Representations as site of the tension between abstract and concrete in mathematical practice: University students at work with a spirograph

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In this paper, we propose to study representations as site of encounter with the lively tension between abstract and concrete in mathematical practice. In so doing, we question visions of practice as a sequence of moves towards abstraction that see mathematical concepts as fixed and disembodied, detached from the physical, and learning in terms of developmental trajectories. Beyond a tendency to reduce abstract and concrete to a dualism conceiving of more of one as less of the other, we are more interested in what learners do and how this mobilises practice as a flow of activity across abstract and concrete. We offer the case of some university students working with a spirograph to experiment with this approach to mathematical practice.

Keywords: Mathematical practice, abstract, concrete, representation, spirograph.

Introduction.

The relevance of representations has been studied in mathematics education research from different points of view. For example, Meira (2002) has looked at mathematical representations as notations-in-use, pointing out that pure referential views of signs and meanings may not give enough emphasis to notational systems as mediational tools that can trigger and sustain mathematical activity. Other studies have investigated how the production of representations by learners does not only support cognition and communication but also transforms the very way that mathematical activity evolves (e.g., Hall, 1996; Nunes, 1997). In this paper, we are interested in this transformative character of the activity within the mathematics classroom, especially when tools are used in practice and allow learners to create and work with representations of mathematical objects. Tools have the potential to transform the teaching and learning of mathematics (see Hoyles, 2018). The ways in which the bodily and the kinaesthetic are involved in interaction with the tool extend the horizon of possibilities that learners come to entertain (Nemirovsky & Ferrara, 2009), and a growing body of research considers notions of embodiment and perceptuo-motor engagement in mathematics to characterise them (Nemirovsky et al., 2013; de Freitas & Sinclair, 2014). Additionally, mathematical activity can be conceptualised as *involving* and *evolving* concepts, showing the role of body and movement in learning mathematics (de Freitas & Ferrara, 2015). We build on these considerations to investigate two main aspects that we see as central in the study of mathematical practice with tools: on the one side, how tools change mathematical practices as tool use constantly reconfigures the activity; on the other side, the ways that tools always demand the body and mobilise mathematics in unexpected ways. We will argue that representations and diagrams are at the core of both aspects as the site of encounter of abstract and concrete in mathematics. We examine these two aspects through the specific case of doing mathematics with a spirograph. The spirograph was invented by a mathematician but it is usually known as a classic toy for children to create diagrams, beautiful drawings that might resemble mandala or recall images of flowers. The mathematics of the spirograph is quite

sophisticated and related to the number of radially symmetric curves that can be made using the tool (Ippolito, 1999). Among the many types of spirograph available nowadays, we consider a very simple spirograph that consists of a couple of plastic ring gears with teeth on the inside of their circumferences and three different-sized pinion gears (or gearwheels) with holes in various positions, which are used to insert a pen and draw (Figure 1a). The class of curves that can be traced rotating the pinion gears inside the ring gears enters that of hypocycloids (an example is shown in Figure 1b). In this report, we draw attention to the mathematical practice of a class of university students that have used this spirograph to examine the mathematics out of the obtainable diagrams. Rather than attending to practice in terms of what the students *can achieve*, we shift focus to what the students *do*, and how this engenders and renders what *is produced* in practice, mainly attending to diagrammatic activity.

a b

Figure 1: (a) The spirograph and (b) an example of hypocycloid drawn with it

Mathematical practice and the tension between abstract and concrete.

