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$\mathbb{Z}/r\mathbb{Z}$ -EQUIVARIANT COVERS OF \mathbb{P}^1 WITH MOVING RAMIFICATION

CARL LIAN AND RICCARDO MOSCHETTI

ABSTRACT. Let $X \rightarrow \mathbb{P}^1$ be a general cyclic cover. We give a simple formula for the number of equivariant meromorphic functions on X subject to ramification conditions at variable points. This generalizes and gives a new proof of a recent result of the second author and Pirola on *hyperelliptic odd covers*.

1. INTRODUCTION

Let \mathcal{H} be a moduli space of branched covers of \mathbb{P}^1 , and let $\overline{\mathcal{H}}$ be its compactification by Harris-Mumford admissible covers, see [HM82]. Let $\phi : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the map remembering the source of a branched cover, possibly with additional marked points (e.g. ramification points).

Pushing forward the fundamental class by ϕ yields a cycle class on $\overline{\mathcal{M}}_{g,n}$, which, by a theorem of Faber-Pandharipande, lies in the tautological ring, see [FP05]. While an algorithm is given to compute this class, it is in general quite impractical to implement, and few explicit formulas are known.

In the case where $\dim(\mathcal{H}) = \dim(\mathcal{M}_{g,n})$, so that ϕ is generically finite, computing this cycle class amounts to computing the degree of ϕ , for which formulas were given (assuming the ramification points lie in distinct fibers) in [L19] using limit linear series and Schubert calculus techniques. By definition, these degrees count maps $h : X \rightarrow \mathbb{P}^1$ out of a fixed curve with ramification conditions imposed at possibly moving (variable) points.

A variant of this problem has been studied in [MP20], in which X is assumed to be hyperelliptic and h is required to be equivariant for the hyperelliptic action. More precisely, it is proven that:

Theorem 1.1. [MP20] *Let X be a general hyperelliptic curve of genus g with Weierstraß points P_1, \dots, P_{2g+2} . Fix integers $n_1, \dots, n_{2g+2} \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{2g+2} n_i = g - 1$. Consider all covers $h : X \rightarrow \mathbb{P}^1$ of degree $4g$ such that:*

- *h is equivariant for the hyperelliptic involution on X and the involution $z \mapsto -z$ on \mathbb{P}^1 ,*
- *$h^{-1}(\infty) = \sum_{i=1}^{2g+2} (2n_i + 1)P_i$,*
- *h is unramified over $0 \in \mathbb{P}^1$, and*
- *all remaining ramification points of h are triple points.*

Then, for any choice of n_i , the number of such covers (called hyperelliptic odd covers) is 2^{2g} .

Here, we have ramification conditions imposed at the *fixed* points P_i , in addition to *moving* (unspecified) triple ramification points away from $0, \infty$. The enumerative count may also be expressed as the degree of the morphism $\phi_{\text{HOC}} : \mathcal{H}_{g,n_i}^{\text{HOC}} \rightarrow \mathcal{H}_g$, where $\mathcal{H}_{g,n_i}^{\text{HOC}}$ is the moduli space of hyperelliptic odd covers for the given choice of integers n_i , \mathcal{H}_g is the stack of hyperelliptic

curves, and ϕ_{HOC} remembers the source curve. The proof expresses such covers as solutions to a certain differential equation and studies these solutions in terms of periods on X .

In this work, we employ a more straightforward technique to give a natural generalization of this problem to arbitrary cyclic covers. Fix a general cyclic cover $\sigma : X \rightarrow \mathbb{P}^1$ of degree r with marked ramification points $P_{i,j} \in X$; the $\mathbb{Z}/r\mathbb{Z}$ -orbits of the marked points are the sets $\{P_{i,j}\}$ where i is fixed and j varies. We will denote by $\mathcal{H}_{g,\mathbb{Z}/r\mathbb{Z},\xi}$ the moduli space parametrizing such σ , where ξ is a collection of elements in $\mathbb{Z}/r\mathbb{Z}$ describing the monodromy over the branch points of σ , see §2.1.

