(b) If $\lambda = \inf \operatorname{spec}(H)$, then λ is an eigenvalue of H of multiplicity one and there exists a corresponding non-negative eigenvector.

Proof of Theorem 4.3. If Λ big enough and $V_0^{\lambda,g}$ small enough, by Proposition 4.4 the operator $H^{\Lambda,g}$ is selfadjoint with non negative spectrum on $\mathcal{H} := L^2(X, \mathbf{C}, d\nu_0)$ for $X := \mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})$. With the choice $V_0^{\Lambda,g} := \inf \operatorname{spec}(H_I^\Lambda + H_{II}^{\Lambda,g} + V^{\Lambda,g})$ we obtain

$$0 = \inf \operatorname{spec}(H^{\Lambda,g}), \tag{148}$$

which, by Theorem 4.7 (b), is a simple eigenvalue for an eigenvector $\Omega_0^{\Lambda,g}$. With the choices

$$H_0 := H^{\Lambda,g} \text{ and } v := 0, \tag{149}$$

the assumptions of Theorem 4.7 are satisfied, because, being H_0 selfadjoint with non-negative spectrum, we have the spectral representation by of the projection valued measure $E : \mathcal{B}(\mathbf{R}) \to \mathcal{L}(\mathcal{H})$ as

$$H_0\varphi = \int_0^{+\infty} \lambda dE(\lambda)\varphi \qquad \text{for } \varphi \in \mathcal{D}(H_0). \tag{150}$$

Therefore, for all $t \geqslant 0$

$$\begin{aligned} &\exp(-tH_0)\varphi = \int_0^{+\infty} \exp(-t\lambda)dE(\lambda)\varphi \\ &\text{for } \varphi \in \mathcal{D}(\exp(-tH_0)) := \left\{\varphi \in \mathcal{H} \,\middle|\, \int_0^{+\infty} |\exp(-t\lambda)|^2 d(E(\lambda)\varphi, \varphi) < +\infty \right\} \end{aligned} \tag{151}$$

(i) For $p = 2$ we have for all $\varphi \in \mathcal{H}$

$$\begin{aligned} \|\exp(-tH_0)\varphi\|_{\mathcal{H}}^2 &= (\exp(-tH_0)\varphi, \exp(-tH_0)\varphi) = (\exp(-2tH_0)\varphi, \varphi) = \\ &= \int_0^{+\infty} \underbrace{\exp(-2t\lambda)}_{\leqslant 1} d(E(\lambda, \varphi, \varphi) \leqslant \|\varphi\|_{\mathcal{H}}^2. \end{aligned} \tag{152}$$

Hence, $\exp(-tH_0)$ is a contraction on L^2.

For $p \in [1, +\infty[$ we have

$$\|\exp(-tH_0)\varphi\|_{L^p}^p = \int_X \left|\int_0^{+\infty} \exp(-t\lambda)dE(\lambda)\varphi\right|^p d\nu \leqslant \underbrace{\left\|\int_0^{+\infty} \underbrace{\exp(-t\lambda)}_{\leqslant 1} dE(\lambda)\right\|_p^p}_{\leqslant 1} \|\varphi\|_p^p. \tag{153}$$

Hence, $\exp(-tH_0)$ is a contraction on L^p for $p \in [1, +\infty[$.

For $p = +\infty$ we have

$$\|\exp(-tH_0)\varphi\|_{L^\infty} = \int_X \sup_X \left|\int_0^{+\infty} \exp(-t\lambda) dE(\lambda)\varphi\right| d\nu \leqslant \underbrace{\left\|\int_0^{+\infty} \underbrace{\exp(-t\lambda)}_{\leqslant 1} dE(\lambda)\right\|_\infty}_{\leqslant 1} \|\varphi\|_\infty. \quad (154)$$

Hence, $\exp(-tH_0)$ is a contraction on L^∞.

(ii) Let us write $\exp(-tH_0)$ as integral operator

$$\exp(-tH_0)\varphi(x) = \int_X K(x,y)\varphi(y)d\nu(y) \quad (155)$$

for an appropriate with integral kernel $K = K(x,y)$. Therefore, for all $\varphi \in \mathcal{H}$

$$(\exp(-tH_0)\varphi, \varphi) = \int_{X^2} K(x,y)|\varphi(y)|^2 d\nu(x)d\nu(y) = \left\|\exp\left(-\frac{t}{2}H_0\right)\varphi\right\|_{\mathcal{H}}^2 \geqslant 0. \quad (156)$$

This can only be true if $K \geqslant 0$. Hence, for $\varphi > 0$, (155) shows that $\exp(-tH_0)\varphi > 0$, meaning that $\exp(-tH_0)\varphi$ is positivity preserving.

(iii) Every element of $\ker(H_0) \subset \mathcal{E}(L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0))$ must be a bounded measurable function on X, because H_0 is an infinite dimensional elliptic operator [BeKo12].

(iv) The choice $v = 0$ satisfies this assumption.

The proof is completed.

□

Remark 4.6. The ground state $\Omega_0^{\Lambda,g}$ of $H^{\Lambda,g}$ has been constructed in $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu_0)$, where ν_0 is the standard Gaussian measure. This is NOT the measure for which we will verify the Osterwalder-Schrader axioms.

4.3 Construction of the Probability Measure under the Ultraviolet and Infrared Cut Offs and Regularization

To construct appropriate probability measures on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ we need results about infinitesimal generators of time inhomogeneous Itô' s diffusions.

Proposition 4.8. *Let $(W_t)_{t\geqslant 0}$ be a M-dimensional standard $\mathbb{P}-$Brownian motion with respect to the filtration $(\mathcal{A}_t)_{t\geqslant 0}$. Let $b : [0, +\infty[\times\mathbf{R}^N \to \mathbf{R}^N$, $\sigma : [0, +\infty[\times\mathbf{R}^N \to \mathbf{R}^{N\times M}$ be Borel measurable and* *locally bounded functions, satisfying*

$$|b(t,x)-b(t,y)|_{\mathbf{R}^N}+|\sigma(t,x)-\sigma(t,y)|_{\mathbf{R}^{N\times M}}\leqslant K\,|x-y|_{\mathbf{R}^N} \tag{157}$$

for a positive constant K, meaning Lipschitz-continuity with respect to x uniform in t. The solution of the SDE

$$\begin{cases} dX_t = b(t,X_t)dt+\sigma(t,X_t)dW_t \\ X_0 = x_0 \in \mathbf{R}^N, \end{cases} \tag{158}$$

is a time inhomogeneous Itô's diffusion, whose infinitesimal generator is given by the PDO with variable coefficients

$$\begin{aligned} L_t &= \frac{1}{2}\sum_{i,i=1}^{N} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_j} \\ \mathcal{D}(L_t) &:= C_0^\infty(\mathbf{R}^N,\mathbf{R}^N)\subset L^2(\mathbf{R}^N,\mathbf{R}^N,d^Nx)\rightarrow L^2(\mathbf{R}^N,\mathbf{R}^N,d^Nx), \end{aligned} \tag{159}$$

where

$$a(t,x):=\sigma(t,x)\sigma(t,x)^\dagger. \tag{160}$$

Conversely, if the operator L_t is elliptic for all $t\geqslant 0$, then for $\sigma(t,x):=a^{\frac{1}{2}}(t,x)$ and $M=N$, there exists an Itô's diffusion as (158) whose transition density $\kappa_t(x_0,x)$ is the heat kernel of L_t, i.e. the solution of

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = L_t u(t,x) \\ u(0,x) = \delta(x-x_0)\in\mathcal{S}'(\mathbf{R}^N,\mathbf{R}^N). \end{cases} \tag{161}$$

It follows that the solution of

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = L_t u(t,x) \\ u(0,x) = f(x)\in C_0^\infty(\mathbf{R}^N,\mathbf{R}^N) \end{cases} \tag{162}$$

is given by

$$u(t,x_0)=\mathbb{E}[f(X_t)|\mathcal{A}_t]=\int_{\mathbf{R}^N} f(x)\kappa_t(x_0,x)d^Nx \tag{163}$$

Proof. See chapters 8.3-8.5 of [CCFI11] and chapters VII.1-VII.2 [RJ99]. □

What is the situation in the infinite dimensional case?

Proposition 4.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be a real separable Hilbert spaces and $\mathcal{L}_{HS}(\mathcal{G},\mathcal{F})$ denote the vector space of all Hilbert-Schmidt operators from $\mathcal{G}$ to $\mathcal{F}$. Let $(W_t)_{t\geqslant 0}$ be a standard $\mathbb{P}-$Wiener process with respect to the filtration $(\mathcal{A}_t)_{t\geqslant 0}$ taking values $\mathcal{G}$ and assume that $b:[0,+\infty[\times\mathcal{F}\rightarrow\mathcal{F}$, $\sigma:[0,+\infty[\times\mathcal{F}\rightarrow$*

$\mathcal{L}_{HS}(\mathcal{G},\mathcal{F})$ be Borel measurable and locally bounded functions, satisfying

$$|b(t,x)-b(t,y)|_{\mathcal{F}}+|\sigma(t,x)-\sigma(t,y)|_{\mathcal{L}_{HS}(\mathcal{G},\mathcal{F})}\leqslant K\,|x-y|_{\mathcal{F}} \tag{164}$$

for a positive constant K, meaning Lipschitz-continuity with respect to x uniform in t. The (strong) solution of the SDE

$$\begin{cases} dX_t = b(t,X_t)dt+\sigma(t,X_t)dW_t \\ X_0 = x_0\in\mathcal{F} \end{cases} \tag{165}$$

is a time inhomogeneous Itô's diffusion. In particular, it satifies the Markov property and thus is a Markov process.

Proof. It is a special case of Theorem 7.4 of [DZ92] for identity covariance of the Wiener process and vanishing linear operator in the drift. The Markov property follows from Theorem 9.8 of [DZ92]. □

We can now proceed with the construction of a probability measure on $\mathcal{S}'_{\perp}(\mathbf{R}^4,\mathbf{C}^{K\times 3})$. Inspired by the treatment of the quantum field associated to a particle in a potential as depicted in chapter 3 of [GJ87], we adapt the ideas therein to the Yang-Mills fields with a cutoff. Using the Feynman-Kač formula, we construct probability measures on $\mathcal{E}$ and $\mathcal{H}$ satisfying those Osterwalder-Schrader axioms implying the reconstruction theorem of quantum mechanics, and thus the selfadjointness and non-negativity of the cutoff Hamilton operator. However, due to the presence of a non-local term in $H_{II}^{\lambda;g}$ created by modified green function G, it is not possible, unless $g=0$ to realize a fibrewise construction over every $x\in\mathbf{R}^3$, and then integrate over x.

If we exclude for the moment the part of the Hamiltonian containing the potential, from Proposition 4.1 formula (104) we can write

$$\begin{split} H_I^{\Lambda}+H_{II}^{\Lambda,g} = \int_{\mathbf{R}^3} d^3x\,\Bigg\{-\frac{1}{2}\left[\frac{\delta}{\delta A_i^a(\varphi_t^{\Lambda}(\cdot-x))}\right]^2+ \\ -\frac{g^2}{2}\left[\int_{\mathbf{R}^3} d^3y\,\partial_i G^{a,b}(\mathbf{A}(\varphi_t^{\Lambda}(\cdot-\cdot));x,y)\varepsilon^{b,c,d}A_k^d(\varphi_t^{\Lambda}(\cdot-y))\frac{\delta}{\delta A_k^c(\varphi_t^{\Lambda}(\cdot-y))}\right]^2\Bigg\}. \end{split} \tag{166}$$

For any probability $\mathbb{P}$, every time $t\in\mathbf{R}$ (seen as parameter) let us now consider the operator h_0^g on the Hilbert space $L^2(L^2_{\perp}(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\mathbb{P})$ defined as

$$\begin{split} &H_0^g := \int_{\mathbf{R}^3} d^3x\,h_0^g(x),\ \text{where} \\ &h_0^g(x) := \left\{-\frac{1}{2}\left[\frac{\delta}{\delta A_i^a(t,x)}\right]^2-\frac{g^2}{2}\left[\int_{\mathbf{R}^3} d^3y\,\partial_i G^{a,b}(\mathbf{A}(t,\cdot);x,y)\varepsilon^{b,c,d}A_k^d(t,y)\frac{\delta}{\delta A_k^c(t,y)}\right]^2\right\}. \end{split} \tag{167}$$