Learning mathematics is often conceptualised as a progression or movement from the concrete to the abstract. This progression amounts to a gradual passage from physical acts and senses to the more fixed and disembodied (ideal) mathematical concepts. In this taken-for-granted idea of school mathematics, the concrete is left behind, often delineating a clear cut between the physical and the conceptual and a directed trajectory for learning. Mathematical practice is thus mainly conceived of as the steps and acts of acquiring concepts and attaining abstract knowledge. Despite their prescriptive habit, traditional images for the attainment of abstraction have been questioned in the literature. More complex pictures have been offered: for example, by troubling the divide of concrete and abstract as properties of entities rather than as relationships between persons and things (e.g., Wilensky, 1991), and by introducing diverse kinds of concrete knowledge (Clements, 2000) or 'hybrid' notions of situated abstraction (Noss et al., 2002) and abstraction in context (Hershkowitz et al., 2001). These studies have in common a tentative discourse on ways in which something can be abstract or concrete for someone in relation to practice, while challenging the idea of whether mathematical abstractions can always be separated from the context of their construction or application. However, they maintain a vision of teaching and learning processes as tending towards the achievement of abstraction. In a similar vein, the nature of expertise has also been a subject of concern, with expert knowledge in mathematics (and science) as commonly described as formal and abstract. Coles and Sinclair (2018) have more recently critiqued assumptions about developmental paths to abstraction and have rather stressed a relational view of abstract and concrete, which challenges what is meaningful in learning mathematics. Whatever approach we take, the tension between abstract and concrete is pervasive in

the doing of mathematics. Following Noss and colleagues (2002), "precisely because mathematical activity is anchored in the artefacts and discourses of the practice, it is recontextualized in ways that sometimes make it difficult to recognize at all" (p. 227). In another work (Nemirovsky et al., 2020), we address the notion of abstraction detaching it from a vision bounded to the dualism abstract-concrete, according to which more of one is simply conceived of as less of the other. Analysing three students at work with a graphing motion sensor, we show how, in the practice with the tool, it is possible to sketch a distinct path for the attainment of abstraction, which involves navigating a surplus of sensible qualities. In this paper, in line with such a non-dualistic perspective on abstraction, we look at the specific work with the representations to shed light onto the ways in which the students (inter)act with mathematical concepts through material activity. In so doing, we want to stress the contextual nature of mathematical practice that emerges from activity within the classroom, instead of reducing it to a sequence of moves toward abstraction. Accordingly, we want to *rethink* mathematical practice as a way of encountering or sustaining the relational dynamics of abstract and concrete in mathematics.

We turn to the French mathematician and philosopher of mathematics G. Châtelet, who has drawn attention to the notion of the virtual as a powerful way of overcoming the so slippery relationship of abstract and concrete and the split of the physical and the mathematical (Châtelet, 1993/2000). Châtelet discusses how the mathematical has a physical aspect to it, by analysing the mathematical practice of scholars in the history of mathematics and physics. The concept of the virtual is that which allows us to reconceive of the mathematical and the physical together, challenging visions that typically associate mathematical thinking to the mind and leave out the body. For Châtelet, the virtual is in the physical world, and its coupling with the actual is at the hearth of the emergence of new mathematical ideas. It pertains to the mobility and indeterminacy at the source of all actions and is mobilised and actualised through activity: the moves, the gestures, the diagrams, and their interplays are those which awake the virtual or potential multiplicities that are implicit in any material surface, and open room for inventiveness in the mathematics classroom (de Freitas & Sinclair, 2014; Sinclair et al., 2013). Briefly speaking, the virtual has to do with all the potential, future alterations of the world, rather than mirroring it as we see it. Borrowing from these ideas, instead of aiming for a clear cut between abstract and concrete in mathematical practice and conceiving of learning as the pursuit of abstraction, we intend to study the *flow*, or movement, of activity across abstract and concrete. We strive to do so in relation to the use of the spirograph, considered as a mathematical instrument (Nemirovsky et al., 2013) that engages kinaesthetically the students' bodies in the creation of mathematical curves. In this context, we look at moves, gestures, diagrams, and their interplays (processes of actualising the virtual) to see how the mathematical activity develops and changes.

Context and method.

