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ORIGINAL ARTICLE

Construction of the log-convex minorant of a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$

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Abstract

We give a simple construction of the log-convex minorant of a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ and consequently extend to the d-dimensional case the well-known formula that relates a log-convex sequence $\{M_p\}_{p\in\mathbb{N}_0}$ to its associated function ω_M, that is, $M_p = \sup_{t>0} t^p \exp(-\omega_M(t))$. We show that in the more dimensional anisotropic case the classical log-convex condition $M_\alpha^2 \le M_{\alpha-e_j}M_{\alpha+e_j}$ is not sufficient: convexity as a function of more variables is needed (not only coordinate-wise). We finally obtain some applications to the inclusion of spaces of rapidly decreasing ultradifferentiable functions in the matrix weighted setting.

KEYWORDS
log-convex sequences, matrix weights, regularization of sequences, ultradifferentiable functions

1 | INTRODUCTION

For a sequence $\{M_p\}_{p\in\mathbb{N}_0}$ of real positive numbers (with $M_0 = 1$ for simplicity; $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$), its *associated function* is defined by

$$\omega_M(t) := \sup_{p\in\mathbb{N}_0} \log\frac{t^p}{M_p}, \qquad t > 0.$$

Mandelbrojt proved in [17, Chap. I] (see also [15]) that if $\lim_{p\to+\infty} M_p^{1/p} = +\infty$ then

$$M_p = \sup_{t>0} \frac{t^p}{\exp \omega_M(t)}, \qquad p \in \mathbb{N}_0, \tag{1.1}$$

if and only if $\{M_p\}_{p\in\mathbb{N}_0}$ is logarithmically convex, that is,

$$M_p^2 \le M_{p-1}M_{p+1}, \qquad \forall p \in \mathbb{N}.$$

We refer also to the recent work [22] where this construction has been studied again in detail, some technical ambiguities have been solved and non-standard cases have been studied as well; see Section 3.

However, to the best of our knowledge this condition has never been generalized to the d-dimensional anisotropic case ($d > 1$), and the reason is that the classical coordinate-wise logarithmic convexity condition (5.8) for a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is not sufficient to obtain the analogous of Equation (1.1) for M_α, as explained in Remark 5.3. The reason is that this is a convexity condition on each variable separately and not on the globality of its variables. Assuming the stronger condition that $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is log-convex on the globality of its variables $\alpha \in \mathbb{N}_0^d$ (see Definition 5.1), we extend Equation (1.1) to $\alpha \in \mathbb{N}_0^d$ instead of $p \in \mathbb{N}_0$ (see Theorem 5.2).

To obtain this result, we construct in Sections 2–4 the (optimal) convex minorant of a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ (then $a_\alpha = \log M_\alpha$ in Section 5). The idea, in the one variable case, takes inspiration from the convex regularization of sequences of Mandelbrojt in [17], which was quite complicated and difficult to export to the more dimensional case. Our construction is made by taking the supremum of hyperplanes approaching from below the given sequence and leads to the notion of convexity for a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ in the sense that $a_\alpha = F(\alpha)$ for a convex function $F : [0, +\infty)^d \to \mathbb{R}$. This condition gives the suitable notion of logarithmic convexity for a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ in order to write it in terms of its associated function as in Equation (5.7).

This result is a very useful tool for working in the anisotropic setting, and we expect several applications, that could be object of future works. Indeed, in the isotropic (or one-dimensional) ultradifferentiable framework the interplay between a given sequence $\mathbf{M} = \{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ and the associated function $\omega_\mathbf{M}$ becomes relevant in many different contexts, for example, very prominent in the proofs of the comparison results by Bonet–Meise–Melikhov [5]. Similarly, this comment applies to other weighted spaces involving weight sequences and weight functions in the sense of Braun–Meise–Taylor as well. In the anistropic setting, very few literature is currently available, see, for example, [4, 7, 10, 18], and one can expect that the new relation between $\mathbf{M}$ and $\omega_\mathbf{M}$ obtained in Theorem 5.2 is becoming crucial when transferring known statements and techniques from the isotropic to the more general anistropic setting (see, for instance, the recent preprint [10], where the analysis of convexity made in this paper is used to obtain inclusion relations for Gelfand–Shilov type spaces based on solid translation-invariant Banach function spaces of bounded type on $\mathbb{R}^d$). It is also reasonable that $\omega_\mathbf{M}$ serves as a standard example of an anisotropic weight function in the sense of Braun–Meise–Taylor, when allowing different growth in different directions and not considering the radial extension of ω to $\mathbb{R}^d$ as it is usual done in the literature. In Section 6, indeed, we obtain inclusion of spaces of rapidly decreasing ultradifferentiable functions in the matrix weighted anisotropic setting, where Theorem 5.2 is crucial (see Remark 6.2 and compare with the isotropic case in [3]). In particular, we characterize the conditions on the matrix weights $\mathcal{M}$ and $\mathcal{N}$ in order to have a continuous inclusion between the spaces $\mathcal{S}_{\{\mathcal{M}\}}/\mathcal{S}_{(\mathcal{M})}$, $\mathcal{S}_{\{\mathcal{N}\}}/\mathcal{S}_{(\mathcal{N})}$, both in the Roumieu and Beurling cases. The advantages of the matrix weighted setting was already enlightened in [19], in order to treat at the same time classes in the sense of Komatsu [15] (estimates of the derivatives with a sequence) and in the sense of Braun et al. [6] (estimates of the derivatives via a weight function). Since then several papers using weight matrices have been published. We mention, for instance, [2, 11, 12, 14], and references therein.

We refer to [5] to compare classes of ultradifferentiable functions as defined by Komatsu [15] and Braun et al. [6]. Concerning the case of anisotropic spaces of Gelfand–Shilov or the ultradifferentiable type, we point out that the word “anisotropic” is used in different meaning in the literature, both for the case when different (isotropic) sequences control the ultradifferentiability and the decay of the function (see, for instance, [1, 8, 20, 21, 23]), and for the situation when the sequence depends on the whole multi-index and not on its length, as it is intended in this paper (see, for instance, [7, 10, 18]).

2 | CONSTRUCTION OF THE CONVEX MINORANT CANDIDATE

Let us recall that a real sequence $\{a_p\}_{p\in\mathbb{N}_0}$ is said to be *convex* if

$$a_p \le \frac{1}{2}a_{p-1} + \frac{1}{2}a_{p+1}, \qquad \forall p \in \mathbb{N}.$$

This is equivalent to say that the polygonal obtained by connecting the points (p, a_p) with $(p+1, a_{p+1})$ by a straight line, for all $p \in \mathbb{N}_0$, is the graph of a convex function, or equivalently that there exists a convex function $F : [0, +\infty) \to \mathbb{R}$ with $F(p) = a_p$ for all $p \in \mathbb{N}_0$.

This suggests for a real sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ the following:

Definition 2.1. We say that a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is *convex* if there exists a convex function $F : [0, +\infty)^d \to \mathbb{R}$ with $F(\alpha) = a_\alpha$ for all $\alpha \in \mathbb{N}_0^d$.

Definition 2.2. The *convex minorant* of a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is the largest convex sequence $\{a^c_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ with $a^c_\alpha \le a_\alpha$ for all $\alpha \in \mathbb{N}_0^d$.

We want to construct the convex minorant of a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d} \subset \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ such that:

(i) $a_\alpha > -\infty, \quad \forall \alpha \in \mathbb{N}_0^d$,
(ii) $\lim_{|\alpha|\to+\infty} \frac{a_\alpha}{|\alpha|} = +\infty, \quad$ for $|\alpha| = \alpha_1 + \cdots + \alpha_d$,
(iii) a_α may be $+\infty$ at most for a finite number of multi-indices $\alpha \in \mathbb{N}_0^d$,
(iv) $a_0 \in \mathbb{R}$.

To this aim, we set

$$\begin{aligned} S &:= \{(\alpha, a_\alpha) : \ \alpha \in \mathbb{N}_0^d\}, \\ \mathcal{L} &:= \{f : \mathbb{R}^d \to \mathbb{R} : \ f \text{ is an affine function}\} \\ &= \{f(x) = \langle k, x\rangle + c : \ (k, c) \in \mathbb{R}^d \times \mathbb{R}\}. \end{aligned}$$

Note that the graphs of the affine functions $f \in \mathcal{L}$ are hyperplanes in $\mathbb{R}^{d+1}$.

Lemma 2.3. *Given $f \in \mathcal{L}$ we have*

$$f(\alpha) > a_\alpha,$$

at most for a finite number of points $\alpha \in \mathbb{N}_0^d$.

Proof. For $|\alpha| \ge 1$ by the Cauchy–Schwarz inequality we have

$$\frac{|f(\alpha)|}{|\alpha|} \le \frac{|\langle k, \alpha\rangle| + |c|}{|\alpha|} \le \frac{\|k\| \cdot \|\alpha\|}{|\alpha|} + |c| \le \|k\| + |c|,$$

since $\|\alpha\| = \sqrt{\alpha_1^2 + \cdots + \alpha_d^2} \le \alpha_1 + \cdots + \alpha_d = |\alpha|$.

On the other hand, assumption (ii) implies

$$\frac{a_\alpha}{|\alpha|} > \|k\| + |c|,$$

for $|\alpha|$ large enough. This implies that only a finite number of $\alpha \in \mathbb{N}_0^d$ may satisfy that $|f(\alpha)| > a_\alpha$. □

Let us now consider

$$\mathcal{L}_S := \{f \in \mathcal{L} : f(\alpha) \le a_\alpha \ \forall \alpha \in \mathbb{N}_0^d\}.$$

The graphs of the functions $f \in \mathcal{L}_S$ are the hyperplanes which lie under S.