Since

$$\left[\frac{\delta}{\delta A_i^a(t,x)}\right]^2 = -\lim_{\Xi\to+\infty}\int_{\mathbf{R}^3\times\mathbf{R}^3} d^3y\, d^3\bar{y}\, \psi^\Xi(y-x)\psi^\Xi(\bar{y}-x)\delta_d^{\bar{d}}\delta_k^{\bar{k}}\frac{\delta}{\delta A_k^d(t,y)}\frac{\delta}{\delta A_{\bar{k}}^{\bar{d}}(t,\bar{y})}, \tag{168}$$

where $(\psi^\Xi)_{\Xi\geqslant 0}\subset\mathcal{S}(\mathbf{R}^3)$ is a delta sequence, i.e. $\mathcal{S}'-\lim_{\Xi\to+\infty}\psi^\Xi=\delta\in\mathcal{S}'(\mathbf{R}^3)$, for which $\psi^\Xi>0$ for all Ξ, and the limit is pointwise on the domain of definition of the operator on the l. h. s. of (168), we can express the operator $h_0^g(x)$ as

$$\begin{aligned} h_0^g(x) = -\lim_{\Xi\to+\infty}\Bigg[&\int_{\mathbf{R}^3} d^3y\, b_k^d(\Xi,x,y,g;\mathbf{A}(t,\cdot))\frac{\delta}{\delta A_k^d(t,y)}+ \\ &+\int_{\mathbf{R}^3\times\mathbf{R}^3} d^3y\, d^3\bar{y}\; a_{k,\bar{k}}^{d,\bar{d}}(\Xi,x,y,\bar{y},g;\mathbf{A}(t,\cdot))\frac{\delta}{\delta A_k^d(t,y)}\frac{\delta}{\delta A_{\bar{k}}^{\bar{d}}(t,\bar{y})}\Bigg], \end{aligned} \tag{169}$$

for appropriate matrix $a(\Xi,x,y,\bar{y},g;\mathbf{A}(t,\cdot))$ and vector $b(\Xi,x,y,g;\mathbf{A}(t,\cdot))$ valued coefficient functions. The horizontal vector $b=[b_k^d]$ has entries ordered by the bi-index (d,k). The quadratic matrix $a=[a_{k,\bar{k}}^{d,\bar{d}}]$ has row index (d,k) and column index $(\bar{d},\bar{k})$. Note that we have written the matrices $\mathbf{A}(t,y)$ and $\mathbf{A}(t,\bar{y})$ in their equivalent column vector forms. The limit in (169) holds pointwise on the domain of definition of h_0^g. Furthermore, equation (169) becomes

$$h_0^g(x) = -\lim_{\Xi\to+\infty}\left\langle b(\Xi,x,g;\mathbf{A}(t,\cdot)),\frac{\delta}{\delta\mathbf{A}(t,\cdot)}\right\rangle_{\mathbf{A}}+\left\langle a(\Xi,x,g;\mathbf{A}(t,\cdot))\frac{\delta}{\delta\mathbf{A}(t,\cdot)},\frac{\delta}{\delta\mathbf{A}(t,\cdot)}\right\rangle_{\mathbf{A}}, \tag{170}$$

where

$$b(\Xi,x,g;\mathbf{A}(t,\cdot)) = y\mapsto[b_k^d(\Xi,x,y,g;\mathbf{A}(t,\cdot))] \tag{171}$$

and

$$\begin{aligned} a(\Xi,x,g;\mathbf{A}(t,\cdot)) &= y\mapsto[a_{k,\bar{k}}^{d,\bar{d}}(\Xi,x,y,g;\mathbf{A}(t,\cdot))] \\ \left(a(\Xi,y,g;\mathbf{A}(t,\cdot)))\frac{\delta}{\delta\mathbf{A}(t,y)}\right)_{\bar{k}}^{\bar{d}} &= \int_{\mathbf{R}^3} d^3\bar{y}\, a_{k,\bar{k}}^{d,\bar{d}}(\Xi,y,\bar{y},g;\mathbf{A}(t,\cdot))\frac{\delta}{\delta A_k^d(t,\bar{y})} \end{aligned} \tag{172}$$

with the notation, for any $\mathbf{B}\in L^2_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})$

$$\langle b_1,b_2\rangle_{\mathbf{B}} := \int_{\mathbf{R}^3} d^3y\; b_1^r(\mathbf{B}(y))b_2^r(\mathbf{B}(y)) \tag{173}$$

for $b_{1,2}\in L^2(\mathbf{C}^{K\times 3},\mathbf{R})$.

Both b and a are functionals of $\mathbf{A}$ and depend on the parameters Ξ. There exists a $g_0\in[0,1[$ (not depending on Λ!), such that, if the coupling constant $g\in[0,g_0[$, then the expression $a(\Xi,x,g;\mathbf{A})$ represents a positive definite operator valued functional of $\mathbf{A}=\mathbf{A}(t,\cdot)$ seen as $\mathbf{C}^{3K}$-valued function in the separable Hilbert space $L^2(\mathbf{R}^3, \mathbf{C}^{3K})$. By Proposition 4.9, we can construct the diffusion $\mathfrak{A}_t = \mathfrak{A}_t(\Xi, x, g)$

$$d\mathfrak{A}_t = b(\Xi, x, g; \mathfrak{A}_t)dt + a^{\frac{1}{2}}(\Xi, x, g; \mathfrak{A}_t)dW_t, \tag{174}$$

where $(W_t)_{t\geqslant 0}$ is the standard Wiener process adapted to the filtration $(\mathcal{A}_t)_{t\geqslant 0}$ in the Hilbert space $L^2(\mathbf{R}^3, \mathbf{C}^{3K})$. Note that $a^{\frac{1}{2}}$ is a Hilbert-Schmidt operator because a is a Hilbert-Schmidt integral operators. Both b and $a^{\frac{1}{2}}$ are Lipschitz-continuous with respect to $\mathbf{A}$, because they are Fréchet differentiable with continuous derivative. The Lipschitz constant does not depend on t, because b and $a^{\frac{1}{2}}$ do not either.

For fixed $x \in \mathbf{R}^3$ the $\mathbf{R}^{3K}$-valued process $\mathfrak{A}_t(\Xi, x, g)$ is a Markov process with stochastic kernel $\kappa_{t,s}(\Xi, x, g)$ such that for all $\mathbf{A}_t \in \mathbf{C}^{3K}$ and all measurable $B \in \mathcal{A}_s$

$$\mathbb{P}[\mathfrak{U}_s(\Xi, g; x) \in B \,|\, \mathfrak{U}_t(\Xi, g, x) = \mathbf{A}_t] = \kappa_{t,s}(\Xi, g, x; \mathbf{A}_t, B). \tag{175}$$

Let $\mathcal{W}_\perp(\mathbf{A}, \mathbf{A}', t, x)$ be the set of continuous paths $\mathbf{A}(s, x)$ in $\mathbf{C}^{3K}$ which take the values $\mathbf{A}(-t/2, x) = \mathbf{A}$ and $\mathbf{A}(+t/2, x) = \mathbf{A}'$ at their endpoints such that their transversal component $\mathbf{A}^{\|}(s, \cdot)$ vanishes for all s. The cylinder sets of $\mathcal{W}_\perp(\mathbf{A}, \mathbf{A}', t, x)$ have the form

$$\begin{aligned} Z_\perp(\mathbf{A}, \mathbf{A}', t, \{I_j\}_j, x) = \big\{\mathbf{A}(s,x) \,\big|\mathbf{A} \in \mathcal{W}_\perp(\mathbf{A}, \mathbf{A}', t, x),\ \mathbf{A}(-t/2, x) = \mathbf{A}, \mathbf{A}(+t/2, x) = \mathbf{A}', \\ \mathbf{A}(t_j, x) \in I_j,\ \text{for all } j = 1, \dots, n\big\}, \end{aligned} \tag{176}$$

where $-t/2 < t_1 < t_2 < \cdots < t_n < t/2$ and I_j are Borel subsets of $\mathbf{C}^{3K}$. On these cylinder sets we can define the measure given by

$$U^{t,x,g,\Xi}_{\mathbf{A},\mathbf{A}'}(Z) := \kappa_{-t/2,t_1}(\Xi, g, x; \mathbf{A}, \cdot) \otimes \kappa_{t_1,t_2}(\Xi, g, x) \otimes \cdots \otimes \kappa_{t_{n-1},t_n}(\Xi, g, x) \otimes \kappa_{t_n,+t/2}(\Xi, g, x; \cdot, \mathbf{A}'), \tag{177}$$

which is countably additive and has a unique extension to the Borel subsets of $\mathcal{W}_\perp(\mathbf{A}, \mathbf{A}', t, x)$, being the tensor product of stochastic kernels.

The proof of Proposition 4.4 contains the proof that the operator $h_0(x)$ on $L^2(\mathbf{C}^{3K}, \mathbf{C})$ is selfadjoint, and from Theorem 4.7, $h^g_0(x)$ has a unique ground state that we denote by $\omega^{x,g}_0 = \omega^{x,g}_0(\mathbf{A}) \in L^2(\mathbf{C}^{K\times 3}, \mathbf{C})$. For any $R > 0$, termed **infrared cutoff**, the expression

$$d\xi^{R,g}_t := \int_{|x|\leqslant R} d^3x \int_{|s|\leqslant \frac{t}{2}} d^1s \int_{\mathbf{R}^{3K}\times\mathbf{R}^{3K}} dU^{s,x,g,R}_{\mathbf{A},\mathbf{A}'}\ \omega^{x,g}_0(\mathbf{A})\, \omega^{x,g}_0(\mathbf{A}') \tag{178}$$

defines a finite measure on $\mathcal{W}_\perp(\mathbf{A}, \mathbf{A}', t, x)$ which pushes forward by means of the inclusion to a finite measure $\xi_t^{g,R}$ on $\mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ first and to $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ next. Note that from (177) to (178) we have chosen $\Xi = R > 0$, which is legitimate, because in our construction they both will tend to infinity. Therefore, inspired by [Ja82], we can introduce the

Definition 4.2 (Infrared/Ultraviolet Cutoff Measure). With

$$
\begin{aligned}
Z_t^{R,g} &:= \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} d\xi_t^{R,g} \\
d\mu_t^{R,g} &:= \frac{1}{Z_t^{R,g,}} d\xi_t^{R,g},
\end{aligned}
\tag{179}
$$

which is a probability measure on on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$, we can define

$$
\begin{aligned}
Z_t^{\Lambda,R,g} &:= \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \left[\exp\left(-\int_{|x|\leqslant R} d^3x \int_{-\frac{t}{2}}^{+\frac{t}{2}} ds\, V^{\Lambda,g}(s,x,\mathbf{A})\right)\right] d\mu_t^{R,g} \\
d\mu_t^{\Lambda,R,g} &:= \frac{1}{Z_t^{\Lambda,R,g}} \left[\exp\left(-\int_{|x|\leqslant R} d^3x \int_{-\frac{t}{2}}^{+\frac{t}{2}} ds\, V^{\Lambda,g}(s,x,\mathbf{A})\right)\right] d\mu_t^{R,g}
\end{aligned}
\tag{180}
$$

as a probability measure on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$.

We remark that $\mu_t^{\Lambda,R,g}$ is not gauge invariant and not translation invariant.

4.4 Infrared Cutoff Removal and Reconstruction of a Selfadjoint Hamiltonian

Definition 4.3 (Ultraviolet Measure). For any measurable $A \subset \mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ let

$$
\begin{aligned}
\mu_t^{\Lambda,g}(A) &:= \limsup_{R\to+\infty} \mu_t^{\Lambda,R,g}(A) \\
\mu^{\Lambda,g}(A) &:= \limsup_{t\to+\infty} \mu_t^{\Lambda,g}(A).
\end{aligned}
\tag{181}
$$

Are $\mu_t^{\Lambda,g}$ and $\mu^{\Lambda,g}$ measures? The answer is yeas and requires several steps. F.i. in [Doo94], chapter IX.10 we can find the proof of

Theorem 4.10 (Vitali-Hahn-Saks). *Let $(X, \mathcal{B})$ be a measurable space and $(\mu_j)_{j\geqslant 0}$ a sequence of probability measures such that $(\mu_j(A))_{j\geqslant 0}$ converges for all measurable $A \in \mathcal{B}$. Then, $\mu(A) := \lim_{j\to+\infty} \mu_j(A)$ defines a probability on $(X, \mathcal{B})$.*

Remark 4.7. In (181) for any A we can always find a sequence $R_j \uparrow +\infty$ as $j \to +\infty$ such that $\mu_{R_j}^{\Lambda,g}(A) \to \mu^{\Lambda,g}(A)$ as $j \to +\infty$. But this sequence can a priori depend on A, so that we cannot apply immediately the Vitali-Hahn-Saks theorem.

Nevertheless, we have

Proposition 4.11. *The expressions $\mu_t^{\Lambda,g}$ and $\mu^{\Lambda,g}$ in (181) define for all $\Lambda \geqslant 0$ big enough, $g \in [0,1[$ small enough and $t > 0$ probability measures on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$.*

Proof. Let us drop the Λ and g to ease the notation. We first prove the claim for the case $V = 0$. We have

$$\mu_t^R(A) = \frac{\xi_t^R(A)}{\xi_t^R(\mathcal{S}')}, \text{ where } \xi_t^R = \int_{\mathcal{S}'} d\xi_t^R \tag{182}$$

By construction (178) for any measurable A the measure $\xi_t^R(A)$ is monotone increasing in R and bounded from above:

$$\xi_t^R(A) \leqslant \xi_t^S(A) \leqslant \xi_t^{+\infty}(A) < +\infty \quad (R \leqslant S). \tag{183}$$

Therefore, for $R \leqslant S$

$$\begin{aligned}
&\frac{\xi_t^R(A)}{\xi_t^{+\infty}(A)} \leqslant \frac{\xi_t^S(A)}{\xi_t^{+\infty}(A)} \\
&\frac{\xi_t^R(A)}{\xi_t^{+\infty}(A)} \underbrace{\frac{\xi_t^R(\mathcal{S}')}{\xi_t^R(\mathcal{S}')}}_{=1} \leqslant \frac{\xi_t^S(A)}{\xi_t^{+\infty}(A)} \underbrace{\frac{\xi_t^R(\mathcal{S}')}{\xi_t^R(\mathcal{S}')}}_{=1} \\
&\frac{\xi_t^R(\mathcal{S}')}{\xi_t^{+\infty}(A)} \mu_t^R(A) \leqslant \frac{\xi_t^S(\mathcal{S}')}{\xi_t^{+\infty}(A)} \mu_t^S(A) \\
&\underbrace{\frac{\xi_t^R(\mathcal{S}')}{\xi_t^S(\mathcal{S}')}}_{\in]0,1]} \leqslant \frac{\mu_t^S(A)}{\mu_t^R(A)}.
\end{aligned} \tag{184}$$

and

$$\mu_t^R(A) \leqslant \mu_t^S(A). \tag{185}$$

We conclude that for any measurable A and for any sequence $R_j \uparrow +\infty$ as $j \to +\infty$, the sequence $(\mu_t^{R_j}(A))_{j \geqslant 0}$ is monotone increasing. By Theorem 4.10 μ_t defines a probability measure on $\mathcal{S}'$. The proof for μ goes analogously, because, by construction, it is the limit of the monotone increasing sequence $(\mu_{t_j})_{j\geqslant 0}$ for $t_j \uparrow +\infty$ as $j \to +\infty$.