We enumerate $\mathbb{Z}/r\mathbb{Z}$ -equivariant maps $h : X \rightarrow \mathbb{P}^1$, where $\mathbb{Z}/r\mathbb{Z}$ acts on the target by multiplication of a coordinate z by r -th roots of unity. We require h to have two types of ramification points:

- ramification at $P_{i,j}$ over $0, \infty \in \mathbb{P}^1$, with common ramification indices (for fixed i) given by integers $|a_i| > 0$; we take $a_i > 0$ whenever $h(P_{i,j}) = 0$ and $a_i < 0$ otherwise.
- ramification away from $0, \infty \in \mathbb{P}^1$ at unspecified (and unordered) points of X occurring in k orbits of size r , with ramification indices given by the multiset $B = \{b_1, \dots, b_k\}$.

We will denote by $\mathcal{H}_{g,\mathbb{Z}/r\mathbb{Z},\xi,B}$ the moduli space parametrizing such h , see §2.2.

Counting $\mathbb{Z}/r\mathbb{Z}$ -equivariant covers amounts to computing the degrees of the forgetful maps $\phi_B : \mathcal{H}_{g,\mathbb{Z}/r\mathbb{Z},\xi,B} \rightarrow \mathcal{H}_{g,\mathbb{Z}/r\mathbb{Z},\xi}$. Under appropriate conditions detailed in §2, these maps are generically quasi-finite and the spaces are non-empty. Our main theorem is as follows (see also Theorem 2.3):

Theorem 1.2. *Under appropriate numerical constraints ensuring that $\mathcal{H}_{g,\mathbb{Z}/r\mathbb{Z},\xi,B}$ and $\mathcal{H}_{g,\mathbb{Z}/r\mathbb{Z},\xi}$ are non-empty of the same dimension (see (1), (2), (3), (4), (5), and Corollary 2.2), the number of $\mathbb{Z}/r\mathbb{Z}$ -equivariant covers $h : X \rightarrow \mathbb{P}^1$ of degree d with ramification indices $|a_i|$ over $0, \infty$ is:*

$$\binom{(t_0 + t_\infty)/r}{t_0/r} \cdot \prod_{j=1}^k (b_j - 1) \cdot \frac{k!}{\prod_{\ell \in L'} c'_\ell!}$$

where:

- t_0, t_∞ are the numbers of points in $h^{-1}(0), h^{-1}(\infty)$, respectively, that are not among the marked ramification points $P_{i,j}$ of σ , and
- L' is the set of distinct ramification indices b_j occurring away from $0, \infty \in \mathbb{P}^1$, and for $\ell \in L'$, c'_ℓ is the number of orbits of ramification points with ramification index ℓ .¹

Among the required numerical constraints is that the (signed) ramification indices a_i of h above $0, \infty$ satisfy a certain congruence condition in terms of the monodromy of σ , see Corollary 2.2. As long as this condition is satisfied, the final formula does not depend on the a_i ; this is also the case in Theorem 1.1.

Theorem 1.1 is the special case of Theorem 1.2 where $r = 2$, $t_0 = 4g$, $t_\infty = 0$, $k = 2g$, and the b_i are all equal to 3. In particular, this covers also the case of elliptic curves with an odd spin structure in degree four, which has been addressed in [FMNP20] and [L19].

For the proof of Theorem 1.2, the key observation is that the equivariant covers in question may be enumerated after passing to the scheme-theoretic quotients by the $\mathbb{Z}/r\mathbb{Z}$ -actions on the source and target, so we reduce to a question of counting maps $h' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with certain

¹We will see later that it is somewhat more natural to consider the set L of distinct integers among $b_j - 1$.

ramification profiles. We address this question in §3: the moving ramification conditions are imposed by way of Segre and Veronese embeddings, and we conclude our main result by analyzing the intersection that arises.

Lastly, we remark that we expect the method to work for equivariant covers for more general finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$, though we have not pursued this here.

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2. SETUP

In this section, we introduce the various moduli spaces and the relations between them. We work over \mathbb{C} .