Note that $\mathcal{L}_S \neq \emptyset$ by Lemma 2.3. As a matter of fact, given $f \in \mathcal{L}$, since $f(\alpha) \le a_\alpha$ except a finite number of points $\alpha_1, \dots, \alpha_\ell$, we have that

$$f - \max_{1\le j\le \ell}\{f(\alpha_j) - a_{\alpha_j}\} \in \mathcal{L}_S.$$

The idea is now to consider the supremum of these hyperplanes which lie under S and project on this set each a_α to construct the desired convex minorant sequence. So, let us first define

$$F(x) := \sup_{f\in\mathcal{L}_S} f(x). \tag{2.1}$$

Note that the above defined $F : \mathbb{R}^d \to \mathbb{R}$ is a convex function since it is the supremum of a set of convex (affine) functions: the epigraph is convex because it is the intersection of convex sets. By construction

$$F(\alpha) \le a_\alpha, \qquad \forall \alpha \in \mathbb{N}_0^d,$$

and we claim that

$$F(x) < +\infty \qquad \forall x \in [0, +\infty)^d.$$

As a matter of fact, given $x \in [0, +\infty)^d$, by assumptions (i) and (iii), we can find $d+1$ points $\alpha_1, \dots, \alpha_{d+1} \in \mathbb{N}_0^d$ such that x is inside (or on the border of) the simplex of vertices $\alpha_1, \dots, \alpha_{d+1}$ and $f(\alpha_j) \le a_{\alpha_j} < +\infty$ for every $f \in \mathcal{L}_S$ and $1 \le j \le d+1$. Then

$$f(x) \le \max_{1\le j\le d+1} f(\alpha_j) < +\infty,$$

by the convexity of f, and hence $F(x) < +\infty$.

Note also that F is continuous on $(0, +\infty)^d$, being a convex function.

For fixed $k \in \mathbb{R}^d$ let us now define

$$\begin{aligned} h_k &:= \sup\{c \in \mathbb{R} : f(x) = \langle k, x\rangle + c \in \mathcal{L}_S\} \\ &= \sup\{c \in \mathbb{R} : \langle k, \alpha\rangle + c \le a_\alpha,\ \forall \alpha \in \mathbb{N}_0^d\}. \end{aligned} \tag{2.2}$$

We have:

Lemma 2.4. *Let $k \in \mathbb{R}^d$ and h_k as in Equation (2.2). Then*

$$h_k = \inf_{\alpha\in\mathbb{N}_0^d}\{a_\alpha - \langle k, \alpha\rangle\} = \min_{\alpha\in\mathbb{N}_0^d}\{a_\alpha - \langle k, \alpha\rangle\}. \tag{2.3}$$

Proof. Let us first remark that $\inf_{\alpha\in\mathbb{N}_0^d}\{a_\alpha - \langle k, \alpha\rangle\}$ is a minimum because assumption (ii) implies that

$$\lim_{|\alpha|\to+\infty}(a_\alpha - \langle k, \alpha\rangle) = +\infty,$$

(see the proof of Lemma 2.3), so that the infimum is attained on a bounded set of $\mathbb{N}_0^d$, which is a finite set.

Let us now set

$$\tilde{h}_k := \inf_{\alpha\in\mathbb{N}_0^d}\{a_\alpha - \langle k,\alpha\rangle\} = \min_{\alpha\in\mathbb{N}_0^d}\{a_\alpha - \langle k,\alpha\rangle\} = a_{\bar{\alpha}} - \langle k,\bar{\alpha}\rangle \tag{2.4}$$

for some minimum point $\bar{\alpha}\in\mathbb{N}_0^d$, and prove that $\tilde{h}_k = h_k$ as defined in Equation (2.2).

Clearly

$$\begin{aligned} &\tilde{h}_k \leq a_\alpha - \langle k,\alpha\rangle, \qquad \forall\alpha\in\mathbb{N}_0^d,\\ \Leftrightarrow\quad &\tilde{h}_k + \langle k,\alpha\rangle \leq a_\alpha, \qquad \forall\alpha\in\mathbb{N}_0^d. \end{aligned}$$

Therefore,

$$f(x) = \tilde{h}_k + \langle k,x\rangle \in \mathcal{L}_S$$

is one of the functions in Equation (2.2) with $c = \tilde{h}_k$ and hence $h_k \geq \tilde{h}_k$.

On the contrary, if $c > \tilde{h}_k$ then Equation (2.4) implies that

$$c + \langle k,\bar{\alpha}\rangle > \tilde{h}_k + \langle k,\bar{\alpha}\rangle = a_{\bar{\alpha}}$$

so that in this case

$$\bar{f}(x) = c + \langle k,x\rangle \notin \mathcal{L}_S$$

and therefore $h_k \leq \tilde{h}_k$. □

Now, we define

$$f_k(x) := h_k + \langle k,x\rangle. \tag{2.5}$$

Proposition 2.5. *Let F be as in Equation (2.1), h_k as in Equation (2.3) and f_k as in Equation (2.5). Then*

$$F(x) = \sup_{k\in\mathbb{R}^d} f_k(x).$$

Proof. Since $f_k\in\mathcal{L}_S$ by Equation (2.3), we clearly have that

$$F(x) \geq \sup_{k\in\mathbb{R}^d} f_k(x).$$

On the other hand, if $f\in\mathcal{L}_S$ then

$$f(x) = c + \langle k,x\rangle$$

for some $c\in\mathbb{R}$, $k\in\mathbb{R}^d$, and hence by Equation (2.2)

$$f(x) \leq f_k(x) \leq \sup_{\ell\in\mathbb{R}^d} f_\ell(x).$$

This finally implies that

$$F(x) = \sup_{f\in\mathcal{L}_S} f(x) \leq \sup_{k\in\mathbb{R}^d} f_k(x).$$ □

We thus have a convex sequence $\{\tilde{a}_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ defined, for F as in Equation (2.1), by

$$\tilde{a}_\alpha := F(\alpha), \tag{2.6}$$

and which can be equivalently defined by

$$\tilde{a}_\alpha := \sup_{k\in\mathbb{R}^d} (\langle k,\alpha\rangle + h_k),$$

for

$$h_k = \inf_{\alpha\in\mathbb{N}_0^d} (a_\alpha - \langle k,\alpha\rangle).$$

We call it the *convex minorant candidate* of $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ since we shall prove in the following sections that it is indeed the convex minorant of $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ (see Corollary 4.2).

The first step in this direction is to clarify the one-dimensional case (in Section 3) and then proceed by descending induction on d for the d-dimensional case (in Section 4). To this aim in the next section, we shall look more closely at the geometric construction in the one-dimensional case, that is particularly clear.

3 | GEOMETRIC CONSTRUCTION OF THE CONVEX MINORANT SEQUENCE IN THE ONE-DIMENSIONAL CASE

The geometric construction in the one-dimensional case can also be found in [22]; we refer to this work and in particular to [22, Sect. 3.3] concerning the comparison with the classical results from [17, Chap. I]. Here, we revisit the construction in the spirit of Section 2 since it is required in the induction argument for the higher-dimensional situation in Section 4.

Let $\{a_\alpha\}_{\alpha\in\mathbb{N}_0}$ satisfy (i)–(iv). In particular, the condition $a_0 \in \mathbb{R}$ will be essential for the first step of the construction. All functions of $\mathcal{L}_S$ are of the form

$$f(x) = c + kx, \qquad \text{with } k \in \mathbb{R} \text{ and } c \le a_0,$$

since $f(\alpha) \le a_\alpha$ for all $\alpha \in \mathbb{N}_0$ implies, in particular, $f(0) = c \le a_0$.

Let us now consider the functions of $\mathcal{L}_S$ of the form

$$f_{a_0,k}(x) = a_0 + kx, \qquad k \in \mathbb{R},$$

and note that

$$F(0) = \sup_{f\in\mathcal{L}_S} f(0) = a_0.$$

The idea is now to rotate (i.e., increasing the slope k) the straight line $y = a_0 + kx$ around the point $(0, a_0) \in S$ until we meet another point $(p, a_p) \in S$. In order to have $f_{a_0,k}(p) = a_p$ we find

$$a_p = a_0 + kp \quad \Leftrightarrow \quad k = \frac{a_p - a_0}{p}.$$

Take then

$$k_0 := \inf_{p\in\mathbb{N}} \frac{a_p - a_0}{p}$$

and note that it is a minimum because of the assumption $a_p/p \to +\infty$ for $p \to +\infty$, which implies that the infimum can be done on a bounded subset of $\mathbb{N}$ and hence on a finite number of indices p:

$$k_0 = \min_{p\in\mathbb{N}} \frac{a_p - a_0}{p} = \frac{a_{p_1} - a_0}{p_1}$$

for some $p_1 \in \mathbb{N}$. Observe that p_1 does not need to be unique; if there is more than one p_1 realizing the minimum, for the construction below it does not matter which one we choose at this step.

Set

$$f_{a_0,k_0}(x) := a_0 + k_0 x. \tag{3.1}$$

We claim that

$$F(x) = f_{a_0,k_0}(x) = a_0 + k_0 x, \qquad \forall x \in [0, p_1],$$

where F is the function defined in Equation (2.1), that is, the geometric construction coincides with the construction made in Section 2, in $[0, p_1]$.