We know consider the general case, when V does not vanish. We have

$$\begin{aligned}
&\mu_t^{\Lambda,R,g}(A) = \frac{\xi_t^{\Lambda,R,g}(A)}{\xi_t^{\Lambda,R,g}(\mathcal{S}')}, \text{ where} \\
&\xi_t^{\Lambda,R,g}(A) := \int_A \left[\exp\left(-\int_{|x|\leqslant R} d^3x \int_{-\frac{t}{2}}^{+\frac{t}{2}} ds\, V^{\Lambda,g}(s,x,\mathbf{A})\right)\right] d\mu_t^{\Lambda,R,g}(\mathbf{A}).
\end{aligned} \tag{186}$$

By construction for any measurable A the measure $\xi_t^{\Lambda,R,g}(A)$ is monotone increasing in R and bounded from above:

$$\xi_t^{\Lambda,R,g}(A) \leqslant \xi_t^{\Lambda,S,g}(A) \leqslant \xi_t^{\Lambda,+\infty,g}(A) < +\infty \quad (R \leqslant S). \tag{187}$$

Therefore, for $R \leqslant S$

$$\begin{aligned}
&\frac{\xi_t^{\Lambda,R,g}(A)}{\xi_t^{\Lambda,+\infty,g}(A)} \leqslant \frac{\xi_t^{\Lambda,S,g}(A)}{\xi_t^{\Lambda,+\infty,g}(A)}\\
&\frac{\xi_t^{\Lambda,R,g}(A)}{\xi_t^{\Lambda,+\infty,g}(A)} \underbrace{\frac{\xi_t^{\Lambda,R,g}(\mathcal{S}')}{\xi_t^{\Lambda,R,g}(\mathcal{S}')}}_{=1} \leqslant \frac{\xi_t^{\Lambda,S,g}(A)}{\xi_t^{\Lambda,+\infty,g}(A)} \underbrace{\frac{\xi_t^{\Lambda,R,g}(\mathcal{S}')}{\xi_t^{\Lambda,R,g}(\mathcal{S}')}}_{=1}\\
&\frac{\xi_t^{\Lambda,R,g}(\mathcal{S}')}{\xi_t^{\Lambda,+\infty,g}(A)} \mu_t^{\Lambda,R,g}(A) \leqslant \frac{\xi_t^{\Lambda,S,g}(\mathcal{S}')}{\xi_t^{\Lambda,+\infty,g}(A)} \mu_t^{\Lambda,S,g}(A)\\
&\underbrace{\frac{\xi_t^{\Lambda,R,g}(\mathcal{S}')}{\xi_t^{\Lambda,S,g}(\mathcal{S}')}}_{\in]0,1]} \leqslant \frac{\mu_t^{\Lambda,S,g}(A)}{\mu_t^{\Lambda,R,g}(A)}.
\end{aligned} \tag{188}$$

and

$$\mu_t^{\Lambda,R,g}(A) \leqslant \mu_t^{\Lambda,S,g}(A). \tag{189}$$

We conclude that for any measurable A and for any sequence $R_j \uparrow +\infty$ as $j \to +\infty$, the sequence $(\mu_t^{\Lambda,R_j,g}(A))_{j\geqslant 0}$ is monotone increasing. By Theorem 4.10 $\mu_t^{\Lambda,g}$ defines a probability measure on $\mathcal{S}'$. The proof for $\mu^{\Lambda,g}$ goes analogously, because, by construction, it is the limit of the monotone increasing sequence $(\mu_{t_j}^{\Lambda,g})_{j\geqslant 0}$ for $t_j \uparrow +\infty$ as $j \to +\infty$.
The proof is completed. □

By Fubini's theorem for distributions (cf. [Tr06]), we can write the measures $\mu_t^{\Lambda,g}$ and $\mu^{\Lambda,g}$ on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ as

$$\begin{aligned}
\mu_t^{\Lambda,g}(\mathbf{A}(s,x)) &= \varrho_t^{\Lambda,g}(\mathbf{A}(\cdot,x))\nu^{\Lambda,g}(\mathbf{A}(s,\cdot))\\
\mu^{\Lambda,g}(\mathbf{A}(s,x)) &= \varrho^{\Lambda,g}(\mathbf{A}(\cdot,x))\nu^{\Lambda,g}(\mathbf{A}(s,\cdot)),
\end{aligned} \tag{190}$$

where $\varrho_t^{\Lambda,g}(\mathbf{A}(\cdot,x)) := \mu_t^{\Lambda,g}(\mathbf{A}(\cdot,x))$ and $\varrho^{\Lambda,g}(\mathbf{A}(\cdot,x)) := \mu^{\Lambda,g}(\mathbf{A}(\cdot,x))$ are probability measures on $\mathcal{S}'_\perp(\mathbf{R}^1, \mathbf{C}^{K\times 3})$, and $\nu^{\Lambda,g}(\mathbf{A}(s,\cdot)) := \mu_t^{\Lambda,g}(\mathbf{A}(s,\cdot)) = \mu^{\Lambda,g}(\mathbf{A}(s,\cdot))$ is a probability measure on $\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})$, respectively. Remark that $\nu^{\Lambda,g}$ does not depend on t by construction.

Theorem 4.12 (Feynman-Kač-Nelson Formula). *The operator $H^{\Lambda,g} = H_I + H_{II}^{\Lambda,g} + V^{\Lambda,g}$ has a domain in the Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^{\Lambda,g})$, has fundamental state $\Omega_0^{\Lambda,g} = \Omega_0^{\Lambda,g}(\mathbf{A}(s,x))$, i.e.*

$H^{\Lambda,g}\Omega_0^{\Lambda,g} = 0$, *and satisfies the Feynman-Kač-Nelson formula*

$$\begin{aligned}\left(\Omega_0^{\Lambda,g}, B_1 e^{-(s_2-s_1)H^{\Lambda,g}} B_2 e^{-(s_3-s_2)H^{\Lambda,g}} \cdot \dots \cdot B_N \Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \\ = \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \Pi_{k=1}^N B_k(\mathbf{A}(s_k,\cdot)) d\mu^{\Lambda,g}(\mathbf{A}),\end{aligned} \tag{191}$$

where the scalar product $(\cdot,\cdot)^{\Lambda,g}$ *on* $\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})$ *is defined as*

$$(\Upsilon,\Theta)^{\Lambda,g} := \int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})} \Upsilon(\mathbf{A})\bar{\Theta}(\mathbf{A}) d\nu^{\Lambda,g}(\mathbf{A}), \tag{192}$$

the functionals $(B_k = B_k(\mathbf{A}))_{k=1,\dots,N}$ *are in* $L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}), \mathbf{C}, d\nu^{\Lambda,g})$, *and* $(s_k)_{k=1,\dots,N}$ *is a partition of the interval* $[\tau, T]$ *defined as* $s_k := \tau + k\frac{T-\tau}{N}$ *for* $\tau := -\frac{t}{2}$ *and* $T := +\frac{t}{2}$.

Proof. We show now that the argument in the proof of Theorem 3.4.1 in [GJ87] can be utilized this set up, Recall from Theorem 4.3 that the operator $H^{\Lambda,g}$ has fundamental state $\Omega_0^{\Lambda,g} = \Omega_0^{\Lambda,g}(\mathbf{A}(s,x))$, i.e. $H^{\Lambda,g}\Omega_0^{\Lambda,g} = 0$. The state $\Omega_0^{\Lambda,g}$ depends neither on Λ nor on g. Note that s is seen as parameter. By adapting the proof of Theorem 3.4.1 in [GJ87], we can write

$$\begin{aligned}\left(\Omega_0^{\Lambda,g}, B_1 e^{-(s_2-s_1)H^{\Lambda,g}} B_2 e^{-(s_3-s_2)H^{\Lambda,g}} \cdot \dots \cdot B_N \Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \\ = \lim_{t\to+\infty} \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \Pi_{k=1}^N B_k(\mathbf{A}(s_k,\cdot)) d\mu_t^{\Lambda,g}(\mathbf{A}).\end{aligned} \tag{193}$$

Since $\mu^{\Lambda,g} = \lim_{t\to+\infty} \mu_t^{\Lambda,g}$, we have

$$\begin{aligned}\left(\Omega_0^{\Lambda,g}, B_1 e^{-(s_2-s_1)H^{\Lambda,g}} B_2 e^{-(s_3-s_2)H^{\Lambda,g}} \cdot \dots \cdot B_N \Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \\ = \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \Pi_{k=1}^N B_k(\mathbf{A}(s_k,\cdot)) d\mu^{\Lambda,g}(\mathbf{A}),\end{aligned} \tag{194}$$

with the same assumptions as for (193).

□

Remark 4.8. Theorem 4.12 for the FKN formula on $3K$-dimensional distributions is consistent with Theorem 3.4.1 in [GJ87] for the FKN formula on 1-dimensional distributions.

Theorem 4.13 (Ultraviolet Measure Properties). *There exists a* $g_0 \in [0,1[$ *not dependent on* Λ, *such that the generating functional*

$$S^{\Lambda,g}(f) = \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} e^{\imath\mathbf{A}(f)} d\mu^{\Lambda,g}(\mathbf{A}), \tag{195}$$

for $f \in \mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ satisfies satisfies the Osterwalder-Schadrer axioms (OS0)-(OS4) and hence the Wightman axioms (W1)-(W8). Note that $S^{\Lambda,g}(f)$ and $A(f)$ are $K \times 3$ complex matrices, and that the exponential is meant componentwise.

Proof.
Without loss of generality we can assume that $S^{\Lambda,g}(f)$ and $A(f)$ are complex numbers throughout this proof, because the general proof can be reconstructed by iterating over the components of the complex $K \times 3$ matrices representing them.

We first prove (OS2) and (OS3), which will be immediately needed to apply the reconstruction theorem of quantum mechanics. (OS2): The invariance of the generating functional $S^{\Lambda,g}$ under time translation and time reflection follows directly from the definition of $\mu^{\Lambda,g}$ in (181). More exactly, by (179) the infrared/ultraviolet cutoff measure $\mu_t^{\Lambda,R,g,0}$ is invariant under the space rotations and reflections in O(3), and hence the ultraviolet cutoff measure in invariant under all space-time rotations and reflections, as well as translations.
(OS3): we have to show that the complex matrix $M^{\Lambda,g} := [M^{\Lambda,g}_{i,j}]$, where

$$M^{\Lambda,g}_{i,j} := S^{\Lambda,g}(\theta f_i - f_j), \tag{196}$$

is positive definite for all choices of $(f_i)_{i=1,\dots,n} \subset \mathbf{S}(\mathbf{R}^4, \mathbf{R})$, such that $\mathrm{supp}(f_i) \subset [0, +\infty[\times \mathbf{R}^3$, and $(\theta f)(s,x) := f(-s,x)$ denotes the time reflection.
With the choice of functionals B_k as

$$B_k(\mathbf{A}) := \exp\left(\frac{\imath}{N}\mathbf{A}\left(\theta(f_i)(s_k,\cdot) - f_j(s_k,\cdot)\right)\right), \tag{197}$$

the r.h.s. of (191) leads to

$$\lim_{N\to+\infty}\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \Pi_{k=1}^N B_k(\mathbf{A}(s_k,\cdot))d\mu^{\Lambda,g}(\mathbf{A}) = \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \exp\left(-\imath\mathbf{A}\left(\theta(f_i) - f_j\right)\right)d\mu^{\Lambda,g}(\mathbf{A}), \tag{198}$$

when $\tau \to -\infty$ and $T \to +\infty$. We have approximated the time integration in $\mathbf{A}(f)$ by means of a

Riemann sum for the partition $(s_k)_{k=1,\ldots,N}$. For the l.h.s of (191) we obtain in the limit

$$\begin{aligned} &\lim_{N\to+\infty}\left(\Omega_0^{\Lambda,g}, B_1 e^{-(t_2-t_1)H^{\Lambda,g}} B_2 e^{-(t_3-t_2)H^{\Lambda,g}}\ldots B_N\Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \\ &= \left(\Omega_0^{\Lambda,g}, \exp\left(-\imath\mathbf{A}\left(\theta(f_i)-f_j\right)\right)\Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \\ &= \left(\exp\left(-\imath\mathbf{A}\left(\theta(f_i)\right)\right)\Omega_0^{\Lambda,g}, \exp\left(-\imath\mathbf{A}\left(f_j\right)\right)\Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \\ &= \left(\exp\left(-\imath\mathbf{A}\left(f_i\right)\right)\Omega_0^{\Lambda,g}, \exp\left(-\imath\mathbf{A}\left(f_j\right)\right)\Omega_0^{\Lambda,g}\right)^{\Lambda,g}, \end{aligned} \tag{199}$$

because the supports of f_i and f_j lie in the time positive half space. Putting (191) with (198) and (199) together shows that the matrix $[M_{i,j}^{\Lambda,g}]$ with entries

$$M_{i,j}^{\Lambda,g} = \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\exp\left(-\imath\mathbf{A}\left(\theta(f_i)-f_j\right)\right)d\mu^{\Lambda,g}(\mathbf{A}) = \left(\exp\left(-\imath\mathbf{A}\left(f_i\right)\right)\Omega_0^{\Lambda,g}, \exp\left(-\imath\mathbf{A}\left(f_j\right)\right)\Omega_0^{\Lambda,g}\right)^{\Lambda,g} \tag{200}$$

is positive definite for all $g \in [0, g_0[$, and, by Proposition 3.8 (or Corollary 3.4.4 in [GJ87]), the reflection positivity axiom (OS3) is fulfilled.

□

We can now prove

Theorem 4.14. *There exists a $g_0 \in [0,1[$ not depending on Λ, such that, if the coupling constant $g \in [0, g_0[$, for the probability measure $\mu^{\Lambda,g}$ on $\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})$ the Hamilton operator $H^{\Lambda,g}$ is selfadjoint for the choice $\nu^{\Lambda,g}$. If the coupling constant g vanishes, both measures $\mu^{\Lambda,0}$ and $\nu^{\Lambda,0}$ are Gaussian, otherwise not.*

Proof. We know that all the assumptions of Theorem 3.7 are satisfied, because we have already verified the Osterwalder-Schrader axioms (OS2) ans (OS3). Hence, the time translation operator $T(t)$ satisfies

$$T(t)^{\curlywedge^{\Lambda,g}} = e^{-t\tilde{H}^{\Lambda,g}}, \tag{201}$$

where $\tilde{H}^{\Lambda,g}$ is a selfadjoint operator on the Hilbert space $L^2\left(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}), d\nu^{\Lambda,g}\right)$. Note that the canonical embedding $\curlywedge$ is defined utilizing the measure $\mu^{\Lambda,g}$, which is highlighted by the superscripts Λ and g in the notation $\curlywedge^{\Lambda,g}$. To conclude the proof we have to show that $H^{\Lambda,g} = \tilde{H}^{\Lambda,g}$. A slight reformulation of (191) provides the equality

$$\left(\Omega_0^{\Lambda,g}, \Psi e^{-sH^{\Lambda,g}}\Phi\Omega_0^{\Lambda,g}\right)^{\Lambda,g} = \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}(0,\cdot)\right)\Phi\left(\mathbf{A}(s,\cdot)\right)d\mu^{\Lambda,g}(\mathbf{A}), \tag{202}$$

where:

- the distribution $\mathbf{A}(s,\cdot)\in\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})$ depends on the parameter $s\in\mathbf{R}^1$,
- the functionals $\Psi,\Phi\in\mathcal{D}(H^{\Lambda,g})\subset L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),\mathbf{C},d\nu^{\Lambda,g})$.