2.1. Cyclic covers. Let X be a smooth, projective, and connected curve of genus g , and let $\sigma : X \rightarrow \mathbb{P}^1$ be a cyclic cover of degree r , ramified over $Q_1, \dots, Q_m \in \mathbb{P}^1$. Let $\xi = (\xi_1, \dots, \xi_m) \in (\mathbb{Z}/r\mathbb{Z})^m$ be the tuple of monodromy elements over the Q_i , and let $\mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi}$ be the moduli space of such covers, which is non-empty and smooth of dimension $m - 3$ as long as $\sum_i \xi_i = 0$ in $\mathbb{Z}/r\mathbb{Z}$ and $\mathrm{gcd}(\{\xi_1, \dots, \xi_m, r\}) = 1$, see [SvZ18, Theorem 3.7 and Proposition 3.20]. We will assume throughout that $m \geq 3$, but the results still hold in the case $m = 2$, with minor modifications. Taking $\xi_i \in \{0, 1, 2, \dots, r - 1\}$, we may express X birationally as the curve given by the equation $y^r = (x - \lambda_1)^{\xi_1} \cdots (x - \lambda_m)^{\xi_m}$, and σ as the projection to the x -coordinate.

For each i , let $f_i = \mathrm{gcd}(\xi_i, r)$, and denote by $P_{i,j} \in X$, for $j = 1, 2, \dots, f_i$, the pre-images of Q_i under σ . Let $e_i = \frac{r}{f_i}$ be the common ramification index of the $P_{i,j}$. By Riemann-Hurwitz, we have

$$(1) \quad 2g - 2 = \sum_{i=1}^m f_i(e_i - 1) - 2r = r(m - 2) - \sum_{i=1}^m f_i.$$

We remark that the hyperelliptic locus $\mathcal{H}_g \subset \mathcal{M}_g$ is the image of the map $\mathcal{H}_{g, \mathbb{Z}/2\mathbb{Z}, (1^{2g+2})} \rightarrow \mathcal{M}_g$ remembering the source of the cover, see [SvZ18, §4.4].

2.2. $\mathbb{Z}/r\mathbb{Z}$ -equivariant covers. Let $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the degree r map totally ramified over $0, \infty$. Consider degree d covers $h : X \rightarrow \mathbb{P}^1$ equivariant under the $\mathbb{Z}/r\mathbb{Z}$ -actions induced by σ and g , so that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & \mathbb{P}^1 \\ \downarrow h & & \downarrow h' \\ \mathbb{P}^1 & \xrightarrow{g} & \mathbb{P}^1 \end{array}$$

where $h' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ also has degree d , and $\sigma : X \rightarrow \mathbb{P}^1$ is a cyclic cover of degree r as defined in the previous section.

For each i , the ramification points $P_{i,j}$ of σ must all map to either 0 or ∞ under h ; let S_0 be the set of i such that $h(P_{i,j}) = 0$ and similarly define S_∞ . For $i \in S_0$, let $a_i > 0$ be the

common ramification index of the $P_{i,j}$ over 0, and for $i \in S_\infty$, let $-a_i > 0$ be the ramification index of $P_{i,j}$ over ∞ .

Suppose in addition that there are t_0 points not among the $P_{i,j}$ mapping to 0 under h and similarly define t_∞ . We assume that all such points are unramified. The sets of pre-images of $0, \infty$ outside of the $P_{i,j}$ are disjoint unions of free $\mathbb{Z}/r\mathbb{Z}$ -orbits, so in particular $r|t_0, t_\infty$. Note that we must have

$$(2) \quad d = t_0 + \sum_{i \in S_0} a_i f_i = t_\infty + \sum_{i \in S_\infty} (-a_i f_i).$$

Suppose now that h has $K = kr$ additional points of ramification, which decompose into k orbits of size r , and that no two such points appear in the same fiber of h . Let $B = \{b_1, \dots, b_k\}$ denote the (unordered) multiset of ramification indices (greater than 1) that appear, and let c_ℓ be the number of the b_1, \dots, b_k equal to $\ell + 1$. Let L be the set of ℓ for which $c_\ell \neq 0$.

Let $\mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi, B}$ be the space of $\mathbb{Z}/r\mathbb{Z}$ -equivariant covers as described above; we suppress the data of the a_i and the values of t_0, t_∞ for notational convenience. We have a quasi-finite and étale map $\delta_B : \mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi, B} \rightarrow \mathcal{P}_{k,r}$, where $\mathcal{P}_{k,r}$ is the smooth variety parametrizing unordered collections of k distinct $\mathbb{Z}/r\mathbb{Z}$ orbits on \mathbb{P}^1 , up to the \mathbb{C}^* action preserving $0, \infty$. In particular, $\mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi, B}$ is smooth of dimension $k - 1$. We are interested in computing degrees of the forgetful maps $\phi_B : \mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi, B} \rightarrow \mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi}$; in order for ϕ_B to expect a non-zero answer, we require

$$(3) \quad k = m - 2.$$

Set $t = t_0 + t_\infty$. By Riemann-Hurwitz, we have

$$(4) \quad 2g + 2d - 2 = \sum_{i=1}^m (|a_i| - 1) f_i + \sum_{j=1}^k r(b_j - 1).$$