On one side, $f_{a_0,k_0} \in \mathcal{L}_S$ by construction and hence

$$F(x) \geq f_{a_0,k_0}(x), \qquad \forall x \in [0, p_1].$$

On the other side, if

$$f(x) = c + kx \in \mathcal{L}_S$$

then we must have

$$f(0) \leq a_0 = f_{a_0,k_0}(0)$$
$$f(p_1) \leq a_{p_1} = f_{a_0,k_0}(p_1)$$

and hence

$$f(x) \leq f_{a_0,k_0}(x), \qquad \forall x \in [0, p_1],$$

since they are affine functions. Therefore,

$$F(x) = \sup_{f\in\mathcal{L}_S} f(x) \leq f_{a_0,k_0}(x), \qquad \forall x \in [0, p_1].$$

We have thus proved that

$$F(x) = a_0 + k_0 x, \qquad \forall x \in [0, p_1],$$

and moreover

$$F(p_1) = a_{p_1}.$$

We can further proceed as in the previous step, considering

$$f_{a_{p_1},k}(x) = a_{p_1} + k(x - p_1)$$

with $f_{a_{p_1},k}(p_1) = a_{p_1} = F(p_1)$. Requiring

$$f_{a_{p_1},k}(p) = a_p \Leftrightarrow a_p = a_{p_1} + k(p - p_1) \Leftrightarrow k = \frac{a_p - a_{p_1}}{p - p_1}$$

we take

$$k_1 := \inf_{p_1 < p \in \mathbb{N}} \frac{a_p - a_{p_1}}{p - p_1} = \min_{p_1 < p \in \mathbb{N}} \frac{a_p - a_{p_1}}{p - p_1} = \frac{a_{p_2} - a_{p_1}}{p_2 - p_1},$$

for some $p_1 < p_2 \in \mathbb{N}$, and set

$$f_{a_{p_1},k_1}(x) := a_{p_1} + k_1(x - p_1).$$

Then, for all $p \in [p_1, p_2]$,

$$\begin{aligned} F(x) = f_{a_{p_1},k_1}(x) &= a_{p_1} + k_1(x - p_1) \\ &= k_1 x + a_{p_1} - k_1 p_1 \\ &= k_1 x + a_{p_1} - \frac{a_{p_2} - a_{p_1}}{p_2 - p_1} p_1 \\ &= k_1 x + \frac{p_2 a_{p_1} - p_1 a_{p_2}}{p_2 - p_1} \\ &= k_1 x + d_{k_1} \end{aligned}$$

with

$$d_{k_1} = \frac{p_2 a_{p_1} - p_1 a_{p_2}}{p_2 - p_1}.$$

Moreover, $F(p_2) = a_{p_2}$. Also in this case, p_2 does not need to be unique, and the choice of p_2 does not affect the next steps.

Going on recursively in the same way we have a geometric construction which coincides with the construction of F in Section 2.

The convex minorant candidate sequence given by $\tilde{a}_p = F(p)$ as defined in Equation (2.6) is in this case the projection of a_p on the segments of lines above defined in each interval $[p_i, p_{i+1}]$:

$$\begin{aligned} \tilde{a}_{p_i} &= a_{p_i}, \qquad \forall i \in \mathbb{N}_0, \\ \tilde{a}_p &= p k_i + d_{k_i} \\ &= \frac{a_{p_{i+1}} - a_{p_i}}{p_{i+1} - p_i} p + \frac{p_{i+1} a_{p_i} - p_i a_{p_{i+1}}}{p_{i+1} - p_i}, \qquad p_i < p < p_{i+1}. \end{aligned}$$

In the one-dimensional case, it is thus immediate by the construction that the convex minorant candidate sequence $\{\tilde{a}_p\}_{p \in \mathbb{N}_0}$ coincides with the convex minorant $\{a_p^c\}_{p \in \mathbb{N}_0}$.

4 | CONVEX MINORANT SEQUENCE IN THE MULTI-DIMENSIONAL CASE

In order to show that the convex minorant candidate sequence $\{\tilde{a}_\alpha\}_{\alpha \in \mathbb{N}_0^d} := \{F(\alpha)\}_{\alpha \in \mathbb{N}_0^d}$ defined in Equation (2.6) is the convex minorant sequence of $\{a_\alpha\}_{\alpha \in \mathbb{N}_0^d}$, we shall now prove that F is the biggest convex function whose epigraph contains S.

Theorem 4.1. *The function F defined in Equation (2.1) coincides with the biggest convex function $g : [0,+\infty)^d \to \mathbb{R}$ such that*

$$g(\alpha) \le a_\alpha, \qquad \forall \alpha \in \mathbb{N}_0^d. \tag{4.1}$$

Proof. Let us first remark that, since F is a convex function such that

$$F(\alpha) \le a_\alpha, \qquad \forall \alpha \in \mathbb{N}_0^d,$$

the largest convex function g which satisfies Equation (4.1) must be larger than F:

$$g(x) \ge F(x), \qquad \forall x \in [0,+\infty)^d. \tag{4.2}$$

In order to prove the opposite inequality, let us first work in the interior of $[0,+\infty)^d$. Fix $x^0 \in (0,+\infty)^d$ and consider $(x^0, y^0) = (x^0, g(x^0))$ on the graph of g.

Since g is convex, its epigraph G_g^+ is a convex subset of $\mathbb{R}^{d+1}$. It follows, as a consequence of the Hahn–Banach theorem, that there is a hyperplane of the form

$$y = \langle k^*, x - x^0\rangle + y^0 = \langle k^*, x\rangle + c^*,$$

for some $k^* \in \mathbb{R}^d$ and $c^* = y^0 - \langle k^*, x^0\rangle$, that leaves the whole set G_g^+ on the same side of the hyperplane. Note that $x^0 \in (0,+\infty)^d$ avoids "vertical" hyperplanes.

Then,

$$f^*(x) := \langle k^*, x\rangle + c^* \in \mathcal{L}_S,$$

since $g(\alpha) \le a_\alpha$ for all $\alpha \in \mathbb{N}_0^d$ by assumption. It follows that

$$F(x^0) = \sup_{f\in\mathcal{L}_S} f(x^0) \ge f^*(x^0) = y^0 = g(x^0),$$

and hence, by the arbitrariness of $x^0 \in (0,+\infty)^d$ and by Equation (4.2), we have that

$$F(x^0) = g(x^0), \qquad \forall x^0 \in (0,+\infty)^d. \tag{4.3}$$

Let us now consider the case $x^0 \in \partial[0,+\infty)^d$, that is, $x_j^0 = 0$ for at least one $1 \le j \le d$. Assume for simplicity $j = d$ and set $x = (x', 0) = (x_1, \dots, x_{d-1}, 0)$ for $x \in [0,+\infty)^{d-1} \times \{0\}$.

The problem in this case is that the graph of g could have in $(x^0, g(x^0))$ a tangent "vertical" hyperplane of the form $\{x_d = 0\}$ and hence not defined by a function of $\mathcal{L}_S$. The idea is to prove that the trace of F on $\{x_d = 0\}$ is a supremum of affine functions on $\mathbb{R}^{d-1}$ whose graph is below $S \cap \{x_d = 0\}$ reducing to the case of dimension $d-1$, and then proceed recursively up to dimension 1, where the assumption that $a_0 \in \mathbb{R}$ guarantees the conclusion.

So, let us define

$$\mathcal{L}_S^0 := \{f(x') = \langle k', x'\rangle + c : \; k' \in \mathbb{R}^{d-1}, c \in \mathbb{R}, \langle k', \alpha'\rangle + c \le a_{(\alpha',0)} \; \forall \alpha' \in \mathbb{N}_0^{d-1}\}.$$

We claim that

$$F(x', 0) = \sup_{f\in\mathcal{L}_S^0} f(x') =: F^0(x'). \tag{4.4}$$

In the following, we use the convention that when $k \in \mathbb{R}^d$ we put $k' \in \mathbb{R}^{d-1}$ with $k = (k', k_d)$. Note first that Proposition 2.5 for the sequence $\{a_{(\alpha',0)}\}_{\alpha'\in\mathbb{N}_0^{d-1}}$ implies that

$$\sup_{f\in\mathcal{L}_S^0} f(x') = \sup_{k'\in\mathbb{R}^{d-1}} (\langle k', x'\rangle + h_{k'}^0)$$

with

$$h^0_{k'} = \min_{\alpha'\in\mathbb{N}_0^{d-1}} (a_{(\alpha',0)} - \langle k',\alpha'\rangle).$$

Moreover, by Proposition 2.5 once more,

$$F(x',0) = \sup_{f\in\mathcal{L}_S} f(x',0) = \sup_{k\in\mathbb{R}^d} (\langle k,(x',0)\rangle + h_k)$$

with

$$h_k = \min_{\alpha\in\mathbb{N}_0^d}(a_\alpha - \langle k,\alpha\rangle) \le \min_{(\alpha',0)\in\mathbb{N}_0^{d-1}\times\{0\}} (a_{(\alpha',0)} - \langle k,(\alpha',0)\rangle) = h^0_{k'}$$

and hence

$$\begin{aligned} F(x',0) \le & \sup_{k\in\mathbb{R}^d} (\langle k,(x',0)\rangle + h^0_{k'}) = \sup_{k'\in\mathbb{R}^{d-1}} (\langle k',x'\rangle + h^0_{k'}) \\ & = \sup_{f\in\mathcal{L}^0_S} f(x') = F^0(x'). \end{aligned}$$

In order to prove the opposite inequality, let us now fix $f\in\mathcal{L}^0_S$. Then,

$$f(x') = \langle k',x'\rangle + c, \quad k'\in\mathbb{R}^{d-1},\ c\in\mathbb{R},$$

and

$$f(\alpha') \le a_{(\alpha',0)} \quad \forall\alpha'\in\mathbb{N}_0^{d-1}. \tag{4.5}$$

We claim that there exists $\Lambda\in\mathbb{R}$ such that

$$\tilde f(x) := f(x') + \Lambda x_d \in \mathcal{L}_s \tag{4.6}$$

for $x=(x',x_d)\in[0,+\infty)^d$.

Indeed, as in Lemma 2.3, from assumption (ii) and $|\langle k',\alpha'\rangle + c| \le \|k'\|\cdot|\alpha| + |c|$ we get that

$$\langle k',\alpha'\rangle + c > a_\alpha,$$

for at most a finite number of points $\alpha=(\alpha',\alpha_d)\in\mathbb{N}_0^d$, with $\alpha_d\ge 1$ (because of Equation (4.5) for $\alpha_d=0$).