We now compute the r.h.s of equation (202) and obtain

$$\begin{aligned}
&\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}(0,\cdot)\right)\Phi\left(\mathbf{A}(s,\cdot)\right)d\mu^{\Lambda,g}(\mathbf{A})=\\
&=\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}(0,\cdot)\right)\Phi\left(\mathbf{A}(s,\cdot)\right)d(\varrho^{\Lambda,g}\otimes\nu^{\Lambda,g})(\mathbf{A})=\\
&=\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}(0,\cdot)\right)\Phi\left(\mathbf{A}(s,\cdot)\right)d\nu^{\Lambda,g}(\mathbf{A}(t,\cdot))d\varrho^{\Lambda,g}(\mathbf{A}(\cdot,x))=\\
&=\int_{\mathcal{S}'_\perp(\mathbf{R}^1,\mathbf{C}^{K\times 3})}\left[\int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}(0,\cdot)\right)\Phi\left(e^{-s\tilde{H}^{\Lambda,g}}\mathbf{A}(0,\cdot)\right)d\nu^{\Lambda,g}(\mathbf{A}(t,\cdot))\right]d\varrho^{\Lambda,g}(\mathbf{A}(\cdot,x))=\\
&=\left(\int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}(0,\cdot)\right)\Phi\left(e^{-s\tilde{H}^{\Lambda,g}}\mathbf{A}(0,\cdot)\right)d\nu^{\Lambda,g}(\mathbf{A})\right)\underbrace{\left(\int_{\mathcal{S}'_\perp(\mathbf{R}^1,\mathbf{C}^{K\times 3})}d\varrho^{\Lambda,g}(\mathbf{A})\right)}_{=1}=\\
&=\int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}\right)(e^{-s\tilde{H}^{\Lambda,g}})\Phi\left(\mathbf{A}\right)d\nu^{\Lambda,g}(\mathbf{A}),
\end{aligned}\tag{203}$$

where we have utilized Fubini's theorem for distributions (cf. [Tr06]), the fact that the integrand does not depend on the time t, and that $\int d\varrho^{\Lambda,g}=1$, being $\varrho^{\Lambda,g}$ a probability measure.

Therefore,

$$\left(\Omega_0^{\Lambda,g},\bar{\Psi}e^{-sH^{\Lambda,g}}\Phi\Omega_0^{\Lambda,g}\right)^{\Lambda,g}=\int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}\right)\overline{e^{-s\tilde{H}^{\Lambda,g}}\Phi\left(\mathbf{A}\right)}d\nu^{\Lambda,g}(\mathbf{A})\tag{204}$$

We now take the derivative $-\left.\frac{d}{ds}\right|_{s:=0}$ on both sides of (204) and obtain

$$\left(\Omega_0^{\Lambda,g},\bar{\Psi}H^{\Lambda,g}\Phi\Omega_0^{\Lambda,g}\right)^{\Lambda,g}=\int_{\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3})}\Psi\left(\mathbf{A}\right)\overline{\tilde{H}^{\Lambda,g}\Phi\left(\mathbf{A}\right)}d\nu^{\Lambda,g}(\mathbf{A}).\tag{205}$$

Equation (205) holds for all $\Psi,\Phi\in\mathcal{D}(H^{\Lambda,g})$, and we conclude that $H^{\Lambda,g}=\tilde{H}^{\Lambda,g}$. The proof is completed.

□

We can now verify the remaining Osterwalder-Schrader axioms.

Proof of Theorem 4.13, continuation.

(OS0): Let us consider a finite set of test functions $f_j\in\mathcal{S}_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})$, $j=1,\dots,n$ and complex numbers $z:=(z_1,z_2,\dots,z_n)\in\mathbf{C}^n$, the complex partial derivative of $S^{\Lambda,g}\left(\sum_{j=1}^n z_jf_j\right)$ with respect to

z_i reads

$$\frac{\partial}{\partial z_i}\left[S^{\Lambda,g}\left(\sum_{j=1}^n z_j f_j\right)\right]=\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\imath\mathbf{A}(f_i)e^{\imath\mathbf{A}\left(\sum_{j=1}^n z_j f_j\right)}d\mu^{\Lambda,g}(\mathbf{A}). \tag{206}$$

By the Cauchy-Schwarz inequality

$$\left|\frac{\partial}{\partial z_i}\left[S^{\Lambda,g}\left(\sum_{j=1}^n z_j f_j\right)\right]\right|^2\leqslant\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|\mathbf{A}(f_i)|^2 d\mu^{\Lambda,g}(\mathbf{A})\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|e^{\imath\mathbf{A}\left(\sum_{j=1}^n z_j f_j\right)}|^2 d\mu^{\Lambda,g}(\mathbf{A}) \tag{207}$$

Since $\lim_{g\to 0^+}\mu^{\Lambda,g}=\mu^{\Lambda,0}$, the positive constant

$$K^{\Lambda,g}=\left\|\frac{d\mu^{\Lambda,g}}{d\mu^{\Lambda,0}}\right\|_{L^\infty(\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3}))} \tag{208}$$

is bounded, and for all g small enough

$$\begin{aligned}\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|\mathbf{A}(f)|^2 d\mu^{\Lambda,g}(\mathbf{A})&\leqslant K^{\Lambda,g}\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|\mathbf{A}(f)|^2 d\mu^{\Lambda,0}(\mathbf{A})\\ \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|e^{\imath\mathbf{A}(f)}|^2 d\mu^{\Lambda,g}(\mathbf{A})&\leqslant K^{\Lambda,g}\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|e^{\imath\mathbf{A}(f)}|^2 d\mu^{\Lambda,0}(\mathbf{A}).\end{aligned} \tag{209}$$

for all $f\in\mathcal{S}_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})$. Being $\mu^{\Lambda,0}$ a Gaussian measure, we obtain

$$\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}e^{\imath\mathbf{A}(f)}d\mu^{\Lambda,0}(\mathbf{A})=e^{-\frac{1}{2}(f,C^\Lambda f)_{L^2(\mathbf{R}^4,\mathbf{C}^{K\times 3})}}, \tag{210}$$

where

$$C^\Lambda=\frac{1}{2}\bigoplus_{\substack{a=1,\dots,K\\ i=1,2,3}}\Bigg(\mathbb{1}_{\mathbf{R}^4}-\Delta_{\mathbf{R}^4}+\underbrace{2V^{\Lambda,0}-\mathbb{1}_{\mathbf{R}^4}}_{\geqslant 0}\Bigg)^{-1}\leqslant\frac{1}{2}\bigoplus_{\substack{a=1,\dots,K\\ i=1,2,3}}\left(\mathbb{1}_{\mathbf{R}^4}-\Delta_{\mathbf{R}^4}\right)^{-1}=:C^\varnothing, \tag{211}$$

because the operator $V^{\Lambda,0}$ is non-negative. By developing the exponential function in both sides of (210) and equating the quadratic term in f, we obtain

$$\begin{aligned}\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|\mathbf{A}(f)|^2 d\mu^{\Lambda,0}(\mathbf{A})&=\left(f,C^\Lambda f\right)_{L^2(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\leqslant\left(f,C^\varnothing f\right)_{L^2(\mathbf{R}^4,\mathbf{C}^{K\times 3})}=\\ &=\|f\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})},\end{aligned} \tag{212}$$

where H^{-1} is the Sobolev space with "differentiability" -1. By comparing all powers of f, we obtain

$$\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}|e^{\imath\mathbf{A}(f)}|^2 d\mu^{\Lambda,0}(\mathbf{A})\leqslant\exp\left(\|f\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\right), \tag{213}$$

Inserting (212), (213) and (209) into (207) leads to

$$\left|\frac{\partial}{\partial z_i}\left[S^{\Lambda,g}\left(\sum_{j=1}^{n} z_j f_j\right)\right]\right|^2 \leqslant \left(K^{\Lambda,g}\right)^2 \|f_i\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \exp\left(\|\sum_{j=1}^{n} z_j f_j\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\right) < +\infty, \tag{214}$$

because all f_j's are in the Schwartz space and a fortiori in the Sobolev space H^{-1}. The analyticity for all $z \in \mathbf{C}^n$ is therefore proved.

(OS1): Since $S^{\Lambda,g}(zf)$ is analytic for all $f \in \mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ and all $z \in \mathbf{C}$ the mean value theorem of differentiation implies that it exists a $z' \in \mathbf{C}$, $|z'| \leqslant |z|$ such that

$$\left|S^{\Lambda,g}(zf) - \underbrace{S^{\Lambda,g}(0)}_{=1}\right| \leqslant \left|\frac{d}{dz_i} S^{\Lambda,g}(z'f)\right| |z|. \tag{215}$$

Therefore, utilizing (214) we obtain

$$\left|S^{\Lambda,g}(zf)\right| \leqslant 1 + |z| K^{\Lambda,g} \|f\|_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \exp\left(\frac{|z|^2}{2}\|f\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\right) \tag{216}$$

We choose $z := 1$

$$\left|S^{\Lambda,g}(f)\right| \leqslant 1 + K^{\Lambda,g} \|f\|_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \exp\left(\frac{1}{2}\|f\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\right), \tag{217}$$

and study the function of the variable $\alpha \in [0, +\infty[$

$$\kappa(\alpha) := \frac{1 + K^{\Lambda,g}\alpha \exp\left(\frac{1}{2}\alpha^2\right)}{\exp(L\alpha^2)}, \tag{218}$$

for a given constant $L > 0$. If $L > \frac{1}{2}$, then

$$\lim_{\alpha\to+\infty} \kappa(\alpha) = 0 \text{ and } \lim_{\alpha\to 0^+} \kappa(\alpha) = 1. \tag{219}$$

Therefore, for any given $K^{\Lambda,g} > 0$ there exists a $L^{\Lambda,g} > 0$ such that for all $\alpha \in [0, +\infty[$

$$1 + K^{\Lambda,g}\alpha \exp\left(\frac{1}{2}\alpha^2\right) \leqslant \exp(L^{\Lambda,g}\alpha^2), \tag{220}$$

which, utilized with $\alpha := \|f\|_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}$ and inserted into (217), leads to

$$\begin{aligned}\left|S^{\Lambda,g}(f)\right| &\leqslant \exp(L^{\Lambda,g}\|f\|^2_{H^{-1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}) \leqslant \exp(L^{\Lambda,g}\|f\|^2_{H^{0}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}) \leqslant \\ &\leqslant \exp(L^{\Lambda,g}(\|f\|_{L^{1}(\mathbf{R}^4,\mathbf{C}^{K\times 3})} + \|f\|^2_{L^{2}(\mathbf{R}^4,\mathbf{C}^{K\times 3})})),\end{aligned} \tag{221}$$

because $\|f\|_{H^{-1}} \leqslant \|f\|_{H^0}$ for all f, and $H^0(\mathbf{R}^4, \mathbf{C}^{K\times 3}) = L^2(\mathbf{R}^4, \mathbf{C}^{K\times 3})$.

(OS4): We have to check that the Euclidean time translation subgroup, which by Theorem 4.14 reads $\{T(t)\}_{t\geqslant 0}$, where $T(t)^{\wedge^{\Lambda,g}} = e^{-tH^{\Lambda,g}}$, acts ergodically on the measure space $(\mathcal{S}'_{\perp}(\mathbf{R}^4, \mathbf{C}^{K\times 3}), d\mu^{\Lambda,g})$, or, equivalently (see [GJ87], Formula 19.7.1), that it satisfies the **cluster property**, i.e.

$$\begin{aligned}\lim_{t\mapsto+\infty}\frac{1}{t}\int_0^t\left[\int_{\mathcal{S}'_{\perp}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Phi(\mathbf{A})T(s)\Psi(\mathbf{A})d\mu^{\Lambda,g}(\mathbf{A})\right]ds = \\ = \int_{\mathcal{S}'_{\perp}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Phi(\mathbf{A})d\mu^{\Lambda,g}(\mathbf{A})\cdot\int_{\mathcal{S}'_{\perp}(\mathbf{R}^4,\mathbf{C}^{K\times 3})}\Psi(\mathbf{A})d\mu^{\Lambda,g}(\mathbf{A}),\end{aligned} \tag{222}$$

for all $\Phi, \Psi \in L^1(\mathcal{S}'_{\perp}(\mathbf{R}^4, \mathbf{C}^{K\times 3}), d\mu^{\Lambda,g})$. In the proof of Theorem 19.7.1 in [GJ87] the cluster property (222) is shown to be equivalent with the uniqueness of the ground state. Hence, we have to show that $\Omega_0^{\Lambda,g}$ is an eigenvector of $H^{\Lambda,g}$ with multiplicity 1, which follows from Theorem 3.3.2 and 3.3.3 in [GJ87], because $A^{\Lambda,g} := e^{-tH^{\Lambda,g}}$ has a strictly positive kernel, being $H^{\Lambda,g} = H_I^{\Lambda} + H_{II}^{\Lambda,g} + V^{\Lambda,g}$ self-adjoint and $V^{\Lambda,g}$ bounded from below by 0 by construction. □

4.5 Ultraviolet Cutoff Removal without Renormalization

Now we remove the regularization given by the ultraviolet cutoff by letting $\Lambda \to +\infty$ and making sure that the properties (OS2) and (OS3), necessary for the reconstruction theorem of quantum mechanics, are maintained. Actually, all Osterwalder-Schrader axioms will be preserved.

Definition 4.4 (**4D-YM-Measure**)**.** For any measurable $A \subset \mathcal{S}'_{\perp}(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ let

$$\mu^g(A) := \limsup_{\Lambda\to+\infty}\mu^{\Lambda,g}(A). \tag{223}$$

Is μ^g a measure? The answer is yes and requires several steps.

Definition 4.5 (**Tightness**)**.** Let $(X, \mathcal{B}(X))$ be a measurable topological space. The collection of probability measures $\mathcal{M}$ over X is **tight** if and only if for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$ for all $\mu \in \mathcal{M}$.