Substituting $2d = t + \sum_{i=1}^m |a_i| f_i$ and $2g - 2 = r(m - 2) - \sum_{i=1}^m f_i$, and $k = m - 2$, we find that

$$(5) \quad t = r \sum_{j=1}^k (b_j - 2).$$

2.3. Congruence condition on a_i . In this section, we show that in order for equivariant covers h to exist, the vanishing orders a_i need to satisfy a certain congruence condition. We give two perspectives, one algebraic and one topological.

Consider the map h' . The fiber over 0 consists of the points Q_i previously defined for $i \in S_0$, with ramification index a_i , and t_0/r points of ramification index r each. Similarly, the fiber over ∞ consists of Q_i for $i \in S_\infty$ with ramification index $-a_i$ and t_∞/r points of ramification index r each. Moreover, it is ramified at k additional (unlabelled) points with ramification index b_j .

We may therefore represent the map h' by the meromorphic function

$$h'(x) = \prod_{i=1}^m (x - \lambda_i)^{a_i} \cdot \left(\frac{P_0(x)}{P_\infty(x)} \right)^r$$

where P_0, P_∞ are non-zero polynomials of degree $t_0/r, t_\infty/r$, respectively. We also allow one of P_0, P_∞ to have strictly smaller degree, which occurs when h' has a pole or zero at $x = \infty$.

Lemma 2.1. *h' lifts to a $\mathbb{Z}/r\mathbb{Z}$ -equivariant cover $h : X \rightarrow \mathbb{P}^1$ if and only if $a_i \equiv \xi_i \pmod{r}$ for all i . If this is the case, the lift is unique, up to composition with the $\mathbb{Z}/r\mathbb{Z}$ -action on the target.*

Proof. In order to lift such an h' , the meromorphic function h must send a general point (x, y) on X to an r -th root of $h'(x)$, that is, such an r -th root must exist in the function field $\mathbb{C}(X) = \mathbb{C}(x)[\sqrt[r]{\prod_{i=1}^m (x - \lambda_i)^{\xi_i}}]$ of X . The field extension $\mathbb{C}(X)/\mathbb{C}(x)$ is Galois with Galois group $G \cong \mathbb{Z}/r\mathbb{Z}$, corresponding to the cyclic cover σ . Write $f = (x - \lambda_i)^{\xi_i}$ and let $\tau \in G$ be a generator such that

$$\tau(\sqrt[r]{f}) = \omega \cdot \sqrt[r]{f},$$

for some primitive root of unity $\omega \in \mathbb{C}$, so that

$$\tau((\sqrt[r]{f})^a) = \omega^a \cdot (\sqrt[r]{f})^a,$$

for $a = 0, 1, \dots, r - 1$.

Suppose now that $h'(x)$ has an r -th root $\alpha \in \mathbb{C}(X)$, so that $\alpha^r \in \mathbb{C}(x)$. Then, $\tau(\alpha)^r = \tau(\alpha^r) \in \mathbb{C}(x)$, so $\tau(\alpha)^r = \alpha^r$. Thus, $\tau(\alpha) = \omega \cdot \alpha$ for some r -th root of unity ω . This immediately implies that α is of the form $f_0 \cdot (\sqrt[r]{f})^a$ for some $f_0 \in \mathbb{C}(x)$, so $h' = f_0^r \cdot f^a$. In particular, modulo r , the m -tuple (a_i) of exponents appearing in h' on the factors $(x - \lambda_i)$ must be a multiple of the m -tuple (ξ_i) of exponents appearing in f .