There exists then

$$L := \min_{\alpha\in\mathbb{N}_0^d}[a_\alpha - (\langle k',\alpha'\rangle + c)].$$

Defining

$$\Lambda := \min(0,L),$$

we have that $\Lambda\le 0$ and hence for all $\alpha=(\alpha',\alpha_d)\in\mathbb{N}_0^d$ we have two cases:

if $\alpha_d=0$, then by Equation (4.5)

$$\langle k',\alpha'\rangle + c + \Lambda\alpha_d \le a_{(\alpha',0)};$$

if $\alpha_d \geq 1$, then by definition of L

$$\langle k', \alpha' \rangle + c + \Lambda\alpha_d \leq \langle k', \alpha' \rangle + c + \Lambda \leq \langle k', \alpha' \rangle + c + L \leq a_\alpha.$$

The two cases above prove Equation (4.6).

Setting now

$$\tilde{k} := (k', \Lambda) \in \mathbb{R}^d \quad \text{and} \quad \tilde{f}(x) := \langle \tilde{k}, x \rangle + c,$$

we have that $\tilde{f} \in \mathcal{L}_S$ and $\tilde{f}(x', 0) = f(x')$.

It follows that

$$\begin{aligned} F^0(x') &= \sup_{f \in \mathcal{L}_S^0} f(x') = \sup_{f \in \mathcal{L}_S^0} \tilde{f}(x', 0) \\ &\leq \sup_{f \in \mathcal{L}_S} f(x', 0) = F(x', 0), \end{aligned}$$

and the equality (4.4) is therefore proved.

This means that we have reduced the problem to prove that $F^0(x')$ is the maximum convex function $g : [0, +\infty)^{d-1} \to \mathbb{R}$ such that

$$g(\alpha') \leq a_{(\alpha',0)} \qquad \forall \alpha' \in \mathbb{N}_0^{d-1}.$$

If $x^0 = (x_1^0, \dots, x_{d-1}^0) \in (0, +\infty)^{d-1}$ the thesis follows from Equation (4.3) (in the case of dimension $d-1$ instead of d). If $x^0 \in \partial[0, +\infty)^{d-1}$, for instance $x^0 = (x_1^0, \dots, x_{d-2}^0, 0)$ we argue as before and thus reduce to determine the biggest convex function $g : [0, +\infty)^{d-2} \to \mathbb{R}$ with

$$g(\alpha_1, \dots, \alpha_{d-2}) \leq a_{(\alpha_1, \dots, \alpha_{d-2}, 0, 0)}, \ \forall (\alpha_1, \dots, \alpha_{d-2}) \in \mathbb{N}_0^{d-2}.$$

Proceeding recursively we are finally led to the one-dimensional case for the sequence $\{a_{(\alpha_1, 0, \dots, 0)}\}_{\alpha_1 \in \mathbb{N}_0}$, where the construction of Section 3 gives the desired maximum convex function whose graph is below S. Note that on $[0, +\infty)$ the problem of the border does not appear since $a_0 \in \mathbb{R}$ by assumption (iv) and the first line through $(0, a_0)$ is not a vertical line, but the graph of

$$f_{a_0, k_0}(x) = a_0 + k_0 x$$

defined in Equation (3.1). In particular, $F(0) = f_{a_0, k_0}(0) = a_0$. □

Corollary 4.2. *Given a sequence* $\{a_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ *satisfying* (*i*)–(*iv*), *its convex minorant sequence* $\{a_\alpha^c\}_{\alpha \in \mathbb{N}_0^d}$ *is defined by*

$$a_\alpha^c = F(\alpha),$$

for F as in Equation (2.1), or equivalently by

$$a_\alpha^c := \sup_{k \in \mathbb{R}^d} (\langle k, \alpha \rangle + h_k), \tag{4.7}$$

for

$$h_k = \inf_{\alpha \in \mathbb{N}_0^d} (a_\alpha - \langle k, \alpha \rangle). \tag{4.8}$$

In particular, $a_0^c = a_0$.

5 | CONSTRUCTION OF THE LOG-CONVEX MINORANT

Let $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ be a sequence of positive real numbers such that

$$\lim_{|\alpha|\to+\infty} M_\alpha^{1/|\alpha|} = +\infty \tag{5.1}$$

(we can also allow $M_\alpha = +\infty$ for finitely many multi-indices $\alpha \neq 0$). We say that the sequence $\mathbf{M} = \{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is *normalized* if $M_0 = 1$.

For a normalized sequence $\mathbf{M} = \{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$, we define its *associated function* $\omega_\mathbf{M}$ by

$$\omega_\mathbf{M}(t) := \sup_{\alpha\in\mathbb{N}_{0,t}^d} \log\frac{|t^\alpha|}{M_\alpha}, \qquad t\in\mathbb{R}^d,$$

with

$$\mathbb{N}_{0,t}^d := \{\alpha\in\mathbb{N}_0^d \,:\, \alpha_j = 0 \text{ if } t_j = 0,\ j = 1,\dots,d\},$$

and the convention that $0^0 = 1$. We observe that here the supremum is made on $\mathbb{N}_{0,t}^d$ in order to ensure that in the definition of the associated function the argument of the logarithm is not 0; we can equivalently write

$$\omega_\mathbf{M}(t) = \sup_{\alpha\in\mathbb{N}_0^d} \log\frac{|t^\alpha|}{M_\alpha}, \qquad t\in\mathbb{R}^d,$$

with the convention that $\log 0 = -\infty$. In what follows (in particular in Section 6), we use the latter expression for convenience.

Condition (5.1) ensures that $\omega_\mathbf{M}(t) < +\infty$ for all $t\in\mathbb{R}^d$ (see [17, Chap. I] or [2, Rem. 1]).

The function $\omega_\mathbf{M}(t)$ is increasing on $(0,+\infty)^d$ in the following sense: $\omega_\mathbf{M}(t) \le \omega_\mathbf{M}(s)$ if $t\le s$ with the order relation $t_j \le s_j$ for all $1\le j\le d$.

Consider then

$$a_\alpha = \log M_\alpha, \qquad \alpha\in\mathbb{N}_0^d. \tag{5.2}$$

By Equation (5.1), the sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ satisfies all assumptions (i)–(iv) of Section 2.

We can thus consider the convex minorant sequence $\{a_\alpha^c\}_{\alpha\in\mathbb{N}_0^d}$ as in Equations (4.7) and (4.8) and call

$$M_\alpha^{\mathrm{lc}} := \exp a_\alpha^c, \qquad \forall\alpha\in\mathbb{N}_0^d,$$

the *log-convex minorant* of $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$.

By the results of the previous section $\{\log M_\alpha^{\mathrm{lc}}\}_{\alpha\in\mathbb{N}_0^d}$ is the largest convex sequence less than or equal to $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ defined in Equation (5.2) and they coincide if $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is log-convex, according to the following Definition.

Definition 5.1. A sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is said to be *log-convex* if there exists a convex function $F : [0,+\infty)^d \to \mathbb{R}$ with $F(\alpha) = \log M_\alpha$ for all $\alpha\in\mathbb{N}_0^d$.

By construction

$$a_\alpha^c = \sup_{k\in\mathbb{R}^d}(\langle k,\alpha\rangle + h_k) = \sup_{k\in\mathbb{R}^d}(\langle k,\alpha\rangle - A(k)), \tag{5.3}$$

where the so-called *trace function* $A(k)$ is given by

$$\begin{aligned} A(k) = -h_k &= -\inf_{\alpha\in\mathbb{N}_0^d}(a_\alpha - \langle k,\alpha\rangle) \\ &= \sup_{\alpha\in\mathbb{N}_0^d}(\langle k,\alpha\rangle - a_\alpha) \\ &= \sup_{\alpha\in\mathbb{N}_0^d}(\langle k,\alpha\rangle - \log M_\alpha) \\ &= \sup_{\alpha\in\mathbb{N}_0^d}\log\frac{\mathrm{e}^{\langle k,\alpha\rangle}}{M_\alpha} \\ &= \sup_{\alpha\in\mathbb{N}_0^d}\log\frac{|(\mathrm{e}^k)^\alpha|}{M_\alpha} \\ &= \omega_{\mathbf{M}}(\mathrm{e}^k), \end{aligned} \tag{5.4}$$

since

$$|(\mathrm{e}^k)^\alpha| = |(\mathrm{e}^{k_1},\dots,\mathrm{e}^{k_d})^\alpha| = |\mathrm{e}^{k_1\alpha_1}\cdots\mathrm{e}^{k_d\alpha_d}| = \mathrm{e}^{\langle k,\alpha\rangle}.$$

From Equations (5.3) and (5.4), we have that

$$\begin{aligned} M_\alpha^{\mathrm{lc}} = \exp a_\alpha^c &= \exp\{\sup_{k\in\mathbb{R}^d}(\langle k,\alpha\rangle - \omega_{\mathbf{M}}(\mathrm{e}^k))\} \\ &= \sup_{k\in\mathbb{R}^d}\frac{|(\mathrm{e}^k)^\alpha|}{\exp\omega_{\mathbf{M}}(\mathrm{e}^k)} \\ &= \sup_{s\in(0,+\infty)^d}\frac{|s^\alpha|}{\exp\omega_{\mathbf{M}}(s)} \\ &= \sup_{s\in(0,+\infty)^d}\frac{s^\alpha}{\exp\omega_{\mathbf{M}}(s)}. \end{aligned} \tag{5.5}$$

In particular, being $M_\alpha^{\mathrm{lc}} \le M_\alpha$ for any normalized sequence of positive real numbers $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$, we have

$$\sup_{s\in(0,+\infty)^d}\frac{s^\alpha}{\exp\omega_{\mathbf{M}}(s)} \le M_\alpha, \qquad \forall\alpha\in\mathbb{N}_0^d, \tag{5.6}$$

and, moreover, $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is log-convex if and only if the equality holds. Hence, we have proved:

Theorem 5.2. *Let $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ be a normalized sequence of positive real numbers satisfying Equation (5.1). Then, the sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is log-convex if and only if*

$$M_\alpha = \sup_{s\in(0,+\infty)^d}\frac{s^\alpha}{\exp\omega_{\mathbf{M}}(s)}, \qquad \forall\alpha\in\mathbb{N}_0^d. \tag{5.7}$$

Remark 5.3. Note that if $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is a log-convex sequence, that is, $\log M_\alpha = F(\alpha)$ for a convex function $F : [0,+\infty)^d \to \mathbb{R}$, and e_j denotes the j-th element of the canonical basis of $\mathbb{R}^d$ with all entries equal to 0 except the j-th entry equal to 1, then

$$F(\alpha) = F\left(\frac{1}{2}(\alpha - e_j) + \frac{1}{2}(\alpha + e_j)\right) \le \frac{1}{2}F(\alpha - e_j) + \frac{1}{2}F(\alpha + e_j),$$

that is,

$$2\log M_\alpha \le \log M_{\alpha-e_j} + \log M_{\alpha+e_j}.$$

This yields the *coordinate-wise log-convexity condition* for a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$:

$$M_\alpha^2 \le M_{\alpha-e_j}M_{\alpha+e_j}, \qquad \alpha\in\mathbb{N}_0^d,\ 1\le j\le d,\ \alpha_j\ge 1. \tag{5.8}$$

This condition is clearly weaker than the condition of logarithmic convexity given in Definition 5.1 since there are functions which are coordinate-wise convex but not convex as functions of more variables. In particular, from Equation (5.5) we have that Equation (5.8) is not sufficient to obtain Equation (5.7); for an explicit example, see Example 5.4. Clearly in the one-dimensional case the two notions of log-convexity coincide, and Equation (5.7) was already known (see [15, Prop. 3.2]).