The mathematical activity we consider in this paper was carried out in the context of a university mathematics education course attended by a group of 30 university students at the first or second year of their master's degree in mathematics. The activity engaged the class during a regular lesson of the course. The students were divided into groups of 5 people and each group worked for about 2 hours on written tasks to be faced using the spirograph. The tasks aimed at guiding the students to explore the tool and one main idea the students had to play with is that with the spirograph one always draws

closed curves. This fact is explainable through elementary concepts of number theory, which allow to model the functioning of the tool depending on the ratio between the numbers of teeth of the chosen ring-wheel combination. With the spirograph used in our activity it is possible to combine six couples (R, W), where R and W indicate the numbers of teeth in the ring gear and in the gearwheel respectively. The mathematical relationships of Table 1 capture the number of 'petals' or 'tips' of the curve created by a specific combination of gears and the number of turns that the wheel covers along the ring to close the curve (lcm: lower common multiple). In Figure 1b we observe 5 petals, drawn with the couple (105, 63): indeed, lcm (105, 63)/63 = 315/63 = 5. Variations on the choice of the hole for the pen modify other characteristics of the curves, like the petals' curvature. For the sake of space, we do not expand our considerations further, but more mathematics can be explored with the spirograph.

Table 1: Some mathematical relationships at play with the spirograph

(R, W)	Number of petals = lcm (R, W)/W	Number of turns = lcm (R, W)/R

The first author was the teacher of the course, and both the authors designed together the tasks and took part in the activity as active observers. Each group was filmed by one author or a research collaborator with a mobile camera for the entire duration of the activity, and all the written productions of the groups were collected. The video and the written protocols constitute the data source of our study. We adopt a micro-ethnographic method (Streeck & Mehus, 2005) to focus our investigation on the students' interactions with the instrument and the corresponding representations.

For this paper, we focus on the work of one group of five students, who have thoroughly explored most of the mathematical ideas, to exemplify the dynamics of abstract and concrete in their activity with the spirograph and how it arises from the students' productive engagement with the instrument and the emerging representations. We point out two main elements which, we believe, speak directly to the lively tension between abstract and concrete in this context: (1) expectations about the 'future' of the curve and (2) interplay between composition and decomposition of gear motions. In the next sections, we first present two episodes and then briefly discuss them.

First episode.

The students (named S1, S2, S3, S4 and S5; three females and two males, Figure 3d) are seated around a table and have at disposal a spirograph, the written worksheet, and blank sheets of paper. We highlight some moments within the first 5 minutes of interaction. Student S3 reads the given description of the spirograph, and the group focuses on the task of describing *what can* be obtained using the tool. The combination of gears (105, 52), randomly selected and used by one student, S2, produces a first diagram (Figure 2a).

The students explore the physical situation: S2 draws moving the pen quite slowly, while his mates carefully gaze closely at him. In several turns, the dialogue develops as follows:

S1: Hm, but it never closes. [Figure 2a]
S2: Perhaps, we don't know.
S3: Well, come on! Go on then (everybody laughs). [S2 accelerates increasing the rotation speed and, after few seconds, removes the pinion gear, stops from drawing, and changes the combination to create another diagram. The student then produces a new diagram, which instead is completed in few turns, using the couple (96, 36). Figure 2b]

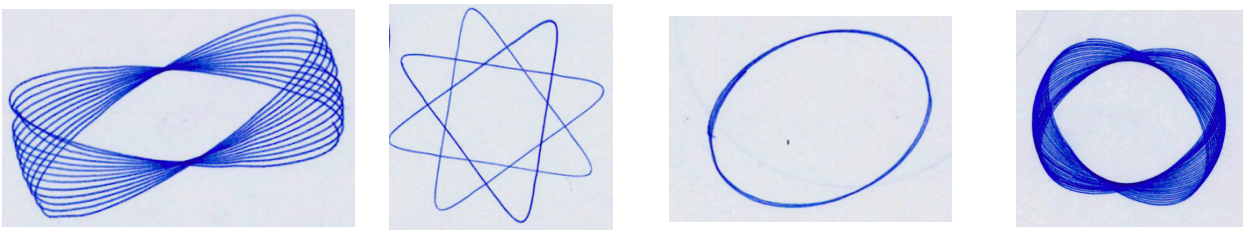
S2: So, it can close.