In f.i. [Bo06] we can find the proof of

Theorem 4.15 (**Prokhorov**)**.** *Let X be a separable metric space, and $\mathcal{P}(X)$ the collection of all probability measures defined on X with its Borel σ-algebra. A family of probability measures $\mathcal{M} \subset \mathcal{P}(X)$ is tight if and only if $\overline{\mathcal{M}}$ is weakly sequentially compact. i.e. if every sequence $(\mu_j)_{j\geqslant 0} \subset \mathcal{M}$ contains a* *subsequence* $(\mu_{j_k})_{k\geqslant 0}$ *weakly converging to a* $\mu \in \overline{\mathcal{M}}$*, meaning by this*

$$\lim_{k\to+\infty} \mathbb{E}^{\mu_{j_k}}[f] = \mathbb{E}^{\mu}[f] \tag{224}$$

for all bounded continuous functions f *on* X*.*

In spite of the fact that $\mathcal{S}'$ is not metrizable, Theorem I.6.5 in [Fe67] implies

Theorem 4.16 (**Fernique**)**.** *Prokhorov's theorem holds true for* $\mathcal{S}'$*.*

Proposition 4.17. *Let* $(X, \mathcal{A}, P)$ *be a probability space and let* $F_\Lambda : X \to [0, +\infty[$ *be a family of uniformly bounded measurable functions indexed by the parameter* $\Lambda \geqslant 0$*. Then, the collection of probability measures* $(P_\Lambda)_{\Lambda\geqslant 0}$ *over* X *defined for any measurable* $A \in \mathcal{A}$

$$P_\Lambda(A) := \frac{\int_A F_\Lambda(x)dP}{\int_X F_\Lambda(x)dP}. \tag{225}$$

is tight.

Proof. We need to show that, for any $\varepsilon > 0$, there exists a compact set $K = K_\varepsilon \subset X$ such that $P_\Lambda(K) > 1 - \varepsilon$ for sufficiently large values of Λ. First, note that since $F_\Lambda(x)$ is uniformly bounded, there exists a constant $M > 0$ such that $F_\Lambda(x) \leqslant M$ for all $x \in X$ and all $\Lambda \geqslant 0$. Then, for any measurable set A in $\mathcal{A}$, we have

$$P_\Lambda(A) = \frac{\int_A F_\Lambda dP}{\int_X F_\Lambda dP} \leqslant \frac{MP(A)}{\int_X F_\Lambda dP}. \tag{226}$$

For any $\varepsilon > 0$, we can choose a compact set $K = K_\varepsilon$ such that

$$P(X\backslash K) \leqslant \varepsilon. \tag{227}$$

Hence we have

$$P_\Lambda(K) = 1 - P_\Lambda(X\backslash K) \geqslant 1 - \frac{MP(X\backslash K)}{\int_X F_\Lambda dP} = 1 - \frac{MP(X\backslash K)}{\Lambda \int_X \frac{F_\Lambda}{\Lambda} dP}. \tag{228}$$

Since $F_\Lambda(x) \leqslant M$ for all $x \in X$ and all $\Lambda \geqslant 0$, it follows by Lebesgue's dominated convergence

$$\lim_{\Lambda\to+\infty} \int_X \frac{F_\Lambda}{\Lambda} dP = 0, \tag{229}$$

and thus

$$\int_X \frac{F_\Lambda}{\Lambda} dP \in [0, 1[\text{ for } \Lambda \text{ big enough.} \tag{230}$$

Inserting (230) into (228) leads to

$$P_\Lambda(K) \geqslant 1 - \frac{M\varepsilon}{\Lambda}. \tag{231}$$

Therefore, the family $(P_\Lambda)_{\Lambda\geqslant 0}$ is tight since, for any $\varepsilon > 0$, we can find a compact set $K = K_\epsilon \subset X$ such that $P_\Lambda(K) > 1 - \varepsilon$ for $\Lambda \geqslant \max(M, \lambda_0)$. The proof is finished.

□

Putting everything together we obtain

Proposition 4.18. *The expression μ^g in (223) defines for all $g \in [0, g_0[$ for a g_0 small enough a probability measures on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$.*

Proof. By (179) and (181) we have

$$\begin{aligned} Z_t^{V=0,R,g} &= \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} d\xi_t^{R,g} \\ d\mu^{V=0,g} &= \lim_{\substack{R\to+\infty \\ t\to+\infty}} \frac{1}{Z_t^{V=0,R,g}} d\xi_t^{R,g}, \end{aligned} \tag{232}$$

and by (180) and (181) we have

$$\begin{aligned} Z_t^{\Lambda,R,g} &= \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \left[\exp\left(-\int_{|x|\leqslant R} d^3x \int_{-\frac{t}{2}}^{+\frac{t}{2}} ds\, V^{\Lambda,g}(s,x,\mathbf{A})\right)\right] d\mu_t^{R,g} \\ d\mu^{\Lambda,g} &= \underbrace{\lim_{\substack{R\to+\infty \\ t\to+\infty}} \left[\frac{1}{Z_t^{\Lambda,R,g}} \left[\exp\left(-\int_{|x|\leqslant R} d^3x \int_{-\frac{t}{2}}^{+\frac{t}{2}} ds\, V^{\Lambda,g}(s,x,\mathbf{A})\right)\right]\right]}_{=:F_\Lambda(\mathbf{A})} d\mu^{V=0,g}. \end{aligned} \tag{233}$$

Now, in (233) we have a fraction whose numerator and denominator both depend on Λ. The denominator is $Z_t^{\Lambda,R,g}$. Numerator and denominator depend on t and R, too, but, as we saw in the proof of Proposition 4.11, the limit of the quotient for $R, t \to +\infty$ is well-defined and finite. By Definition 4.1 and (104) $V^{\Lambda,g}(t,x,\mathbf{A})$ is a fourth degree polynomial in Λ. By (144) the potential $V^{\Lambda,g}(t,x,\mathbf{A}) \geqslant 0$ for Λ big enough. Therefore,

- $\frac{\text{numerator}}{\Lambda^4}$ converges to a strictly positive constant (depending on $\mathbf{A}$) for $\Lambda \to +\infty$,
- $\frac{\text{denominator}}{\Lambda^4}$ converges to strictly positive constant for $\Lambda \to +\infty$,
- the quotient $\frac{\text{numerator}}{\text{denominator}}$ converges to strictly positive constant (depending on $\mathbf{A}$) for $\Lambda \to +\infty$

Hence $F_\Lambda(\mathbf{A})$ is bounded in Λ. Since the $V^{\Lambda,g}(t,x,\mathbf{A}) \geqslant 0$ for Λ big enough, the numerator is smaller than 1 for all $\mathbf{A} \in \mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ and for Λ big enough. Therefore $F_\Lambda(\mathbf{A})$ is bounded in both Λ and

$\mathbf{A}$, and the family $(F_\Lambda)_{\Lambda\geqslant 0}$ is uniformly bounded on $\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})$. By Proposition 4.17 the family of probabilities $(\mu^{\Lambda,g})_{\Lambda\geqslant 0}$ is tight. By Theorem 4.16 the family $(\mu^{\Lambda,g})_{\Lambda\geqslant 0}$ has a weakly convergent subsequence to a probability, which by (223) is μ^g. The proof is finished.

□

We studied the original renormalization and ultraviolet cutoff removal techniques for the φ_3^4 introduced by Glimm-Jaffe ([GJ73]) for finite volume and extended by Feldman-Osterwalder ([FO76]) and Magnen-Sénéor ([MS76]) for infinite volume and small positive values of the coupling constant using small cluster expansion methods. Finally the work of Seiler-Simon ([SeSi76]) allowed to extend the existence result any positive value of the coupling constant (this is claimed in [GJ87]) even though we could not find a clear statement in Seiler-Simon's paper). One notices that the model parameter have been made ultraviolet cutoff level dependent in order to produce counter terms which eliminate divergences in the integrals; by mean of an a-priori estimate on the Schwinger functions the Osterwalder-Schrader axioms are then inferred. Similarly, a renormalization involving the bare coupling constant is needed for Φ_4^4 in [GK85] and [FMRS87]. With our definition of the ultraviolet measure the situation is different, because we have no divergences to compensate, and the bare coupling constant g must not be made dependent on the ultraviolet cutoff level Λ. Later we will see that the running of the coupling constant guarantees asymptotic freedom of the Yang-Mills model.

In Magnen-Rivasseau-Sénéor's construction of a Yang-Mills measure in four dimensions ([MRS93]) only the ultraviolet but not the infrared cutoff is removed, while maintaining gauge invariance. In their construction the ultraviolet cutoff is implemented as a multiplication of the fields on the momentum space with the regularization of the characteristic function of a domain converging towards $\mathbf{R}^3$ as the cutoff parameter tends to $+\infty$. That way they create a divergence which they compensate by renormalization and running the bare coupling constant to obtain asymptotic freedom as in Chapter III.5 of ([Riv91]). An important difference to our model is that in the present construction the ultraviolet cutoff is implemented as an application of the fields seen as distributions on the position space to a delta sequence in $\mathbf{R}^3$ for fixed time t with respect to the cutoff parameter. This way the problem of the non-existence of products of tempered distributions is circumvented, and no divergenges appear in the limit for the cutoff parameter tending to $+\infty$. Moreover, once the infrared cutoff in the Magnen-Rivasseau-Sénéor model is removed, the mass gap is killed, as we will see in Subsection 4.8, because the limit of the renormalized coupling constant vanishes.

Corollary 4.19 (4D-YM-Measure Properties). *There exists a $g_0\in[0,1[$, such that for all $g\in[0,g_0[$ the generating functional*

$$S^g(f)=\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})}e^{\imath\mathbf{A}(f)}d\mu^g(\mathbf{A}),\tag{234}$$

for $f \in \mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ satisfies the Osterwalder-Schadrer axioms (OS0)-(OS4) and hence the Wightman axioms (W1)-(W8). Note that $S^{\Lambda,g}(f)$ and $A(f)$ are $K \times 3$ complex matrices, and that the exponential is meant componentwise.

Proof.

Without loss of generality we can assume that $S^{\Lambda,g}(f)$ and $A(f)$ are complex numbers throughout this proof, because the general proof can be reconstructed by iterating over the components of the complex $K \times 3$ matrices representing them.

First we prove that

$$S^{\Lambda,g}(f) \to S^g(f) \quad (\Lambda \to +\infty) \tag{235}$$

locally uniformly in $f \in \mathcal{S}_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$. We have

$$\begin{aligned} S^g(f) &= \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} e^{\imath \mathbf{A}(f)} d\mu^g(\mathbf{A}) \\ S^{\Lambda,g}(f) &= \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} e^{\imath \mathbf{A}(f)} \frac{d\mu^{\Lambda,g}}{d\mu^g} d\mu^g(\mathbf{A}), \end{aligned} \tag{236}$$

and hence

$$\begin{aligned} &\left|S^{\Lambda,g}(f) - S^g(f)\right|^2 = \\ &= \left|\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} e^{\imath \mathbf{A}(f)} \left(1 - \frac{d\mu^{\Lambda,g}}{d\mu^g}\right) d\mu^g(\mathbf{A})\right|^2 \leqslant \\ &\leqslant \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \left|e^{\imath \mathbf{A}(f)}\right|^2 d\mu^g(\mathbf{A}) \int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \left|1 - \frac{d\mu^{\Lambda,g}}{d\mu^g}\right|^2 d\mu^g(\mathbf{A}) \leqslant \\ &\leqslant \exp(2L^{\Lambda,g}(\|f\|_{L^1(\mathbf{R}^4,\mathbf{C}^{K\times 3})} + \|f\|^2_{L^2(\mathbf{R}^4,\mathbf{C}^{K\times 3})})) \underbrace{\int_{\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3})} \left|1 - \frac{d\mu^{\Lambda,g}}{d\mu^g}\right|^2 d\mu^g(\mathbf{A})}_{\to 0 \quad (\Lambda \to +\infty)} \end{aligned} \tag{237}$$

As we saw in (220) the constant $L^{\Lambda,g}$ is bounded in Λ if and only if the positive constant

$$K^{\Lambda,g} = \left\|\frac{d\mu^{\Lambda,g}}{d\mu^{\Lambda,0}}\right\|_{L^\infty(\mathcal{S}'_\perp(\mathbf{R}^4,\mathbf{C}^{K\times 3}))} \tag{238}$$

is bounded in Λ. By Definition 4.1 and (104) we have

$$\frac{V^{\Lambda,g}}{\Lambda^4} = O_g(1) \quad (\Lambda \to +\infty). \tag{239}$$

By Definitions 4.2 and 4.3 it follows that

$$\mu^{\Lambda,g} = O_g(1) \quad (\Lambda \to +\infty) \tag{240}$$

and hence

$$K^{\Lambda,g} = O_g(1) \quad (\Lambda \to +\infty) \tag{241}$$

as we needed for inequality (237) to prove the f-locally uniform convergence of $S^{\Lambda,g}(f)$ towards $S^g(f)$ for $\Lambda \to +\infty$.

The Osterwalder-Schadrer axioms (OS0)-(OS4) for μ^g follow now from the proof of Theorem 4.13 because we have proved that the constants occurring in the inequalities for (OS0) (214) and (OS1) (221) are bounded in Λ; the invariance (OS2) holds true for μ^g because it does for $\mu^{A,g}$ for all $\Lambda \geqslant 0$; the reflexion positivity (OS3) holds true because $[M^{\Lambda,g}_{i,j}]$ defined in (200) is positive definite for all $\Lambda \geqslant 0$; the ergodicity property (OS4) is fulfilled, because the cluster property (222) holds true for all $\Lambda \geqslant 0$, and, therefore in the limit for $\Lambda \to +\infty$. The proof is completed.

□

By Fubini's theorem for distributions (cf. [Tr06]), we can write the measure μ^g on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$ as

$$\mu^g(\mathbf{A}(s,x)) = \varrho^g(\mathbf{A}(\cdot,x))\nu^g(\mathbf{A}(s,\cdot)), \tag{242}$$

where $\varrho^g_t(\mathbf{A}(\cdot,x)) := \mu^{\Lambda,g}_t(\mathbf{A}(\cdot,x))$ and $\nu^g(\mathbf{A}(s,\cdot)) := \mu^g(\mathbf{A}(s,\cdot))$ are probability measures on $\mathcal{S}'_\perp(\mathbf{R}^1, \mathbf{C}^{K\times 3})$, and $\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})$, respectively.