Now, suppose that $a_i \equiv a\xi_i \pmod{r}$ for all i . Then, up to multiplication by an r -th root of unity (corresponding to the cyclic action on \mathbb{P}^1), we need $h(x, y) = y^a \cdot \prod_{i=1}^m (x - \lambda_i)^{(a_i - a\xi_i)/r} \cdot \left(\frac{P_0(x)}{P_\infty(x)}\right)$. However, in order for such an h to be $\mathbb{Z}/r\mathbb{Z}$ -equivariant, we need $a \equiv 1 \pmod{r}$. \square

We may also see the necessary condition $a_i \equiv \xi_i \pmod{r}$ from the point of view of monodromy as follows. Suppose first that $a_i > 0$, and let γ_z be a small loop with basepoint $z_0 \in \mathbb{P}^1$ winding a_i times counterclockwise around 0; the orientation of \mathbb{P}^1 is given by the complex structure. Then, given a basepoint $y_0 \in \mathbb{P}^1$ such that $h'(y_0) = z_0$, γ_z lifts under h' uniquely to a loop γ_y winding once around λ_i . Next, given a basepoint $x_0 \in X$ such that $\sigma(x_0) = y_0$, γ_y lifts uniquely to a path in X travelling from x_0 to $\tau^{\xi_i} \cdot x_0$, where $\tau \in \mathbb{Z}/r\mathbb{Z}$ is the generator acting on $T_{x_0}X$ by $e^{2\pi i/r}$.

On the other hand, we may also lift γ_z to X via the maps g and h : letting $y'_0 = h(x_0)$, we find that γ_z lifts under g uniquely to a path γ'_y in \mathbb{P}^1 travelling from y'_0 to $\tau^{a_i} \cdot y'_0$. Now, we must have $h(\tau^{\xi_i} \cdot x_0) = \tau^{a_i} \cdot y'_0$, from which we conclude $a_i \equiv \xi_i \pmod{r}$ from the equivariance of h .

If $a_i < 0$ instead, we may follow the same argument, except now γ_z will be a path with winding number $-a_i$ around ∞ , and the monodromy of g around ∞ is the opposite of that around 0.

We have therefore shown that:

Corollary 2.2. *The space of equivariant covers $\mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi, B}$ is empty unless $a_i \equiv \xi_i \pmod{r}$ for all i .*

Henceforth we assume this congruence condition on the a_i .

2.4. Reduction to h' . The main idea which allows us to count the number of $\mathbb{Z}/r\mathbb{Z}$ -equivariant covers $h : X \rightarrow \mathbb{P}^1$, is that we can count maps $h' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $h' \circ \sigma = g \circ h$ instead. In this section we formulate this statement in terms of moduli spaces.

Let \mathcal{H}_{rat} be the moduli space of covers $h' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with ramification profiles dictated by h upon passing to the scheme-theoretic $\mathbb{Z}/r\mathbb{Z}$ -quotients. That is, the zeroes and poles of h' are given by the λ_i and the roots of P_0, P_∞ and have orders exactly $|a_i|, r$, respectively, and h' is branched over k additional *distinct* points with orders governed by B .

The source \mathbb{P}^1 has a fixed choice of coordinate x , whereas the target \mathbb{P}^1 has fixed points $0, \infty$, but the functions h' are taken only up to a constant factor. That is, two covers are considered isomorphic if they agree up to scaling on the target.

Now, we have a quasi-finite and étale morphism $\delta_{\text{rat}} : \mathcal{H}_{\text{rat}} \rightarrow \mathcal{P}_k$, where \mathcal{P}_k is the space of collections of k unordered branch points on the target, up to scaling. Thus, \mathcal{H}_{rat} is smooth of dimension $k - 1 = m - 3$.

On the other hand, we have a morphism $\phi_{\text{rat}} : \mathcal{H}_{\text{rat}} \rightarrow (\mathbb{P}^1)^{m-3}$ remembering the λ_i for $i \geq 4$. The source and target have the same dimension, and by generic smoothness, the general fiber of ϕ_{rat} consists of a finite number of reduced points equal to the degree of ϕ_{rat} . The above discussion shows that $\deg(\phi_{\text{rat}}) = \deg(\phi_B)$.

2.5. Main result. We now restate the main theorem.

Theorem 2.3. *Assume the numerical conditions (1), (2), (3), (4), (5), as well as the congruence condition given in Corollary 2.2, so that the map $\phi_B : \mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi, B} \rightarrow \mathcal{H}_{g, \mathbb{Z}/r\mathbb{Z}, \xi}$ has non-empty source and target of the same dimension. Then, the degree of ϕ_B is*

$$\binom{t/r}{t_0/r} \cdot \prod_{j=1}^k (b_j - 1) \cdot \frac{k!}{\prod_{\ell \in L} c_\ell!}.$$

In the case of hyperelliptic odd covers for which all Weierstraß points live over ∞ (with fixed ramification indices a_i), we have $r = 2$, $m = 2g + 2$, $\xi_i \equiv a_i \equiv 1 \pmod{2}$ for all i , $d = t = t_0 = 4g$, $k = 2g$, and $b_j = 3$ for all j . Thus, $L = \{2\}$ and $c_2 = 2g$, so we find that $\deg(\phi) = 2^{2g}$, recovering the main result of [MP20].