S3: Try with the most inner [hole], like where the spiral begins, but I don't know (to S2). [S2 returns to the initial combination (105, 52), but changes the hole: after nearly four complete rotations around the ring gear, he stops. Figure 2c]

S3: Always an ellipse. [S2 changes the hole again, maintaining the same couple of gears]. Anyway, a fixed ellipse always occurs. [Several trials are made, then S2 creates a new diagram. Figure 2d]

S1: Ah, no, we had to go on (laughs).

For 2 more minutes, the students draw diagrams exploring the change of the hole while using the same couple of gears. Then, they start to discuss a collective answer for the task.



a b c d

Figure 2: Four diagrams created by the students within the first 5 minutes of work

Second episode.

After 5 more minutes, the students still discuss how to describe what they have created:

S2: It's the orbit of a, a satellite, that is, an epicycle (draws circles repeatedly with the pen) isn't it? That is, it turns around itself, around a…

S3: Yes, after all it's like the orbit of a planet. [The girls focus again on the written, S2 starts drawing with the spirograph]

S4: What were you telling about the orbit (to S2)?

S2: The orbit of a satellite (mimes inverted commas).

S4: This (points to one of the curves on paper)?

S2: All of them. [S3: Yes]. That is, they rotate around a centre and around themselves, more or less (sketches a new diagram. Figure 3a). That is, for me, we could obtain this thing. [S4 laughs]. Yes! That is, this is rotating around itself and around a centre, and it's the orbit, in the sense, hm, of the moon around the sun, here there's the earth (points to the inner circle of Figure 3a, one gear is positioned onto the diagram) and it's this one, and this one rotates around itself (moves the gear from the diagram and makes it rotate around itself).

S3: Try to go very slow (to S2) and let's see what happens […] that is, when you turn, let's go slowly, so that we can see when this [the wheel] has done a turn on itself, and what's been drawn meanwhile. [S2 marks the point where the pinion gear is tangent to the ring gear. He slowly begins to draw and stops the pen when the mark is touching the ring again. Figure 3b]

S2: This is a complete turn on itself (points to the drawn curve segment. Figure 3c).

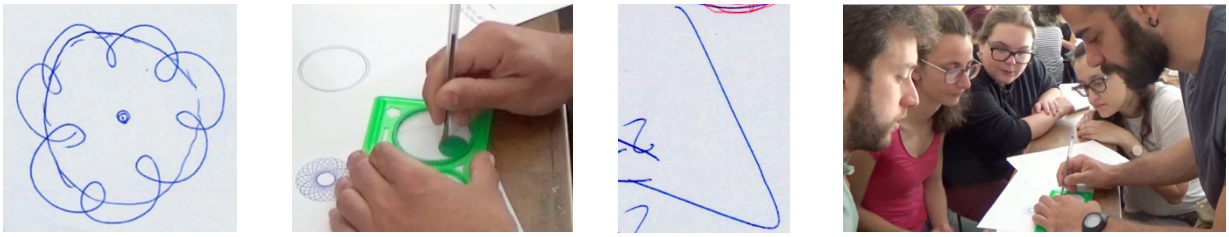
S4: That doesn't give us any information. [S3 and S1 giggle, while agreeing]

S2: But, if we decompose it (moves the gearwheel to the side), if we had a complete turn on itself (makes the gearwheel turning on itself) without composing it with the other motion (mimes a rotation with the pen over the ring gear), we will have a circumference. If this (points to the wheel) turns on itself…

S3-S4: Yes, but we've to consider it must turn around…

S2: Yes, yes (nods), but, I mean, what I was saying is that, well, it turns around itself and does a circle (draws a circle around the gearwheel). Turns around this, a point (points the pen in the middle of the ring gear), well, and it does a circle (draws a

circle inside the ring gear). The composition of these two circles gives you, well, of these two motions (points to both the drawn circles).



a b c d

Figure 3: Four moments of the group work in the second episode

Discussion and conclusion.