Corollary 4.20 (Ultraviolet Cutoff Removal). *There exists a $g_0 \in]0,1[$ such that, if the bare coupling constant $g \in [0, g_0[$, then, for any choice of the regularizing mollifier, the probability measures $\mu^{\Lambda,g}$ and $\nu^{\Lambda,g}$ converge for $\Lambda \to +\infty$ to the probability measures μ^g on $\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3})$, and ν^g on $\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3})$. The regularized Hamiltonian $H^{\Lambda,g}$ converges pointwise on a dense domain to a selfadjoint non negative operator H^g on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu^g)$. If the coupling constant g vanishes, both measures μ^0 and ν^0 are Gaussian, otherwise not. The domain of definition is*

$$\mathcal{D}(H^g) := \left\{\Psi \in L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{R}^{K\times 3}), \mathbf{C}, d\nu^g) \,\middle|\, H\Psi \in L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{R}^{K\times 3}), \mathbf{C}, d\nu^g)\right\}. \tag{243}$$

Moreover, the operator H^g can be decomposed on $\mathcal{D}(H^g) \cap L^2(L^2_\perp(\mathbf{R}^3, \mathbf{R}^{K\times 3}), \mathbf{C}, d\nu^g)$ as

$$H^g = H_I + H^g_{II} + V^g - V^g_0, \tag{244}$$

where

$$H_I = -\frac{1}{2}\int_{\mathbf{R}^3} d^3x \left[\frac{\delta}{\delta A_i^a(t,x)}\right]^2$$

$$H_{II}^g = -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, \partial_i G^{a,b}(\mathbf{A}(t,y);x,y)\varepsilon^{b,c,d} A_k^d(t,y)\frac{\delta}{\delta A_k^c(t,y)}\right]^2 \tag{245}$$

$$V^g = \int_{\mathbf{R}^3} d^3x\, |R^{\nabla^{\mathbf{A}}}(t,x)|^2$$

for $\mathbf{A} \in L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x)$, *where* V_0^g *is a real constant which will be chosen later so that the ground state* Ω_0^g *satisfies*

$$H^g\Omega_0^g = 0. \tag{246}$$

Proof. By Corollary 4.19 the 4D-YM-Measure μ^g satisfies the Osterwalder-Schrader axioms for $g \in [0, g_0[$. By Theorem 3.7 we can reconstruct a selfadjoint operator $\tilde{H}^g$ on $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), \mathbf{C}, d\nu^g)$, and, as in the proof of Theorem 4.14 from (205) it follows that

$$\lim_{\Lambda\to+\infty} H^{\Lambda,g} = \lim_{\Lambda\to+\infty} \tilde{H}^{\Lambda,g} = \tilde{H}^g, \tag{247}$$

where the pointwise convergence on the projective limit $\bigcap_{\Lambda\geqslant 0}\mathcal{D}(H^{\Lambda,g})$ is meant.
For $\mathbf{A} \in L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x)$ and $(t,x) \in \mathbf{R}^4$ fixed we saw in (144) that

$$\lim_{\Lambda\to+\infty} V^{\Lambda,g}(t,x,\mathbf{A}) = |R^{\nabla^{\mathbf{A}}}(t,x)|^2 \tag{248}$$

pointwise, and thus

$$\lim_{\Lambda\to+\infty}\int_{\mathbf{R}^3} d^3x\, V^{\Lambda,g}(t,x,\mathbf{A}) = \int_{\mathbf{R}^3} d^3x\, |R^{\nabla^{\mathbf{A}}}(t,x)|^2 =: V^g. \tag{249}$$

By taking the limit on both side for the equation with (247) with $V_0^{\Lambda,g}$ chosen as in Theorem 4.3

$$H^{\Lambda,g} = H_I^\Lambda + H_{II}^{\Lambda,g} + V^{\Lambda,g} - V_0^{\Lambda,g} \tag{250}$$

leads to

$$H^g = H_I + H_{II}^g + V^g - V_0^g = \tilde{H}^g, \tag{251}$$

with the definitions for $\mathbf{A} \in L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x)$

$$H_I := -\frac{1}{2}\int_{\mathbf{R}^3} d^3x \left[\frac{\delta}{\delta A_i^a(t,x)}\right]^2$$

$$H_{II}^g := -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, \partial_i G^{a,b}(\mathbf{A}(t,y);x,y)\varepsilon^{b,c,d} A_k^d(t,y)\frac{\delta}{\delta A_k^c(t,y)}\right]^2. \tag{252}$$

Hence, H^g is selfadjoint, because $\tilde{H}^g$ is, and the property (246) of the ground state Ω_0^g follows from Theorem 4.7. The proof is completed.

□

Remark 4.9. Note that H is selfadjoint for all choices of the coupling constant $g \in [0, g_0[$. As we will see, the non vanishing of the g-contribution in H is essential for the proof of the existence of a positive mass gap.

4.6 Gauge Invariance

We want to prove that the construction of the Hamiltonian in Subsection 4.1 is gauge invariant. That for we show that, if we repeat the construction for a principal fibre bundle subject to a gauge transformation preserving the Coulomb gauge, we obtain an Hamiltonian which is unitary equivalent with the original one and has, in particular, the same spectrum.

Definition 4.6 (**Gauge Transformation**)**.** Let P be a principal fibre bundle over a manifold M and $\pi : P \to M$ be the projection. An automorphism of P is a diffeomorphism $f : P \to P$ such that $f(pg) = f(p)g$ for all $g \in G$, $p \in P$. A **gauge transformation** of P is an automorphism $f : P \to P$ such that $\pi(p) = \pi(f(p))$ for all $p \in P$. In other words f induces a well defined diffeomorphism $\bar{f} : M \to M$ given by $\bar{f}(\pi(p)) = \pi(f(p))$.

Following section 3.3 of [Bl05] we notice that the Lagrangian density on the principal fibre bundle P on which we define the Yang-Mills connection is a G-invariant functional on the space of 1-jets of maps from P to the fibre of the vector bundle V associated with P induced by the representation $\rho : G \to \mathrm{GL}(\mathbf{C}^K)$. Hence, the position variable $\mathbf{A}$ occurring in the Lagrangian density and its Legendre transform, the Hamiltonian density takes value in $\mathbf{C}^{3K}$, which is the fibre of the complex vector bundle V. We want to analyze how the position variable behaves if the principal fibre bundle is subject to a gauge transformation.

Proposition 4.21. *Let f be a gauge transformation of the principal fiber bundle P and ω a connection.*

Then, $\omega^f := (f^{-1})^*\omega$ *is a connection on* P. *They have the local representation on* $\pi^{-1}(U)$

$$\begin{aligned} \omega_p &= ad_{\zeta(p)^{-1}} \circ \pi^* A + \zeta^*\theta \\ \omega_p^f &= ad_{\zeta(p)^{-1}} \circ \pi^* A^f + \zeta^*\theta, \end{aligned} \tag{253}$$

and

$$A^f = ad_\phi \circ (A - \phi^*\theta), \tag{254}$$

where:

- $\pi : P \to M$ *is the projection of the principal fibre bundle* P *onto its base space* M,
- $U \subset M$ *is an open subset of the base space,*
- $\psi : \pi^{-1}(U) \to U \times G$ *is a local trivialization of* $\pi^{-1}(U) \subset P$, *that is a* G*-equivariant diffeomorphism such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \psi\ } & U \times G \\ \big\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle pr_1} & \\ U & & \end{array} \tag{255}$$

  This means that $\psi(p) = (\pi(p), \zeta(p))$, *where* $\zeta : \pi^{-1}(U) \to G$ *is a fibrewise diffeomorphism satisfying* $\zeta(pg) = \zeta(p)g$ *for all* $g \in G$.

- *the trivialization map* $\psi(f(p)) = (\pi(p), \zeta(f(p)))$ *let us define* $\bar{\phi} : \pi^{-1}(U) \to G$ *by* $\bar{\phi}(p) := \zeta(f(p))\zeta(p)^{-1}$, *whence* $\bar{\phi}(p) = \phi(\pi(p))$ *for a well defined function* ϕ *in virtue of the equivariance of* ψ and f.
- *The Maurer-Cartan form is the* $\mathfrak{g}$*-valued* 1*-form defined by* $\theta_g := (L_{g^{-1}})_* : T_gG \to T_eG = \mathfrak{g}$.
- A *and* A^f *are* $\mathfrak{g}$*-valued* 1*-forms on* M *introduced in Remark 2.1.*

Proof. These are collected results from Proposition 3.3 and Proposition 3.22 in [Bau14].

□

Remark 4.10. For matrix groups equation (254) becomes

$$A^f = \phi A \phi^{-1} - d\phi\phi^{-1} \tag{256}$$

For the Yang-Mills construction we denote the $K \times 3$ matrices of the local representation of A and its gauge transformation A^f by $\mathbf{A}$ and $\mathbf{A}^f$, which is in line with the notation utilized so far for the position variable and introduced in Theorem 2.4. To avoid confusion we drop the dependence on the coupling constant.

Theorem 4.22. *Let f be a gauge transform preserving the Coulomb gauge for the Yang-Mills construction and let H^{Λ} and $H^{\Lambda;f}$ be the cutoff Hamilton operators for the quantized Yang-Mills equation, before and after the gauge transform, as shown in Proposition 102 and Theorem 4.14. Let U be the operator on $L^2(\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^{\Lambda})$ induced by the gauge transform as*

$$U\Psi(\mathbf{A}) := \Psi(\mathbf{A}^f). \tag{257}$$

Then, U is a unitary operator in $L^2(\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^{\Lambda})$ and H^{Λ} and $H^{\Lambda;f}$ are unitary equivalent:

$$H^{\Lambda;f} = UH^{\Lambda}U^{-1}. \tag{258}$$

Moreover, the same holds true for the operator H and H^f, where the cutoff is removed as shown in Corollary 4.20. The operator U is unitary in $L^2(\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu)$ and

$$H^f = UHU^{-1}. \tag{259}$$

Proof. First, we remark that U maps $L^2(\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu)$ onto itself, because it preserves the Coulomb gauge. Next, we prove that U is unitary. For all $\Psi, \Phi \in L^2(\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^{\Lambda})$

$$\begin{aligned}(U\Psi, U\Phi) &= \int_{\mathcal{S}'_{\perp}(\mathbf{R}^3,\mathbf{R})} \Psi(\mathbf{A}^f)\Phi(\mathbf{A}^f)d\nu^{\Lambda}(\mathbf{A}) = \\ &= \int_{\mathcal{S}'_{\perp}(\mathbf{R}^3,\mathbf{R})} \Psi(\mathbf{A})\Phi(\mathbf{A}) \underbrace{\left|\frac{\partial \mathbf{A}^f}{\partial \mathbf{A}}\right|^{-1}}_{=1} d\nu^{\Lambda}(\mathbf{A}) = \\ &= (\Psi, \Phi)\end{aligned} \tag{260}$$

The change of variable is given by equation (254) which is affine in $\mathbf{A}$ because the adjoint representation is linear in A, which means $(\mathrm{ad}_g)_* = \mathrm{ad}_g$ for all $g \in G$. Moreover, since

$$\mathrm{ad}_g(A) = (L_g)_* A (R_{g^{-1}})_*, \tag{261}$$

the Jacobi determinant reads

$$\det\left((\mathrm{ad}_g)_*\right) = \det\left(\mathrm{ad}_g\right) = \det\left((L_g)_* 1_{\mathbf{R}^{3K}} (R_{g^{-1}})_*\right) = \det\left(1_{\mathbf{R}^{3K}}\right) = 1. \tag{262}$$

Note that the change of variable respects the fibre of the vector bundle V, and hence the change of variable formula for the integral is the one of finite dimensional analysis.

Next, we prove the unitary equivalence of the Hamilton operators before and after the gauge transform. Their definitions read

$$\begin{aligned} H^{\Lambda} &= H\left(\mathbf{A}(\varphi_t^{\Lambda}(\cdot - \cdot)), \frac{1}{\imath}\frac{\delta}{\delta \mathbf{A}(\varphi_t^{\Lambda}(\cdot - \cdot))}\right) \\ H^{\Lambda;f} &= H\left(\mathbf{A}^f(\varphi_t^{\Lambda}(\cdot - \cdot)), \frac{1}{\imath}\frac{\delta}{\delta \mathbf{A}^f(\varphi_t^{\Lambda}(\cdot - \cdot))}\right) \end{aligned} \tag{263}$$

For appropriate $\Psi, \Phi \in L^2(\mathcal{S}'_{\perp}(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^{\Lambda})$ we have

$$\begin{aligned} &(H^{\Lambda;f}\Psi, \Phi) = \\ &= \int_{\mathcal{S}'_{\perp}(\mathbf{R}^3,\mathbf{R})} H\left(\mathbf{A}^f(\varphi_t^{\Lambda}(\cdot - \cdot)), \frac{1}{\imath}\frac{\delta}{\delta \mathbf{A}^f(\varphi_t^{\Lambda}(\cdot - \cdot))}\right)\Psi(\mathbf{A})\bar{\Phi}(\mathbf{A})d\nu^{\Lambda}(\mathbf{A}) = \\ &= \int_{\mathcal{S}'_{\perp}(\mathbf{R}^3,\mathbf{R})} H\left(\mathbf{A}(\varphi_t^{\Lambda}(\cdot - \cdot)), \frac{1}{\imath}\frac{\delta}{\delta \mathbf{A}(\varphi_t^{\Lambda}(\cdot - \cdot))}\right)U^{-1}\Psi(\mathbf{A})U^{-1}\bar{\Phi}(\mathbf{A})\underbrace{\left|\frac{\partial \mathbf{A}^f}{\partial \mathbf{A}}\right|^{-1}}_{=1}d\nu^{\Lambda}(\mathbf{A}) = \\ &= (H^{\Lambda}U^{-1}\Psi, U^{-1}\Phi), \end{aligned} \tag{264}$$

leading to

$$H^{\Lambda;f} = UH^{\Lambda}U^{-1} \tag{265}$$

on the corresponding domains.
The proof for the Hamilton operators where the cutoffs have been removed are formally the same.

□

We can therefore conclude that the spectrum of the Hamilton operator for the quantized Yang-Mills problem is gauge invariant.