Suppose instead that r is arbitrary and $B = \{2, \dots, 2\}$, that is, the moving ramification conditions are generic (and thus vacuous), we must have $t = 0$ by (5). Then, in the notation of §2.3, we must have simply $h'(x) = \prod_{i=1}^m (x - \lambda_i)^{a_i}$, and indeed Theorem 2.3 gives $\deg(\phi_B) = 1$.

2.6. Theta characteristics. We comment here on one additional aspect of the case $r = 2$ which plays an important role in the computation of [MP20], namely that of the theta characteristic associated with a hyperelliptic odd cover, and how the situation changes in our more general setting.

Recall that if $h : X \rightarrow \mathbb{P}^1$ is any cover for which all ramification indices are odd, one may naturally associate to h a theta characteristic on X , that is, a square root of the canonical bundle (see [ACGH85, Appendix B] and [Mum71] for the basic theory of theta characteristics). Namely, we have

$$\omega_X \cong h^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_X(2R).$$

where $2R$ denotes the ramification divisor of h , which by assumption is a square. Thus, the divisor

$$\Theta_h := h^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_X(R)$$

is a theta characteristic on X .

Now, suppose X is hyperelliptic and h is equivariant for the hyperelliptic action, that is, h is a point of $\mathcal{H}_{g, \mathbb{Z}/2\mathbb{Z}, (1^{2g+2}), B}$. Assume further that the $|a_i|$ are all odd and that all b_j are equal to 3. This situation generalizes that of [MP20] in that Weierstraß points are allowed over 0.

In this case, we can write

$$\begin{aligned} h^{-1}(0) &= \sum_{i \in S_0} (2n_i + 1)P_i + \frac{t_0}{2}H, \\ h^{-1}(\infty) &= \sum_{i \in S_\infty} (2n_i + 1)P_i + \frac{t_\infty}{2}H \end{aligned}$$

where H is the hyperelliptic divisor on X .

If we denote by R' the ramification divisor outside $h^{-1}(0)$ and $h^{-1}(\infty)$, we have $\deg(R') = 4k$, where as before, k is the number of $\mathbb{Z}/r\mathbb{Z}$ -orbits of branch points of outside 0 and ∞ . In particular, $R' = 2kH$. Thus, we find, up to linear equivalence,

$$\begin{aligned} \Theta_h &= kH + \sum_{i \in S_0, S_\infty} n_i P_i - \sum_{i \in S_0} (2n_i + 1)P_i - \frac{t_0}{2}H \\ &= \left(k - \frac{t_0}{2}\right)H + \sum_{i \in S_\infty} n_i P_i - \sum_{i \in S_0} (n_i + 1)P_i. \end{aligned}$$

In the covers of [MP20], we have $k = 2g$, $t_0 = 4g$, $t_\infty = 0$, and $S_0 = \emptyset$, so in fact Θ_h is *effective*. In our more general situation in which S_0 may be non-empty, note that even if we retain the conditions $(t_0, t_\infty) = (4g, 0)$ and $k = 2g$, any (possibly non-effective) theta characteristic of X arises as above. Indeed, if

$$\Theta = \sum_{i=1}^{2g+2} c_i P_i,$$

then we may take S_∞ to be the set of i such that $c_i \geq 0$, and $n_i = c_i$ for $i \in S_\infty$ and $n_i = -c_i - 1$ for $i \in S_0$.

3. COUNTING MEROMORPHIC FUNCTIONS ON \mathbb{P}^1

In this section, we carry out the proof of Theorem 2.3. We have reduced to counting the possible meromorphic functions

$$h'(x) = \prod_{i=1}^m (x - \lambda_i)^{a_i} \cdot \left(\frac{P_0(x)}{P_\infty(x)} \right)^r,$$

where we impose the condition that, away from the λ_i and the roots of P_0, P_∞ , the cover h' has ramification indices (at distinct branch points) given by the multi-set B . We parametrize choices of P_0, P_∞ by a product of projective spaces, and impose the ramification conditions from B in terms of a combination of Segre and Veronese embeddings.