Example 5.4. The function of two variables $F(x,y)=(x+1)^2(y+1)^2$ is coordinate-wise convex but not convex in $[0,+\infty)^2$, and the sequence defined by $M_\alpha := e^{F(\alpha)-1}$ is normalized and satisfies Equation (5.1), but does not satisfy Equation (5.7).

Remark 5.5. We finally observe that we can easily rewrite the results of this section for a sequence $\mathbf{M}$ that is not normalized, analogously to [22]. In this case, we can define the associated function $\omega_{\mathbf{M}}$ as

$$\omega_{\mathbf{M}}(t) = \sup_{\alpha\in\mathbb{N}_{0,t}^d}\log\frac{M_0|t^\alpha|}{M_\alpha}, \qquad t\in\mathbb{R}^d,$$

and, by considering the (normalized) sequence $\widetilde{\mathbf{M}}$ defined by $\widetilde{M}_\alpha := M_\alpha/M_0$, formula (5.7) becomes

$$M_\alpha = M_0\sup_{s\in(0,+\infty)^d}\frac{s^\alpha}{\exp\omega_{\mathbf{M}}(s)}, \qquad \forall\alpha\in\mathbb{N}_0^d.$$

6 | CHARACTERIZATION OF INCLUSION RELATIONS OF SPACES OF RAPIDLY DECREASING ULTRADIFFERENTIABLE FUNCTIONS

In this section, we propose an application of the previous results to the characterization of inclusion relations of anisotropic ultradifferentiable classes. Such an application is based on results from [2]. We observe in particular that in [2] some conditions are imposed on the weights in order that the corresponding space contains Hermite functions. We do not discuss here the question if these conditions are optimal or if some of them could be relaxed still obtaining nontrivial spaces satisfying the same inclusion relations; we observe however that the nontriviality of the spaces is a very delicate question, and, by considering weights that oscillate around the, say, ‘limit case’ for nontriviality, it happens that all Hermite functions disappear from the corresponding space (see [3, Sec. 3], where an explicit example is given). For other applications giving inclusion results of anisotropic ultradifferentiable spaces we refer to the recent preprint [10].

Let us first recall the notion of weight matrices (anisotropic framework) as considered in [2, Sec. 3].

A *weight matrix* $\mathcal{M}$ is the set

$$\mathcal{M} := \{(\mathbf{M}^{(\lambda)})_{\lambda>0} : \mathbf{M}^{(\lambda)} = (M_\alpha^{(\lambda)})_{\alpha\in\mathbb{N}_0^d}, M_0^{(\lambda)}=1, M_\alpha^{(\lambda)}\le M_\alpha^{(\kappa)}\ \forall\alpha\in\mathbb{N}_0^d\ \forall 0<\lambda\le\kappa\}.$$

Denoting by $\|\cdot\|_\infty$ the supremum norm we consider the following spaces of matrix-weighted global ultradifferentiable functions of Roumieu/Beurling type

$$\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) := \{f\in C^\infty(\mathbb{R}^d) :\ \exists\lambda,h,C>0 :\ \sup_{\alpha,\beta\in\mathbb{N}_0^d}\frac{\|x^\alpha\partial^\beta f\|_\infty}{h^{|\alpha+\beta|}M_{\alpha+\beta}^{(\lambda)}}\le C\}$$

$$\mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d) := \{f\in C^\infty(\mathbb{R}^d) :\ \forall\lambda,h>0\,\exists C_{h,\lambda}>0 :\ \sup_{\alpha,\beta\in\mathbb{N}_0^d}\frac{\|x^\alpha\partial^\beta f\|_\infty}{h^{|\alpha+\beta|}M_{\alpha+\beta}^{(\lambda)}}\le C_{h,\lambda}\}$$

endowed with the inductive limit topology in the Roumieu case and the projective limit topology in the Beurling case (see [2, Sec. 3]).

In order to characterize the inclusion of spaces of this type, given two weight matrices $\mathcal{M} = \{(\mathbf{M}^{(\lambda)})_{\lambda>0}\}$ and $\mathcal{N} = \{(\mathbf{N}^{(\lambda)})_{\lambda>0}\}$ we introduce the following relations:

$$\mathcal{M}\{\preceq\}\mathcal{N} \quad \text{if} \quad \forall\lambda > 0\exists\kappa > 0\exists C \geq 1 \text{ s.t. } M_\alpha^{(\lambda)} \leq C^{|\alpha|}N_\alpha^{(\kappa)}\ \forall\alpha \in \mathbb{N}_0^d$$

$$\mathcal{M}(\preceq)\mathcal{N} \quad \text{if} \quad \forall\lambda > 0\exists\kappa > 0\exists C \geq 1 \text{ s.t. } M_\alpha^{(\kappa)} \leq C^{|\alpha|}N_\alpha^{(\lambda)}\ \forall\alpha \in \mathbb{N}_0^d$$

$$\mathcal{M} \lhd \mathcal{N} \quad \text{if} \quad \forall\lambda > 0\forall\kappa > 0\forall h > 0\exists C \geq 1 \text{ s.t. } M_\alpha^{(\lambda)} \leq Ch^{|\alpha|}N_\alpha^{(\kappa)}\ \forall\alpha \in \mathbb{N}_0^d.$$

We shall also need the following conditions for a weight matrix $\mathcal{M} = \{(\mathbf{M}^{(\lambda)})_{\lambda>0}\}$ (see [2]; cf. also [16]), in the Roumieu setting

$$\forall\lambda > 0\exists\kappa \geq \lambda, B, C, H > 0 : \quad \alpha^{\alpha/2}M_\beta^{(\lambda)} \leq BC^{|\alpha|}H^{|\alpha+\beta|}M_{\alpha+\beta}^{(\kappa)}\ \forall\alpha,\beta \in \mathbb{N}_0^d \tag{6.1}$$

$$\forall\lambda > 0\exists\kappa \geq \lambda, A \geq 1 : \quad M_{\alpha+e_j}^{(\lambda)} \leq A^{|\alpha|+1}M_\alpha^{(\kappa)}\ \forall\alpha \in \mathbb{N}_0^d, 1 \leq j \leq d \tag{6.2}$$

$$\forall\lambda > 0\exists\kappa \geq \lambda, A \geq 1 : \quad M_\alpha^{(\lambda)}M_\beta^{(\lambda)} \leq A^{|\alpha+\beta|}M_{\alpha+\beta}^{(\kappa)}\ \forall\alpha,\beta \in \mathbb{N}_0^d \tag{6.3}$$

and in the Beurling case

$$\forall\lambda > 0\exists 0 < \kappa \leq \lambda, H > 0 : \forall C > 0\exists B > 0 : \quad \alpha^{\alpha/2}M_\beta^{(\kappa)} \leq BC^{|\alpha|}H^{|\alpha+\beta|}M_{\alpha+\beta}^{(\lambda)}\ \forall\alpha,\beta \in \mathbb{N}_0^d \tag{6.4}$$

$$\forall\lambda > 0\exists 0 < \kappa \leq \lambda, A \geq 1 : \quad M_{\alpha+e_j}^{(\kappa)} \leq A^{|\alpha|+1}M_\alpha^{(\lambda)}\ \forall\alpha \in \mathbb{N}_0^d, 1 \leq j \leq d \tag{6.5}$$

$$\forall\lambda > 0\exists 0 < \kappa \leq \lambda, A \geq 1 : \quad M_\alpha^{(\kappa)}M_\beta^{(\kappa)} \leq A^{|\alpha+\beta|}M_{\alpha+\beta}^{(\lambda)}\ \forall\alpha,\beta \in \mathbb{N}_0^d. \tag{6.6}$$