4.7 Spectral Bounds

We prove now that the Hamilton operator has a mass gap, being the sum of three non negative selfadjoint operators, one of which has a mass gap and having all the same ground state, the vacuum.

Proposition 4.23. *The spectra of H_I, H_{II}^g and V^g are:*

$$\begin{aligned} \mathrm{spec}(H_I) &= [0, +\infty[\\ \mathrm{spec}(H_{II}^g) &= \{0\} \cup [\eta, +\infty[, \ \textit{for a } \eta > 0 \\ \mathrm{spec}(V^g) &= [0, +\infty[. \end{aligned} \tag{266}$$

Moreover $\eta = O(g^{2(n+1)})$ for any $n \in \mathbf{N}_0$, where g is the bare coupling constant.

Now we can compute the lower bound of the spectrum of the Hamilton operator.

Proof of Proposition 4.23. The proof mimics the proof of Proposition 4.4. We will construct generalized eigenvectors to show that the spectra have only a continuous part depicted as in (266). First, we analyze the operator H_I, which can be seen as

$$H_I = -\frac{1}{2} \int_{\mathbf{R}^3} d^3x \Delta_{\mathbf{A}(t,x)}. \tag{267}$$

Let $x \in \mathbf{R}^3$ and $t \in \mathbf{R}$ now be fixed. For any $R > 0$ the Laplace operator $\Delta_{\mathbf{A}}$ on $[-\frac{R}{2}, +\frac{R}{2}]^{3K}$ under Dirichlet boundary conditions has a discrete spectral resolution $(\lambda_k, \psi_k)_{k \geqslant 0}$, where $\lambda_k = -\frac{\pi^2}{R^2}(k+1)$, and $\psi_k = \psi_k(\mathbf{A}) \in C_0^\infty([-\frac{R}{2}, +\frac{R}{2}]^{3K}, \mathbf{C})$. We can extend ψ_k outside the cube by setting its value to 0, obtaining an approximated eigenvector for the approximated eigenvalue λ_k, which is in line with the fact that the Laplacian on $L^2(\mathbf{R}^{3K}, \mathbf{C})$ has solely a continuous spectrum, which is $]-\infty, 0]$. The functional

$$\Psi_k^{\mathbf{A};x_0}(\bar{\mathbf{A}}) := \delta(\bar{\mathbf{A}} - \mathbf{A})\delta(x - x_0)\psi_k(\mathbf{A}) \tag{268}$$

for $x_0 \in \mathbf{R}^3, k \in \mathbf{N}$ and $\mathbf{A} \in \mathbf{R}^{3K}$ is a generalized eigenvector in $\mathcal{E}'(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^g)$ for the operator H_I on the rigged Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu^g)$ for the generalized eigenvalue $\frac{\pi^2}{R^2}(k+1)$, which, by Theorem A.2, is an element of the continuous spectrum of the non negative operator H_I. By varying the generalized eigenvalue over k and R, the claim about the spectrum follows.

Next, we analyze the multiplication operator

$$V^g = \int_{\mathbf{R}^3} d^3x |R^{\nabla^{\mathbf{A}}}(t,x)|^2, \tag{269}$$

where $R^{\nabla^{\mathbf{A}}}$ is the curvature operator associated to the connection $\mathbf{A}$. Let $\mathbf{A} \in L^2_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}, d^3x)$ now be fixed. Any non zero $\psi \in C_0^\infty(\mathbf{R}^{3K}, \mathbf{C})$ is eigenvector of the multiplication with the non negative real $|R^{\nabla^{\mathbf{A}}}(t,x)|^2$. The functional

$$\Psi^{\mathbf{A};x_0}(\bar{\mathbf{A}}) := \delta(\bar{\mathbf{A}} - \mathbf{A})\delta(x - x_0)\psi_k(\mathbf{A}) \tag{270}$$

where $(\psi_k)_{k\geqslant 0}$ is an orthonormal basis of $L^2(\mathbf{R}^{3K},\mathbf{C})$, is a generalized eigenvector in $\mathcal{E}'(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),d\nu^g)$ for the operator V^g on the rigged Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3}),d\nu^g)$ for the generalized eigenvalue $V^g(t,x_0,\mathbf{A})$, which, by Theorem A.2, is an element of the continuous spectrum of the non negative operator V^g and the claim about its spectrum follows.

Finally, we analyze the operator H_{II}, which we can write for any $\mathbf{A}\in L^2_\perp(\mathbf{R}^3,\mathbf{C}^{K\times 3},d^3x)$ as

$$H^g_{II}=-\frac{g^2}{2}\int_{\mathbf{R}^3}d^3x\left[\int_{\mathbf{R}^3}d^3y\,D^a_i(\mathbf{A};x,y)\right]^2, \tag{271}$$

for the operator $D=D(\mathbf{A};\mathbf{x},\mathbf{y})$ defined as

$$D^a_i(\mathbf{A};x,y):=\partial_i G^{a,b}(\mathbf{A}(t,y);x,y)\varepsilon^{b,c,d}A^d_k(t,y)\frac{\delta}{\delta A^c_k(t,y)}. \tag{272}$$

Let $x_0,y_0\in\mathbf{R}^3$ now be fixed. We set

$$f^{a,c}_{i,k}(\mathbf{A};x_0,y_0):=\partial_i G^{a,b}(\mathbf{A}(t,y_0);x_0,y_0)\varepsilon^{b,c,d}A^d_k(t,y_0) \tag{273}$$

and apply Lemma 4.5 and Lemma 4.6. Assuming that for all indices c,k

$$\int_{-\infty}^{+\infty}dA^c_k\,f^{a,c}_{i,k}(\mathbf{A}(t,y_0);x_0,y_0)^{-1}<+\infty \tag{274}$$

uniformly in $\mathbf{A}$, we can find a diffeomeorphism $\mathbf{B}:\mathbf{R}^{3K}\to\mathbf{R}^{3K}$ in the form of formula (118), such that for any $R>0$ the operator $D^a_i(\mathbf{A}(t,y_0);x_0,y_0)^2$ on $B^{-1}([-\frac{R}{2},+\frac{R}{2}]^{3K})$ under Dirichlet boundary conditions has a discrete spectral resolution $(\lambda^a_{i,s}(x_0,y_0),\psi^a_{i,s}(\mathbf{A}(t,y_0);x_0,y_0))_{s\geqslant 0}$, where

$$\lambda^a_{i,s}(x_0,y_0)=-\sum_{j=1}^{3}\sum_{c=1}^{K}\frac{\pi^2k^2_{j,c,s}}{\left[\int_{-\frac{R}{2}}^{+\frac{R}{2}}dB^c_j\,g^{a,c}_{i,j}(B^c_j;x_0,y_0)^{-1}\right]^2}, \tag{275}$$

where $k_{j,c,s}\in\mathbf{Z}^*$ for all indices $s\in\mathbf{N}_0$, $j\in\{1,2,3\}$ and $c\in\{1,\dots,K\}$, and, by Lemma 4.6 we defined

$$g^{a,l}_{i,j}(B^l_j;x_0,y_0):=\left(\sum_{k=1}^{3}\sum_{c=1}^{K}L^{a,c}_{i,k}(x_0,y_0)\right)e^{{B^l_j}^2} \tag{276}$$

for

$$\begin{aligned}L^{a,c}_{i,j}(x_0,y_0)&=\left[\sup_{\mathbf{A}}\left[\int_{-\infty}^{A^c_j}d\bar{A}^c_j\,f^{a,c}_{i,j}(\mathbf{A};x_0,y_0)^{-1}+K^{a,c}_{i,j}(\mathbf{A};x_0,y_0)\right]\right]^{-1}\\ K^{a,c}_{i,j}(\mathbf{A}\,;x_0,y_0)&:=-\inf_{A^c_j}\int_{-\infty}^{A^c_j}d\bar{A}^c_j\,f^{a,c}_{i,j}(\mathbf{A};x_0,y_0)^{-1}.\end{aligned} \tag{277}$$

Note that for any $R > 0$ the operator $D_i^a(\mathbf{A}(t, y_0); x_0, y_0)$ on $B^{-1}([-\frac{R}{2}, +\frac{R}{2}]^{3K})$ under Dirichlet boundary conditions has a discrete spectral resolution with the same eigenvectors but other eigenvalues $(\zeta_{i,s}^a(x_0, y_0), \psi_{i,s}^a(\mathbf{A}(t, y_0); x_0, y_0))_{s\geqslant 0}$, where

$$\zeta_{i,s}^a(x_0, y_0) = -\sum_{j=1}^{3}\sum_{c=1}^{K} \frac{\imath\pi k_{j,c,s}^2}{\int_{-\frac{R}{2}}^{+\frac{R}{2}} dB_j^c\, g_{i,j}^{a,c}(B_j^c; x_0, y_0)^{-1}}, \tag{278}$$

where $k_{j,c,s} \in \mathbf{Z}^*$ for all indices $s \in \mathbf{N}_0$, $j \in \{1, 2, 3\}$ and $c \in \{1, \ldots, K\}$.
Since $B^{-1}([-\frac{R}{2}, +\frac{R}{2}]^{3K}) \uparrow \mathbf{R}^{3K}$ for $R \uparrow +\infty$, we can extend $\psi_{i,s}^a(\cdot; x_0, y_0)$ outside the cube by setting its value to 0, obtaining an approximated eigenvector for the approximated eigenvalue $\lambda_{i,s}^a(x_0, y_0)$ for the operator $D_i^a(\mathbf{A}; x_0, y_0)^2$ on $L^2(\mathbf{R}^{3K}, \mathbf{C})$, which means that $\lambda_{i,s}^a(x_0, y_0) \in \mathrm{spec}_c(D_i^a(\mathbf{A}; x_0, y_0)^2)$. For fixed i, s and a the functional

$$\Psi_{i,k}^{a,x_0,y_0,\mathbf{A}}(\bar{\mathbf{A}}) := \delta(\bar{\mathbf{A}} - \mathbf{A})\delta(x - x_0)\delta(y - y_0)\delta(\bar{y} - y_0)\psi_{i,k}^a(\mathbf{A}; x_0, y_0) \tag{279}$$

is a generalized eigenvector in $\mathcal{E}'(\mathcal{S}'_\perp(\mathbf{R}^3, \mathbf{C}^{K\times 3}), d\nu)$ for the operator

$$\begin{aligned} H_{i,II}^{a,g} &:= -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, D_i^a(\mathbf{A}; x, y)\right]^2 = \\ &= -\frac{g^2}{2}\int_{\mathbf{R}^3} d^3x \left[\int_{\mathbf{R}^3} d^3y\, D_i^a(\mathbf{A}; x, y)\right]\left[\int_{\mathbf{R}^3} d^3\bar{y}\, D_i^a(\mathbf{A}; x, \bar{y})\right] \end{aligned} \tag{280}$$

on the rigged Hilbert space $L^2(\mathcal{S}'_\perp(\mathbf{R}^4, \mathbf{C}^{K\times 3}), \mu)$ for the strictly positive generalized eigenvalue

$$\lambda_{i,s}^{a,g}(x_0, y_0) = g^2\sum_{j=1}^{3}\sum_{c=1}^{K} \frac{\frac{\pi^2}{2}k_{j,c,s}^2}{\left[\int_{-\frac{R}{2}}^{+\frac{R}{2}} dB_j^c\, g_{i,j}^{a,c}(B_j^c; x_0, y_0)^{-1}\right]^2}, \tag{281}$$

which, by Theorem A.2, is an element of the continuous spectrum of the operator $H_{i,II}^a$. We still have to prove to check that $\lambda_{i,s}^a(x_0, y_0)$ is bounded away from 0 uniformly in R. By Proposition 2.2 and Corollary 2.3, for every $n \in \mathbf{N}_0$ there is a constant $c_n > 0$, bounded in n such that

$$\int_{-\frac{R}{2}}^{+\frac{R}{2}} dA_j^c(t, y_0)[\partial_i G^{a,b}(\mathbf{A}; x_0, y_0)\varepsilon^{b,c,d}A_j^d(t, y_0)]^{-1} \leqslant c_n g^{2n}\int_{-\infty}^{+\infty} dA_j^c(t, y_0)\,\frac{1}{1 + |A_j^d(t, y_0)|^{2n+1}}. \tag{282}$$

Therefore, inserting (282) into (277) and (278) leads to the spectral lower bound

$$\lambda_{i,s}^{a,g}(x_0, y_0) \geqslant C_n\, g^{2(n+1)} \tag{283}$$

for all $i \in 1, 2, 3$, $s \in \mathbf{N}_0$ and for all $n \in \mathbf{N}_0$, for an appropriate constant C_n bounded in n. Since the collection of generalized eigenvectors obtained by varying (279) over k, x_0, and y_0 is complete in the sense of Theorem A.2 (ii), by varying the generalized eigenvalue (281) over $k_{j,c,s}$ and R, the claim about the spectrum follows for $\eta = O(g^{2(n+1)})$. The proof is complete.

□

Lemma 4.24. *let A and B be two self adjoint operators on the Hilbert space $\mathcal{H}$, such that $\mathcal{D}(A+B)$ is dense in $\mathcal{H}$, $\mathrm{spec}(A) \subset [0, +\infty[$, $\mathrm{spec}(B) \subset \{0\} \cup [\eta, +\infty[$ for a $\eta > 0$, and 0 is an eigenvalue of finite multiplicity for the eigenvector Ω_0 for both A and B. Then, $\mathrm{spec}(A + B) \subset \{0\} \cup [\eta, +\infty[$, i.e. the spectral gap of B is maintained.*

Proof. The spectral gap of $A + B$ reads

$$\begin{aligned}\eta(A+B) = \inf_{\psi \in \langle\Omega_0\rangle^\perp \cap \mathcal{D}(A+B)} \frac{(\psi, (A+B)\psi)}{(\psi, \psi)} &\geqslant \inf_{\psi \in \langle\Omega_0\rangle^\perp \cap \mathcal{D}(A+B)} \frac{(\psi, B\psi)}{(\psi, \psi)} = \\ = \inf_{\psi \in \langle\Omega_0\rangle^\perp \cap \mathcal{D}(B)} \frac{(\psi, B\psi)}{(\psi, \psi)} &= \eta(B) = \eta > 0.\end{aligned} \tag{284}$$

□

Counterexample 4.1. Let H_1 be a non negative selfadjoint operator on the Hilbert space $\mathcal{H}_1$, and H_2 a non negative selfadjoint operator on the Hilbert space $\mathcal{H}_2$. Both operators have 0 a simple eigenvalue for the eigenvectors $\Omega_1 \in \mathcal{H}_1$ and $\Omega_2 \in \mathcal{H}_2$. Moreover let H_2 have a spectral gap $\mathrm{spec}(H_2) \subset \{0\} \cup [\eta, +\infty[$ for a $\eta > 0$. Let us define

$$\begin{aligned}\mathcal{H} &:= \mathcal{H}_1 \otimes \mathcal{H}_2 \\ H &:= H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2.\end{aligned} \tag{285}$$

The operator H is selfadjoint and non negative on the Hilbert space $\mathcal{H}$, and has 0 has eigenvalue. But it has no spectral gap. The reason is that 0, as an eigenvalue of $H_1 \otimes \mathbb{1}$ and $\mathbb{1} \otimes H_2$ has infinite multiplicity, leading to a clustering of elements of $\mathrm{spec}(H)$ near 0. Lemma 4.24 cannot be applied.