3.1. The map ψ . Let $\mathbb{P}^{t_0/r}, \mathbb{P}^{t_\infty/r}$ be the projective spaces parametrizing non-zero polynomials P_0, P_∞ of degrees (at most) $t_0/r, t_\infty/r$, up to scaling, and let $U \subset \mathbb{P}^{t_0/r} \times \mathbb{P}^{t_\infty/r}$ be the open locus consisting of polynomials P_0, P_∞ satisfying the following properties:

- P_0, P_∞ have no roots in common,

- P_0, P_∞ do not have roots at any of the λ_i ,
- P_0, P_∞ have no double roots.

Here, we regard $x = \infty$ as a possible root of P_0, P_∞ . Thus, the statement that P_0, P_∞ have no roots in common includes the statement that at most one of them has zero leading coefficient, and the statement that they have no double roots includes the statement that the first two coefficients of either polynomial do not both vanish.

If $(P_0, P_\infty) \in U$, the ramification divisor for h' away from the λ_i and the roots of P_0, P_∞ is given by the roots of the polynomial

$$\begin{aligned} \psi_{h'}(x) &:= \frac{1}{\prod_{i=1}^m (x - \lambda_i)^{a_i - 1}} \cdot \frac{P_\infty^{r+1}}{P_0^{r-1}} \cdot \frac{d}{dx} h'(x) \\ &= \sum_{i=1}^m a_i \cdot \frac{\prod_{i=1}^m (x - \lambda_i)}{x - \lambda_i} \cdot P_0 \cdot P_\infty + r \prod_{i=1}^m (x - \lambda_i) \cdot (P_0' P_\infty - P_0 P_\infty') \end{aligned}$$

The degree of $\psi_{h'}(x)$ is $b = \sum_j (b_j - 1)$ (note that the leading terms of the two summands cancel), unless ∞ is among the additional ramification points of h' , in which the ramification index at ∞ is $b - \deg(\psi_{h'}) + 1$.

Note that the definition of $\psi_{h'}(x)$ makes sense even for $(P_0, P_\infty) \notin U$, and that the roots of $\psi_{h'}(x)$ away from λ_i and the roots of P_0, P_∞ , taken with multiplicity, still control the ramification of h' . In this case, $\psi_{h'}(x)$ will in general have extraneous roots among λ_i and the roots of P_0, P_∞ .

We have always that $\psi_{h'}(x)$ is not identically zero, or else h' is constant, which is impossible even for $(P_0, P_\infty) \in U$ because the a_i are not divisible by r . Therefore, we may make the following definition.

Definition 3.1. We define $\psi : \mathbb{P}^{t_0/r} \times \mathbb{P}^{t_\infty/r} \rightarrow \mathbb{P}^b$ by $\psi((P_0, P_\infty)) = \psi_{h'}(x)$.

The map ψ is the composition of a Segre embedding followed by a linear projection whose base locus does not meet the image of $\mathbb{P}^{t_0/r} \times \mathbb{P}^{t_\infty/r}$, so its degree is equal to that of the Segre embedding, which is $\binom{t/r}{t_0/r}$.

3.2. The map ρ . We want to intersect ψ with the locus of polynomials that arise as a product of the form $\prod_{\ell \in L} R_\ell(x)^\ell$, where R_ℓ has degree c_ℓ . This is the image of the map defined below:

Definition 3.2. We define the map

$$\rho : \prod_{\ell \in L} \mathbb{P}^{c_\ell} \rightarrow \mathbb{P}^b$$

by sending $(R_\ell)_{\ell \in L} \mapsto \prod_{\ell \in L} R_\ell^{c_\ell}$.

The map ρ is the composition of the ℓ -Veronese embedding on the factor corresponding to $\ell \in L$, followed by a Segre embedding, followed by a linear projection. Because ρ has no base locus, its degree is the degree of the composition of the first two maps, which is

$$\prod_{\ell \in L} \ell^{c_\ell} \cdot \frac{(\sum_{\ell \in L} c_\ell)!}{\prod_{\ell \in L} c_\ell!} = \prod_{j=1}^k (b_j - 1) \cdot \frac{k!}{\prod_{\ell \in L} c_\ell!}.$$

3.3. Intersection of ψ and ρ . Theorem 2.3 now follows from Bézout's Theorem, assuming that, for λ_i general, the intersections of ψ and ρ correspond exactly to points in the corresponding fiber of \mathcal{H}_{rat} , and that this intersection is transverse. These statements are verified below, completing the proof of the main result.