Note that Equation (6.1) for $\beta = 0$ implies that $\mathbf{M}^{(\kappa)}$ satisfies Equation (5.1) for some $\kappa > 0$, and hence for all $\kappa' \geq \kappa$. Similarly, Equation (6.4) implies that $\mathbf{M}^{(\lambda)}$ satisfies Equation (5.1) for all $\lambda > 0$ (see [2, Rem. 3]). Since condition (5.1) ensures that the associate weight function $\omega_{\mathbf{M}}$ is finite, the above remarks are essential to recall, from [2, Thm. 1], that if the weight matrix $\mathcal{M} = \{(\mathbf{M}^{(\lambda)})_{\lambda>0}\}$ satisfies Equations (6.1) and (6.2) then $\mathcal{S}_{\{\mathcal{M}\}}$ is isomorphic to the sequence space

$$\Lambda_{\{\mathcal{M}\}} := \{\mathbf{c} = (c_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbb{C}^{\mathbb{N}_0^d} \ :\ \exists\lambda, h > 0 \text{ s.t. } \sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha|e^{\omega_{\mathbf{M}^{(\lambda)}}(\alpha^{1/2}/h)} < +\infty\}, \tag{6.7}$$

and similarly in the Beurling case, for a weight matrix $\mathcal{M}$ having Equations (6.4) and (6.5), the space $\mathcal{S}_{(\mathcal{M})}$ is isomorphic to

$$\Lambda_{(\mathcal{M})} := \{\mathbf{c} = (c_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbb{C}^{\mathbb{N}_0^d} \ :\ \forall\lambda, h > 0 \ \sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha|e^{\omega_{\mathbf{M}^{(\lambda)}}(\alpha^{1/2}/h)} < +\infty\}, \tag{6.8}$$

where $\alpha^{1/2} := (\alpha_1^{1/2}, \dots, \alpha_d^{1/2})$. We recall from [2] that in both cases the Hermite functions $(H_\alpha(x))_{\alpha\in\mathbb{N}_0^d}$ are an absolute Schauder basis in $\mathcal{S}_{\{\mathcal{M}\}}$ (resp., in $\mathcal{S}_{(\mathcal{M})}$), and the isomorphism $T : \mathcal{S}_{\{\mathcal{M}\}} \to \Lambda_{\{\mathcal{M}\}}$ (resp., $T : \mathcal{S}_{(\mathcal{M})} \to \Lambda_{(\mathcal{M})}$) is given by

$$Tf = (\xi_\alpha(f))_{\alpha\in\mathbb{N}_0^d} := \left(\int_{\mathbb{R}^d} f(x)H_\alpha(x)\,dx\right)_{\alpha\in\mathbb{N}_0^d} \tag{6.9}$$

for $f \in \mathcal{S}_{\{\mathcal{M}\}}$ (resp., $f \in \mathcal{S}_{(\mathcal{M})}$). It will also be useful to write the sequence spaces (6.7) and (6.8) as follows (see [2]):

$$\Lambda_{\{\mathcal{M}\}} = \{\mathbf{c} = (c_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbb{C}^{\mathbb{N}_0^d} \ : \ \exists j \in \mathbb{N}\,\text{s.t.} \sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{M}^{(j)}}(\alpha^{1/2}/j)} < +\infty\} \tag{6.10}$$

$$\Lambda_{(\mathcal{M})} = \{\mathbf{c} = (c_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbb{C}^{\mathbb{N}_0^d} \ : \ \forall j \in \mathbb{N} \sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{M}^{(1/j)}}(j\alpha^{1/2})} < +\infty\}. \tag{6.11}$$

We say that a weight matrix $\mathcal{M} = \{(\mathbf{M}^{(\lambda)})_{\lambda>0}\}$ is *log-convex* if $\{M_\alpha^{(\lambda)}\}_{\alpha\in\mathbb{N}_0^d}$ is log-convex (according to Definition 5.1) for all $\lambda > 0$.

Let us remark that in the one-dimensional case the assumption of log-convexity, together with $M_0 = 1$, implies both Equations (6.3) and (6.6) with $A = 1$ (we are here considering the sequence case for simplicity, that is, the case when in the weight matrix $\mathcal{M}$ all the $\mathbf{M}^{(\lambda)}$, $\lambda > 0$, coincide) since the convex function $F(p) = \log M_p$ has increasing difference quotient and hence

$$\frac{\log M_{p+q} - \log M_p}{q} \geq \frac{\log M_q - \log M_0}{q} = \frac{\log M_q}{q}.$$

On the contrary, in the more-dimensional case log-convexity and $M_0 = 1$ do not imply Equation (6.3)/(6.6), not even under the additional conditions (6.1)/(6.4) and (6.2)/(6.5). Take, for instance, $M_\alpha = \alpha^{\alpha/2} e^{\max\{\alpha_1^2,\alpha_2^2\}}$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ (with the convention $0^0 := 1$). It is easy to check that it is log-convex (since $\log M_\alpha$ is the sum of convex functions) and satisfies Equations (6.1) and (6.2). However, taking $\alpha = (n, 0)$ and $\beta = (0, n)$ condition (6.3) is not valid for $n \to +\infty$.

The above remarks explain why conditions (6.3) and (6.6), that we shall need in Theorem 6.1, are not required in the one-dimensional/isotropic case (see [3]).

Theorem 6.1. *Let $\mathcal{M} = \{(\mathbf{M}^{(\lambda)})_{\lambda>0}\}$ and $\mathcal{N} = \{(\mathbf{N}^{(\lambda)})_{\lambda>0}\}$ be two weight matrices and assume that $\mathcal{M}$ is log-convex. Then, the following characterizations hold:*

(i) Let $\mathcal{M}$ satisfy Equations (6.1)–(6.3) and $\mathcal{N}$ satisfy Equations (6.1) and (6.2). Then, the following are equivalent:
  (a) $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\mathcal{N}\}}(\mathbb{R}^d)$ holds with continuous inclusion;
  (b) $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\mathcal{N}\}}(\mathbb{R}^d)$ holds as sets;
  (c) $\mathcal{M}\{\preceq\}\mathcal{N}$.
(ii) Let $\mathcal{M}$ satisfy Equations (6.1)–(6.3) and $\mathcal{N}$ satisfy Equations (6.4) and (6.5). Then, the following are equivalent:
  (a) $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d)$ holds with continuous inclusion;
  (b) $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d)$ holds as sets;
  (c) $\mathcal{M} \lhd \mathcal{N}$.
(iii) Let $\mathcal{M}$ satisfy Equations (6.4)–(6.6) and $\mathcal{N}$ satisfy Equations (6.4) and (6.5). Then, the following are equivalent:
  (a) $\mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d)$ holds with continuous inclusion;
  (b) $\mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d)$ holds as sets;
  (c) $\mathcal{M}(\preceq)\mathcal{N}$.

Proof. The implications (c) ⇒ (a), in all the three cases, clearly follow from the definition of the spaces (hypotheses (6.1)–(6.3)/(6.4)–(6.6) and log-convexity of $\mathcal{M}$ are not needed at this step).

We now prove the other implications.

(i): It is enough to prove that (b) ⇒ (c) in order to have equivalence of the three conditions. Let us then assume that $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\mathcal{N}\}}(\mathbb{R}^d)$ and prove that $\mathcal{M}\{\preceq\}\mathcal{N}$.

By the already mentioned isomorphism (6.9) with the sequence space (6.10), we have that

$$\Lambda_{\{\mathcal{M}\}} \simeq \mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\mathcal{N}\}}(\mathbb{R}^d) \simeq \Lambda_{\{\mathcal{N}\}}. \tag{6.12}$$

Let us first choose $\bar{h} \in \mathbb{N}$, $\bar{h} \geq 2$, such that $\mathbf{M}^{(h)}$ satisfies Equation (5.1), and hence $\omega_{\mathbf{M}^{(h)}}(t) < +\infty$ for all $t \in \mathbb{R}^d$, for all $h \geq \bar{h}$ (this is possible because of assumption (6.1)). For some fixed (but arbitrary) $j \in \mathbb{N}$, $j \geq \bar{h}$, let us consider the sequence $\mathbf{c} := (c_\alpha)_{\alpha \in \mathbb{N}_0^d} \in \mathbb{C}^{\mathbb{N}_0^d}$ defined by

$$c_\alpha := e^{-\omega_{\mathbf{M}^{(j)}}(\bar{\alpha}^{1/2}/j)} \tag{6.13}$$

for $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_d)$ with

$$\bar{\alpha}_k = \begin{cases} \alpha_k & \text{if } \alpha_k \neq 1 \\ 0 & \text{if } \alpha_k = 1. \end{cases} \tag{6.14}$$

Let us prove that $\mathbf{c} \in \Lambda_{\{\mathcal{M}\}}$; we must find $h \in \mathbb{N}$ and $C > 0$ such that

$$|c_\alpha| = e^{-\omega_{\mathbf{M}^{(j)}}(\bar{\alpha}^{1/2}/j)} \leq C e^{-\omega_{\mathbf{M}^{(h)}}(\alpha^{1/2}/h)} \qquad \forall \alpha \in \mathbb{N}_0^d,$$

that is,

$$\omega_{\mathbf{M}^{(h)}}\left(\frac{\sqrt{\alpha_1}}{h}, \dots, \frac{\sqrt{\alpha_d}}{h}\right) \leq \log C + \omega_{\mathbf{M}^{(j)}}\left(\frac{\sqrt{\bar{\alpha}_1}}{j}, \dots, \frac{\sqrt{\bar{\alpha}_d}}{j}\right). \tag{6.15}$$

By assumption (6.3) we choose $h \in \mathbb{N}$, $h \geq j$, and $A \geq 1$ such that

$$M_\alpha^{(j)} M_\beta^{(j)} \leq A^{|\alpha+\beta|} M_{\alpha+\beta}^{(h)}, \qquad \forall \alpha, \beta \in \mathbb{N}_0^d. \tag{6.16}$$

For such a choice of h and from the definition of the associated function

$$\omega_{\mathbf{M}^{(h)}}\left(\frac{\sqrt{\alpha_1}}{h}, \dots, \frac{\sqrt{\alpha_d}}{h}\right) = \sup_{\beta_1, \dots, \beta_d \in \mathbb{N}_0} \log \frac{\alpha_1^{\beta_1/2} \cdots \alpha_d^{\beta_d/2}}{h^{\beta_1 + \cdots + \beta_d} M_{(\beta_1, \dots, \beta_d)}^{(h)}}. \tag{6.17}$$