Corollary 4.25. *The spectrum of the Hamiltonian H^g contains 0 as a simple eigenvalue for the vacuum eigenstate, and satisfies*

$$\mathrm{spec}(H^g) \subset \{0\} \cup [\eta, +\infty[, \text{ for a } \eta > 0, \tag{286}$$

and $\eta = O(g^{2(n+1)})$ for any $n \in \mathbf{N}_0$, where g is the bare coupling constant. Hence, there is no mass gap only if $g > 0$, and, in particular, the group G must be non-abelian.

Proof. By Proposition 4.23 the operators H_I, H^g_{II} and V^g are positive semidefinite and so is H^g. By Proposition 4.20 the ground state Ω_0 is the eigenvector of finite multiplicity for the eigenvalue 0 for all these four positive semidefinite operators. By Lemma 4.24 the spectral gap of H^g is bounded from below by the spectral gap of H^g_{II}:

$$\eta(H^g) \geqslant \eta(H^g_{II}) =: \eta, \tag{287}$$

and $\eta = O(g^{2(n+1)})$ for any $n \in \mathbf{N}_0$ holds true by Proposition 4.20. □

4.8 Running of the Coupling Constant by Renormalization and Asymptotic Freedom

Till now all of our considerations referred to the *bare coupling constant*, which we now denote by g_0. We can repeat the classical and quantum mechanical construction of Section 2 and Section 4 for the running coupling constant $g = g(\mu)$, where μ is the energy scale, instead of the bare coupling constant g_0. To treat the non-trivial behaviour of $g(\mu)$ we have to renormalize running fields A and constants g by an appropriate scaling of the bare quantities A_0 and g_0. Following [Ti08] Chapter 21.9 we introduce renormalization constants Z_3 and Z_g and the transform

$$A_{0}{}_j^a = \sqrt{Z_3}A_j^a \qquad g_0 = Z_g\mu^\epsilon g, \tag{288}$$

and choose the values of the constants as

$$Z_3 := 1 - C_2(G)\frac{1}{\epsilon}\frac{\alpha_s}{4\pi} \qquad Z_g := 1 - \frac{1}{\epsilon}\frac{b_0}{2}\alpha_s, \tag{289}$$

where $\alpha_s := \frac{g^2}{4\pi}$, $b_0 := \frac{11}{12\pi}C_2(G)$, and $C_2(G)$ is the quadratic casimir operator in the adjoint representation of the Lie algebra of the group G. The parameter b_0 is positive. The parameter ϵ will be let to converge to 0 at the end of the calculation.

Since the bare coupling constant knows nothing about the energy scale μ,

$$\frac{dg_0}{d\mu} = 0, \tag{290}$$

which, by mean of (288) leads to the Gellman-Low equation

$$\beta(\alpha_s) = \frac{-\epsilon\alpha_s}{1 - \frac{b_0}{\epsilon}\alpha_s} = -b_0\alpha_s^2 + O(\epsilon, \alpha_s) \qquad (\epsilon, \alpha_s \to 0), \tag{291}$$

where we have defined $\beta(g) := \frac{d\alpha_s}{dt}$ for $t := \log(\mu^2)$. With the choice $\epsilon := 0$ we can easily solve the Gellman-Low equation and obtain the implicit

$$\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(M)} = b_0 \log \frac{\mu^2}{M^2}, \tag{292}$$

where M is an integration constant, which we choose such that $\lim_{\mu\to M^+} \alpha_s(\mu) = +\infty$, so that

$$\alpha_s(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{M^2}} \qquad (\mu \in]M, +\infty[), \tag{293}$$

which is in line with [We05] page 156. Therefore,

$$\lim_{\mu\to+\infty} g(\mu) = 0. \tag{294}$$

This phenomenon, termed **asymptotic freedom** was discovered by Gross and Wilczek ([GW73], and independently by Politzer ([Po73]).

The running mass gap is

$$\eta(\mu) = O(g^{2n+1}(\mu)), \tag{295}$$

for any $n \in \mathbf{N}_0$, where

$$g(\mu) = \sqrt{\frac{4\pi}{b_0 \log \frac{\mu^2}{M^2}}}. \tag{296}$$

With Corollaries 4.19, 4.20 and 4.25 we have therefore proved

Theorem 4.26. *In the case of a running coupling constant $g = g(\mu)$ the construction of the 4D-YM-measure satisfies Wightman axioms (W1)-(W8) and the spectrum of the running Hamilton operator H contains 0 as simple eigenvalue for the vacuum eigenstate. If $g(\mu) > 0$, there exists a constant $\eta = \eta(\mu) > 0$ such that spec$(H^{g(\mu)}) \subset \{0\} \cup [\eta(\mu), +\infty[$. Moreover $\eta = O(g^{2(n+1)}(\mu))$ for any $n \in \mathbf{N}_0$. In particular, the group G must be non-abelian. The mass gap tends to 0 if the energy scale becomes arbitrary large.*

Remark 4.11. As an application we see that, once the infrared cutoff in the Magnen-Rivasseau-Sénéor model is removed, the mass gap is zero, because the limit of the running coupling constant vanishes, and the Hamiltonian has the same spectral properties as in the case of electrodynamics.

5 Conclusion

We have quantized Yang-Mills equations for the positive light cone in the Minkowskian $\mathbf{R}^{1,3}$ obtaining field maps satisfying Wightman axioms of Constructive Quantum Field Theory. Moreover, the spectrum of the corresponding Hamilton operator is positive and bounded away from zero except for the case of the vacuum state which has vanishing energy level. The construction is invariant under gauge transforms preserving the Coulomb gauge.

A Spectral Theory in Rigged Hilbert Spaces

Rigged Hilbert spaces have been introduced in mathematical physics to utilize Dirac calculus for the spectral theory of operators appearing in quantum mechanics (see [Ro66], [ScTw98] and [Ma01], [Ma08]). Within that framework the role of distributions to provide a rigorous foundation to generalized eigenvectors and eigenvalues is highlighted by the Gel'fand-Kostyuchenko-Vilenkin spectral theorems for unitary and selfadjoint operators (see [GV64], [Sp19], [An07] and [Ze09] Chapter 12.2.4).

Definition A.1. Let $\mathcal{F}$ be a vector space on which are defined two inner products. We say the inner products are **compatible** if every sequence in $\mathcal{F}$ which is Cauchy with respect to both inner products and converges to $\varphi \in \mathcal{F}$ with respect to one inner product also converges to φ with respect to the other inner product.

Definition A.2. A Frechét space $\mathcal{F}$ is a **countably Hilbert space** if its topology can be induced by a countable system of pairwise compatible inner products $((\cdot,\cdot)_k)_{k\geqslant 0}$. Without loss of generality we can assume that

$$(\varphi,\varphi)_0 \leqslant (\varphi,\varphi)_1 \leqslant (\varphi,\varphi)_2 \leqslant \dots \tag{297}$$

for all $\varphi \in \mathcal{F}$. We denote by $\mathcal{F}_k$ the completion of $\mathcal{F}$ with respect to $(\cdot,\dot{)}_k$.

Proposition A.1. *There is a decreasing chain*

$$\mathcal{F} \subset \dots \subset \mathcal{F}_k \subset \mathcal{F}_{k-1} \dots \subset \mathcal{F}_1 \subset \mathcal{F}_0, \tag{298}$$

where every inclusion is a linear injective continuous operator of norm 1*, and an increasing sequence of dual spaces*

$$\mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \dots \subset \mathcal{F}'_{k-1} \subset \mathcal{F}'_k \subset \dots \subset \mathcal{F}', \tag{299}$$

where every inclusion is a linear injective continuous operator of norm 1*.*
For $k \geqslant l$ we denote the embedding $\mathcal{F}_k \subset \mathcal{F}_l$ by T^k_l.

Remark A.1. Since by Riesz's Lemma every Hilbert space is isomorphic to its dual, it follows that for any $k, l \geqslant 0$ $\mathcal{F}_k$ is embedded in $\mathcal{F}_l$ and viceversa, but the two Hilbert spaces are not necessarily isomorphic.

Definition A.3. A countably Hilbert spaceis **nuclear** if for every $l \in \mathbf{N}_0$, there exists a $k \geqslant l$ such that the mapping $T_l^k : \mathcal{F}_k \to \mathcal{F}_l$ has the form

$$T_l^k(\varphi) = \sum_{j=0}^{\infty} \lambda_j (\varphi, \varphi_j)\psi_j = \sum_{j=0}^{\infty} \lambda_j F_j(\varphi)\psi_j, \tag{300}$$

where $(\varphi_j)_{j\geqslant 0} \subset \mathcal{F}_k$, $(\psi_j)_{j\geqslant 0} \subset \mathcal{F}_l$, and $(Fj)_{j\geqslant 0} \subset \mathcal{F}'_k$ are orthonormal bases and the λ_js are positive numbers such that $\sum_{j=0}^{\infty} \lambda_j < +\infty$.

Definition A.4. A **rigged Hilbert space** $\mathcal{H}$ is a nuclear countably Hilbert space equipped with yet another inner product $(\cdot, \cdot)$ which is continuous in both variables.

Definition A.5. Let A be a linear operator on a locally convex topological vector space $\mathcal{F}$. A linear functional $F \in \mathcal{F}'$ is a **generalized eigenvector** of A if there exists a scalar λ such that $F(A\varphi) = \lambda F(\varphi)$ for all $\varphi \in \mathcal{F}$. We call λ the eigenvalue of the eigenvector F. In other words, a generalized eigenvector of A is an eigenvector of the adjoint $A' : \mathcal{F}' \to \mathcal{F}'$. We say the set of all generalized eigenvectors $(F_\iota)_{\iota\in I}$ of A is **complete** if $F_\iota(\varphi) = 0$ for all $\iota \in I$ implies $\varphi = 0$.

Theorem A.2 (Gel'fand-Kostyuchenko-Vilenkin). *A selfadjoint operator A on a rigged Hilbert space $\mathcal{F} \subset \mathcal{H} \subset \mathcal{F}'$, where $\mathcal{D}(A) \subset \mathcal{F}$, has a complete system of orthonormal generalized eigenvectors with real eigenvalues. The spectrum of A then reads* $\operatorname{spec}(A) = (\lambda_\iota)_{\iota\in I}$*, where for all $\iota \in I$ the real number $\lambda\iota$ is either an eigenvalue, i.e. $F_\iota \in \mathcal{H}$, or an element of the continuous spectrum, i.e. $F_\iota \notin \mathcal{H}$. For every $\varphi, \psi \in \mathcal{F}$ we have*

$$(\varphi, \psi) = \int_{\mathbf{R}} d\xi(\iota) F_\iota(\varphi)\overline{F_\iota(\psi)}, \tag{301}$$

for a measure ξ on $\mathbf{R}$.

Corollary A.3. *If an operator A on a rigged Hilbert space $\mathcal{F} \subset \mathcal{H} \subset \mathcal{F}'$, where $\mathcal{D}(A) \subset \mathcal{F}$, has a complete system of generalized eigenvectors with real eigenvalues, then it is selfadjoint.*

Proof. Choose the measure ξ such that the r.h.s. of (301) is equal to the scalar product in $\mathcal{H}$. Then, A is selfadjoint with respect to this scalar product. □

Example A.1 (Schwartz's Space). The inclusion

$$\mathcal{S}(\mathbf{R}^N) \subset L^2(\mathbf{R}^N, d^N x) \subset \mathcal{S}'(\mathbf{R}^N) \tag{302}$$

defines a rigged Hilbert space, as can be seen with the definition of the compatible scalar products

$$(\varphi, \psi)_k := \sum_{|\alpha|\leqslant k} \int_{\mathbf{R}^N} d^N x (1+|x|^2)^{\frac{k}{2}} \partial^\alpha \varphi(x) \overline{\partial^\alpha \psi(x)}. \tag{303}$$

for all $k \in \mathbf{N}_0$

Example A.2 (**Kubo-Takenaka Construction**). we follow [Ku96] Chapter 4.2. Let ν_0 be the standard Gaussian probability measure on $\mathcal{S}'(\mathbf{R}^N)$. By Proposition 3.6 every $\varphi \in L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0)$ can be written as

$$\varphi = \sum_{j=0}^{\infty} \theta_W(f_j), \tag{304}$$

where $f_j \in \mathcal{H}_s^j := S_j(L^2(\mathbf{R}^N, d^N x)^{\otimes j})$ is an element of the Bosonic Fock space and θ_W the Wiener-Itô-Segal isomorphism. For any $k \in \mathbf{N}_0$ we can define a norm and an associated scalar product as

$$\|\varphi\|_k^2 := \sum_{j=0}^{\infty} j! \|f_j\|_k^2, \tag{305}$$

where $\|\cdot\|_k$ is the norm $\mathcal{H}_s^j$ induced by the norm $\|\cdot\|_k$ in $L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0)$ defined as (303.)

Proposition A.4. *Let*

$$\begin{aligned} &\mathcal{E}(L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0)) := \left\{\varphi \in L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0) | \, \|\varphi\|_k < +\infty\right\} \\ &\mathcal{E}'(L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0)) : \text{ dual space of } (\mathcal{F}). \end{aligned} \tag{306}$$

Then, we have a rigged Hilbert space

$$\mathcal{E}(L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0)) \subset L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0) \subset \mathcal{E}'(L^2(\mathcal{S}'(\mathbf{R}^N), d\nu_0)). \tag{307}$$

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