Proposition 3.3. *Suppose the $\lambda_i \in \mathbb{C}$ are general and consider the intersection of ψ, ρ in \mathbb{P}^b . Then, any cover $h' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by a point of intersection between ψ, ρ is a point of \mathcal{H}_{rat} .*

Proof. Suppose we have $h'(x) = \prod_{i=1}^m (x - \lambda_i)^{a_i} \cdot \left(\frac{P_0(x)}{P_\infty(x)} \right)^r$ defining a point of intersection of ψ, ρ outside of \mathcal{H}_{rat} . We claim that the corresponding map $h' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, possibly of degree less than d , has strictly fewer than k branch points away from $0, \infty$, and thus varies in a family of dimension strictly less than that of \mathcal{H}_{rat} , so is avoided by a general choice of λ_i . (If $k = 1$, then we find that h' may only have two branch points, and hence two ramification points, but we see that this is impossible when $m \geq 3$, as $r \neq a_i$.)

To see this, note first that if h' has k distinct ramification points away from $0, \infty$, then the ramification indices must be given exactly by the elements of B . Now, we claim that any choices of P_0, P_∞ that define a cover h' outside of \mathcal{H}_{rat} increases the total amount of ramification of h' over 0 and ∞ compared to the change in degree, which cannot happen if the ramification away from $0, \infty$ is unchanged.

On one hand, suppose that the degree of h' is d , so that P_0, P_∞ have no roots in common and no roots at the λ_i . Then, the only way h' can fail to define a point of \mathcal{H}_{rat} is if some of the ramification imposed by the λ_i and P_0, P_∞ coalesces, either by allowing P_0, P_∞ to have multiple roots or roots at λ_i . However, this increases the total amount of ramification of h' .

On the other hand, if P_0, P_∞ are chosen to decrease the degree of h' , any such choice will decrease the degree of h' strictly more than the amount of ramification lost. For example, if P_0, P_∞ have a linear factor in common, the degree of h' drops by r , which must decrease the total ramification by $2r$, but the amount of ramification lost over $0, \infty$ is only $2r - 2$. Similarly, if P_0 or P_∞ has a root at λ canceling a factor of $(x - \lambda)$, we get a similar contradiction.

Therefore, if the λ_i are chosen generally, the ramification for all covers in the intersection of ψ, ρ is as expected. \square

Proposition 3.4. *Suppose the $\lambda_i \in \mathbb{C}$ are general. Then, the intersection of ψ, ρ is transverse.*

Proof. Suppose ψ, ρ have a tangent vector. We then obtain a 1-parameter deformation

$$\tilde{h}' = \prod_{i=1}^m (x - \lambda_i)^{a_i} \cdot \left(\frac{P_0 + \epsilon Q_0}{P_\infty + \epsilon Q_\infty} \right)^r$$

of the meromorphic function h' , which yields a 1-parameter deformation as a point of \mathcal{H}_{rat} . By construction, \tilde{h}' keeps the λ_i constant on the target. Because the λ_i are general and ϕ_{rat} is generically unramified, \tilde{h}' must in fact be the trivial deformation on \mathcal{H}_{rat} .

This means that, as meromorphic functions on $\mathbb{P}^1 \times \text{Spec } \mathbb{C}[\epsilon]/\epsilon^2$, we have

$$\prod_{i=1}^m (x - \lambda_i)^{a_i} \cdot \left(\frac{P_0 + \epsilon Q_0}{P_\infty + \epsilon Q_\infty} \right)^r = \mu \prod_{i=1}^m (x - \lambda_i)^{a_i} \cdot \left(\frac{P_0}{P_\infty} \right)^r$$

for a scalar $\mu \in \mathbb{C}[\epsilon]/\epsilon^2$. We find that Q_0, Q_∞ must be scalar multiples of P_0, P_∞ , corresponding to the trivial tangent vector on ψ , and the ramification remains unchanged to first order, corresponding to the trivial tangent vector on ρ . This completes the proof. \square

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