In the first episode, the students use the spirograph to draw and discuss their expectations about the behaviour of the curves they are (partially or completely) creating. In the initial moment, right after few turns of the pinion gear, the possibility of a 'never-ending going on' intuitively reflects the possibility of non-closure of the curve in the diagram (a delicate, crucial point of the mathematical modelling of the instrument). The trial with a different combination of gears makes a different possibility emerge, that of a closed curve (Figure 2b). The new diagram embeds the end of the drawing movement and suggests it as a possible future also for the first curve, so that the students return to the initial combination of gears but changing the hole for the pen. The recognition of a seemingly elliptical shape (Figure 2c) and the association of a known curve with the "always" possible curve implicates that the students change the hole again, involving the creation of a new curve which do no longer resemble an ellipse. At this point, the possibility for the first curve to close and for the movement to arrive at an end is brought forth again, even beyond the material act of drawing. The iterative nature of the movement that creates the initial diagram, together with the time needed to close it, suggests a possibility for the movement to be repeated unlimitedly. The need of continuing the movement is therefore mapped onto the possible future(s) of the curve. We see how the tension between abstract and concrete is at play in the diagrams as possibilities of movement or possibilities of mathematical creation are encountered by the students and emerge out of the interaction with different combinations of gears or holes. Activity is reconfigured each time these possibilities are changed. The first episode is full of mathematical expectations that emerge from the physical movement, the combination/coordination of the gears and the material activity with them. This deeply explorative phase is characterised by a flow across *what is there* and *what might be*.

In the second episode, once a description must be produced, a new, more explanatory phase begins, in which the students attempt to make sense of the instrument by composing and decomposing its parts and the drawing movements. This happens by shifting attention to the fact that two movements occur simultaneously, each along a circular trajectory: that of the gearwheel along the ring gear, and that of the gearwheel on itself. The students closely observe this behaviour by moving slowly, so that they can decompose the curve "in the making", relating it to the internal gear's rotations. This is done to better unfold the previous reference by S2 to the planet/satellite dynamics, which pointed out that none of the two movements can be thought without the other. We see how the change in pace gives the students a different feeling for the curve, helping them crystallize how movements partake in the

creation of the diagram. In line with Châtelet's vision of diagrams as site for contractions and expansions to be realised, in this brief excerpt, the diagrammatic activity with the tool unfolds through the ways in which the students use, reimagine, and rethink the tool, the curve and the composed movements. We have tracked three aspects of the work with the representations: the need of slowing down, the interpretation in terms of planets and satellites, the decomposition of parts that cannot be separated. We can see how these aspects speak directly to the lively dynamics of abstract and concrete that make mathematical practice entangled with the materiality of the tool and inseparable from the fleeting perception of movement. They express the expansions and contractions proposed by Châtelet, capturing the ways in which the students navigate the surplus of sensible qualities that arise in the experience of/with the instrument, and by shifting the activity towards the mathematics of the spirograph, they come to constitute mathematics as experience. Mathematical activity is imbued with possible futures and changes inextricably as it is intertwined with the kinaesthetic and the bodily. The diagrams that emerge are not static characters of the activity but generative of doubts, conjectures, and new meanings.

In this paper, we have offered a non-prescriptive and non-static vision of mathematical practice. We attended to practice as a way to encounter or sustain the tension between abstract and concrete and we focused on its contextual nature and what the students do, rather than what they can achieve. In this dynamic vision, diagrams operate as a locus of practicing and the abstract/concrete relationship. Drawing on Châtelet, we can say that the practice of mathematics is not of mathematics that *can* be practiced, but of mathematics that is *really* in the process of being practiced. Knowledge is of course potential in mathematics, but above all it actualizes itself there. We therefore recognize the complex nature of learning and the way that knowledge emerges from practice and material activity. Diagrams, representations, are intensive sites of actualization(s). Further research is necessary to investigate whether this approach to practice can open the ground for studying the tension between abstract and concrete in other contexts.

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