Let us now assume, without loss of generality, that the entries of α equal to 1 (if there are some) are in the first positions, that is, $(\alpha_1, \dots, \alpha_d) = (1, \dots, 1, \alpha_{s+1}, \dots, \alpha_d)$, for some $s \geq 0$, with $\alpha_k \neq 1$ for $s + 1 \leq k \leq d$. By Equation (6.16) we have that

$$M_{(\beta_1, \dots, \beta_d)}^{(h)} \geq \frac{1}{A^{\beta_1 + \cdots + \beta_d}} M_{(\beta_1, \dots, \beta_s, 0, \dots, 0)}^{(j)} M_{(0, \dots, 0, \beta_{s+1}, \dots, \beta_d)}^{(j)}$$

and hence from Equation (6.17)

$$\begin{aligned} \omega_{\mathbf{M}^{(h)}}\left(\frac{\alpha^{1/2}}{h}\right) \leq & \sup_{\beta_1, \dots, \beta_d \in \mathbb{N}_0} \left(\log \frac{\alpha_{s+1}^{\beta_{s+1}/2} \cdots \alpha_d^{\beta_d/2}}{\left(\frac{h}{A}\right)^{\beta_{s+1} + \cdots + \beta_d} M_{(0, \dots, 0, \beta_{s+1}, \dots, \beta_d)}^{(j)}} \right. \\ & \left. + \log \frac{(A/h)^{\beta_1 + \cdots + \beta_s}}{M_{(\beta_1, \dots, \beta_s, 0, \dots, 0)}^{(j)}} \right) \\ \leq & \sup_{\beta_{s+1}, \dots, \beta_d \in \mathbb{N}_0} \log \frac{\alpha_{s+1}^{\beta_{s+1}/2} \cdots \alpha_d^{\beta_d/2}}{\left(\frac{h}{A}\right)^{\beta_{s+1} + \cdots + \beta_d} M_{(0, \dots, 0, \beta_{s+1}, \dots, \beta_d)}^{(j)}} \\ & + \sup_{\beta_1, \dots, \beta_s \in \mathbb{N}_0} \log \frac{\left(\frac{A}{h}\right)^{\beta_1 + \cdots + \beta_s}}{M_{(\beta_1, \dots, \beta_s, 0, \dots, 0)}^{(j)}}. \end{aligned}$$

Eventually enlarging h so that $h \geq Aj$ we thus have

$$\omega_{\mathbf{M}^{(h)}}\left(\frac{\alpha^{1/2}}{h}\right) \leq \sup_{\beta_{s+1},\dots,\beta_d\in\mathbb{N}_0} \log \frac{\alpha_{s+1}^{\beta_{s+1}/2}\cdots\alpha_d^{\beta_d/2}}{j^{\beta_{s+1}+\cdots+\beta_d} M^{(j)}_{(0,\dots,0,\beta_{s+1},\dots,\beta_d)}} + \omega_{\bar{\mathbf{M}}^{(j)}}\left(\frac{A}{h}\right)$$
$$\leq \omega_{\mathbf{M}^{(j)}}\left(\frac{\bar{\alpha}^{1/2}}{j}\right) + C_{j,h} \tag{6.18}$$

for $\{\bar{M}^{(j)}_\beta\}_{\beta\in\mathbb{N}_0^s}$ the sequence given by $\bar{M}^{(j)}_\beta = M^{(j)}_{(\beta_1,\dots,\beta_s,0,\dots,0)}$ and $C_{j,h} = \omega_{\bar{\mathbf{M}}^{(j)}}(A/h)$. Note that this constant is finite because if $|(\beta_1,\dots,\beta_s)| \to +\infty$ then also $|(\beta_1,\dots,\beta_s,0,\dots,0)| \to +\infty$ and hence

$$\left(\bar{M}^{(j)}_\beta\right)^{1/|\beta|} = \left(M^{(j)}_{(\beta_1,\dots,\beta_s,0,\dots,0)}\right)^{1/(\beta_1+\cdots+\beta_s)} \longrightarrow +\infty$$

by Equation (5.1). Inequality (6.18) proves Equation (6.15) and hence $\mathbf{c} \in \Lambda_{\{\mathcal{M}\}}$.

From Equation (6.12), we have that $\mathbf{c} \in \Lambda_{\{\mathcal{N}\}}$, and therefore there exist $\ell \in \mathbb{N}$ and $C \geq 1$ such that

$$e^{-\omega_{\mathbf{M}^{(j)}}(\bar{\alpha}^{1/2}/j)} = |c_\alpha| \leq C e^{-\omega_{\mathbf{N}^{(\ell)}}(\alpha^{1/2}/\ell)}, \qquad \forall \alpha \in \mathbb{N}_0^d,$$

that is,

$$\omega_{\mathbf{N}^{(\ell)}}(\alpha^{1/2}/\ell) \leq \log C + \omega_{\mathbf{M}^{(j)}}(\bar{\alpha}^{1/2}/j), \qquad \forall \alpha \in \mathbb{N}_0^d. \tag{6.19}$$

Fix now $t \in (0,+\infty)^d$ and set $\mathbf{1} := (1,\dots,1) \in \mathbb{R}^d$. There exists $\alpha \in \mathbb{N}_0^d$ with $\alpha^{1/2} < t \leq (\alpha+\mathbf{1})^{1/2}$, that is, $\alpha_k < t_k^2 \leq \alpha_k + 1$ for all $1 \leq k \leq d$. From Equation (2), we have

$$\omega_{\mathbf{N}^{(\ell)}}(t/\ell) \leq \omega_{\mathbf{N}^{(\ell)}}((\alpha+\mathbf{1})^{1/2}/\ell) \leq \log C + \omega_{\mathbf{M}^{(j)}}((\overline{\alpha+\mathbf{1}})^{1/2}/j)$$
$$\leq \log C + \omega_{\mathbf{M}^{(j)}}(\alpha^{1/2}) \leq \log C + \omega_{\mathbf{M}^{(j)}}(t)$$

because $\omega_{\mathbf{M}^{(j)}}$ is increasing on $(0,+\infty)^d$ and $(\overline{\alpha+\mathbf{1}})^{1/2}/j \leq \alpha^{1/2}$ since $\overline{\alpha_k+1} = 0$ if $\alpha_k = 0$ and $(\overline{\alpha_k+1})^{1/2}/j = \sqrt{\alpha_k+1}/j \leq \sqrt{\alpha_k}$ if $\alpha_k \geq 1$ and $j \geq \bar{h} \geq 2$. From Equations (5.6) and (5.7), we thus have that

$$M^{(j)}_\alpha = \sup_{t\in(0,+\infty)^d} \frac{t^\alpha}{\exp \omega_{\mathbf{M}^{(j)}}(t)} \leq C \sup_{t\in(0,+\infty)^d} \frac{t^\alpha}{\exp \omega_{\mathbf{N}^{(\ell)}}(t/\ell)}$$
$$= C \sup_{s\in(0,+\infty)^d} \frac{(s\ell)^\alpha}{\exp \omega_{\mathbf{N}^{(\ell)}}(s)} \leq C\ell^{|\alpha|} N^{(\ell)}_\alpha.$$

We have thus proved that

$$\forall j \in \mathbb{N}, j \geq \bar{h}, \exists \ell \in \mathbb{N}, C \geq 1: \quad M^{(j)}_\alpha \leq C\ell^{|\alpha|} N^{(\ell)}_\alpha \quad \forall \alpha \in \mathbb{N}_0^d.$$

Since $M^{(j)}_\alpha \leq M^{(\bar{h})}_\alpha$ for $j < \bar{h}$ and the sequences in the weight matrices are normalized, we have proved that $\mathcal{M}\{\preceq\}\mathcal{N}$.

(ii): As in the previous point, it is enough to prove (b) ⇒ (c). Assuming

$$\Lambda_{\{\mathcal{M}\}} \simeq \mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d) \simeq \Lambda_{(\mathcal{N})}$$

we have to prove that $\mathcal{M} \lhd \mathcal{N}$.

We choose $\mathbf{c} = (c_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbb{C}^{\mathbb{N}^d}$ as in Equation (6.13) so that $\mathbf{c} \in \Lambda_{\{\mathcal{M}\}} \subseteq \Lambda_{(\mathcal{N})}(\mathbb{R}^d)$, that is,

$$\forall j \in \mathbb{N}, j \geq \bar{h}, \forall \ell \in \mathbb{N}\, \exists C \geq 1: \ e^{-\omega_{\mathbf{M}^{(j)}}(\bar{\alpha}^{1/2}/j)} = |c_\alpha| \leq C e^{-\omega_{\mathbf{N}^{(1/\ell)}}(\alpha^{1/2}\ell)} \ \forall \alpha \in \mathbb{N}_0^d. \tag{6.20}$$

As in the case (i), for any $t \in (0,+\infty)^d$ we can choose $\alpha \in \mathbb{N}_0^d$ with $\alpha^{1/2} < t \leq (\alpha+\mathbf{1})^{1/2}$ so that from Equation (6.20):

$$\begin{aligned}\omega_{\mathbf{N}^{(1/\ell)}}(t\ell) \leq\ & \omega_{\mathbf{N}^{(1/\ell)}}((\alpha+\mathbf{1})^{1/2}\ell) \leq \log C + \omega_{\mathbf{M}^{(j)}}(\overline{(\alpha+\mathbf{1})}^{1/2}/j)\\ \leq\ & \log C + \omega_{\mathbf{M}^{(j)}}(\alpha^{1/2}) \leq \log C + \omega_{\mathbf{M}^{(j)}}(t).\end{aligned}$$

It follows that for all $j \in \mathbb{N}$, $j \geq \bar h$, and for all $\ell \in \mathbb{N}$ there exists $C \geq 1$ such that

$$\begin{aligned}M_\alpha^{(j)} &= \sup_{t\in(0,+\infty)^d} \frac{t^\alpha}{\exp\omega_{\mathbf{M}^{(j)}}(t)} \leq C \sup_{t\in(0,+\infty)^d} \frac{t^\alpha}{\exp\omega_{\mathbf{N}^{(1/\ell)}}(t\ell)}\\ &= C \sup_{s\in(0,+\infty)^d} \frac{(s/\ell)^\alpha}{\exp\omega_{\mathbf{N}^{(1/\ell)}}(s)} \leq C\frac{1}{\ell^{|\alpha|}}N_\alpha^{(1/\ell)},\end{aligned}$$

because of Equations (5.6) and (5.7). Since $M_\alpha^{(j)} \leq M_\alpha^{(\bar h)}$ for $j < \bar h$ we finally obtain

$$\forall j \in \mathbb{N}\,\forall \ell \in \mathbb{N}\,\exists C \geq 1: \quad M_\alpha^{(j)} \leq C\frac{1}{\ell^{|\alpha|}}N_\alpha^{(1/\ell)} \quad \forall \alpha \in \mathbb{N}_0^d. \tag{6.21}$$

Now, it is obvious that Equation (6.21) implies condition $\mathcal{M} \lhd \mathcal{N}$. In fact, it is enough to take $\ell \geq \max\{1/\kappa, 1/h\}$ for any given $\kappa, h > 0$ in the definition of $\mathcal{M} \lhd \mathcal{N}$.

(iii): We start by proving that (a) ⇒ (c). Assuming

$$\Lambda_{(\mathcal{M})} \simeq \mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d) \simeq \Lambda_{(\mathcal{N})}$$

we have to prove that $\mathcal{M}(\preceq)\mathcal{N}$. By the continuity of the inclusion and Equation (6.11) we have

$$\begin{aligned}&\forall \ell \in \mathbb{N}\,\exists h \in \mathbb{N}, C \geq 1 \text{ s.t. } \forall \mathbf{c} \in \Lambda_{(\mathcal{M})}\\ &\sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{N}^{(1/\ell)}}(\ell\alpha^{1/2})} \leq C \sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{M}^{(1/h)}}(h\alpha^{1/2})}.\end{aligned} \tag{6.22}$$

Let us now fix $\alpha \in \mathbb{N}_0^d$ and assume as before, without loss of generality, that $(\alpha_1,\dots,\alpha_d) = (1,\dots,1,\alpha_{s+1},\dots,\alpha_d)$, for some $s \geq 0$, with $\alpha_k \neq 1$ for $s+1 \leq k \leq d$. Setting $\bar\alpha$ as in Equation (6.14), by Equation (6.6) there exist $j \in \mathbb{N}$, $j \geq h$, and $A \geq 1$ such that

$$\begin{aligned}\omega_{\mathbf{M}^{(1/h)}}(h\alpha^{1/2}) &= \sup_{\beta\in\mathbb{N}_0^d} \log \frac{h^{\beta_1+\dots+\beta_d}\alpha_1^{\beta_1/2}\cdots\alpha_d^{\beta_d/2}}{M_\beta^{(1/h)}}\\ &\leq \sup_{\beta\in\mathbb{N}_0^d} \log \frac{h^{\beta_1+\dots+\beta_d}\cdot 1\cdot \alpha_{s+1}^{\beta_{s+1}/2}\cdots\alpha_d^{\beta_d/2}}{(1/A)^{\beta_1+\dots+\beta_d} M^{(1/j)}_{(\beta_1,\dots,\beta_s,0,\dots0)} M^{(1/j)}_{(0,\dots,0,\beta_{s+1},\dots,\beta_d)}}\\ &\leq \sup_{\beta\in\mathbb{N}_0^d} \log \frac{(Ah)^{\beta_1+\dots+\beta_s}}{M^{(1/j)}_{(\beta_1,\dots,\beta_s,0,\dots0)}}\\ &\quad + \sup_{\beta\in\mathbb{N}_0^d} \log \frac{(Ah)^{\beta_{s+1}+\dots+\beta_d}\alpha_{s+1}^{\beta_{s+1}/2}\cdots\alpha_d^{\beta_d/2}}{M^{(1/j)}_{(0,\dots,0,\beta_{s+1},\dots,\beta_d)}}\\ &\leq \omega_{\bar{\mathbf{M}}^{(1/j)}}(Ah) + \omega_{\mathbf{M}^{(1/j)}}(j\bar\alpha^{1/2})\end{aligned} \tag{6.23}$$

for $\{\bar{M}^{(1/j)}_\beta\}_{\beta\in\mathbb{N}_0^s}$ defined by $\bar{M}^{(1/j)}_\beta := M^{(1/j)}_{(\beta_1,\dots,\beta_s,0,\dots,0)}$, where we have chosen $j \geq Ah$, taking into account that the associate function is increasing on $(0,+\infty)^d$. Note also that if $j \geq j_0$ then $\mathbf{M}^{(1/j)} \leq \mathbf{M}^{(1/j_0)}$ and hence $\omega_{\mathbf{M}^{(1/j)}} \geq \omega_{\mathbf{M}^{(1/j_0)}}$. Since $\omega_{\bar{\mathbf{M}}^{(1/j)}}(Ah)$ is a new constant depending on ℓ (A and j depend on h that depends on ℓ), substituting Equation (6.23) into Equation (6.22) we have that

$$\forall \ell \in \mathbb{N}\, \exists j \in \mathbb{N}, C \geq 1 \text{ s.t. } \forall \mathbf{c} \in \Lambda_{(\mathcal{M})}$$
$$\sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{N}^{(1/\ell)}}(\ell\alpha^{1/2})} \leq C \sup_{\alpha\in\mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{M}^{(1/j)}}(j\bar{\alpha}^{1/2})}. \tag{6.24}$$

For $\beta \in \mathbb{N}_0^d$ we now consider $\mathbf{c}^\beta = (c^\beta_\alpha)_{\alpha\in\mathbb{N}_0^d}$ defined by

$$c^\beta_\alpha := \delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Clearly, $\mathbf{c}^\beta \in \Lambda_{(\mathcal{M})}$ since

$$\sup_{\alpha\in\mathbb{N}_0^d} |c^\beta_\alpha| e^{\omega_{\mathbf{M}^{(1/h)}}(h\alpha^{1/2})} = e^{\omega_{\mathbf{M}^{(1/h)}}(h\beta^{1/2})} < +\infty$$

for every $\beta \in \mathbb{N}_0^d$ and $h \in \mathbb{N}$, by assumption (6.4) which implies that $\mathbf{M}^{(1/h)}$ satisfies Equation (5.1) and hence $\omega_{\mathbf{M}^{(1/h)}}(t) < +\infty$ for all $t \in \mathbb{R}^d$.

We apply Equation (6.24) to the sequences $\mathbf{c}^\beta$ and get that for all $\ell \in \mathbb{N}$ there exist $j \in \mathbb{N}$ and $C \geq 1$ such that for all $\beta \in \mathbb{N}_0^d$ we have

$$e^{\omega_{\mathbf{N}^{(1/\ell)}}(\ell\beta^{1/2})} \leq C e^{\omega_{\mathbf{M}^{(1/j)}}(j\bar{\beta}^{1/2})}$$

or equivalently

$$\omega_{\mathbf{N}^{(1/\ell)}}(\ell\beta^{1/2}) \leq \log C + \omega_{\mathbf{M}^{(1/j)}}(j\bar{\beta}^{1/2}). \tag{6.25}$$

If $t \in (0,+\infty)^d$ then there exist $\beta \in \mathbb{N}_0^d$ with $\beta^{1/2} < t \leq (\beta + \mathbf{1})^{1/2}$ so that, from Equation (6.25):

$$\begin{aligned} \omega_{\mathbf{N}^{(1/\ell)}}(\ell t) \leq {} & \omega_{\mathbf{N}^{(1/\ell)}}(\ell(\beta+\mathbf{1})^{1/2}) \leq \log C + \omega_{\mathbf{M}^{(1/j)}}(j(\overline{\beta+\mathbf{1}})^{1/2}) \\ \leq {} & \log C + \omega_{\mathbf{M}^{(1/j)}}(2j\beta^{1/2}) \leq \log C + \omega_{\mathbf{M}^{(1/j)}}(2jt) \end{aligned}$$

since the associate function is increasing on $(0,+\infty)^d$ and $(\overline{\beta+\mathbf{1}})^{1/2} \leq 2\beta^{1/2}$.

It finally follows from Equations (5.6) and (5.7) that for all $\ell \in \mathbb{N}$ there exist $j \in \mathbb{N}$ and $C \geq 1$ such that

$$\begin{aligned} N^{(1/\ell)}_\alpha \geq {} & \sup_{t\in(0,+\infty)^d} \frac{t^\alpha}{\exp \omega_{\mathbf{N}^{(1/\ell)}}(t)} = \sup_{s\in(0,+\infty)^d} \frac{(s\ell)^\alpha}{\exp \omega_{\mathbf{N}^{(1/\ell)}}(s\ell)} \\ \geq {} & \frac{\ell^{|\alpha|}}{C} \sup_{s\in(0,+\infty)^d} \frac{s^\alpha}{\exp \omega_{\mathbf{M}^{(1/j)}}(2js)} = \frac{\ell^{|\alpha|}}{C} \sup_{t\in(0,+\infty)^d} \frac{\left(\frac{t}{2j}\right)^\alpha}{\exp \omega_{\mathbf{M}^{(1/j)}}(t)} \\ = {} & \frac{1}{C}\left(\frac{\ell}{2j}\right)^{|\alpha|} M^{(1/j)}_\alpha \end{aligned}$$

and we have thus proved that $\mathcal{M}(\preceq)\mathcal{N}$, since the sequences $\mathbf{M}^{(1/j)}$ and $\mathbf{N}^{(1/\ell)}$ are normalized.

Finally, the implication $(b) \Rightarrow (a)$ in (iii) follows from the closed graph theorem by De Wilde [9]; we also refer to [13, Prop. 4.5/Rem. 4.6]. □

Remark 6.2. In Theorem 6.1, we used Theorem 5.2 to characterize the inclusion relations of the spaces for any dimension d, which was not possible in the analogous results [3, Theorems 4.4–4.6], where we needed $d = 1$ in one implication since formula (5.7) was not available for the general anisotropic case.

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The authors declare no conflicts of interest.

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The paper has no associated data.

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